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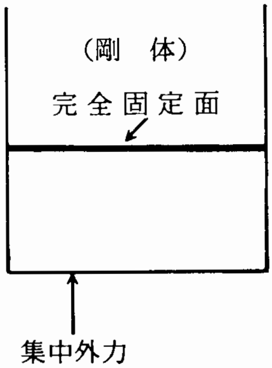
長方形領域での Navier 方程式混合境界値問題の解

新潟歯学部　桜　岡　　充

静弾性学の基礎方程式である Navier 方程式に関しては，解の存在定理に関する仕事や，Boussinesq-Papkovich の解に代表されるように，解の一般形を見出そうとする仕事が数多くなされている。然るに，解析的な形の解はその広汎な有用性にもかかわらず，見出されているのは半無限弾性体とか，球，円板などの，それも特殊な境界条件の場合に限られている。ここでは二次元長方形領域について一辺が完全に固定され，その対辺に集中応力が加わった場合の解析解を Schwarz-Christoffel 変換を用いて求めた結果を報告する。今回は特に，固定面に現れる応力分布を説明することにしたい。これは単純化しすぎたモデルではあるが，この弾性体を齲歯治療に用いられた矩形状の鋳造補綴冠に見立てるならば，歯質とクラウンの接着面に現れる応力分布の目安を与えるものである。



Mixed Boundary Solutions of Two-Dimensional Navier Equation in a Rectangular Region

Abstract

An analytical solution of elastostatic problem in two-dimensional rectangular region is derived in the case of an mixed boundary condition. The stress destribution which arises on the surface, on which the displacement is fixed, is illustrated explicitly.

One of the most powerful approach to two-dimensional elasto-static problem seems the method of complex potential [1) 2)], developed by Kolosov, Muskhelishvili and Stevenson. The essentials of their arguments are as follows;

Let us introduce the complex ζ-plane, $\zeta=\xi+i\eta$, which represents the individual point inside the elastic body, instead of orthogonal coordinates (ξ, η). The stress component $\sigma_{\xi\xi}$, $\sigma_{\xi\eta}$, $\sigma_{\eta\eta}$, and the components u_ξ, u_η of the displacement vector are written as

$$2\mu(u_\xi+iu_\eta)=\kappa\varphi(\zeta)-\zeta\varphi'^*(\zeta^*)-\psi^*(\zeta^*) \quad , \tag{1 a}$$

$$\sigma_{\xi\xi}+\sigma_{\eta\eta}=2[\varphi'(\zeta)+\varphi'^*(\zeta^*)] \quad , \tag{1 b}$$

$$\sigma_{\xi\xi}-\sigma_{\eta\eta}+2i\sigma_{\xi\eta}=-2[\zeta\varphi''^*(\zeta^*)+\psi'^*(\zeta^*)] \quad , \tag{1 c}$$

where φ and ψ are analytic function in the ζ-plane, and μ is shear modulus of the elastic body. κ is given by

$$\kappa=3-4\nu \quad , \tag{2}$$

where ν denotes Poisson's ratio of our elastic body*).

In effect, Eqs. (1) have satisfied the equations of equilibrium or Navier equation, the two-dimensional elastostatic problem is reduced to find analytic functions which satisfy certain condition on the boundaries of the elastic body.

For the application of this representation, Muskhelishelishvili [2)] have yielded the expressions for the stress and displacement in an orthogonal coordinate system $z=x+iy$, which we take to be defined by the conformal transformation $\zeta=\zeta(z)$.

*) In this paper, we write down the formulae for a state of plane strain. They are true for a state of plane stress if, instead of ν, a "modified Poisson's ratio" $\nu'=\nu/(1+\nu)$ is substituted [3)].

That is

$$2\mu(u_x+iu_y)=\left|\frac{d\zeta}{dz}\right|\cdot\left(\frac{d\zeta}{dz}\right)^{-1}\{\kappa\varphi[\zeta(z)]-\zeta(z)\cdot\varphi'^{*}[\zeta(z)^{*}]$$
$$-\psi^{*}[\zeta(z)^{*}]\}\quad, \tag{3 a}$$
$$\sigma_{xx}+\sigma_{yy}=-2\{\Phi[\zeta(z)]+\Phi^{*}[\zeta(z)^{*}]\}\quad, \tag{3 b}$$
$$\sigma_{xx}-\sigma_{yy}-2i\sigma_{xy}=-2\left(\frac{d\zeta}{dz}\right)^{*-1}\cdot\frac{d\zeta}{dz}\cdot\{\zeta^{*}(z^{*})\Phi'[\zeta(z)]+\Psi[\zeta(z)]\}, \tag{3 c}$$

where $\Phi(\zeta)$ and $\Psi(\zeta)$ denote $\varphi'(\zeta)$ and $\psi'(\zeta)$, respectively.

Here, we suppose that the elastic body occupies the half-plane Im $(\zeta)\leqq 0$, i.e., lower half-plane, in which the complex potentials are defined. In this case, Muskhelishvili provided a beautiful method for the analytic continuation of the complex potential function onto the upper half-plane, through the unloaded part of the boundary (real axis). His method leads us to the following forms for the complex potential $\Psi(\zeta)$:

$$\Psi(\zeta)=-\Phi(\zeta)-\Phi^{*}(\zeta)-\zeta\cdot\Phi'(\zeta)\qquad \text{in all-plane.} \tag{4}$$

Then from Eqs. (1), we get a perspicuous expressions of the displacement and the stress components in terms of the single function $\Phi(\zeta)$ as follows:

$$2\mu\cdot(u_\xi+iu_\eta)=\kappa\varphi(\zeta)+\varphi(\zeta^{*})-(\zeta-\zeta^{*})\Phi'(\zeta)\quad, \tag{5 a}$$
$$\sigma_{\xi\xi}+\sigma_{\eta\eta}=2[\Phi(\zeta)+\Phi^{*}(\zeta^{*})]\quad, \tag{5 b}$$
$$\sigma_{\xi\xi}-\sigma_{\eta\eta}-2i\sigma_{\xi\eta}=2[\Phi(\zeta)+\Phi^{*}(\zeta)+(\zeta-\zeta^{*})\Phi'(\zeta)]\quad, \tag{5 c}$$

In the result, the two-dimensional elastostatic problem is reduced to find the sectionally holomorphic function which satisfies certain conditions on the boundaries.

Now, in this paper, we intend to derive an analytical solution inside the rectangular body in the case of mixed boundary conditions. The problem tackled here is illustrated in Fig. 1. Attach the upper surface OA of $2a\times 2b$ rectangular elastic body OABC to a perfectly rigid body, then the displacement of the surface of our elastic body is zero. An external concentrated load acts vertically at a point D on the lower surface BC. The stress and strain arise inside the rectangle OABC. Particularly, in this paper, we intend to give a full account of the normal and shear stresses which appear on a rigidly fixed surface OA.

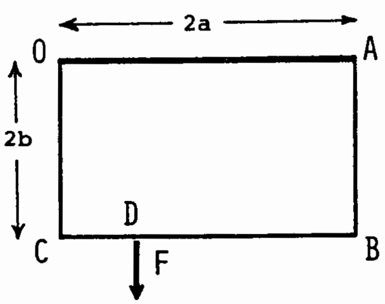


Fig. 1 $2a\times 2b$ rectangular elastic body. The surface OA is rigidly fixed.

The essential idea of the method of finding the solution, which we adopt here, is to transform the known solution of a simple region but with mixed boundaries into our region OABC. Such being the case, we consider a body occupying the lower half-plane, Im $(\zeta)\leq 0$, illustrated in Fig. 2A. The elastostatic solution we should obtain obeys the following boundary conditions:

$$2\mu\cdot(u_\xi+iu_\eta)=0 \quad \text{for} \quad \xi<\zeta_A' \quad , \tag{6 a}$$

$$\sigma_{\eta\eta}+i\sigma_{\xi\eta}=-F\cdot\delta(\xi-\zeta_D') \quad \text{for} \quad \xi>\zeta_A' \quad . \tag{6 b}$$

This has been solved by Sakuraoka and Murata, and given as[4]

$$\Phi_0(\zeta)=\frac{F}{2\pi i}\cdot\frac{I}{\zeta_D'-\zeta}\cdot\frac{X_0(\zeta)}{X_0(\zeta_D')} \quad , \tag{7}$$

$$X_0(\zeta)=[\zeta-\zeta_A']^{-(1/2+i\beta)} \quad ;$$

where β denotes $(\log\kappa)/2\pi$. From Eqs. (1), at the boundary of the elastic body, one can immediately write down the following relations:

$$\frac{\partial}{\partial\xi}[2\mu(u_\xi+iu_\eta)]|_{\eta\to 0-}=\kappa\Phi_0^-(\xi)+\Phi_0^+(\xi) \quad , \tag{8 a}$$

$$[\sigma_{\eta\eta}-i\sigma_{\xi\eta}]|_{\eta\to 0-}=\Phi_0^-(\xi)-\Phi_0^+(\xi) \quad , \tag{8 b}$$

where $\Phi_0^-(\xi)$ [$\Phi_0^+(\xi)$] denotes the limit of the function $\Phi_0(\zeta)$ as ζ tends to real axis, $\zeta=\xi$, from the lower [upper] plane. Thus the solution (7) obviously satisfies the boundary condition (6).

Next we introduce the conformal transformation which carries the lower half-plane in which the complex potential is known, onto the actual region in which we are to get the solution. That is, we take the Schwarz-Christoffel transformation which maps the lower half of the ζ-plane on to the interior of the rectangular region OABC of the complex z-plane[5]. It is illustrated in Fig. 2 in the case of $a=2$ and $b=1$. The differential equation for the mapping function $\zeta=\zeta(z)$ and its solution are

$$d\zeta/dz=2\cdot e^{3\pi i/2}[(\zeta-\zeta_A')(\zeta-\zeta_B')(\zeta-\zeta_C')]^{1/2} \tag{9}$$

and

$$\zeta(z)=-\wp(z;4a,4bi) \quad , \tag{10}$$

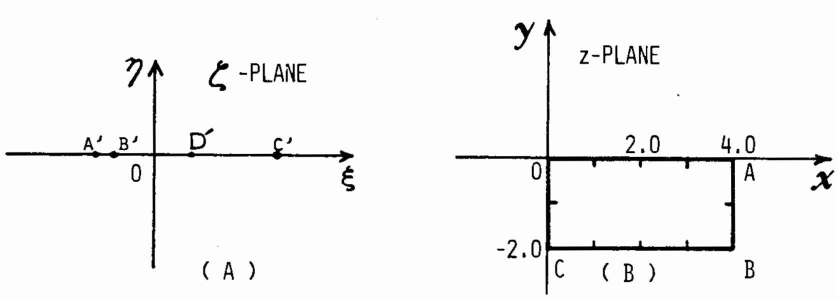


Fig. 2 Schwarz-Christoffel transformation between ζ and z.

where $\wp(z;\ 4a,\ 4bi)$ is the Weierstrass elliptic function having dual periodicity equal to $4a$ and $4bi$. The explicit expression for $\wp(z;\ 4a,\ 4bi)$ is

$$\wp(z; 4a, 4bi) = z^{-2} + \sum_{m,n}{}' [(z-4am-4bni)^{-2} - (4am+4bni)^{-2}] \quad , \qquad (10')$$

where $\sum'_{n,m}$ denotes omission of the term $m=n=0$ from the double summations over m and n. As shown in Fig. 2, the three corners A, B, and C of the rectangle are mapped points from the points A', B', and C', respectively. The original point O of the z-plane comes from the infinity of the ζ-plane. Considering the fact that the mapping (10) transforms a line-element $d\zeta$ in original ζ-plane to dz which is $1/|d\zeta/dz|$ times as long as $|d\zeta|$, we arrive at following form for the complex potential $\Phi[\zeta(z)]$ for the inside of our rectangular region:

$$\Phi[\zeta(z)] = (d\zeta/dz)\Phi_0[\zeta(z)] \quad . \qquad (11)$$

Now, substituting Eqs. (4) and (11) to Eqs. (5), we can immediately write down the stresses in our rectangular region OABC. It is shown that this solution satisfies the boundary condition as follows;

1) on the segment OA:

$$\frac{\partial}{\partial x}[2\mu(u_x + iu_y)]|_{y\to 0-} = \frac{d\zeta}{dz} \cdot \{\kappa\Phi(\zeta) + \Phi(\zeta^*)\}|_{\eta\to 0-} = 0 \quad .$$

2) on the segment AB:

$$\sigma_{xx} + i\sigma_{xy}|_{x\to 2a-0} = [\Phi(\zeta) - \Phi(\zeta^*)]_{\eta\to 0-} = 0 \quad .$$

3) on the segment BC:

$$\begin{aligned}\sigma_{yy} - i\sigma_{xy}|_{y\to -2b+0} &= [\Phi(\zeta) - \Phi(\zeta^*)]|_{\eta\to 0-} \\ &= -|d\zeta/dz| \cdot [\Phi_0(\zeta) - \Phi_0(\zeta^*)]|_{\eta\to 0-} \\ &= F \cdot |d\zeta/dz| \cdot \delta(\xi - \zeta_D') = F \cdot \delta(x - x_D) \quad ,\end{aligned}$$

and the proofs on the segment CO are the same on the segment AB. The explict expressions of the normal and shear stress, σ_{yy} and σ_{xy}, which appear on a rigidly fixed surface OA are

$$\begin{aligned}\sigma_{yy}/F = [2\cosh\pi\beta/\pi] \cdot \{(\zeta_D' - \zeta_A')[\zeta_B' - \zeta(x)][\zeta_C' - \zeta(x)][\zeta_D' - \zeta(x)]^{-2}\}^{1/2} \\ \times \cos\{\beta\log(\zeta_D' - \zeta_A')/[\zeta_A' - \zeta(x)]\} \quad , \qquad (12\text{a})\end{aligned}$$

$$\sigma_{xy}/F = -[\sigma_{yy}/F] \cdot \tan\{\beta \cdot \log(\zeta_D' - \zeta_A')/[\zeta_A' - \zeta(x)]\} \quad . \qquad (12\text{b})$$

For nummerical calculation, we take the case of plane stress and take, for example, $a=2$cm, $b=1$cm and $\nu=0.393$. The results are illustrated in Figs. 3–6, in regard to some loading points of the external concentrated force F. From the figures, we can draw an interesting conclusion concerning with two quantities. The one is the point (x_n) at which the normal stress is largest, and the other the point (x_s) at which the direction of shear stress changes. As a matter of course they disagree with x_D, in general. In Fig. 7, we illustrate x_n and x_s as a function of x_D. From Eqs. (12), however, it is easy to see that x_n depends the Poisson's ratio ν, but x_s does not. Thus x_s does not depend on the properties of elastic materials but depends only on the shape of rectangle. This fact may be a fine

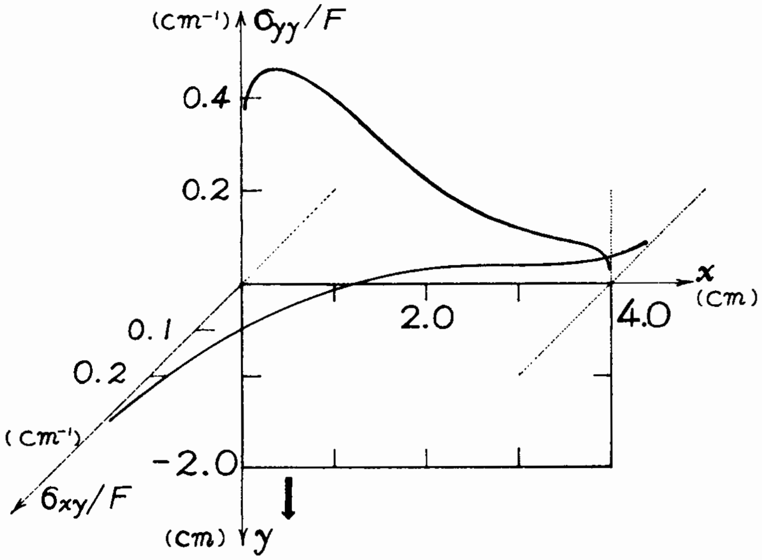


Fig 3 The stress destributions which arise on the rigidly fixed surface OA. Arrow represents the loaded point x_D, i.e., $x_D=0.5$cm.

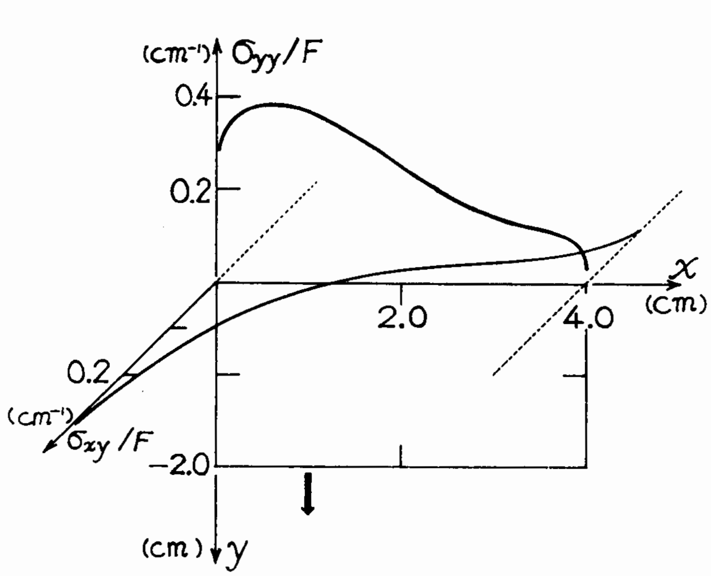


Fig. 4 The same as Fig. 3 except $x_D=1.0$cm.

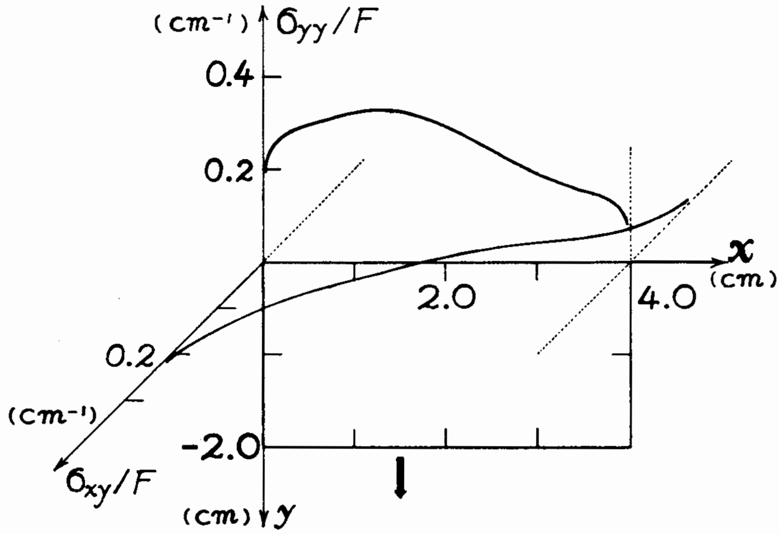


Fig. 5 The same as Fig. 3 except x_D=1.5cm.

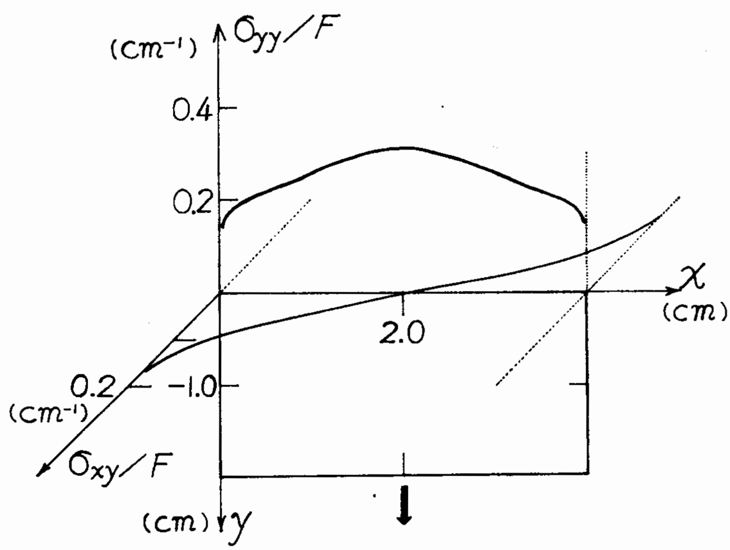


Fig. 6 The same as Fig. 3 except x_D=2.0cm.

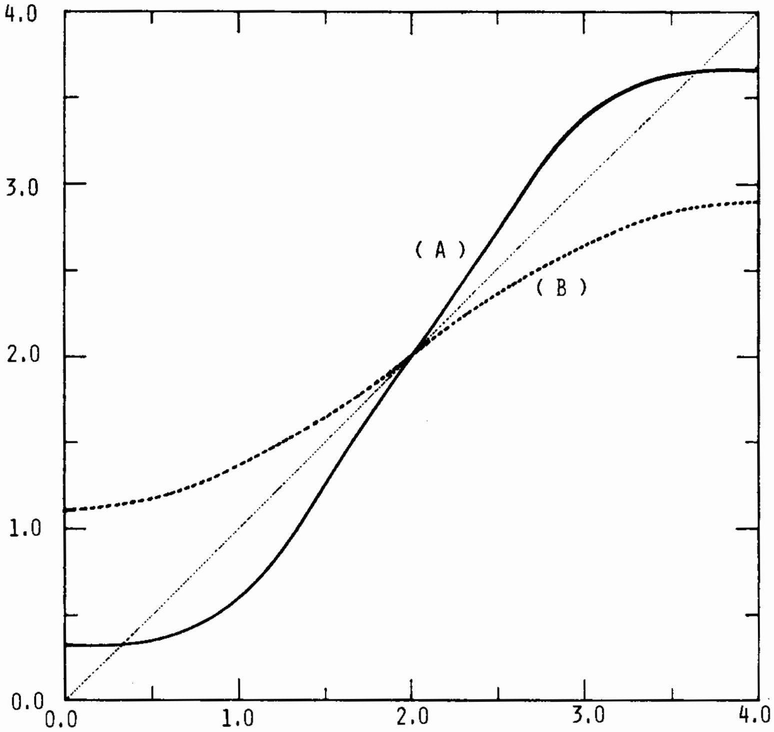


Fig. 7 The curves of x_n (A) and x_s (B) as functions of the loading point x_D (the axis of abscissas).

fruit of our analytical solutions in contrast to the numerical method such as finite element method [4) 6) 7)].

The preferable features of such an analytical solution, in comparison with the numerical calculations such as the finite element method and the method of variational principles [8)], may be followings:

1) A careful study of the analytical solution of this sort may give us a probe of the intuitive conjecture for the estimated stress state with more complex boundaries.

2) The analytical solution may be a great help in direct examining the principle of Saint-Venant [9)].

3) When all is said and done, the numerical method can give only the approximate value to the rigorous solution after all.

Nevertheless, it is a matter of course that, it is very rarely the case that analytical solution is actually found, for elasticity the mumerical method provides as the most powerful and practical approach as ever.

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