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Universal Features of Dimensional Reduction Schemes from General Covariance Breaking

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Abstract

Many features of dimensional reduction schemes are determined by the breaking of higher dimensional general covariance associated with the selection of a particular subset of coordinates. By investigating residual covariance we introduce lower dimensional tensors, that successfully generalize to one side Kaluza-Klein gauge fields and to the other side extrinsic curvature and torsion of embedded spaces, thus fully characterizing the geometry of dimensional reduction. We obtain general formulas for the reduction of the main tensors and operators of Riemannian geometry. In particular, we provide what is probably the maximal possible generalization of Gauss, Codazzi and Ricci equations and various other standard formulas in Kaluza-Klein and embedded spacetimes theories. After general covariance breaking, part of the residual covariance is perceived by effective lower dimensional observers as an infinite dimensional gauge group. This reduces to finite dimensions in Kaluza-Klein and other few remarkable backgrounds, all characterized by the vanishing of appropriate lower dimensional tensors.

Key words: Dimensional reduction, Kaluza-Klein theories, embedded spaces.
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1 Introduction

In many different situations, ranging from low to high energy physics, we are interested in –or have access to– only a part of the coordinates describing a

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given physical system. The problem is finding the effective dynamics that drive the interesting variables by reducing the uninteresting ones or, vice versa, given the effective dynamics of accessible variables, introducing extra coordinates that simplify the overall dynamical picture. Dimensional reduction may be induced by a number of very different mechanisms which in general leave a track in the lower dimensional dynamics. Typical examples are Kaluza-Klein and brane-world reduction in high energy physics, quantum dots/lines/surfaces in semiconductor physics, magnetic confinement in plasma physics and so on. However, there are features that only depend on the selection of the 'interesting' coordinates and not on the specific mechanism under consideration. In this paper we focus on these universal features that depend on the selection of a subset of coordinates and not on specific reduction schemes. It should be stressed, that even if in certain cases –like brane-worlds and quantum lines/surfaces– the effective lower dimensional configuration space can be identified with a regularly embedded metric submanifold, this is not the general situation. A classical example is provided by Kaluza-Klein theories where the physical spacetime is obtained by identifying higher dimensional points connected by a special class of diffeomorphisms that will eventually be identified with gauge transformations. The resulting quotient space can not be given the structure of metric submanifold. The classical theory of embeddings [1,2] is not enough for describing the general situation. In this paper we further investigate the geometry of coordinate separation with emphasis on residual general covariance and provide a unifying framework that successfully generalizes the theory of metric submanifolds.

The paper is organized as follows. In Section 2 we find that dimensional reduction is completely characterized by lower dimensional tensors, generalizing, on the one hand, Kaluza-Klein gauge fields [3,4,5] and, on the other, extrinsic curvature and torsion –i.e. second and normal fundamental forms– of embedded spaces [1,2]. In terms of these we obtain in Section 3 general reduction formulas for the Riemann tensor, Ricci tensor, scalar curvature, geodesics equations, Laplace and Dirac operators, providing what is probably the maximal possible generalization of Gauss, Codazzi, Ricci equations [6] and various other standard identities in embeddings and Kaluza-Klein theories. These equations also represent the natural starting point to investigate higher dimensional unification scenarios in which physics is allowed to fully depend on all the introduced coordinates. In Section 4 special attention is given to induced gauge structures. We show how residual general covariance in the reduced variables always emerges in the effective dynamics as gauge covariance. The induced gauge group is in general infinite dimensional and reduces to finite dimensions in Kaluza-Klein and a few other remarkable backgrounds, all characterized by the vanishing of appropriate lower dimensional tensors. Finally, in Section 5 a discussion of the findings is presented and concluding remarks are made.

For the shake of concreteness we tackle the problem from the viewpoint of higher dimensional unification. We consider a higher dimensional (HD) spacetime $\mathbf{M}_D$ parameterized by D continuous coordinates $\mathbf{x}^I$, $I = 0, 1, ..., D-1$, endowed with a pseudo-Riemannian metric $\mathbf{g}_{IJ}$. In addition to the coordinate system, we set up reference frames at each spacetime point $\mathbf{r}_A^{\ I}(\mathbf{x})$, $A = 0, 1, ..., D-1$, $\mathbf{r}_A^{\ I}\mathbf{r}_B^{\ J}\mathbf{g}_{IJ} = \eta_{AB}$. Physical laws are assumed to be covariant under general coordinates transformations and local redefinitions of reference frames [7]

$$\mathbf{x}^I \to \mathbf{x}'^I(\mathbf{x}), \;\; \mathbf{r}_A^{\ I} \to \mathbf{\Lambda}_A^{\ B}(\mathbf{x})\mathbf{r}_B^{\ I} \tag{1}$$

At low energies the spacetime M_d (e.g. $d = 4$) is parameterized by d continuous coordinates x^μ, $\mu = 0, 1, ..., d-1$, and reference frames are made up of d reference vectors $r_\alpha^{\ \mu}$, $\alpha = 0, 1, ..., d-1$. Physical laws are covariant under (electroweak and strong) gauge transformations, other than general coordinates transformations $x^\mu \to x'^\mu(x)$ and local redefinitions of reference frames $r_\alpha^{\ \mu} \to \Lambda_\alpha^{\ \beta}(x)r_\beta^{\ \mu}$. The original motivation for considering higher dimensional unification is the hope that HD covariance can account for lower dimensional (LD) gauge symmetries other than LD spacetime covariance. To make contact with LD physics, we split HD coordinates in two groups $\mathbf{x}^I = (x^\mu, y^i)$ with $\mu = 0, 1, ..., d-1$, $i = 1, 2, ..., c \equiv D-d$. We refer to x^μ and y^i as *external* and *internal* coordinates, respectively. Consequently, reference frames split in four blocks $\mathbf{r}_\alpha^{\ \mu} \equiv r_\alpha^{\ \mu}$, ..., $\mathbf{r}_{a+d-1}^{\ i+d-1} \equiv \rho_a^{\ i}$ with $\alpha = 0, 1, ..., d-1$, $a = 1, 2, ..., c$. As we are willing to make no a priori hypothesis on specific reduction mechanisms, we proceed by noticing that the minimal assumption that drives us to recover the desired LD spacetime covariance is that the HD transformation group (1) is effectively broken down to

$$\begin{cases} x^\mu \to x'^\mu(x) \\ y^i \to y'^i(x,y) \end{cases} \qquad \begin{cases} r_\alpha^{\ \mu} \to \Lambda_\alpha^{\ \beta}(x) r_\beta^{\ \mu} \\ \vdots \\ \rho_a^{\ i} \to \Lambda_a^{\ b}(x,y)\rho_b^{\ i} \end{cases} \tag{2}$$

We take this as a characterization of dimensional reduction. In working out the consequences that it implies, as a check of our results and to make contact with the most important applications, we constantly specialize in appropriate subsections to Kaluza-Klein theories [3,4,5] and spacetimes embedded in a flat [1] higher dimensional space [2]. While in the former case the topology reduces to that of a direct product and in the latter the system is localized on a submanifold, in the general case the structure of the HD spacetime is more complex. In correspondence to every choice of external coordinates x^μ, the internal coordinates y^i span a c-dimensional *internal spacetime* M_c^x regularly

[1] The assumption of flatness is clearly not necessary and is made because it is common in applications and to keep our explanatory formulas as simple as possible.

embedded in $\mathbf{M}_D$. Every internal spacetime M_c^x has to be identified with a point of the d-dimensional *external spacetime* M_d and may posses a geometry –and even a topology– that vary from point to point. Strictly speaking, M_d can not be identified with the effective spacetime before internal coordinates have been completely removed. In spite of this we will talk about LD external metric, curvature or general tensors, with the bona fide assumption that internal coordinate dependence will be eventually removed from the effective theory. Clearly, any realistic reduction mechanism will eventually involve such a removal. However we will not address this issue in this paper.

2 The Geometry of Dimensional Reduction

The HD spacetime $\mathbf{M}_D$ is endowed with standard pseudo-Riemannian geometry.

2.1 Tensors

HD *tensors* $\mathbf{t}_{...I...}^{\;\;...J...}$ transform according to

$$\mathbf{t}_{...I...}^{\;\;...J...} \rightarrow ...\ \mathbf{J}_I^{\;K} ...\ \mathbf{t}_{...K...}^{\;\;...L...}\ ...\ \mathbf{J}^{-1}{}_L^{\;J} ...$$

with $\mathbf{J}_I^{\;J} = \frac{\partial \mathbf{x}^J}{\partial \mathbf{x}'^I}$ the Jacobian matrix associated with the transformation of HD coordinates.

LD *external tensors* $t_{...\mu...}^{\;\;...\nu...}$ and LD *internal tensors* $t_{...i...}^{\;\;...j...}$, respectively carrying external and internal indices, transform according to

$$\begin{aligned} t_{...\mu...}^{\;\;...\nu...} &\rightarrow ...\ J_\mu^{\;\kappa} ...\ t_{...\kappa...}^{\;\;...\lambda...}\ ...\ J^{-1}{}_\lambda^{\;\nu} ... \\ t_{...i...}^{\;\;...j...} &\rightarrow ...\ J_i^{\;k} ...\ t_{...k...}^{\;\;...l...}\ ...\ J^{-1}{}_l^{\;j} ... \end{aligned}$$

with $J_\mu^{\;\nu} = \frac{\partial x^\nu}{\partial x'^\mu}$ and $J_i^{\;j} = \frac{\partial y^j}{\partial y'^i}$ the Jacobian matrices associated with the transformations of x^μ and y^i respectively. LD *hybrid tensors* $t_{...\mu...i...}^{\;\;...\nu...j...}$, carrying internal and external indices that transform with $J_\mu^{\;\nu}$ and $J_i^{\;j}$, respectively, will also be considered.

When HD covariance is broken from (1) to (2), $\mathbf{J}_I{}^J$ takes the block non-diagonal form

$$\mathbf{J}_I{}^J(\mathbf{x}') = \begin{pmatrix} J_\mu{}^\nu(x') & \frac{\partial y^j}{\partial x'^\mu}(x', y') \\ 0 & J_i{}^j(x', y') \end{pmatrix} \tag{3}$$

The off-diagonal block makes covariant external $\mathbf{t}_{\dots\mu\dots}$, contravariant internal $\mathbf{t}^{\dots i\dots}$ and analogous hybrid components $\mathbf{t}_{\dots\mu\dots}^{\dots j\dots}$ of HD tensors, in non-covariant LD objects. On the other hand, contravariant external $\mathbf{t}^{\dots\mu\dots}$, covariant internal $\mathbf{t}_{\dots i\dots}$ and analogous hybrid components $\mathbf{t}_{\dots i\dots}^{\dots\nu\dots}$ of HD tensors, transform like LD tensors. As an explicit example, external and internal components of a HD covariant vector $\mathbf{v}_I$ transform like

$$\mathbf{v}_\mu \rightarrow J_\mu{}^\kappa \mathbf{v}_\kappa + \frac{\partial y^k}{\partial x'^\mu}\mathbf{v}_k \quad \text{and} \quad \mathbf{v}_i \rightarrow J_i{}^k \mathbf{v}_k$$

so that $\mathbf{v}_\mu$ can not be identified with an external vector, while $\mathbf{v}_i \equiv v_i$ transforms like a LD internal vector. External and internal components of a HD contravariant vector $\mathbf{v}^I$ transform according to

$$\mathbf{v}^\mu \rightarrow \mathbf{v}^\kappa J^{-1}{}_\kappa{}^\mu \quad \text{and} \quad \mathbf{v}^i \rightarrow \mathbf{v}^\kappa \frac{\partial y'^i}{\partial x^\kappa} + \mathbf{v}^k J^{-1}{}_k{}^i$$

so that $\mathbf{v}^\mu \equiv v^\mu$ can be identified with a LD contravariant external vector, while $\mathbf{v}^i$ is not a LD vector.
When constructed from HD tensors, LD tensors are in general functions of external and internal coordinates. In internal directions the x^μ dependence just labels the internal space M_c^x under consideration. In external directions the y^i dependence will be eventually removed.

2.2 Metric

The most general parameterization of the HD spacetime metric $\mathbf{g}_{IJ}$ covariant under (2) reads

$$\mathbf{g}_{IJ} = \begin{pmatrix} g_{\mu\nu} + h_{kl}a_\mu^k a_\nu^l & a_\mu^k h_{kj} \\ h_{il}a_\nu^l & h_{ij} \end{pmatrix} \tag{4}$$

with $g_{\mu\nu}(x, y)$, $h_{ij}(x, y)$ and $a_\mu^i(x, y)$ functions of external and internal coordinates that transform according to

$$g_{\mu\nu} \to J_\mu{}^\kappa J_\nu{}^\lambda g_{\kappa\lambda} \tag{5}$$

$$h_{ij} \to J_i{}^k J_j{}^l h_{kl} \tag{6}$$

$$a^i_\mu \to J_\mu{}^\kappa \left(a^k_\kappa J^{-1}{}_k{}^i - \partial_\kappa y'^i \right) \tag{7}$$

The square matrices $g_{\mu\nu}$ and h_{ij} respectively transform like LD external and internal tensors and can be identified with metrics on M_d (after y^i removal) and M^x_c. The rectangular matrix a^i_μ transforms like a LD hybrid tensor up to an inhomogeneous term reminding the transformation rule of a gauge potential. By means of a^i_μ it is also possible to construct a genuine LD hybrid tensor

$$f^i_{\mu\nu} = \partial_\mu a^i_\nu - \partial_\nu a^i_\mu - a^j_\mu \partial_j a^i_\nu + a^j_\nu \partial_j a^i_\mu \tag{8}$$

appearing as the associated gauge curvature [2]. It is well known, that this is more than a similarity in Kaluza-Klein [3,4,5] and embedded spacetime [8] theories, where (7) precisely corresponds to the transformation rule of a G^{KK} or $SO(c)$ gauge potential. On the other hand, apparently unnoticed is the fact that (7) always corresponds to the transformation rule of a vector potential. To see this explicitly, we read x-dependent internal coordinate transformations (2) as the actions of the internal diffeomorphism group $\mathcal{D}iff_c$ on M^x_c

$$y^i \to \exp\{\xi^k(x,y)\partial_k\} y^i \tag{9}$$

with $\xi^k(x,y)$ an appropriate internal vector. By introducing the operator-valued external covariant vector

$$a_\mu = -i a^i_\mu \partial_i \tag{10}$$

and denoting by $T = \exp\{-\xi^k(x,y)\partial_k\}$ the inverse of the operator generating the transformation, it is straightforward to check that (7) can be rewritten in the familiar gauge transformation form

$$a_\mu \to T a_\mu T^{-1} + i T(\partial_\mu T^{-1}) \tag{11}$$

The off-diagonal term of the HD metric has to be identified with a vector potential taking values in the internal diffeomorphism algebra of $diff_c$. The associated curvature $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu]$ corresponds to the operator associated to $f^i_{\mu\nu}$

$$f_{\mu\nu} = -i f^i_{\mu\nu} \partial_i \tag{12}$$

[2] The vanishing of $f^i_{\mu\nu}$ implies the existence of an internal coordinate transformation setting $a^i_\mu = 0$. In general relativity –identifying space-like coordinates with external variables and time with the internal coordinate– the vanishing of $f^i_{\mu\nu}$ characterizes static gravitational fields.

and transforms in the adjoint representation

$$f_{\mu\nu} \to T f_{\mu\nu} T^{-1} \tag{13}$$

General coordinate transformations do not preserve lengths and angles, so that the operator T is in general not unitary. The vanishing of the divergence of ξ^i makes T formally unitary, a condition always met in Kaluza-Klein and embedded spacetime theories.

Kaluza-Klein: The HD spacetime $\mathbf{M}_D = \mathrm{M}_d \times \mathcal{K}_c$ is the product manifold of a Lorentz space M_d and the internal space $\mathcal{K}_c$ admitting an isometry group G^{KK}. The metric ansatz reads

$$\mathbf{g}_{IJ} = \begin{pmatrix} \mathrm{g}_{\mu\nu} + \mathrm{A}_\mu^{\mathsf{a}} \mathrm{A}_\nu^{\mathsf{b}} \mathrm{K}_{\mathsf{a}}^k \mathrm{K}_{\mathsf{b}}^l \kappa_{kl} & \mathrm{A}_\mu^{\mathsf{a}} \mathrm{K}_{\mathsf{a}}^k \kappa_{kj} \\ \kappa_{il} \mathrm{A}_\nu^{\mathsf{a}} \mathrm{K}_{\mathsf{a}}^l & \kappa_{ij} \end{pmatrix} \tag{14}$$

with $\mathrm{g}_{\mu\nu}(x)$ a metric on M_d, $\kappa_{ij}(y)$ a metric on $\mathcal{K}_c$, $\mathrm{K}_{\mathsf{a}}^k(y)$ Killing vector fields on $\mathcal{K}_c$ and $\mathrm{A}_\mu^{\mathsf{a}}(x)$ identified with the gauge potential taking values in the algebra of G^{KK}. By assumption $L_{\mathrm{K}_{\mathsf{a}}} \kappa = 0$, equivalently $(\partial_i \mathrm{K}_{\mathsf{a}}^k)\kappa_{kj} + (\partial_j \mathrm{K}_{\mathsf{a}}^k)\kappa_{ik} + \mathrm{K}_{\mathsf{a}}^k \partial_k \kappa_{ij} = 0$ or $\nabla_i \mathrm{K}_{\mathsf{a}j} + \nabla_j \mathrm{K}_{\mathsf{a}i} = 0$. Allowed internal coordinate transformations are generated by Killing vector fields $\xi^k(x,y) = \epsilon^{\mathsf{a}}(x)\mathrm{K}_{\mathsf{a}}^k(y)$. Because of the above identity, $\nabla_i \mathrm{K}_{\mathsf{a}}^i = 0$, so that T is unitary. The transformation rule (11) yields for $\mathrm{A}_\mu^{\mathsf{a}}$ the G^{KK} gauge potential transformation rule, which infinitesimally takes the standard form

$$\mathrm{A}_\mu^{\mathsf{a}} \to \mathrm{A}_\mu^{\mathsf{a}} + \mathrm{A}_\mu^{\mathsf{b}} \epsilon^{\mathsf{c}} c_{\mathsf{bc}}^{\mathsf{a}} - \partial_\mu \epsilon^{\mathsf{a}} \tag{15}$$

The corresponding curvature is related to (8) by

$$f_{\mu\nu}^i = \left(\partial_\mu \mathrm{A}_\nu^{\mathsf{c}} - \partial_\nu \mathrm{A}_\mu^{\mathsf{c}} - c_{\mathsf{ab}}^{\mathsf{c}} \mathrm{A}_\mu^{\mathsf{a}} \mathrm{A}_\nu^{\mathsf{b}}\right) \mathrm{K}_{\mathsf{c}}^i = \mathrm{F}_{\mu\nu}^{\mathsf{c}} \mathrm{K}_{\mathsf{c}}^i \tag{16}$$

Embedded spacetime: The HD spacetime $\mathbf{M}_D \equiv \mathbb{R}^D$ is reduced to a Lorentz space M_d. Denoting be x^μ the coordinates on M_d, by $\mathbf{t}_\mu$ the associated tangent vectors and by $\mathbf{n}_i(x)$ a smooth assignment of c orthonormal vectors, $\mathbf{n}_i \cdot \mathbf{n}_j = 0$, $\mathbf{n}_i \cdot \mathbf{t}_\mu = 0$, coordinates are adapted by parameterizing internal directions by the distances y^i along the geodesics leaving M_d with velocity $\mathbf{n}_i$. In adapted coordinates the flat HD metric reads

$$\mathbf{g}_{IJ} = \begin{pmatrix} g_{\mu\nu} + \mathrm{A}_{\mu m}{}^k \mathrm{A}_{\mu n}{}^l y^m y^n \eta_{kl} & \mathrm{A}_{\mu m}{}^k y^m \eta_{kj} \\ \eta_{il} \mathrm{A}_{\mu n}{}^l y^n & \eta_{ij} \end{pmatrix} \tag{17}$$

where $g_{\mu\nu} = \mathrm{g}_{\mu\nu} + 2\mathrm{II}_{k\mu\nu} y^k + \mathrm{II}_{k\mu\kappa} \mathrm{II}_{l\nu}{}^\kappa y^k y^l$ with $\mathrm{g}_{\mu\nu}(x) = \mathbf{t}_\mu \cdot \mathbf{t}_\nu$ the induced metric and $\mathrm{II}_{i\mu\nu}(x) = \mathbf{t}_\mu \cdot \partial_\nu \mathbf{n}_i$ the extrinsic curvature (or *second fundamental form*) of the embedding; η_{ij} is a (pseudo-)Euclidean metric in extra directions; $\mathrm{A}_{\mu ij}(x) = \mathbf{n}_i \cdot \partial_\mu \mathbf{n}_j$ is the extrinsic torsion (or *normal fundamental form*) of the embedding [1]. The off-diagonal blocks of (17) are proportional to the Killing vectors generating (pseudo-)rotations around the point $y^i = 0$ in the flat internal space. However, the metric is not Kaluza-Klein because of terms that make $g_{\mu\nu}$ explicitly dependant on y^i. Allowed internal coordinate transformations correspond to the x-dependent (pseudo-)rotation $\mathbf{n}_i \rightarrow \Lambda_i{}^j(x)\mathbf{n}_j$ and are generated by the Killing vector fields $\xi^k(x,y) = y^l \omega_l{}^k(x)$ with $\omega_{kl} = -\omega_{lk}$. $\nabla_k y^l \omega_l{}^k = \omega_k{}^k = 0$ so that T is unitary. Under (2) $\mathrm{A}_{\mu i}{}^j$ transform like a $SO(c)$ gauge potential

$$\mathrm{A}_{\mu i}{}^j \rightarrow \Lambda_i{}^k \mathrm{A}_{\mu k}^{\ l} \Lambda^{-1}{}_l{}^j - \Lambda_i{}^k \partial_\mu \Lambda^{-1}{}_k{}^j \tag{18}$$

The associated curvature is related to (8) by

$$f^i_{\mu\nu} = \left(\partial_\mu \mathrm{A}_{\nu j}{}^i - \partial_\nu \mathrm{A}_{\mu j}{}^i - [\mathrm{A}_\mu, \mathrm{A}_\nu]_j{}^i\right) y^j = \mathrm{F}_{\mu\nu j}{}^i y^j \tag{19}$$

Denoting by $\mathbf{g}$ the HD metric determinant and by g and h the LD metric determinants, we have that $\mathbf{g} = gh$. The HD volume element factorizes in the product of LD volume elements $|\mathbf{g}|^{1/2} = |g|^{1/2}|h|^{1/2}$. The HD inverse metric $\mathbf{g}^{IJ}$ can be evaluated in general terms as

$$\mathbf{g}^{IJ} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\kappa} a^j_\kappa \\ -a^i_\lambda g^{\lambda\nu} & h^{ij} + a^i_\kappa a^j_\lambda g^{\kappa\lambda} \end{pmatrix}$$

with $g^{\mu\nu}$ and h^{ij} the inverses of the LD metrics.
The parameterization (4) is particularly convenient in connecting HD with LD geometrical quantities. It generalizes the Kaluza-Klein and embedded spacetime metric ansätze, to the case where no a priori symmetries or special submanifold have been introduced.

2.3 Connections and Curvature Tensors

The HD covariant derivative induced by $\mathbf{g}_{IJ}$ is denoted by $\boldsymbol{\nabla}_I$ and acts on tensors as

$$\boldsymbol{\nabla}_I \mathbf{t}^{\ \cdots}_{\ldots J \ldots} = \partial_I \mathbf{t}^{\ \cdots}_{\ldots J \ldots} + \ldots - \boldsymbol{\Gamma}_{IJ}{}^K \mathbf{t}^{\ \cdots}_{\ldots K \ldots} + \ \ldots$$

where $\boldsymbol{\Gamma}_{IJ}{}^K = \frac{1}{2}\mathbf{g}^{KL}\left(\partial_I \mathbf{g}_{LJ} + \partial_J \mathbf{g}_{IL} - \partial_L \mathbf{g}_{IJ}\right)$ are the intrinsic connection coefficients; by definition $\boldsymbol{\nabla}_K \mathbf{g}_{IJ} = 0$ and $\boldsymbol{\nabla}_K |\mathbf{g}|^{1/2} = 0$. The commutator of two covariant derivatives

$$[\boldsymbol{\nabla}_I, \boldsymbol{\nabla}_J]\, \mathbf{t}_{...K...}^{\;\cdots} = ... - \mathbf{R}_{IJK}{}^{L}\mathbf{t}_{...L...}^{\;\cdots} + ...$$

defines the intrinsic curvature tensor

$$\mathbf{R}_{IJK}{}^{L} = \partial_I \boldsymbol{\Gamma}_{JK}{}^{L} - \partial_J \boldsymbol{\Gamma}_{IK}{}^{L} - \boldsymbol{\Gamma}_{IK}{}^{H}\boldsymbol{\Gamma}_{JH}{}^{L} + \boldsymbol{\Gamma}_{JK}{}^{H}\boldsymbol{\Gamma}_{IH}{}^{L} \tag{20}$$

We also denote the Ricci tensor by $\mathbf{R}_{IJ} = \mathbf{R}_{IKJ}{}^{K}$ and the scalar curvature by $\mathbf{R} = \mathbf{g}^{IJ}\mathbf{R}_{IJ}$. The covariant derivative $\boldsymbol{\nabla}_I$ and the associated curvature tensor $\mathbf{R}_{IJK}{}^{L}$ completely characterize the geometry of the HD spacetime $\mathbf{M}_D$. We now consider analogous quantities for the LD internal spaces M_c^x and external space M_d.

2.3.1 Internal connection and curvatures

The LD *internal covariant derivative* ∇_i induced by the metric tensor h_{ij}

$$\nabla_i t_{...j...}^{\;\cdots} = \partial_i t_{...j...}^{\;\cdots} + ... - \mathit{\Gamma}_{ij}{}^{k} t_{...k...}^{\;\cdots} + ... \tag{21}$$

with *internal intrinsic connection coefficients*

$$\mathit{\Gamma}_{ij}{}^{k} = \frac{1}{2}h^{kl}(\partial_i h_{lj} + \partial_j h_{il} - \partial_l h_{ij}) \tag{22}$$

is covariant under (2) when acting either on LD internal, external or hybrid tensors. As a consequence, new LD tensors can be generated by the action of ∇_i. The commutator

$$[\nabla_i, \nabla_j]\, t_{...k...}^{\;\cdots} = ... - R_{ijk}{}^{l} t_{...l...}^{\;\cdots} + ...$$

defines the *internal intrinsic curvature*

$$R_{ijk}{}^{l} = \partial_i \mathit{\Gamma}_{jk}{}^{l} - \partial_j \mathit{\Gamma}_{ik}{}^{l} - \mathit{\Gamma}_{ik}{}^{m}\mathit{\Gamma}_{jm}{}^{l} + \mathit{\Gamma}_{jk}{}^{m}\mathit{\Gamma}_{im}{}^{l} \tag{23}$$

Internal Ricci tensor and scalar curvature are defined like in higher dimensions. The internal metric h_{ij} and the internal volume element $|h|^{1/2}$ are parallel transported $\nabla_k h_{ij} = 0$, $\nabla_k |h|^{1/2} = 0$. The internal covariant derivative, however, is not compatible with the external metric structure as $\nabla_i g_{\mu\nu} \neq 0$. External indices can not be raised, lowered or contracted regardless to the position of ∇_i. To overcome the problem we extend the action of ∇_i to external indices. We define an *internal total covariant derivative* $\nabla_i^{\rm tot}$ by

$$\nabla_i^{\rm tot} t_{\dots\mu\dots j\dots}^{\;\;\dots} = \nabla_i t_{\dots\mu\dots j\dots}^{\;\;\dots} + \dots - \hat{E}_{i\mu}{}^{\nu} t_{\dots\nu\dots j\dots}^{\;\;\dots} \cdots \tag{24}$$

with *internal extrinsic connection coefficients* $\hat{E}_{i\mu}{}^{\nu}$ chosen so that $\nabla_k^{\rm tot} g_{\mu\nu} = 0$ (also implying $\nabla_k^{\rm tot} |g|^{1/2} = 0$). This requirement fixes the symmetric part of the extrinsic connection to $\hat{E}_{i(\mu\nu)} = \frac{1}{2}\partial_i g_{\mu\nu}$, living the antisymmetric part completely arbitrary. It is possible and even natural to include in $\hat{E}_{i\mu\nu}$ a term proportional to the hybrid tensor $f_{i\mu\nu}$. Different choices correspond to different internal extrinsic geometries. In Section 3, equation (76), we will see that the internal extrinsic connection induced by HD geometry corresponds to the choice $\hat{E}_{i[\mu\nu]} = \frac{1}{2} f_{i\mu\nu}$. We therefore set

$$\hat{E}_{i\mu}{}^{\nu} = \frac{1}{2}\left(\partial_i g_{\mu\kappa} + f_{i\mu\kappa}\right) g^{\kappa\nu} \tag{25}$$

Under coordinate redefinitions (2), $\hat{E}_{i\mu}{}^{\nu}$ transforms like a genuine LD hybrid tensor

$$\hat{E}_{i\mu}{}^{\nu} \rightarrow J_i{}^{j} J_{\mu}{}^{\kappa} \hat{E}_{j\kappa}{}^{\lambda} J^{-1}{}_{\lambda}{}^{\nu} \tag{26}$$

Kaluza-Klein: The symmetric part of the internal extrinsic connection vanishes identically; the antisymmetric part reduces to the gauge curvature

$$\hat{E}_{i\mu\nu} = \frac{1}{2} \mathrm{F}^{\mathsf{c}}_{\mu\nu} \mathrm{K}_{\mathsf{c}i} \tag{27}$$

Embedded spacetime: The internal extrinsic connection equals the second fundamental form $\mathrm{II}_{i\mu\nu}$ of M_d plus a term linear in y^i

$$\hat{E}_{i\mu\nu} = \mathrm{II}_{i\mu\nu} + \frac{1}{2}\left(\mathrm{II}_{i\mu\kappa}\mathrm{II}_{j\nu}{}^{\kappa} + \mathrm{II}_{i\nu\kappa}\mathrm{II}_{j\mu}{}^{\kappa} - \mathrm{F}_{\mu\nu ij}\right) y^j \tag{28}$$

On M_d the linear term vanishes and $\hat{E}_{i\mu\nu}$ coincides with the second fundamental form $\hat{E}_{i\mu\nu}|_{y=0} \equiv \mathrm{II}_{i\mu\nu}$.

The hybrid tensor $\hat{E}_{i\mu\nu}$ reduces to the gauge curvature of the external space in Kaluza-Klein backgrounds and to the extrinsic curvature –second fundamental form– of the external spacetime in embedded spacetime models. In Section 3 we will see that $\hat{E}_{i\mu\nu}$ enters the general equations of Subsection 3.2 relating higher and lower dimensional curvatures in the very same way as the second fundamental form enters Gauss, Codazzi and Ricci equations. For these reasons we will also refer to $\hat{E}_{i\mu\nu}$ as to the *external fundamental form*. The 'hat' is introduced to remind us that M_d is not in general an embedded object and $\hat{E}_{i\mu\nu}$ is not a fundamental form in the standard sense of embedding theory. The commutator of two total internal covariant derivatives

$$\left[\nabla_i^{\text{tot}}, \nabla_j^{\text{tot}}\right] t_{\ldots\mu\ldots k\ldots} = \ldots - R_{ijk}{}^{l} t_{\ldots\mu\ldots l\ldots}\ldots - F_{ij\mu}{}^{\nu} t_{\ldots\nu\ldots k\ldots} + \ldots$$

defines the *internal extrinsic curvature*

$$F_{ij\mu}{}^{\nu} = \partial_i \hat{E}_{j\mu}{}^{\nu} - \partial_j \hat{E}_{i\mu}{}^{\nu} - \hat{E}_{i\mu}{}^{\kappa} \hat{E}_{j\kappa}{}^{\nu} + \hat{E}_{j\mu}{}^{\kappa} \hat{E}_{i\kappa}{}^{\nu} \tag{29}$$

carrying two internal and two external indices. A direct computation allows to rewrite $F_{ij\mu\nu}$ as

$$F_{ij\mu\nu} = \frac{1}{2}\partial_i f_{j\mu\nu} - \frac{1}{2}\partial_j f_{i\mu\nu} + \hat{E}_{i\mu}{}^{\kappa} \hat{E}_{j\nu\kappa} - \hat{E}_{j\mu}{}^{\kappa} \hat{E}_{i\nu\kappa} \tag{30}$$

2.3.2 External connection and curvatures

The definition of a covariant differentiation along external direction is less straightforward. The derivative ∇_μ associated with the external metric $g_{\mu\nu}$ is not a covariant LD object. Difficulties already emerge at the scalar level. The allowed external coordinate dependence of internal coordinate redefinitions produces an inhomogeneous term in the transformation rule of partial derivatives

$$\partial_\mu \to \partial'_\mu = J_\mu{}^{\nu} \left(\partial_\nu + \frac{\partial y^i}{\partial x^\nu} \partial_i \right) \tag{31}$$

The problem can be resolved by adding a counter term proportional to a^i_μ which also transform inhomogeneously. The derivative operator

$$\hat{\partial}_\mu = \partial_\mu - i a_\mu \tag{32}$$

transforms like a genuine LD external vector when acting on scalars

$$\hat{\partial}_\mu \to \hat{\partial}'_\mu = J_\mu{}^{\nu} \hat{\partial}_\nu \tag{33}$$

On the other hand, the commutator of two hatted derivatives is no longer vanishing

$$\left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = -i f_{\mu\nu}$$

Differentiation is extended to LD external tensors by introducing the generalized Christoffel symbols

$$\hat{\Gamma}_{\mu\nu}{}^{\kappa} = \frac{1}{2} g^{\kappa\lambda} (\hat{\partial}_\mu g_{\lambda\nu} + \hat{\partial}_\nu g_{\mu\lambda} - \hat{\partial}_\lambda g_{\mu\nu}) \tag{34}$$

where ordinary derivatives are replaced by hatted ones in the standard definition. Generalized Christoffel symbols transform like proper connection symbols. The *external covariant derivative* $\hat{\nabla}_\mu$

$$\hat{\nabla}_\mu t_{\dots\nu\dots}^{\ \dots} = \hat{\partial}_\mu t_{\dots\nu\dots}^{\ \dots} + \dots - \hat{\Gamma}_{\mu\nu}{}^{\kappa} t_{\dots\kappa\dots}^{\ \dots} \tag{35}$$

is covariant under (2) when acting on LD external tensors. New LD tensors can be generated by the action of $\hat{\nabla}_\mu$ on external tensors. The commutator

$$\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] t_{\dots\kappa\dots}^{\ \dots} = \dots - \hat{R}_{\mu\nu\kappa}{}^{\lambda} t_{\dots\lambda\dots}^{\ \dots} + \dots - f^i_{\mu\nu} \nabla_i^{\mathrm{tot}} t_{\dots\kappa\dots}^{\ \dots}$$

defines a genuine *external intrinsic curvature* tensor as

$$\hat{R}_{\mu\nu\kappa}{}^{\lambda} = \hat{\partial}_\mu \hat{\Gamma}_{\nu\kappa}{}^{\lambda} - \hat{\partial}_\nu \hat{\Gamma}_{\mu\kappa}{}^{\lambda} - \hat{\Gamma}_{\mu\kappa}{}^{\rho} \hat{\Gamma}_{\nu\rho}{}^{\lambda} + \hat{\Gamma}_{\nu\kappa}{}^{\rho} \hat{\Gamma}_{\mu\rho}{}^{\lambda} + f^i_{\mu\nu} \hat{E}_{i\kappa}{}^{\lambda} \tag{36}$$

External Ricci and scalar curvatures are defined as usual by contraction $\hat{R}_{\mu\nu} = \hat{R}_{\mu\kappa\nu}{}^{\kappa}$ and $\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}$. It is worth noticing that $\hat{R}_{\mu\nu\kappa}{}^{\lambda}$, $\hat{R}_{\mu\nu}$ and $\hat{R}$ are reducible tensors.

Kaluza-Klein: In Kaluza-Klein theories $\hat{R}$ does not correspond with the scalar curvature R associate with the four dimensional metric $\mathrm{g}_{\mu\nu}(x)$. Equation (36) yields

$$\hat{R} = \mathrm{R} + \mathrm{F}^{\mathsf{a}}_{\mu\nu} \mathrm{F}^{\mathsf{a}\mu\nu}/2 \tag{37}$$

with gauge indices contracted with the group metric.

Embedded spacetime: The corresponding equation in embedded spacetime theories is more complicated involving, apart from the gauge field $\mathrm{F}_{\mu\nu i}{}^{j}$, the external fundamental forms $\hat{E}_{i\mu\nu}$. Specializing to $y^i = 0$ we obtain

$$\hat{R} = \mathrm{R} + \mathcal{O}(y) \tag{38}$$

with R the intrinsic curvature associated with the metric induced on the submanifold.

The external metric $g_{\mu\nu}$ and the external volume element $|g|^{1/2}$ are parallel transported $\hat{\nabla}_\kappa g_{\mu\nu} = 0$, $\hat{\nabla}_\kappa |g|^{1/2} = 0$. On the other hand, it is not even possible to ask wether the external derivative is compatible with internal metric structures, because $\hat{\nabla}_\mu$ is not covariant when acting on internal and hybrid tensors. Both problems can be resolved by extending the action of $\hat{\nabla}_\mu$ to internal indices. We define the *external total covariant derivative* $\hat{\nabla}^{\rm tot}_\mu$ by

$$\hat{\nabla}^{\rm tot}_\mu t_{\ldots\kappa\ldots k\ldots} = \hat{\nabla}_\mu t_{\ldots\kappa\ldots k\ldots} + \ldots - \hat{C}_{\mu k}{}^{l} t_{\ldots\kappa\ldots l\ldots} + \ldots \tag{39}$$

where the *external extrinsic connection coefficients* $\hat{C}_{\mu k}{}^{l}$ are determined by the requirement of covariance and by the compatibility condition $\hat{\nabla}^{\rm tot}_\mu h_{ij} = 0$ (also implying $\hat{\nabla}^{\rm tot}_\mu |h|^{1/2} = 0$). We obtain

$$\hat{C}_{\mu k}{}^{l} = \partial_k a^l_\mu + E_{\mu k m} h^{ml} \tag{40}$$

where

$$E_{\mu ij} = \frac{1}{2}[\hat{\partial}_\mu h_{ij} - (\partial_i a^k_\mu) h_{kj} - (\partial_j a^k_\mu) h_{ik}] \tag{41}$$

transforms like a genuine LD hybrid tensor. At any given external point $E_{\mu ij}|_x$ corresponds to the standard second fundamental form describing the embedding of M^x_c in $\mathbf{M}_D$. $E_{\mu ij}$ generalizes the notion of second fundamental form to the whole foliation of the HD spacetime in internal spaces. For this reason we refer to $E_{\mu ij}$ as the *internal fundamental form*. For later use we also rewrite $E_{\mu ij}$ as

$$E_{\mu ij} = \frac{1}{2}(\partial_\mu h_{ij} - \nabla_i a_{\mu j} - \nabla_j a_{\mu i}) \tag{42}$$

and note that the following identity holds

$$\hat{\nabla}^{\rm tot}_\mu E_{\nu ij} - \hat{\nabla}^{\rm tot}_\nu E_{\mu ij} = \frac{1}{2}\left(\nabla_i f_{j\nu\mu} + \nabla_j f_{i\nu\mu}\right) \tag{43}$$

Under the residual general covariance group (2) external extrinsic connection coefficients transform like a genuine $GL(c)$ connection

$$\hat{C}_{\mu k}{}^{l} \to J_\mu{}^{\nu}(J_k{}^{m} \hat{C}_{\nu m}{}^{n} J^{-1}{}_n{}^{l} - J_k{}^{m} \hat{\partial}_\nu J^{-1}{}_m{}^{l}) \tag{44}$$

Kaluza-Klein: By virtue of the identity $(\partial_i \mathrm{K}^k_{\mathsf{a}})\kappa_{kj} + (\partial_j \mathrm{K}^k_{\mathsf{a}})\kappa_{ik} + \mathrm{K}^k_{\mathsf{a}}\partial_k \kappa_{ij} = 0$ the internal fundamental form vanishes identically

$$E_{\mu ij} = 0 \tag{45}$$

The embedding of each $\mathcal{K}_c$ is *totally geodesic* [1]. The external extrinsic connection coefficients only depends on off-diagonal blocks of the metric

$$\hat{C}_{\mu k}{}^{l} = \mathrm{A}^{\mathsf{a}}_{\mu}(\partial_k \mathrm{K}^l_{\mathsf{a}}) \tag{46}$$

Embedded spacetime: Since the internal metric η_{ij} does not depend on coordinates and $A_{\mu kl}$ are antisymmetric in internal indices the embedding of internal spaces is again *totally geodesic*

$$E_{\mu ij} = 0 \tag{47}$$

The external extrinsic connection reduces to the normal fundamental form of the embedding

$$\hat{C}_{\mu k}{}^{l} = \mathrm{A}_{\mu k}{}^{l} \tag{48}$$

The commutator of two external total covariant derivatives yields the associated curvature forms

$$\left[\hat{\nabla}^{\text{tot}}_{\mu}, \hat{\nabla}^{\text{tot}}_{\nu}\right] t_{\ldots\kappa\ldots k\ldots} = \ldots - \hat{R}_{\mu\nu\kappa}{}^{\lambda} t_{\ldots\lambda\ldots k\ldots} + \\ + \ldots - \hat{F}_{\mu\nu k}{}^{l} t_{\ldots\kappa\ldots l\ldots} + \ldots - f^i_{\mu\nu} \nabla^{\text{tot}}_i t_{\ldots\kappa\ldots k\ldots}$$

where the *external extrinsic curvature* tensor, carrying two external and two internal indices, is defined as

$$\hat{F}_{\mu\nu k}{}^{l} = \hat{\partial}_{\mu}\hat{C}_{\nu k}{}^{l} - \hat{\partial}_{\nu}\hat{C}_{\mu k}{}^{l} - \hat{C}_{\mu k}{}^{m}\hat{C}_{\nu m}{}^{l} + \hat{C}_{\nu k}{}^{m}\hat{C}_{\mu m}{}^{l} + f^i_{\mu\nu}\Gamma_{ik}{}^{l} \tag{49}$$

With the help of (43) a straightforward computation allows to evaluate $\hat{F}_{\mu\nu k}{}^{l}$ directly in terms of $f^i_{\mu\nu}$ and $E_{\mu ij}$ as

$$\hat{F}_{\mu\nu kl} = \frac{1}{2}\partial_k f_{l\mu\nu} - \frac{1}{2}\partial_l f_{k\mu\nu} + E_{\mu k}{}^{i}E_{\nu li} - E_{\nu k}{}^{i}E_{\mu li} \tag{50}$$

a formula that closely resembles (30).

2.3.3 Hybrid curvatures

The commutator of external and internal total covariant derivatives defines one more curvature tensor that describes the tangling of M_d and M_c^x in $\mathbf{M}_D$

$$\left[\hat{\nabla}_\mu^{\rm tot}, \nabla_i^{\rm tot}\right] t_{\ldots\kappa\ldots k\ldots} = \ldots - \hat{H}_{\mu i\kappa}{}^{\lambda} t_{\ldots\lambda\ldots k\ldots} + \ldots - H_{\mu ik}{}^{l} t_{\ldots\kappa\ldots l\ldots} + \\ + \ldots + \hat{E}_{i\mu}{}^{\nu}\hat{\nabla}_\nu^{\rm tot} t_{\ldots\kappa\ldots k\ldots} - E_{\mu i}{}^{j}\nabla_j^{\rm tot} t_{\ldots\kappa\ldots k\ldots}$$

where the two *hybrid curvature* tensors $\hat{H}_{\mu i\kappa}{}^{\lambda}$ and $H_{\mu ik}{}^{l}$ have the form

$$\hat{H}_{\mu i\kappa}{}^{\lambda} = \hat{\partial}_\mu \hat{E}_{i\kappa}{}^{\lambda} - \partial_i \hat{\Gamma}_{\mu\kappa}{}^{\lambda} - \hat{\Gamma}_{\mu\kappa}{}^{\nu}\hat{E}_{i\nu}{}^{\lambda} + \hat{E}_{i\kappa}{}^{\nu}\hat{\Gamma}_{\mu\nu}{}^{\lambda} - (\partial_i a_\mu^j)\hat{E}_{j\kappa}{}^{\lambda} \tag{51}$$

and

$$H_{\mu ik}{}^{l} = \hat{\partial}_\mu \Gamma_{ik}{}^{l} - \partial_i \hat{C}_{\mu k}{}^{l} - \hat{C}_{\mu k}{}^{j}\Gamma_{ij}{}^{l} + \Gamma_{ik}{}^{j}\hat{C}_{\mu j}{}^{l} - (\partial_i a_\mu^j)\Gamma_{jk}{}^{l} \tag{52}$$

A direct computation allows to reexpress the hybrid curvatures in terms of the sole fundamental forms $\hat{E}_{i\mu\nu}$ and $E_{\mu ij}$ as

$$\hat{H}_{\mu i\kappa\lambda} = \hat{\nabla}_\lambda^{\rm tot}\hat{E}_{i\kappa\mu} - \hat{\nabla}_\kappa^{\rm tot}\hat{E}_{i\lambda\mu} + E_{\lambda i}{}^{k}\hat{E}_{k\mu\kappa} + E_{\kappa k}{}^{k}\hat{E}_{k\mu\lambda} + f_{\kappa\lambda}^{j}E_{\mu ik} \tag{53}$$

$$H_{\mu ikl} = \nabla_k^{\rm tot}E_{\mu li} - \nabla_l^{\rm tot}E_{\mu ki} + \hat{E}_{k\mu}{}^{\nu}E_{\nu li} - \hat{E}_{l\mu}{}^{\nu}E_{\nu ki} \tag{54}$$

Therefore, the four LD tensors $\hat{R}_{\mu\nu\kappa}{}^{\lambda}$, $R_{ijk}{}^{l}$, $E_{\mu ij}$, $\hat{E}_{i\mu\nu}$ give a complete characterization of the intrinsic and extrinsic geometry of external and internal spaces. Note that $f_{i\mu\nu}$ is the antisymmetric part of $\hat{E}_{i\mu\nu}$ and a_μ^i is related to it by (8). It is curious that in spite of the different role played by external and internal coordinates the formalism is symmetric under their interchange. The symmetry is substantial only when $f_{\mu\nu}^i \equiv 0$ and $\mathbf{M}_D$ double foliates in internal and external directions.

2.4 Reference Frames

Besides standard tensor calculus in holonomic coordinates, there is a second formalism that allows to successfully deal with geometrical problems: the tetrad (in four dimensions) or reference frame formalism. Among other things, it allows to clarify the role of gauge invariance for the gravitational field [9] and is indispensable to deal with general relativistic interactions of spinors. In this section we show that the reference frame formalism is also the natural

language to deal with dimensional reduction problems.
In the HD spacetime, we consider pseudo-orthogonal covariant and contravariant *reference frames* $\mathbf{r}_I{}^A$ and $\mathbf{r}_A{}^I$, decomposing the metric and its inverse as $\mathbf{g}_{IJ} = \mathbf{r}_I{}^A \mathbf{r}_J{}^B \eta_{AB}$, $\mathbf{g}^{IJ} = \mathbf{r}_A{}^I \mathbf{r}_B{}^J \eta^{AB}$. In terms of the metric parametrization (4)

$$\mathbf{r}_I{}^A = \begin{pmatrix} r_\mu{}^\alpha & a_\mu^k \rho_k{}^a \\ 0 & \rho_i{}^a \end{pmatrix}, \quad \mathbf{r}_A{}^I = \begin{pmatrix} r_\alpha{}^\mu & -r_\alpha{}^\kappa a_\kappa^i \\ 0 & \rho_a{}^i \end{pmatrix} \tag{55}$$

with $r_\mu{}^\alpha$, $r_\alpha{}^\mu$ and $\rho_i{}^a$, $\rho_a{}^i$ decomposing the LD metrics, $r_\mu{}^\alpha r_\nu{}^\beta \eta_{\alpha\beta} = g_{\mu\nu}$, $\rho_i{}^a \rho_j{}^b \eta_{ab} = h_{ij}$ etc. Reference vectors are determined up to point dependent pseudo-rotations expressing observer's freedom of arbitrarily choosing the reference frame. Hence, reference frames transform as holonomic vectors under general coordinate transformations and like pseudo-Euclidean vectors under pseudo-rotations. The theory is covariant under

$$\mathbf{r}_A{}^I \to \mathbf{r}_A{}^J J^{-1}{}_J{}^I, \quad \mathbf{r}_A{}^I \to \mathbf{\Lambda}_A{}^B \mathbf{r}_B{}^I$$

with $\mathbf{\Lambda}_A{}^B(\mathbf{x})$ any point dependent, pseudo-orthogonal matrix, $\mathbf{\Lambda}_A{}^C \mathbf{\Lambda}_B{}^D \eta_{CD} = \eta_{AB}$. When coordinate invariance is broken, local pseudo-orthogonal transformations get restricted to the block diagonal form

$$\mathbf{\Lambda}_A{}^B(\mathbf{x}) = \begin{pmatrix} \Lambda_\alpha{}^\beta(x) & 0 \\ 0 & \Lambda_a{}^b(x,y) \end{pmatrix} \tag{56}$$

with $\Lambda_\alpha{}^\beta(x)$ and $\Lambda_a{}^b(x,y)$ lower dimensional pseudo-orthogonal matrices satisfying $\Lambda_\alpha{}^\gamma \Lambda_\alpha{}^\delta \eta_{\gamma\delta} = \eta_{\alpha\beta}$ and $\Lambda_a{}^c \Lambda_b{}^d \eta_{cd} = \eta_{ab}$. The LD vectors $r_\alpha{}^\mu$ and $\rho_a{}^i$ correctly transform as LD reference frames

$$r_\alpha{}^\mu \to r_\alpha{}^\kappa J^{-1}{}_\kappa{}^\mu, \;\; r_\alpha{}^\mu \to \Lambda_\alpha{}^\beta r_\beta{}^\mu, \;\; \rho_a{}^i \to \rho_a{}^k J^{-1}{}_k{}^i, \quad \rho_a{}^i \to \Lambda_a{}^b \rho_b{}^i$$

We fix the following notation for Kaluza-Klein and embedded spacetime models

Kaluza-Klein: LD reference frames are denoted by

$$r_\mu{}^\alpha = \mathrm{r}_\mu{}^\alpha(x), \quad \rho_i{}^a = k_i{}^a(y) \tag{57}$$

with $\mathrm{g}_{\mu\nu} = \mathrm{r}_\mu{}^\alpha \mathrm{r}_\nu{}^\beta \eta_{\alpha\beta}$ and $\kappa_{ij} = k_i{}^a k_j{}^b \eta_{ab}$.

Embedded spacetime: LD reference frames are chosen as

$$r_\mu{}^\alpha = (\delta_\mu^\kappa + y^i \mathrm{II}_{i\mu}{}^\kappa)\mathrm{t}_\kappa{}^\alpha(x), \;\; \rho_i{}^a = \mathrm{n}_i{}^a(x) \tag{58}$$

with $g_{\mu\nu} = t_\mu{}^\alpha t_\nu{}^\beta \eta_{\alpha\beta}$ and $\eta_{ij} = n_i{}^a n_j{}^b \eta_{ab}$.

2.5 More on Tensors

Instead of specifying HD tensors by giving their components with respect to the holonomic coordinate system, we can specify them by giving their projections on the reference frame

$$\mathbf{t}_{...A...}^{...B...} = ...\mathbf{r}_A{}^I ... \, \mathbf{t}_{...I...}^{...J...} \, ...\mathbf{r}_J{}^B ...$$

These quantities are invariant under general coordinate transformations and transform like pseudo-Euclidean tensor components under point dependent reference frame redefinition

$$\mathbf{t}_{...A...}^{...B...} \to ...\mathbf{\Lambda}_A{}^C ... \, \mathbf{t}_{...C...}^{...D...} \, ...\mathbf{\Lambda}^{-1}{}_D{}^B ...$$

LD external, internal and hybrid tensor components $t_{...\alpha...}^{...\beta...}$, $t_{...a...}^{...b...}$ and $t_{...\alpha...a...}^{...\beta...b...}$ are introduced with analogous conventions and transformation properties

$$\begin{aligned} t_{...\alpha...}^{...\beta...} &\to ...\Lambda_\alpha{}^\gamma ... \, t_{...\gamma...}^{...\delta...} \, ...\Lambda^{-1}{}_\delta{}^\beta ... \\ t_{...a...}^{...b...} &\to ...\Lambda_a{}^c ... \, t_{...c...}^{...d...} \, ...\Lambda^{-1}{}_d{}^b ... \\ t_{...\alpha...a...}^{...\beta...b...} &\to ...\Lambda_\alpha{}^\gamma ...\Lambda_a{}^c ... \, t_{...\gamma...c...}^{...\delta...d...} ... \, \Lambda^{-1}{}_\delta{}^\beta ...\Lambda^{-1}{}_d{}^b ... \end{aligned}$$

It is readily checked that, when HD covariance is broken *pseudo-orthogonal components of HD tensors transform like (pseudo-)orthogonal components of LD tensors.* For example, external and internal components of a HD covariant vector $\mathbf{v}_A = \mathbf{r}_A{}^I \mathbf{v}_I$ transform like

$$\mathbf{v}_\alpha \to \Lambda_\alpha{}^\beta \mathbf{v}_\beta \quad \text{and} \quad \mathbf{v}_a \to \Lambda_a{}^b \mathbf{v}_b$$

so that $v_\alpha \equiv \mathbf{v}_\alpha$ and $v_a \equiv \mathbf{v}_a$ may be identified with the components of two LD external and internal vectors. A HD rank-two covariant tensor $\mathbf{b}_{AB}$ produces an external $b_{\alpha\beta} \equiv \mathbf{b}_{\alpha\beta}$, an internal $b_{ab} \equiv \mathbf{b}_{ab}$ and two hybrid $b_{\alpha b} \equiv \mathbf{b}_{\alpha b}$, $b'_{\alpha b} \equiv \mathbf{b}_{b\alpha}$ LD rank-two covariant tensors. This makes the use of pseudo-orthogonal reference frames particularly convenient in investigating dimensional reduction problems.

The whole machinery of calculus on manifolds is readily transposed in the reference frame formalism by defining a covariant derivative acting on both, curved and flat spacetime indices

$$\mathbf{D}_I \mathbf{t}_{...A...} = \boldsymbol{\nabla}_I \mathbf{t}_{...A...} + ... - \boldsymbol{\Omega}_{I,A}{}^B \mathbf{t}_{...B...} + ... \tag{59}$$

with connection coefficients $\boldsymbol{\Omega}_{I,AB} = (\boldsymbol{\nabla}_I \mathbf{r}_A^K)\mathbf{r}_B^L \mathbf{g}_{KL}$. With these conventions $\mathbf{D}_I \mathbf{r}_A^J \equiv 0$. The commutator of two covariant derivatives yields the intrinsic curvature tensor

$$\mathbf{R}_{IJAB} = \partial_I \boldsymbol{\Omega}_{J,AB} - \partial_J \boldsymbol{\Omega}_{I,AB} - \boldsymbol{\Omega}_{I,A}{}^C \boldsymbol{\Omega}_{J,CB} + \boldsymbol{\Omega}_{J,A}{}^C \boldsymbol{\Omega}_{I,CB} \tag{60}$$

which is related to (20) by contraction with reference frames given by $\mathbf{R}_{IJKL} = \mathbf{R}_{IJAB}\mathbf{r}_K^A \mathbf{r}_L^B$. In LD internal and external spaces we proceed along the very same lines.

2.6.1 Internal connection and curvatures

On internal spaces, we define an *internal total covariant derivative* D_i^{tot} as

$$D_i^{\text{tot}} t_{...\alpha...a...} = \nabla_i^{\text{tot}} t_{...\alpha...a...} + ... - \Omega_{i,a}{}^b t_{...\alpha...b...} + + ... - A_{i,\alpha}{}^\beta t_{...\beta...a...} + ... \tag{61}$$

with connection coefficients $\Omega_{i,ab} = (\nabla_i^{\text{tot}} \rho_a{}^k)\rho_b{}^l h_{kl}$ and $A_{i,\alpha\beta} = (\nabla_i^{\text{tot}} r_\alpha{}^\kappa) r_\beta{}^\lambda g_{\kappa\lambda}$. Under coordinate redefinitions $\Omega_{i,ab}$ and $A_{i,\alpha\beta}$ transform like genuine internal tensors. Under local (pseudo-)rotations of reference frames, $\Omega_{i,ab}$ transforms like an $SO(c)$ gauge connection while $A_{i,\alpha\beta}$ behave like a tensor

$$\Omega_{i,a}{}^b \rightarrow \Lambda_a{}^c \Omega_{i,c}{}^d \Lambda_d{}^b - \Lambda_a{}^c(\partial_i \Lambda_c{}^b) \tag{62}$$

$$A_{i,\alpha}{}^\beta \rightarrow \Lambda_\alpha{}^\gamma A_{i,\gamma}{}^\delta \Lambda_\delta{}^\beta \tag{63}$$

With these conventions $D_i^{\text{tot}} \rho_a{}^j = 0$ and $D_i^{\text{tot}} r_\alpha{}^\mu = 0$. The commutator of two total internal covariant derivatives yields the intrinsic and extrinsic curvature tensors

$$R_{ijab} = \partial_i \Omega_{j,ab} - \partial_j \Omega_{i,ab} - \Omega_{i,a}{}^c \Omega_{j,cb} + \Omega_{j,a}{}^c \Omega_{i,cb} \tag{64}$$

and

$$F_{ij\alpha\beta} = \partial_i A_{j,\alpha\beta} - \partial_j A_{i,\alpha\beta} - A_{i,\alpha}{}^\gamma A_{j,\gamma\beta} + A_{j,\alpha}{}^\gamma A_{i,\gamma\beta} \tag{65}$$

which are related to (23) and (29) by contraction with LD reference frames, $R_{ijkl} = R_{ijab}\rho_k{}^a\rho_l{}^b$ and $F_{ij\kappa\lambda} = F_{ij\alpha\beta}r_\kappa{}^\alpha r_\lambda{}^\beta$.

2.6.2 External connection and curvatures

On the external space, we define an *external total covariant derivative* $\hat{D}_\mu^{\text{tot}}$ as

$$\hat{D}_\mu^{\text{tot}} t_{\ldots\alpha\ldots a\ldots} = \hat{\nabla}_\mu^{\text{tot}} t_{\ldots\alpha\ldots a\ldots} + \ldots - \hat{\Omega}_{\mu,\alpha}{}^\beta t_{\ldots\beta\ldots a\ldots} + \ldots - \hat{A}_{\mu,a}{}^b t_{\ldots\alpha\ldots b\ldots} + \ldots \quad (66)$$

with connection coefficients $\hat{\Omega}_{\mu,\alpha\beta} = (\hat{\nabla}_\mu^{\text{tot}} r_\alpha{}^\kappa) r_\beta{}^\lambda g_{\kappa\lambda}$ and $\hat{A}_{\mu,ab} = (\hat{\nabla}_\mu^{\text{tot}}\rho_a{}^k)\rho_b{}^l h_{kl}$. Under coordinate transformations $\hat{\Omega}_{\mu,\alpha\beta}$ and $\hat{A}_{\mu,ab}$ behaves like genuine external tensors. Under local redefinition of reference frames $\hat{\Omega}_{\mu,\alpha\beta}$ and $\hat{A}_{\mu,ab}$ transform as $SO(d)$ and $SO(c)$ connections respectively

$$\hat{\Omega}_{\mu,\alpha}{}^\beta \to \Lambda_\alpha{}^\gamma \hat{\Omega}_{\mu,\gamma}{}^\delta \Lambda_\delta{}^\beta - \Lambda_\alpha{}^\gamma(\hat{\partial}_\mu \Lambda_\gamma{}^\beta) \quad (67)$$

$$\hat{A}_{\mu,a}{}^b \to \Lambda_a{}^c \hat{A}_{\mu,c}{}^d \Lambda_d{}^b - \Lambda_a{}^c(\hat{\partial}_\mu \Lambda_c{}^b) \quad (68)$$

As above $\hat{\nabla}_\mu^{\text{tot}} r_\alpha{}^\nu = 0$ and $\hat{\nabla}_\mu^{\text{tot}}\rho_a{}^i = 0$. The commutator of two external total covariant derivative again yields the intrinsic and extrinsic curvature tensors

$$\hat{R}_{\mu\nu\alpha\beta} = \hat{\partial}_\mu \hat{\Omega}_{\nu,\alpha\beta} - \hat{\partial}_\nu \hat{\Omega}_{\mu,\alpha\beta} - \hat{\Omega}_{\mu,\alpha}{}^\gamma \hat{\Omega}_{\nu,\gamma\beta} + \hat{\Omega}_{\nu,\alpha}{}^\gamma \hat{\Omega}_{\mu,\gamma\beta} + f^i_{\mu\nu} A_{i,\alpha\beta} \quad (69)$$

and

$$\hat{F}_{\mu\nu ab} = \hat{\partial}_\mu \hat{A}_{\nu,ab} - \hat{\partial}_\nu \hat{A}_{\mu,ab} - \hat{A}_{\mu,a}{}^c \hat{A}_{\nu,cb} + \hat{A}_{\nu,a}{}^c \hat{A}_{\mu,cb} + f^i_{\mu\nu}\Omega_{i,ab} \quad (70)$$

again related to (36) and (49) by contraction with LD reference frames, $\hat{R}_{\mu\nu\kappa\lambda} = \hat{R}_{\mu\nu\alpha\beta} r_\kappa{}^\alpha r_\lambda{}^\beta$ and $\hat{F}_{\mu\nu kl} = \hat{F}_{\mu\nu ab}\rho_k{}^a\rho_l{}^b$.

2.6.3 Hybrid curvatures

The commutator of total external and internal derivative yields the hybrid curvatures

$$\hat{H}_{\mu i\alpha\beta} = \hat{\partial}_\mu A_{i,\alpha\beta} - \partial_i \hat{\Omega}_{\mu,\alpha\beta} - \hat{\Omega}_{\mu,\alpha}{}^\gamma A_{i,\gamma\beta} + A_{i,\alpha}{}^\gamma \hat{\Omega}_{\mu,\gamma\beta} - (\partial_i a^j_\mu) A_{j,\alpha\beta} \quad (71)$$

and

$$H_{\mu iab} = \hat{\partial}_\mu \Omega_{i,ab} - \partial_i \hat{A}_{\mu,ab} - \hat{A}_{\mu,a}{}^c \Omega_{i,cb} + \Omega_{i,a}{}^c \hat{A}_{\mu,cb} - (\partial_i a^j_\mu)\Omega_{j,ab} \quad (72)$$

related to (51) and (52) by contraction with LD reference frames and that can be rewritten in terms of the pseudo-orthogonal components of fundamental forms

$$E_{\gamma ab} = r_\gamma{}^\kappa \rho_a{}^i \rho_b{}^j E_{\kappa ij}, \quad \hat{E}_{c\alpha\beta} = \rho_c{}^k r_\alpha{}^\mu r_\beta{}^\nu \hat{E}_{k\mu\nu} \tag{73}$$

Nothing has really changed; the pseudo-Euclidean tensors $\hat{R}_{\alpha\beta\gamma\delta}$, R_{abcd}, $E_{\gamma ab}$ and $\hat{E}_{c\alpha\beta}$ completely characterize the geometry of dimensional reduction.

3 Reducing Geometry

We are now in position to write down general equations that relate the higher and lower dimensional geometries. In holonomic coordinates this task requires very long and tedious calculations with results that are not always transparent. Instead, within the reference frames formalism, it is almost straightforward to establish the desired relations. The formulas obtained in this section extend and unify well known identities of Kaluza-Klein and submanifold theories.

3.1 Connection coefficients

In the reference frames formalism, HD connection coefficients directly relate to LD intrinsic connection coefficients, fundamental forms and extrinsic connection coefficients in the following way

$$\mathbf{r}_\gamma{}^I \mathbf{\Omega}_{I,\alpha\beta} = r_\gamma{}^\mu \hat{\Omega}_{\mu,\alpha\beta} \tag{74}$$

$$\mathbf{r}_c{}^I \mathbf{\Omega}_{I,ab} = \rho_c{}^i \Omega_{i,ab} \tag{75}$$

$$\mathbf{r}_\gamma{}^I \mathbf{\Omega}_{I,a\beta} = \hat{E}_{a\gamma\beta} \tag{76}$$

$$\mathbf{r}_c{}^I \mathbf{\Omega}_{I,\alpha b} = E_{\alpha cb} \tag{77}$$

$$\mathbf{r}_\gamma{}^I \mathbf{\Omega}_{I,ab} = r_\gamma{}^\mu \hat{A}_{\mu,ab} \tag{78}$$

$$\mathbf{r}_c{}^I \mathbf{\Omega}_{I,\alpha\beta} = \rho_c{}^i A_{i,\alpha\beta} \tag{79}$$

Analogous equations connecting HD Christoffel symbols with LD quantities are much more complicated. By means of relations (74)-(79) it is straightforward to relate HD to LD curvatures, geodesic equations and geometric operators.

Gauss, Codazzi and Ricci equations give relations between HD curvature and LD curvatures, second and normal fundamental forms of a submanifold and provide, at the same time, integrability conditions for a subspace to be embeddable in a HD spacetime [1]. They are important in a variety of physical applications, especially in general relativity. Recently, they have been extended to foliations and applied to the analysis of embedded spacetimes [6]. Equations of an apparently different nature relating HD curvature to LD curvatures and gauge fields are also the key ingredient of Kaluza-Klein unification schemes [3,4,5]. Both set of equations are special cases of the general equations relating the HD Riemann tensor $\mathbf{R}_{ABCD}$ to LD Riemann tensors $\hat{R}_{\alpha\beta\gamma\delta}$, R_{abcd} and fundamental forms $E_{\gamma ab}$, $\hat{E}_{c\alpha\beta}$. The symmetries of the Riemann tensor allow only six independent projections on external/internal directions.

3.2.1 Gauss type equations

The external components of the HD Riemann tensor are related to the external intrinsic curvature and fundamental forms by an equation which is formally identical to the Gauss equation for an embedded space

$$\mathbf{R}_{\alpha\beta\gamma\delta} = \hat{R}_{\alpha\beta\gamma\delta} + \hat{E}_{a\alpha\gamma}\hat{E}^{a}{}_{\beta\delta} - \hat{E}_{a\beta\gamma}\hat{E}^{a}{}_{\alpha\delta} \tag{80}$$

In spite of this analogy it is worth remarking that the external space M_d is not an embedded object, $\hat{R}_{\alpha\beta\gamma\delta}$ is not a standard Riemannian curvature tensor and $\hat{E}_{c\alpha\beta}$ has an antisymmetric part keeping truck of the gauge field $f^i_{\mu\nu}$. The internal components of the HD Riemann tensor are related to the internal intrinsic curvature and the fundamental forms yielding again an equation formally identical to the Gauss equation for an embedded space

$$\mathbf{R}_{abcd} = R_{abcd} + E_{\alpha ac}E^{\alpha}{}_{bd} - E_{\alpha bc}E^{\alpha}{}_{ad} \tag{81}$$

This time the analogy is more than formal. For every given value x^μ of the external coordinates the internal space M^x_c is an embedded object in $\mathbf{M}_D$. In this case, $R_{abcd}|_x$ is the relative Riemann tensor and $E_{\gamma ab}|_x$ is the second fundamental form so that (81) correspond to a genuine Gauss equation for the embedding.

3.2.2 Codazzi type equations

HD Riemann tensor components with three indices of one sort and one index of the other, are related to the LD hybrid curvatures and the fundamental forms. These terms yield the generalization of the Codazzi equation for the external space M_d

$$\mathbf{R}_{\alpha b\gamma\delta} = \hat{H}_{\alpha b\gamma\delta} + E_{\gamma b}{}^{a}\hat{E}_{a\alpha\delta} - E_{\delta b}{}^{a}\hat{E}_{a\alpha\gamma} \tag{82}$$

and for the foliation of $\mathbf{M}_D$ in the internal spaces M_c^x

$$\mathbf{R}_{\alpha bcd} = H_{\alpha bcd} - \hat{E}_{c\alpha}{}^{\alpha}E_{\alpha bd} + \hat{E}_{d\alpha}{}^{\alpha}E_{\alpha bc} \tag{83}$$

The explicit appearance of the hybrid curvatures $\hat{H}_{\alpha b\gamma\delta}$ and $H_{\alpha bcd}$ can be eliminated by means of (53) and (54), giving the Codazzi equations in their more familiar form

$$\mathbf{R}_{\alpha b\gamma\delta} = \hat{D}_{\delta}^{\mathrm{tot}}\hat{E}_{b\gamma\alpha} - \hat{D}_{\gamma}^{\mathrm{tot}}\hat{E}_{b\delta\alpha} + f_{\gamma\delta}^{i}E_{\alpha ib} \tag{84}$$

$$\mathbf{R}_{\alpha bcd} = D_{c}^{\mathrm{tot}}E_{\alpha db} - D_{d}^{\mathrm{tot}}E_{\alpha cb} \tag{85}$$

The interpretation of these equations requires the same caution used for generalized Gauss equations. While (85) are genuine Codazzi equations for the embedded spaces M_c^x, (84) correspond to standard Codazzi equations only when $f_{\mu\nu}^{i} = 0$ and the external space M_d reduce to an embedded object.

3.2.3 Ricci type equations

HD Riemann tensor components with the first two indices of one sort and the last two indices of the other, relate the LD extrinsic curvatures (49), (29) to the hybrid tensor (8), yielding a single equation

$$\mathbf{R}_{\alpha\beta cd} = \hat{F}_{\alpha\beta cd} + F_{cd\alpha\beta} - \frac{1}{2}r_{\alpha}{}^{\mu}r_{\beta}{}^{\nu}\rho_{c}{}^{k}\rho_{d}{}^{l}\left(\partial_k f_{l\mu\nu} - \partial_l f_{k\mu\nu}\right) \tag{86}$$

This generalizes the Ricci equation for both, the external space M_d and the foliation in internal subspaces M_c^x. The explicit appearance of the external extrinsic curvature $\hat{F}_{\alpha\beta cd}$ or of the internal extrinsic curvature $F_{cd\alpha\beta}$ (or of both of them), can be removed by means of (50) and (30). It respectively yields the standard form of the Ricci equation for the external space M_d

$$\mathbf{R}_{\alpha\beta cd} = \hat{F}_{\alpha\beta cd} + \hat{E}_{c\alpha}{}^{\gamma}\hat{E}_{d\beta\gamma} - \hat{E}_{c\beta}{}^{\gamma}\hat{E}_{d\alpha\gamma} \tag{87}$$

and for the foliation in internal spaces M_c^x

$$\mathbf{R}_{\alpha\beta cd} = F_{cd\alpha\beta} + E_{\alpha c}{}^{a}E_{\beta da} - E_{\beta c}{}^{a}E_{\alpha da} \tag{88}$$

Once again, a little caution in the interpretation of (86), or (87), or (88), is necessary.

3.2.4 The sixth equation

The remaining group of HD Riemann tensor components relates the fundamental forms to their total covariant derivatives, yielding an equation that has no equivalent in the theory of embedding

$$\mathbf{R}_{\alpha b\gamma d} = \hat{D}^{\text{tot}}_{\alpha} E_{\gamma bd} + D^{\text{tot}}_{b} \hat{E}_{d\alpha\gamma} + E_{\alpha b}{}^{a} E_{\gamma ad} + \hat{E}_{b\alpha}{}^{\beta} \hat{E}_{d\beta\gamma} \tag{89}$$

This equation appears as a further integrability condition for the tangling of M_d and M^x_c in $\mathbf{M}_D$ and consists a new result obtained by this approach.

3.3 Ricci curvatures

By contracting the generalized Gauss, Codazzi, Ricci equations and (89) we easily obtain the external

$$\mathbf{R}_{\alpha\beta} = \hat{R}_{\alpha\beta} + \hat{D}^{\text{tot}}_{\alpha} E_{\beta c}{}^{c} + E_{\alpha cd} E_{\beta}{}^{dc} + D^{\text{tot}}_{c} \hat{E}^{c}{}_{\alpha\beta} + \hat{E}_{c\alpha\beta} \hat{E}^{c}{}_{\gamma}{}^{\gamma} \tag{90}$$

hybrid

$$\mathbf{R}_{\alpha b} = \hat{D}^{\text{tot}}_{\alpha} \hat{E}_{b\gamma}{}^{\gamma} - \hat{D}^{\text{tot}}_{\gamma} \hat{E}_{b\alpha}{}^{\gamma} - f^{c}_{\alpha\gamma} \hat{E}^{\gamma}{}_{cb} + D^{\text{tot}}_{b} E_{\alpha c}{}^{c} - D^{\text{tot}}_{c} E_{\alpha b}{}^{c} \tag{91}$$

and internal

$$\mathbf{R}_{ab} = R_{ab} + \hat{D}^{\text{tot}}_{\gamma} E^{\gamma}{}_{ab} + E_{\gamma ab} E^{\gamma}{}_{c}{}^{c} + D^{\text{tot}}_{a} \hat{E}_{b\gamma}{}^{\gamma} + \hat{E}_{a\gamma\delta} \hat{E}_{b}{}^{\delta\gamma} \tag{92}$$

components of the HD Ricci tensor. From the viewpoint of pure higher dimensional gravity these equations display the most general kind of LD matter that can be obtained in induced-matter theories [10].

3.4 Scalar curvatures

The eventual contraction of equations (90), (91), (92) yields the identity connecting the HD scalar curvature with LD intrinsic and extrinsic curvatures, lying at the heart of Lagrangian reduction of HD Einstein gravity. We display it in standard tensor formalism

$$\mathbf{R} = \hat{R} + 2\nabla_i \hat{E}^i{}_\mu{}^\mu + \hat{E}_{i\mu}{}^\mu \hat{E}^i{}_\nu{}^\nu + \hat{E}_{i\mu\nu}\hat{E}^{i\nu\mu} + \\ + R + 2\hat{\nabla}_\mu E^\mu{}_i{}^i + E_{\mu i}{}^i E^\mu{}_j{}^j + E_{\mu ij}E^{\mu ji} \tag{93}$$

This equation generalizes well known relations holding in Kaluza-Klein and submanifold theories.

Kaluza-Klein: In virtue of (27), (37) and (45) equation (93) reduces to

$$\mathbf{R} = \mathrm{R} + R + \frac{1}{4}\mathrm{F}^{\mathsf{a}}_{\mu\nu}\mathrm{F}^{\mathsf{a}\mu\nu} \tag{94}$$

with R the standard scalar curvature associated with the four dimensional metric $\mathrm{g}_{\mu\nu}(x)$.

Embedded spacetime: By recalling (28), (38), (47) and the fact that we are considering spacetimes embedded in flat HD spacetime equation (93) evaluated at $y^i = 0$ reproduces the well known identity

$$\mathrm{R} + \mathrm{II}_{i\mu}{}^\mu \mathrm{II}^i{}_\nu{}^\nu - \mathrm{II}_{i\mu\nu}\mathrm{II}^{i\mu\nu} = 0 \tag{95}$$

that relates intrinsic and extrinsic curvature scalars for a submanifold embedded in a HD flat space.

By means of equations (93), (90), (91), (92) and the equations in Subsection 3.2 it is also possible to obtain general reduction formulas for the conformal Weyl tensor, which also plays an important role in the analysis of dimensional reduction [11].

3.5 Geodesic motion

Free motion in HD spacetime is described by geodesic equations

$$\ddot{\mathbf{x}}^K + \mathbf{\Gamma}^K_{IJ}\dot{\mathbf{x}}^I\dot{\mathbf{x}}^J = 0 \tag{96}$$

where $\dot{\mathbf{x}}^I = d\mathbf{x}^I/d\tau$ is the HD velocity vector. As discussed in Subsection 2.1 external contravariant components of HD vectors behave like LD vectors, so that $\dot{\mathbf{x}}^\mu = \dot{x}^\mu$ is identified with the LD external velocity. On the other hand internal contravariant components do not, so that $\dot{\mathbf{x}}^i = \dot{y}^i$ is not a LD object. The definition of a LD vector once again involves a^i_μ

$$\hat{\dot{y}}^i = \dot{y}^i + a^i_\mu \dot{x}^\mu$$

HD geodesic equations split in two groups that separately transform under the residual covariance group (2). The first group describes a no longer free motion in external directions and its coupling to internal variables through the fundamental forms, is given by

$$\ddot{x}^\kappa + \hat{\Gamma}^\kappa_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + 2\hat{E}_i{}^\kappa{}_\mu\hat{\dot{y}}^i\dot{x}^\mu - E^\kappa{}_{ij}\hat{\dot{y}}^i\hat{\dot{y}}^j = 0 \tag{97}$$

The second group takes in to account internal motion and its dynamical interaction with external variables

$$\dot{\hat{\dot{y}}}^k + \Gamma^k_{ij}\hat{\dot{y}}^i\hat{\dot{y}}^j + (\partial_i a^k_\mu)\dot{x}^\mu\hat{\dot{y}}^i + 2E_\mu{}^k{}_i\dot{x}^\mu\hat{\dot{y}}^i - \hat{E}^k{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \tag{98}$$

(the first three terms of the left hand side can be recast in the LD covariant expression $\dot{x}^\mu\hat{\nabla}^{\mathrm{tot}}_\mu\hat{\dot{y}}^k + \hat{\dot{y}}^i\nabla^{\mathrm{tot}}_i\hat{\dot{y}}^k - E_\mu{}^k{}_i\hat{\dot{y}}^i\dot{x}^\mu$). The interaction between internal and external motion vanishes if and only if the fundamental forms identically vanish, $\hat{E}_{i\mu\nu} = 0$ and $E_{\mu ij} = 0$. Specializing to Kaluza-Klein and embedded spacetime models we obtain:

Kaluza-Klein: Taking into account (27), (45) equations (97), (98) reduce to

$$\ddot{x}^\kappa + \Gamma^\kappa_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \mathrm{q_a}\mathrm{F^a}_\mu{}^\kappa\dot{x}^\mu = 0, \quad \dot{\mathrm{q}}_\mathsf{a} - c^\mathsf{c}_{\mathsf{ab}}\dot{x}^\mu\mathrm{A}^\mathsf{b}_\mu\mathrm{q_c} = 0 \tag{99}$$

where $\mathrm{q_a} = \mathrm{K}_{\mathsf{a}i}(y)\hat{\dot{y}}^i$. The first equation describes the external motion of a particle of vector charge $\mathrm{q_a}$ in the possibly non-Abelian gauge field $\mathrm{F^a}_{\mu\nu}$. The second equation describes the rotation of the charge-vector in the group space. In the case of a one dimensional Abelian group $\mathrm{q_1}$ is constant in time and the first equation reduces to the classical Lorentz equation of a charged particle moving on a manifold in an electromagnetic field.

Embedded spacetime: In a neighborhood of radius ϵ of a submanifold the equations (28), (47) allow to rewrite (97), (98) in the form

$$\begin{aligned} &\ddot{x}^\kappa + \Gamma^\kappa_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \tfrac{1}{2}\mathrm{L}_{ij}\mathrm{F}_\mu{}^{\kappa ij}\dot{x}^\mu + 2\hat{\dot{y}}^i\mathrm{II}_{i\mu}{}^\kappa\dot{x}^\mu + \mathcal{O}(\epsilon) = 0 \\ &\dot{\mathrm{L}}^{ij} - \dot{x}^\mu\mathrm{A}_\mu{}^i{}_k\mathrm{L}^{kj} + \dot{x}^\mu\mathrm{A}_\mu{}^j{}_k\mathrm{L}^{ki} + \mathcal{O}(\epsilon) = 0 \end{aligned} \tag{100}$$

where $\mathrm{L}^{ij} = y^i\hat{\dot{y}}^j - y^j\hat{\dot{y}}^i$ is the angular momentum in internal directions and $\mathrm{II}_{i\mu\nu}(x)$ the second fundamental form of the embedding. Higher order terms in ϵ can be neglected only if some physical mechanism constrains the system in a sufficiently small neighborhood of the submanifold. As in standard Kaluza-Klein theories the first equation describes the external motion of a particle of charge $\frac{1}{2}\mathrm{L}_{ij}$ in the gauge field $\mathrm{F}_{\mu\nu}{}^{ij}$; the non-trivial dependence of external metric on internal coordinates produces the extra term $2\hat{y}^i \mathrm{II}_{i\mu}{}^{\kappa}\dot{x}^\mu$ making geodesics to drift away from the submanifold. The second equation describes the precession of internal angular momentum produced by the extrinsic torsion of the embedding.

Equations (97), (98) can also be obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}h_{ij}\left(\dot{y}^i + a^i_\mu\dot{x}^\mu\right)\left(\dot{y}^j + a^j_\nu\dot{x}^\nu\right) \tag{101}$$

For later considerations it is also useful to write the corresponding Hamiltonian

$$\mathcal{H} = \frac{1}{2}g^{\mu\nu}\left(p_\mu - a^i_\mu\pi_i\right)\left(p_\nu - a^j_\nu\pi_j\right) + \frac{1}{2}h^{ij}\pi_i\pi_j \tag{102}$$

with $p_\mu = \partial\mathcal{L}/\partial\dot{x}^\mu$, $\pi_i = \partial\mathcal{L}/\partial\dot{y}^i$ the momenta conjugated to external and internal coordinates respectively. Internal momenta π_i correctly transform as LD vectors, while LD external covariant momenta have to be defined as $\hat{p}_\mu \equiv p_\mu - a^i_\mu\pi_i$.

3.6 Geometric operators

We now consider the dimensional reduction of Laplace and Dirac operators.

3.6.1 Laplace operator

In every local coordinate frame the HD scalar Laplace operator $\boldsymbol{\Delta}$ takes the form

$$\boldsymbol{\Delta} = |\mathbf{g}|^{-1/2}\partial_I\mathbf{g}^{IJ}|\mathbf{g}|^{1/2}\partial_J \tag{103}$$

$\boldsymbol{\Delta}$ is Hermitian with respect to the standard scalar product constructed by means of the HD covariant measure $|\mathbf{g}|^{1/2}d\mathbf{x} = |g|^{1/2}|h|^{1/2}dxdy$. By rewriting the operator in terms of covariant derivatives, recalling the inverse metric decomposition and the relations (74)-(79) between HD and LD connection coefficients, we obtain the most general decomposition covariant under (2)

$$\begin{aligned}\boldsymbol{\Delta} = {} & |g|^{-1/2}\left(\hat{\partial}_\mu+\frac{1}{2}E_{\mu i}{}^{i}\right)g^{\mu\nu}|g|^{1/2}\left(\hat{\partial}_\nu+\frac{1}{2}E_{\nu i}{}^{i}\right)+\\ & +|h|^{-1/2}\left(\partial_i+\frac{1}{2}\hat{E}_{i\mu}{}^{\mu}\right)h^{ij}|h|^{1/2}\left(\partial_j+\frac{1}{2}\hat{E}_{j\mu}{}^{\mu}\right)+\\ & -\frac{1}{2}\hat{\nabla}_\mu E^{\mu}{}_{i}{}^{i}-\frac{1}{4}E_{\mu i}{}^{i}E^{\mu}{}_{j}{}^{j}-\frac{1}{2}\nabla_i\hat{E}^{i}{}_{\mu}{}^{\mu}-\frac{1}{4}\hat{E}_{i\mu}{}^{\mu}\hat{E}^{i}{}_{\nu}{}^{\nu}\end{aligned} \tag{104}$$

The first righthand side term of this equation corresponds to the LD external Laplace operator

$$\Delta^{\text{ext}} = |g|^{-1/2}\partial_\mu g^{\mu\nu}|g|^{1/2}\partial_\nu \tag{105}$$

(Hermitian with respect to the external scalar product constructed by means of the LD volume element $|g|^{1/2}dx$) with partial derivatives ∂_μ replaced by the HD Hermitian operators

$$\hat{\partial}_\mu + \frac{1}{2}E_{\mu i}{}^{i} = \partial_\mu + \left(\partial_\mu \ln |h|^{1/4}\right) - ia_\mu - \frac{1}{2}\nabla_i a^i_\mu$$

The total derivative $\partial_\mu \ln |h|^{1/4}$ takes into account the different normalization of HD and LD states. It amounts to the rescaling $\Delta^{\text{ext}} \to |h|^{-1/4}\Delta^{\text{ext}}|h|^{1/4}$. The Hermitian internal operator $a_\mu - \frac{i}{2}\nabla_i a^i_\mu$ enters the expression as a gauge potential. The second righthand term of (104) corresponds to the LD internal Laplace operator

$$\Delta^{\text{int}} = |h|^{-1/2}\partial_i h^{ij}|h|^{1/2}\partial_j \tag{106}$$

(Hermitian with respect to the internal scalar product constructed by means of the LD volume element $|h|^{1/2}dy$) with partial derivatives ∂_i replaced by

$$\partial_i + \frac{1}{2}\hat{E}_{i\mu}{}^{\mu} = \partial_i + \left(\partial_i \ln |g|^{1/4}\right)$$

As above, the total derivative amounts to the rescaling $\Delta^{\text{int}} \to |g|^{-1/4}\Delta^{\text{int}}|g|^{1/4}$, necessary to correct the different normalization of HD and LD states. The remaining terms in the righthand side of (104) are identified with a scalar potential induced by dimensional reduction. They are know to produce observable effects in low energy physics [8]. Specializing to Kaluza-Klein and embedded spacetime models we obtain:

Kaluza-Klein: By recalling (14), (27), (37), (45), the fact that $\nabla_i \mathrm{K}^i_{\mathsf{a}} = 0$ and assuming that the external metric only depends on external coordinates, (104) reduces to the well known expression

$$\boldsymbol{\Delta}^{\text{KK}} = |\mathrm{g}|^{-1/2}\left(\partial_\mu - i\mathrm{A}^{\mathsf{a}}_\mu\hat{\mathrm{K}}_{\mathsf{a}}\right)\mathrm{g}^{\mu\nu}|\mathrm{g}|^{1/2}\left(\partial_\nu - i\mathrm{A}^{\mathsf{a}}_\nu\hat{\mathrm{K}}_{\mathsf{a}}\right) + |\kappa|^{-1/2}\partial_i\kappa^{ij}|\kappa|^{1/2}\partial_j \tag{107}$$

where $\hat{\mathrm{K}}_{\mathsf{a}} = -i\mathrm{K}^i_{\mathsf{a}}\partial_i$ are infinite dimensional Hermitian generators of the isometry algebra.

Embedded spacetime: By recalling (17), (28), (38), (47), the fact that $\nabla_i \mathrm{A}_j{}^i y^j = \mathrm{A}_i{}^i = 0$ and after rescaling fields and operators by

$$\begin{aligned} \mathbf{\Psi} &\to |g|^{1/4}|\mathrm{g}|^{-1/4}\mathbf{\Psi} \\ \mathbf{\Delta} &\to |g|^{1/4}|\mathrm{g}|^{-1/4}\mathbf{\Delta}\ |g|^{-1/4}|\mathrm{g}|^{1/4} \end{aligned} \tag{108}$$

in a neighborhood of radius ϵ of M_d (104) reduces to

$$\begin{aligned} \mathbf{\Delta}^{\mathrm{emb}} = |\mathrm{g}|^{-1/2}\left(\partial_\mu - \tfrac{i}{2}\mathrm{A}_\mu{}^{ij}\mathrm{L}_{ij}\right)\mathrm{g}^{\mu\nu}|\mathrm{g}|^{1/2}\left(\partial_\nu - \tfrac{i}{2}\mathrm{A}_\mu{}^{kl}\mathrm{L}_{kl}\right) + \\ + \tfrac{1}{2}\mathrm{II}_{i\mu\nu}\mathrm{II}^{i\mu\nu} - \tfrac{1}{4}\mathrm{II}_{i\mu}{}^{\mu}\mathrm{II}^{i}{}_{\nu}{}^{\nu} + \partial^i\partial_i + \mathcal{O}(\epsilon) \end{aligned} \tag{109}$$

where $\mathrm{L}_{ij} = -i(y_i\partial_j - y_j\partial_i)$ are orbital angular momentum operators in internal directions.

3.6.2 Dirac operator

The HD Dirac operator $\not{\mathbf{D}}$ acts on $2^{[D/2]}$-dimensional Dirac fermions. In every local coordinate frame $\not{\mathbf{D}}$ is written in terms of HD gamma matrices $\boldsymbol{\gamma}^A$, reference frames, partial derivatives, pseudo-orthogonal connection coefficients and spin pseudo-orthogonal generators $\mathbf{\Sigma}^{AB} = -\frac{i}{4}[\boldsymbol{\gamma}^A, \boldsymbol{\gamma}^B]$ as

$$\not{\mathbf{D}} = \boldsymbol{\gamma}^C \mathbf{r}_C^I \left(\partial_I - \frac{i}{2}\mathbf{\Omega}_{I,AB}\mathbf{\Sigma}^{AB}\right) \tag{110}$$

$\not{\mathbf{D}}$ is Hermitian with respect to the measure constructed by means of Dirac adjoint and HD covariant volume element $|\mathbf{g}|^{1/2}d\mathbf{x}$. HD gamma matrices $\boldsymbol{\gamma}^A$ can be decomposed in terms of LD external γ^α and internal γ^a gamma matrices as

$$\boldsymbol{\gamma}^\alpha = \gamma^\alpha \otimes \mathbf{1}^{\mathrm{int}}\ , \quad \boldsymbol{\gamma}^a = \gamma^{\mathrm{ext}} \otimes \gamma^a$$

where here and in what follows, $\mathbf{1}^{\mathrm{ext}}$, γ^{ext} and $\mathbf{1}^{\mathrm{int}}$, γ^{int} denote identity and chiral matrices in external and internal spin spaces, respectively. Correspondingly, the HD spin generators $\mathbf{\Sigma}^{AB}$ decompose in terms of LD external $\Sigma^{\alpha\beta} = -\frac{i}{4}[\gamma^\alpha, \gamma^\beta]$ and internal $\Sigma^{ab} = -\frac{i}{4}[\gamma^a, \gamma^b]$ ones as

$$\mathbf{\Sigma}^{\alpha\beta} = \Sigma^{\alpha\beta} \otimes \mathbf{1}^{\mathrm{int}}, \quad \mathbf{\Sigma}^{\alpha b} = \frac{i}{2}\gamma^{\mathrm{ext}}\gamma^\alpha \otimes \gamma^b, \quad \mathbf{\Sigma}^{ab} = \mathbf{1}^{\mathrm{ext}} \otimes \Sigma^{ab}$$

By recalling the reference frames decomposition (55), the relation between HD and LD connection coefficients (74)-(79), suppressing –as customary– tensor product symbols and spin identity matrices, we obtain the most general LD decomposition covariant under (2)

$$
\begin{aligned}
\not{D} = \gamma^\gamma {r_\gamma}^\mu \left(\hat{\partial}_\mu + \frac{1}{2}{E_{\mu i}}^i - \frac{i}{2}\hat{A}_{\mu,ab}\Sigma^{ab} - \frac{i}{2}\hat{\Omega}_{\mu,\alpha\beta}\Sigma^{\alpha\beta}\right) + \\
+ \gamma^{\text{ext}}\gamma^c {\rho_c}^i \left(\partial_i + \frac{1}{2}{\hat{E}_{i\mu}}{}^\mu - \frac{i}{2}A_{i,\alpha\beta}\Sigma^{\alpha\beta} - \frac{i}{2}\Omega_{i,ab}\Sigma^{ab}\right) + \\
+ \frac{i}{2}\gamma^{\text{ext}}\gamma^c f_{c\alpha\beta}\Sigma^{\alpha\beta}
\end{aligned}
\tag{111}
$$

The first righthand side term reproduces the four dimensional Dirac operator

$$
\not{D}^{\text{ext}} = \gamma^\gamma {r_\gamma}^\mu \left(\partial_\mu - \frac{i}{2}\Omega_{\mu,\alpha\beta}\Sigma^{\alpha\beta}\right) \tag{112}
$$

with connection coefficients replaced by hatted ones and partial derivatives ∂_μ replaced by

$$
\hat{\partial}_\mu + \frac{1}{2}{E_{\mu i}}^i - \frac{i}{2}\hat{A}_{\mu,ab}\Sigma^{ab}
$$

As in the scalar case, the total derivative hidden in the trace of the second fundamental form $\frac{1}{2}{E_{\mu i}}^i$ corrects the different HD and LD normalization, while the operator gauge potential $a_\mu - \frac{i}{2}\nabla_i a^i_\mu$ is now supplemented by the Hermitian internal spin matrix $\frac{1}{2}\hat{A}_{\mu,ab}\Sigma^{ab}$. The second righthand side term corresponds to γ^{ext} times the internal Dirac operator

$$
\not{D}^{\text{int}} = \gamma^c {\rho_c}^i \left(\partial_i - \frac{i}{2}\Omega_{i,ab}\Sigma^{ab}\right) \tag{113}
$$

with partial derivatives replaced by

$$
\partial_i + \frac{1}{2}{\hat{E}_{i\mu}}{}^\mu - \frac{i}{2}A_{i,\alpha\beta}\Sigma^{\alpha\beta}
$$

Once again $\frac{1}{2}{\hat{E}_{i\mu}}{}^\mu$ remedies the different states normalization, while the Hermitian external spin matrix $\frac{1}{2}A_{i,\alpha\beta}\Sigma^{\alpha\beta}$ enters the expression as a gauge potential. The third righthand term $\frac{i}{2}\gamma^{\text{ext}}\gamma^c f_{c\alpha\beta}\Sigma^{\alpha\beta}$ is an induced Pauli term. Specializing to Kaluza-Klein and embedded spacetime models we obtain:

Kaluza-Klein By recalling (57), (27), (37), (45), we have $\nabla_i \mathrm{K}^i_{\mathsf{a}} = 0$ and assuming that the external metric only depends on external coordinates, (111) reduces to the well known Kaluza-Klein decomposition of the Dirac operator

$$\begin{aligned} \not{\mathcal{D}}^{\mathrm{KK}} &= \gamma^{\alpha}\mathrm{r}_{\alpha}^{\ \mu}\left(\partial_{\mu} - i\mathrm{A}_{\mu}^{\mathsf{a}}\hat{\mathrm{K}}_{\mathsf{a}} - \tfrac{i}{2}\Omega_{\mu,\alpha\beta}\Sigma^{\alpha\beta}\right) + \\ &+\gamma^{\mathrm{ext}}\gamma^{a}k_{a}^{\ i}\left(\partial_{i} - \tfrac{i}{2}\Omega_{i,ab}\Sigma^{ab}\right) + \tfrac{i}{2}\gamma^{\mathrm{ext}}\gamma^{i}\mathrm{F}^{\mathsf{a}}_{\alpha\beta}\mathrm{K}_{\mathsf{a}i}\Sigma^{\alpha\beta} \end{aligned} \tag{114}$$

where

$$\hat{\mathrm{K}}_{\mathsf{a}} = -i\mathrm{K}^{i}_{\mathsf{a}}\partial_{i} + \frac{1}{2}\left[k_{a}^{\ i}(\partial_{i}\mathrm{K}^{j}_{\mathsf{a}})k_{bj} - \mathrm{K}^{i}_{\mathsf{a}}(\partial_{i}k_{a}^{\ j})k_{bj}\right]\Sigma^{ab} \tag{115}$$

are infinite dimensional Hermitian generators of the isometry group algebra.

Embedded spacetime By recalling (58), (28), (38), (47), that $\nabla_{i}\mathrm{A}_{j}^{\ i}y^{j} = 0$ and by rescaling fields and operators by

$$\begin{aligned} \mathbf{\Psi} &\to |g|^{1/4}|\mathrm{g}|^{-1/4}\mathbf{\Psi} \\ \not{\mathcal{D}} &\to |g|^{1/4}|\mathrm{g}|^{-1/4}\not{\mathcal{D}}\,|g|^{-1/4}|\mathrm{g}|^{1/4} \end{aligned} \tag{116}$$

we obtain the following expression for the Dirac operator in neighborhood of radius ϵ of M_{d}

$$\not{\mathcal{D}}^{\,\mathrm{emb}} = \gamma^{\alpha}\mathrm{t}_{\alpha}^{\ \mu}\left(\partial_{\mu} - \tfrac{i}{2}\mathrm{A}_{\mu}^{\ ij}\mathrm{J}_{ij} - \tfrac{i}{2}\Omega_{\mu,\alpha\beta}\Sigma^{\alpha\beta}\right) + \gamma^{\mathrm{ext}}\gamma^{i}\partial_{i} + \mathcal{O}(\epsilon) \tag{117}$$

with $\mathrm{J}_{ij} = \mathrm{L}_{ij} + \Sigma_{ij}$ the total angular momentum in internal directions.

3.6.3 Higher spin operators

HD higher spin operators decompose in the very same way as the sum of LD spin operators, with partial derivatives replaced by 'gauge covariant' ones and the possible addition of scalar potential terms. In particular, external partial derivatives ∂_{μ} are replaced by

$$\hat{\partial}_{\mu} + \frac{1}{2}E_{\mu i}^{\ \ i} - \frac{i}{2}\hat{A}_{\mu,ab}\mathrm{S}^{ab} \tag{118}$$

with S^{ab} appropriate internal spin generators.

4 Gauge Symmetries from Higher Dimensional Covariance

One of the most interesting features of dimensional reduction is the possibility of geometrically inducing gauge structures in the effective LD dynamics.

In this section we see that, after general covariance breaking, residual internal coordinates and reference frame transformations are always perceived by effective LD observers as gauge transformations. We also discuss conditions for the induced gauge group to be finite dimensional, providing a covariant characterization of Kaluza-Klein and other few remarkable backgrounds.
Gauge fields are identified by their coupling to matter and by their transformation rules. From the point of view of classical equation of motion, a quick look to (102) shows that $a^i_\mu \pi_i$ enters the Hamiltonian as a gauge potential. To make the gauge structure explicit we may rewrite the interaction term as $\mathbf{tr}(q a_\mu)$ with $q = i y^j \pi_j$ a suitable charge operator and $a_\mu = -i a^i_\mu \partial_i$ the gauge connection introduced in Subsection 2.2. The corresponding curvature $f_{\mu\nu} = -i f^i_{\mu\nu}\partial_i$ enters the third term of equation (97). From the operatorial/quantum viewpoint, after adapting the state measure to the external spacetime by the scale transformation

$$\mathbf{\Psi} \to |h|^{1/4}\mathbf{\Psi} \quad \text{and} \quad \mathbf{\Delta} \to |h|^{1/4}\mathbf{\Delta}|h|^{-1/4}, \;\; \not{\mathbf{D}} \to |h|^{1/4}\not{\mathbf{D}}|h|^{-1/4} \;\; \dots \tag{119}$$

expressions (104), (111) and (118) taken by Laplace, Dirac and higher spin operators show that

$$\mathcal{A}_\mu = -i a^i_\mu \left(\partial_i - \frac{i}{2}\Omega_{i,ab}\mathrm{S}^{ab}\right) - \frac{i}{2}\nabla_i a^i_\mu + \frac{1}{2}(\partial_\mu \rho_a{}^k)\rho_{bk}\mathrm{S}^{ab} \tag{120}$$

couples to effective LD degrees of freedom as a gauge potential. Under the residual covariance group $\mathcal{A}_\mu$ transforms like a gauge potential:

– internal diffeomorphisms

$$y^i \to \exp\{\xi^k(x,y)\partial_k\} y^i \tag{121}$$

make a^i_μ and hence $\mathcal{A}_\mu$ to transform like

$$\mathcal{A}_\mu \to T\mathcal{A}_\mu T^{-1} + iT(\partial_\mu T^{-1}) \tag{122}$$

with $T = \exp\{-\xi^k\partial_k\}$
– internal reference frame redefinitions

$$\rho_a{}^i \to \Lambda_a{}^b(x,y)\rho_b{}^i \tag{123}$$

make $(\partial_\mu \rho_a{}^k)\rho_{bk}$ and hence $\mathcal{A}_\mu$ to transform like

$$\mathcal{A}_\mu \to \mathit{\Lambda}\mathcal{A}_\mu \mathit{\Lambda}^{-1} + i\mathit{\Lambda}(\partial_\mu \mathit{\Lambda}^{-1}) \tag{124}$$

with $\mathit{\Lambda} = \exp\{\frac{i}{2}\Lambda_{ab}\mathrm{S}^{ab}\}$

The commutator of two gauge covariant derivatives defines the operator $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$. A direct computation yields

$$\mathcal{F}_{\mu\nu} = -if^i_{\mu\nu}\left(\partial_i - \frac{i}{2}\Omega_{i,ab}\mathrm{S}^{ab}\right) - \frac{i}{2}\nabla_i f^i_{\mu\nu} + \frac{1}{2}(\nabla_a f_{b\mu\nu})\mathrm{S}^{ab} + E_{\mu a}{}^{c} E_{\nu bc}\mathrm{S}^{ab} \quad (125)$$

Under internal diffeomorphisms and reference frames redefinitions $\mathcal{F}_{\mu\nu}$ correctly transforms like a gauge curvature

$$\mathcal{F}_{\mu\nu} \to T\mathcal{F}_{\mu\nu}T^{-1} \quad \text{and} \quad \mathcal{F}_{\mu\nu} \to \Lambda\mathcal{F}_{\mu\nu}\Lambda^{-1} \quad (126)$$

$\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ are Hermitian operators –i.e. infinite dimensional matrices– acting on internal tensors/spinors. After HD covariance braking, residual internal covariance is perceived by effective LD observers as an infinite dimensional gauge group, with internal coordinate and spin playing the role of –one of the many possible choices of– gauge indices. The gauge curvature $\mathcal{F}_{\mu\nu}$ receives contributions from two independent LD tensors: $f^i_{\mu\nu}$ and $E_{\mu ij}$. In general, the two contributions are simultaneously active producing an effective infinite-dimensional gauge group. In some special backgrounds the gauge group may reduce to finite dimensions.

4.1 $E_{\mu ij} = 0$: *gauge structures related to the isometric structure of internal spaces*

Let us first consider the case where the internal fundamental form $E_{\mu ij}$ vanishes identically while $f^i_{\mu\nu}$ is arbitrary. This requirement is equivalent to the statement that the induced gauge structure is of the Kaluza-Klein type. We have already seen in Subsection 2.3.2, equations (45) and (47), that gauge structures of the Kaluza-Klein type imply $E_{\mu ij} = 0$. To prove the inverse, we note that under the vanishing of the internal second fundamental form equation (43) implies that

$$\nabla_i f_{j\mu\nu} + \nabla_j f_{i\mu\nu} = 0 \quad (127)$$

In every internal space the vector $f^i_{\mu\nu}$ is Killing. In principle the Killing structure of M^x_c can depend on the external point x. However, the fact that $f^i_{\mu\nu}$ belongs to the Killing algebra also implies that a^i_μ takes values on the same algebra up to a pure gauge term. It is therefore possible to choose internal coordinates in which a^i_μ is Killing, $\nabla_i a_{\mu j} + \nabla_j a_{\mu i} = 0$. In such adapted coordinate frames equation (42) implies $\partial_\mu h_{ij} = 0$, that is $h_{ij}(x, y) = \kappa_{ij}(y)$. Thus, the intrinsic geometry of internal spaces does not depend on the external spacetime point. Having the same intrinsic and extrinsic geometry, all internal spaces are isomorphic: $\mathrm{M}^x_c \equiv \mathcal{K}_c$. By choosing a Killing vector basis $\mathrm{K}^i_{\mathsf{a}}(y)$, $\mathsf{a} = 1, ..., n$, for the isometry algebra $iso(\mathcal{K}_c) \equiv \mathrm{g}_{\mathrm{KK}}$, $[\mathrm{K}_{\mathsf{a}}, \mathrm{K}_{\mathsf{b}}]^i = k_{\mathsf{ab}}^{\mathsf{c}}\mathrm{K}^i_{\mathsf{c}}$, the off-diagonal metric term a^i_μ and the antisymmetric hybrid tensor $f^i_{\mu\nu}$ can be expanded as in (16) or (19) by

$$a^i_\mu(x,y) = \mathrm{A}^{\mathsf{a}}_\mu(x)\mathrm{K}^i_{\mathsf{a}}(y), \quad f^i_{\mu\nu}(x,y) = \mathrm{F}^{\mathsf{a}}_{\mu\nu}(x)\mathrm{K}^i_{\mathsf{a}}(y)$$

with $\mathrm{F}^{\mathsf{c}}_{\mu\nu} = \partial_\mu \mathrm{A}^{\mathsf{c}}_\nu - \partial_\nu \mathrm{A}^{\mathsf{c}}_\mu - k_{\mathsf{ab}}{}^{\mathsf{c}}\mathrm{A}^{\mathsf{a}}_\mu \mathrm{A}^{\mathsf{b}}_\nu$. The gauge potential (120) and the gauge field (125) acting on spin-s matter take then the standard Kaluza-Klein form

$$\mathcal{A}_\mu = \mathrm{A}^{\mathsf{a}}_\mu(x)\hat{\mathrm{K}}_{\mathsf{a}} \tag{128}$$

$$\mathcal{F}_{\mu\nu} = \mathrm{F}^{\mathsf{a}}_{\mu\nu}(x)\hat{\mathrm{K}}_{\mathsf{a}} \tag{129}$$

with

$$\hat{\mathrm{K}}_{\mathsf{a}} = \mathrm{K}^i_{\mathsf{a}}\left(-i\partial_i - \frac{1}{2}\Omega_{i,ab}\mathrm{S}^{ab}\right) + \frac{1}{2}(\nabla_a \mathrm{K}_{\mathsf{a}b})\mathrm{S}^{ab}$$

spin-s valued Hermitian differential operators closing the finite-dimensional algebra $iso(\mathcal{K}_c)$, $[\hat{\mathrm{K}}_{\mathsf{a}}, \hat{\mathrm{K}}_{\mathsf{b}}] = -ik_{\mathsf{ab}}{}^{\mathsf{c}}\hat{\mathrm{K}}_{\mathsf{c}}$.
The theory is still covariant under the whole residual covariance group (2). However, a generic diffeomorphism $T = \exp\{-\xi^i(x,y)\partial_i\}$ will bring a^i_μ outside $iso(\mathcal{K}_c)$. To keep the group structure of the background explicit it is necessary to work in adapted coordinates. This is achieved by restricting the allowed covariance group to Killing transformations, that is, by restricting attention to $\xi^i(x,y) = \epsilon^{\mathsf{a}}(x)\mathrm{K}^i_{\mathsf{a}}(y)$ as standard in Kaluza-Klein theories. In arbitrary coordinate frames, Kaluza-Klein gauge structures are completely characterized by the LD covariant condition

$$E_{\mu ij} = 0 \tag{130}$$

Kaluza-Klein backgrounds, in the strict sense, further require the independence of the induced external metric on internal coordinates, a condition enforced by the vanishing of the symmetric part of the external fundamental form $\hat{E}_{i(\mu\nu)} = 0$. By contrast, with diffeomorphisms, the Killing algebra of a manifold is always finite-dimensional having dimension at most $c(c+1)/2$ [12]. As a consequence, in the Kaluza-Klein context, at least two internal dimensions are necessary to produce non-Abelian gauge structures. Thus, a minimum of seven extra-dimensions is required to realize the Standard Model group $U(1) \times SU(2) \times SU(3)$ [13].

4.2 $E_{\mu ij} - \frac{1}{c}E_{\mu k}{}^k h_{ij} = 0$: gauge structures related to the conformal structure of internal spaces

Let us now weaken the Kaluza-Klein condition by requiring the proportionality of the internal fundamental form $E_{\mu ij}$ to the internal metric h_{ij}, $E_{\mu ij} =$

$\frac{1}{c}E_{\mu k}{}^{k}h_{ij}$ (this condition is trivial in $c = 1$). Assuming this, equation (43) implies that

$$\nabla_i f_{j\mu\nu} + \nabla_j f_{i\mu\nu} = \frac{2}{c}(\nabla_k f^k_{\mu\nu})h_{ij} \tag{131}$$

In every internal space the internal vector $f^i_{\mu\nu}$ belongs to the conformal algebra of the manifold. As above, the fact that $f^i_{\mu\nu}$ belongs to an algebra implies that also a^i_μ belongs to the same algebra up to a pure gauge term. It is then possible to adapt internal coordinates in such a way that $\nabla_i a_{\mu j} + \nabla_j a_{\mu i} = \frac{2}{c}(\nabla_k a^k_\mu)h_{ij}$. Equation (42) implies that $|h|^{1/c}\partial_\mu |h|^{-1/c}h_{ij} = 0$. Hence, in the adapted coordinate system $h_{ij}(x, y) = \lambda(x)\mathrm{c}_{ij}(y)$ for some conformal factor $\lambda(x)$ and some internal metric $\mathrm{c}_{ij}(y)$. All internal spaces are conformal to a given manifold $\mathcal{C}_c$. Choosing a basis $\mathrm{C}^i_{\mathsf{a}}(y)$, $\mathsf{a} = 1, ..., n$ for the conformal algebra $conf(\mathcal{C}_c)$, $[\mathrm{C}_{\mathsf{a}}, \mathrm{C}_{\mathsf{b}}]^i = c_{\mathsf{ab}}^{\mathsf{c}}\mathrm{C}^i_{\mathsf{c}}$, a^i_μ and $f^i_{\mu\nu}$ can be expanded as

$$a^i_\mu(x, y) = \mathrm{A}^{\mathsf{a}}_\mu(x)\mathrm{C}^i_{\mathsf{a}}(y), \quad f^i_{\mu\nu}(x, y) = \mathrm{F}^{\mathsf{a}}_{\mu\nu}(x)\mathrm{C}^i_{\mathsf{a}}(y)$$

where again $\mathrm{F}^{\mathsf{c}}_{\mu\nu} = \partial_\mu \mathrm{A}^{\mathsf{c}}_\nu - \partial_\nu \mathrm{A}^{\mathsf{c}}_\mu - c_{\mathsf{ab}}^{\mathsf{c}}\mathrm{A}^{\mathsf{a}}_\mu \mathrm{A}^{\mathsf{b}}_\nu$. Also (120) and (125) take the standard gauge potential and gauge curvature form

$$\mathcal{A}_\mu = \mathrm{A}^{\mathsf{a}}_\mu(x)\hat{\mathrm{C}}_{\mathsf{a}} \tag{132}$$
$$\mathcal{F}_{\mu\nu} = \mathrm{F}^{\mathsf{a}}_{\mu\nu}(x)\hat{\mathrm{C}}_{\mathsf{a}} \tag{133}$$

where the spin-s valued Hermitian operators $\hat{\mathrm{C}}_{\mathsf{a}}$ take now the slightly more complicated form

$$\hat{\mathrm{C}}_{\mathsf{a}} = \mathrm{C}^i_{\mathsf{a}}\left(-i\partial_i - \frac{1}{2}\Omega_{i,ab}\mathrm{S}^{ab}\right) + \frac{1}{2}(\nabla_a \mathrm{C}_{\mathsf{a}b})\mathrm{S}^{ab} - \frac{i}{2}\nabla_i \mathrm{C}^i_{\mathsf{a}} \tag{134}$$

It is readily checked that the $\hat{\mathrm{C}}_{\mathsf{a}}$ do not depend on external coordinates and close $conf(\mathcal{C}_c)$, $[\hat{\mathrm{C}}_{\mathsf{a}}, \hat{\mathrm{C}}_{\mathsf{b}}] = -ic_{\mathsf{ab}}^{\mathsf{c}}\hat{\mathrm{C}}_{\mathsf{c}}$.
As in the previous case, gauge invariance is only explicit when the allowed covariance group is restricted to conformal transformations $\xi^i(x, y) = \epsilon^{\mathsf{a}}(x)\mathrm{C}^i_{\mathsf{a}}(y)$, while in arbitrary coordinates the background is completely characterized by the LD covariant condition

$$E_{\mu ij} - \frac{1}{c}E_{\mu k}{}^{k}h_{ij} = 0 \tag{135}$$

The conformal algebra of a manifold contains the isometry algebra as subalgebra and is always finite dimensional with maximal dimension $(c+1)(c+2)/2$.

As a consequence non-Abelian gauge fields may be induced even with a single internal dimension.

Example: To check this explicitly we consider a one-dimensional internal space with topology of a circle parameterized by the internal coordinate $\theta \in [-\pi, \pi]$. The corresponding conformal algebra $so(2,1)$ is generated by the vector fields $\mathrm{C}_1^\theta = 1$, $\mathrm{C}_2^\theta = \sin\theta$ and $\mathrm{C}_3^\theta = \cos\theta$. Assuming the off-diagonal term of the HD metric to be of the form

$$a_\mu^\theta(x,\theta) = A_\mu^1(x) + A_\mu^2(x)\sin\theta + A_\mu^3(x)\cos\theta \tag{136}$$

the vector field (125) rewrites like in (133) with

$$\hat{\mathrm{C}}_1 = -i\frac{\partial}{\partial\theta}, \;\; \hat{\mathrm{C}}_2 = -i\sin\theta\frac{\partial}{\partial\theta} - \frac{i}{2}\cos\theta, \;\; \hat{\mathrm{C}}_3 = -i\cos\theta\frac{\partial}{\partial\theta} + \frac{i}{2}\sin\theta$$

which are easily checked to close the $so(2,1)$ algebra

$$[\hat{\mathrm{C}}_1, \hat{\mathrm{C}}_2] = i\hat{\mathrm{C}}_3, \;\; [\hat{\mathrm{C}}_2, \hat{\mathrm{C}}_3] = -i\hat{\mathrm{C}}_1, \;\; [\hat{\mathrm{C}}_3, \hat{\mathrm{C}}_1] = i\hat{\mathrm{C}}_2$$

We should remark, however, that $so(2,1)$ is the only non-Abelian Lie algebra that can be embedded in $\mathit{diff}(\mathrm{M}_1)$.

4.3 $f_{\mu\nu}^i = 0$: gauge structures related to the local freedom of choosing internal reference frames

Eventually, we consider the case where the antisymmetric hybrid tensor $f_{\mu\nu}^i$ vanishes identically while $E_{\mu ij}$ is arbitrary. Under these circumstances it is always possible to choose internal coordinates in such a way that the off-diagonal block of the HD metric vanishes identically $a_\mu^i = 0$. In such adapted coordinate systems the internal fundamental form reduces to the external derivative of the internal metric

$$E_{\mu ij} = \frac{1}{2}\partial_\mu h_{ij} \tag{137}$$

The gauge potential (120) and the gauge curvature (125) acting on spin-s matter take the form of standard (pseudo-)orthogonal gauge fields, with internal spin generators S^{ab} playing the role of gauge algebra generators

$$\mathcal{A}_\mu = \frac{1}{2}(\partial_\mu {\rho_a}^k)\rho_{bk}\mathrm{S}^{ab} \tag{138}$$

$$\mathcal{F}_{\mu\nu} = \frac{1}{4}{\rho_a}^i{\rho_b}^j(\partial_\mu h_{ik})h^{kl}(\partial_\mu h_{jl})\mathrm{S}^{ab} \tag{139}$$

We see that LD gauge structures can be induced even when the off-diagonal block of the HD metric vanishes identically, but they only act on matter carrying spin. As in the previous cases, the theory is still covariant under the whole residual covariance group. The gauge structure emerges explicitly only when adapted coordinates are introduced and the covariance group is restricted to (pseudo-)rotations of internal reference frames. In generic coordinate systems the background is fully characterized by the LD covariant condition

$$f^i_{\mu\nu} = 0 \tag{140}$$

Under these circumstances a minimum of three internal dimensions is required to generate non-Abelian gauge structures, while ten extra dimensions naturally provide the background for $SO(10)$ grand unification [14]. Internal gauge indices like isospin and color can be nicely understood as internal spin indices and a complete matter unification can be achieved in terms of a single fourteen-dimensional spinor [15].

5 Discussion and Conclusion

The selection of a subset of coordinates –with the relative general covariance breaking– does not imply in itself neither the selection of a reduced space nor a dimensional reduction procedure. However, it determines the geometrical features of all reduction schemes leading to that subset of coordinates as residual coordinates. By investigating invariant/covariant quantities under the residual transformation group we constructed LD tensors that fully characterize the geometry of the coordinate choice and hence of the associated dimensional reduction schemes. These allow to see in the same light reduction procedures that seems otherwise totally unrelated, like Kaluza-Klein models –where the system is totally delocalized in internal directions– and embedded spacetimes –where, on the contrary, the system gets localized at an internal space point. Most of the formulas of Kaluza-Klein and embedded spacetime theories do not depend on the averaging procedure employed, but only on the geometry of the coordinate choice. In this paper we presented general formulas for the reduction of the main tensors and operators of Riemannian geometry. In particular, the reduction of the HD Riemann tensor provides what is probably the maximal possible generalization of Gauss, Codazzi and Ricci equations. Our work also sheds some new light on the nature of geometrically induced gauge structures, tracing their origin to residual general covariance in internal directions.

We conclude by remarking that –from the separation of radial and angular coordinates in the two-body problem to the latest theories of everything– *adapting*, *selecting an appropriate subset* and *exactly or effectively separating* coordinates is such a basic procedure in solving physical problems, that it is unthinkable to compile even a partial list of the papers where particular adapted/reduced expressions of geometric tensors, equations and operators have been obtained. Our hope is that the formulas presented in this paper may be of help and save some tedious computational work to all researcher working on some adapting coordinates problem.

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