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The Horváth's spaces $\mathscr{S}'_k$ and the Fourier transform

Benito Juan González* and Emilio Ramon Negrín

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de La Laguna (ULL), Campus de Anchieta, ES-38271 La Laguna (Tenerife), España

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Abstract. In this paper, we establish new properties for the Fourier transform over the space of distributions $\mathscr{S}'_k$ introduced by Horváth. We prove Abelian theorems for the Fourier transform over the space $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$. Continuity properties and some results concerning regular distributions are studied. We also prove that the Fourier transform is an injection from $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, into $\mathscr{O}_C^{-2k-1}$, where this space denotes the union of the spaces $\mathscr{S}_{k^*}^{-2k-1}$, as k^* varies in $\mathbb{Z}$, which have been given by Horváth. The convolution over $\mathscr{S}'_k$ for certain regular distributions and its relation with the usual convolution product of functions is exhibited. Finally, some illustrative examples are considered.
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1. Introduction and preliminaries

In this paper, we deal with new interesting results concerning the Fourier transform over the space of distributions $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$. This space is the dual of the space of functions $\mathscr{S}_k$ introduced by Horváth in [9], on which the authors have also published several papers (see [2], [3] and [5], amongst others).

The spaces $\mathscr{S}_k$ [9, p. 90] are related to the spaces $\dot{\mathscr{B}}$ introduced by Laurent Schwartz [16, p. 199], which are used in the theory of partial differential equations.

In Section 2, we analyse the asympotic behaviour of the Fourier transform over the space $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$. These types of results are also known as Abelian theorems, which have been studied in several works (see [6], [7], [13], [14] and [17], amongst others). Specifically, we establish some results in which one shows the behaviour of the Fourier transform of any distribution in $\mathscr{S}'_k$, when its domain variable approaches infinity. Its development has applications in different branches of mathematical analysis, such as the case of integral transforms, the summability of the Fourier series, as well as in the field of distribution spaces and generalized functions due to their usefulness in various fields, such as PDEs, number theory, and others. Abelian theorems on distributional transforms were first established by Zemanian in [21], (see also [6], [7], [10] and [18], amongst others).

In Section 3, we study properties of the regular distribution generated by the function given by the Fourier transform of the distribution $f \in \mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$.

*Corresponding author. *Email addresses:* `bjglez@ull.es` (B. J. González), `enegrin@ull.es` (E. R. Negrín)

In Section 4, we prove that the image of $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, via the Fourier transform, is contained in the space $\mathscr{O}_C^{-2k-1}$ and it is an injection. For this purpose we use the definition given by Horváth of the space $\mathscr{O}_C^m$, for m a positive integer, as the union of the spaces $\mathscr{S}_k^m$ (see [9, example 11, p. 90]).

Section 5 is devoted to the convolution of the Fourier transform over the space $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, for certain regular distributions over this space and its relation with the usual convolution product of functions.

Finally, some illustrative examples are shown.

Throughout this paper we shall use the terminology and notation of [9]. Thus, $\mathbb{N}$ denotes the set of all non-negative integers.

The Fourier transform on $\mathbb{R}^n$ of a complex-valued function $f \in L^1(\mathbb{R}^n)$ is given by

$$\int_{\mathbb{R}^n} f(x)e^{ixy}dx, \quad y \in \mathbb{R}^n. \tag{1}$$

In [2], we studied the Fourier transform over the spaces of distributions $\mathscr{S}'_k$. In this sense we recall that (see [9, example 12, p. 90]) if k is a fixed integer, by $\mathscr{S}_k$ we denote the vector spaces of all functions ϕ defined on $\mathbb{R}^n$ which possess continuous partial derivatives of all orders and satisfy the condition that if $p \in \mathbb{N}^n$ and $\varepsilon > 0$, then there exists $A(\phi, p, \varepsilon) > 0$ such that

$$\left|(1+|x|^2)^k \partial^p \phi(x)\right| \leq \varepsilon, \quad \text{for} \quad |x| > A(\phi, p, \varepsilon).$$

For every $p \in \mathbb{N}^n$, on $\mathscr{S}_k$ Horváth defines the seminorms

$$q_{k,p}(\phi) = \max_{x \in \mathbb{R}^n} \left|(1+|x|^2)^k \partial^p \phi(x)\right|.$$

The space $\mathscr{S}_k$ equipped with the countable family of seminorms $(q_{k,p})$ is a Fréchet space. As usual, $\mathscr{S}'_k$ denotes the dual of the space $\mathscr{S}_k$.

This topology coincides with the initial topology of this space with respect to the linear mapping

$$\mathscr{S}_k \longrightarrow \dot{\mathscr{B}}, \quad \phi \longrightarrow (1+|x|^2)^k \phi,$$

(see [11, p. 87]).

Observe that for $k \in \mathbb{Z}$, $k < 0$, and each $y \in \mathbb{R}^n$, one has $e^{ixy} \in \mathscr{S}_k$.

Furthermore, we recall that [9, p. 173] if m is a positive integer and k an arbitrary integer, the vector space of all complex-valued ϕ defined on $\mathbb{R}^n$ is denoted by $\mathscr{S}_k^m$, whose partial derivatives $\partial^p \phi$ exist and are continuous for $|p| \leq m$m and which satisfies the following condition: given $p \in \mathbb{N}^n$ with $|p| \leq m$ and $\varepsilon > 0$, there exists $A(\phi, p, \varepsilon) > 0$ such that

$$\left|(1+|x|^2)^k \partial^p \phi(x)\right| \leq \varepsilon, \quad \text{for} \quad |x| > A(\phi, p, \varepsilon).$$

For every $p \in \mathbb{N}^n$ with $|p| \leq m$, the family of seminorms on $\mathscr{S}_k^m$ is defined by

$$q_{k,p}(\phi) = \max_{x \in \mathbb{R}^n} \left|(1+|x|^2)^k \partial^p \phi(x)\right|.$$

Let m be a positive integer. We denote by $\mathscr{O}_C^m$ the union of the spaces $\mathscr{S}_k^m$ as k varies in $\mathbb{Z}$ (see [12, p. 63]).

The Fourier transform of a member $f \in \mathscr{S}_k'$ is defined by

$$(\mathscr{F} f)(y) = F(y) = \left\langle f(x), e^{ixy} \right\rangle, \quad y \in \mathbb{R}^n. \tag{2}$$

Moreover, in [2, Theorem 2.1] it is established that for all $f \in \mathscr{S}_k'$ one has that

$$\langle f, \mathscr{F}\phi \rangle = \int_{\mathbb{R}^n} \left\langle f(x), e^{ixy} \right\rangle \phi(y) dy, \tag{3}$$

where $\mathscr{S}$ is the space of rapidly decreasing functions, and $\mathscr{F}\phi$ denotes the classical Fourier transform (1) of the function ϕ.

Formula (3) proves that function (2) represents the usual distributional Fourier transform [16, Chapter VII, Section 6, p. 248], when it acts over distributions $f \in \mathscr{S}_k'$ and functions in $\mathscr{S}$.

For previous studies of Fourier transform over spaces of distributions, we refer to [15], [19] and [20], amongst others.

2. Abelian theorems for the Fourier transform of distributions of $\mathscr{S}_k'$

For $f \in \mathscr{S}_k'$, $k \in \mathbb{Z}$, $k < 0$, we consider the Fourier transform of f by means of the function given by (2).

Lemma 1. *Let $f \in \mathscr{S}_k'$, $k \in \mathbb{Z}$, $k < 0$, and let F be defined by (2). Then there exist $C > 0$ and a nonnegative integer m, all depending on f, such that*

$$|F(y)| \leq C \max_{|p| \leq m} |y|^{|p|}, \quad p \in \mathbb{N}^n. \tag{4}$$

Proof. Set $f \in \mathscr{S}_k'$, $k \in \mathbb{Z}$, $k < 0$. From [9, Proposition 2, p. 97], there exist $C > 0$ and a nonnegative integer m, all depending on f, such that

$$|< f, \phi >| \leq C \max_{|p| \leq m} \max_{x \in \mathbb{R}^n} \left| (1 + |x|^2)^k \partial^p \phi(x) \right|, \tag{5}$$

for all $\phi \in \mathscr{S}_k$.

Now, we have

$$\begin{aligned} |F(y)| &\leq C \max_{|p| \leq m} \max_{x \in \mathbb{R}^n} \left| (1 + |x|^2)^k \partial^p e^{ixy} \right| \\ &= C \max_{|p| \leq m} \max_{x \in \mathbb{R}^n} \left| (1 + |x|^2)^k (iy)^p e^{ixy} \right| \leq C \max_{|p| \leq r} \max_{x \in \mathbb{R}^n} \left\{ (1 + |x|^2)^k |y^p| \right\}. \end{aligned}$$

Since for $p = (p_1, p_2, \ldots, p_n) \in \mathbb{N}^n$ we get

$$|(iy)^p| = |(iy_1)^{p_1} (iy_2)^{p_2} \cdots (iy_n)^{p_n}| \leq |y|^{p_1} \cdots |y|^{p_n} = |y|^{|p|},$$

inequality (4) follows. □

The smallest integer r which verifies inequality (5) is defined as the order of the distribution f (cf. [16, Théorème XXIV, p. 88]).

The next result establishes an Abelian theorem for the Fourier transform of distributions in $\mathscr{S}_k'$, $k \in \mathbb{Z}$, $k < 0$.

Theorem 1 (Abelian theorem)**.** *Let f be a member in $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, of order $r \in \mathbb{N}$, and let $F(y)$ be given by (2). Then for any $\gamma > 0$ one has*

$$\lim_{|y| \to +\infty} \{|y|^{-r-\gamma} F(y)\} = 0.$$

Proof. From Lemma 1 one has

$$|F(y)| \leq C \max_{|p| \leq r} |y|^{|p|},$$

Thus, for $|y| \geq 1$,

$$|F(y)| \leq C \max_{|p| \leq r} |y|^{|p|} \leq C|y|^r.$$

So, the result follows. □

Now let f be a locally integrable function defined on $\mathbb{R}^n$ such that $(1+|x|^2)^{-k} f(x)$ is Lebesgue integrable on $\mathbb{R}^n$ for $k \in \mathbb{Z}$, $k < 0$. One has that f gives rise to a regular distribution T_f on $\mathscr{S}'_k$ of order $r = 0$ through

$$< T_f, \phi >= \int_{\mathbb{R}^n} f(x)\phi(x)dx, \quad \forall \phi \in \mathscr{S}_k.$$

In fact, taking into account that

$$|\langle T_f, \phi \rangle| \leq \int_{\mathbb{R}^n} \left|(1+|x|^2)^{-k} f(x)\right| \left|(1+|x|^2)^k \phi(x)\right| dx$$

$$\leq q_{k,0}(\phi) \int_{\mathbb{R}^n} \left|(1+|x|^2)^{-k} f(x)\right| dx,$$

it follows that $T_f \in \mathscr{S}'_k$ and its order is $r = 0$.

Thus, we have

$$F(y) =< T_f(x), e^{ixy} >= \int_{\mathbb{R}^n} f(x) e^{ixy} dx, \quad \phi \in \mathscr{S}_k. \tag{6}$$

So one concludes

Corollary 1. *Let f be a locally integrable function defined on $\mathbb{R}^n$ such that $(1+|x|^2)^{-k} f(x)$ is Lebesgue integrable on $\mathbb{R}^n$ for $k \in \mathbb{Z}$, $k < 0$, and let F be given by (6). Then for any $\gamma > 0$ one has*

$$\lim_{|y| \to +\infty} \left\{|y|^{-\gamma} F(y)\right\} = 0.$$

For the case $p = 1$, $k < 0$, one has the well-known result that the Fourier transform is a continuous operator from $L^1\left(\mathbb{R}^n, (1+|x|^2)^{-k} dx\right) \subseteq L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$.

Moreover, one has

Proposition 1. *If $f \in L^p\left(\mathbb{R}^n, (1+|x|^2)^{-k}dx\right)$, $k \in \mathbb{Z}$, $k < 0$, $1 \le p < \infty$ and*

$$(\mathscr{F} f)(y) = \int_{\mathbb{R}^n} f(x) e^{ixy} dx, \quad y \in \mathbb{R}^n,$$

then for $1 \le p < 1 - 2k/n$ the operator $\mathscr{F}$ is bounded from $L^p\left(\mathbb{R}^n, (1+|x|^2)^{-k}dx\right)$ into $L^\infty(\mathbb{R}^n)$.

Proof. Denote by $\|\cdot\|_p$ the norm of the space $L^p\left(\mathbb{R}^n, (1+|x|^2)^{-k}dx\right)$. By Hölder's inequality, one has

$$\begin{aligned}(\mathscr{F} f)(y) &\le \int_{\mathbb{R}^n} |f(x)|\, dx = \int_{\mathbb{R}^n} |f(x)|\,(1+|x|^2)^{-k/p}(1+|x|^2)^{k/p} dx \\ &\le \left(\int_{\mathbb{R}^n} |f(x)|^p\,(1+|x|^2)^{-k} dx\right)^{1/p} \left(\int_{\mathbb{R}^n} (1+|x|^2)^{kp'/p} dx\right)^{1/p'} \\ &= \|f\|_p \left(\int_{\mathbb{R}^n} (1+|x|^2)^{kp'/p} dx\right)^{1/p'}, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.\end{aligned}$$

So,

$$\text{ess supp}_{y\in\mathbb{R}^n} |(\mathscr{F} f)(y)| \le \|f\|_p \cdot \text{ess supp}_{y\in\mathbb{R}^n} \left\{ \left(\int_{\mathbb{R}^n} (1+|x|^2)^{kp'/p} dx\right)^{1/p'} \right\}.$$

Now, making use of spherical coordinates $(\rho, \theta_1, \ldots, \theta_{n-1})$ we have

$$\int_{\mathbb{R}^n} \left(1+|x|^2\right)^{kp'/p} dx = \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \int_0^\infty (1+\rho^2)^{kp'/p} \rho^{n-1} d\rho,$$

and this integral converges for $k < -\dfrac{n}{2}(p-1)$, that is, $1 \le p < 1 - 2k/n$, $k \in \mathbb{Z}$, $k < 0$. Thus concludes the proof. $\square$

3. Regular distributions versus Fourier transform over $\mathscr{S}'_k$

In [2, Theorem 2.1], it was proved that for $f \in \mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, the functional T_F is a member of $\mathscr{S}'$, where F is the function $F(y) =< f(x), e^{ixy} >$, $y \in \mathbb{R}^n$. Now, as a consequence of Theorem 1 above, we can be more explicit and prove that T_F is a member of $\mathscr{S}'_{k_*}$, where $k_* \in \mathbb{Z}$ is such that $k_* > \dfrac{r+n}{2}$, and where r denotes the order of f.

Before we prove this result, we need to extend Proposition 2.1 of [3] from $k \in \mathbb{Z}$, $k < 0$ to all $k \in \mathbb{Z}$. In fact,

Proposition 2. *Let f be a locally integrable function defined on $\mathbb{R}^n$ such that $(1+|x|^2)^{-k} f(x)$ is Lebesgue integrable on $\mathbb{R}^n$ for some $k \in \mathbb{Z}$. Then the linear functional over $\mathscr{S}'_k$ given by*

$$< T_f, \phi >= \int_{\mathbb{R}^n} f(x)\phi(x) dx, \quad \phi \in \mathscr{S}_k,$$

is a member of $\mathscr{S}'_k$.

Proof. For $\phi \in \mathscr{S}_k$, one has

$$|\langle T_f, \phi\rangle| \leq \int_{\mathbb{R}^n} \left|(1+|x|^2)^{-k} f(x)\right| \left|(1+|x|^2)^{k}\phi(x)\right| dx$$
$$\leq q_{k,0}(\phi) \int_{\mathbb{R}^n} \left|(1+|x|^2)^{-k} f(x)\right| dx.$$

From the hypothesis, the continuity of T_f follows immediately. □

Now, we have the next result

Theorem 2. *Let $f \in \mathscr{S}_k'$, $k \in \mathbb{Z}$, $k < 0$, and let r be the order of f. Denotes by $F(y) = < f(x), e^{ixy} >$, $y \in \mathbb{R}^n$. Then the functional T_F given by $\langle T_F, \phi\rangle = \int_{\mathbb{R}^n} F(y)\phi(y)dy$ is a member of $\mathscr{S}_{k_*}'$, where $k_* \in \mathbb{Z}$ is such that $k_* > \dfrac{r+n}{2}$.*

Proof. Making use of (4) and taking into account that r is the order of f one has

$$\int_{\mathbb{R}^n} (1+|y|^2)^{-k_*} |F(y)|\, dy \leq C \int_{\mathbb{R}^n} (1+|y|^2)^{-k_*} \max_{|p|\leq r} |y|^{|p|} dy, \quad p \in \mathbb{N}^n. \tag{7}$$

From the use of spherical coordinates, the right-hand side of equation (7) becomes

$$C \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \int_0^{\infty} (1+\rho^2)^{-k_*} \max_{|p|\leq r} \left(\rho^{|p|}\right) \rho^{n-1} d\rho,$$

which converges for $k_* > \dfrac{r+n}{2}$. From this fact the result follows. □

Concerning regular distributions on $\mathscr{S}_k'$, $k \in \mathbb{Z}$, and having into account that its order is $r = 0$, one obtains

Corollary 2. *Let f be a locally integrable function defined on $\mathbb{R}^n$ such that $(1 + x^2)^{-k} f(x)$ is Lebesgue integrable on $\mathbb{R}^n$ for some $k \in \mathbb{Z}$, $k < 0$. Denotes*

$$F(y) = < T_f, e^{ixy} > = \int_{\mathbb{R}^n} f(x) e^{ixy} dx, \quad y \in \mathbb{R}^n,$$

(the classical Fourier transform of f).

Then the functional T_F is a member of $\mathscr{S}_{k_*}'$, where $k_* \in \mathbb{Z}$, $k_* > \dfrac{n}{2}$.

Proof. Since T_f is a regular distribution, one has $r = 0$. Now, the conclusion follows from Theorem 2. □

4. The Fourier transform as an injection from $\mathscr{S}_k'$ into $\mathscr{O}_C^{-2k-1}$

In [3, Proposition 2.2], it was proved that if $f \in \mathscr{S}_k'$, $k \in \mathbb{Z}$, $k < 0$, and $F(y) = \langle f(x), e^{ixy}\rangle$, then the partial derivatives $\partial^m F(y)$ exists on $\mathbb{R}^n$ for $m \in \mathbb{N}^n$, $|m| \leq -2k-1$, and one has

$$\partial^m F(y) = \langle f(x), (ix)^m e^{ixy}\rangle, \quad y \in \mathbb{R}^n.$$

Now, we establish the next

Proposition 3. *Let f be a member of $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, and $F(y) = \langle f(x), e^{ixy} \rangle$. Then for any $m \in \mathbb{N}^n$, $|m| \leq -2k - 1$, $\partial^m F(y)$ is continuous in $\mathbb{R}^n$.*

Proof. We now prove that

$$\lim_{h \to 0} \left(\partial^m F(y+h) - \partial^m F(y) \right) = 0, \quad y \in \mathbb{R}^n, \ h \in \mathbb{R}^n.$$

Since $f \in \mathscr{S}'_k$, it suffices to prove that

$$(ix)^m \left(e^{ix(y+h)} - e^{ixy} \right) \to 0,$$

as $h \to 0$ in $\mathscr{S}_k$.

For this, we consider

$$\max_{x \in \mathbb{R}^n} \left| \left(1 + |x|^2\right)^k \partial_x^p \left((ix)^m \left(e^{ix(y+h)} - e^{ixy} \right) \right) \right|, \quad p \in \mathbb{N}^n. \tag{8}$$

By the Leibniz rule, this expression is equal to

$$\begin{aligned} &\max_{x \in \mathbb{R}^n} \left| \left(1 + |x|^2\right)^k \sum_{j \leq p} \binom{p}{j} \partial_x^j (ix)^m \partial_x^{p-j} \left(e^{ix(y+h)} - e^{ixy} \right) \right| \\ &\quad \leq \sum_{j \leq p} \binom{p}{j} \max_{x \in \mathbb{R}^n} \left(\left(1 + |x|^2\right)^k \left| \partial_x^j (ix)^m \right| \left| \partial_x^{p-j} \left(e^{ix(y+h)} - e^{ixy} \right) \right| \right). \end{aligned}$$

Observe that, applying a process similar to that followed in the proof of Proposition 2.1 of [2], one has

$$\left| \partial_x^{p-j} \left(e^{ix(y+h)} - e^{ixy} \right) \right| \leq \left((|p| - |j|)(|y| + |h|)^{|p|-|j|-1} + (|y| + |h|)^{|p|-|j|} n|x| \right) |h|,$$

and if we suppose that $|h| \leq 1$, we get

$$\left| \partial_x^{p-j} \left(e^{ix(y+h)} - e^{ixy} \right) \right| \leq \left((|p| - |j|)(|y| + 1)^{|p|-|j|-1} + (|y| + 1)^{|p|-|j|} n|x| \right) |h|.$$

On the other hand, with $m = (m_1, , \ldots, m_n) \in \mathbb{N}^n$ and $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$, one has

$$\begin{aligned} \partial^j (ix)^m &= \partial^j \left((ix_1)^{m_1} \cdots (ix_n)^{m_n} \right) \\ &= i^{|m|} \left(\frac{m_1!}{(m_1 - j_1)!} x_1^{m_1 - j_1} \cdots \frac{m_n!}{(m_n - j_n)!} x_n^{m_n - j_n} \right). \end{aligned}$$

Thus, for $j \leq m$,

$$\left| \partial^j (ix)^m \right| \leq \frac{m!}{(m-j)!} |x_1|^{m_1 - j_1} \cdots |x_n|^{m_n - j_n} = \frac{m!}{(m-j)!} |x|^{|m|-|j|},$$

and $\partial^j (ix)^m = 0$ for $j \nleq m$.

Hence, expression (8) is less than or equal to

$$\sum_{j\le p}\binom{p}{j}\max_{x\in\mathbb{R}^n}\Big\{\left(1+|x|^2\right)^k\frac{m!}{(m-j)!}|x|^{|m|-|j|} \\ \times\Big((|p|-|j|)(|y|+1)^{|p|-|j|-1}+(|y|+1)^{|p|-|j|}n|x|\Big)|h|\Big\}. \tag{9}$$

Note that for $k\in\mathbb{Z}$, $k<0$, and $|m|\le -2k-1$, expression (9) is bounded by

$$|h|\sum_{j\le p}M_{y,m,p,j},$$

where $M_{y,m,p,j}\ge 0$. This concludes the proof. □

From this proposition one has that if $f\in\mathscr{S}'_k$, $k\in\mathbb{Z}$, $k<0$, and $F(y)=\langle f(x),e^{ixy}\rangle$, then the partial derivatives $\partial^m F$ of the function F are continuous on $\mathbb{R}^n$ for $|m|\le -2k-1$. That is, $F\in\mathscr{C}^{-2k-1}(\mathbb{R}^n)$, $k\in\mathbb{Z}$, $k<0$.

We can now state the following

Theorem 3. *The Fourier transform understood as an automorphism of $\mathscr{S}'$ is a linear injective map from $\mathscr{S}'_k$ onto $\mathscr{O}_C^{-2k-1}$, $k\in\mathbb{Z}$, $k<0$.*

Proof. Let f be a member of $\mathscr{S}'_k$. By Proposition 3 above, the function $F(y)=\langle f(x),e^{ixy}\rangle$ is of class $\mathscr{C}^{-2k-1}(\mathbb{R}^n)$, $k\in\mathbb{Z}$, $k<0$.

By Proposition 2.2 in [3], one has

$$|\partial^m F(y)|=\left|\langle f(x),(ix)^m e^{ixy}\rangle\right|,\quad y\in\mathbb{R}^n\quad m\in\mathbb{N}^n,\quad |m|\le -2k-1.$$

By virtue of [9, Proposition 2, p. 97], there exists $C>0$ and $l\in\mathbb{N}$, all depending on f such that

$$\begin{aligned}|\partial^m F(y)| &\le C\cdot\max_{|p|\le l}\left(\max_{x\in\mathbb{R}^n}\left|\left(1+|x|^2\right)^k\partial_x^p\left(x^m e^{ixy}\right)\right|\right)\\ &\le C\cdot\max_{|p|\le l}\left(\sum_{j\le p}\binom{p}{j}\max_{x\in\mathbb{R}^n}\left|\left(1+|x|^2\right)^k\partial_x^{p-j}x^m\partial^j\left(e^{ixy}\right)\right|\right)\\ &= C\cdot\max_{|p|\le l}\left(\sum_{j\le p}\binom{p}{j}\max_{x\in\mathbb{R}^n}\left|\left(1+|x|^2\right)^k\partial_x^{p-j}x^m\left((iy)^j e^{ixy}\right)\right|\right),\end{aligned} \tag{10}$$

where, since $|y^j|\le|y|^{|j|}$, equation (10) is bounded above by

$$C\cdot\max_{|p|\le l}\left(\sum_{j\le p}\binom{p}{j}|y|^{|j|}\max_{x\in\mathbb{R}^n}\left|\left(1+|x|^2\right)^k\partial_x^{p-j}x^m\right|\right). \tag{11}$$

Thus, since $|m|\le -2k-1$, there exists a constant $M_{m,p,j}\ge 0$ such that

$$\binom{p}{j}\cdot\max_{x\in\mathbb{R}^n}\left|\left(1+|x|^2\right)^k\partial_x^{p-j}x^m\right|=M_{m,p,j},$$

and expression (11) can be written as

$$C \cdot \max_{|p|\leq r} \left\{ \sum_{j\leq p} M_{m,p,j} |y|^{|j|} \right\},$$

which is for $|y| \geq 1$ less than or equal to

$$C \cdot |y|^r \cdot \max_{|p|\leq r} \left\{ \sum_{j\leq p} M_{m,p,j} \right\},$$

where r denotes the order of f.

Therefore, one has

$$\left|(1+|y|^2)^{k_*} \partial^m F(y)\right| \to 0, \quad |y| \to +\infty,$$

whenever $2k_* + r < 0$. From this fact one has that

$$F \in \bigcup_{k_* \in \mathbb{Z}} \mathscr{S}_{k_*}^{-2k-1} = \mathscr{O}_C^{-2k-1}.$$

Finally, from [2, Corollary 3.1] it follows that the mentioned map $\mathscr{S}'_k \hookrightarrow \mathscr{O}_C^{-2k-1}$ is injective. □

5. Regular distributions versus convolution on $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$

In this section, we deal with the convolution for the Fourier transform over the space $\mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$ for certain regular distributions over this space.

Under certain restrictions, the convolution of members in $\mathscr{S}'_k$ corresponds to the usual convolution of ordinary functions.

Proposition 4. *Let f, g be locally integrable functions defined on $\mathbb{R}^n$ such that $(1+|x|^2)^{-k} f(x)$ and $(1+|x|^2)^{-k} g(x)$ are Lebesgue integrable on $\mathbb{R}^n$ for some $k \in \mathbb{Z}$, $k < 0$. Then the linear functional $T_f * T_g$ given by*

$$\langle T_f * T_g, \phi \rangle = \langle T_f(x), \langle T_g(y), \phi(x+y) \rangle \rangle, \quad \phi \in \mathscr{S}_k,$$

is equal to T_{f*g}, where $f * g$ denotes the usual convolution of the functions f and g.

Proof. In [1], it was established that for $f, g \in \mathscr{S}'_k$, $k \in \mathbb{Z}$, $k < 0$, the convolution $f * g$ given by

$$\langle f * g, \phi \rangle = \langle f(x), \langle g(y), \phi(x+y) \rangle \rangle, \quad \phi \in \mathscr{S}_k,$$

is a member of $\mathscr{S}'_k$.

Now, if f and g are functions satisfying the above hypothesis, one has that $T_f, T_g \in \mathscr{S}'_k$, and then

$$\langle T_f * T_g, \phi \rangle = \langle T_f(x), \langle T_g(y), \phi(x+y) \rangle \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(y) \phi(x+y) dy dx. \quad (12)$$

In order to show that $f(x)g(y)\phi(x+y)$ is integrable as a function of (x,y), we consider the following inequality:

$$
\begin{aligned}
&|f(x)g(y)\phi(x+y)| \\
&\quad = \left| \frac{f(x)g(y)}{(1+|x|^2)^k(1+|y|^2)^k} \right| \frac{(1+|x|^2)^k(1+|y|^2)^k}{(1+|x+y|^2)^k} \left|(1+|x+y|^2)^k \phi(x+y)\right|.
\end{aligned}
$$

Note that for $k<0$, [1, Lemma 2.1], one has that

$$\frac{(1+|x|^2)^k(1+|y|^2)^k}{(1+|x+y|^2)^k} \leq 4^{-k}.$$

Moreover, the function

$$\frac{f(x)g(y)}{(1+|x|^2)^k(1+|y|^2)^k}$$

is integrable on $\mathbb{R}^{2n}$ and $(1+|x+y|^2)^k\phi(x+y)$ is bounded on $\mathbb{R}^{2n}$. Therefore, by using Fubini's theorem, the repeated integral (12) becomes a double integral in (x,y) given by

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)\phi(x+y)dydx. \tag{13}$$

Next, applying the change of variables $u=x$, $v=x+y$, whose Jacobian is equal to one, it follows that (13) becomes

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v-u)\phi(v)dudv,$$

which, by applying Fubini's theorem again, can be written in the form:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(u)g(v-u)du\phi(v)dv.$$

Here one observes that

$$\int_{\mathbb{R}^n} f(u)g(v-u)du = (f*g)(v)$$

is the usual convolution product of f and g.

Let us now see that

$$(1+|v|^2)^{-k}\int_{\mathbb{R}^n} f(u)g(v-u)du$$

is integrable on $\mathbb{R}^n$ from which $T_{f*g} \in \mathscr{S}'_k$, $k\in\mathbb{Z}$, $k<0$.

Now we consider

$$\int_{\mathbb{R}^n}(1+|v|^2)^{-k}\int_{\mathbb{R}^n} f(u)g(v-u)dudv.$$

The change of variables $x=u$, $y=v-u$ yields

$$\int_{\mathbb{R}^n}(1+|x+y|^2)^{-k}\int_{\mathbb{R}^n} f(x)g(y)dxdy.$$

Taking into account that

$$\begin{aligned}
&\left|(1+|x+y|^2)^{-k}f(x)g(y)\right| \\
&\qquad = \left|\frac{f(x)g(y)}{(1+|x|^2)^k(1+|y|^2)^k}\right| \frac{(1+|x|^2)^k(1+|y|^2)^k}{(1+|x+y|^2)^k}(1+|x+y|^2)^k(1+|x+y|^2)^{-k},
\end{aligned}$$

the fact

$$\frac{(1+|x|^2)^k(1+|y|^2)^k}{(1+|x+y|^2)^k} \leq 4^{-k}$$

for $k < 0$, [1, Lemma 2.1], and that from the hypotheses $(1+|x|^2)^{-k}f(x)$ and $(1+|x|^2)^{-k}g(x)$ are Lebesgue integrable on $\mathbb{R}^n$, it follows that

$$\frac{f(x)g(y)}{(1+|x|^2)^k(1+|y|^2)^k}$$

is integrable on $\mathbb{R}^{2n}$, and therefore, $T_f*T_g = T_{f*g}$, where $f*g$ is the usual convolution of the functions f and g. $\square$

6. Some examples

6.1. Example 1

Assume that f is a locally integrable function on $\mathbb{R}^n$ such that $(1+|x|^2)^{-k}f(x)$ is integrable on $\mathbb{R}^n$, $k \in \mathbb{Z}$, $k < 0$, and set $g(x) = 2^{-n}e^{-(|x_1|+|x_2|+\cdots+|x_n|)}$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Observe that

$$\begin{aligned}
&2^{-n}\int_{\mathbb{R}^n}(1+|x|^2)^{-k}e^{-(|x_1|+|x_2|+\cdots+|x_n|)}dx \\
&\quad \leq \frac{1}{2}\int_{-\infty}^{+\infty}(1+x_1^2)^{-k}e^{-|x_1|}dx_1\frac{1}{2}\int_{-\infty}^{+\infty}(1+x_2^2)^{-k}e^{-|x_2|}dx_2\cdots\frac{1}{2}\int_{-\infty}^{+\infty}(1+x_n^2)^{-k}e^{-|x_n|}dx_n
\end{aligned}$$

since

$$1+|x|^2 = 1+x_1^2+x_2^2+\cdots+x_n^2 \leq (1+x_1^2)(1+x_2^2)\cdots(1+x_n^2),$$

and thus, with $k<0$,

$$(1+|x|^2)^{-k} \leq (1+x_1^2)^{-k}(1+x_2^2)^{-k}\cdots(1+x_n^2)^{-k}.$$

This shows that $(1+|x|^2)^{-k}e^{-(|x_1|+|x_2|+\cdots+|x_n|)}$ is Lebesgue integrable on $\mathbb{R}^n$ and thus $T_g \in \mathscr{S}'_k$.

Set

$$(f*g)(w) = 2^{-n}\int_{\mathbb{R}^n}f(x)e^{-(|w_1-x_1|+|w_2-x_2|-\cdots-|w_n-x_n|)}dx,\ w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n.$$

Then $T_{f*g} \in \mathscr{S}'_k$ and $T_f * T_g = T_{f*g}$.

Taking into account that

$$\left(\mathcal{F}\left(T_{f*g}\right)\right)(y) = \left(\mathcal{F}\left(T_f*T_g\right)\right)(y) = \left(\mathcal{F}\left(T_f\right)\right)(y)\left(\mathcal{F}\left(T_g\right)\right)(y), \quad y \in \mathbb{R}^n,$$

we get

$$\left(\mathcal{F}\left(2^{-n}\int_{\mathbb{R}^n} f(x)e^{-(|w_1-x_1|+|w_2-x_2|-\cdots-|w_n-x_n|)}dx\right)\right)(y) = (\mathcal{F}f)(y)\,(\mathcal{F}g)(y).$$

Next,

$$\begin{aligned}(\mathcal{F}g)(y) &= 2^{-n}\int_{\mathbb{R}^n} e^{-(|x_1|+|x_2|+\cdots+|x_n|)}e^{ixy}dx\\ &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x_1|}e^{ix_1y_1}dx_1\cdots\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x_n|}e^{ix_ny_n}dx_n, \quad y=(y_1,\ldots,y_n)\in\mathbb{R}^n.\end{aligned}$$

Moreover, for $x>0$ one has

$$\frac{1}{2}\int_0^{+\infty} e^{-x}e^{ixy}dx = \frac{1}{2}\frac{1}{1-iy},$$

and for $x<0$ one has

$$\frac{1}{2}\int_{-\infty}^{0} e^{x}e^{ixy}dx = \frac{1}{2}\frac{1}{1+iy};$$

therefore,

$$\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x|}e^{ixy}dx = \frac{1}{1+y^2}.$$

Thus,

$$(\mathcal{F}g)(y) = \frac{1}{(1+y_1^2)\cdots(1+y_n^2)}, \quad y=(y_1,\ldots,y_n)\in\mathbb{R}^n,$$

and then,

$$\begin{aligned}&\left(\mathcal{F}\left(\int_{\mathbb{R}^n} f(x)e^{-(|w_1-x_1|+\cdots+|w_n-x_n|)}dx\right)\right)(y)\\ &= \frac{2^n}{(1+y_1^2)\cdots(1+y_n^2)}\,(\mathcal{F}f)(y), \quad y=(y_1,\ldots,y_n)\in\mathbb{R}^n,\end{aligned}$$

for f a locally integrable function on $\mathbb{R}^n$ such that $(1+|x|^2)^{-k}f(x)$ is integrable on $\mathbb{R}^n$, $k\in\mathbb{Z}$, $k<0$, and where $\mathcal{F}$ denotes the classical Fourier transform.

6.2. Example 2

Consider the function $g(x)=e^{-|x|^2}=e^{-(x_1^2+\cdots+x_n^2)}$, for $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$. One has that $T_g\in\mathscr{S}_k'$. Moreover,

$$\begin{aligned}(\mathcal{F}g)(y) &= \int_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_n^2)}e^{ix_1y_1+\cdots+x_ny_n}dx\\ &= \int_{-\infty}^{+\infty} e^{ix_1y_1}e^{-x_1^2}dx_1\cdots\int_{-\infty}^{+\infty} e^{ix_ny_n}e^{-x_n^2}dx_n. \qquad (14)\end{aligned}$$

Next, from the formula

$$\frac{1}{\sqrt{2\pi c}}\int_{-\infty}^{+\infty} e^{\nu x}e^{-x^2/2c}dx = e^{c\nu^2/2}, \quad c>0,$$

there follows

$$\int_{-\infty}^{+\infty} e^{ix_jy_j}e^{-x_j^2}dx_j = \sqrt{\pi}e^{-y_j^2/4}, \quad 1\leq j\leq n,$$

and then (14) is equal to

$$\pi^{n/2}e^{-\frac{1}{4}|y|^2}.$$

Thus, as in Example 6.1, one has

$$\left(\mathcal{F}\left(\int_{\mathbb{R}^n} f(x)e^{-|w-x|^2}dx\right)\right)(y) = \pi^{n/2}e^{-\frac{1}{4}|y|^2}\left(\mathcal{F}f\right)(y), \quad y\in\mathbb{R}^n,$$

where f is a locally integrable function on $\mathbb{R}^n$ such that $(1+|x|^2)^{-k}f(x)$ is integrable on $\mathbb{R}^n$, $k\in\mathbb{Z}$, $k<0$, and $\mathcal{F}$ denotes the classical Fourier transform.

References

[1] B. J. González, E. R. Negrín, *Convolution over the spaces $\mathscr{S}_k'$*, J. Math. Anal. Appl. **190**(1995), 829–843.

[2] B. J. González, E. R. Negrín, *Fourier transform over the spaces $\mathscr{S}_k'$*, J. Math. Anal. Appl. **194**(1995), 780–798.

[3] B. J. González, E. R. Negrín, *On the structure of invertible elements of the convolution algebra $\mathscr{S}_k'$*, J. Math. Anal. Appl. **195**(1995), 506–516.

[4] B. J. González, *A representation for the Fourier transform of members in $\mathscr{S}_k'$*, Math. Japonica **45**(1997), 363–368.

[5] B. J. González, E. R. Negrín, *On an Inversion Formula for the Fourier Transform on Distributions by Means of Gaussian Functions*, Filomat **32**(2018), 2327–2334.

[6] B. J. González, E. R. Negrín, *Abelian Theorems for Distributional Kontorovich-Lebedev and Mehler-Fock Transforms of General Order*, Banach J. Math. Anal. **13**(2019), 524-537.

[7] N. Hayek, B. J. González, *Abelian theorems for the generalized index ${}_2F_1$-transform*, Rev. Acad. Canaria Cienc. **4**(1992), 23–29.

[8] N. Hayek, B. J. González, E. R. Negrín, *Distributional representation of a Fourier inversion formula*, Integral Transforms Spec. Funct. **16**(2005), 21–28.

[9] J. Horváth, *Topological Vector Spaces and Distributions, Vol. I*, Addison-Wesley, Reading, MA, 1966.

[10] J. K. John, M. S. Chaudhary, *Stieltjes transform of Banach-space-valued distributions and Abelian theorems*, Bull. Calcutta Math. Soc. **86**(1994), 109–118.

[11] N. Ortner, P. Wagner, *Distribution-Valued Analytic Functions*, SWK, Hamburg, 2013.

[12] N. Ortner, P. Wagner, *On the spaces $\mathscr{O}_C^m$ of John Horvth*, J. Math. Anal. Appl. **415**(2014), 62–74.

[13] S. Pilipović, B. Stanković, J. Vindas, *Asymptotic Behavior of Generalized Functions*, World Scientific Publishing Company, Singapore, 2012.

[14] A. Prasad, P. Kumar, *Abelian theorems for fractional wavelet transform,* Asian-Eur. J. Math. **10**(2017), 1750019, 15 pp.

[15] W. RUDIN, *Functional analysis*, Second edition, International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.

[16] L. SCHWARTZ, *Théorie des Distributions*, Publ. Inst. Math. Univ. Strasbourg **9–10**, Hermann, Paris, 1966.

[17] H. M. SRIVASTAVA, B. J. GONZÁLEZ, E. R. NEGRÍN, *A new class of Abelian theorems for the Mehler–Fock transforms,* Russ. J. Math. Phys., **24**, (2017), 124–126 (see also Errata Russ. J. Math. Phys. **24**(2017), 278–278).

[18] B. STANKOVIĆ, *Abelian theorems for integral transforms of distributions*, Integral Transforms Spec. Funct. **1**(1993), 61–73.

[19] F. TRÈVES, *Topological vector spaces, distributions and kernels*, Academic Press, New York-London, 1967.

[20] A. H. ZEMANIAN, *Distribution theory and transform analysis. An introduction to generalized functions, with applications*, McGraw-Hill Book Co., New York-Toronto-London-Sydney, 1965.

[21] A. H. ZEMANIAN, *Some Abelian theorems for the distributional Hankel and K transformations*, SIAM J. Appl. Math. **14**(1966), 1255–1265.

[22] A. H. ZEMANIAN, *Generalized Integral Transformations*, Pure Appl. Math. **18**, Wiley, New York, 1968.