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The electromagnetic energy–momentum tensor

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Abstract
We clarify the relation between canonical and metric energy–momentum tensors. In particular, we show that a natural definition arises from Noether's theorem which directly leads to a symmetric and gauge invariant tensor for electromagnetic field theories on an arbitrary spacetime of any dimension.

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For many decades, a suitable definition for the energy–momentum tensor has been under investigation. This is not a technical point, not only because T^{ab} should provide meaningful physical conserved quantities, but also because it is the source of Einstein's gravitational field equations.

In flat spacetime, the canonical energy–momentum tensor arises from Noether's theorem by considering the conserved currents associated with translation invariance. However, only for scalar fields does the energy–momentum tensor constructed in this way turn out to be symmetric. Moreover, for Maxwell's theory, it breaks the gauge symmetry. Of course, it is possible to correct it by a symmetrization procedure [1], although this looks like an *ad hoc* prescription.

On the other hand, a completely different approach leads to the metric energy–momentum tensor (see for example [2]) which is, by definition, symmetric and gauge invariant.

The aim of this paper is to clarify the relation between these tensors.

Let us consider a field theory where the Lagrangian $\mathscr{L}$ is a local function of F_{ab}, the exterior derivative $\partial_a A_b - \partial_b A_a$, of a one-form field A_b, and the metric tensor g_{ab}, defined on a (semi-) Riemannian manifold of dimension $n \geqslant 2$.

The field equations are obtained by requiring that the action

$$S = \int_\Omega \mathscr{L}(F_{ab}, g_{ab})\sqrt{|g|}\,\mathrm{d}^n x \tag{1}$$

be stationary under arbitrary variations of the fields δA_b in the interior of any compact region Ω. Thus, one obtains

$$\nabla_a \left(\frac{\partial \mathscr{L}}{\partial F_{ab}}\right) = 0 \tag{2}$$

where ∇_a is the covariant derivative associated with the Levi-Civita connection.

Needless to say, even in flat spacetime we are allowed to use curvilinear coordinates, so the action (1) must be invariant under general coordinate transformations. This requires $\mathscr{L}$ to be a scalar function. Thus, its Lie derivative with respect to any vector field ξ^a, $\mathcal{L}_\xi \mathscr{L}$, must satisfy

$$\mathcal{L}_\xi \mathscr{L} - \nabla_a(\mathscr{L})\xi^a = 0. \tag{3}$$

Now, taking into account that the Lagrangian $\mathscr{L}$ depends on the coordinates only through the tensor fields F_{ab} and g_{ab}, we have

$$\mathcal{L}_\xi \mathscr{L} = \frac{\partial \mathscr{L}}{\partial F_{ab}} \mathcal{L}_\xi F_{ab} + \frac{\partial \mathscr{L}}{\partial g_{ab}} \mathcal{L}_\xi g_{ab}. \tag{4}$$

But, for any tensor field F_{ab} of type (0, 2), it holds that

$$\mathcal{L}_\xi F_{ab} = \xi^c \nabla_c F_{ab} + F_{cb} \nabla_a \xi^c + F_{ac} \nabla_b \xi^c. \tag{5}$$

Thus

$$\frac{\partial \mathscr{L}}{\partial F_{ab}} \mathcal{L}_\xi F_{ab} = 2 \frac{\partial \mathscr{L}}{\partial F_{ab}} (\nabla_a F_{cb} \xi^c + F_{cb} \nabla_a \xi^c) = 2 \frac{\partial \mathscr{L}}{\partial F_{ab}} \nabla_a (F_{cb} \xi^c) \tag{6}$$

where we have used the identity

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0 \tag{7}$$

($\mathrm{d}\boldsymbol{F} = \mathrm{d}(\mathrm{d}\boldsymbol{A}) = 0$), and the obvious antisymmetry of the (2, 0) tensor field $\frac{\partial \mathscr{L}}{\partial F_{ab}}$.

Now, for fields satisfying the equations of motion (2), (6) reads

$$\frac{\partial \mathscr{L}}{\partial F_{ab}} \mathcal{L}_\xi F_{ab} = 2 \nabla_a \left(\frac{\partial \mathscr{L}}{\partial F_{ab}} F_{cb} \xi^c \right). \tag{8}$$

So, from (3), (4) and (8), we get for any vector field ξ^a

$$\nabla_a \left(2 \frac{\partial \mathscr{L}}{\partial F_{ac}} F^b{}_c \xi_b \right) + \frac{\partial \mathscr{L}}{\partial g_{ab}} \mathcal{L}_\xi g_{ab} - \nabla_a(\mathscr{L})\xi^a = 0. \tag{9}$$

Moreover, applying (5) to the metric tensor, we have

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \tag{10}$$

and so, (9) can be rewritten as

$$\nabla_a \left(2 \frac{\partial \mathscr{L}}{\partial F_{ac}} F^b{}_c \xi_b - g^{ab} \mathscr{L} \xi_b \right) + \left(\frac{\partial \mathscr{L}}{\partial g_{ab}} + \frac{1}{2} g^{ab} \mathscr{L} \right) \mathcal{L}_\xi g_{ab} = 0. \tag{11}$$

We define the 'true' canonical energy–momentum tensor as

$$T^{ab}_{\mathscr{C}} := -2 \frac{\partial \mathscr{L}}{\partial F_{ac}} F^b{}_c + g^{ab} \mathscr{L} \tag{12}$$

and the metric one as

$$T^{ab}_{\mathscr{M}} := 2 \frac{\partial \mathscr{L}}{\partial g_{ab}} + g^{ab} \mathscr{L}. \tag{13}$$

Note that, by definition, $T^{ab}_{\mathscr{M}}$ is a symmetric (2, 0) tensor.

In terms of these tensors, (11) reads

$$\nabla_a \left(T^{ab}_{\mathscr{C}} \xi_b \right) - \tfrac{1}{2} T^{ab}_{\mathscr{M}} \mathcal{L}_\xi g_{ab} = 0. \tag{14}$$

This last equation, a rewritten form of (3), which holds for any vector field ξ^a, has several important consequences. In fact, we shall obtain all the results of this work by using it in four different ways.

(i) Let us restrict attention to the case where ξ^a is a Killing vector field, i.e. a generator of an infinitesimal isometry, so $\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0$. From (14), one directly obtains the Noether current $\mathcal{J}^a_\xi$ associated with this symmetry,

$$\nabla_a \mathcal{J}^a_\xi = \nabla_a \left(T^{ab}_{\mathscr{C}} \xi_b \right) = 0 \tag{15}$$

for, in this case, the last term in (14) clearly vanishes. But this is just the beginning.

(ii) At any point of the manifold, we can choose Riemannian normal coordinates x^α (i.e. a local inertial coordinate system). Moreover, we can choose for ξ_b any set of n linear independent covectors with constant components in this coordinate system; for instance, the dual basis covectors $\mathrm{d}x^\alpha_b$. So, (14) reads

$$\partial_\alpha \big(T^{\alpha\beta}_{\mathscr{C}} \big) \xi_\beta + T^{\alpha\beta}_{\mathscr{C}} \partial_\alpha \xi_\beta - T^{\alpha\beta}_{\mathscr{M}} \partial_\alpha \xi_\beta = \partial_\alpha \big(T^{\alpha\beta}_{\mathscr{C}} \big) \xi_\beta = 0 \tag{16}$$

because of the vanishing of Christoffel symbols and partial derivatives of ξ_β. Hence, we get $\nabla_\alpha T^{\alpha\beta}_{\mathscr{C}} = \partial_\alpha T^{\alpha\beta}_{\mathscr{C}} = 0$. But this is a tensor relation, then[1]

$$\nabla_a T^{ab}_{\mathscr{C}} = 0. \tag{17}$$

(iii) Coming back to (14), we rewrite it as

$$\nabla_a \left(\left(T^{ab}_{\mathscr{C}} - T^{ab}_{\mathscr{M}} \right) \xi_b \right) + \nabla_a \left(T^{ab}_{\mathscr{M}} \right) \xi_b = 0. \tag{18}$$

Now, we integrate (18) over any compact region Ω, taking arbitrary vector fields ξ^a vanishing everywhere except in its interior. The first contribution may be transformed into an integral over the boundary which vanishes, as ξ^a is zero there. Since the second term must therefore be zero for arbitrary ξ^a, it follows that

$$\nabla_a T^{ab}_{\mathscr{M}} = 0. \tag{19}$$

(iv) Now, coming back to (14) written as in (18), we see that the diffeomorphism invariance of the action yields not only $\nabla_a T^{ab}_{\mathscr{C}} = \nabla_a T^{ab}_{\mathscr{M}} = 0$, but also

$$\nabla_a \left(\left(T^{ab}_{\mathscr{C}} - T^{ab}_{\mathscr{M}} \right) \xi_b \right) = \left(T^{ab}_{\mathscr{C}} - T^{ab}_{\mathscr{M}} \right) \nabla_a \xi_b = 0 \tag{20}$$

for any covector field ξ_b. Therefore, since $\nabla_a \xi_b$ is arbitrary, we conclude that both tensors coincide,

$$T^{ab}_{\mathscr{C}} = T^{ab}_{\mathscr{M}}. \tag{21}$$

We have thus shown that

$$\nabla_a T^{ab}_{\mathscr{C}} = 0 \qquad \nabla_a T^{ab}_{\mathscr{M}} = 0 \qquad \text{and} \qquad T^{ab}_{\mathscr{C}} = T^{ab}_{\mathscr{M}} \tag{22}$$

follow as a consequence of the diffeomophism invariance of the action.

Some comments are in order. We want to point out that $T^{ab}_{\mathscr{C}}$ has nothing to do with Killing vectors. $T^{ab}_{\mathscr{C}}$ depends only on the fields, their derivatives and the metric, and $\nabla_a T^{ab}_{\mathscr{C}} = 0$ is always true, even when the metric has no isometry at all. But, of course, a tensor by itself does not give rise to any conserved quantity[2] so, in order to construct conserved quantities, it is necessary to have a Killing vector at hand to construct the current $\mathcal{J}^a_\xi = T^{ab}_{\mathscr{C}} \xi_b$.

The $T^{ab}_{\mathscr{C}}$ we define in (12) arises naturally from Noether's theorem taking into account that the Lagrangian is a local function of F_{ab}. It is important to realize that, as shown by (15), if spacetime admits a Killing vector we obtain from $T^{ab}_{\mathscr{C}}$ a conserved current $\mathcal{J}^a_\xi$. Thus,

[1] Of course, it also follows directly from the equations of motion (2), for $\nabla_a T^{ab}_{\mathscr{C}} = 2 \frac{\partial \mathscr{L}}{\partial F_{ac}} \nabla_a F^b{}_c - g^{ab} \nabla_a \mathscr{L} = 2 \frac{\partial \mathscr{L}}{\partial F_{ac}} \nabla_a F^b{}_c - g^{ab} \frac{\partial \mathscr{L}}{\partial F_{cd}} \nabla_a F_{cd} = 2 \frac{\partial \mathscr{L}}{\partial F_{ac}} \nabla_a F^b{}_c - 2 \frac{\partial \mathscr{L}}{\partial F_{ac}} \nabla_a F^b{}_c = 0$, where we have used identity (7).

[2] For $\nabla_a T^{ab} = \frac{\partial_a (\sqrt{-g} T^{ab})}{\sqrt{-g}} + T^{ac} \Gamma^b{}_{ca}$.

for instance, the $n(n+1)/2$ currents in Minkowski spacetime are obtained from $T^{ab}_{\mathscr{C}}$, by contracting it with the corresponding Killing vectors.

In Minkowski spacetime, the canonical energy–momentum tensor is defined as (see for example [3, 4])

$$T_c^{ab} := -\frac{\partial \mathscr{L}}{\partial \nabla_a A_c} \nabla^b A_c + g^{ab} \mathscr{L} \tag{23}$$

perhaps as a simple generalization of its expression for scalar fields. It seems to us that this definition is in some sense unnatural, for $\mathscr{L}$ depends on the $n(n-1)/2$ components of $\mathrm{d}\boldsymbol{A}$ and not on the n^2 derivatives $\nabla_a A_b$.

T_c^{ab} defined as in (23) is neither symmetric nor gauge invariant. Of course, in flat spacetime, $\nabla_a T_c^{ab} = 0$ holds. But, it is worth noting that this is not even true for curved spacetime, for

$$\begin{aligned}
\nabla_a T_c^{ab} &= -2 \frac{\partial \mathscr{L}}{\partial F_{ac}} \nabla_a \nabla^b A_c + g^{ab} \nabla_a \mathscr{L} \\
&= -2 \frac{\partial \mathscr{L}}{\partial F_{ac}} \nabla_a \nabla^b A_c + g^{ab} \frac{\partial \mathscr{L}}{\partial F_{cd}} \nabla_a F_{cd} \\
&= -2 \frac{\partial \mathscr{L}}{\partial F_{ac}} \nabla_a \nabla_c A^b = -\frac{\partial \mathscr{L}}{\partial F_{ac}} R^b_{dac} A^d
\end{aligned} \tag{24}$$

where, again, we have used the identity (7), and R^b_{dac} is the Riemann curvature tensor. So, T_c^{ab} is neither symmetric nor gauge invariant, and also $\nabla_a T_c^{ab}$ vanishes only when spacetime is flat.

Moreover, in flat spacetime, for a Killing field ξ_b it holds that $\nabla_a \left(T_c^{ab} \xi_b\right) = T_c^{[ab]} \nabla_a \xi_b$, so the current $T_c^{ab} \xi_b$ is conserved only for constant ξ_b, for T_c^{ab} is not symmetric. Then we get from T_c^{ab} only n currents associated with the constant Killing vectors (translations). A similar result holds for curved spacetime, even though $\nabla_a T_c^{ab} \neq 0$. In fact, if there exists a constant Killing vector ($\nabla_a \xi^b = 0$) we have

$$\nabla_a \left(T_c^{ab} \xi_b\right) = \nabla_a T_c^{ab} \xi_b = \frac{\partial \mathscr{L}}{\partial F_{ac}} A^d R^b{}_{dac} \xi_b = 2 \frac{\partial \mathscr{L}}{\partial F_{ac}} A^d \nabla_c \nabla_a \xi_d = 0 \tag{25}$$

and so, we get a conserved current for each constant Killing vector ξ_b.

Note that $T^{ab}_{\mathscr{C}} = T^{ab}_{\mathscr{M}}$ means that for any scalar Lagrangian depending on the tensor fields F_{ab} and g_{ab}

$$\frac{\partial \mathscr{L}}{\partial F_{ac}} F^b{}_c = -\frac{\partial \mathscr{L}}{\partial g_{ab}}. \tag{26}$$

Moreover, as the right-hand side is a symmetric tensor field, so is the left-hand side. It is worth bearing in mind that (26) holds off-shell too. In fact, from (3) we have, for any field configuration,

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} \mathscr{L} - \nabla_b(\mathscr{L}) \xi^b &= \frac{\partial \mathscr{L}}{\partial F_{ac}} (\mathcal{L}_{\boldsymbol{\xi}} F_{ac} - \nabla_b F_{ac} \xi^b) + \frac{\partial \mathscr{L}}{\partial g_{ab}} \mathcal{L}_{\boldsymbol{\xi}} g_{ab} \\
&= 2 \left(\frac{\partial \mathscr{L}}{\partial F_{ac}} F^b{}_c + \frac{\partial \mathscr{L}}{\partial g_{ab}} \right) \nabla_a \xi_b = 0
\end{aligned} \tag{27}$$

and the vector field ξ^b is completely arbitrary.

For the sake of clarity, let us consider the translation invariance in flat spacetime. Taking Cartesian coordinates we compute $\partial_a \mathscr{L}$ in two different ways. First, thinking of $\mathscr{L}$ as a function of F_{ab}, as it actually is, we get

$$\partial_a \mathscr{L} = \frac{\partial \mathscr{L}}{\partial F_{cd}} \partial_a F_{cd} = 2 \frac{\partial \mathscr{L}}{\partial F_{cd}} \partial_c F_{ad} = 2\partial_c \left(\frac{\partial \mathscr{L}}{\partial F_{cd}} F_{ad} \right). \tag{28}$$

In the second step we used $\partial_{[a} F_{ab]} = 0$ and the field equations in the third. Thus, we get the conservation law $\partial_a T^{ab}_{\mathscr{C}} = 0$.

On the other hand, if we consider $\mathscr{L}$ as a function of $\partial_a A_b$

$$\partial_a \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \partial_c A_d} \partial_a \partial_c A_d = \frac{\partial \mathscr{L}}{\partial \partial_c A_d} \partial_c \partial_a A_d = \partial_c \left(\frac{\partial \mathscr{L}}{\partial \partial_c A_d} \partial_a A_d \right) \tag{29}$$

where we have commuted the partial derivatives. And this is the conservation $\partial_a T^{ab}_c = 0$.

Clearly, the first computation still holds in curved spacetime. But the second one fails, for covariant derivatives acting on one-forms do not commute, and we get (24).

For scalar fields, our arguments remain valid. The only change to be made is the definition $T^{ab}_{\mathscr{C}} := -\frac{\partial \mathscr{L}}{\partial \partial_a \phi} \partial^b \phi + g^{ab} \mathscr{L}$. Equation (14) still holds and all the results follow as above. For general tensor fields, the dependence of the Lagrangian on the affine connection as well as the noncommutativity between ∇_a and $\mathcal{L}_\xi$ makes the computation more involved [5].

Summarizing, we have shown that, properly defined as in (12), the canonical energy–momentum tensor $T^{ab}_{\mathscr{C}}$ is symmetric, gauge invariant and coincides with $T^{ab}_{\mathscr{M}}$. Moreover, it is the one which arises naturally from Noether's theorem when the metric has isometries, and all the currents are written as $\mathcal{J}^a_\xi = T^{ab}_{\mathscr{C}} \xi_b$. For these reasons, we call $T^{ab}_{\mathscr{C}}$ the 'true' canonical energy–momentum tensor.

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