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$\tau \to \pi\pi\pi\nu_\tau$ decays and the $a_1(1260)$ off-shell width revisited

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ABSTRACT

The $\tau \to \pi\pi\pi\nu_\tau$ decay is driven by the hadronization of the axial-vector current. Within the resonance chiral theory, and considering the large-N_C expansion, this process has been studied in Ref. [1] (D. Gómez Dumm, A. Pich, J. Portolés, 2004). In the light of later developments we revise here this previous work by including a new off-shell width for the lightest a_1 resonance that provides a good description of the $\tau \to \pi\pi\pi\nu_\tau$ spectrum and branching ratio. We also consider the role of the $\rho(1450)$ resonance in these observables. Thus we bring in an overall description of the $\tau \to \pi\pi\pi\nu_\tau$ process in excellent agreement with our present experimental knowledge.

1. Introduction

The decays of the τ lepton represent an outstanding laboratory for the analysis of various topics in particle physics. In particular, τ decays into hadrons allow to study the hadronization of vector and axial-vector currents, thus they can be used to determine intrinsic properties of the hadron resonances that govern the dynamics of these processes [1–7].

At very low energies, typically $E \ll M_\rho$ [where M_ρ is the mass of the $\rho(770)$ meson], chiral perturbation theory (χPT) [8,9] is an appropriate effective theory of QCD. However, in general this approach cannot be extended to the intermediate energy range, in which the dynamics of resonant states plays a major role. This is the case of hadron tau decays: these processes happen to be driven by hadron resonances, and the corresponding energy spectrum extends over a region where these resonances reach their on-shell peaks. In consequence, χPT is not directly applicable to the study of the whole spectrum but only to the very low energy domain [10]. A standard way of dealing with these decays is to use $\mathcal{O}(p^2)$ χPT to fix the normalization of the amplitudes in the low energy region, including the effects of vector and axial-vector meson resonances by modulating the amplitudes with *ad hoc* Breit–Wigner functions [2,6]. However, it has been shown that in the low energy limit this model is not consistent with $\mathcal{O}(p^4)$ χPT, a fact that leads to question the outcomes that could arise from this procedure [1,11].

The significant amount of experimental data on τ decays, in particular, $\tau \to \pi\pi\pi\nu_\tau$ branching ratios and spectra [12], encourages an effort to carry out a theoretical analysis within a model-independent framework capable to provide information on the hadronization of the involved QCD currents. A step in this direction has been done in Ref. [1], where we have analyzed $\tau \to \pi\pi\pi\nu_\tau$ decays within the resonance chiral theory (RχT) [13,14]. This procedure amounts to build an effective Lagrangian in which resonance states are treated as active degrees of freedom. Though the analysis in Ref. [1] allows to reproduce the experimental data on $\tau \to \pi\pi\pi\nu_\tau$ by fitting a few free parameters in this effective Lagrangian, it soon would be seen that the results of this fit are not compatible with theoretical expectations from short-distance QCD constraints [15]. We believe that the inconsistency can be attributed to the usage of an ansatz for the off-shell width of the lightest a_1 resonance, which was introduced *ad hoc* in Ref. [1]. The aim of this work is to reanalyse $\tau \to \pi\pi\pi\nu_\tau$ processes within the same general scheme, now considering the energy-dependent width of the a_1 state within a proper RχT framework.

2. Theoretical framework

The construction of the effective Lagrangian in RχT is basically ruled by the approximate chiral symmetry of QCD, which drives

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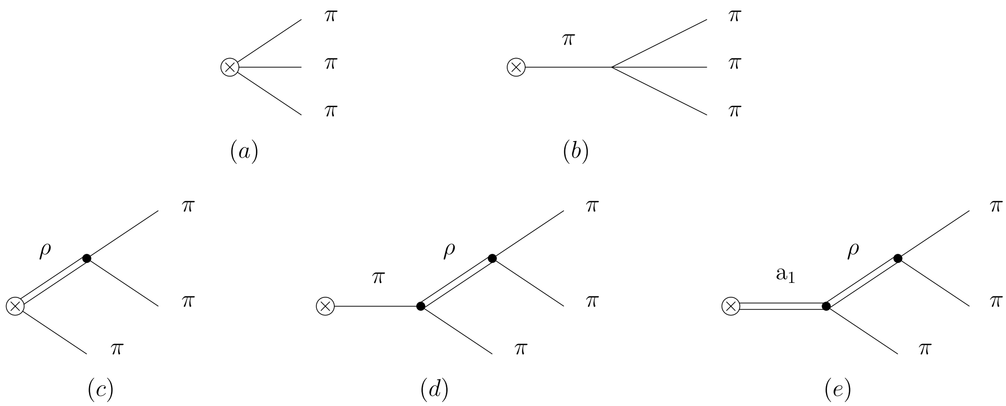

Fig. 1. Diagrams contributing to the hadron axial-vector form factors F_i: (a) and (b) contribute to F_1^{χ}, (c) and (d) to F_1^{R} and (e) to F_1^{RR}.

the interaction of light pseudoscalar mesons, and the $SU(3)_V$ assignments of the resonance multiplets [13,14]. An additional ingredient to be taken into account is the expansion in $1/N_C$, where N_C is the number of colors in QCD [16]. At the leading order in this expansion one should only consider tree-level diagrams given by a local Lagrangian with a spectrum of infinite zero-width resonant states. However, since light resonances reach their on-shell peaks in the energy region spanned by $\tau \to \pi\pi\pi\nu_\tau$, the corresponding resonance widths (that only appear at next-to-leading order in the large-N_C expansion) have to be also included. Moreover, as shown in Ref. [1], it is possible to obtain an adequate description of the experimental data by including just the lowest multiplets of vector and axial-vector resonances in the theory.

We will work out $\tau \to \pi\pi\pi\nu_\tau$ decays considering exact isospin symmetry. In this limit the processes are driven only by the axial-vector current, and appear to be dominated by the contributions of the lightest ρ and a_1 resonances. The corresponding effective Lagrangian in RχT reads:

$$\mathcal{L}_{\mathrm{R}\chi\mathrm{T}} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle + \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + i\frac{G_V}{\sqrt{2}}\langle V_{\mu\nu} u^\mu u^\nu\rangle + \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu} f_-^{\mu\nu}\rangle + \mathcal{L}_{\mathrm{kin}}^V + \mathcal{L}_{\mathrm{kin}}^A + \sum_{i=1}^{5} \lambda_i \mathcal{O}_{\mathrm{VAP}}^i, \tag{1}$$

where all coupling constants are real. The notation is that of Refs. [1,13]. Here F stands for the decay constant of the pion in the chiral limit, and the operators $\mathcal{O}_{\mathrm{VAP}}^i$ are given by:

$$\begin{aligned}
\mathcal{O}_{\mathrm{VAP}}^1 &= \langle [V^{\mu\nu}, A_{\mu\nu}]\chi_-\rangle, \\
\mathcal{O}_{\mathrm{VAP}}^2 &= i\langle [V^{\mu\nu}, A_{\nu\alpha}] h_\mu^\alpha\rangle, \\
\mathcal{O}_{\mathrm{VAP}}^3 &= i\langle [\nabla^\mu V_{\mu\nu}, A^{\nu\alpha}] u_\alpha\rangle, \\
\mathcal{O}_{\mathrm{VAP}}^4 &= i\langle [\nabla^\alpha V_{\mu\nu}, A_\alpha{}^\nu] u^\mu\rangle, \\
\mathcal{O}_{\mathrm{VAP}}^5 &= i\langle [\nabla^\alpha V_{\mu\nu}, A^{\mu\nu}] u_\alpha\rangle.
\end{aligned} \tag{2}$$

Nonets of spin 1 resonances V and A are described here using the antisymmetric tensor formulation, which is consistent with the usage of the χPT Lagrangian for light pseudoscalar mesons up to $\mathcal{O}(p^2)$ [14].

In the Standard Model, the decay amplitudes for $\tau^- \to \pi^+\pi^-\pi^-\nu_\tau$ and $\tau^- \to \pi^-\pi^0\pi^0\nu_\tau$ decays can be written as

$$\mathcal{M}_\pm = -\frac{G_F}{\sqrt{2}} V_{ud} \bar{u}_{\nu_\tau} \gamma^\mu (1-\gamma_5) u_\tau T_{\pm\mu}, \tag{3}$$

where $V_{ud} \simeq \cos\theta_C$ is an element of the Cabibbo–Kobayashi–Maskawa matrix, and $T_{\pm\mu}$ is the hadron matrix element of the axial-vector QCD current A_μ,

$$T_{\pm\mu}(p_1, p_2, p_3) = \langle \pi_1(p_1)\pi_2(p_2)\pi^\pm(p_3)\big| A_\mu e^{i\mathcal{L}_{\mathrm{QCD}}}|0\rangle, \tag{4}$$

as there is no contribution of the vector current to these processes in the isospin limit. Outgoing states $\pi_{1,2}$ correspond here to π^- and π^0 for upper and lower signs in $T_{\pm\mu}$, respectively. The hadron tensor can be written in terms of three form factors, F_1, F_2 and F_P, as [7]:

$$T^\mu = V_1^\mu F_1 + V_2^\mu F_2 + Q^\mu F_P, \tag{5}$$

where

$$\begin{aligned}
V_1^\mu &= \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right)(p_1 - p_3)_\nu, \\
V_2^\mu &= \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right)(p_2 - p_3)_\nu, \\
Q^\mu &= p_1^\mu + p_2^\mu + p_3^\mu.
\end{aligned} \tag{6}$$

In this way, the terms involving the form factors F_1 and F_2 have a transverse structure in the total hadron momenta Q_μ, and drive a $J^P = 1^+$ transition. Meanwhile F_P accounts for a $J^P = 0^-$ transition that carries pseudoscalar degrees of freedom and vanishes with the square of the pion mass. Its contribution to the spectral function of $\tau \to \pi\pi\pi\nu_\tau$ goes like m_π^4/Q^4 and, accordingly, it is very much suppressed with respect to those coming from F_1 and F_2. We will not consider it in the following.

The evaluation of the form factors F_1 and F_2 within in the context of RχT has been carried out in Ref. [1]. One has:

$$F_{\pm i} = \pm\left(F_i^{\chi} + F_i^{\mathrm{R}} + F_i^{\mathrm{RR}}\right), \quad i = 1, 2, \tag{7}$$

where the different contributions correspond to the diagrams in Fig. 1. In terms of the Lorentz invariants Q^2, $s = (p_1 + p_3)^2$, $t = (p_2 + p_3)^2$ and $u = (p_1 + p_2)^2$ (notice that $u = Q^2 - s - t + 3m_\pi^2$) these contributions are given by [1]

$$\begin{aligned}
F_1^{\chi}(Q^2, s, t) &= -\frac{2\sqrt{2}}{3F}, \\
F_1^{\mathrm{R}}(Q^2, s, t) &= \frac{\sqrt{2} F_V G_V}{3F^3}\left[\frac{3s}{s - M_V^2} - \left(\frac{2G_V}{F_V} - 1\right)\left(\frac{2Q^2 - 2s - u}{s - M_V^2} + \frac{u - s}{t - M_V^2}\right)\right],
\end{aligned}$$

$$F_1^{RR}(Q^2,s,t) = \frac{4F_AG_V}{3F^3}\frac{Q^2}{Q^2-M_A^2}\left[-(\lambda'+\lambda'')\frac{3s}{s-M_V^2} + H(Q^2,s)\frac{2Q^2+s-u}{s-M_V^2} + H(Q^2,t)\frac{u-s}{t-M_V^2}\right], \tag{8}$$

where

$$H(Q^2,x) = -\lambda_0\frac{m_\pi^2}{Q^2} + \lambda'\frac{x}{Q^2} + \lambda'', \tag{9}$$

λ_0, λ' and λ'' being linear combinations of the λ_i couplings in Eq. (1) that can be read in Ref. [1]. Bose symmetry under the exchange of the two identical pions in the final state implies that the form factors F_1 and F_2 are related by $F_2(Q^2,s,t) = F_1(Q^2,t,s)$.

Besides the pion decay constant F, the above results for the form factors F_i depend on six combinations of the coupling constants in the Lagrangian $\mathcal{L}_{R\chi T}$, namely F_V, F_A, G_V, λ_0, λ' and λ'' and the masses M_V, M_A of the vector and axial-vector nonets. All of them are in principle unknown parameters. However, it is clear that $\mathcal{L}_{R\chi T}$ does not represent an effective theory of QCD for arbitrary values of the couplings. Though the determination of the effective parameters from the underlying theory is still an open problem, one can get information on the couplings by assuming that the resonance region — even when one does not include the full phenomenological spectrum — provides a bridge between the chiral and perturbative regimes [14]. This is implemented by matching the high energy behaviour of Green functions (or related form factors) evaluated within the resonance theory with asymptotic results obtained in perturbative QCD [14,15,17–22]. In the $N_C \to \infty$ limit, and within the approximation of only one nonet of vector and axial-vector resonances, the analysis of the two-point Green functions $\Pi_{V,A}(q^2)$ and the three-point Green function VAP of QCD currents with only one multiplet of vector and axial-vector resonances lead to the following constraints [23]:

(i) By demanding that the two-pion vector form factor vanishes at high momentum transfer one obtains the condition $F_VG_V = F^2$ [14].
(ii) The first Weinberg sum rule [24] leads to $F_V^2 - F_A^2 = F^2$, and the second Weinberg sum rule gives $F_V^2M_V^2 = F_A^2M_A^2$ [13].
(iii) The analysis of the VAP Green function [15] gives for the coupling combinations λ_0, λ' and λ'' entering the form factors in Eq. (8) the following results:

$$\lambda' = \frac{F^2}{2\sqrt{2}F_AG_V} = \frac{M_A}{2\sqrt{2}M_V}, \tag{10}$$

$$\lambda'' = \frac{2G_V-F_V}{2\sqrt{2}F_A} = \frac{M_A^2-2M_V^2}{2\sqrt{2}M_VM_A}, \tag{11}$$

$$4\lambda_0 = \lambda'+\lambda'' = \frac{M_A^2-M_V^2}{\sqrt{2}M_VM_A}, \tag{12}$$

where the second equalities in Eqs. (10) and (11) are obtained using the above relations (i) and (ii).

As mentioned above, M_V and M_A stand for the masses of the vector and axial-vector resonance nonets, in the chiral and large-N_C limits. A phenomenological analysis carried out in this limit [20] shows that M_V is well approximated by the $\rho(770)$ mass, whereas for the axial mass one gets $M_{a_1}^{1/N_C} \equiv M_A = 998(49)$ MeV (which differs appreciably from the presently accepted value of M_{a_1}).

In addition, one can require that the $J=1$ axial spectral function in $\tau \to \pi\pi\pi\nu_\tau$ vanishes for large momentum transfer. This can be seen from the asymptotic behaviour of the axial-vector current correlator $\Pi_A(Q^2)$ [25], taking into account that each intermediate state carrying the appropriate quantum numbers yields a positive contribution to $\mathrm{Im}\,\Pi_A(Q^2)$. In fact, it is found that this constraint leads to the relations in Eqs. (10) and (11), showing the consistency of the procedure.

The above constraints allow in principle to fix all six free parameters entering the form factors F_i in terms of the vector and axial-vector masses M_V, M_A. However the form factors in Eq. (8) include zero-width ρ and a_1 propagator poles, which lead to divergent phase-space integrals in the calculation of $\tau \to \pi\pi\pi\nu_\tau$ decay widths. As stated above, in order to regularize the integrals one should take into account the inclusion of resonance widths, which means to go beyond the leading order in the $1/N_C$ expansion. In order to account for the inclusion of NLO corrections we perform the substitutions:

$$\frac{1}{M_{R_j}^2-q^2} \longrightarrow \frac{1}{M_j^2-q^2-iM_j\Gamma_j(q^2)}. \tag{13}$$

Here $R_j = V, A$, while the subindex $j = \rho, a_1$ on the right-hand side stands for the corresponding physical state.

The substitution in Eq. (13) implies the introduction of additional theoretical inputs, in particular, the behaviour of resonance widths off the mass shell, to which now we turn.

3. Energy-dependent widths of resonances

In general, it is seen that resonances with wide energy-dependent widths modify the dynamics of the processes in a non-trivial manner. Moreover, up to now a definite way to obtain those widths directly from QCD is lacking. The problem has been addressed in detail in Ref. [26], where off-shell widths of resonances have been studied in the context of RχT. In that work, vector meson resonances are analysed through the two-point correlator of the vector current, defining the resonance width as the imaginary part of the pole generated by the resummation of loop diagrams that have absorptive contributions in the s-channel. The widths obtained in this way are shown to satisfy the requirements of analyticity, unitarity and chiral symmetry prescribed by QCD. According to this definition, the energy-dependent width of the $\rho(770)$ resonance is given by [26]:

$$\Gamma_\rho(s) = \frac{M_\rho s}{96\pi F^2}\left[\sigma_\pi^3\theta(s-4m_\pi^2) + \frac{1}{2}\sigma_K^3\theta(s-4m_K^2)\right], \tag{14}$$

where $\sigma_P = \sqrt{1-4m_P^2/s}$. Incidentally it can be seen that an analogous calculation for the $K^*(892)$ state leads to:

$$\Gamma_{K^*}(s) = \frac{M_{K^*}s}{128\pi F^2}\left[\lambda^{3/2}(1,m_K^2/s,m_\pi^2/s)\theta(s-(m_K+m_\pi)^2) + \lambda^{3/2}(1,m_K^2/s,m_\eta^2/s)\theta(s-(m_K+m_\eta)^2)\right], \tag{15}$$

where $\lambda(a,b,c) = (a+b-c)^2 - 4ab$.

In principle, one could apply the same definition in order to evaluate the energy-dependent width of the a_1 resonance. However, this involves a complex two-loop calculation within the resonance theory and this is beyond our present reach. In Ref. [1] we proposed an oversimplified approach in which the a_1 width was written in terms of three parameters, namely the on-shell width $\Gamma_{a_1}(M_{a_1}^2)$, the mass M_{a_1} and an exponent α that rules the asymptotic behaviour:

$$\Gamma_{a_1}^{I}(Q^2) = \Gamma_{a_1}(M_{a_1}^2)\frac{\phi(Q^2)}{\phi(M_{a_1}^2)}\left(\frac{M_{a_1}^2}{Q^2}\right)^\alpha\theta(Q^2-9m_\pi^2), \tag{16}$$

where

$$\phi(Q^2) = Q^2 \int \mathrm{d}s\,\mathrm{d}t\,\{V_1^2|\mathrm{BW}_\rho(s)|^2 + V_2^2|\mathrm{BW}_\rho(t)|^2 + 2(V_1\cdot V_2)\,\mathrm{Re}[\mathrm{BW}_\rho(s)\,\mathrm{BW}_\rho(t)^*]\}. \quad (17)$$

Here V_1, V_2, s and t are defined as in the previous section, the integral extends over the 3π phase space and the function $\mathrm{BW}_\rho(Q^2)$ is the usual Breit–Wigner for the $\rho(770)$ resonance, with the energy-dependent width $\Gamma_\rho(q^2)$ given by Eq. (14).

The analysis in Ref. [1], which was previous to the determination of the short-distance constraints from the three-point VAP Green function [15], left λ_0 unconstrained. In this way, from the phenomenological analysis of experimental data on $\tau\to\pi\pi\pi\nu_\tau$ processes the following set of values was obtained: $\lambda_0\simeq 12$, $\alpha\simeq 2.5$, $M_{a_1}\simeq 1.2$ GeV and $\Gamma_{a_1}(M_{a_1}^2)\simeq 0.48$ GeV. On the other hand, if one takes into account the relation in Eq. (12), using $M_V = M_{\rho(770)}$ and $M_A\simeq 1$ GeV one gets $\lambda_0\simeq 0.09$, which differs drastically from the result quoted above (obtained from the fit). In fact, as commented already in Ref. [1], the determination of λ_0 from the fit was undermined. The reason is that in the $\tau\to\pi\pi\pi\nu_\tau$ amplitude, without the constraint in Eq. (12), λ_0 appears always together with a suppression factor m_π^2/Q^2 [see Eqs. (8) and (9)], thus the dependence of the amplitude on this parameter should be small. The result $\lambda_0\simeq 12$ is likely to be an artifact to cure a wrong behaviour of the amplitude.

In this work we stick to the short-distance constraints ruled by the VAP Green function, thus we assume λ_0 as given by Eq. (12). In fact, it will be seen that this value of λ_0 is perfectly compatible with a proper description of $\tau\to\pi\pi\pi\nu_\tau$ phenomenology. The new ingredients are the adoption of an adequate definition for the energy-dependent width of the a_1 resonance, and the inclusion of a small effect arising from the presence of a vector resonance $\rho(1450)$ that we will consider in Section 4.

We propose here a new parameterization of the a_1 width that is compatible with the RχT framework used throughout our analysis. As stated, to proceed as in the ρ meson case, one faces the problem of dealing with a resummation of two-loop diagrams in the two-point correlator of axial-vector currents. However, it is still possible to obtain a definite result by considering the correlator up to the two-loop order only. The width can be defined in this way by calculating the imaginary part of the diagrams through the well-known Cutkosky rules.

Let us focus on the transversal component, $\Pi_T(Q^2)$, of the two-point Green function:

$$\Pi_{\mu\nu}^{33} = i\int \mathrm{d}^4x\, e^{iQ\cdot x}\langle 0|T[A_\mu^3(x)A_\nu^3(0)]|0\rangle = (Q^2 g_{\mu\nu} - Q_\mu Q_\nu)\Pi_T(Q^2) + Q_\mu Q_\nu \Pi_L(Q^2), \quad (18)$$

where $A_\mu^i = \bar{q}\gamma_\mu\gamma_5\frac{\lambda^i}{2}q$. We will assume that the transversal contribution is dominated by the π^0 and the neutral component of the a_1 triplet: $\Pi_T(Q^2)\simeq \Pi^{\pi^0}(Q^2) + \Pi^{a_1}(Q^2)$. Following an analogous procedure to the one in Ref. [26], we write $\Pi^{a_1}(Q^2)$ as the sum

$$\Pi^{a_1}(Q^2) = \Pi_{(0)}^{a_1} + \Pi_{(1)}^{a_1} + \Pi_{(2)}^{a_1} + \cdots, \quad (19)$$

where $\Pi_{(0)}^{a_1}$ corresponds to the tree level amplitude, $\Pi_{(1)}^{a_1}$ to a two-loop order contribution, $\Pi_{(2)}^{a_1}$ to a four-loop order contribution, etc. The diagrams to be included are those which have an absorptive part in the s-channel. The first two terms are represented by diagrams (a) and (b) in Fig. 2, respectively, where effective vertices denoted by a square correspond to the sum of the diagrams in Fig. 1. Solid lines in the diagram (b) of Fig. 2 correspond to any set of light pseudoscalar mesons that carry the appropriate quantum numbers to be an intermediate state.

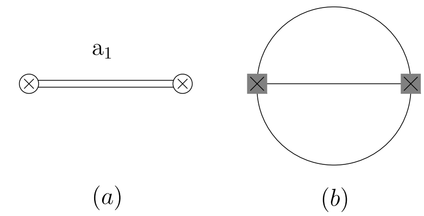

Fig. 2. Diagrams contributing to the transverse part of the correlator of axial-vector currents in Eq. (19). Diagram (a) gives $\Pi_{(0)}^{a_1}$ and diagram (b) provides $\Pi_{(1)}^{a_1}$. The squared axial-vector current insertion in (b) corresponds to the sum of the diagrams in Fig. 1. The double line in (a) indicates the a_1 resonance intermediate state. Solid lines in (b) indicate any Goldstone bosons that carry the appropriate quantum numbers.

The first term of the expansion in Eq. (19) arises from the coupling driven by F_A in the effective Lagrangian (1). We find

$$\Pi_{(0)}^{a_1} = -\frac{F_A^2}{M_{a_1}^2 - Q^2}. \quad (20)$$

Thus, if the series in Eq. (19) can be resummed one should get

$$\Pi^{a_1}(Q^2) = -\frac{F_A^2}{M_{a_1}^2 - Q^2 + \Delta(Q^2)}, \quad (21)$$

and the energy dependent width of the a_1 resonance can be defined by

$$M_{a_1}\Gamma_{a_1}(Q^2) = -\,\mathrm{Im}\,\Delta(Q^2). \quad (22)$$

Now if we expand $\Pi^{a_1}(Q^2)$ in powers of Δ and compare term by term with the expansion in Eq. (19), from the second term we obtain

$$\Delta(Q^2) = -\frac{(M_{a_1}^2 - Q^2)}{\Pi_{(0)}^{a_1}}\Pi_{(1)}^{a_1}. \quad (23)$$

The off-shell width of the a_1 resonance will be given then by

$$\Gamma_{a_1}(Q^2) = \frac{(M_{a_1}^2 - Q^2)}{M_{a_1}\Pi_{(0)}^{a_1}}\,\mathrm{Im}\,\Pi_{(1)}^{a_1}. \quad (24)$$

As stated, $\Pi_{(1)}^{a_1}$ receives the contribution of various intermediate states. These contributions can be calculated within our theoretical RχT framework from the effective Lagrangian in Eq. (1). In particular, for the intermediate $\pi^+\pi^-\pi^0$ state one has

$$\Pi_{(1)}^{a_1}(Q^2) = \frac{1}{6Q^2}\int \frac{\mathrm{d}^4p_1}{(2\pi)^4}\frac{\mathrm{d}^4p_2}{(2\pi)^4} T_{1^+}^{\mu} T_{1^+\mu}^{*} \times \prod_{i=1}^{3}\frac{1}{p_i^2 - m_\pi^2 + i\epsilon}, \quad (25)$$

where $p_3 = Q - p_1 - p_2$, and T_{1^+} is the 1^+ piece of the hadron tensor in Eq. (5),

$$T_{1^+}^{\mu} = V_1^{\mu}F_1 + V_2^{\mu}F_2. \quad (26)$$

When extended to the complex plane, the function $\Pi_{(1)}^{a_1}(z)$ has a cut in the real axis for $z\geqslant 9m_\pi^2$, where $\mathrm{Im}\,\Pi_{(1)}^{a_1}(z)$ shows a discontinuity. The value of this imaginary part on each side of the cut can be calculated according to the Cutkosky rules as:

$$\mathrm{Im}\,\Pi_{(1)}^{a_1}(Q^2\pm i\epsilon) = \mp\frac{i}{2}\frac{1}{6Q^2}\int\frac{\mathrm{d}^4p_1}{(2\pi)^4}\frac{\mathrm{d}^4p_2}{(2\pi)^4}T_{1^+}^{\mu}T_{1^+\mu}^{*} \times \prod_{i=1}^{3}(-2i\pi)\theta(p_i^0)\delta(p_i^2 - m_\pi^2), \quad (27)$$

with $p_3 = Q - p_1 - p_2$ and $Q^2 > 9m_\pi^2$. After integration of the delta functions one finds

$$\mathrm{Im}\, \Pi^{a_1}_{(1)}\left(Q^2 \pm i\epsilon\right) = \pm \frac{1}{192 Q^4} \frac{1}{(2\pi)^3} \int \mathrm{d}s\, \mathrm{d}t\, T^\mu_{1^+} T^*_{1^+\mu}, \tag{28}$$

where the integrals extend over a three-pion phase space with total momentum squared Q^2. Therefore, the contribution of the $\pi^+\pi^-\pi^0$ state to the a_1 width will be given by

$$\Gamma^\pi_{a_1}\left(Q^2\right) = \frac{-1}{192(2\pi)^3 F_A^2 M_{a_1}} \left(\frac{M^2_{a_1}}{Q^2} - 1\right)^2 \int \mathrm{d}s\, \mathrm{d}t\, T^\mu_{1^+} T^*_{1^+\mu}. \tag{29}$$

In the same way one can proceed to calculate the contribution of the intermediate states $K^+K^-\pi^0$, $K^0\bar{K}^0\pi^0$, $K^-K^0\pi^+$ and $K^+\bar{K}^0\pi^-$. The corresponding hadron tensors $T^K_{1^+}$ can be obtained from Ref. [27]. Additionally one could consider the contribution of $\eta\pi\pi$ and $\eta\eta\pi$ intermediate states. However, these are suppressed by tiny branching ratios [28,29] and will not be taken into account.

In this way we have[1]

$$\Gamma_{a_1}\left(Q^2\right) = \Gamma^\pi_{a_1}\left(Q^2\right)\theta\left(Q^2 - 9m_\pi^2\right) + \Gamma^K_{a_1}\left(Q^2\right)\theta\left(Q^2 - (2m_K + m_\pi)^2\right), \tag{30}$$

where

$$\Gamma^{\pi,K}_{a_1}\left(Q^2\right) = \frac{-S}{192(2\pi)^3 F_A^2 M_{a_1}} \times \left(\frac{M^2_{a_1}}{Q^2} - 1\right)^2 \int \mathrm{d}s\, \mathrm{d}t\, T^{\pi,K\mu}_{1^+} T^{\pi,K*}_{1^+\mu}. \tag{31}$$

Here $\Gamma^\pi_{a_1}(Q^2)$ recalls the three-pion contributions and $\Gamma^K_{a_1}(Q^2)$ collects the contributions of the $KK\pi$ channels. In Eq. (31) the symmetry factor $S = 1/n!$ reminds the case with n identical particles in the final state. It is also important to point out that, contrarily to the width we proposed in Ref. [1] [$\Gamma^{\mathrm{I}}_{a_1}(Q^2)$, in Eq. (16)], the on-shell width $\Gamma_{a_1}(M^2_{a_1})$ is now a prediction and not a free parameter.

An additional point to be taken into account are the off-shell widths of the vector meson resonances entering the form factors in $T^{\pi,K}_{1^+}$. Once again, since these resonances reach their on-shell peaks in the phase-space integrals, it is necessary to go beyond the leading $1/N_C$ limit and include the corresponding energy-dependent widths. The involved resonances for the $\pi\pi\pi$ and $KK\pi$ intermediate states (always sticking to the approximation of taking only the lowest nonets) are the $\rho(770)$, $K^*(892)$, $\omega(782)$ and $\phi(1020)$ vector mesons. For the ρ and K^* we will consider the energy-dependent widths in Eqs. (14) and (15). Since resonances $\omega(892)$ and $\phi(1020)$ are very narrow, the energy dependence is irrelevant, and for our purposes we can take the experimental on-shell widths quoted by the PDG [28].

4. The contribution of the $\rho(1450)$

It turns out that, though some flexibility is allowed around the predicted values for the parameters, the region between 1.5–2.0 GeV^2 of the three-pion spectrum is still poorly described by the scheme we have proposed here. This is not surprising as the $\rho(1450)$, acknowledgeably rather wide, arises in that energy region. We find that it is necessary to include, effectively, the role of a $\rho' \equiv \rho(1450)$, in order to recover good agreement with the experimental data. The ρ' belongs to a second, heavier, multiplet of vector resonances that we have not considered in our procedure. Its inclusion would involve a complete new set of analogous operators to the ones already present in $\mathcal{L}_{\mathrm{R}\chi\mathrm{T}}$, Eq. (1), with the corresponding new couplings. This is beyond the scope of our analysis. However we propose to proceed by performing the following substitution in the $\rho(770)$ propagator:

$$\frac{1}{M_\rho^2 - q^2 - iM_\rho\Gamma_\rho(q^2)} \longrightarrow \frac{1}{1+\beta_{\rho'}}\left[\frac{1}{M_\rho^2 - q^2 - iM_\rho\Gamma_\rho(q^2)} + \frac{\beta_{\rho'}}{M_{\rho'}^2 - q^2 - iM_{\rho'}\Gamma_{\rho'}(q^2)}\right], \tag{32}$$

where as a first approximation the ρ' width is given by the decay into two pions:

$$\Gamma_{\rho'}\left(q^2\right) = \Gamma_{\rho'}\left(M^2_{\rho'}\right)\frac{M_{\rho'}}{\sqrt{q^2}}\left(\frac{p(q^2)}{p(M^2_{\rho'})}\right)^3 \theta\left(q^2 - 4m_\pi^2\right),$$
$$p(x) = \frac{1}{2}\sqrt{x - 4m_\pi^2}. \tag{33}$$

For the numerics we use the values $M_{\rho'} = 1.465$ GeV and $\Gamma_{\rho'}(M^2_{\rho'}) = 400$ MeV as given in Ref. [28]. We find that a good agreement with the spectrum, $\mathrm{d}\Gamma/\mathrm{d}Q^2$, measured by ALEPH [12] is reached for the set of values:

$$F_V = 0.180\ \mathrm{GeV}, \qquad F_A = 0.149\ \mathrm{GeV}, \qquad \beta_{\rho'} = -0.25,$$
$$M_V = 0.775\ \mathrm{GeV}, \qquad M_{K^*} = 0.8953\ \mathrm{GeV},$$
$$M_{a_1} = 1.120\ \mathrm{GeV}, \tag{34}$$

that we call Set 1. The corresponding width is $\Gamma(\tau \to \pi\pi\pi\nu_\tau) = 2.09 \times 10^{-13}$ GeV, in excellent agreement with the experimental figure $\Gamma(\tau \to \pi\pi\pi\nu_\tau)|_{\mathrm{exp}} = (2.11 \pm 0.02) \times 10^{-13}$ GeV [28]. From F_V and F_A in Eq. (34), and the second Weinberg sum rule we can also determine the value of $M_A = F_V M_V / F_A \simeq 0.94$ GeV, a result consistent with the one obtained in Ref. [20]. If, instead, we do not include the ρ' contribution, the best agreement with experimental data is reached for the values of Set 2:

$$F_V = 0.206\ \mathrm{GeV}, \qquad F_A = 0.145\ \mathrm{GeV}, \qquad \beta_{\rho'} = 0,$$
$$M_V = 0.775\ \mathrm{GeV}, \qquad M_{K^*} = 0.8953\ \mathrm{GeV},$$
$$M_{a_1} = 1.115\ \mathrm{GeV}, \tag{35}$$

though the branching ratio is off by 15%. A comparison between the results for the $\tau \to \pi\pi\pi\nu_\tau$ spectra obtained from Sets 1, 2 and the data provided by ALEPH is shown in Fig. 3. Notice that we have corrected the results provided by Set 2 by a normalization factor of 1.15 in order to compare the shapes of the spectra. Though it is difficult to assign an error to our numerical values, by comparing Set 1 and Set 2 we consider that a 15% should be on the safe side. Notice, however, that the error appears to be much smaller in the case of M_{a_1}.

The value that we get for $M_{a_1} = 1.120$ GeV differs from the one we got in Ref. [1], namely $M_{a_1} = 1.203(3)$ GeV (the error only includes the fit procedure). The disparity is mainly an outcome of the different off-shell width of the a_1 that we introduce in this Letter and that we consider much more appropriate. It has to be taken into account that our definition of M_{a_1} is the one given by Eq. (13) that constitutes and approach consistent with the features

[1] It is important to stress that we do not intend to carry out the resummation of the series in Eq. (19). In fact, our expression in Eq. (24) would correspond to the result of the resummation if this series happens to be geometric, which in principle is not guaranteed [26].

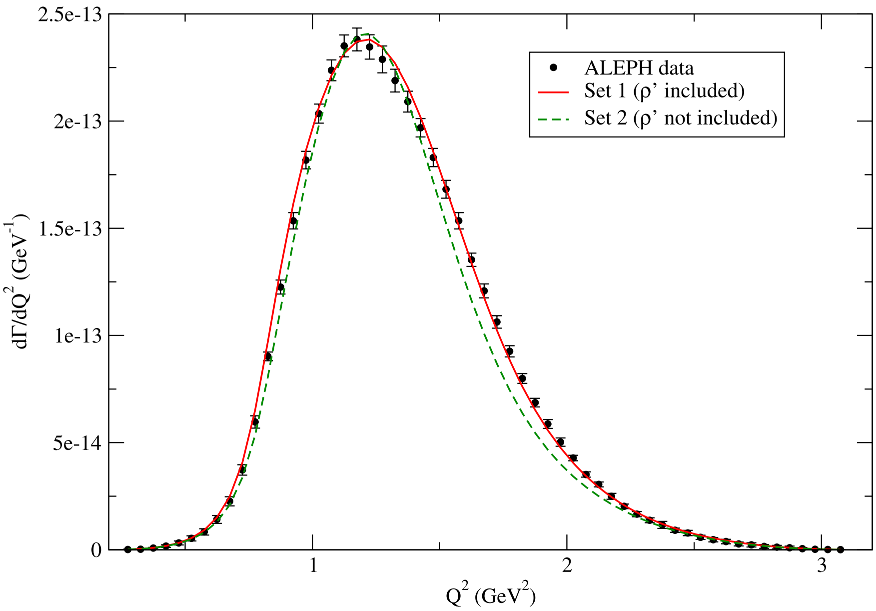

Fig. 3. Comparison between the theoretical $M^2_{3\pi}$-spectra of the $\tau^- \to \pi^+\pi^-\pi^-\nu_\tau$ with ALEPH data [12]. Set 1 corresponds to the values of the parameters: $F_V = 0.180$ GeV, $F_A = 0.149$ GeV, $M_{a_1} = 1.120$ GeV, $\beta_{\rho'} = -0.25$, $M_A \simeq 0.94$ GeV. Set 2 corresponds to the values of the parameters: $F_V = 0.206$ GeV, $F_A = 0.145$ GeV, $M_{a_1} = 1.150$ GeV, $\beta_{\rho'} = 0$, i.e. without the inclusion of the ρ'. In the case of Set 2 the overall normalization of the spectrum has been corrected by a 15% to match the experimental data.

of our scheme. The physical pole mass could show a correction that, within our procedure, is a next-to-leading effect in the $1/N_C$ expansion. It is also necessary to point out that masses and on-shell widths collected in the PDG [28] also rely on models of the form factors or scattering amplitudes.

For Set 1 the width of the a_1 is $\Gamma_{a_1}(M^2_{a_1}) = 0.483$ GeV, which, incidentally, is in agreement with the figure we got in Ref. [1] from a fit to the data. The value of $\Gamma_{a_1}(M^2_{a_1})$ quoted in the PDG (2008) [28] goes from 250 MeV up to 600 MeV.

Our preferred set of values in Eq. (34) satisfies reasonably well all the short distance constraints pointed out in Section 2, with a deviation from Weinberg sum rules of at most 10%, perfectly compatible with deviations due to the single resonance approximation.

5. Conclusions

The data available in $\tau \to \pi\pi\pi\nu_\tau$ decays provide an excellent benchmark to study the hadronization of the axial-vector current and, consequently, the properties of the $a_1(1260)$ resonance. In this Letter we give a description of those decays within the framework of resonance chiral theory and the large-N_C limit of QCD that: (1) Satisfies all constraints of the asymptotic behaviour, ruled by QCD, of the relevant two- and three-point Green functions; (2) Provides an excellent description of the branching ratio and spectrum of the $\tau \to \pi\pi\pi\nu_\tau$ decays.

Though this work was started in Ref. [1], later achievements showed that a deeper comprehension of the dynamics was needed in order to enforce the available QCD constraints. To achieve a complete description we have defined a new off-shell width for the a_1 resonance in Eq. (30), which is one of the main results of this work. Moreover we have seen that the inclusion of the $\rho(1450)$ improves significantly the description of the observables. In passing we have also obtained the mass value $M_{a_1} = 1.120$ GeV and the on-shell width $\Gamma_{a_1}(M^2_{a_1}) = 0.483$ GeV.

With the description of the off-shell width obtained in this work we can now consider that the hadronization of the axial-vector current within our scheme is complete and it can be applied in other hadron channels of tau decays.

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