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LETTER

Entanglement properties and momentum distributions of hard-core anyons on a ring

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Abstract. We study the one-particle von Neumann entropy of a system of N hard-core anyons on a ring. The entropy is found to have a clear dependence on the anyonic parameter which characterizes the generalized fractional statistics described by the anyons. This confirms that the entanglement is a valuable quantity for investigating topological properties of quantum states. We derive the generalization to anyonic statistics of the Lenard formula for the one-particle density matrix of N hard-core bosons in the large N limit and extend our results by a numerical analysis of the entanglement entropy, providing additional insight into the problem under consideration.

Keywords: quantum integrability (Bethe ansatz), quantum wires (theory), entanglement in extended quantum systems (theory)

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In recent years an intensive research activity has been devoted to the study of entanglement in many-body states. Initially, this effort was mostly motivated by the fact that quantum correlated many-body states, which appear in various solid-state models, can be valuable resources for information processing and quantum computation [1, 2]. The theory of entanglement is now attracting even more attention because of its fundamental implications for the development of new efficient numerical methods for quantum systems [3]–[5] and for the characterization of quantum critical phases [6]–[8].

Generally speaking, entanglement measures non-local properties of composite quantum systems and it can provide information additional to that obtained by investigating local observables or traditional correlation functions. In this respect entanglement might be a sensitive probe into the topological properties of quantum states. A particularly significant quantity is the entanglement entropy S_A, which is defined in a bipartite system $A \cup B$ and quantified as the von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \ln \rho_A$ associated with the reduced density matrix ρ_A of a subsystem A. In two-dimensional systems a firm connection between topological order and entanglement entropy has been established in [9, 10], where the entanglement entropy was defined by spatial partitioning. Recent studies on Laughlin states [11, 12] have considered the entanglement entropy associated with particle partitioning [11, 12]. Also in this case, the entanglement entropy turns out to reveal important aspects of the topological order in Laughlin states.

The two-dimensional case is of particular interest due to the existence of models whose elementary excitations exhibit generalized fractional statistics. Anyons, the particles obeying such statistics, play a fundamental role in the description of the fractional quantum Hall effect [13]. Although this concept is essentially two-dimensional, anyons can also occur in one-dimensional (1D) systems [14]–[20], where statistics and interactions are inextricable, leading to strong short-range correlations. The 1D anyonic models have proven useful for studying persistent charge and magnetic currents in 1D mesoscopic rings [15]. This possibility and their own pure theoretical interest led us to investigate the effects of the anyonic statistics on the entanglement entropy in the present letter. A discussion about quantum statistics and entanglement in a two-fermion system was introduced in [21] and extended to the case of two bosons in [22]. A mechanism of spin-space entanglement transfer based on the indistinguishability of two particles was proposed in [23] and shown to depend on the statistics (either fermionic or bosonic) of the particles involved.

In this letter, we consider a system of N hard-core anyons on a ring which is the direct anyonic generalization of the Tonks–Girardeau gas. It offers a convenient framework for studying topological effects: the many-body ground state is known and its behaviour under the exchange of two particles interpolates between bosons and fermions. We carry out an analytical and numerical analysis of the dependence of the one-particle von Neumann entropy on the statistical parameter which determines the symmetry of the many-body state. We derive the large N asymptotic expression of the anyonic one-particle density matrix. This asymptotic form generalizes the one obtained for hard-core bosons [24, 25] and provides a one-parameter family of zero-temperature momentum distributions interpolating between hard-core boson and free fermion distributions. Our results show that particle entanglement depends in a non-trivial manner on the statistics and, as such, may prove to be relevant to the study of topological properties of many-body quantum states.

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Let us consider a 1D system of anyons confined on a ring of length L interacting with each other via a repulsive δ-function potential. The model is defined by the Hamiltonian

$$H = -\sum_{i}^{N} \frac{\partial^2}{\partial x_i^2} + \gamma \sum_{1 \le i < j \le N} \delta(x_i - x_j). \tag{1}$$

The N-anyon wavefunction $\Psi^\theta(x_1, x_2, \ldots, x_N)$ exhibits a generalized symmetry under the exchange of particles:

$$\Psi^\theta(\cdots x_i, x_{i+1} \cdots) = \mathrm{e}^{\mathrm{i}(\theta-\pi)\varepsilon(x_{i+1}-x_i)} \Psi^\theta(\cdots x_{i+1}, x_i \cdots), \tag{2}$$

where $\varepsilon(x) = -1$ (or 1) if $x > 0$ ($x < 0$), $\varepsilon(0) = 0$ and θ is the anyonic parameter ($-\pi < \theta \le \pi$), defined as in [15]. For $\theta = 0$ this model describes free fermions while, for $\theta = \pi$, it reduces to the Lieb–Liniger Bose gas.

As first discussed in [16], the problem of 1D anyons with contact interactions allows for an exact Bethe ansatz solution which shows that the Hamiltonian (1) has the same energy spectrum as a 1D interacting Bose gas with anyonic statistics-dependent effective coupling in the moving frame. Very recently, a detailed analysis of the low-energy properties of this model has been carried out in [17]–[19]. It was shown that the low-temperature thermodynamics of 1D anyons with a δ-function potential coincides with that of a gas of ideal particles obeying Haldane statistics: the interplay between the anyonic parameter θ and the coupling constant γ determines a continuous range of these generalized statistics. These studies have shown that, for strong coupling, the dispersion relations of the anyon gas remain linear in the thermodynamic limit and the finite size corrections of the ground state energy found a central charge $c = 1$.

In the case of spatial partitioning, the subsystem A being a block of size l, conformal field theory results [26, 27] predict the entanglement entropy (block entropy) $S_A(l)$ to scale as $S_A(l) \sim \frac{1}{3}\ln l$. A similar behaviour is predicted also in some class of strongly random spin chains [28]–[30]. The dependence on the coupling constant and on the anyonic parameter is expected to show up in the sub-leading terms which, to the best of our knowledge, remain unknown. Below we will demonstrate that the one-particle entanglement entropy $S_1^\theta(N)$ of N anyons depends on the anyonic parameter for finite N. Furthermore, we will show that, in the asymptotic limit $N \gg 1$, the dependence of the entanglement entropy on the anyonic statistics appears in the sub-leading term.

Let us consider the limit of hard-core anyons, i.e. $\gamma \to \infty$. As recently shown in [20], the Fermi–Bose mapping method for one-dimensional hard-core bosons [31] can be generalized to an anyon–fermion mapping (AF). Imposing the exclusion principle, i.e. the vanishing of the many-body wavefunction when two particles occupy the same position, the AF mapping reads [20]

$$\Psi_0^\theta(x_1, \ldots, x_N) = \left[\prod_{1 \le i < j \le N} A(x_i - x_j)\right] \Psi_0^F(x_1, \ldots, x_N), \tag{3}$$

where $\Psi_0^F(x_1, \ldots, x_N)$ is the N free fermion ground state function and $A(x_i - x_j) = \mathrm{e}^{\mathrm{i}\theta}$ for $x_i < x_j$ and $A(x_i - x_j) = 1$ for $x_i > x_j$. In the following, we restrict ourselves to the case where N is odd, which corresponds to a non-degenerate ground state. The topological properties of the N-anyon wavefunction are encoded in the factor $\prod_{1 \le i < j \le N} A(x_i, x_j)$ which gives the statistical phase $\mathrm{e}^{\mathrm{i}\theta P}$ resulting from the P exchanges needed for the

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particle positions to be brought to the ordering $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq L$. The boundary periodic conditions of the wavefunction allow in general an overall phase, coming for example via a non-zero magnetic flux penetrating the ring [15]. However, in order to simplify some analytical manipulations and make the comparison between different statistics more direct, we require the wavefunction to be single-valued $\Psi^\theta(x_1, \ldots, x_i + L, \ldots, x_N) = \Psi^\theta(x_1, \ldots, x_i, \ldots, x_N)$. This condition restricts the anyonic parameter to being an integer multiple of $2\pi/(N-1)$, $\theta = (2\pi/(N-1))n$. Since the values of θ become dense in the limit $N \to \infty$ in which we are interested, this choice will not affect the generality of our results.

The definition of criteria of entanglement between systems of identical particles is in general a subtle issue (see for example [21, 32]) since the symmetry properties of the many-body wavefunction forbid the occurrence of factorized states. For a pure state of two particles, the Von Neumann entropy of the reduced density matrix of one particle has been shown to remain a good entanglement measure [22]. The same measure has been used to compare the entanglement between two bosons and two fermions trapped in a 1D harmonic potential [33] and to study the entanglement properties of some fractional quantum Hall liquids [11, 12, 34].

Inspired by these results, we are interested here in computing the entropy $S_1^\theta(N) = -\mathrm{Tr}(\rho_N^\theta \ln \rho_N^\theta)$, where $\rho_N^\theta(x - x')$ is the one-particle reduced density matrix:

$$\rho_N^\theta(x - x') = \int_0^L \ldots \int_0^L \prod_{i=2}^N \mathrm{d}x_i \left[\bar{\Psi}^\theta(x, x_2, \ldots, x_N)\Psi^\theta(x', x_2, \ldots, x_N)\right], \tag{4}$$

normalized such that $\rho_N(0) = 1$. The von Neumann entropy measures the uncertainty in attributing a state to the subsystem under consideration. In our case, its value is directly related to the momentum state occupation distribution. Indeed, using the fact that the one-particle density matrix is diagonal in the momentum space, the entanglement entropy reads

$$S_1^\theta(N) = -\sum_{n=-\infty}^{\infty} c_N^\theta(n) \ln c_N^\theta(n), \tag{5}$$

where $c_N^\theta(n) = 1/L \int_0^L \rho_N^\theta(x) \cos(2\pi/Lnx)$ is the momentum occupation in the ground state. We can read $c_N^\theta(n)$ as the probability of the one-particle subsystem being in the state with momentum $k_n = 2\pi/Ln$. For free bosons, for example, one has $c_N^{\mathrm{frbos}}(n) = \delta_{n,0}$ and the one-particle entanglement entropy is identically zero. For free fermions ($\theta = 0$), instead, the one-particle subsystem occupies with the same probability $c_N^0(n) = 1/N$ the states with momentum k_n, where $-(N-1)/2 \leq n \leq (N-1)/2$. This leads to the well known result for free fermions $S_1^0 = \ln N$.

In the limit of infinite interaction, the statistical effects are greatly suppressed, as can be seen from the anyonic wavefunction (3) which reflects the fundamental similarities between strongly interacting anyons and non-interacting fermions in one dimension. The mapping (3) between Fermi and anyon eigenfunctions preserves all scalar products and thus the energy spectrum and all the probability distributions involving the norm of the wavefunction are identical. Nevertheless, $c_N^\theta(n)$ strongly depends on θ, as they do not need to be in different momentum states. This is apparent in the drastic difference between the momentum distributions of the free fermions and the hard-core bosons.

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Analytical expressions of $c^{\pi}(n)$ for small ($|n| \ll 1$) and large ($|n| \gg 1$) momenta have been found in the thermodynamic limit [35]–[39]. These results show a $|n|^{-1/2}$ singularity at $n=0$, reflecting the tendency towards Bose–Einstein condensation. The corresponding result for finite N is more cumbersome. Using the $N \gg 1$ asymptotic result for $\rho^{\pi}(x)$ [24],

$$\rho^{\pi}(x) \sim \rho_{\infty} N^{-1/2} |\sin \pi x/L|^{-1/2}, \tag{6}$$

with $\rho_{\infty} = G(3/2)^4/\sqrt{2}$ and $G(z)$ the Barnes G-function[6], $c_n(N)$ for $N \gg n$ was shown to behave like [25]

$$c_n^{\pi}(N) \sim \frac{\rho_{\infty}}{\sqrt{\pi}} \frac{\Gamma(n+1/4)}{\Gamma(n+3/4)} N^{-1/2}, \tag{7}$$

where $\Gamma(z)$ is the standard Gamma function. We generalized the above results to anyonic statistics. Representing the density matrix (4) in terms of a Toeplitz $N-1 \times N-1$ determinant,

$$N\rho_N^{\theta}(x) = \det{}_{N-1}\left[\phi_{k,l}\right](x) \tag{8}$$

where

$$\phi_{k,l} = \int_0^{2\pi} \mathrm{d}s \frac{2\mathrm{e}^{\mathrm{i}(k-l)s}}{\pi} A\left(s - \frac{2\pi x}{L}\right) \sin\left(\frac{s}{2} - \frac{\pi x}{L}\right) \sin\left(\frac{s}{2}\right), \tag{9}$$

we were able to compute the asymptotic form of ρ_N^{θ} using the Fisher–Hartwig conjecture [40, 41]. Note that the above representation is consistent with the requirement $\rho_N^{\theta}(x+L) = \rho^{\theta}(x)$. For $N \gg 1$ the one-particle anyon density matrix reads

$$\rho_N^{\theta}(x) \sim (2N)^{-1/2-2\beta(\theta)^2} G\left(1+\frac{\theta}{2\pi}\right)^2 G\left(2-\frac{\theta}{2\pi}\right)^2 \times \mathrm{e}^{-\mathrm{i}2\beta(\theta)(N\pi x/L-1/2)} \left|\sin\left(\frac{\pi x}{L}\right)\right|^{-1/2-2\beta(\theta)^2} \tag{10}$$

where $\beta(\theta) = \theta/(2\pi) - 1/2$ for $0 \le \theta \le \pi$ and $\beta(\theta) = -\beta(-\theta)$ for $-\pi \le \theta \le 0$. We see that equation (6) is recovered for $\theta = \pi$. Details of the derivation and a more complete discussion of this result will be presented elsewhere [42][7]. In the case of the generating function (9) the analysis of the behaviour of the Toeplitz determinant is subtle and the Fisher–Hartwig formula remains a conjecture. The validity of equation (10) has thus to be compared to the numerical evaluation of the determinant (8) for finite N. For small θ, the convergence to the asymptotic result is quite slow and the formula provides a rough estimate for finite N, as can be seen for $N = 121$ and $\theta = \pi/60$ (figure 1(a)). The similarity increases greatly with θ and the agreement is already perfect with $\theta = \pi/2$ for $N = 61$ (figure 1(b)) and for $\theta = 9\pi/10$, close to hard-core bosons, for $N = 21$ (figure 1(c)). From equation (10), one can see that, compared to the bosonic case, the main effect of the anyonic statistics is to introduce an oscillating term (figures 1(b), (c)). It is interesting to note that the same term characterizes in general the one-dimensional anyonic systems, as has been shown for free anyons by means of field theory methods [44, 45]. The

[6] $G(z+1) = (2\pi)^{z/2} \exp(-(z+(\gamma_{\mathrm{E}}+1)z^2)/2) \prod_{k=1}^{\infty}(1+z/k)^k \exp(-z+z^2/(2k))$ where γ_{E} is the Euler constant.

[7] We have detected an error in the derivation the sine exponent thanks to comparison with the findings of [43].

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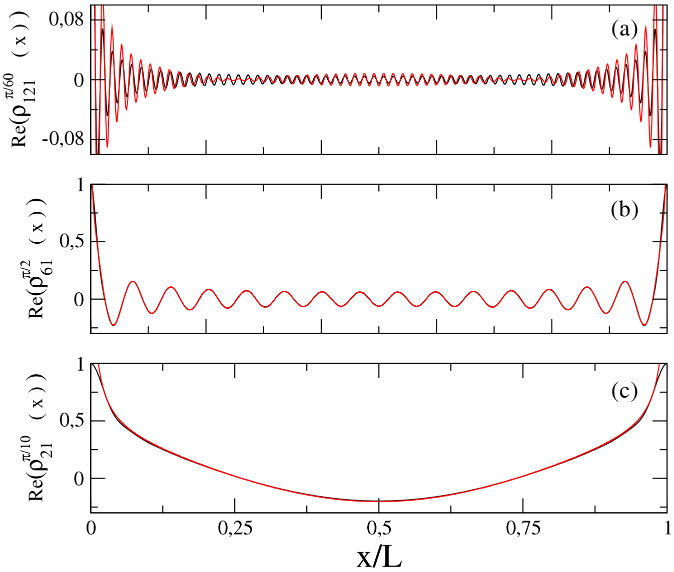

Figure 1. Comparison between the numerical computation (red/dark grey curve) and asymptotic equation (black curves) for the one-particle anyon density for (a) $N = 121, \theta = \pi/60$, (b) $N = 61, \theta = \pi/2$ and (c) $N = 21, \theta = 9\pi/10$.

oscillations produced by the anyonic statistics can produce measurable effects in current noise experiments in fractional quantum Hall fluids and can be used to probe the fractional statistics in topological liquids [46]. Further, the equation (10) provides also the explicit dependence on the anyonic parameter of the factor in front of the x-dependent terms. In general this factor cannot be determined by field theory approaches.

The Fourier coefficients of (10) can be computed analytically [47]. The asymptotic behaviour of $c_n^\theta(N)$ for $N \gg n$ reads

$$\begin{aligned} c_n^\theta(N) \sim \frac{1}{\pi} N^{-1/2-2\beta(\theta)^2} G\left(1+\frac{\theta}{2\pi}\right)^2 G\left(2-\frac{\theta}{2\pi}\right)^2 \\ \times \Gamma\left(1/2-2\beta(\theta)^2\right) \sin\left(\pi(1/2+2\beta(\theta)^2\right) \\ \times \frac{\Gamma(n'+1/4+\beta(\theta)/4+\beta(\theta)^2)}{\Gamma(n'+3/4+\beta(\theta)/4+\beta(\theta)^2)} \end{aligned} \qquad (11)$$

where $n' = n + \lfloor \beta(\theta)(N+1/2) \rfloor$ with $\lfloor x \rfloor$ being the integer part of x. The exact momentum distribution has been obtained numerically by performing the Fourier transform over the Toeplitz determinant. Note that the moment distributions gives a total current $\sum_n n c_n^\theta(N) = 0$, as expected in the case $\theta = (2\pi)/(N-1)n$ under consideration [15]. A comparison between the formula (11) and the numerical results is shown in figure 2. From equation (10), the number of particles occupying the low-energy state ($n' = 0$) scales as $N^{\alpha(\theta)-1}$, thus ruling out the possibility of anyon condensation predicted in the case of free anyons [45]. For $n \gg N$, the bosonic momentum distribution c_n^π decays like n^{-4} [25, 48, 49]. We expect this to be true for anyonic statistics as well (see figure 2). Under this assumption, the terms $c_n^\theta(N)$ for $n \gg N$ will not contribute significantly to the entanglement entropy. The main contribution to $S_1^\theta(N)$ can be extracted by using equation (11) in its range of validity. This gives roughly a $\propto \ln N$ leading order behaviour

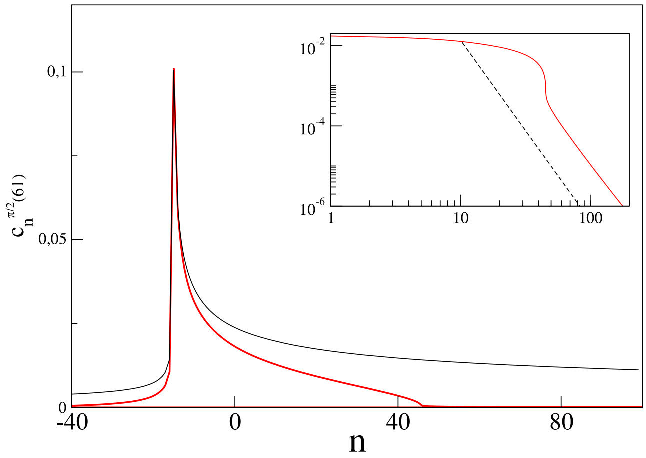

Figure 2. $c_n^{\theta}(N)$ obtained from numerical analysis and equation (11) (black line) versus n for $\theta = \pi/2$ and $N = 61$. The inset is the same plot for $n > 0$ on a log–log scale (red/dark grey curve), while the dashed curve is a visual guide, proportional to n^{-4}.

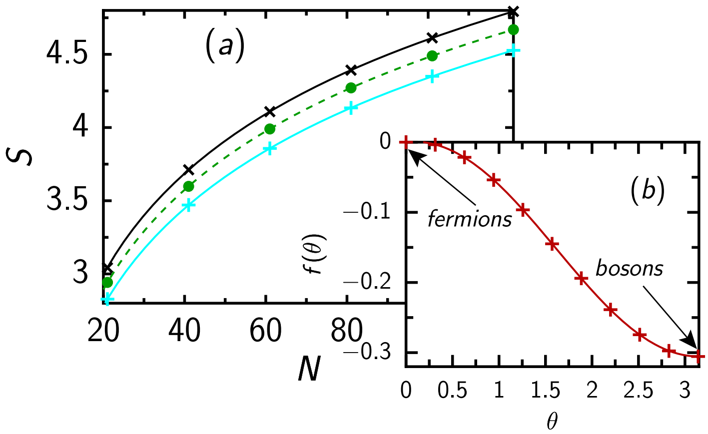

Figure 3. (a) Entanglement entropy as a function of N for $\theta = \pi$ (plus signs), $\theta = \pi/2$ (dots), $\theta = \pi/10$ (crosses) in log–linear scale, fitted according to equation (12). (b) $f(\theta)$ obtained by numerical integration (plus signs) and the corresponding sine fit (plain line).

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for the large N asymptotics. However, the numerical results show that, for finite N, the crossover region, between the asymptotic (11) and the power law, is important, especially near the fermionic point $\theta = 0$ (see figure 2).

We have determined numerically the value of the one-particle von Neumann entropy $S_1^{\theta}(N)$ for different values of N (figures 3(a), (b)). We were able to explicitly verify that, in the proximity of the bosonic point, the one-particle subsystem occupies with high probability the momentum states with $n' \approx 0$. For instance, if we consider $N = 61$ and $-4 \leq n' \leq 4$, this probability is $\sim$0.8 for $\theta = \pi$ and $\sim$0.3 for $\theta = \pi/2$. As already said before, the entanglement entropy measures the uncertainty of attributing a state to the subsystem. We thus expect the entanglement entropy to decrease from free fermions to hard-core bosons. The numerical data (see figure 3) suggest the entanglement entropy to scale as the logarithm of the number of particles plus some negative correction $f(\theta)$ depending only on θ ($f(0) = 0$). The deviation from this behaviour is below 1% already for $N > 81$ and it is found to be well fitted by the power law function $\propto N^{-1/2}$. On the basis of the above discussions we have fitted our data with the following guess:

$$S_1^\theta(N) \approx \ln N + f(\theta) + \frac{\kappa(\theta)}{\sqrt{N}}. \tag{12}$$

We have checked that, among the possible choices, the above scaling formula is the one which fits better the numerical results. Equation (12) is reminiscent of the scaling behaviour of the block entropy $S_A(l)$ where the anyonic dependence is expected to appear in the sub-leading order. The anyonic-parameter-dependent function $f(\theta)$ was determined numerically for system sizes for which the last term of this expansion cannot be neglected. However, in the thermodynamic limit, only $f(\theta)$ will remain relevant in our discussion. The results of our numerical analysis are displayed in figure 3, where panel (a) shows the data as fitted by equation (12) for three values of the anyonic parameter and panel (b) is the resulting $f(\theta)$. This function decreases monotonically from free fermions to hard-core bosons where it respectively takes the values $f(0) = 0$ and $f(\pi) \approx -0.3$, and from figures 3(b) it can be seen to be well fitted by a sine function.

To conclude, we have investigated analytically and numerically the one-particle von Neumann entropy and the momentum distributions of N hard-core anyons on a ring. We have determined the asymptotic expressions for the one-particle density matrix and for the momentum distributions. Numerical results show the entanglement entropy exhibits a simple and non-trivial dependence on the anyonic parameter, making it a suitable tool for studying the topological properties of many-body quantum states.

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