q-entropies and the entanglement dynamics of two-qubits interacting with an environment.

A. Hamadou-Ibrahim
Physics Department, University of Pretoria, Pretoria 0002, South Africa

A. R. Plastino
Physics Department, University of Pretoria, Pretoria 0002, South Africa
National University La Plata, UNLP-CREG-CONICET
C. C. 727 - 1900 La Plata - Argentina and
Instituto Carlos I de Física Teórica, Universidad de Granada, Granada, Spain, EU

A. Plastino
Exact Sciences Fac. - CCT-CONICET,
National University La Plata
C. C. 727 - 1900 La Plata - Argentina
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We investigate entropic aspects of the quantum entanglement dynamics of two-qubits systems interacting with an environment. In particular we consider the detection, based on the violation of classical entropic inequalities involving q-entropies, of the phenomenon of entanglement disappearance and subsequent entanglement revival during the alluded two-qubits' evolution.

Keywords: q-Entropies, Entanglement dynamics

1. INTRODUCTION

Entanglement and decoherence are two closely related quantum phenomena that lie at the heart of the present understanding of the fabric of Nature [1–4]. There is nowadays wide consensus that entanglement constitutes the most distinctive feature of the quantum mechanical description of the physical world. The multiple manifestations of quantum entanglement are currently the focus of intense and increasing research efforts. From the point of view of the foundations of physics, entanglement plays an important role, for example, in connection with the origin of the classical macroscopic world from a quantum mechanical background [4], and also in justifying the main tenets of equilibrium statistical mechanics [5]. On the other hand, the creation and manipulation of multi-partite entangled states have remarkable technological applications, such as quantum computation [2, 3] and quantum metrology [6]. The phenomenon of decoherence comprises, basically, a family of effects arising from the interaction (and the accompanying entanglement-development) between quantum systems and their environments [3, 4]. Physical systems in Nature are usually immersed in an environment and inevitably interact with it to some extent. The concomitant entanglement developed between the system and the environment leads to the suppression of typical quantum features of the system, such as the interference between different system's states. This process constitutes the basic ingredient of the "decoherence program" for explaining the quantum-to-classical transition [4].

The (internal) amount of entanglement exhibited by a composite quantum system undergoing decoherence tends to decrease as the decoherence process takes place. This decay of entanglement has recently attracted the interest of many researchers [7, 8] because it constitutes one of the main obstacles that have to be overcome in order to develop quantum technologies based upon the controlled manipulation of entangled states [3]. It has been shown that in some cases entanglement can completely disappear in finite times. This phenomenon is known as *entanglement sudden death* (ESD) [7–10] and has been observed experimentally by Almeida et al. [11]. It is of clear relevance to study and understand ESD and related phenomena occurring during the evolution of open quantum systems, because the actual implementation of quantum computation and other quantum information tasks depend on the longevity of entanglement in multiqubits states.

There have been many developments in recent years concerning the application of q-entropies to the study of several physical systems and processes. The two principal members of the family of q-entropies are the Tsallis entropy $S_q^{(T)}$ [12] and the related Rényi entropy $S_q^{(R)}$ [13]. Research activity on applications of these entropic measures to physics (as contrasted to their restricted, sole application to information theory) started in earnest after Tsallis proposal in 1988 of a thermostatistical formalism based on the $S_q^{(T)}$ entropy [12]. After Tsallis' 1988 pioneering work, the $S_q^{(T)}$ measure has been successfully utilized in connection with several problems both in the classical [14–23] and the quantum regimes [24–28]. Tsallis entropy is nowadays thought to be of relevance for the study (among others) of systems governed by non linear Fokker-Planck equations [15, 16]; systems exhibiting a scale-invariant occupancy of phase space [17, 18]; systems with anomalous thermostatting dynamics [19]; non equilibrium scenarios characterized by temperature fluctuations [20]; systems exhibiting weak chaos [21]; many body systems with interactions of long range relative to the system's size [22]; and biological ecosystems [23]. Last, but certainly not least, several authors have explored the relationships between the q-entropies and the phenomenon of quantum entanglement [25–32]. Of course, the above list is far from complete. For other interesting applications of Tsallis entropy see [14] and references therein.

It is worth stressing that, even if the main motivation for studying the properties of Tsallis entropy has been its application to the development of a generalized thermo-statistics for the study of various kinds of out of equilibrium many body systems, the relevance of Tsallis entropy is not restricted to those applications. Due to the wealth of interesting and usefull results obtained during the last twenty years concerning the properties of the $S_q^{(T)}$ measure, Tsallis entropy can now be regarded as part of the standard tool-kit of scientists and engineers that use probabilistic or information-related concepts in their work. A remarkable illustration of this is given by the numerous applications of Tsallis measure to the field of quantum information (and, in particular to the study of quantum entanglement). In this regard, it is highly significant that some recent monographs on quantum information or quantum entanglement [1, 2] discuss the Tsallis entropic measure, even if these works don't deal with "nonextensive thermostatistics" at all. Concerning the applications of Tsallis entropy to the theory of quantum entanglement, a cautionary comment is in order. One should not be dog matic. The future is always open: even if most current applications of the S_q entropies to the study of quantum entanglement don't have any direct relationship with nonextensive thermostatistics, it might happen that in the future someone finds that there is some deep connection between quantum non-locality and nonextensive themrostatistics.

When analysing the effects originating on the interactions between a quantum mechanical system and its surroundings the system must be regarded as an open quantum system. In order to succeed in the development of useful quantum information processing devices it is crucial to achieve a complete characterization and understanding, from all possible points of view, of the aforementioned effects arising from the interaction with the environment. The aim of the present work is to explore some entropic manifestations, expressed in terms of q-entropies, of the entanglement dynamics of couples of qubits each interacting with a reservoir in a regime where the non-Markovian effects are important. In other words, we are going to consider reservoirs whose correlation times are greater than, or of the same order as, the relaxation time over which the state of the system changes [36]. In particular, we are going to investigate the possibility of detecting the disappearance of entanglement (related to the celebrated phenomenon of "Entanglement sudden death") and its subsequent revival, using the violation of the classical entropic inequalities associated with different q-entropies.

2. QUANTUM ENTANGLEMENT AND Q-ENTROPIES

Given a composite quantum system AB consisting of two subsystems A and B, the entropies associated with the composite system as a whole, $S[AB]$, and the entropies associated with the subsystems, $S[A]$ and $S[B]$, allow for a direct, information-theoretical way of characterizing the entanglement exhibited by certain quantum states. A very intuitive property of classical composite systems is that the global entropy of the complete system is always larger or equal than the individual entropies associated with each of the subsystems. However, this is not always true in the case of composite quantum systems. For instance, when we have such a composite system in a pure quantum state the entropies of its subsystems are in general not zero, in spite of the fact that the entropy of the global system vanishes. Indeed, for pure states of bi-partite quantum systems the entropy of one of the subsystems constitutes a valuable quantitative measure for the amount of entanglement exhibited by the state.

In the case of mixed states of bi-partite quantum systems it is also the case that, unlike what happens with classical systems, the entropy of a sub-system may be greater than the entropy of the global system if the composite system is in an entangled state. However, the situation is for mixed states more complicated than it is for pure states. All non-entangled states comply with the classical entropic inequalities, but some entangled states also verify those inequalities. This means that if, for a given state, the entropy of a subsystem is larger than the entropy of the complete system, then we know for sure that the state under consideration is entangled. However, if the state complies with the entropic classical inequalities, we cannot be sure that the state is separable.

On the basis of the above considerations, it is interesting to consider entropic differences of the form

$$D_q = S_q[A] - S_q[AB] \tag{1}$$

as indicators of entanglement. If D_q is positive, then one can conclude that the state is entangled, but in the case when D_q is negative or zero, one cannot conclude that the state is separable, for there are entangled states for which the value of D_q is negative. Thus, in the case of mixed states, the quantities D_q lead to sufficient (but not necessary) criteria for entanglement.

Quantum entanglement gives rise to various non-classical and counterintuitive properties of entangled states, such as the violation of Bell inequalities [2] or the violation of the aforementioned classical entropic inequalities. However, not all entangled states exhibit all these non-classical properties. Consequently, it is of considerable interest not only to determine the amount of entanglement present in quantum states, but also to characterize which entangled states do have (and which do not) the different non-classical features. The exploration of which states do not comply with the classical entropic inequalities is of special interest for the following reasons:

- As already mentioned, the violations of the classical entropic inequalities constitute the most straightforward entropic or information-theoretical manifestations of entanglement: a part of a physical system having a larger entropy than the whole system.

- In the case $q = 2$ the quantity $S_q[A] + S_q[B] - 2S_q[AB]$ (evaluated using Tsallis q-entropies) is a lower bound for the squared concurrence of the state and is *an experimentally measurable quantity* [33]. For quantum states verifying the equality $S_2[A] = S_2[B]$ the alluded measurable quantity coincides with D_2. The time dependent states that we are going to study in the present contribution comply with the last equality.

- A concrete experimental procedure for detecting violations of the clasical entropic inequality (based on Renyi entropy with $q = 2$) for pairs of polarization-entangled photons has been succesfuly implemented recently [34].

- The separability criteria based on q-entropic inequalities constitute prototipe examples of nonlinear separability criteria, which are nowadays the focus of intense reserach activity. In particular, the entropic criteria are stronger than criteria based on Bell-CHSH inequalities [34, 35].

The second of the above points is particularly important because the amount of entanglement (as measured, for instance, by the squared concurrence) is not a directly measurable quantity. It is therefore important, both from the practical and the theoretical points of view, to investigate in detail the properties of experimentally measurable indicators of entanglement, especially if they also establish lower bounds for the amount of entanglement.

We are going to consider entropic differences like (1) based on q-entropies [13] . We will use the Tsallis' entropies

$$S_q^{(T)} = \frac{1}{q-1}(1 - Tr(\hat{\rho}^q)) \tag{2}$$

and the Rényi entropies

$$S_q^{(R)} = \frac{1}{1-q}\ln(Tr(\hat{\rho}^q)) \tag{3}$$

In the limit $q \rightarrow 1$, both these entropic measures become the Von Neumann entropy given by

$$S_1 = -Tr(\hat{\rho}\ln\hat{\rho}) \tag{4}$$

and in the limit $q \rightarrow \infty$ the Rényi entropies becomes

$$S_\infty^{(R)}(\rho) = -\ln(\lambda_m) \tag{5}$$

where λ_m is the maximum eigenvalue of the density matrix ρ.

The case $q = 2$ is of particular interest because the q-entropies $S_2^{(T)}$ and $S_2^{(R)}$ offer many advantages for both numerical and analytical studies. In part this is due to the fact that to evaluate these entropies it is not necessary to diagonalize the density matrix. The q-entropic measure $S_2^{(T)}[\rho]$ is usually referred to as the linear entropy of the density matrix ρ in the literature, and has proven to be very useful in the field of quantum information theory (see [37, 38] and references therein). Furthermore, as already explained, the entropic difference $D_2^{(T)}$ is closely related to a recently advanced experimentally measurable entanglement indicator [33].

3. SYSTEM STUDIED

In order to study the q-entropic characterizations of the entanglement dynamics of a two-qubit system interacting with an environment we are going to use the paradigmatic model discussed in [9] (this model was previously studied by Garraway [39] who provided its analytical solution). The model is described by a "qubit + reservoir" Hamiltonian of the form

$$H = \omega_0\sigma_+\sigma_- + \sum_k \omega_k b_k^\dagger b_k + (\sigma_+ B + \sigma_- B^\dagger) \tag{6}$$

where $B = \sum_k g_k b_k$, ω_0 stands for the transition frequency of the two-level system (that is, a qubit) and $\sigma_\mp$ denotes the system's raising and lowering operators. The reservoir consists of a set of field modes, $b_k^\dagger$ and b_k being respectively the creation and annihilation operators corresponding to the k-mode. These field modes are characterized by frequencies ω_k and coupling constants g_k with the two-level system. The Hamiltonian (6) may describe, for instance, a qubit consisting of the excited and ground electronic states of a two-level atom that interacts with the quantized electromagnetic modes associated with a high-Q cavity. The effective spectral density of the reservoir is assumed to be of the form

$$J(\omega) = \frac{1}{2\pi}\frac{\gamma_0\lambda^2}{(\omega-\omega_0)^2+\lambda^2}, \tag{7}$$

where γ_0 and λ are positive parameters with dimensions of inverse time (see [9] for details).

The dynamics of the single qubit is then described by the density matrix

$$\rho(t) = \begin{pmatrix} \rho_{11}(0)P_t & \rho_{10}(0)\sqrt{P_t} \\ \rho_{01}(0)\sqrt{P_t} & \rho_{00}(0)+\rho_{11}(0)(1-P_t) \end{pmatrix}, \tag{8}$$

where $\rho_{ij}(0)$ are the initial density matrix elements of the qubit and the function P_t is given by

$$P_t = e^{-\lambda t}\left[\cos\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sin\left(\frac{dt}{2}\right)\right]^2 \tag{9}$$

with

$$d = \sqrt{2\gamma_0\lambda - \lambda^2}. \tag{10}$$

The time evolution of two non-interacting qubits, each of them individually evolving according to (8), is then given by a time dependent density matrix whose elements with respect to the computational basis

$$B = \{|1\rangle \equiv |11\rangle, |2\rangle \equiv |10\rangle, |3\rangle \equiv |01\rangle, |4\rangle \equiv |00\rangle\} \tag{11}$$

are [9]

$$\begin{aligned}
\rho_{11}^T(t) &= \rho_{11}^T(0)P_t^2, \\
\rho_{22}^T(t) &= \rho_{22}^T(0)P_t + \rho_{11}^T(0)P_t(1-P_t), \\
\rho_{33}^T(t) &= \rho_{33}^T(0)P_t + \rho_{11}^T(0)P_t(1-P_t), \\
\rho_{44}^T(t) &= 1 - [\rho_{11}^T + \rho_{22}^T + \rho_{33}^T], \\
\rho_{12}^T(t) &= \rho_{12}^T(0)P_t^{3/2}, \rho_{13}^T(0) = \rho_{13}^T(0)P_t^{3/2}, \\
\rho_{14}^T(t) &= \rho_{14}^T(0)P_t, \rho_{23}^T(0) = \rho_{23}^T(0)P_t, \\
\rho_{24}^T(t) &= \sqrt{P_t}[\rho_{24}^T(0) + \rho_{13}^T(0)P_t(1-P_t)], \\
\rho_{34}^T(t) &= \sqrt{P_t}[\rho_{34}^T(0) + \rho_{12}^T(0)P_t(1-P_t)]
\end{aligned} \tag{12}$$

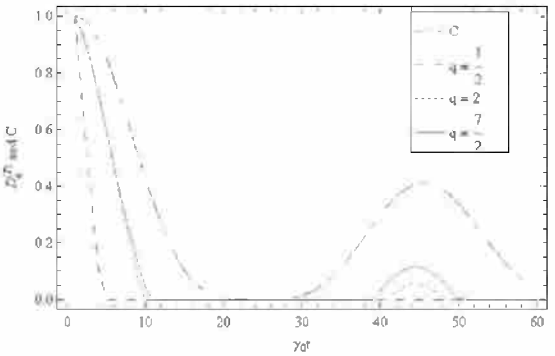

FIG. 1: Plots of the concurrence C and of the quantity $\frac{D_q^{(T)}}{D_q^{(T)}(t=0)}$ against $\gamma_0 t$ for different values of q for the Tsallis' entropy. All depicted quantities are non-dimensional.

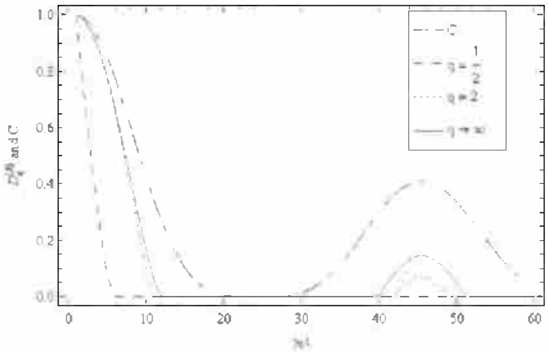

FIG. 2: Plots of the concurrence C and of the quantity $D_q^{(R)}$ against $\gamma_0 t$ for different values of q for the Rényi entropy. All depicted quantities are non-dimensional.

with $\rho_{ij}^T(t) = \rho_{ji}^{T*}(t)$. That is, the matrix $\rho^T(t)$ is Hermitian.

It is also possible to obtain equations similar to (12) describing the evolution of the density matrix elements describing the dynamics of a set of N non-interacting qubits each of them interacting with its "own" reservoir, but in the present work we are going to restrict our considerations to the two-qubit case.

4. ENTANGLEMENT DYNAMICS AND q-ENTROPIES

In the model under consideration it has been observed that for some initial states, entanglement disappears at a finite time and afterwards the entanglement gets revived [9]. Here we want to investigate the q-entropic counterpart of this behaviour. In order to do that we calculate, for the initial Bell state

$$\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big). \tag{13}$$

and the strong non-Markovian regime corresponding to $\lambda = 0.01\gamma_0$, the time evolution of the entropic difference D_q, both for the Tsallis and the Rényi entropies.

The results are shown in Figure 1 and Figure 2, where the time evolutions of the concurrence C and of the D_q quantities are shown for the Tsallis and the Rényi entropies, respectively. In these figures the entropic differences D_q are plotted against the non-dimensional variable $\gamma_0 t$. Since we use the quantities D_q as entanglement indicators (and $D_q > 0$ is a sufficient but not necessary condition for entanglement) in Figures 1 and 2 we set $D_q = 0$ (indicating that no entanglement is detected by this quantity) whenever D_q becomes negative.

It is plain from Figures 1 and 2 that the first entanglement disappearance and its subsequent revival can be observed in the behaviour of D_q. It also transpires from Figure 2 (and it is suggested by Figure 1) that the limit case $q \to \infty$ is the most favourable for these purposes, which is fully consistent with several previous studies by other researchers [29–32]. Indeed, it is observed in Figure 2 that the Rényi based entropic difference $D_\infty^{(R)}$ is the one that detects the presence of entanglement for the largest time periods, both prior to the disappearance of entanglement and also during the first entanglement revival. During the initial decay of entanglement $D_\infty^{(R)}$ detects entanglement (that is, the concomitant classical entropic inequality does not hold) for approximately one half of the time interval where the two-qubits are entangled. During the first entanglement revival, the classical inequality is not verified for approximately one third of the time interval corresponding to non-vanishing entanglement. In both cases the classical entropic inequality starts to be verified when the squared concurrence falls below a value around 0.2. Note that in the figures we plotted C and not C^2, in order to make it easier to see the points where entanglement disappears. However, had we plotted C^2 instead of C, it would be clear that $2D_2^{(T)}$ does not constitute too bad a lower bound for C^2.

An analytical expression for the entropic difference $D_q^{(R)}$ associated with Rényi entropy in the limit $q \to \infty$ can be obtained for the initial Bell state (13), and it is given by

$$D_\infty^{(R)} = \ln\left[\frac{1 - P_t - P_t^2 + \sqrt{1 - 2P_t + P_t^2}}{2 - P_t}\right]. \tag{14}$$

The entropic differences corresponding to $q = 1/2$ are considerably less efficient as entanglement indicators during the aforementioned processes of entanglement decay and revival. On the other hand, it can also be appreciated in Figures 1 and 2 that the cases $q = 2$ and $q = 7/2$, even being not as good as the case $q = \infty$, are almost as efficient as this limit case. This feature of the $q = 2$ entropic differences is of particular interest because, in the present case, $D_2^{(T)}$ coincides with a recently discovered experimentally measurable entanglement indicator [33].

None of the entropic differences D_q based on either the Tsallis or the Rényi entropies are able to detect the second entanglement revival occurring after its second "death". However, these later events are less important than the first entanglement revival, because the actual amount of entanglement exhibited by the two-qubits system during the second entanglement "resurrection" is rather small and, consequently, of limited practical relevance.

5. CONCLUSIONS

We have explored some entropic manifestations of the entanglement dynamics of a two-qubits system interacting with an environment. We have investigated the time behaviour of entropic differences $D_q = S_q[A] - S_q[AB]$ between the q-entropy of one of the alluded qubits and the q-entropy of the two-qubits. The quantity D_q is an entanglement indicator in the sense that $D_q > 0$ is a sufficient criterion for entanglement. We computed the time evolution of D_q for various values of q, both for the Tsallis and for the Rényi entropies. Classical entropic inequalities are violated for (approximately) one half of the time interval corresponding to the initial entanglement decay, and for one third of the duration of the first entanglement revival. In both cases the classical inequalities are verified when the concurrence falls below a value around 0.2. It is interesting to note that this behaviour shows some similarities with the behaviour exhibited for this system by the Bell inequalities. It was found in [10] that the Bell inequalities are satisfied (and, consequently, they don't detect entanglement) during an appreciable part of the first entanglement revival.

We found that the limit case $q \to \infty$ constitutes the most favourable one for detecting the first decay of entanglement and the subsequent entanglement revival, in agreement with results obtained previously by other researchers in different contexts [29–32]. However, the case $q = 2$ proved to be almost as good as the limit case $q \to \infty$ (particularly during the first phase of entanglement decay). This is specially relevant because, in the present case, $D_2^{(T)}$ constitutes an experimentally accessible indicator of entanglement.

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