

**Remnant group of local Lorentz transformations in  $f(T)$  theories**Rafael Ferraro<sup>1,2,\*</sup> and Franco Fiorini<sup>3,†</sup><sup>1</sup>*Instituto de Astronomía y Física del Espacio (IAFE, CONICET-UBA), Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina*<sup>2</sup>*Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina*<sup>3</sup>*Instituto Balseiro, Universidad Nacional de Cuyo, R8402AGP Bariloche, Argentina*

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It is shown that the extended teleparallel gravitational theories, known as  $f(T)$  theories, inherit some *on shell* local Lorentz invariance associated with the tetrad field defining the spacetime structure. We discuss some enlightening examples, such as Minkowski spacetime and cosmological (Friedmann-Robertson-Walker and Bianchi type I) manifolds. In the first case, we show that the absence of gravity reveals itself as an incapability in the selection of a preferred parallelization at a local level, due to the fact that the infinitesimal local Lorentz subgroup acts as a symmetry group of the frame characterizing Minkowski spacetime. Finite transformations are also discussed in these examples and, contrary to the common lore on the subject, we conclude that the set of tetrads responsible for the parallelization of these manifolds is quite vast and that the remnant group of local Lorentz transformations includes one- and two-dimensional Abelian subgroups of the Lorentz group.

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**I. INTRODUCTION**

In spaces with absolute parallelism, the geometry of a given spacetime is encoded in the tetrad field  $\mathbf{e}^a$ . This global basis of the tangent bundle constitutes a preferred reference frame which defines the spacetime structure  $(\mathcal{T}(\mathcal{M}), \mathbf{e}^a)$ . In general, any tetrad  $\mathbf{e}^a$  which serves as a global frame leads to a certain Lorentzian geometry, characterized by  $\mathbf{g} = \eta_{ab} \mathbf{e}^a \otimes \mathbf{e}^b$ . The common belief concerning the geometrical structure of gravitational theories in such spaces is based on the notion of absolute parallelism characterizing Weitzenböck spacetime. According to this description, a preferred reference frame emerges as the agent which defines the spacetime structure by means of a parallelization process. In principle, the notion of parallelism so obtained should be defined only with arbitrariness of making global Lorentz transformations of the preferred frame, for this special tetrad field dictates what an autoparallel is: a curve will be autoparallel if its tangent vector has constant components with respect to the global preferred frame. However, from a purely mathematical point of view, it has been known for a long time that if a given space is parallelizable, the vector fields carrying out such a parallelization are not unique [1]. On physical grounds, and for some gravitational theories constructed out of the concept of absolute parallelism [like  $f(T)$

gravity, the one concerned in the present work], this means that, apart from the freedom to perform *global* Lorentz transformations to a given global frame, certain *local* boosts and rotations will act as a symmetry group of the theory.

Among the gravitational theories relaying on absolute parallelism, the so-called  $f(T)$  gravity [2–5] has been an object of considerable study in the last few years (see, for instance, [6–9] and references contained therein). Since the very beginning, it was realized that the local Lorentz symmetry is not present in these theories [2,10–12], and as a consequence of this, preferred reference frames emerge as the agent encoding the gravitational field [13,14]. It is our concern now to show that, besides the global symmetry always present in any theory constructed upon the notion of absolute parallelism, these preferred frames are defined with the arbitrariness of making certain *local* Lorentz transformations. The admissible group of local Lorentz transformations depends on the particular spacetime under consideration: for a given frame  $\mathbf{e}^a$  representing a solution of the  $f(T)$  motion equations, there exist a subgroup  $\mathcal{A}(\mathbf{e}^a)$  of the Lorentz group which officiates as a symmetry group. The presence of a restricted local invariance of this sort have been occasionally documented in the literature; see, for instance, Ref. [15] regarding the theory exposed in [16].

The study of the group  $\mathcal{A}(\mathbf{e}^a)$  is mandatory for at least two important reasons. On one hand, the knowledge of  $\mathcal{A}(\mathbf{e}^a)$  allows us to obtain new solutions of the motion equations from the old. This is particularly important if they involve the matching of different tetrads, as it happens in stellar and wormhole models, where we have two different spacetimes which must be smoothly matched on a certain

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hypersurface. On the other hand, in order to perform a correct counting of degrees of freedom, detailed information about the symmetries of the theory under consideration becomes fundamental. These two constitute the main motivations of the present work, and it is expected that the techniques involved in this article might serve for answering similar questions in other theories relying on absolute parallelism, for instance, in Born-Infeld gravity [17,18], and in the extensions of Gauss-Bonnet gravity in the teleparallel context [19,20].

In order to understand the nature of  $\mathcal{A}(\mathbf{e}^a)$  in the context of  $f(T)$  gravity, we first set down the preliminary geometrical concepts in Sec. II. After these ingredients are presented there, we expose general relativity (GR) and its teleparallel equivalent (TEGR) in Sec. III. The behavior of  $f(T)$  theories under local Lorentz transformation is then thoroughly discussed in Sec. IV, followed by a number of important examples which crystallize the concepts of the former sections. These examples are the central point of Sec. V. Finally, we establish our conclusions in Sec. VI.

## II. GEOMETRICAL SETTING

The theories where gravity is regarded as the geometry of the spacetime rest on two basic concepts of differential geometry: *torsion*  $\mathbf{T}^i$  and *curvature*  $\mathbf{R}^i{}_j$ ,

$$\mathbf{T}^i \doteq D\mathbf{E}^i \doteq d\mathbf{E}^i + \omega^i{}_j \wedge \mathbf{E}^j. \quad (1)$$

$$\mathbf{R}^i{}_j \doteq d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \quad (2)$$

Torsion  $\mathbf{T}^i$  is the covariant derivative of the 1-forms constituting a local basis  $\{\mathbf{E}^i\}$  of the cotangent space. The covariant derivative  $D$  is defined by endowing the manifold with a *spin connection*, which is a set of 1-forms  $\omega^i{}_j$  taking care of additional tensor characteristics of the object under differentiation.  $D$  is an exterior derivative on  $p$ -forms preserving their tensor-valued features. For instance,  $\mathbf{T}^i$  is a vector-valued 2-form; it transforms as  $\mathbf{T}^{\prime i} = \Lambda^{\prime i}{}_i \mathbf{T}^i$  under the change of basis  $\mathbf{E}^{\prime i} = \Lambda^{\prime i}{}_i \mathbf{E}^i$ . This is so because the spin connection transforms as

$$\omega^{\prime i}{}_j = \Lambda^{\prime i}{}_i \omega^i{}_j \Lambda^{\prime j}{}_j + \Lambda^{\prime i}{}_k d\Lambda^k{}_j \quad (3)$$

(matrices  $\Lambda^{\prime i}{}_i, \Lambda^{\prime i}{}_i$  are inverses of each other; the dual basis in the tangent space transforms as  $\mathbf{E}_{\prime i} = \Lambda^i{}_{\prime i} \mathbf{E}_i$ ). Analogously,  $\mathbf{R}^i{}_j$  is tensorial in the indices  $i, j$ . However,  $\mathbf{R}^i{}_j$  cannot be thought of as the covariant derivative of  $\omega^i{}_j$  because the connection is not a tensor.  $\mathbf{R}^i{}_j$  can be covariantly differentiated to obtain the (second) Bianchi identity,

$$D\mathbf{R}^i{}_j = d\mathbf{R}^i{}_j + \omega^i{}_k \wedge \mathbf{R}^k{}_j - \omega^k{}_j \wedge \mathbf{R}^i{}_k \equiv 0. \quad (4)$$

Besides, by differentiating the torsion we obtain the first Bianchi identity:

$$D\mathbf{T}^i - \mathbf{R}^i{}_j \wedge \mathbf{E}^j \equiv 0. \quad (5)$$

In gravitational theories of geometrical character, we choose an orthonormal basis or tetrad  $\{\mathbf{e}^a = e^a{}_\mu dx^\mu\}$  and the spin connection  $\{\omega^a{}_b\}$  to play the role of potentials for describing the gravitational fields (torsion and curvature). The assumed orthonormality of the tetrad establishes the link tetrad metric:

$$\eta^{ab} = g^{\mu\nu} e^a{}_\mu e^b{}_\nu, \quad \mathbf{g} = \eta_{ab} \mathbf{e}^a \otimes \mathbf{e}^b. \quad (6)$$

This link is invariant under local Lorentz transformations  $\mathbf{e}^{a'} = \Lambda^{a'}{}_b(x) \mathbf{e}^b$  (i.e., those linear transformations preserving orthonormality). On the other hand, the spin connection is assumed to be *metric*, which means the vanishing of the covariant derivative of the *Lorentz* tensor-valued 0-form  $\eta_{ab}$ :

$$0 = D\eta_{ab} = d\eta_{ab} - \omega^c{}_a \eta_{cb} - \omega^c{}_b \eta_{ac}, \quad (7)$$

i.e.,

$$\omega_{ba} = -\omega_{ab} \quad (8)$$

(Lorentz tensor indexes  $a, b, \dots$  are lowered with  $\eta_{ab}$ ). This property also implies

$$D\epsilon_{abcd} = 0, \quad (9)$$

where  $\epsilon_{abcd}$  is the Levi-Civita symbol, which is a tensor under Lorentz transformations.

General relativity is a theory of gravity where the connection is metric and torsionless; it is the Levi-Civita connection  $\overset{L}{\omega}{}^i{}_j$ :

$$d\mathbf{E}^i + \overset{L}{\omega}{}^i{}_j \wedge \mathbf{E}^j = 0, \quad \overset{L}{\omega}{}_{ba} = -\overset{L}{\omega}{}_{ab}. \quad (10)$$

These relations can be solved for the Levi-Civita connection in terms of the exterior derivative of the tetrad:

$$(\overset{L}{\omega}{}_{ab})_c = \frac{1}{2} [(d\mathbf{e}_a)_{bc} + (d\mathbf{e}_b)_{ca} - (d\mathbf{e}_c)_{ab}]. \quad (11)$$

For connections differing from the Levi-Civita connection it is convenient to introduce the contorsion as the set of 1-forms expressing such a difference:

$$\mathbf{K}^i{}_j \doteq \omega^i{}_j - \overset{L}{\omega}{}^i{}_j. \quad (12)$$

Although connections are not tensors, the nontensorial term in the transformation (3) is equal for any connection. Therefore, the difference between connections is a tensor. Some useful properties of the contorsion tensor can be consulted in Appendix A.

### III. EINSTEIN-HILBERT AND TEGR LAGRANGIANS

In this section we will suppress the symbol of wedge product, since no confusion exists provided that the order between  $p$ -forms is preserved.

The Einstein-Hilbert Lagrangian is the Lorentz scalar-valued 4-form defined as

$$L_{\text{EH}} = \frac{1}{4\kappa} \epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \overset{L}{\mathbf{R}}{}^{cd}, \quad (13)$$

where  $\kappa = 8\pi G$ . Property 4 of Appendix A implies that

$$\begin{aligned} L_{\text{EH}} &= \frac{1}{4\kappa} \epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b (\mathbf{R}{}^{cd} - \overset{L}{D}\mathbf{K}{}^{cd} - \mathbf{K}{}^c{}_e \mathbf{K}{}^{ed}) \\ &= \frac{1}{4\kappa} [\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b (\mathbf{R}{}^{cd} - \mathbf{K}{}^c{}_e \mathbf{K}{}^{ed}) - \overset{L}{D}(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^{cd})], \end{aligned} \quad (14)$$

where we have used that the Levi-Civita connection is metric ( $\overset{L}{D}\epsilon_{abcd} = 0$ ) and torsionless ( $\overset{L}{D}\mathbf{e}^a = 0$ ). Moreover,  $\overset{L}{D}(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^{cd}) = d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^{cd})$ , because  $\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^{cd}$  is a Lorentz scalar. So the last term in (14) is a boundary term that can be suppressed:

$$L = \frac{1}{4\kappa} \epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b (\mathbf{R}{}^{cd} - \mathbf{K}{}^c{}_e \mathbf{K}{}^{ed}). \quad (15)$$

The Lagrangian (15) now contains an arbitrary connection  $\omega^{cd}$ ; however, it does not provide any dynamics for  $\omega^{cd}$ . In fact, the Lagrangian (15) is the Einstein-Hilbert Lagrangian (13) modulo a boundary term. Since  $\omega^{cd}$  is not contained in the Einstein-Hilbert Lagrangian, we conclude that the variation of (15) with respect to  $\omega^{cd}$  will produce a boundary term to compensate for the variation of the suppressed boundary term appearing in (14):

$$\delta_\omega L = \frac{1}{4\kappa} d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \delta\omega^{cd}). \quad (16)$$

So  $\omega^{cd}$  enters the Lagrangian (15) as a *dummy* variable to be chosen in an arbitrary way. TEGR chooses  $\omega^{cd}$  to be zero, which is the Weitzenböck connection (for the form Weitzenböck connection acquires in a coordinate basis, see Appendix B). So,  $\mathbf{R}{}^{cd}$  vanishes and  $\mathbf{K}{}^c{}_e = -\overset{L}{\omega}{}^c{}_e[\mathbf{e}]$  becomes linear and homogeneous in derivatives of the tetrad. Then

$$L_{\text{TEGR}} = -\frac{1}{4\kappa} \epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^c{}_e[\mathbf{e}] K{}^{ed}[\mathbf{e}]. \quad (17)$$

Thus, the freezing of  $\omega^{cd}$  throws the Lagrangian into a form quadratic in first derivatives of the tetrad [see (11)].

However, we cannot freeze a connection without paying a price. Although  $\omega^{cd}$  is a dummy dynamical variable in (15), it plays the important role of making (15) a Lorentz scalar-valued volume [i.e., (15) is invariant under local Lorentz transformations of the tetrad]. This is because  $\mathbf{K}{}^c{}_e$  is a Lorentz tensor as long as it is a difference between connections. By eliminating  $\omega^{cd}$  from the Lagrangian, we are depriving  $\mathbf{K}{}^c{}_e$  of its tensorial character;  $\mathbf{K}{}^c{}_e$  becomes a connection,  $\mathbf{K}{}^c{}_e[\mathbf{e}] = -\overset{L}{\omega}{}^c{}_e[\mathbf{e}]$ , which only keeps a tensorial behavior under *global* Lorentz transformations of the tetrad [ $d\Lambda^k{}_j = 0$  in (3)]. Actually this is not a serious problem in (17) because a local Lorentz transformation of the tetrad just generates a boundary term, as could be imagined. In fact, let us perform a local Lorentz transformation on both sides of Eq. (14) for  $\omega^{cd} = 0$ ; since  $L_{\text{EH}}$  is not sensitive to a local Lorentz transformation, then one obtains

$$\delta_\Lambda L_{\text{TEGR}} = \frac{1}{4\kappa} \delta_\Lambda d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^{cd}[\mathbf{e}]), \quad (18)$$

i.e.,

$$\delta_\Lambda L_{\text{TEGR}} = \frac{1}{4\kappa} d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \eta^{de} \Lambda^c{}_e d\Lambda^{e'}{}_e). \quad (19)$$

Therefore TEGR dynamics does not care about the local orientation of the tetrad, meaning that TEGR, just like GR, is only involved with the dynamics of the locally invariant metric tensor (6). Moreover, a boundary term could be added to the action for balancing the behavior of  $L_{\text{TEGR}}$  in (18). In fact, we can build the strictly local Lorentz invariant action

$$\begin{aligned} S_{\text{TEGR}}[\mathbf{e}] &= -\frac{1}{4\kappa} \int_U \epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^c{}_e[\mathbf{e}] K{}^{ed}[\mathbf{e}] \\ &\quad - \frac{1}{4\kappa} \int_{\partial U} \epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}{}^{cd}[\mathbf{e}], \end{aligned} \quad (20)$$

where  $\mathbf{K}{}^c{}_e[\mathbf{e}] = -\overset{L}{\omega}{}^c{}_e[\mathbf{e}]$ . The Lagrangian (17) is usually written as

$$L_{\text{TEGR}} = (2\kappa)^{-1} T \Omega, \quad (21)$$

where  $\Omega$  is the metric volume  $\mathbf{e}^0 \mathbf{e}^1 \mathbf{e}^2 \mathbf{e}^3 = \det[e^a_\mu] dx^0 dx^1 dx^2 dx^3$ , and

$$T = K^c{}_{ec} K^{ed}{}_d - K^c{}_{ed} K^{ed}{}_c \quad (22)$$

is the so-called Weitzenböck scalar. In principle,  $T$  remains invariant only under *global* Lorentz transformations of the tetrad, since  $\mathbf{K}{}^c{}_e$  has been deprived of its tensor character. Expression (19) was obtained also in [21] by independent

means in the context of metric affine gravity. For more details about TEGR written in the usual index notation, see Appendix B.

#### IV. LORENTZ INVARIANCE OF $f(T)$ THEORIES

An  $f(T)$  theory consists in a deformation of the TEGR Lagrangian, as much as an  $f(R)$  theory is a deformation of the Einstein-Hilbert Lagrangian. The teleparallel Lagrangian density  $\mathcal{L}_{\text{TEGR}} = (2\kappa)^{-1}eT$  is deformed to  $\mathcal{L} = (2\kappa)^{-1}ef(T)$ . The dynamical equations for  $f(T)$  theories are

$$4e^{-1}\partial_\mu[ef'(T)S_a^{\mu\nu}] + 4f'(T)e_a^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\mu\nu} - f(T)e_a^\nu = -2\kappa e_a^\lambda T^\nu{}_\lambda, \quad (23)$$

where  $T^\nu{}_\lambda$  is the energy-momentum tensor (matter is assumed to couple the metric as usual), and  $S_a^{\mu\nu}$  is a quantity linear in the torsion that is defined in the Appendix. The great advantage of field equations (23) with respect to the ones coming from  $f(R)$  gravity is that they are of second order in derivatives of the dynamical field  $\mathbf{e}^a$ .

As an essential feature of  $f(T)$  theories, the variation (19)—which is essentially the variation of  $T$ , since the volume does not vary under Lorentz transformations—is trapped in the argument of function  $f$  instead of being a boundary term to rule out. This feature means that the action is sensitive to local Lorentz transformations, which implies that  $f(T)$  theories contain dynamics not only for the metric, but also for some other degrees of freedom related to the orientation of the tetrad.<sup>1</sup> Actually, because of Eq. (19),  $f(T)$  theories are invariant only under Lorentz transformations of the tetrad accomplishing

$$d(\epsilon_{abcd}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \eta^{de}\Lambda^c{}_{f'}d\Lambda^{f'}{}_e) = 0. \quad (24)$$

Of course, *global* Lorentz transformations ( $d\Lambda^{f'}{}_e = 0$ ) do fulfill Eq. (24). We wonder whether Eq. (24) has some room for a subset of *local* Lorentz transformations. This issue is essential for understanding the nature of the new degrees of freedom added in an  $f(T)$  theory [23].

We shall denote  $\mathcal{A}(\mathbf{e}^a)$  as the set of those local Lorentz transformations which fulfill Eq. (24) for a given frame  $\mathbf{e}^a$ , i.e., for a given solution of the field equations (23).  $\mathcal{A}(\mathbf{e}^a)$  is thus the set of local Lorentz transformations admissible by a certain spacetime  $\mathbf{e}^a$ , so it is defined on shell. By virtue of the nonlinear character of (24), it is clear that the set  $\mathcal{A}(\mathbf{e}^a)$  does not form a Lie group in general; in fact, if  $\Lambda$  and  $\Lambda'$  belong to  $\mathcal{A}(\mathbf{e}^a)$ , then the product  $\Lambda\Lambda'$  does not necessarily belong to  $\mathcal{A}(\mathbf{e}^a)$ . Nevertheless, if we have an element of  $\mathcal{A}(\mathbf{e}^a)$  then the inverse transformation is also in  $\mathcal{A}(\mathbf{e}^a)$ ;

<sup>1</sup>The presence of new degrees of freedom is also a feature characteristic of  $f(R)$  theories (see, for instance, Ref. [22]).

actually, since  $\Lambda^c{}_{f'}\Lambda^{f'}{}_e = \delta^c_e$  we can then replace  $\Lambda^c{}_{f'}d\Lambda^{f'}{}_e$  for  $-\Lambda^{f'}{}_e d\Lambda^c{}_{f'}$  in Eq. (24).

Let us investigate now under what circumstances the set  $\mathcal{A}(\mathbf{e}^a)$  becomes a Lie group. In order to do so, we shall write Lorentz transformations as

$$\Lambda^a{}_{b'} = \exp\left[\frac{1}{2}\sigma^{gh}(x)(M_{gh})^a{}_{b'}\right], \quad (25)$$

where  $\sigma^{cd}(x)$  are the parameters of the transformation, and  $M_{cd}$  are six matrices labeled by antisymmetric indices that generate the Lorentz group. The  $M_{cd}$ 's satisfy the algebra

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}. \quad (26)$$

The components of matrices  $M_{cd}$  are

$$(M_{cd})^a{}_{b'} = \delta^a_c \eta_{db'} - \delta^a_d \eta_{cb'}. \quad (27)$$

In terms of the boost generators  $K_\alpha = M_{0\alpha}$  and rotation generators  $J_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma}M^{\beta\gamma}$ , the algebra (26) is

$$\begin{aligned} [J_\alpha, J_\beta] &= \epsilon_{\alpha\beta\gamma}J^\gamma \\ [K_\alpha, K_\beta] &= -\epsilon_{\alpha\beta\gamma}J^\gamma \\ [K_\alpha, J_\beta] &= \epsilon_{\alpha\beta\gamma}K^\gamma. \end{aligned} \quad (28)$$

For infinitesimal Lorentz transformations, the expression (25) takes the form

$$\Lambda^a{}_{b'} = \delta^a_{b'} + \frac{1}{2}\sigma^{gh}(x)(M_{gh})^a{}_{b'} + O(\sigma^2). \quad (29)$$

In this case we obtain

$$\begin{aligned} \Lambda^c{}_{f'}d\Lambda^{f'}{}_e &\simeq -\frac{1}{2}d\sigma^{gh}(M_{gh})^c{}_e \\ &= -\frac{1}{2}d\sigma^{gh}(\delta^c_g\eta_{he} - \delta^c_h\eta_{ge}) = \eta_{ge}d\sigma^{gc}, \end{aligned} \quad (30)$$

where we have used  $\sigma^{gh} = -\sigma^{hg}$ . Therefore, Eq. (24) becomes

$$d(\epsilon_{abcd}\mathbf{e}^a \wedge \mathbf{e}^b \wedge d\sigma^{cd}) = 0, \quad (31)$$

or, equivalently,

$$\epsilon_{abcd}d(\mathbf{e}^a \wedge \mathbf{e}^b) \wedge d\sigma^{cd} = 0. \quad (32)$$

As expected, expression (32) is linear in  $\sigma^{cd}$  which means that the composition of two local infinitesimal transformations belonging to  $\mathcal{A}(\mathbf{e}^a)$  satisfies Eq. (32) at the lowest order in the differential of their parameters.

We found it very convenient to classify the solutions of the motion equations (23) according to the number of closed 2-forms they involve. In this manner, a given solution  $\mathbf{e}^a$  of Eq. (23) will be called an *n-closed-area frame* (*n*-CAF), if it satisfies  $d(\mathbf{e}^a \wedge \mathbf{e}^b) = 0$  for *n* of the six different pairs (*a*-*b*) ( $0 \leq n \leq 6$ ). Clearly, from Eq. (32), we have that if  $\mathbf{e}^a$  is a 6-CAF, then all the infinitesimal parameters  $\sigma^{cd}$  remain free. This important result just states that for a 6-CAF, we have  $SO(3,1)_{\text{inf}} \subset \mathcal{A}(\mathbf{e}^a)$ , where  $SO(3,1)_{\text{inf}}$  stands for the infinitesimal Lorentz subgroup.

Regarding *finite* transformations, from Eq. (24) it can be proved that if two commuting local Lorentz transformations belong to  $\mathcal{A}(\mathbf{e}^a)$  then their composition is also an element of  $\mathcal{A}(\mathbf{e}^a)$ . Therefore, from the set  $\mathcal{A}(\mathbf{e}^a)$  of those local Lorentz transformations solving Eq. (24), we can extract Abelian subgroups of the Lorentz group. Notoriously, the result (30), which says that  $\Lambda^c_{f'} d\Lambda^f_e$  is exact at the infinitesimal level, is also valid for separate finite boosts and rotations. As a matter of fact, finite boosts in a given direction and rotations in a given plane are one-parameter Lorentz transformations of the form  $\Lambda = \exp[\sigma M]$ , where *M* is  $K_\alpha$  or  $J_\alpha$  depending on the case; therefore it is  $\Lambda^{-1} d\Lambda = M d\sigma$ . For instance, we have

$$\Lambda_{K_3}^{-1} d\Lambda_{K_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} d\sigma, \quad (33)$$

$$\Lambda_{J_3}^{-1} d\Lambda_{J_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} d\sigma. \quad (34)$$

Thus, Eq. (32), which was obtained in the context of infinitesimal transformations, remains valid also for separated *finite* boosts and rotations. In particular, if  $\mathbf{e}^a$  is a 6-CAF, then Eq. (24) will be satisfied for any local boost or rotation. This remark seems to indicate that the finite local transformations will be organized in 6 Abelian subgroups of dimension 1 (each corresponding to a boost in a given direction or a rotation in a given plane). However we will show below that for a given *n*-CAF, a number  $\lfloor \frac{n}{2} \rfloor$  of two-dimensional Abelian subgroups of the type  $\{K_\alpha, J_\alpha\}$  can also appear (here  $\lfloor \rfloor$  refers to the floor function). For  $n \geq 4$  their appearance will actually be unavoidable.

In order to proceed constructively, let us begin by considering a 1-CAF such that, let us say,  $d(\mathbf{e}^0 \wedge \mathbf{e}^3) = 0$ . This property implies that the local parameter  $\sigma^{12}$  can be freely chosen without affecting the fulfillment of Eq. (32). As said, this result is also valid for *finite* local rotations generated by  $M_{12} = -J_3$ . In fact, Eq. (34) shows that the exact matrix-valued 1-form  $\Lambda_{(J_3)}^{-1} d\Lambda_{(J_3)}$  only contributes to Eq. (24) through the components (1-2);

however, such a contribution is canceled whenever  $d(\mathbf{e}^0 \wedge \mathbf{e}^3)$  vanishes. We then get a one-dimensional subgroup of finite local transformations (the subgroup of rotations about  $x^3$ ). This reasoning is applicable to any of the other possible closed areas as well.

In general, for an *n*-CAF one could expect *n* one-dimensional subgroups of finite local transformations. However, if  $n \geq 2$  there is a more interesting case. Let us consider the case  $d(\mathbf{e}^0 \wedge \mathbf{e}^3) = 0 = d(\mathbf{e}^1 \wedge \mathbf{e}^2)$ . Then Eq. (32) is accomplished by local transformations generated by combinations of  $M_{12}$  and  $M_{03}$  (i.e.,  $J_3$  and  $K_3$ ). Since these *commuting* generators preserve the closedness of *both* areas, we can expect that the result remains valid for finite local transformations generated by  $M_{12}$  and  $M_{03}$ . In fact, if  $\Lambda$  is

$$\Lambda = \begin{pmatrix} \cosh \sigma & 0 & 0 & \sinh \sigma \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ \sinh \sigma & 0 & 0 & \cosh \sigma \end{pmatrix}, \quad (35)$$

then it will be

$$\Lambda^{-1} d\Lambda = \begin{pmatrix} 0 & 0 & 0 & d\sigma \\ 0 & 0 & -d\alpha & 0 \\ 0 & d\alpha & 0 & 0 \\ d\sigma & 0 & 0 & 0 \end{pmatrix}. \quad (36)$$

So two independent local parameters  $\sigma(x^\mu)$  and  $\alpha(x^\mu)$  can be chosen without affecting the fulfillment of Eq. (24), because they contribute just to terms that are canceled by the vanishing of  $d(\mathbf{e}^0 \wedge \mathbf{e}^3)$  and  $d(\mathbf{e}^1 \wedge \mathbf{e}^2)$ . So we get a two-dimensional Abelian subgroup (we have  $\lfloor \frac{n}{2} \rfloor = 1$  in this case). Schematically, we have then

$$\begin{aligned} d(\mathbf{e}^0 \wedge \mathbf{e}^1) = 0 = d(\mathbf{e}^2 \wedge \mathbf{e}^3) &\rightarrow \{K_1, J_1\} \\ d(\mathbf{e}^0 \wedge \mathbf{e}^2) = 0 = d(\mathbf{e}^1 \wedge \mathbf{e}^3) &\rightarrow \{K_2, J_2\} \\ d(\mathbf{e}^0 \wedge \mathbf{e}^3) = 0 = d(\mathbf{e}^1 \wedge \mathbf{e}^2) &\rightarrow \{K_3, J_3\}. \end{aligned} \quad (37)$$

However, there exist other types of 2-CAFs, for instance, the one having  $d(\mathbf{e}^0 \wedge \mathbf{e}^1) = 0 = d(\mathbf{e}^0 \wedge \mathbf{e}^2)$ . This 2-CAF will lead to free  $M_{01} = K_1$  and  $M_{02} = K_2$ , but these do not commute. So, the appearance or not of a two-dimensional Abelian subgroup of the Lorentz group in a 2-CAF depends on the closed areas it involves. It can be checked that this is also true for a 3-CAF. In this case, if the 3-CAF involves the proper closed areas, we also expect just one two-dimensional Abelian subgroup ( $\lfloor \frac{3}{2} \rfloor = 1$ ).

If  $n \geq 4$  the emergence of two-dimensional Abelian subgroups is unavoidable. In the case  $n = 4, 5$  we shall obtain two of them, and for  $n = 6$  we will obtain the maximum number of such groups, i.e., three; these will be just  $\{K_1, J_1\}$ ,  $\{K_2, J_2\}$ , and  $\{K_3, J_3\}$ . It should be noticed

that these subgroups cannot be combined in a larger group: only one of them can be locally applied to the solution  $\mathbf{e}^a$  while the rest of the symmetries remain global. This is so because the local action of one of them will affect the closedness of the rest of the closed areas.

For  $n \geq 4$ , let us consider a case including the subgroups  $\{K_1, J_1\}, \{K_2, J_2\}$  [i.e.,  $d(\mathbf{e}^0 \wedge \mathbf{e}^1) = 0 = d(\mathbf{e}^2 \wedge \mathbf{e}^3)$  and  $d(\mathbf{e}^0 \wedge \mathbf{e}^2) = 0 = d(\mathbf{e}^1 \wedge \mathbf{e}^3)$ ]. In such a case there is another way of organizing the subgroups. In fact, we can introduce the Abelian subgroups  $\{A^{(1)}, A^{(2)}\}, \{B^{(1)}, B^{(2)}\}$ , where

$$\begin{aligned} A^{(1)} &\doteq K_1 + J_2, & A^{(2)} &\doteq K_2 - J_1, \\ B^{(1)} &\doteq K_1 - J_2, & B^{(2)} &\doteq K_2 + J_1. \end{aligned} \quad (38)$$

The Lorentz transformation generated by  $A^{(1)}$  is  $\Lambda_{A^{(1)}} = \exp[\sigma A^{(1)}]$ . So the matrix  $\Lambda^c{}_f d\Lambda^f{}_e$  in Eq. (24) is

$$\Lambda_{A^{(1)}}^{-1} d\Lambda_{A^{(1)}} = A^{(1)} d\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} d\sigma. \quad (39)$$

The rest of the cases are obtained by using the matrices

$$\begin{aligned} A^{(2)} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\ B^{(1)} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ B^{(2)} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (40)$$

As can be seen, the contributions of any of these local transformations to Eq. (24) will be canceled by the closedness of the areas (0–1), (2–3), (0–2), and (1–3). It is worth noticing that  $\{A^{(1)}, A^{(2)}\}$  ( $\{B^{(1)}, B^{(2)}\}$ ) constitute the Abelian sector of the little group for massless particles traveling towards decreasing (increasing) values of  $x^3$  [24].<sup>2</sup>

<sup>2</sup>The little group also includes the rotations generated by  $J_3$ . Thus, its algebra gets the form of the algebra of translations and rotations in the Euclidean plane. It has been proved that  $A^{(1,2)}, B^{(1,2)}$  generate gauge transformations of the electromagnetic field [25,26].

We conclude this section with two remarks. Of course, the classification of tetrads through the number of closed areas they contain is not invariant under global Lorentz transformations. Actually we can use the always admissible global Lorentz transformations to maximize the number  $n$  for the tetrad under consideration. Besides, the scheme of  $n$ -CAFs does not exhaust the chances of obtaining a local invariance for a given solution of the equations of motion (23). For instance, even if  $\mathbf{e}^0 \wedge \mathbf{e}^1$  were not closed,  $\sigma^{23}$  could admit a limited dependence on the coordinates without destroying the validity of Eq. (32). This means, on one hand, that even for a 0-CAF the possibility of a restricted local invariance is still present, and on the other, that the remnant group for a given  $n$ -CAF can be larger than that considered in the paragraphs above. This restricted local invariance depends on the form of each solution and it should be considered in each particular case, as we will show in the next section.

## V. EXAMPLES

In this section we will offer a number of simple but quite important examples that will help to visualize the ideas displayed in the preceding paragraphs.

### A. Minkowski spacetime

Perhaps one of the most important cases to be analyzed should be Minkowski spacetime, because it approximately represents the geometrical arena where our daily experience takes place. For this reason it is our concern now to figure out what kind of local Lorentz transformations we are free to perform in the Euclidean frame (see below), in order to be unable to distinguishing them from the outcomes of experiments performed in our local lab.

The Euclidean frame  $\mathbf{e}^a = \delta^a{}_b dx^b$  is a global smooth basis for Minkowski spacetime (the  $x^b$ 's refer to  $x^{0,\alpha}$ , where  $x^\alpha$  are Cartesian coordinates). Since  $T^a = d\mathbf{e}^a = 0$ , the Weitzenböck scalar is identically null, and the Euclidean frame is a vacuum solution of Eq. (23) for any  $f(T)$  function smooth at  $T = 0$  [14].<sup>3</sup>

The Euclidean frame is perhaps the best example of a 6-CAF. Therefore,  $f(T)$  theories that are smooth at  $T = 0$  do not distinguish among locally related orthonormal frames in Minkowski spacetime. In other words, the absence of gravity in  $f(T)$  theories is revealed as an incapability in the selection of a preferred parallelization at a local level.

<sup>3</sup>Other  $f(T)$  deformations of GR, such as the ones used for describing the late time cosmic speed-up [for instance,  $f(T) = T + \alpha/T$ ], do not have Minkowski spacetime as a vacuum solution. Instead, they lead to a constant but non-null  $T$ , and so, to a de Sitter or anti-de Sitter spacetime.

## B. Cosmological spacetimes

### 1. Spatially flat FRW spacetimes

The diagonal frame  $\mathbf{e}^0 = dt$ ,  $\mathbf{e}^\alpha = a(t)\delta_i^\alpha dx^i$  is a solution to Eq. (23) for flat Friedmann-Robertson-Walker (FRW) spacetimes [13]. This frame is a 3-CAF since  $d(\mathbf{e}^0 \wedge \mathbf{e}^\alpha) = 0$ ,  $\forall \alpha$ . Because of the comments made in the last section, we expect  $\mathcal{A}(\mathbf{e}^a)$  to include at least three one-dimensional Abelian subgroups of the Lorentz group. Actually, Eq. (24) is accomplished for any local rotation  $\sigma^{\beta\gamma}(x^\alpha)$  of the diagonal frame  $\mathbf{e}^a$ , because for every pair  $(0\alpha)$ , we have a pair  $(\beta\gamma)$  (since these last two are different from  $\alpha$ ), and there are three such pairs. Then,  $\mathcal{A}(\mathbf{e}^a)$  includes the three Abelian subgroups of rotation about a given axis.

A nice example of the behavior discussed at the end of the last section (i.e., the presence of an admissible transformation even though  $\mathbf{e}^\alpha \wedge \mathbf{e}^\beta$  is not closed) can be shown as follows. Since we have

$$d(\mathbf{e}^\alpha \wedge \mathbf{e}^\beta) = 2a\dot{a}dt \wedge dx^\alpha \wedge dx^\beta, \quad (41)$$

we note that three Lorentz boosts  $\sigma^{0\gamma}(t, x^\alpha, x^\beta)$  of  $\mathbf{e}^a$  will also lead to an equivalent solution of the dynamical equations (23) (take note that  $\gamma \neq \alpha \neq \beta$ ). This is so because the 1-form  $d\sigma^{0\gamma}$  in Eq. (32) does not contain a term proportional to  $dx^\gamma$ , so the wedge product  $d(\mathbf{e}^\alpha \wedge \mathbf{e}^\beta) \wedge d\sigma^{0\gamma}$  is null. Then, for this particular 3-CAF, we have that  $\mathcal{A}(\mathbf{e}^a)$  contains not three, but six independent generators  $\sigma^{0\gamma}(t, x^\alpha, x^\beta)$  and  $\sigma^{\beta\gamma}(x^\mu)$ . Nevertheless, it is important to realize that the three one-dimensional Abelian subgroups of boosts in a given direction [generated by  $\sigma^{0\gamma}(t, x^\alpha, x^\beta)$ ], are constrained to possess a restricted dependence on the spacetime coordinates  $x^\mu$ . For instance, if we consider a boost in the  $t-x$  plane, we have that the generator can depend just on  $(t, y, z)$ . For boosts in the remaining planes, analogous comments are in order.

Because of this, even though this particular 3-CAF does not allow the emergence of a two-dimensional Abelian subgroup of the form  $\{K_\alpha(x^\mu), J_\alpha(x^\mu)\}$  [as explained in the paragraph below Eq. (37)], we still expect three Abelian subgroups of dimension 2 with restricted coordinate dependence contained in  $\mathcal{A}(\mathbf{e}^a)$ . Precisely, these are the ones generated by

$$\begin{aligned} & \{K_x(t, y, z), J_x(x^\mu)\} \\ & \{K_y(t, x, z), J_y(x^\mu)\} \\ & \{K_z(t, x, y), J_z(x^\mu)\}. \end{aligned} \quad (42)$$

Spatially flat FRW cosmological models admit, hence, an infinite number of proper tetrads, organized in the Abelian subgroups of the Lorentz group just mentioned. This strong result seems to suggest that some claims present in the literature regarding superluminal propagating modes and

nonuniqueness of time evolution in  $f(T)$  theories [27,28] should be revised in light of the new developments here introduced.

### 2. Spatially curved FRW spacetime

The parallelization of closed and open FRW universes is much less obvious. In these cases we can write the line element as

$$ds^2 = dt^2 - a^2(t)k^2[d(k\psi)^2 + \sin^2(k\psi)(d\theta^2 + \sin^2\theta d\phi^2)], \quad (43)$$

where  $(\psi, \theta, \phi)$  are standard hyperspherical coordinates on the three-sphere. The parameter  $k$  appearing in (46) takes the values  $k=1$  for the spatially spherical universe and  $k=i$  for the spatially hyperbolic one.

In Ref. [13] it was shown how one can find a global frame for spatially curved FRW spacetimes, i.e., a global basis that turns the dynamical equations (23) into a consistent system of differential equations for the scale factor  $a(t)$ . It reads

$$\mathbf{e}^0 = dt, \quad \mathbf{e}^\alpha = a(t)\mathbf{E}^\alpha, \quad (44)$$

where the 1-forms  $\mathbf{E}^\alpha$  are

$$\begin{aligned} \frac{\mathbf{E}^1}{k} &= -k \cos\theta d\psi + \sin\theta \sin(k\psi) \cos(k\psi) d\theta \\ &\quad - \sin^2(k\psi) \sin^2\theta d\phi, \\ \frac{\mathbf{E}^2}{k} &= k \sin\theta \cos\phi d\psi \\ &\quad - \sin^2(k\psi) [\sin\phi - \cot(k\psi) \cos\theta \cos\phi] d\theta \\ &\quad - \sin^2(k\psi) \sin\theta [\cot(k\psi) \sin\phi + \cos\theta \cos\phi] d\phi, \\ \frac{\mathbf{E}^3}{k} &= -k \sin\theta \sin\phi d\psi \\ &\quad - \sin^2(k\psi) [\cos\phi + \cot(k\psi) \cos\theta \sin\phi] d\theta \\ &\quad - \sin^2(k\psi) \sin\theta [\cot(k\psi) \cos\phi - \cos\theta \cos\phi] d\phi. \end{aligned} \quad (45)$$

Many fewer local symmetries are left in this case, because the frame (44) is just a 0-CAF. Nonetheless, since  $d(\mathbf{e}^0 \wedge \mathbf{e}^\alpha) = dt \wedge d\mathbf{e}^\alpha$ , we can say that time-dependent rotations  $\sigma^{\beta\gamma}(t)$  are authorized by Eq. (32). Thus, we get three one-dimensional Abelian subgroups composed of time-dependent rotations about a given axis.

### 3. Bianchi type I models

Homogeneous and anisotropic Bianchi type I models are described by the line element

$$ds^2 = dt^2 - a_1^2(t)dx^2 - a_2^2(t)dy^2 - a_3^2(t)dz^2. \quad (46)$$

The manifold topology is  $R^4$ , so a proper parallelization is given by the frame  $\mathbf{e}^0 = dt$ ,  $\mathbf{e}^1 = a_1 dx$ ,  $\mathbf{e}^2 = a_2 dy$ ,  $\mathbf{e}^3 = a_3 dz$ . Although Bianchi type I spacetimes contain less isometries than FRW cosmologies, we can easily check that  $d(\mathbf{e}^0 \wedge \mathbf{e}^\alpha) = 0$ ,  $\forall \alpha$ , so we are in the presence of a 3-CAF once again, and the same comments of Sec. VB 1 are in order.

## VI. CONCLUDING COMMENTS

For special kinds of frames, the so-called  $n$ -CAFs (which include Minkowski spacetime and a wide variety of cosmological models), we have obtained in Sec. IV a number of results regarding certain conditions for a local Lorentz transformation that belongs to the set  $\mathcal{A}(\mathbf{e}^a)$ . In particular, for 6-CAFs, we have concluded the following:

- (1) Any infinitesimal local Lorentz transformation belongs to  $\mathcal{A}(\mathbf{e}^a)$ .
- (2) Regarding finite transformations, we have that  $\langle K_\alpha(x^\mu), J_\alpha(x^\mu) \rangle \subset \mathcal{A}(\mathbf{e}^a)$ , where  $K_\alpha(x^\mu), J_\alpha(x^\mu)$  are the six generators of the two-dimensional Abelian subgroups of the Lorentz group. In particular, six one-dimensional Abelian subgroups are included in  $\mathcal{A}(\mathbf{e}^a)$  (boosts in a given direction and rotations about a given axis).

As a direct consequence, we see that there are infinitely many adequate tetrads representing Minkowski spacetime in  $f(T)$  gravity. This result does not mean that any tetrad giving rise to the Minkowski metric is a solution of the  $f(T)$  motion equations in vacuum. For instance, the tetrad

$$\mathbf{e}^0 = dt, \quad \mathbf{e}^1 = dr, \quad \mathbf{e}^2 = r d\theta, \quad \mathbf{e}^3 = r \sin \theta d\phi, \quad (47)$$

corresponding to  $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$ , is not a solution of the vacuum field equations, because it fails to be a basis of the tangent space at  $r = 0$ , and so, it is not a parallelization of Minkowski spacetime. This is so because (47) can be obtained from the Euclidean frame by means of a local Lorentz transformation which is not an element of  $\mathcal{A}(\mathbf{e}^a)$ . Precisely, the Euclidean frame in spherical coordinates stands as (just change coordinates in  $\mathbf{e}^a = \delta^a_b dx^b$ )

$$\begin{aligned} \mathbf{e}^0 &= dt, \\ \mathbf{e}^1 &= \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi, \\ \mathbf{e}^2 &= \sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi, \\ \mathbf{e}^3 &= \cos \theta dr - r \sin \theta d\theta. \end{aligned} \quad (48)$$

The tetrad (47) can be obtained from (48) by means of a local rotation which, however, does not satisfy Eq. (24) because the involved generators do not commute.

For  $n$ -CAFs (flat FRW and Bianchi type I models of Sec. V being in this category), the picture is more restrictive, and the statements made for 6-CAFs change to

- (1') Some infinitesimal local Lorentz transformations, generated by  $n$  one-dimensional subgroups of the infinitesimal Lorentz group, belong to  $\mathcal{A}(\mathbf{e}^a)$ .
- (2') Regarding finite transformations, a number  $\lfloor n/2 \rfloor$  of two-dimensional Abelian subgroups of the form  $K_\alpha(x^\mu), J_\alpha(x^\mu)$  might arise, depending on the particular closed area involved. For  $n \geq 4$  these Abelian subgroups will actually exist. In particular,  $n$  one-dimensional Abelian subgroups will be included always in  $\mathcal{A}(\mathbf{e}^a)$ . Sometimes, depending on the specific form of the  $n$ -CAF, an additional (restricted) Lorentz invariance can exist [see, e.g., Eq. (42)].

Finally, we would like to mention some remaining open questions of conceptual guise. One of these concerns the relationship between the isometries of a given spacetime  $(\mathcal{T}(\mathcal{M}), \mathbf{e}^a)$ , and its remnant set  $\mathcal{A}(\mathbf{e}^a)$ . Perhaps it would be plausible to think that an increase in the number of isometries will lead to an enlargement of the set  $\mathcal{A}(\mathbf{e}^a)$ . It should be clear from the examples examined above that this is not actually true. All FRW spacetimes have the same number of isometries, whereas the set  $\mathcal{A}(\mathbf{e}^a)$  is considerably larger for spatially flat models. More drastically, curved FRW spacetimes have a notoriously smaller  $\mathcal{A}(\mathbf{e}^a)$  compared with the less symmetric Bianchi type I models.

Presumably, the answer to this issue underlies the global properties of the cited spacetimes and not only in their local geometry. As a matter of fact, flat FRW and Bianchi type I spacetimes both have topology  $R^4$ , and they are both represented by 3-CAFs. In turn, due to that fact that (let us say) closed FRW spacetimes have topology  $R \times S^3$ , we should expect a more involved global behavior concerning the parallelization process, which reflects itself in the fact that the frame (45) is just a 0-CAF.

As a final remark, we can comment on an important result obtained in Refs. [29,30]. There it was shown that where the connection is other than the Levi-Civita connection, the notion of an inertial reference frame can still be defined locally by means of local normal frames. This is a realization of the equivalence principle in theories with torsion, which means that in a spacetime with an arbitrary (though metric compatible) connection, we still recover the Minkowskian behavior locally. It would be interesting to figure out under what circumstances this property will still hold for (torsional) theories of gravity in which the Lorentz symmetry is not fully present, in the sense discussed in this work. By virtue of the result here obtained, the existence of locally inertial frames would assure (locally) the full Lorentz symmetry of any spacetime arising as a solution of the  $f(T)$  field equations, and so a well-behaved causal structure at a local level [31].

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## APPENDIX:

### 1. On the contorsion tensor

Some properties of the contorsion tensor can be enumerated as follows:

- (1) The equation of geodesics in an arbitrary connection is  $(DU/D\tau)^i = \mathbf{K}^i_j(\mathbf{U})U^j$ , so the contorsion represents the gravitational force.
- (2)  $\mathbf{T}^i = \mathbf{K}^i_j \wedge \mathbf{E}^j$ , then  $(\mathbf{T}^i)_{jk} = (\mathbf{K}^i_k)_j - (\mathbf{K}^i_j)_k$  [combine Eqs. (1), (10) and (12)].
- (3) If  $\overset{L}{D}$  is the covariant derivative associated with the Levi-Civita connection (10), then it results in  $D\mathbf{K}^i_j - \overset{L}{D}\mathbf{K}^i_j = 2\mathbf{K}^i_k \wedge \mathbf{K}^k_j$ .
- (4)  $\mathbf{R}^i_j - \overset{L}{\mathbf{R}}^i_j = \overset{L}{D}\mathbf{K}^i_j + \mathbf{K}^i_k \wedge \mathbf{K}^k_j = D\mathbf{K}^i_j - \mathbf{K}^i_k \wedge \mathbf{K}^k_j$ .
- (5)  $\mathbf{K}_{ba} = -\mathbf{K}_{ab}$  [use (8)].
- (6)  $K_{abc} \doteq (\mathbf{K}_{ab})_c = -\frac{1}{2}[(\mathbf{T}_a)_{bc} + (\mathbf{T}_b)_{ca} - (\mathbf{T}_c)_{ab}] = -\frac{1}{2}[(D\mathbf{e}_a)_{bc} + (D\mathbf{e}_b)_{ca} - (D\mathbf{e}_c)_{ab}]$  [use Property 2 and Eq. (1)].

### 2. TEGR in usual language

The Weitzenböck connection is defined in a given orthonormal basis  $\{\mathbf{e}^a\}$  as  $\omega^{cd} = 0$ . Since  $\mathbf{e}^a = e^a_\mu dx^\mu$ , one realizes that the transformation between coordinate and orthonormal bases uses the coefficients  $\Lambda^a_\mu = e^a_\mu$ . According to (3), if the connection vanishes in the basis  $\{\mathbf{e}^a\}$ , then it transforms to a coordinate basis as  $(\omega^\mu)_\lambda = e^\mu_a \partial_\lambda e^a_\nu$ , which is the familiar form of Weitzenböck connection. In Weitzenböck spacetime, it is  $T^\mu = Ddx^\mu = \omega^\mu_\nu \wedge dx^\nu = e^\mu_a \partial_\lambda e^a_\nu dx^\lambda \wedge dx^\nu$ , i.e.,

$$T^\mu_{\lambda\nu} = e^\mu_a (\partial_\lambda e^a_\nu - \partial_\nu e^a_\lambda). \quad (\text{A1})$$

To recover the familiar form of  $L_{\text{TEGR}}$  one writes  $\mathbf{K}^c_e = \mathbf{K}^c_{ef} \mathbf{e}^f$ , so

$$L_{\text{TEGR}} = -\frac{1}{4\kappa} K^c_{ef} K^{ed}_g \epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{e}^f \mathbf{e}^g. \quad (\text{A2})$$

In this expression one recognizes the volume 4-form  $\Omega$ ,

$$\mathbf{e}^a \mathbf{e}^b \mathbf{e}^f \mathbf{e}^g = \epsilon^{abfg} \Omega = \epsilon^{abfg} e dx^0 dx^1 dx^2 dx^3, \quad (\text{A3})$$

where  $e \doteq \det[e^a_\mu]$ . We use the identity  $\epsilon^{abfg} \epsilon_{abcd} = -2(\delta^f_c \delta^g_d - \delta^g_c \delta^f_d)$  to obtain

$$L_{\text{TEGR}} = \frac{1}{2\kappa} (K^c_{ec} K^{ed}_d - K^c_{ed} K^{ed}_c) \Omega. \quad (\text{A4})$$

According to property 4 of Appendix A it is  $K^c_{ec} = -T^c_{ec}$ ,  $K^{ed}_d = -T^d_{d^e}$  (we exploited the antisymmetry of torsion). Also,  $K^c_{ed} K^{ed}_c = K^c_{[ed]} K^{ed}_c = (-1/2) T^c_{ed} K^{ed}_c$ . Then,

$$\begin{aligned} K^c_{ec} K^{ed}_d - K^c_{ed} K^{ed}_c &= T^c_{ec} T^d_{d^e} + \frac{1}{2} T^c_{ed} K^{ed}_c \\ &= \frac{1}{2} T^c_{ed} (T_a{}^{ae} \delta^d_c - T_a{}^{ad} \delta^e_c + K^{ed}_c) \\ &= T^c_{ed} S_c{}^{ed}, \end{aligned}$$

where

$$S_c{}^{ed} \doteq \frac{1}{2} K^{ed}_c + T_a{}^{a[e} \delta^d]_c = \frac{1}{2} K^{ed}_c + K_a{}^{a[e} \delta^d]_c. \quad (\text{A5})$$

The quantity  $T^c_{ed} S_c{}^{ed}$  is the Weitzenböck scalar  $T$ . All these quantities behave tensorially under local Lorentz transformations whenever the spin connection is not frozen to zero. Otherwise, they are tensors just under global Lorentz transformations.

The boundary term in (20) contributes  $-(4\kappa)^{-1} \times d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}^{cd})$  to the Lagrangian. This exact 4-form can be rewritten in terms of a four-divergence. Notice that

$$d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}^{cd}) = d(\epsilon_{abcd} K^{cd}_e \mathbf{e}^a \mathbf{e}^b \mathbf{e}^e),$$

where

$$\mathbf{e}^a \mathbf{e}^b \mathbf{e}^e = -\epsilon^{abef} \Omega(\mathbf{e}_f).$$

Therefore

$$\begin{aligned} d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}^{cd}) &= d(2(\delta^e_c \delta^f_d - \delta^f_c \delta^e_d) K^{cd}_e \Omega(\mathbf{e}_f)) \\ &= 4d(K^{cd}_e \Omega(\mathbf{e}_d)) = 4\text{div}(K^{cd}_e \mathbf{e}_d) \Omega \\ &= \frac{4}{e} \partial_\mu (e K^{cd}_e e^\mu_d) \Omega. \end{aligned} \quad (\text{A6})$$

According to (11) it is

$$\begin{aligned} K^{cd}_c[\mathbf{e}] &= -(d\mathbf{e}^c)^d_c = \eta^{db} \partial_\lambda e^c_\nu (e^\lambda_c e^b_\nu - e^b_\nu e^\lambda_c) \\ &= \eta^{db} e^\lambda_c e^b_\nu (\partial_\lambda e^c_\nu - \partial_\nu e^c_\lambda). \end{aligned}$$

By comparing with Eq. (A1), one obtains

$$d(\epsilon_{abcd} \mathbf{e}^a \mathbf{e}^b \mathbf{K}^{cd}) = \frac{4}{e} \partial_\mu (e T^{\lambda\mu}) \Omega.$$

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