International Journal of Geometric Methods in Modern Physics
Vol. 13 (2016) 1650113 (10 pages)

DOI: 10.1142/S0219887816501139

Quaternionic structures, supertwistors and fundamental superspaces

Diego Julio Cirilo-Lombardo*

Universidad de Buenos Aires
Consejo Nacional de Investigaciones
Cientificas y Tecnicas (CONICET)
National Institute of Plasma Physics (INFIP)
Facultad de Ciencias Exactas y Naturales
Cuidad Universitaria Buenos Aires 1428, Argentina

and

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
141980 Dubna, Russia
diego777jcl@gmail.com

Victor N. Pervushin

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
141980 Dubna, Russia

Received 21 March 2016
Accepted 29 May 2016
Published

Superspace is considered as space of parameters of the supercoherent states defining the basis for oscillator-like unitary irreducible representations of the generalized superconformal group $SU(2m, 2n\,|\,2N)$ in the field of quaternions $\mathbb{H}$. The specific construction contains naturally the supertwistor one of the previous work by Litov and Pervushin [1] and it is shown that in the case of extended supersymmetry such an approach leads to the separation of a class of superspaces and its groups of motion. We briefly discuss this particular extension to the domain of quaternionic superspaces as nonlinear realization of some kind of the affine and the superconformal groups with the final end to include also the gravitational field [6] (this last possibility to include gravitation, can be realized on the basis of Ref. 12 where the coset $\frac{Sp(8)}{SL(4R)} \sim \frac{SU(2,2)}{SL(2C)}$ was used in the non supersymmetric case). It is shown that this quaternionic construction avoid some unconsistencies appearing at the level of the generators of the superalgebras (for specific values of p and q; $p + q = N$) in the twistor one.

Keywords: Supergeometry; coherent states; almost complex structures; twistors; suerspaces.

*Corresponding author.

1. Introduction

There are three main approaches to the construction of supersymmetric theories [2]. In the first the supersymmetry is realized on fields, while in the second directly on the superspace. The third approach is based on the assumption of the simplest structure elements of superspace called supertwistors [1, 3].

The particular geometrical environment of this approach, plus the explicit covariant formulation appears as the main advantage over the standard component one. The most important problem of the superfield approach is the explicit formulation in terms of unconstrained superfields [3]. One way to deal with this issue is with twistors. If one begins with twistors, the compact complex version of the Minkowski superspace M is naturally realized as a flag space. The geometry of the flat case corresponding to $N = 1$ (the minimal number of odd coordinates) turns into the geometry of the simple supergravity model of Ogievetsky and Sokatchev [4] after a convenient twist.

In this paper, because we are interested in the number of parameters of the super-Lorentz transformations and the number of the Goldstone modes, we extend the supertwistor construction of Ref. 1 to the quaternionic one. We expect that this particular extension, that is justified by the Ogievetsky theorem [5] to the domain of superspaces, will be bring us the correct number of fields of the standard model as the simultaneous nonlinear realization of some kind of the affine and the superconformal groups [6–9], with the final end to include also the gravitational field [6] (this last possibility to include gravitation, can be realized on the basis of the reference [12] where the coset $\frac{Sp(8)}{SL(4R)} \sim \frac{SU(2,2)}{SL(2C)}$ was used in the non-supersymmetric csase). Consequently, we close this paper with a short discussion about the kinds of possible supergroups able to support a twistor and a quaternionic structure.

2. Quaternionic Construction

The link between a (super) twistor space $\mathbb{T}$ and the (super) quaternionic one $\mathbb{H}$ is through the natural symplectic structure of $\mathbb{H}$ (see in other contexts, for example [11]). Because the minimal quaternionic realization is in a 2×2 complex matrix structure (e.g. $R \otimes SU(2)$), the supertwistor construction only can be implemented for an even number of components in a matrix realization as we will show below. Now we will construct the quaternionic superextension analog to the twistorial one in paper [1].

2.1. *Supertwistors*

In the twistor theory our starting point is a complex space $\mathbb{C}M \sim \mathbb{C}_{2,4}(T)$. By using the correspondence with null twistors we can successfully separate from it, the real Minkowski space M invariant with respect to the conformal group.

The complex space CM in its compactified form, also contains S^4. For this space, the twistor correspondence can be introduced as follows. Reality of the space S^4 doesn't follows, as in the Minkowski case, from the nullification of some kind on twistors. It follows from the invariance under an antilinear mapping $\rho : \mathbb{C}^4 \to \mathbb{C}^4, \rho^2 = -1$. Then, is clear that this mapping represents the multiplication by the standard quaternionic unit due $\mathbb{C}^4 \sim \mathbb{H}$. Because the space S^4 is not invariant under $SU(2,2)$ we have to take the covering group of the complexified $SU(2,2)$ group. The direct possibility is, for example, $SL(4,\mathbb{C})$ which covers simultaneously $SO(6,\mathbb{C})$ and S^4 being consequently invariant with respect to $SL(2,\mathbb{H})$: its real form which covers $SO(5,1)$.

When we pass to supertwistor space there exists similar mapping in analogy with the ρ in the non-susy case but only for N even. Then, if we assume $N = 2(p+q)$ the quaternionic structure can be introduced. This can be realized taking into account that the quaternions can be written as $\mathbb{H} = \mathbb{C} \otimes \mathbb{C}$ e.g.: the first quaternionic unit in the field of $\mathbb{H}$ is identified with the imaginary unit in $\mathbb{C}$. Consequently, any $q \in \mathbb{H}$ can be written as

$$q = a + b\widehat{i_2}, \quad a, b \in \mathbb{C}(\widehat{i_1}), \tag{1}$$

where $\mathbb{C}(\widehat{i_1})$ is the complex space with the first quaternionic generator as imaginary unit.

To make explicit connection between the quaternionic structures in supertwistor spaces, we only need to introduce 2×2 matrices by each element of the standard supertwistors operators. For instance, is clear that under conjugation any supertwistor(standard notation, internal fermion indices dropped) (ω, π, θ) goes to $(\overline{\omega} = \epsilon\omega^*, \overline{\pi} = \epsilon\pi^*, \overline{\theta} = \epsilon\theta^*)$ where:

$$\epsilon = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{2}$$

Remark 1. In the pure twistor theory we need to use of the correspondence with null twistors in order to separate from the twistor complex space $C_{2,4}(T)$ the real Minkowski space M invariant with respect to the conformal group. Because in the quaternionic case the twistor constructions don't follows from nullification of some kind of twistors (as in Minkowski space) but through the invariance under an antilinear mapping $\rho : \mathbb{C}^4 \to \mathbb{C}^4, \rho^2 = -1, (\mathbb{C}^4 \sim \mathbb{H})$, we have not singularities of the super-generators of the theory. Also is possible to have an alternative consistent quantum-field theoretical construction to the ligh-cone one.

2.2. *L and R subspaces*

Although in the simplest case in four spacetime dimensions, it is neccesary to find the elements from L and R subspaces invariant under the ρ map. We are interested in the $(2,0)$ and $(2,N)$ subspaces in the field of the quaternions. As is the standard supertwistor case, we can establish an incidendence condition to determining $(2, 0)$ subspace in full quaternionic form as follows:

$$q = wp, \quad s = \overline{\eta}p, \tag{3}$$

where q, p and s are quaternions (or higher-dimensional quaternionic matrices) constructed as in the previous paragraph, and w and $\overline{\eta}$ are quaternions obtained via correspondence:

$$a + b\widehat{i_2} \leftrightarrow \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}. \tag{4}$$

As it is clear, we get the quaternionic superspace $\mathbb{H}^{(1\,|\,N)}$ with projective coordinates on $P(\mathbb{H}^{(2\,|\,N)})$ defined as $qp^{-1} = w$, $sp^{-1} = \overline{\eta}$. For example, if the supergroup $SL(2, \mathbb{H}\,|\,N)$ acts on $\mathbb{H}^{(2\,|\,N)}$, knowing that $SO(5,1) \subset SL(2, \mathbb{H}\,|\,N, \mathbb{H})$, then we obtain the transformations of the quaternionic superspace including the S^4 (Euclidean) conformal transformations. Let us to consider the complexification of the $SU(2,2\,|\,2N)$ supergroup, namely the supergroup $SL(2, \mathbb{H}\,|\,N) \sim SL(4, \mathbb{C}\,|\,2N)$ and let us to take its real form which preserves the reality of the map ρ in the sense, for example, which maps the given subspace S^4 onto itself. The infinitesimal transformation from can be written as:

$$\begin{pmatrix} \delta\omega_\alpha \\ \delta\pi^{\alpha'} \\ \delta\theta_i \end{pmatrix} = \begin{pmatrix} l^\alpha_\beta + \dfrac{(D+G)}{2}\delta^\alpha_\beta & a^{\alpha\beta'} & \psi^\alpha_j \\ b_{\alpha'\beta} & -k^{\beta'}_{\alpha'} - \dfrac{(D-G)}{2}\delta^{\beta'}_{\alpha'} & \varphi_{\alpha' j} \\ \xi^i_\beta & \chi^{i\beta'} & S^i_j + \dfrac{2G}{N}\delta^i_j \end{pmatrix} \begin{pmatrix} \omega^\beta \\ \pi_{\beta'} \\ \theta_j \end{pmatrix}, \tag{5}$$

where l, a, b, k are quaternion-valuated parameters. These parameters of $SL(2, \mathbb{H}\,|\,N)$ are restricted by the requirement of conservation of ρ-invariant subspaces. Finally, due to the quaternion-twistor correspondence, these transformations can be related with the quaternion-valuated spaces E^N_R, E^N_L and E_0.

2.3. *Quaternionic superspace*

We know that the equations relating the B_0 and B_1 parts of the superspace in the case of supertwistors can be in terms of quaternions as follows.

(i) The fundamental representation can be descomposed, in principle, as in the case of [1] as

$$\mathcal{U} = t \cdot h, \tag{6}$$

where h is an element of the maximal compact subgroup $S(U(2m) \times U(2n))$ and t of the corresponding coset space $\frac{SU(2m;2n)}{S(U(2m)\times U(2n))}$. Explicitly

$$h = \exp\left[i \begin{pmatrix} \chi & 0 \\ 0 & \varepsilon \end{pmatrix}\right] = \begin{pmatrix} \mu & 0 \\ 0 & \upsilon \end{pmatrix} \tag{7}$$

and

$$t = \exp\left[i\begin{pmatrix} 0 & \nu \\ \overline{\nu} & 0 \end{pmatrix}\right] = \frac{1}{\sqrt{1-\overline{Z}Z}}\begin{pmatrix} \mathbb{I} & Z \\ \overline{Z} & \mathbb{I} \end{pmatrix}, \tag{8}$$

where

$$Z_B^A = \left[\frac{\mathrm{Tanh}\sqrt{\overline{\nu}\nu}}{\sqrt{\overline{\nu}\nu}}\nu\right]_B^A \tag{9}$$

with ν_B^A is a quaternionic matrix that can be generically represented by a complex $(2(m+p)\times 2(n+q))$-matrix $\nu^{i\alpha}(i = 1\cdots 2(m+p); \alpha = 1\cdots 2(n+q))$ with

$$\nu^{i\alpha} = \mathcal{J}^{ij}\mathcal{J}^{\alpha\beta}\overline{\nu}_{j\beta}, \tag{10}$$

where $\mathcal{J}^{ij}, \mathcal{J}^{\alpha\beta}$ are the $(2m+2p; 2n+2q)$ matrices (obvious generalization of the standard $\epsilon^{\alpha\beta}$ that introduces the corresponding symplectic structure [10]) such that:

$$\mathcal{J} = -\mathcal{J}^T = -\mathcal{J}^\dagger = -\mathcal{J}^{-1}. \tag{11}$$

The convenient representation is the canonical symplectic matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix} \tag{12}$$

consequently $\nu^{i\alpha}$ has $4(m+p)(n+q)$ degrees of freedom.

(ii) The metric constraint is invariant under $Usp(m\,|\,p) \times Usp(n\,|\,q) \approx SU(2m\,|\,2p) \times SU(2n\,|\,2q)$ transformations

$$\nu' = g\cdot\nu\cdot h, \tag{13}$$

where g, h are unitary matrices of dimension $2(m+p), 2(n+q)$ respectively.

$$g\cdot\mathcal{J}_m\cdot g^T = \mathcal{J}_m, \quad h\cdot\mathcal{J}_n\cdot h^T = \mathcal{J}_n. \tag{14}$$

This constraints and the unitarity condition are the defining propierties of the Usp-groups, even in the SUSY case. The Lie algebra is spanned by independent complex $2m\times 2n$-dimensional generators $A = \mathcal{J}\cdot X$, with

$$A = -A^\dagger \quad \text{or} \quad X = \mathcal{J}\cdot X^\dagger\cdot\mathcal{J} \tag{15}$$

and

$$A\cdot J + J\cdot A^\dagger = 0 \quad \text{or} \quad X = X^T. \tag{16}$$

3. The Super-Hilbert Space

The starting point is the Lie superalgebra coming from the Poisson structure of the Manifold:

$$[f,g] = f\frac{\overleftarrow{\partial}}{\partial \widetilde{T}^R}(\omega^{-1})^R_S\frac{\overrightarrow{\partial}}{\partial T_S}g \tag{17}$$

we obtain the following set of Poisson bracket relations between the quaternionic valuated variables:

$$[\mu^\alpha_m, \lambda_{\beta n}] = \delta^\alpha_\beta \delta_{mn}, \quad [\overline{\lambda}_{\dot{\beta}n}, \overline{\mu}^{\dot{\alpha}}_m] = \delta^{\dot{\alpha}}_{\dot{\beta}}\delta_{nm}, \tag{18}$$

$$[-\xi_{ir}, \xi^j_s] = -\delta^j_i\delta_{rs}, \quad [-\overline{\eta}_{kt}, \eta^l_u] = -\delta^l_k\delta_{tu}. \tag{19}$$

Therefore, for generalized quaternion-valuated supertwistors T, $\widetilde{T}$ we directly have

$$[\widetilde{T}^R, T_S\} = \delta^R_S \tag{20}$$

with

$$T_R = \begin{pmatrix} \overline{\xi}_A \\ \eta_M \end{pmatrix} \widetilde{T}^R = (\xi^A, -\overline{\eta}^M) \tag{21}$$

such that

$$\overline{\xi}_A, \eta_M \in \mathbb{H}_n. \tag{22}$$

The superspace Z has the following general form (we follow notation from [1])

$$Z^{\dagger A}_B = \overbrace{\begin{pmatrix} X^a_m & \theta^i_m \\ \chi^a_l & \lambda^i_l \end{pmatrix}}^{2n+2q} \Bigg\} 2m+2p \tag{23}$$

it represents the space of parameters of quaternionic supercoherent states, depending on the structure of the supercoset space. The above quaternionic supermatrix acts over the following quaternionic supervectors

$$\xi^A = (a^c, -\xi^i); \quad \overline{\xi}_A = \begin{pmatrix} a^\dagger_c \\ \xi^\dagger_i \end{pmatrix} \Bigg\} 2n+2q, \tag{24}$$

$$\overline{\eta}^M = \overbrace{(b^{\dagger m}, \eta^{\dagger l})}^{2m+2p}; \quad \eta_M = \begin{pmatrix} b_m \\ \eta_l \end{pmatrix}, \tag{25}$$

where we have defined

$$a^\dagger_c = \frac{1}{\sqrt{2}}(\lambda_\alpha + \overline{\mu}^{\dot{\alpha}}), \tag{26}$$

$$b_m = -\frac{1}{\sqrt{2}}(\lambda_\alpha - \overline{\mu}^{\dot{\alpha}}). \tag{27}$$

The explicit superfield coherent state reads as

$$|\Phi^{A\cdots}_{B\cdots}(Z)\rangle = e^{\overline{\eta}^M Z^{\dagger A}_M \overline{\xi}_A} |\Phi^{A\cdots}_{B\cdots}\rangle_0, \tag{28}$$

$$= \exp[b^{\dagger m}(X^a_m a^\dagger_a + \theta^i_m \xi^\dagger_i) + \eta^{\dagger l}(\chi^a_l a^\dagger_a + \lambda^i_l \xi^\dagger_i)] |\Phi^{A\cdots}_{B\cdots}\rangle_0. \tag{29}$$

The Grassmann character of the matrix coefficients, restricts the number of terms in (28) (see appendix):

$$\Phi^{A\cdots}_{B\cdots}(Z) = \sum_{n_1=0}^{2p} \frac{1}{n_1!}(b^{\dagger m}\theta^i_m \xi^\dagger_i)^{n_1} \sum_{n_2=0}^{2q} \frac{1}{n_2!}(\eta^{\dagger l}\chi^a_l a^\dagger_a)^{n_2}$$

$$\times \sum_{n_3=0}^{\min(p,q)} \frac{1}{n_3!}(\eta^{\dagger l}\lambda^i_l \xi^\dagger_i)^{n_3} f^{A\cdots}_{B\cdots}(x), \tag{30}$$

where

$$f^{A\cdots}_{B\cdots}(x) = e^{b^{\dagger m} X^a_m a^\dagger_a} |\Phi^{A\cdots}_{B\cdots}\rangle_0. \tag{31}$$

4. Quaternionic Supercoherent States

4.1. *Some examples*

(i) The basis in the simplest cases is the superalgebra $U(1,1\,|\,1,\mathbb{H})$ that contains as subgroups $U(1,1;\mathbb{H}) \sim SO(1,4)$ and $U(1,\mathbb{H}) \sim SO(2)$. It is determined by infinitesimal transformations preserving the scalar product $\overline{q}_a q_a - \overline{q}_b q_b + \overline{\eta} e_2 \eta$ where η is a standard Grassmann quaternion and e_2 is the B_1 part of the supermetric.

(ii) For example, a more complicated case is in the field of $\mathbb{H}$ with $p = 2$ and $q = 2$, (that is the quaternionic analog of the $SU(2,2\,|\,8)$ with $N = 8 = p + q \rightarrow p = 4, q = 4$). In this case we have the following scalar quaternionic superfield:

$$\Phi(Z) = f(x) + \overline{\theta}^i_m \overline{\chi}^a_k f(x)^{\{mk\}}_{(ia)} + \overline{\theta}^i_m \overline{\theta}^j_n \overline{\chi}^a_k \overline{\chi}^b_l f(x)^{\{mnkl\}}_{(ijab)} \tag{32}$$

which is a supermultiplet with helicities ranging up to $|s| = 2$ and the multiplicities of the quaternionic $N = 4$ (complex $N = 8$) Maxwell supermultiplet. There is one more condition to be fulfilled by the quaternionic wave function:

$$\frac{1}{2}\widetilde{T}T|\Phi^{A\cdots}_{B\cdots}(Z)\rangle = 0 \tag{33}$$

because, in this particular case, $\widetilde{T}T$ is a $U(1,1\,|\,p+q,\mathbb{H})$ invariant quantity, this condition transforms into:

$$\left[\widehat{s} - \frac{1}{2}(F_\xi + F_\eta) + \frac{1}{4}(p+q)\right] |\Phi^{A\cdots}_{B\cdots}(Z)\rangle = 0, \tag{34}$$

where $\widehat{s}$ is the $U(1,1,\mathbb{H})$-invariant helicity operator and F_ξ, F_η are the quaternion-valuated fermion number operators. The above condition plus the annihilation of the vacuum by all L^- and R^- operators determine the structure of the lowest states univoquely as follows:

$$|\Phi^{A\cdots}_{B\cdots}\rangle_0 = \exp[b^{\dagger m}(X^a_m a^\dagger_a + \theta^i_m \xi^\dagger_i + \eta^{\dagger l}(\chi^a_l a^\dagger_a + \lambda^i_l \xi^\dagger_i)]a^\dagger_{\{a}a^\dagger_{b\}}b^{\dagger\{m}b^{\dagger n\}}|0,0\rangle. \quad (35)$$

Remark 2. Notice that the quaternion-Casimir given by expression (33) shows that for $p = q$ in this $\mathbb{H}$-formulation we have fundamental representation in a sharp contrast with the purely $\mathbb{C}$-supertwistor one (where the last term into the helicity operator goes as $(p - q)$)(see [1]).

5. Concluding Remarks and Outlook

The results concerning to this preliminary work can be enumerated in the following points:

(1) We have been extended the supertwistor construction [1] to the quaternionic one.
(2) We are capable to extend, accordling to this new quaternionic description, the number of fermionic fields beyond the pure supertwistorial one avoiding (because the division ring structure) the singularities in the representation of the supergenerators.
(3) Some corresponding cosets (with some examples) have been identified, remaining they as a characteristic subset of the quaternionic superextensions with the full supertwistor propierties (e.g. arising from the super light cone structure).
(4) We have obtained the corresponding coherent super-quaternionic states spanning the super-Hilbert spaces.

In a separate paper [9], the interesting cosets from which we are able to perform the nonlinear realization in order to obtain the super-analog of the Borisov–Ogievetsky one [6] (e.g. to obtain the corresponding number of Goldstone fields for the standard model) will be discussed and an alternative to the light cone construction will be perfomed. Moreover, the more important task that remains is to perform explicitly the super-analog of the Borisov–Ogievetsky nonlinear realization in the same way as [12] developing consequently the same analysis and physical construction as [13].

Appendix A. Grassmann Quaternion

Theorem A.1. *A Grassmann quaternion, as in the case of the standard one with coefficient* $s \in \mathbb{R}$ *can be written as* $\Psi = A(r)e^{i\theta\Sigma}$ *with* A *and* Σ *matrices depending of four different* (*anticommuting*) *Grassmann numbers, with* $\Sigma^2 = 1$.

Proof. A Grassmann quaternion is described by the following matrix

$$\Psi = \begin{pmatrix} \psi_0 + i\psi_3 & -\psi_2 + i\psi_1 \\ \psi_2 + i\psi_1 & \psi_0 - i\psi_3 \end{pmatrix} \quad (A.1)$$

with ψ_a $(a = 0 \cdots 3)$ Grassmann numbers. We introduce polar coordinates as

$$\left.\begin{aligned} \psi_0 &= r\cos\theta = r\cdot 1 \\ \psi_1 &= r\sin\theta\sin\phi\cos\chi = r\cdot\theta\cdot\phi\cdot 1 \\ \psi_2 &= r\sin\theta\sin\phi\sin\chi = r\cdot\theta\cdot\phi\cdot\chi \\ \psi_3 &= r\sin\theta\cos\phi = r\cdot\theta\cdot 1 \end{aligned}\right\} \text{Grassmann coefficients,} \tag{A.2}$$

where r, θ, ϕ, χ are also Grasmann numbers. Then (35) can be written as follows:

$$\Psi = A(r)e^{i\theta\Sigma} \tag{A.3}$$

with

$$A(r) = r\sigma_0, \quad \sigma_0 = \mathbb{I}_{2\times 2}, \tag{A.4}$$

$$\Sigma = \begin{pmatrix} \cos\phi & \sin\phi e^{i\chi} \\ \sin\phi e^{-i\chi} & -\cos\phi \end{pmatrix} = \begin{pmatrix} 1 & \phi(1+i\chi) \\ \phi(1-i\chi) & -1 \end{pmatrix}, \tag{A.5}$$

where the last matrix in the RHS of the above expression coming from the Grassmann properties of the corresponding coefficients. Notice that automatically $\Sigma^2 = 1$, consequently the concrete construction proposed in this paper is faithful and consistent with the $\mathcal{J} - matrix$ and the quaternionic supervectors ξ^A and η_M. □

Acknowledgments

The authors would like to thank A.B. Arbuzov, A.A. Zheltukhin, and A.E. Pavlov for useful discussions. D.J.C-L. is grateful to the JINR Directorate for hospitality and to CONICET-Argentina for financial support. This work is in memory of the professor and friend Boris Moiseevich Zupnik, one of the main researchers in the supersymmetry, supergravity and other areas of the modern mathematical physics, that pass away recently.

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