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Domain wall interactions due to vacuum Dirac field fluctuations in $2+1$ dimensions

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We evaluate quantum effects due to a two-component Dirac field in $2+1$ spacetime dimensions, coupled to domain-wall-like defects with a smooth shape. We show that these effects induce nontrivial contributions to the (shape-dependent) energy of the domain walls. For a single defect, we study the divergences in the corresponding self-energy, and also consider the role of the massless zero mode—corresponding to the Callan-Harvey mechanism—by coupling the Dirac field to an external gauge field. For two defects, we show that the Dirac field induces a nontrivial, Casimir-like effect between them, and we provide an exact expression for that interaction in the case of two straight-line parallel defects. As is the case for the Casimir *interaction* energy, the result is finite and unambiguous.

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I. INTRODUCTION

The study of effects due to fermionic fields in the background of defects has been a subject of general interest in rather different areas, from the behavior of textures in superfluid phases of He_3 [1] to cosmic strings [2]. A representative of these phenomena is the Callan-Harvey effect [3], where a Fermi field in an odd-dimensional spacetime couples to a defect, the latter corresponding to a mass term which changes sign along a domain wall. Under these circumstances, the Dirac field spawns a localized zero mode which may be capable of carrying currents when coupled to a gauge field.

A similar phenomenon happens also in a nonrelativistic two-dimensional electron gas in the presence of a magnetic field in the regime of the quantum Hall effect, where the zero modes become the so-called chiral edge states [4].

In this paper we evaluate yet another effect due to the interplay between the fermionic field and domain walls, this time involving not just the presence of the zero mode, but also the quantum fluctuations of the fermionic field on top of the domain walls. This effect amounts to the emergence of nontrivial contributions to the vacuum energy as a function of the geometry of the wall(s). One should indeed expect this kind of effect, since the existence of the walls produces a geometry-dependent distortion of the vacuum fluctuation, a fertile setup for the induction of Casimir-like effects. Indeed, the quantum vacuum interaction between kinks of the sine-Gordon equation has been evaluated in Ref. [5] using the TGTG formula [6]. This force is attractive, and it should be considered as a small quantum correction to the well-known repulsive classical force between kinks [7].

Having in mind its potential application to graphene [8], where some effective continuum models correspond to Dirac fields in $2+1$ dimensions coupled to space-dependent masses [9], we evaluate here some effects due to the vacuum (i.e., zero-temperature) quantum fluctuations of a Dirac field in the presence of domain walls [10]. Since there is no reason *a priori* to assume that just the zero modes are relevant to this effect, we will include all the modes.

Effective continuum models for graphene using Dirac fields generally involve not just a single two-component field, but an even number of them, which amounts to putting the Dirac field in a reducible representation of the Poincaré group in $2+1$ dimensions. Nevertheless, since these models can be constructed in terms of decoupled two-component fields in a rather straightforward way, we will consider just the latter, i.e., spinors in an irreducible representation. Besides, the two-component case is relevant to other applications in condensed matter physics, like the above-mentioned quantum Hall effect case.

The domain wall energy is usually described by means of an effective Landau-Ginzburg-like functional of its shape (assumed to be smooth). That functional could, at least in principle, be obtained from a detailed microscopic model for the system. In this context, contributions depending on, for example, the crystal structure, should be quite relevant. In this work, we focus on contributions to the domain wall energy, which should appear on top of the ones coming from the lattice structure. Besides, we also obtain a result corresponding to two domain walls, whereby the Dirac field is shown to induce an attractive, Casimir-like force. To the best of our knowledge, this force does not have an analogue within the context of the phenomenological model, since it is not a contact interaction which could be incorporated by means of a local term into the energy.

For static-wall configurations, we shall see that the effective action predicts the existence of an effective interaction between domain-wall-like defects in the mass (the "pseudogap"). We also study the dependence of this interaction as a function of the geometry of the defects, at

least under some simplifying assumptions and for particular cases.

This paper is organized as follows. In Sec. II we describe the kind of model that we consider in this article, introduce our notation and conventions, and define its effective action. As a warm-up exercise, in Sec. III we consider the case of a single domain wall, namely, of a static mass that changes sign along a single spatial curve, having constant values (and the same module) everywhere else. The corresponding self-energy is divergent, including the contributions corresponding to small deformations of a rectilinear wall. We also verify, by coupling the field to an external gauge field, that the massless fermion mode is properly taken into account in our approach.

Then, in Sec. IV we deal with the situation of a static mass that changes sign on two spatial curves, also having the same constant value elsewhere. We compute exactly the interaction energy for the particular case of two straight lines, showing that the force between defects is always attractive. A perturbative expansion to treat cases when the shape of one of the rectilinear walls is slightly perturbed can be implemented. An Appendix contains the details corresponding to the first- and second-order terms.

Section V contains our conclusions.

II. THE SYSTEM

The system that we consider in this work is defined in terms of an Euclidean action $\mathcal{S}$, given by

$$\mathcal{S}(\overline{\psi},\psi,M)=\int d^3x\overline{\psi}(x)\mathcal{D}\psi(x), \tag{1}$$

with $\mathcal{D}\equiv\not{\partial}+M(\mathbf{x})$, for a Dirac field $\psi(x)$, $\overline{\psi}(x)$ in the presence of a space-dependent mass $M(\mathbf{x})$, in $2+1$ dimensions. We have adopted the convention that x denotes the three Euclidean spacetime coordinates, x_0, x_1, x_2, while $\mathbf{x}$ corresponds to just x_1 and x_2.

In the representation we adopt for Dirac's algebra, γ_μ, $\mu=0,\ 1,\ 2$, are Hermitian 2×2 matrices, satisfying $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$. Indices from the middle of the greek alphabet, like μ, ν, …, run from 0 to 2, while roman ones can take the values 1 and 2. The Dirac field has two spinorial components, and it can be used as a building block for higher, reducible representations (this is indeed the usual situation in graphene models).

Regarding the specific form of the mass $M(\mathbf{x})$, we shall restrict ourselves in this work to configurations such that $|M(\mathbf{x})|=m=$ constant almost everywhere, changing sign along a two-dimensional spacetime region $\mathcal{U}$ which, for static domain walls, has the form

$$\mathcal{U}=\mathcal{C}\times\mathbb{R}, \tag{2}$$

where $\mathcal{C}$ is a one-dimensional region contained on $\mathbb{R}^2$, the x_1, x_2 plane. For static configurations, $\mathcal{C}$ is assumed, in this paper, to correspond to either a single curve, or to two disjoint regular curves. In the next sections, we discuss these two cases separately.

We conclude this section by introducing the effective action, $\Gamma(M)$:

$$e^{-\Gamma(M)}=\mathcal{Z}(M), \tag{3}$$

where

$$\mathcal{Z}(M)=\int\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-\mathcal{S}(\overline{\psi},\psi,M)} \tag{4}$$

is the Euclidean vacuum-to-vacuum transition amplitude. From $\Gamma(M)$ one can obtain the vacuum energy E:

$$E=\lim_{T\to\infty}\frac{\Gamma(M)}{T}, \tag{5}$$

where T denotes the extent of the time-like coordinate x_0 (regarded temporarily as finite but tending to infinity to extract the vacuum energy).

We also note that $\mathcal{Z}(M)$ can be formally written in terms of a fermionic determinant; indeed,

$$\mathcal{Z}(M)=\det\mathcal{D}. \tag{6}$$

Note that the effective action is real; indeed, since we work in the Euclidean formalism, any imaginary part of $\Gamma(M)$, had it existed, should have been parity violating (since the imaginary part of the action is parity violating). But no parity-violating functional (either local or nonlocal) can be constructed in terms of just a scalar function and its derivatives. On the other hand, having in mind its application to graphene, the reducible representations used in that context for the fermions are such that the action is explicitly real.

Thus, having the above discussion in mind, we can write

$$\begin{aligned}\mathcal{Z}(M)&=\det\mathcal{D}=\det\mathcal{D}^\dagger=[\det\mathcal{D}^\dagger\det\mathcal{D}]^{1/2}\\&=(\det\mathcal{H})^{1/2}=(\det\check{\mathcal{H}})^{1/2},\end{aligned} \tag{7}$$

where

$$\mathcal{H}\equiv\mathcal{D}^\dagger\mathcal{D},\qquad\check{\mathcal{H}}\equiv\mathcal{D}\mathcal{D}^\dagger. \tag{8}$$

Taking into account our assumptions about $M(x)$, we see that

$$\begin{aligned}\mathcal{H}&=-\partial^2+m^2-\not{\partial}M(x),\\\check{\mathcal{H}}&=-\partial^2+m^2+\not{\partial}M(x).\end{aligned} \tag{9}$$

Since results that can be expressed in terms of Γ are independent of the sign of M, one can work with either $\mathcal{H}$ or $\check{\mathcal{H}}$; in the remainder of this paper, we use the former. Thus,

$$\Gamma(M) = -\frac{1}{2}\log\det\mathcal{H} = -\frac{1}{2}\mathrm{Tr}\log\mathcal{H}. \quad (10)$$

III. A SINGLE DOMAIN WALL

We consider here a time-independent M, i.e., $M = M(\mathbf{x})$, which has a domain wall defect along a single closed curve $\mathcal{C}$. In other words, $M(\mathbf{x})$ jumps from $-m$ to $+m$ when crossing $\mathcal{C}$ (physical observables, like the vacuum energy, are independent of the sign of the jump).

In order to gain some insight into the nature of the system, let us first define the domain wall implicitly, in terms of a *smooth* function $F(\mathbf{x})$, assumed to vanish with a nonzero gradient on $\mathcal{C}$. Then we set $M(\mathbf{x}) = m\sigma(F(\mathbf{x}))$ ($m \geq 0$), where σ denotes the sign function. Thus, we see that the operator $\mathcal{H}$ introduced in Eq. (9) has the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \quad (11)$$

with

$$\mathcal{H}_0 = -\partial^2 + m^2, \qquad \mathcal{H}_I = -2m\delta[F(\mathbf{x})]\gamma_j\partial_j F(\mathbf{x}), \quad (12)$$

where δ denotes Dirac's δ function, and $\mathcal{H}_0$ is proportional to the (omitted) 2×2 unit matrix. The appearance of a δ function is a consequence of the assumption of zero width for the domain wall.

By an application of the co-area formula, we note that $\mathcal{H}_I$ can also be rendered in the form $\mathcal{H}_I(\mathbf{x}) = \gamma_i\mathcal{A}_i(\mathbf{x})$, where

$$\mathcal{A}_i(\mathbf{x}) = -2m\int d\tau\left|\frac{d\mathbf{y}(\tau)}{d\tau}\right|\delta[\mathbf{x}-\mathbf{y}(\tau)]\hat{n}_i(\tau), \quad (13)$$

where $\tau \to \mathbf{y}(\tau)$ is a parametrization of $\mathcal{C}$, and $\hat{n}_i(\tau)$ is the ith component of $\hat{\mathbf{n}}(\tau)$, the unit normal[1] to $\mathcal{C}$ at the point $\mathbf{y}(\tau)$. Thus,

$$\mathcal{A}_i(\mathbf{x}) = \epsilon_{ij}\chi_j(\mathbf{x}),$$
$$\chi_i(\mathbf{x}) = -2m\int d\tau\delta[\mathbf{x}-\mathbf{y}(\tau)]\frac{dy_i(\tau)}{d\tau}. \quad (14)$$

A. Effective action and self-energy

As it should have been expected, it is far from trivial to calculate the effective action (and therefore the self-energy) exactly for an arbitrary closed curve $\mathcal{C}$. Besides, one should expect the existence of infinities, due to the assumption that the defects have zero width. These infinities can nevertheless be regulated by the introduction of a UV cutoff which, in coordinate space, may be regarded as a nonvanishing width for the defect. In the next section, when dealing with the interaction between two defects, no cutoff dependence is expected in the interaction energy, since this object is independent of the respective self-energies.

One can attempt to implement different approximation schemes in order to calculate Γ; the choice is determined, in the end, by the nature of the configuration being studied. The case we consider here corresponds to small-amplitude deviations with respect to a linear defect. More precisely, we assume that $F(\mathbf{x}) = x_2 - \varphi(x_1)$, expanding the effective action in powers of φ, which is the deviation with respect to the $x_2 = 0$ straight line.

We first note that, with this choice of F

$$\mathcal{H}_I = -2m\delta(x_2-\varphi(x_1))(-\varphi'(x_1)\gamma_1+\gamma_2), \quad (15)$$

where a prime denotes a derivative with respect to the argument. Denoting now by Γ_i the order-i term in the expansion for Γ,

$$\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \cdots, \quad (16)$$

we also have the corresponding expansion for the energy:

$$E = E_0 + E_1 + E_2 + \cdots, \quad (17)$$

where E_0 amounts to an uninteresting infinite constant independent of φ, which can be interpreted as coming from a linear energy density. The divergence is present already at the level of the energy density, which is cutoff dependent.

Regarding the first- and second-order terms, we see that, as a consequence of assuming that the functional expansion exists, they can be written as follows:

$$E_1 = \int dx_1\mathcal{E}^{(1)}(x_1)\varphi(x_1),$$
$$E_2 = \frac{1}{2}\int dx_1\int dy_1\mathcal{E}^{(2)}(x_1,y_1)\varphi(x_1)\varphi(y_1). \quad (18)$$

Since the functional expansion coefficients $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are independent of φ, they must be translation-invariant objects. Thus, $\mathcal{E}^{(1)} =$ constant and $\mathcal{E}^{(2)}(x_1,y_1) = \mathcal{E}^{(2)}(x_1-y_1)$.

The first-order term then has the form

$$E_1 = \mathcal{E}^{(1)}\int dx_1\varphi(x_1); \quad (19)$$

namely, it depends only on the average value of the deformation φ. Since this average value could be changed just by performing a rigid translation of the defect along the x_2 direction, and the energy cannot change under such a shift, we conclude that $\mathcal{E}^{(1)}$ (and therefore E_1) vanishes. We have checked this explicitly by evaluating Γ_1 which, recalling Eq. (10), is given by

[1]Results for Γ are independent of the choice (inwards or outwards) for the normal.

$$\Gamma_1 = -\frac{1}{2}\mathrm{Tr}[(\mathcal{H}_0)^{-1}\mathcal{H}_1]. \tag{20}$$

Here, $\mathcal{H}_l$ consistently denotes the order-l term in an expansion of $\mathcal{H}_l$. The ones appearing here are

$$\begin{aligned}\mathcal{H}_0 &= -\partial^2 + m^2 - 2m\gamma_2\delta(x_2),\\ \mathcal{H}_1 &= 2m[\gamma_1\delta(x_2)\varphi'(x_1) + \gamma_2\delta'(x_2)\varphi(x_1)].\end{aligned} \tag{21}$$

The inverse of $\mathcal{H}_0$ is given by

$$\begin{aligned}(\mathcal{H}_0)^{-1}(x,y) &= \int \frac{d^2k_\parallel}{(2\pi)^2}\, e^{ik_\parallel\cdot(x_\parallel - y_\parallel)} G(k_\parallel; x_2, y_2),\\ G(k_\parallel; x_2, y_2) &= G^+(k_\parallel; x_2, y_2)\mathcal{P}^+ + G^-(k_\parallel; x_2, y_2)\mathcal{P}^-,\\ G^\pm(k_\parallel; x_2, y_2) &= \frac{1}{2\Omega(k_\parallel)}\left[e^{-\Omega(k_\parallel)|x_2 - y_2|}\right.\\ &\quad \left. \pm \frac{m}{\Omega(k_\parallel)\mp m}\, e^{-\Omega(k_\parallel)(|x_2| + |y_2|)}\right],\end{aligned} \tag{22}$$

with $\mathcal{P}^\pm = \frac{1\pm\gamma_2}{2}$, $v_\parallel \equiv (v_0, v_1)$, and $\Omega(k_\parallel) = \sqrt{k_\parallel^2 + m^2}$.

Then, after some algebra, we find

$$\begin{aligned}\mathcal{E}^{(1)} &= -m\int dx_2 \int \frac{d^2k_\parallel}{(2\pi)^2}\\ &\quad \times \{[G^+(k_\parallel; x_2, y_2) - G^-(k_\parallel; x_2, y_2)]\delta'(y_2)\}|_{y_2\to x_2}\\ &= 0,\end{aligned} \tag{23}$$

as expected.

We conclude this section by dealing with the second-order term, which is made up of two contributions:

$$\begin{aligned}\Gamma_2 &= \Gamma_{2,1} + \Gamma_{2,2},\\ \Gamma_{2,1} &= -\frac{1}{2}\mathrm{Tr}[(\mathcal{H}_0)^{-1}\mathcal{H}_2],\\ \Gamma_{2,2} &= \frac{1}{4}\mathrm{Tr}[(\mathcal{H}_0)^{-1}\mathcal{H}_1(\mathcal{H}_0)^{-1}\mathcal{H}_1].\end{aligned} \tag{24}$$

The first contribution involves $\mathcal{H}_2$,

$$\mathcal{H}_2 = -m[\gamma_1\delta'(x_2)\partial_1(\varphi(x_1))^2 + \gamma_2\delta''(x_2)(\varphi(x_1))^2], \tag{25}$$

and yields an energy with the form

$$E_{2,1} = \mathcal{E}^{(2,1)}\int dx_1(\varphi(x_1))^2, \tag{26}$$

where

$$\begin{aligned}\mathcal{E}^{(2,1)} &= \frac{m}{2}\int dx_2 \int \frac{d^2k_\parallel}{(2\pi)^2}\\ &\quad \times \{[G^+(k_\parallel; x_2, y_2) - G^-(k_\parallel; x_2, y_2)]\delta''(y_2)\}|_{y_2\to x_2}.\end{aligned} \tag{27}$$

An explicit evaluation shows that the object above is quadratically divergent in the ultraviolet, namely, introducing a UV cutoff Λ,

$$\mathcal{E}^{(2,1)} = c\frac{(m\Lambda)^2}{2}, \tag{28}$$

where c is a dimensionless constant, which depends upon the regularization approach. Thus, we conclude that the role of this term amounts to introducing a mass density proportional to $(m\Lambda)^2$ for the collective degree of freedom.

The remaining term, $E_{2,2}$, can be evaluated and represented in Fourier space, with the result being a nonlocal quadratic functional in φ:

$$E_{2,2} = \frac{1}{2}\int \frac{dk_1}{2\pi}\tilde{\mathcal{E}}^{(2,2)}(k_1)|\tilde{\varphi}(k_1)|^2. \tag{29}$$

From this expression, we can extract its local piece, quadratic in derivatives, which is logarithmically divergent:

$$E_{2,2} = m^2\int \frac{d^2k_\parallel}{k_\parallel^2}\int dx_1(\varphi'(x_1))^2, \tag{30}$$

which may be thought of as generating a "tension" for the domain wall. Note that the existence of an infrared divergence in the momentum integral can only proceed from the existence of a massless field, which we can readily identify here as corresponding to the one predicted by the Callan-Harvey mechanism.

The aim of the previous calculation was to illustrate that, in an effective theory describing the dynamics of the wall, vacuum fluctuations renormalize both the mass and the tension of the defect. A complete calculation in the framework of a renormalizable field theory should include a quantum scalar field whose classical part is responsible for a smooth domain wall. The limit of a sharp domain wall for the self-energy could be ill defined [11].

B. Coupling to an external gauge field

When coupling the Dirac field to an external Abelian gauge field A_μ, we have to perform the following change in the operator $\mathcal{D}$:

$$\mathcal{D} \to \not{\partial} + ie\not{A}(x) + M(\mathbf{x}). \tag{31}$$

Then, assuming a rectilinear defect, the term of second order in A_μ, $\Gamma^{(2)}(A)$, will have the structure

$$\Gamma^{(2)}(A)=\frac{1}{2}\int d^3x\int d^3y A_\mu(x)\Pi_{\mu\nu}(x,y)A_\nu(y). \quad (32)$$

It is convenient to perform a Fourier transform of the objects above with respect to the x_0 and x_1 coordinates (the defect breaks translation invariance along the x_2 axis):

$$\Gamma^{(2)}(A)=\frac{1}{2}\int\frac{d^2k_\parallel}{(2\pi)^2}\int dx_2 \times\int dy_2\tilde{A}_\mu(-k_\parallel;x_2)\tilde{\Pi}_{\mu\nu}(k_\parallel;x_2,y_2)\tilde{A}_\nu(k_\parallel;y_2). \quad (33)$$

A straightforward calculation shows that the vacuum polarization tensor $\tilde{\Pi}_{\mu\nu}$ is given by

$$\tilde{\Pi}_{\mu\nu}(k_\parallel;x_2,y_2)=-e^2\int\frac{d^2p_\parallel}{(2\pi)^2} \times \mathrm{tr}[S(p_\parallel;y_2,x_2)\gamma_\mu S(p_\parallel+k_\parallel;x_2,y_2)\gamma_\nu], \quad (34)$$

with

$$S(p_\parallel;x_2,y_2)=(-\gamma_2\partial_{x_2}-i\not{p}_\parallel+m\sigma(x_2))G(p_\parallel;x_2,y_2). \quad (35)$$

Having in mind to study the response of the system to an electric field along the direction and location of the defect, we consider the components $\tilde{\Pi}_{\alpha\beta}$, with α,β in the 0,1 range, and set $x_2=y_2=0$. After a rather lengthy calculation, we see that the only surviving contributions to the vacuum polarization lead to

$$\tilde{\Pi}_{\mu\nu}(k_\parallel;x_2,x_2)=\frac{e^2}{4}\int\frac{d^2p_\parallel}{(2\pi)^2}[p_\alpha(p+k)_\beta+p_\beta(p+k)_\alpha -\delta_{\alpha\beta}p_\parallel\cdot(p_\parallel+k_\parallel)] \times\left[\frac{1}{(\Omega(p_\parallel)-m)(\Omega(p_\parallel+k_\parallel)-m)} +\frac{1}{(\Omega(p_\parallel)+m)(\Omega(p_\parallel+k_\parallel)+m)}\right]. \quad (36)$$

It can be seen that the leading contribution proceeds from the first term on the second line above. In particular, for large m

$$\tilde{\Pi}_{\mu\nu}(k_\parallel;x_2,x_2) \sim\left(\frac{em}{2}\right)^2\int\frac{d^2p_\parallel}{(2\pi)^2} \times\frac{p_\alpha(p+k)_\beta+p_\beta(p+k)_\alpha-\delta_{\alpha\beta}p_\parallel\cdot(p_\parallel+k_\parallel)}{p_\parallel^2(p_\parallel+k_\parallel)^2}, \quad (37)$$

which is the expression for the vacuum polarization in $1+1$ dimensions, due to a massless fermion field. Thus we have verified, in a concrete example, the presence of this mode in our treatment of the problem.

IV. TWO DOMAIN WALLS

In this section, the mass is assumed to have a purely spatial dependence, with two domain-wall like defects, i.e., zero-width regions of space where the mass passes through zero. One of those regions will be assumed to correspond to a straight line, hereafter denoted by L, defined by $x_2=0$. The other defect, R, is assumed to correspond to a curve which represents a small departure from a line which is parallel to L. We assume that it can be defined in terms of a single function $\varphi(x_1)$, which specifies the distance, along x_2, of each point in R to L. Thus,

$$M(\mathbf{x})=m\sigma(x_2)\sigma(x_2-\varphi(x_1)), \quad (38)$$

where, as before, σ denotes the sign function. m is a positive constant which defines the constant value of the absolute value of $M(\mathbf{x})$, as well as half the height of the jump in the mass at each defect.

The assumption about R being a small departure from a straight line parallel to L is made more precise by assuming that the above introduced function $\varphi(x_1)$ can be written as $\varphi(x_1)=a+\eta(x_1)$, with $a>0$ and $|\eta(x_1)|\ll a$.

The effective action $\Gamma(M)$ can then be expanded in powers of η,

$$\Gamma=\Gamma_0+\Gamma_1+\Gamma_2+\cdots, \quad (39)$$

where the index denotes the order in η of the corresponding term. We will evaluate here the zeroth order, and describe the calculation of the first and second orders in the Appendix.

The zeroth order corresponds to setting $\eta=0$, so that the walls L and R will be located at $x_2=0$ and $x_2=a$, respectively. The system then has translation invariance along x_1, as well as time independence. The effective action to this order will then diverge, since it will be proportional to the extent of the time interval, T, and to L_1, the length of the system along x_1, which should tend to infinity. As usual, one can take care of this divergence by considering the effective action per unit time and per unit length, a quantity which we shall denote by $\mathcal{E}_0(a)$ and which has the dimensions of energy per unit length. This quantity—a function of a and m—contains the information about the interaction energy between the two domain walls, in particular on the part of that function which does depend on a. Self-energy contributions are a independent and will be discarded. In other words, since the force per unit length between L and R is proportional to (minus) the derivative of Γ_0 with respect to a, we only keep the terms which contribute to that observable.

$\Gamma_0(a)$ is formally given by a functional determinant:

$$e^{-\Gamma_0(a)} = \det[\not{\partial} + m(x_2)] = \det[-\not{\partial} + M_0(x_2)], \quad (40)$$

where the second equality is a consequence of the reality of the energy.

Then,

$$e^{-\Gamma_0(a)} = \{\det[(-\not{\partial} + M_0(x_2))(\not{\partial} + M_0(x_2))]\}^{\frac{1}{2}}. \quad (41)$$

Then we see, by Fourier transforming the dependence on the x_0 and x_1 coordinates, that

$$\mathcal{E}_0(a) = \lim_{T,L_1\to\infty} \frac{\Gamma_0}{TL_1} = -\frac{1}{2}\int \frac{d^2k_\parallel}{(2\pi)^2} \log\det \mathcal{K}, \quad (42)$$

where $k_\parallel \equiv (k_0, k_1)$, and $\mathcal{K}$ denotes a functional matrix operator acting on functions of x_2:

$$\begin{aligned}\mathcal{K} &= -\partial_2^2 + k_\parallel^2 + m^2 - \gamma_2\partial_2 M_0(x_2) \\ &= -\partial_2^2 + \Omega^2(k_\parallel) + 2m\gamma_2[\delta(x_2) - \delta(x_2 - a)]. \end{aligned} \quad (43)$$

As expected, the problem has been reduced to the calculation of a reduced fermionic determinant involving a nontrivial dependence on x_2 only. Besides, the 2×2 matrix structure can be straightforwardly dealt with, decomposing the problem into two scalar ones:

$$\begin{aligned}\mathcal{E}_0(a) = -\frac{1}{2}\int \frac{d^2k_\parallel}{(2\pi)^2} \{&\log(-\partial_2^2 + \Omega^2(k_\parallel) + 2m[\delta(x_2) \\ &- \delta(x_2 - a)]) + \log(-\partial_2^2 + \Omega^2(k_\parallel) \\ &- 2m[\delta(x_2) - \delta(x_2 - a)])\}. \end{aligned} \quad (44)$$

Therefore, we reduced the calculation of the interaction energy between domain walls mediated by a Dirac field, to that of computing the Casimir energy for scalar fields in the presence of two δ potentials of different strength. This calculation has been previously considered elsewhere [12,13]. Since it involves operators acting nontrivially on only one coordinate, they can be evaluated using the Gelfand-Yaglom theorem, in an identical fashion to the one presented in Ref. [12]. Following the method applied in that reference, we get the same contribution for each scalar problem (each one is independent of the sign of m). The expression for the energy density thus becomes

$$\mathcal{E}_0(a) = -\int \frac{d^2k}{(2\pi)^2} \log\left[1 + \frac{m^2}{k_\parallel^2} e^{-2\Omega(k_\parallel)a}\right] + \cdots, \quad (45)$$

where the ellipsis means "modulo terms independent of a." Indeed (see Ref. [12]), the energy is obtained by integrating the force, and therefore possibly infinite terms independent of a are neglected.

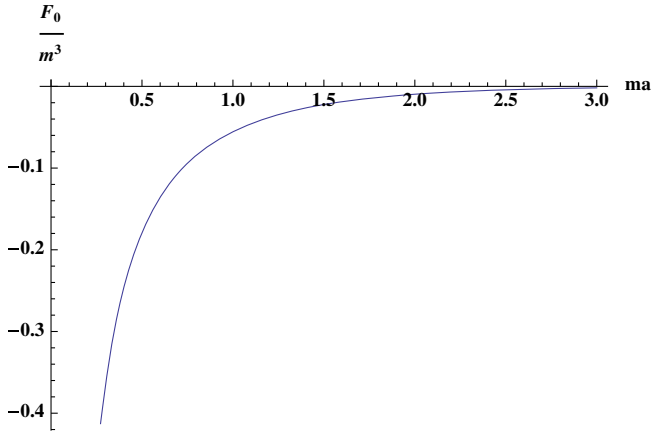

FIG. 1. Dimensionless force per unit length $\mathcal{F}_0/m^3$ as a function of the dimensionless distance ma between planar domain walls.

From the interaction energy density we can compute the force per unit length $\mathcal{F}_0 = -\partial\mathcal{E}_0/\partial a$, which is unambiguously defined and reads

$$\frac{\mathcal{F}_0(a)}{m^3} = -\frac{1}{2\pi x^3}\int_0^\infty du \frac{\sqrt{u+x^2}}{1 + \frac{u}{x^2} e^{2\sqrt{u+x^2}}}, \quad (46)$$

where $x = ma$. Dividing the force by m^3, we obtain a dimensionless function that depends on the dimensionless variable ma. As the explicit form of the integral cannot be given analytically, in Fig. 1 we present a plot of the force per unit length. Note that it is a monotonous function of the distance, and that it is always attractive.

The behaviors of the force both at short ($ma \ll 1$) and large ($ma \gg 1$) distances can be studied analytically. At short distances, one can expand the integrand in Eq. (46) for $x \ll 1$ and then perform the integral. The result is that $\mathcal{F}_0(a)/m^3$ diverges as $-1/(2\pi ma)$ in this limit. In the opposite limit, the force vanishes exponentially as $-ma\exp(-2ma)/\pi$. This exponential behavior is typical for the vacuum force associated to massive fields. It can be obtained analytically by approximating $\sqrt{u+x^2} \simeq x$ in Eq. (46), performing the integral up to a maximum value of u of order one, and then expanding the result for $x \gg 1$.

It is interesting to remark that the structure of the result for the vacuum force between domain walls, Eq. (45), is similar to those obtained for the Casimir effect for massive fermions between planar boundaries [14] or in the presence of δ potentials [15].

It is at first sight surprising that the force between domain walls diverges as $a \to 0$ since, naively, the δ potential in Eq. (44) vanishes in this limit. Although this would certainly be the case for smeared potentials, it is not true in the presence of δ functions. A simple example illustrates an analogous situation: when considering the electrostatic interaction between point charges $+q$ and $-q$

separated by a distance a, both the energy and the force diverge as $a \to 0$, even though there are no charges at all when a strictly vanishes. Of course everything would be smooth when considering spheres of nonvanishing radii instead of point charges. In our case, the use of δ functions to model the domain walls is a mere idealization of the actual situation. From a physical point of view, the domain walls will have a nonvanishing width σ. Therefore, at distances $a \sim \sigma$ the use of smeared potentials is unavoidable, and the results presented here are valid only in the limit $a \gg \sigma$.

V. CONCLUSIONS

We have computed the effects of quantum fluctuations of a Dirac scalar field in $2+1$ dimensions on domain wall defects. For a single defect, the vacuum energy is highly divergent. We can understand the origin of the divergences as follows. Had we considered a theory in which the fermion field is coupled to a dynamical scalar field whose classical part generates a smooth domain wall, the vacuum polarization of the fermion field would produce a renormalization of the mass of the scalar field along with a finite correction to the mass of the domain wall. In the present paper, there is an additional source of divergences, because we are assuming a zero-width domain wall. For a nonplanar wall, we have found that the vacuum energy contains divergences that are proportional to φ^2 and to φ'^2. This indicates that in a dynamical model for the wall, there would be a renormalization of the mass and of the tension of the defect. We have seen that part of this renormalization appears to be due to the fluctuations of the fermionic zero mode. We have verified this in an independent fashion: by computing the vacuum polarization tensor on the domain wall, we found that the virtual effects due to this mode are, indeed, present.

For two domain walls, we have shown that vacuum energy induces a Casimir-like force between defects. For planar walls, the force could be computed using standard techniques based on the Gelfand-Yaglom theorem; the result shows that it is always attractive. At short distances, it is inversely proportional to the distance, while it vanishes exponentially at large distances. The divergences that occur in the vacuum energy for a single defect are not present in the force, which is moreover unambiguously defined.

We remark that in the graphene case one should multiply our result corresponding to the attractive force by the proper number of two-component fermions.

We have also obtained explicit expressions for the interaction energy between a planar wall and a slightly deformed wall (see the Appendix). As for the usual Casimir effect, in this case the energy is a nonlocal functional of the deformation.

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APPENDIX: SECOND ORDER EXPANSION OF THE EFFECTIVE ACTION

In this appendix we compute the first- and second-order terms in the expansion of the effective action given in Eq. (39).

1. First order

The calculation of the first-order term does not give a new result, but it can be used as a consistency check for the previous calculation. Indeed, the contribution of first order in η has the form

$$\Gamma_1 = -\mathrm{Tr}[(\not{\partial} + M(x_2))^{-1} M_1], \tag{A1}$$

where $M_1(\mathbf{x}) = -2m\delta(x_2 - a)\eta(x_1)$. By taking the functional trace, after some algebra we see that the first-order term in units of energy per unit length is

$$\begin{aligned}\mathcal{E}_1 &= \lim_{T,L\to\infty} \frac{\Gamma_1}{TL_1} \\ &= 2m\eta_0 \int \frac{d^2k_\parallel}{(2\pi)^2} \\ &\quad \times \mathrm{tr}[\langle x_2|(\gamma_2\partial_2 + i\not{k}_\parallel + M_0(x_2))^{-1}|y_2\rangle]|_{x_2=y_2=a},\end{aligned} \tag{A2}$$

where "tr" denotes the trace over Dirac indices. On the other hand, $\eta_0 \equiv \frac{1}{L_1}\int dx_1 \eta(x_1)$ is the mean value of η.

Since the previous expression depends on η only through the constant η_0, it is not sensitive to the details of its local space dependence. Therefore, it can be obtained from the zeroth-order expression. Indeed, one should have the relation

$$\mathcal{E}_1 = \mathcal{E}_0(a + \eta_0) - \mathcal{E}_0(a) + \mathcal{O}(\eta_0^2), \tag{A3}$$

so that the first-order term we are about to calculate should be compared with the one obtained by evaluating the derivative of the zeroth-order term with respect to a and multiplying by η_0.

One can show that

$$\begin{aligned}&\langle x_2|(\gamma_2\partial_2 + i\not{k}_\parallel + M_0(x_2))^{-1}|y_2\rangle \\ &\quad = [-\gamma_2\partial_{x_2} - i\not{k}_\parallel + M_0(x_2)]\langle x_2|\mathcal{K}^{-1}|y_2\rangle,\end{aligned} \tag{A4}$$

where $\mathcal{K}$ is the operator introduced in the calculation of the zeroth-order term.

The inverse of the scalar operator above can be obtained by using standard techniques, and the result obtained by inserting it into the expression for the first-order term is consistent with the relation obtained between it and the derivative of the zeroth-order term.

2. Second order

The second order-term Γ_2 receives two different contributions:

$$\Gamma_2 = \Gamma_2^a + \Gamma_2^b, \tag{A5}$$

where

$$\Gamma_2^a = \frac{1}{2}\mathrm{Tr}[(\not{\partial} + M_0(x_2))^{-1} M_1 (\not{\partial} + M_0(x_2))^{-1} M_1] \tag{A6}$$

and

$$\Gamma_2^b = -\mathrm{Tr}[(\not{\partial} + M_0(x_2))^{-1} M_2]. \tag{A7}$$

It may be seen that Γ_2^b can, like the first-order term, be derived from the knowledge of the zeroth-order term. In other words, it is only sensitive to the average value of η. Thus, we shall concentrate on Γ_2^a, since it is the only one that contains new information to this order.

We see that

$$\Gamma_2^a = \frac{1}{2}(2m)^2 \int_{x_\parallel y_\parallel} \mathrm{tr}[\langle x|(\not{\partial} + M_0(x_2))^{-1}|y\rangle \eta(y_1) \times \langle y|(\not{\partial} + M_0(x_2))^{-1}|x\rangle \eta(x_1)]|_{x_2=y_2=a}. \tag{A8}$$

The system is now time independent but translation invariance along x_1 is not necessarily preserved. Thus, Γ_2^a will produce a contribution to the energy (total, not the linear density) E_2, which in Fourier space can be written as follows:

$$E_2 = \frac{1}{2}\int dk_1 |\tilde{\eta}(k_1)|^2 f_2(k_1), \tag{A9}$$

with

$$f_2(k) = 4m^2 \int \frac{d^2 p_\parallel}{(2\pi)^2} \mathrm{tr}[\tilde{G}(p_\parallel; a, a)\tilde{G}(p_\parallel + k_\parallel; a, a)], \tag{A10}$$

where we have introduced

$$\tilde{G}(p_\parallel; x_2, y_2) = \int d^2 x_\parallel e^{-i p_\parallel \cdot x_\parallel} \langle x_\parallel, x_2|(\not{\partial} + M_0(x_2))^{-1}|0_\parallel, y_2\rangle. \tag{A11}$$

We can obtain a more explicit expression for $\tilde{G}(p_\parallel; x_2, y_2)$, as follows:

$$\tilde{G}(p_\parallel; x_2, y_2) = [(-\partial_{x_2} - i\not{p}_\parallel + M_0(x_2))\mathcal{G}_+(p_\parallel; x_2, y_2)\mathcal{P}_+ + (\partial_{x_2} - i\not{p}_\parallel + M_0(x_2))\mathcal{G}_-(p_\parallel; x_2, y_2)\mathcal{P}_-], \tag{A12}$$

where

$$\mathcal{G}^\pm(p_\parallel; x_2, y_2) = \langle x_2|[-\partial_{x_2}^2 + p_\parallel^2 \mp (\delta(x_2) - \delta(x_2 - a))]^{-1}|y_2\rangle, \tag{A13}$$

and $\mathcal{P}_\pm \equiv \frac{1\pm\gamma_2}{2}$.

A rather lengthy but otherwise straightforward calculation shows that

$$\tilde{G}(p_\parallel; x_2, y_2) = -\frac{i\not{p}_\parallel}{2\Omega(p_\parallel)} + \frac{m}{2p_\parallel^2}\frac{1 - e^{-2\Omega(p_\parallel)a}}{1 + \frac{m^2}{p_\parallel^2}e^{-2\Omega(p_\parallel)a}} \times \left[-i\not{p}_\parallel\left(\frac{m}{\Omega(p_\parallel)} - \gamma_2\right) + \Omega(p_\parallel)\frac{e^{-2\Omega a}}{1 - e^{-2\Omega a}}\right]. \tag{A14}$$

Evaluating the Dirac trace, we see that the kernel f_2 is given by

$$\begin{aligned} f_2(k_\parallel) = 2m^2 \int \frac{d^2 p_\parallel}{(2\pi)^2} \Bigg\{ & p_\parallel \cdot (p_\parallel + k_\parallel)\left[\frac{m^2}{p_\parallel^2 (p_\parallel + k_\parallel)^2}\left(1 - \frac{m^2}{\Omega(p_\parallel)\Omega(p_\parallel + k_\parallel)}\right)\right. \\ & \times \left(1 - \left(1 + \frac{m^2}{p_\parallel^2}\right)B(p_\parallel)\right)\left(1 - \left(1 + \frac{m^2}{(p_\parallel + k_\parallel)^2}\right)B(p_\parallel + k_\parallel)\right) \\ & - \frac{1}{\Omega(p_\parallel)\Omega(p_\parallel + k_\parallel)}\left(1 + \frac{m^2}{p_\parallel^2}\left(1 - \left(1 + \frac{m^2}{p_\parallel^2}\right)B(p_\parallel)\right)\right. \\ & \left.\left. + \frac{m^2}{(p_\parallel + k_\parallel)^2}\left(1 - \left(1 + \frac{m^2}{(p_\parallel + k_\parallel)^2}\right)B(p_\parallel + k_\parallel)\right)\right)\Bigg] \\ & + \frac{m^2}{p_\parallel^2 (p_\parallel + k_\parallel)^2}\Omega(p_\parallel)\Omega(p_\parallel + k_\parallel)B(p_\parallel)B(p_\parallel + k_\parallel)\Bigg\}, \end{aligned} \tag{A15}$$

where we have introduced $B(p_\parallel) = (e^{2\Omega(p_\parallel)a} + \frac{m^2}{p_\parallel^2})^{-1}$. We have checked that this kernel is indeed finite, so that the expansion is, at least up to this order, well defined.

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