Representations of Yang-Mills algebras

By Estanislao Herscovich and Andrea Solotar*

Abstract

The aim of this article is to describe families of representations of the Yang-Mills algebras $\mathrm{YM}(n)$ ($n \in \mathbb{N}_{\geq 2}$) defined in by A. Connes and M. Dubois-Violette [CD]. We first describe some irreducible finite dimensional representations. Next, we provide families of infinite dimensional representations of $\mathrm{YM}(n)$, big enough to separate points of the algebra. In order to prove this result, we prove and use that all Weyl algebras $A_r(k)$ are epimorphic images of $\mathrm{YM}(n)$.

2000 Mathematics Subject Classification: 13N10, 16S32, 17B56, 70S15, 81T13.

Keywords: Yang-Mills, orbit method, representation theory, homology theory.

1. Introduction

This article is devoted to the study of the representation theory of Yang-Mills algebras. Very little is known on this subject. Our goal is to describe families of representations which, although they do not cover the whole category of representations, are large enough to distinguish elements of the Yang-Mills algebra.

In order to describe our results in more detail, let us recall the definition of Yang-Mills algebras by A. Connes and M. Dubois-Violette in [CD]. Given a positive integer $n \geq 2$, the Lie Yang-Mills algebra over an algebraically closed field k of characteristic zero is

$$\mathfrak{ym}(n) = \mathfrak{f}(n)/\langle\{\sum_{i=1}^{n}[x_i,[x_i,x_j]] : j = 1, \dots, n\}\rangle,$$

*This work has been supported by the projects UBACYTX169 and X212, CONICET-CNRS, PICS 3410 and PIP-CONICET 5099. The first author is a CONICET fellow. The second author is a research member of CONICET (Argentina) and a Regular Associate of ICTP Associate Scheme.

where $\mathfrak{f}(n)$ is the free Lie algebra in n generators $x_1, \dots, x_n$. Its associative enveloping algebra $\mathcal{U}(\mathfrak{ym}(n))$ will be denoted $\mathrm{YM}(n)$. It is, for each n, a cubic Koszul algebra of global dimension 3 and we shall see that it is noetherian if and only if $n = 2$. The notion of N-Koszul algebra is clearly developed in [Ber1].

The first instance of Yang-Mills theory in physics is through Maxwell's equations for the charge free situation which gives a representation of the Yang-Mills equations.

In general, the Yang-Mills equations we consider are equations for covariant derivatives on bundles over the affine space $\mathbb{R}^n$ provided with a pseudo-Riemannian metric g. Any complex vector bundle of rank m over $\mathbb{R}^n$ is trivial and every connection on such bundle is given by a $M_m(\mathbb{C})$-valued 1-form $\sum_{i=1}^n A_i dx^i$. The corresponding covariant derivative is given by $\nabla_i = \partial_i + A_i$. The Yang-Mills equations for the covariant derivative are thus

$$\sum_{i,j=1}^{n} g^{i,j}[\nabla_i, [\nabla_j, \nabla_k]] = 0,$$

where $g^{-1} = (g^{i,j})$.

Also, Yang-Mills equations have been recently studied due to their applications to the gauge theory of D-branes and open string theory (cf. [Ne, Mov, Doug]).

In [HKL], the authors discuss a superized version of Yang-Mills algebras.

Our main result may be formulated as follows:

Theorem 1.1. *Given $n \geq 3$ and $r \geq 1$, the Weyl algebra $A_r(k)$ is an epimorphic image of* $\mathrm{YM}(n)$*.*

The key ingredient of the proof is the existence of a Lie ideal in $\mathfrak{ym}(n)$ which is free as Lie algebra. This ideal has already been considered by M. Movshev in [Mov]. It allows us to define morphisms from the Yang-Mills algebras onto the Weyl algebras, making use for this of the Kirillov orbit method.

Once this is achieved, the categories of representations of *all* Weyl algebras, that have been extensively studied by V. Bavula and V. Bekkert in [BB], are also representations of the Yang-Mills algebras. Thus, we provide several families of representations of the Yang-Mills algebras. However, an easy argument using Gelfand-Kirillov dimension shows that this construction does not provide all representations.

The contents of the article are as follows. In section 2 we recall the definition and elementary properties of Yang-Mills algebras and we also study the subcategory of nilpotent finite dimensional representations, describing completely those which are finite dimensional and irreducible.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

Section 3 is devoted to the description of the Lie ideal $\mathfrak{tym}(n)$. We give complete proofs of the fact that it is a free Lie algebra in itself. This involves the construction of a model of the graded associative algebra $\mathrm{TYM}(n) = \mathcal{U}(\mathfrak{tym}(n))$, which permits us to replace the bar complex of $\mathrm{TYM}(n)$ by the quasi-isomorphic bar complex of the model of $\mathrm{TYM}(n)$.

Finally, in section 4 we prove our main result, Corollary 4.5, and describe the families of representations appearing in this way.

Throughout this article k will denote an algebraically closed field of characteristic zero. Given an ordered basis $\{v_1, \dots, v_n\} \subset V$ of the k-vector space V, $\{v_1^*, \dots, v_n^*\} \subset V^*$ will denote its dual basis. Also, we shall identify $\mathfrak{g}$ with its image inside $\mathcal{U}(\mathfrak{g})$ via the canonical morphism $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$.

Given an associative or Lie k-algebra A and a subgroup G of $\mathbb{Z}$, we shall denote ${}_A^G\mathrm{Mod}$, Mod_A^G, ${}_A^G\mathrm{mod}$ and mod_A^G the categories of G-graded left and right A-modules, and finite dimensional G-graded left and right A-modules, respectively, and ${}_{\mathfrak{g}\mathrm{nil}}\mathrm{mod}$ will be the category of finite dimensional nilpotent left $\mathfrak{g}$-modules.

We will also denote ${}_k^G\mathrm{Alg}$ and ${}_k^G\mathrm{LieAlg}$ the categories of G-graded associative and Lie algebras, respectively. We notice that if G is trivial, each definition yields the non-graded case.

We would like to thank Jacques Alev, Jorge Vargas and Michel Dubois-Violette for useful comments and remarks. We are indebted to Mariano Suárez-Álvarez for a careful reading of the manuscript, suggestions and improvements.

2. Generalities and finite dimensional modules

In this first section we fix notations and recall some elementary properties of Yang-Mills algebras. We also study the category of finite dimensional representations, describing some of the irreducible ones.

Let n be a positive integer such that $n \geq 2$ and let $\mathfrak{f}(n)$ be the free Lie algebra with generators $\{x_1, \dots, x_n\}$. This Lie algebra is trivially provided with a locally finite dimensional $\mathbb{N}$-grading.

The quotient Lie algebra

$$\mathfrak{ym}(n) = \mathfrak{f}(n)/\langle\{\sum_{i=1}^{n}[x_i, [x_i, x_j]] : 1 \leq j \leq n\}\rangle,$$

is called the *Yang-Mills algebra with n generators*. This definition apparently differs from the one given in [CD], where the authors consider the quotient of the free Lie algebra $\mathfrak{f}(n)$ by the Lie ideal K generated by the relations

$$\{\sum_{i,j=1}^{n} g^{i,j}[x_i, [x_j, x_k]] : 1 \leq k \leq n\},$$

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

for $g = (g_{i,j})$ an invertible symmetric matrix in $M_n(\mathbb{R})$ and $g^{-1} = (g^{i,j})$. The matrix g uniquely defines a nondegenerate symmetric bilinear form on the complex vector space generated by $\{x_1, \dots, x_n\}$. One may choose an orthonormal basis for this bilinear form. It is not difficult to see that in fact the ideal K does not depend on the choice of the basis. So, when $k = \mathbb{C}$ our definition coincides with the one given in [CD] once we have fixed a basis such that $g = \mathrm{id}$. For some physical applications, one may be interested in choosing a basis such that g is Lorentzian.

The $\mathbb{N}$-grading of $\mathfrak{f}(n)$ induces an $\mathbb{N}$-grading of $\mathfrak{ym}(n)$, which is also locally finite dimensional. We denote $\mathfrak{ym}(m)_i$ the i-th homogeneous component, so that

$$\mathfrak{ym}(n) = \bigoplus_{j\in\mathbb{N}} \mathfrak{ym}(n)_j, \tag{2.1}$$

and put $\mathfrak{ym}(n)^l = \bigoplus_{j=1}^{l} \mathfrak{ym}(n)_j$. The Lie ideal

$$\mathfrak{tym}(n) = \bigoplus_{j\geq 2} \mathfrak{ym}(n)_j \tag{2.2}$$

will be of considerable importance in the sequel.

The universal enveloping algebra $\mathcal{U}(\mathfrak{ym}(n))$ will be denoted by $\mathrm{YM}(n)$ and it is called the *(associative) Yang-Mills algebra with n generators*. Let $V(n)$ be $\mathrm{span}_k\langle\{x_1, \dots, x_n\}\rangle$ and $R(n)$ be $\mathrm{span}_k\langle\{\sum_{i=1}^{n}[x_i,[x_i,x_j]] : 1 \leq j \leq n\}\rangle \subset V(n)^{\otimes 3}$, it turns out that

$$\mathrm{YM}(n) \simeq TV(n)/\langle R(n)\rangle.$$

We shall also consider the universal enveloping algebra of the Lie ideal $\mathfrak{tym}(n)$, which will be denoted $\mathrm{TYM}(n)$. Occasionally, we will omit the index n in order to simplify the notation if it is clear from the context.

It is easy to see that the Yang-Mills algebra $\mathrm{YM}(n)$ is a domain for any $n \in \mathbb{N}$, since it is the enveloping algebra of a Lie algebra (cf. [Dix], Corollary 2.3.9, (ii), p. 76).

The first example of Yang-Mills algebra appears when $n = 2$. We shall see in the sequel that it is in fact quite different from the other cases.

EXAMPLE 2.1. *Let $n = 2$. In this case, $\mathfrak{ym}(2)$ is isomorphic to the* Heisenberg Lie algebra $\mathfrak{h}_1$, *with generators x, y, z, and relations $[x, y] = z$, $[x, z] = [y, z] = 0$. The isomorphism is given by $x_1 \mapsto x$, $x_2 \mapsto y$.*

Alternatively, $\mathfrak{ym}(2) \simeq \mathfrak{n}_3$, where $\mathfrak{n}_3$ is the Lie algebra of strictly upper triangular 3×3 matrices with coefficients in k. The isomorphism is now given by $x_1 \mapsto e_{12}$, $x_2 \mapsto e_{23}$.

We see that $\mathrm{YM}(2)$ *is a noetherian algebra, since $\mathfrak{ym}(2)$ is finite dimensional. Furthermore, since $\mathcal{U}(\mathfrak{h}_1) \simeq A(2, -1, 0)$, the Yang-Mills algebra with*

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

two generators is isomorphic to a down-up algebra, already known to be noetherian.

We shall consider two different but related gradings on $\mathfrak{ym}(n)$. On the one hand, the grading given by (2.1) will be called the *usual grading* of the Yang-Mills algebra $\mathfrak{ym}(n)$. On the other hand, following [Mov], we shall also consider the *special grading* of the Yang-Mills algebra $\mathfrak{ym}(n)$, for which it is a graded Lie algebra concentrated in even degrees with each homogeneous space $\mathfrak{ym}(n)_j$ in degree $2j$. In this case, the Lie ideal $\mathfrak{tym}(n)$ given in (2.2) is also concentrated in even degrees (strictly greater than 2). We will see that in fact $\mathfrak{tym}(n)$ is isomorphic (as graded Lie algebras) to the graded free Lie algebra on a graded vector space $W(n)$, i.e., $\mathfrak{tym} \simeq \mathfrak{f}_{gr}(W(n))$ (cf. [Mov], [MS] and Section 3).

These gradings of the Lie algebra $\mathfrak{ym}(n)$ induce respectively the *usual grading* and the *special grading* on the associative algebra $\mathrm{YM}(n)$. This last one corresponds to taking the graded universal enveloping algebra of the graded Lie algebra $\mathfrak{ym}(n)$.

In order to understand the relation among the graded and non-graded cases we present the following proposition for which we omit the proof:

Proposition 2.2. *The following diagram of functors, where $\mathcal{O}$ denote the corresponding forgetful functors,*

$$\begin{tikzcd}
{}^{2\mathbb{Z}}_{k}\mathrm{Mod} \arrow[rr, "\mathfrak{f}_{gr}"] \arrow[dr, "T_{gr}"] \arrow[dd, "\mathcal{O}"] & & {}^{2\mathbb{Z}}_{k}\mathrm{LieAlg} \arrow[dl, "\mathcal{U}_{gr}"'] \arrow[dd, "\mathcal{O}"] \\
& {}^{2\mathbb{Z}}_{k}\mathrm{Alg} \arrow[dd, "\mathcal{O}"] & \\
{}_{k}\mathrm{Mod} \arrow[rr, "\mathfrak{f}"] \arrow[dr, "T"] & & {}_{k}\mathrm{LieAlg} \arrow[dl, "\mathcal{U}"] \\
& {}_{k}\mathrm{Alg} &
\end{tikzcd}$$

is commutative.

Remark 2.3. *The forgetful functors ${}^{2\mathbb{Z}}_{k}\mathrm{LieAlg} \to {}_{k}\mathrm{LieAlg}$ and ${}^{2\mathbb{Z}}_{k}\mathrm{Alg} \to {}_{k}\mathrm{Alg}$ preserve and reflect free objects.*

The Yang-Mills algebra is not nilpotent in general, since its lower central series

$$\mathfrak{ym}(n) = \mathcal{C}^0(\mathfrak{ym}(n)) \supset \mathcal{C}^1(\mathfrak{ym}(n)) \supset \cdots \supset \mathcal{C}^k(\mathfrak{ym}(n)) \supset \dots$$

is not finite as we will show below. However, it is *residually nilpotent*, that is, $\cap_{m\in\mathbb{N}}\mathcal{C}^m(\mathfrak{ym}(n)) = 0$. This fact is a direct consequence of the following: $\mathcal{C}^m(\mathfrak{ym}(n))$ is included in $\oplus_{j\geq m+1}\mathfrak{ym}(n)_j$, since $\mathfrak{ym}(n)$ is graded. Also

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

$\mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n))$ is a finite dimensional nilpotent Lie algebra for every $m \in \mathbb{N}_0$, since the lower central series of $\mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n))$ is

$$\mathcal{C}^0(\mathfrak{ym})/\mathcal{C}^m(\mathfrak{ym}) \supset \mathcal{C}^1(\mathfrak{ym})/\mathcal{C}^m(\mathfrak{ym}) \supset \cdots \supset \mathcal{C}^m(\mathfrak{ym})/\mathcal{C}^m(\mathfrak{ym}) = 0.$$

Let us study in detail the ideals appearing in the lower central series. Since the ideal of $\mathfrak{f}(n)$

$$I(n) = \langle \{\sum_{i=1}^{n} [x_i, [x_i, x_j]] : 1 \leq j \leq n\} \rangle$$

is homogeneous, then

$$I(n) = \bigoplus_{j \in \mathbb{N}} I(n)_j = \bigoplus_{j \in \mathbb{N}} (I(n) \cap \mathfrak{f}(n)_j).$$

We also notice that in the free Lie algebra,

$$\mathcal{C}^k(\mathfrak{f}(n)) = \bigoplus_{j \geq k+1} \mathfrak{f}(n)_j,$$

so that

$$\begin{aligned}\mathcal{C}^k(\mathfrak{ym}(n)) = \mathcal{C}^k(\mathfrak{f}(n))/(I(n) \cap \mathcal{C}^k(\mathfrak{f}(n))) &= \bigoplus_{j \geq k+1} \mathfrak{f}(n)_j/(I(n) \cap \mathfrak{f}(n)_j) \\ = \bigoplus_{j \geq k+1} \mathfrak{f}(n)_j/I(n)_j = \bigoplus_{j \geq k+1} \mathfrak{ym}(n)_j.\end{aligned}$$

Hence, there exists a canonical k-linear isomorphism $j_l : \mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)) \to \mathfrak{ym}(n)^l$. As a consequence, we see that if $\mathfrak{ym}(n)$ is not finite dimensional, then $\mathcal{C}^k(\mathfrak{ym}(n)) \neq 0$ for $k \in \mathbb{N}_0$, and $\mathfrak{ym}(n)$ is not nilpotent. We shall prove below that the Yang-Mills algebra $\mathfrak{ym}(n)$ is finite dimensional if and only if $n = 2$ (cf. Remark 3.14).

Since we are interested in studying representations of the Yang-Mills algebra we prove the following useful lemma.

LEMMA 2.4. *For each $l \in \mathbb{N}$, the surjective Lie algebra homomorphism $\pi_l : \mathfrak{ym}(n) \twoheadrightarrow \mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$, induces a surjective algebra homomorphism $\Pi_l : \mathcal{U}(\mathfrak{ym}(n)) \twoheadrightarrow \mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))$. Let $K_l = \mathrm{Ker}(\Pi_l)$. Then, the fact that $\cap_{l \in \mathbb{N}} \mathcal{C}^l(\mathfrak{ym}(n)) = 0$ implies that*

$$K = \bigcap_{l \in \mathbb{N}} K_l = 0.$$

Proof. The Poincaré-Birkhoff-Witt Theorem says that, given a Lie algebra $\mathfrak{g}$, there exists a canonical k-linear isomorphism $\gamma : S(\mathfrak{g}) \xrightarrow{\simeq} \mathcal{U}(\mathfrak{g})$ given by symmetrization (cf. [Dix], 2.4.5). We shall denote by $\epsilon_{\mathfrak{g}} : S(\mathfrak{g}) \to k$ the augmentation of $S(\mathfrak{g})$ given by the canonical projection over the field k.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

Since the functor $S(-)$ is a left adjoint to the forgetful functor of commutative k-algebras into k-modules, it preserves colimits. In particular, we obtain from the decomposition $\mathfrak{ym}(n) = \mathfrak{ym}(n)^l \oplus \mathcal{C}^l(\mathfrak{ym}(n))$ that $S(\mathfrak{ym}(n))$ is isomorphic to $S(\mathfrak{ym}(n)^l) \otimes S(\mathcal{C}^l(\mathfrak{ym}(n)))$, via the k-algebra isomorphism t induced by the k-linear map $v + w \mapsto v \otimes 1 + 1 \otimes w$, where $v \in \mathfrak{ym}(n)^l$, $w \in \mathcal{C}^l(\mathfrak{ym}(n))$. The inverse of t is multiplication $v \otimes w \mapsto vw$.

The surjective k-linear map $\pi_l : \mathfrak{ym}(n) \twoheadrightarrow \mathfrak{ym}(n)/\mathcal{C}^l\mathfrak{ym}(n))$ induces a surjective k-algebra homomorphism $P_l : S(\mathfrak{ym}(n)) \twoheadrightarrow S(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))$. Analogously, the k-linear isomorphism $j_l : \mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)) \to \mathfrak{ym}(n)^l$ induces a k-algebra isomorphism $J_l : S(\mathfrak{ym}(n)/\mathcal{C}^l\mathfrak{ym}(n))) \to S(\mathfrak{ym}(n)^l)$. The composition $J_l \circ P_l$ coincides with $(\mathrm{id}_{\mathfrak{ym}(n)^l} \otimes \epsilon_{\mathcal{C}^l(\mathfrak{ym}(n))}) \circ t$, and hence has kernel $t^{-1}(S(\mathfrak{ym}(n)^l) \otimes S_+(\mathcal{C}^l(\mathfrak{ym}(n)))) = S(\mathfrak{ym}(n)^l)S_+(\mathcal{C}^l(\mathfrak{ym}(n)))$.

On the other hand, the following diagram

$$\begin{tikzcd}
S(\mathfrak{ym}(n)) \arrow[r, "P_l"] \arrow[d, "\gamma"] & S(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))) \arrow[d, "\gamma"] \\
\mathcal{U}(\mathfrak{ym}(n)) \arrow[r, "\Pi_l"] & \mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))
\end{tikzcd}$$

is commutative.

As a consequence, $K_l = \gamma(S(\mathfrak{ym}(n)^l)S_+(\mathcal{C}^l(\mathfrak{ym}(n))))$. Whence, using the fact that γ is bijective,

$$\begin{aligned} K = \bigcap_{l\in\mathbb{N}} K_l &= \bigcap_{l\in\mathbb{N}} \gamma(S(\mathfrak{ym}(n)^l)S_+(\mathcal{C}^l(\mathfrak{ym}(n)))) \\ &= \gamma(\bigcap_{l\in\mathbb{N}} S(\mathfrak{ym}(n)^l)S_+(\mathcal{C}^l(\mathfrak{ym}(n)))) = 0. \end{aligned}$$

The last equality can be proved as follows: taking into account that

$$\big(\bigcap_{l\in\mathbb{N}} S(\mathfrak{ym}(n)^l)S_+(\mathcal{C}^l(\mathfrak{ym}(n)))\big) \cap S(\mathfrak{ym}(n)^l) = 0,$$

for all $l \in \mathbb{N}$, and, since $S(\mathfrak{ym}(n)) = \bigcup_{l\in\mathbb{N}} S(\mathfrak{ym}(n)^l)$, then

$$\bigcap_{l\in\mathbb{N}} S(\mathfrak{ym}(n)^l)S_+(\mathcal{C}^l(\mathfrak{ym}(n))) = 0.$$

□

Let

$$\psi : \mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)) \to \mathfrak{gl}(V),$$

be a representation of the quotient $\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$. It provides a representation of $\mathfrak{ym}(n)$ simply by composition with the canonical projection π_l. Given a morphism f between two representations V and W of the quotient

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

$\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$, it induces a morphism between the corresponding representations of the algebra $\mathfrak{ym}(n)$ in a functorial way. Hence, it yields a k-linear functor

$$I_l : {}_{\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))}\mathrm{Mod} \to {}_{\mathfrak{ym}(n)}\mathrm{Mod},$$

which also restricts to the full subcategories of finite dimensional modules, denoted by i_l. Moreover, since the map $\pi_l : \mathfrak{ym}(n) \to \mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$ is onto, the change-of-rings functors are fully faithful.

Analogously, given $l, m \in \mathbb{N}$, such that $l \leq m$, the homomorphism of Lie algebras $\pi_{l\leq m} : \mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n)) \to \mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$ induced by the canonical projection gives a k-linear functor

$$I_{l\leq m} : {}_{\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))}\mathrm{Mod} \to {}_{\mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n))}\mathrm{Mod},$$

which restricts to the full subcategories of finite dimensional modules. We shall denote this restriction by $i_{l\leq m}$. It is clear that $I_{m\leq p} \circ I_{l\leq m} = I_{l\leq p}$ and $I_m \circ I_{l\leq m} = I_l$.

Remark 2.5. *The functors $I_{l\leq m}$ and I_l ($l, m \in \mathbb{N}$) preserve irreducible modules.*

The following proposition concerning the categories of finite dimensional modules is easy to prove.

Proposition 2.6. *Let $\phi : \mathfrak{ym}(n) \to \mathfrak{gl}(V)$ be a finite dimensional nilpotent representation of $\mathfrak{ym}(n)$. Then, there exist $m \in \mathbb{N}$ and a homomorphism of Lie algebras $\phi' : \mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n)) \to \mathfrak{gl}(V)$, such that $\phi = \phi' \circ \pi_m$.*

Proof. Since $\mathrm{Im}(\phi)$ is a finite dimensional nilpotent subalgebra of $\mathfrak{gl}(V)$, $\mathrm{Ker}(\phi)$ is finite codimensional and there exists $m \in \mathbb{N}$ such that $\mathrm{Ker}(\phi) \supset \oplus_{j\geq m}\mathfrak{ym}(n)_j$. Since $\oplus_{j\geq m}\mathfrak{ym}(n)_j \supset \mathcal{C}^m(\mathfrak{ym}(n))$, we get $\mathrm{Ker}(\phi) \supset \mathcal{C}^m(\mathfrak{ym}(n))$. Therefore ϕ induces a morphism

$$\phi' : \mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n)) \to \mathfrak{gl}(V)$$

satisfying $\phi = \phi' \circ \pi_m$. □

Corollary 2.7. *The category ${}_{\mathfrak{ym}(n)\,nil}\mathrm{mod}$ of nilpotent finite dimensional modules over $\mathfrak{ym}(n)$ is the filtered colimit in the category of k-linear categories of the categories ${}_{\mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n))}\mathrm{mod}$ of finite dimensional modules over $\mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n))$.*

Proof. Let $\mathcal{C}$ be a k-linear category and let $F_l : {}_{\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))}\mathrm{mod} \to \mathcal{C}$ be a collection of k-linear functors indexed by $l \in \mathbb{N}$ satisfying that $F_m \circ I_{l\leq m} = F_l$, for $l, m \in \mathbb{N}$, $l \leq m$. We shall define a k-linear functor $F : {}_{\mathfrak{ym}(n)\mathrm{nil}}\mathrm{mod} \to \mathcal{C}$.

If M is a finite dimensional nilpotent representation of $\mathfrak{ym}(n)$ given by $\phi : \mathfrak{ym}(n) \to \mathfrak{gl}(M)$, using Proposition 2.6 we see that there exists $l \in \mathbb{N}$

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

such that $\phi = \phi_l \circ \pi_l$ with $\phi_l : \mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)) \to \mathfrak{gl}(M)$, and hence M can be considered as a module over $\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$, denoted M_l. We define $F(M) = F_l(M_l)$.

The functor F is well-defined. Suppose that for another $m \in \mathbb{N}$ there exists $\phi_m : \mathfrak{ym}(n)/\mathcal{C}^m(\mathfrak{ym}(n)) \to \mathfrak{gl}(M)$ such that $\phi = \phi_m \circ \pi_m$, then for $m \geq l$ there is a diagram

$$\begin{tikzcd}
 & \mathfrak{ym}/\mathcal{C}^m(\mathfrak{ym}) \arrow[dr, "\phi_m"] \arrow[ddd, "\pi_{l\leq m}"'] & \\
 & & \mathfrak{gl}(M) \\
\mathfrak{ym} \arrow[uur, "\pi_m"] \arrow[urr, "\phi"] \arrow[dr, "\pi_l"] & & \\
 & \mathfrak{ym}/\mathcal{C}^l(\mathfrak{ym}) \arrow[uur, "\phi_l"'] &
\end{tikzcd}$$

From the definitions, all faces are commutative except maybe the one implying $\phi_l \circ \pi_{l\leq m} = \phi_m$. Since π_m is surjective, the previous equality is satisfied if and only if $\phi_l \circ \pi_{l\leq m} \circ \pi_m = \phi_m \circ \pi_m$. But

$$\phi_l \circ \pi_{l\leq m} \circ \pi_m = \phi_l \circ \pi_l = \phi = \phi_m \circ \pi_m,$$

so $M_m = I_{k\leq m}(M_k)$ and $F_m(M_m) = F_m \circ I_{l\leq m}(M_l) = F_l(M_l)$.

Let M and N be two finite dimensional nilpotent representations of $\mathfrak{ym}(n)$ by means of $\phi : \mathfrak{ym}(n) \to \mathfrak{gl}(M)$ and $\psi : \mathfrak{ym}(n) \to \mathfrak{gl}(N)$ and let $f : M \to N$ be a module homomorphism. By Proposition 2.6 there exists $l \in \mathbb{N}$ such that $\phi = \phi_l \circ \pi_l$ and $\psi = \psi_l \circ \pi_l$. As before, we shall denote M_l and N_l these modules. It follows directly from the definitions that f is also a morphism of $\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$-modules, denoted f_l. Take $F(f) = F_l(f_l)$. It is clearly well-defined. □

The previous proposition says that every irreducible finite dimensional nilpotent $\mathfrak{ym}(n)$-module is in fact an irreducible $\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$-module, for some $l \in \mathbb{N}$. Since the latter is finite dimensional nilpotent, it suffices to find irreducible finite dimensional modules over this kind of Lie algebras. It is in fact well-known that these representations are one dimensional (cf. [Dix], Coro. 1.3.13). The following lemma provides a description of them.

Lemma 2.8. *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra over k. Every irreducible finite dimensional representation of $\mathfrak{g}$ is one dimensional. Furthermore, the set of isomorphism classes of irreducible finite dimensional representations of $\mathfrak{g}$ is parametrized by $(\mathfrak{g}/\mathcal{C}^1(\mathfrak{g}))^*$.*

Proof. Let V be an irreducible finite dimensional $\mathfrak{g}$-module of dimension $n \geq 1$ defined by $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. Since V is simple, Lie's theorem tells us that

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

V contains a common eigenvector v for all the endomorphisms in $\mathfrak{g}$ (cf. [Hum], Thm. 4.1). Then, the non trivial submodule $k.v \subset V$ should coincide with V, for V is irreducible, so $\dim_k(V) = 1$.

On the other hand, since $\mathfrak{gl}(V) \simeq k$, the morphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ takes values in an abelian Lie algebra, so $\mathcal{C}^1(\mathfrak{g}) \subset \mathrm{Ker}(\phi)$, which in turn implies that ϕ induces a morphism $\bar{\phi} : \mathfrak{g}/\mathcal{C}^1(\mathfrak{g}) \to k$. Conversely, given a linear form $\bar{\phi} \in (\mathfrak{g}/\mathcal{C}^1(\mathfrak{g}))^*$, we have a Lie algebra homomorphism $\phi : \mathfrak{g} \to k \simeq \mathfrak{gl}(k)$, whose kernel contains $\mathcal{C}^1(\mathfrak{g})$. Thus, k turns out to be a irreducible representation of $\mathfrak{g}$. The lemma is proved. □

Notice that any 2-dimensional representation of $\mathfrak{ym}(n)$ is nilpotent.

From the previous lemma and Proposition 2.6, taking into account that an irreducible finite dimensional solvable $\mathfrak{ym}(n)$-module is also nilpotent, we obtain the following theorem:

Theorem 2.9. *Every irreducible finite dimensional solvable representation of $\mathfrak{ym}(n)$ is of dimension 1. Moreover, the set of isomorphism classes of irreducible finite dimensional solvable representations of the Yang-Mills algebra $\mathfrak{ym}(n)$ is parametrized by $(\mathfrak{ym}(n)/\mathcal{C}^1(\mathfrak{ym}(n)))^*$.*

Remark 2.10. *The previous theorem is not only true for Yang-Mills algebras but also for any Lie algebra $\mathfrak{g}$ provided with a locally finite dimensional $\mathbb{N}$-grading, taking into account that the set of isomorphism classes of irreducible finite dimensional solvable representations is parametrized by $(\mathfrak{g}/\mathcal{C}^1(\mathfrak{g}))^*$.*

3. The ideal $\mathfrak{tym}(n)$

In this section we shall study in detail the ideal $\mathfrak{tym}(n)$ of the Lie Yang-Mills algebra and its associative version $\mathrm{TYM}(n)$. This ideal is the key point of the construction of the family of representations we shall define. In order to achieve this construction we need to prove that $\mathrm{TYM}(n)$ is a free algebra. We shall then perform detailed computations in order to prove this fact, for which we shall make use of the bar construction for augmented differential graded algebras (or A_∞-algebras). We suggest [Lef, Kel] as a reference. Although some of the results of this section are mentioned in [MS], our aim here is to give detailed proofs of the results that we will need later.

The starting point is to define k-linear morphisms d_i, $i = 1, \dots, n$, given by

$$\begin{aligned} d_i &: V(n) \to V(n) \\ d_i(x_j) &= \delta_{ij}. \end{aligned} \tag{3.1}$$

They can be uniquely extended to derivations d_i, $i = 1, \dots, n$, on $TV(n)$. Since, for all $i, j, k = 1, \dots, n$,

$$d_i([x_j, x_k]) = d_i(x_j x_k - x_k x_j) = \delta_{ij} x_k + x_j \delta_{ik} - \delta_{ik} x_j - x_k \delta_{ij} = 0, \tag{3.2}$$

we see that each d_i induces a derivation on $TV(n)/\langle R(n)\rangle = \mathrm{YM}(n)$, which we will also denote d_i.

The following proposition characterizes the algebra $\mathcal{U}(\mathfrak{tym}(n))$ as a subalgebra of $\mathcal{U}(\mathfrak{ym}(n)) = \mathrm{YM}(n)$.

Proposition 3.1. *The inclusion* $\mathrm{inc} : \mathfrak{tym}(n) \hookrightarrow \mathfrak{ym}(n)$ *induces a monomorphism* $\mathcal{U}(\mathrm{inc}) : \mathcal{U}(\mathfrak{tym}(n)) \hookrightarrow \mathcal{U}(\mathfrak{ym}(n))$ *whose image is* $\bigcap_{i=1}^n \mathrm{Ker}(d_i)$.

Proof. The first statement is a direct consequence of the Poincaré-Birkhoff-Witt Theorem (cf. [Dix], Sec. 2.2.6, Prop. 2.2.7). As it is usual, we will identify $\mathcal{U}(\mathfrak{tym}(n))$ with its image by $\mathcal{U}(\mathrm{inc})$ in $\mathcal{U}(\mathfrak{ym}(n))$.

Let us prove the second statement. On the one hand, if $z \in \mathfrak{tym}(n) = [\mathfrak{ym}(n), \mathfrak{ym}(n)]$, then $d_i(z) = 0$, $i = 1, \dots, n$. Since $\mathfrak{tym}(n)$ generates the algebra $\mathcal{U}(\mathfrak{tym}(n))$ and d_i is a derivation, each d_i vanishes in $\mathcal{U}(\mathfrak{tym}(n))$. Hence,

$$\mathcal{U}(\mathfrak{tym}(n)) \subseteq \bigcap_{i=1}^{n} \mathrm{Ker}(d_i).$$

We next choose an ordered basis of $\mathfrak{tym}(n)$ as k-vector space, denoted by $\mathcal{B}' = \{y_j : j \in J\}$. As a consequence, the set $\mathcal{B} = \{x_1, \dots, x_n\} \cup \mathcal{B}'$ is an ordered basis of $\mathfrak{ym}(n)$. By the Poincaré-Birkhoff-Witt Theorem, given $z \in \mathcal{U}(\mathfrak{ym}(n))$,

$$z = \sum_{\substack{j_1, \dots, j_l \in J \text{ not equal} \\ (r_1, \dots, r_n, s_1, \dots, s_l) \in \mathbb{N}_0^{n+l}}} c^{j_1, \dots, j_l}_{(r_1, \dots, r_n, s_1, \dots, s_l)} x_1^{r_1} \dots x_n^{r_n} y_{j_1}^{s_1} \dots y_{j_l}^{s_l}.$$

Since $d_i(x_j^p) = p x_j^{p-1} \delta_{ij}$ and $d_i(y_{j_m}) = 0$, $m = 1, \dots, l$,

$$d_i(z) = \sum_{\substack{j_1, \dots, j_l \in J \text{ not equal} \\ (r_1, \dots, r_n, s_1, \dots, s_l) \in \mathbb{N}_0^{n+l}}} c^{j_1, \dots, j_l}_{(r_1, \dots, r_n, s_1, \dots, s_l)} r_i x_1^{r_1} \dots x_i^{r_i - 1} \dots x_n^{r_n} y_{j_1}^{s_1} \dots y_{j_l}^{s_l}.$$

Let us suppose that $d_i(z) = 0$, for all $i = 1, \dots, n$. By the Poincaré-Birkhoff-Witt Theorem, we obtain that $r_i = 0$, for all $i = 1, \dots, n$. This in turn implies that $z \in \mathcal{U}(\mathfrak{tym}(n))$, and the other inclusion is proved. □

We shall consider the following filtration on the algebra $\mathrm{YM}(n)$

$$F^j = \begin{cases} 0 & \text{if } j = 0, \\ \{z \in \mathrm{TYM}(n) : d_i(z) \in F^{j-1}, \forall i, 1 \leq i \leq n\} & \text{if } j \in \mathbb{N}. \end{cases}$$

By the previous proposition, $F^1 = \mathrm{TYM}(n)$.

Lemma 3.2. *The filtration* $\{F^j\}_{j \in \mathbb{N}_0}$ *defined on* $YM(n)$ *is increasing, multiplicative, exhaustive, Hausdorff and such that* $x_i \in F^2$, *for all* $i = 1, \dots, n$.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

Proof. Cf. [MS], Lemma 28. □

The following result is implicit in the analysis of [MS].

LEMMA 3.3. *Let A be an associative k-algebra with unit. Let ∂_i, $i = 1, \dots, n$ be the usual derivations on $k[t_1, \dots, t_n]$ and define the derivations $D_i = \mathrm{id}_A \otimes \partial_i$ on the algebra $A[t_1, \dots, t_n] \simeq A \otimes_k k[t_1, \dots, t_n]$. Then, given polynomials $p_1, \dots, p_n \in A[t_1, \dots, t_n]$ such that $D_i(p_j) = D_j(p_i)$, for all $i, j = 1, \dots, n$, there exists a polynomial $P \in A[t_1, \dots, t_n]$ such that $D_i(P) = p_i$, $i = 1, \dots, n$.*

Proof. The classical proof for $A = k$ (cf. [Cou], Lemma 2.2) works as well in this context. □

Let $\mathrm{Gr}_{F^\bullet}(\mathrm{YM}(n))$ be the associated graded algebra of $\mathrm{YM}(n)$ provided with the filtration $\{F^j\}_{j \in \mathbb{N}_0}$, and let us denote $\bar{z}$ the class in $\mathrm{Gr}_{F^\bullet}(\mathrm{YM}(n))$ of an element $z \in \mathrm{YM}(n)$.

The derivations d_i, $i = 1, \dots, n$, induce morphisms on the associated graded algebra, which are also derivations and we shall denote them in the same way.

Furthermore, F^1/F^0 is a subalgebra of $\mathrm{Gr}_{F^\bullet}(\mathrm{YM}(n))$, isomorphic as an algebra to $F^1 = \mathrm{TYM}(n)$. From now on, we shall make use of this identification.

LEMMA 3.4. *The algebra $\mathrm{Gr}_{F^\bullet}(\mathrm{YM}(n))$ satisfies the following properties:*

(i) The elements $\bar{x}_i$, $i = 1, \dots, n$, commute with each other and with the subalgebra F^1/F^0.

(ii) The elements $\bar{x}_i$, $i = 1, \dots, n$, and F^1/F^0 generate $\mathrm{Gr}_{F^\bullet}(\mathrm{YM}(n))$.

(iii) The algebra generated by the elements $\bar{x}_i$ and F^1/F^0 is isomorphic to $(F^1/F^0) \otimes_k k[t_1, \dots, t_n]$. Hence, there exists an isomorphism of algebras $(F^1/F^0) \otimes_k k[t_1, \dots, t_n] \simeq \mathrm{Gr}_{F^\bullet}(\mathrm{YM}(n))$.

Proof. Cf. [MS], Lemma 29. Observe that Lemma 3.3 is necessary to prove (ii) and (iii). □

The projection $\mathfrak{ym}(n) \to \mathfrak{ym}(n)/\mathfrak{tym}(n) \simeq V(n)$ provides an action of the Lie Yang-Mills algebra $\mathfrak{ym}(n)$ on the symmetric algebra $S(V(n))$ such that the restricted action of $\mathfrak{tym}(n)$ on $S(V(n))$ is trivial.

In the same way as we have done for the Yang-Mills algebra, we define the *special grading* of $S(V(n))$: we consider $V(n)$ concentrated in degree 2 and we identify $S(V(n))$ and $S_{gr}(V(n))$, where the latter is the symmetric algebra in the category of graded k-vector spaces. The *usual grading* of $S(V(n))$ is given by considering $V(n)$ concentrated in degree 1.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

If the Yang-Mills algebra $\mathfrak{ym}(n)$ and the symmetric algebra $S(V(n))$ are provided with the special grading, then $S(V(n))$ is a graded module over $\mathfrak{ym}(n)$.

We will compute the cohomology of $\mathfrak{ym}(n)$ with coefficients on $S(V(n))$ because it will be used in order to prove weak convergence of the spectral sequence defined in Corollary 3.6.

PROPOSITION 3.5. *The Lie homology of $\mathfrak{ym}(n)$ with coefficients in the module $S(V(n))$ is given by*

$$H_\bullet(\mathfrak{ym}(n), S(V(n))) = \begin{cases} k, & \text{if } \bullet = 0, \\ 0, & \text{if } \bullet \geq 2. \end{cases}$$

The homology in degree 1 is the direct sum of vector spaces H_1^p ($p \in \mathbb{N}_0$), where

$$\dim_k(H_1^p) = \begin{cases} n\frac{(n+p-1)!}{(n-1)!} - \frac{(n+p)!}{(n-1)!(p+1)!}, & \text{if } p \leq 1, \\ n\frac{(n+1)!}{(n-1)!2!} - \frac{(n+2)!}{(n-1)!3!} - n, & \text{if } p = 2, \\ n\frac{(n+p-1)!}{(n-1)!p!} - \frac{(n+p)!}{(n-1)!(p+1)!} - n\frac{(n+p-3)!}{(n-1)!(p-2)!} + \frac{(n+p-4)!}{(n-1)!(p-3)!}, & \text{if } p \geq 3. \end{cases}$$

Proof. We will now denote V instead of $V(n)$.

We shall use the Koszul resolution (9) of $\mathcal{U}(\mathfrak{ym}(n))$ described by [CD] for homological computations. The complex $(C_\bullet(\mathrm{YM}(n), S(V)), d_\bullet)$ computing the homology $\mathrm{Tor}_\bullet^{\mathcal{U}(\mathfrak{ym}(n))}(k, S(V)) \simeq H_\bullet(\mathfrak{ym}(n), S(V))$ is given by tensoring the Koszul resolution (9) of [CD] with $S(V)$ over $\mathrm{YM}(n)$

$$0 \longrightarrow S(V)[-4] \xrightarrow{d_3} (S(V) \otimes V)[-2] \xrightarrow{d_2} S(V) \otimes V \xrightarrow{d_1} S(V) \longrightarrow 0, \tag{3.3}$$

where the differentials are

$$d_3(w) = \sum_{i=1}^{n} x_i.w \otimes x_i,$$
$$d_2(w \otimes x_i) = \sum_{j=1}^{n} (x_j^2.w \otimes x_i - x_i x_j.w \otimes x_j),$$
$$d_1(w \otimes x_i) = x_i.w,$$

for $S(V)$ is commutative. We have shifted some terms of the complex so the complex (3.3) is composed of homogeneous morphisms of degree 0.

We immediately get that $H_\bullet(\mathfrak{ym}(n), S(V)) = 0$ if $\bullet > 3$.

The complex 3.3 is the direct sum of the following subcomplexes of finite dimensional k-vector spaces

(3.4)

$$0 \longrightarrow S^{p-1}(V)[-4] \xrightarrow{d_3^{p-1}} (S^p(V) \otimes V)[-2] \xrightarrow{d_2^p} S^{p+2}(V) \otimes V \xrightarrow{d_1^{p+2}} S^{p+3}(V) \longrightarrow 0,$$

where $p \in \mathbb{Z}$ and we consider $S^p(V) = 0$, if $p < 0$. We define

$$H_\bullet^p(\mathfrak{ym}(n), S(V)) = \begin{cases} \mathrm{Ker}(d_\bullet^p)/\mathrm{Im}(d_{\bullet+1}^{p-1}), & \text{if } \bullet \neq 1, \\ \mathrm{Ker}(d_\bullet^p)/\mathrm{Im}(d_{\bullet+1}^{p-2}), & \text{if } \bullet = 1. \end{cases}$$

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

Notice that d_3^p is injective for $p \in \mathbb{N}_0$. This is proved as follows. If $d_3^p(w) = 0$, then $x_i.w = 0$, for all $i = 1, \dots, n$, so $w = 0$, because $\{x_i\}_{i=1,\dots,n}$ is a basis of V and $S(V)$ is entire. Hence, $H_3(\mathfrak{ym}(n), S(V)) = 0$.

On the other hand, the morphism d_1^p is surjective for $p \in \mathbb{N}_0$, since, if $w \in S^{p+1}(V)$, then there exists i such that $w = x_i.w'$ ($p + 1 > 0$!), i.e. $w = d_1(w')$. The morphism d_1 being homogeneous of degree 1, if $w \in S^0(V) = k$, there does not exist $w' \in S(V) \otimes V$ such that $d_1(w') = w$. As a consequence, $H_0(\mathfrak{ym}(n), S(V)) = k$.

We will now show that $H_2(\mathfrak{ym}(n), S(V)) = 0$. Let now $w = \sum_{i=1}^{n} w_i \otimes x_i \in S^p(V) \otimes V$ (where $w_i \in S^p(V)$ for $i = 1, \dots, n$) be in the kernel of the morphism d_2^p. Then $0 = d_2(w) = \sum_{i,j=1}^{n} (w_i x_j^2 \otimes x_i - w_i x_i x_j \otimes x_j)$, and using that $\{x_i\}_{i=1,\dots,n}$ is a basis of V, it turns out that $\sum_{j=1}^{n} (w_i x_j^2 - w_j x_i x_j) = 0$, for $i = 1, \dots, n$. This is equivalent to

$$w_i \sum_{j=1}^{n} x_j^2 = x_i \sum_{j=1}^{n} x_j w_j,\ i = 1, \dots, n.$$

Since $S(V)$ is a unique factorization domain and the elements x_i are prime, this identity implies that x_i divides w_i, for $i = 1, \dots, n$.

Let w_i' be such that $w_i = x_i w_i'$. We can rewrite the previous equation as follows

$$\sum_{j=1}^{n} x_j^2 (w_i' - w_j') = 0, \forall\, i = 1, \dots, n.$$

Fix i_1, i_2 such that $1 \leq i_1 < i_2 \leq n$. Then, $\sum_{j=1}^{n} x_j^2 (w_{i_1}' - w_{i_2}') = 0$, and $S(V)$ being entire, we see that $w_{i_1}' = w_{i_2}'$, for all $i_1, i_2, 1 \leq i_1 < i_2 \leq n$. Let us call this element w'. Hence, $w_i = x_i w'$.

Now, $d_3(w') = \sum_{i=1}^{n} w' x_i \otimes x_i = \sum_{i=1}^{n} w_i \otimes x_i = w$. We conclude that $H_2(\mathfrak{ym}(n), S(V)) = 0$.

We finally compute the homology in degree 1.

Since d_1^p is surjective, we have that

$$\begin{aligned} \dim_k(\operatorname{Ker}(d_1^p)) &= \dim_k(S^p(V) \otimes V) - \dim_k(S^{p+1}(V)) \\ &= n\frac{(n+p-1)!}{(n-1)!p!} - \frac{(n+p)!}{(n-1)!(p+1)!}. \end{aligned}$$

On the other hand, as $H_2(\mathfrak{ym}(n), S(V)) = 0$ we know that $\operatorname{Ker}(d_2^{p-2}) = \operatorname{Im}(d_3^{p-3})$. Moreover, injectivity of d_3^{p-3} yields that

$$\begin{aligned} \dim_k(\operatorname{Im}(d_2^{p-2})) &= \dim_k(S^{p-2}(V) \otimes V) - \dim_k(S^{p-3}(V)) \\ &= n\frac{(n+p-3)!}{(n-1)!(p-2)!} - \frac{(n+p-4)!}{(n-1)!(p-3)!}, \end{aligned}$$

for $p \geq 3$.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

If $p = 0, 1$, then $\mathrm{Im}(d_2^{p-2}) = \{0\}$ and in case $p = 2$, $S^{p-3}(V) = \{0\}$, so

$$\dim_k(\mathrm{Im}(d_2^{p-2})) = \dim_k(S^{p-2}(V) \otimes V) = n.$$

The proposition is proved. □

Corollary 3.6. *The filtration* $\{F^p C_\bullet(\mathrm{YM}(n), S(V))\}_{p\in\mathbb{Z}}$ *of the complex* $(C_\bullet(\mathrm{YM}(n), S(V)), d_\bullet)$ *of* (3.3) *given by*

$$(3.5)\quad F^p C_\bullet(\mathrm{YM}(n), S(V))$$
$$= (0 \longrightarrow S^{\geq -p}(V) \xrightarrow{d_3} S^{\geq -p}(V)\otimes V \xrightarrow{d_2} S^{\geq -p}(V)\otimes V \xrightarrow{d_1} S^{\geq -p}(V) \longrightarrow 0),$$

is increasing, exhaustive and Hausdorff, and the spectral sequence associated to this filtration weakly converges to the homology $H_\bullet(\mathfrak{ym}(n), S(V))$ *of the complex* $(C_\bullet(\mathrm{YM}(n), S(V)), d_\bullet)$.

Proof. We shall indicate the consecutive steps of the spectral sequence, following the standard construction detailed in [Wei], Sec. 5.4. First, notice that, since d_1 and d_3 are homogeneous morphisms of degree 1, and d_2 is homogeneous of degree 2, then

$$d_i(F^p C_\bullet(\mathrm{YM}(n), S(V))) \subseteq F^{p-1} C_\bullet(\mathrm{YM}(n), S(V)),$$

for $i = 1, 2, 3$, and

$$d_2(F^p C_\bullet(\mathrm{YM}(n), S(V))) \subseteq F^{p-2} C_\bullet(\mathrm{YM}(n), S(V)).$$

This in turns implies that the differentials $d^0_{p,q}$ are 0. Besides, as

$$E^0_{p,q} = F^p C_{p+q}(\mathfrak{ym}(n), S(V))/F^{p-1} C_{p+q}(\mathfrak{ym}(n), S(V)),$$

the spectral sequence is concentrated in the set of (p, q) such that $0 \leq p+q \leq 3$, and $p \leq 0$, and

$$E^0_{p,q} = \begin{cases} S^{-p}(V), & \text{if } q = -p, \\ S^{-p}(V) \otimes V, & \text{if } q = -p+1, \\ S^{-p}(V) \otimes V, & \text{if } q = -p+2, \\ S^{-p}(V), & \text{if } q = -p+3, \\ 0 & \text{in other case.} \end{cases}$$

We may picture the terms $E^0_{\bullet,\bullet}$ as it appears in Figure 3.1.

Since $d^0_{p,q} = 0$ and $E^0_{p,q} = E^1_{p,q}$, it turns out that

$$d^1_{p,q} = \begin{cases} 0, & \text{if } q = -p, \\ d_1^{-p}, & \text{if } q = -p+1, \\ 0, & \text{if } q = -p+2, \\ d_3^{-p}, & \text{if } q = -p+3, \\ 0 & \text{in other case.} \end{cases}$$

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

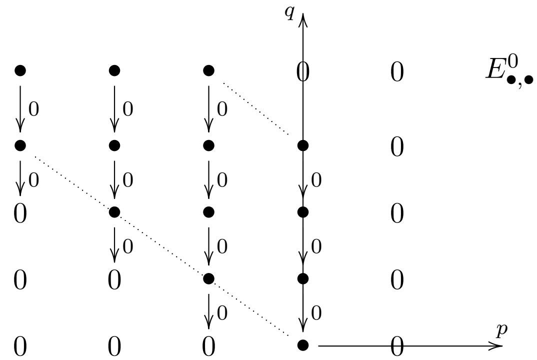

Figure 3.1: Zeroth term $E^0_{\bullet,\bullet}$ of the spectral sequence. The dotted lines indicate the limits wherein the spectral sequence is concentrated.

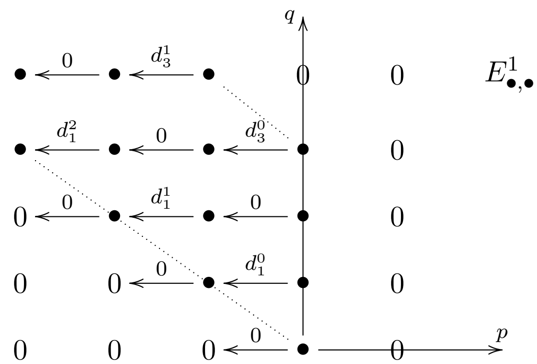

Figure 3.2: First term $E^1_{\bullet,\bullet}$ of the spectral sequence.

In consequence, the second step of the spectral sequence is

$$E^2_{p,q} = \begin{cases} S^{-p}(V)/\mathrm{Im}(d_1^{-p+1}), & \text{if } q = -p, \\ \mathrm{Ker}(d_1^{-p}), & \text{if } q = -p+1, \\ \mathrm{Im}(d_3^{-p+1}), & \text{if } q = -p+2, \\ \mathrm{Ker}(d_3^{-p}), & \text{if } q = -p+3, \\ 0 & \text{in other case,} \end{cases}$$

and the differentials are

$$d^2_{p,q} = \begin{cases} d_2^{-p}, & \text{if } q = -p+2, \\ 0 & \text{if not.} \end{cases}$$

In this case,

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

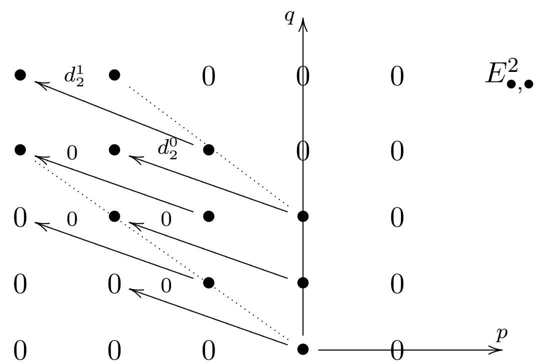

Figure 3.3: Step $E^2_{\bullet,\bullet}$ of the spectral sequence.

As a consequence, the third step of the spectral sequence is

$$E^3_{p,q} = \begin{cases} H_0^{-p}(\mathfrak{ym}(n), S(V)), & \text{if } q = -p, \\ H_1^{-p}(\mathfrak{ym}(n), S(V)), & \text{if } q = -p+1, \\ H_2^{-p}(\mathfrak{ym}(n), S(V)), & \text{if } q = -p+2, \\ H_3^{-p}(\mathfrak{ym}(n), S(V)), & \text{if } q = -p+3, \\ 0 & \text{in other case.} \end{cases}$$

Hence, the spectral sequence is weakly convergent because of Proposition 3.5. □

We shall consider the graded algebra $\mathrm{YM}(n) \otimes \Lambda^\bullet V(n)$ with the usual multiplication and the grading given by putting $\mathrm{YM}(n)$ in degree zero and the usual grading of $\Lambda^\bullet V(n)$. We define a differential d of degree 1 on this algebra by the formula

$$d(z \otimes w) = \sum_{i=1}^{n} d_i(z) \otimes (x_i \wedge w),$$

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

where $z \in \mathrm{YM}(n)$ and $w \in \Lambda^\bullet V(n)$. The identity $d \circ d = 0$ is immediate. The map d is a graded derivation, since

$$\begin{aligned} d((z \otimes w)(z' \otimes w')) &= d(zz' \otimes w \wedge w') \\ &= \sum_{i=1}^{n} d_i(zz') \otimes (x_i \wedge w \wedge w') \\ &= \sum_{i=1}^{n} (d_i(z)z' + zd_i(z')) \otimes (x_i \wedge w \wedge w') \\ &= (\sum_{i=1}^{n} d_i(z) \otimes (x_i \wedge w))(z' \otimes w') \\ &\quad + (-1)^{|w|}(z \otimes w)(\sum_{i=1}^{n} d_i(z')) \otimes (x_i \wedge w') \\ &= d(z \otimes w)(z' \otimes w') + (-1)^{|z \otimes w|}(z \otimes w)d(z' \otimes w'). \end{aligned}$$

If $\epsilon_{\mathfrak{ym}(n)}$ is the augmentation of $\mathrm{YM}(n)$ and ϵ' is the augmentation of the exterior algebra $\Lambda^\bullet V(n)$, we shall consider the augmentation of $\mathrm{YM}(n) \otimes \Lambda^\bullet V(n)$ given by the usual formula $\epsilon = \epsilon_{\mathfrak{ym}(n)} \otimes \epsilon'$. Hence, we have defined a structure of augmented differential graded algebra on $\mathrm{YM}(n) \otimes \Lambda^\bullet V(n)$.

The next result says that this augmented differential graded algebra is in fact a model for $\mathrm{TYM}(n)$.

Proposition 3.7. *If we consider the algebra* $\mathrm{TYM}(n)$ *as an augmented differential graded algebra concentrated in degree zero, with zero differential and augmentation* $\epsilon_{\mathfrak{tym}(n)}$*, then the morphism*

$$\begin{aligned} \mathrm{inc} : \mathrm{TYM}(n) &\to \mathrm{YM}(n) \otimes \Lambda^\bullet V(n) \\ z &\mapsto z \otimes 1 \end{aligned}$$

is a quasi-isomorphism of augmented differential graded algebras and we will write $\mathrm{TYM}(n) \simeq_q \mathrm{YM}(n) \otimes \Lambda^\bullet V(n)$*.*

Proof. The map inc is a morphism of graded algebras, as we can easily see. It also commutes with differentials by Proposition 3.1. Furthermore, the same proposition also implies that inc induces an isomorphism between $\mathrm{TYM}(n)$ and $H^0(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n)) = Z^0(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n))$, since $z \in Z^0(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n))$ if and only if $z = v \otimes 1$, with $v \in \mathrm{YM}(n)$ and $d(z) = \sum_{i=1}^{n} d_i(v) \otimes x_i = 0$, which in turn happens if and only if $d_i(v) = 0$, for all $i = 1, \dots, n$.

Note that inc also commutes with the augmentations.

We shall now proceed to prove that inc induces an isomorphism in cohomology. It is thus necessary to compute the cohomology of the underlying cochain complex $(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n), d)$. We will write the cochain complex

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

$(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n), d)$ as a chain complex in the usual way $C_\bullet = \mathrm{YM}(n) \otimes \Lambda^{n-\bullet} V(n)$.

We consider $(C_\bullet, d)$ provided with the filtration $\{F_\bullet C\}_{\bullet \in \mathbb{Z}}$ defined as follows

$$F_p C_q = F^{p+n-q} \otimes \Lambda^{n-q} V(n).$$

Notice that $\{F_\bullet C\}_{\bullet \in \mathbb{Z}}$ is an increasing, bounded below and exhaustive filtration and $d(F_p C_q) \subseteq F_{p-2} C_{q-1}$. Hence, $\{F_\bullet C\}_{\bullet \in \mathbb{Z}}$ is a filtration of complexes and in turn it induces a spectral sequence whose second term is

$$\begin{aligned} E^2_{p,q} = F_p C_{p+q} / F_{p-1} C_{p+q} &= (F^{n-q} \otimes \Lambda^{n-p-q} V(n)) / (F^{n-q-1} \otimes \Lambda^{n-p-q} V(n)) \\ &\simeq (F^{n-q}/F^{n-q-1}) \otimes \Lambda^{n-(p+q)} V(n) \\ &= \mathrm{Gr}_{F^\bullet}(YM(n))^{n-q} \otimes \Lambda^{n-(p+q)} V(n) \\ &\simeq \mathrm{TYM}(n) \otimes S^{n-q}(V(n)) \otimes \Lambda^{n-(p+q)} V(n), \end{aligned}$$

where the last isomorphism follows from the last item of Lemma 3.4. The differential $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$ can be written in the following simple way, where $V = V(n)$, $\mathrm{TYM} = \mathrm{TYM}(n)$ and $r = p + q$

$$\begin{tikzcd} E^2_{p,q} \arrow[r, "d^2_{p,q}"] \arrow[d, equal] & E^2_{p-2,q+1} \arrow[d, equal] \\ (F^{n-q}/F^{n-q-1}) \otimes \Lambda^{n-r} V \arrow[r, "\bar{d}"] \arrow[d, equal] & (F^{n-q-1}/F^{n-q-2}) \otimes \Lambda^{n-r+1} V \arrow[d, equal] \\ \mathrm{TYM} \otimes S^{n-q}(V) \otimes \Lambda^{n-r} V \arrow[r, "d'_{p,q}"] & \mathrm{TYM} \otimes S^{n-q}(V) \otimes \Lambda^{n-r+1} V. \end{tikzcd}$$

The morphism $\bar{d}$ is induced by d and $d'_{p,q}$ is the map given by

$$d'_{p,q}(z \otimes v \otimes w) = \sum_{i=1}^{n} z \otimes \partial_i(v) \otimes (x_i \wedge w).$$

We see that this complex is exact except in case $p = 0$ and $q = n$. This follows from the fact that the differential $d'_{p,q}$ is the $\mathrm{TYM}(n)$-linear extension of the differential of the de Rham complex of the algebra $S(V(n))$, whose cohomology is zero, except in degree zero, where it equals k (cf. [Wei], Coro. 9.9.3). This implies that the spectral sequence collapses in the third step, for the unique non zero element is $E^3_{0,n} = \mathrm{TYM}(n)$. As the filtration is bounded below and exhaustive, the Classical Convergence Theorem tells us that the spectral sequence is convergent and it converges to the homology of the complex $(C_\bullet, d_\bullet)$ (cf. [Wei], Thm. 5.5.1).

Finally, $H^\bullet(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n)) = H_{n-\bullet}(C) = 0$, if $\bullet \neq 0$, and $H^0(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n)) = H_n(C) = \mathrm{TYM}(n)$. □

Let $B^+(-)$ denote the bar construction for augmented differential graded algebras or more generally, for A_∞-algebras (cf. [Lef], Notation 2.2.1.4). From the previous proposition we obtain that $B^+(\mathrm{TYM}(n))$ is quasi-isomorphic to $B^+(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))$. Since the underlying complex of the bar complex of an augmented (graded) algebra coincides with the normalized (graded) Hochschild bar complex with coefficients in the (graded) bimodule k (cf. [Lef], Proof of Lemma 2.2.1.9), the respective Hochschild homologies of differential graded algebras with coefficients in k are isomorphic, i.e.

$$\begin{aligned} H_\bullet(\mathrm{TYM}(n),k) &= H_\bullet(B^+(\mathrm{TYM}(n))) \\ &\simeq H_\bullet(B^+(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))) \\ &= H_\bullet(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n),k). \end{aligned}$$

We define another filtration on the augmented differential graded algebra $\mathrm{YM}(n)\otimes\Lambda V(n)$ by the formula

$$F_p(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n)) = \mathrm{YM}(n)\otimes\Lambda^{\bullet\geq p}V(n). \tag{3.6}$$

It may be easily verified that $F_p(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))$ is decreasing, bounded, multiplicative and compatible with differentials, i.e. $d(F_p(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n)))\subseteq F_p(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))$. Furthermore,

$$d(F_p(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n)))\subseteq F_{p+1}(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n)).$$

Notice that $\epsilon(F_p(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n)))=0$, if $p\geq 1$.

As a consequence, the associated graded algebra to this filtration is an augmented differential graded algebra, provided with zero differential and augmentation induced by ϵ.

As before, we will consider $\mathrm{YM}(n)$ as an augmented differential graded algebra concentrated in degree zero and augmentation $\epsilon_{\mathfrak{ym}(n)}$, and $\Lambda^\bullet V(n)$ as an augmented differential graded algebra with the usual grading (i.e. given by $\bullet$) and the usual augmentation, both with zero differential. The associated graded algebra $\mathrm{Gr}_{F_\bullet}(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))$ is the tensor product (in the category of augmented differential graded algebras) of the algebras $\mathrm{YM}(n)$ and $\Lambda^\bullet V(n)$.

The filtration (3.6) induces a decreasing and bounded above filtration of coaugmented coalgebras on

$$B^+(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n)),$$

which we denote $F_\bullet(B^+(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n)))$ (cf. [Lef], Sec. 1.3.2). By its very definition,

$$\mathrm{Gr}_{F_\bullet}(B^+(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))) = B^+(\mathrm{Gr}_{F_\bullet}(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))).$$

Since the associated graded algebra $\mathrm{Gr}_{F_\bullet}(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))$ is the tensor product of the algebras $\mathrm{YM}(n)$ and $\Lambda^\bullet V(n)$, then

$$B^+(\mathrm{Gr}_{F_\bullet}(\mathrm{YM}(n)\otimes\Lambda^\bullet V(n))) \simeq_q B^+(\mathrm{YM}(n))\otimes B^+(\Lambda^\bullet V(n)).$$

Again using that $B^+(\mathrm{YM}(n))$ is quasi-isomorphic as a graded k-vector space to the tensor product over $\mathrm{YM}(n)^e$ of any projective resolution of graded $\mathrm{YM}(n)^e$-modules of k with k, we obtain that $B^+(\mathrm{YM}(n)) \simeq_q C_\bullet(\mathrm{YM}(n), k)$.

Besides, since $\Lambda^\bullet V(n)$ is a Koszul algebra, $B^+(\Lambda^\bullet V(n))$ is quasi-isomorphic to $S^\bullet(V(n))$, where we consider $S^\bullet(V(n))$ as a differential graded algebra provided with the usual grading (i.e. given by $\bullet$) and zero differential (cf. [PP], Sec. 2.1, Example; Sec. 2.3).

Finally,

$$\mathrm{Gr}_{F_\bullet}(B^+(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n))) \simeq_q C_\bullet(\mathrm{YM}(n), k) \otimes S(V(n)).$$

Notice that the space on the right is the zero step of the weakly convergent spectral sequence in Corollary 3.6.

Moreover, the spectral sequence associated to the decreasing, exhaustive, Hausdorff and bounded above filtration $F_\bullet(B^+(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n)))$ is convergent. From the Comparison Theorem for spectral sequences (cf. [Wei], Thm. 5.2.12), it follows that

$$H_\bullet(B^+(\mathrm{YM}(n) \otimes \Lambda^\bullet V(n))) \simeq H_\bullet(C_\bullet(\mathrm{YM}(n), S(V(n)))).$$

By Proposition 3.7

$$\begin{aligned} H_\bullet(\mathfrak{tym}(n), k) \simeq H_\bullet(\mathrm{TYM}(n), k)) &\simeq H_\bullet(B^+(\mathrm{TYM}(n))) \\ &\simeq H_\bullet(C_\bullet(\mathrm{YM}(n), S(V(n)))). \end{aligned} \tag{3.7}$$

We now recall some useful results concerning cohomology of algebras.

Proposition 3.8. *Let k be a commutative ring with unit and let $\mathfrak{g}$ be a Lie algebra over k. If $H^\bullet(\mathfrak{g}, M) = 0$, for all $\mathfrak{g}$-module M and $\bullet \geq 2$. Then $\mathfrak{g}$ is a free Lie algebra.*

Let W be a k-module. If M is a bimodule over the free algebra $T(W)$, then $H_\bullet(T(W), M) = 0$ and $H^\bullet(T(W), M) = 0$, for all $\bullet \geq 2$.

Proof. Cf. [Wei], Prop. 9.1.6. and Ex. 7.6.3. □

Remark 3.9. *From Proposition 3.8, we immediately see that if the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is free, then $\mathfrak{g}$ is also free. The converse is also clear.*

Theorem 3.10. *Let W be a graded k-vector space concentrated in degree 1, let I be an homogeneous ideal generated in degrees greater or equal to 2 and let $A = T(W)/I$. We may write $I = \bigoplus_{m \in \mathbb{N}_{\geq 2}} I_m$, where I_m denotes the m-th homogeneous component. Then, for each $m \geq 2$, it is possible to choose a*

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

k-linear subspace $R_m \subseteq I_m$ such that

$$I_2 = R_2,$$
$$I_m = R_m \oplus \left(\sum_{i+j+l=m, 2\leq j<m} W^{\otimes i} \otimes R_j \otimes W^{\otimes l} \right).$$

Taking $R = \bigoplus_{m\in\mathbb{N}_{\geq 2}} R_m$, the homology is

$$H_0(A,k) = \mathrm{Tor}_0^A(k,k) = k,$$
$$H_1(A,k) = \mathrm{Tor}_1^A(k,k) = V,$$
$$H_2(A,k) = \mathrm{Tor}_2^A(k,k) = R.$$

Proof. Cf. [Ber2], Prop. 2.5. □

Using Propositión 3.5, we obtain that $H_2(\mathrm{TYM}(n),k) = 0$. If we consider the vector space $V(n)$ as a graded vector space concentrated in degree 2, the algebra $\mathrm{TYM}(n) = \mathcal{U}(\mathfrak{tym}(n))$ is graded, and Theorem 3.10 implies that $\mathrm{TYM}(n)$ is a free graded algebra, so a fortiori, $\mathrm{TYM}(n)$ is a free algebra, when we forget the grading. Furthermore, from Remark 3.9 it yields that $\mathfrak{tym}(n)$ is a free Lie algebra, in the graded sense or not, due to Remark 2.3.

We have proved the following theorem, which is a key result in order to study representations of the Yang-Mills algebra.

Theorem 3.11. *The Lie algebra $\mathfrak{tym}(n)$ is free and the same holds for the associative algebra $\mathrm{TYM}(n)$.*

The last paragraphs of this section are devoted to a description of the graded vector space which generates the above free algebra.

Let us call $W(n)$ the graded k-vector space satisfying $\mathfrak{f}_{gr}(W(n)) \simeq \mathfrak{tym}(n)$, or equivalently $T_{gr}(W(n)) \simeq \mathrm{TYM}(n)$, where we are using the special grading. By Theorem 3.10, we see that $H_1(\mathrm{TYM}(n),k) \simeq W(n)$ as graded vector spaces, since all morphisms considered in this section are homogeneous of degree 0.

Using that $H_\bullet(\mathrm{TYM}(n),k))$ is isomorphic to $H_\bullet(\mathfrak{ym}(n), S(V(n)))$, by (3.7), and that Proposition 3.5 implies the following isomorphism of graded vector spaces

$$H_1(\mathfrak{ym}(n), S(V(n))) \simeq \sum_{p\in\mathbb{N}} H_1^p(\mathfrak{ym}(n), S(V(n)))$$

where $H_1^p(\mathfrak{ym}(n), S(V(n)))$ lives in degree $p+1$, we immediately see that the homogeneous component $W(n)_m$ of degree m of $W(n)$ is isomorphic to $H_1^{m-1}(\mathfrak{ym}(n), S(V(n)))$.

In particular, $W(2) \simeq H_1(\mathfrak{ym}(2), S(V(2))) = H_1^1(\mathfrak{ym}(2), S(V(2))) \simeq k[-2]$ (as k-vector spaces) and $W(n)$ is infinite dimensional if $n \geq 3$. For instance,

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

it follows easily from Proposition 3.5 that for $n = 3$,

$$W(3) \simeq H_1(\mathfrak{ym}(3), S(V(3))) = \bigoplus_{p\in\mathbb{N}_0} H_1^p(\mathfrak{ym}(3), S(V(3))), \tag{3.8}$$

where $\dim_k(H_1^p(\mathfrak{ym}(3), S(V(3)))) = 2p+1$ for $p \in \mathbb{N}$ and $H_1^0(\mathfrak{ym}(3), S(V(3)))$ is zero.

We recall the following definition.

Definition 3.12. *(cf. [PP], [La], Ch. X, §6) The* Hilbert series *of a $\mathbb{Z}$-graded k-vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ (also called* Poincaré series*) is the formal power series in $\mathbb{Z}[[t, t^{-1}]]$ given by*

$$h_V(t) = \sum_{n\in\mathbb{Z}} \dim_k(V_n)t^n.$$

We will also denote it simply $V(t)$.

We will now compute the Hilbert series $W(n)(t)$ of $W(n)$ provided with the usual grading. In order to do so, we must notice that if

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a short exact sequence of graded vector spaces with homogeneous morphisms f and g of degree 0, then

$$M(t) = M''(t) + M'(t).$$

The previous identity implies that that the Hilbert series is an *Euler-Poincaré map* (cf. [La], Ch. III, §8) when considering the category of graded vector spaces with homogeneous morphisms of degree 0. As a consequence, if we consider the *Euler-Poincaré characteristic* of a complex $(C_\bullet, d_\bullet)$ of graded vector spaces with homogeneous morphisms, which is defined as

$$\chi(C)(t) = \sum_{i\in\mathbb{Z}} (-1)^i C_i(t),$$

it turns out that it coincides with the Euler-Poincaré characteristic of its homology (cf. [La], Ch. XX, §3, Thm. 3.1).

On the one hand, the Euler-Poincaré characteristic of the complex (3.3) is

$$\begin{aligned}\chi_{C_\bullet(\mathrm{YM}(n),S(V(n)))(t)} &= S(V(n))(t) - ntS(V(n))(t)\\ &\quad + nt^3S(V(n))(t) - t^4S(V(n))(t)\\ &= \frac{1 - nt + nt^3 - t^4}{(1-t)^n},\end{aligned}$$

since $S(V(n))(t) = (1-t)^{-n}$.

On the other hand, since the morphisms of the complex (3.3) are homogeneous of degree 0, its Euler-Poincaré characteristic coincides with the one of its homology.

Since there are isomorphisms of graded vector spaces

$$\begin{aligned} &H_0(\mathfrak{ym}(n), S(V(n))) \simeq k, \\ &H_1(\mathfrak{ym}(n), S(V(n))) \simeq W(n), \\ &H_2(\mathfrak{ym}(n), S(V(n))) = H_3(\mathfrak{ym}(n), S(V(n))) = 0, \end{aligned}$$

the Euler-Poincaré characteristic of the homology is

$$\chi_{H_\bullet(C_\bullet(\mathrm{YM}(n),S(V(n))))(t)} = 1 - W(n)(t).$$

Finally,

$$W(n)(t) = \frac{(1-t)^n - 1 + nt - nt^3 + t^4}{(1-t)^n}.$$

We have thus proved the following proposition.

Proposition 3.13. *The Hilbert series of the graded vector space $W(n)$ with the usual grading is given by*

$$W(n)(t) = \frac{(1-t)^n - 1 + nt - nt^3 + t^4}{(1-t)^n}.$$

It is trivially verified that the term of degree m of $W(n)(t)$ coincides with the computation of $\dim_k(W(n)_m)$ obtained from Proposition 3.5.

Observe that for $n = 2$, the Yang-Mills algebra YM(2) is noetherian. This is a direct consequence of the isomorphism $\mathrm{YM}(2) \simeq \mathcal{U}(\mathfrak{h}_1)$ (cf. Example 2.1). Since $\mathfrak{h}_1$ is finite dimensional, then $S(\mathfrak{h}_1)$ is noetherian, so $\mathcal{U}(\mathfrak{h}_1)$ is also noetherian, because its associated graded algebra $S(\mathfrak{h}_1)$ is (cf. [Dix], Corollary 2.3.8, p. 76).

However, for $n \geq 3$, Yang-Mills algebras $\mathrm{YM}(n)$ are non noetherian. In order to prove this fact we proceed as follows.

On the one side, $\mathrm{YM}(n) \supset \mathrm{TYM}(n)$ is a (left and right) free extension of algebras, which is not finite but finitely generated (cf. [Wei], Coro. 7.3.9). One set of generators of the extension $\mathrm{YM}(n) \supset \mathrm{TYM}(n)$ is $\{x_1, \dots, x_n\}$.

On the other side, since $\mathrm{TYM}(n)$ is a free algebra with an infinite set of generators if $n \geq 3$ (cf. Thm. 3.11, Prop. 3.13), it is not noetherian, which implies that $\mathrm{YM}(n)$ is not noetherian. This is a direct consequence of the following simple fact: If $A \supset B$ is an extension of algebras such that A is a right (resp. left) free B-module and $I \triangleleft B$ is a left (resp. right) ideal, then $A.I$ is a left (resp. right) ideal of A that satisfies that $A.I \cap B = I$. This property directly yields that if B is not left (resp. right) noetherian, then A is not left (resp. right) noetherian.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

Let us thus show the previously stated property for free extensions of algebras. We shall prove it in the case that A is a right free B-module and $I \lhd B$ is a left ideal, the other being analogous.

The inclusion $I \subseteq A.I \cap B$ is clear. We will prove the other one. Let $\mathcal{B} = \{a_j\}_{j \in J}$ be a basis of A as a right B-module. We assume without loss of generality that $1 = a_{j_0} \in \mathcal{B}$. Since every element of A can be written as $\sum_{j \in J} a_j b_j$, with $b_j \in B$, an element $x \in A.I \cap B$ may be written as $x = \sum_{j \in J} a_J c_j$, where $c_j \in I$. But $x \in B$, so $c_{j_0} = x$ and $a_j = 0$, for all $j \in J$, $j \neq j_0$. Hence, $x = a_{j_0} c_{j_0} = 1 c_{j_0} = c_{j_0} \in I$.

Remark 3.14. *The fact that* $\mathrm{YM}(n) = \mathcal{U}(\mathfrak{ym}(n))$ *is not noetherian for* $n \geq 3$ *implies that* $\mathfrak{ym}(n)$ *is not finite dimensional for* $n \geq 3$.

4. Main theorem: relation between Yang-Mills algebras and Weyl algebras

The aim of this last section is to prove that all the Weyl algebras $A_r(k)$ ($r \in \mathbb{N}$) are epimorphic images of all Yang-Mills algebras $\mathrm{YM}(n)$ for $n \geq 3$. In order to do so we make intensive use of $\mathrm{TYM}(n)$. As a consequence, the representations of all $A_r(k)$ are also representations of $\mathrm{YM}(n)$ ($r \in \mathbb{N}$, $n \geq 3$). These families of representations have been previously studied by Bavula and Bekkert in [BB] and are enough to separate points of $\mathrm{YM}(n)$.

Since $\mathrm{YM}(3)$ is a quotient of $\mathrm{YM}(n)$, as an algebra, for every $n \geq 3$, it will be sufficient to prove that that the Weyl algebras $A_r(k)$ are epimorphic images of $\mathrm{YM}(3)$. Our first step is to give explicit bases for the quotients $\mathfrak{ym}(3)/\mathcal{C}^j(\mathfrak{ym}(3))$, for $j = 1, 2, 3, 4$. The elements of these bases will be useful while defining the epimorphisms onto $A_r(k)$.

Using the fact that $\mathrm{YM}(n)$ is Koszul, Connes and Dubois-Violette obtain the Hilbert series of the Yang-Mills algebra $\mathrm{YM}(n)$ (cf. [CD], Corollary 3)

$$h_{\mathrm{YM}(n)}(t) = \frac{1}{(1-t^2)(1-nt+t^2)}.$$

Using the Poincaré-Birkhoff-Witt Theorem it is possible to deduce the dimensions of the homogeneous spaces of the Yang-Mills algebra $N(n)_j = \dim_k(\mathfrak{ym}(n)_j)$ ($j \in \mathbb{N}$) from the equality

$$\prod_{j \in \mathbb{N}} \left(\frac{1}{1-t^j} \right)^{N(n)_j} = h_{\mathrm{YM}(n)}(t).$$

Connes and Dubois-Violette find in [CD] the direct formula ($j \geq 3$)

$$N(n)_j = \frac{1}{j} \sum_{k=1}^{j} \mu(\frac{j}{k})(t_1^k + t_2^k),$$

where t_1 y t_2 are the roots of the polynomial $t^2 - nt + 1 = 0$ and $\mu(x)$ is the Möbius function.

For $\mathfrak{ym}(3)$, the sequence of dimensions $N(3)_j$ ($j \in \mathbb{N}$) is (cf. [Sl], sequence A072337)

$$3, 3, 5, 10, 24, 50, 120, 270, 640, 1500, 3600, 8610, 20880, 50700, 124024, \dots$$

An ordered basis for the quotient algebra $\mathfrak{ym}(3)/\mathcal{C}^1(\mathfrak{ym}(3))$ of the Yang-Mills algebra is given by

$$\mathcal{B}_1 = \{x_1, x_2, x_3\}.$$

For the quotient $\mathfrak{ym}(3)/\mathcal{C}^2(\mathfrak{ym}(3))$, a possible ordered basis is

$$\mathcal{B}_2 = \{x_1, x_2, x_3, x_{12}, x_{13}, x_{23}\},$$

where $x_{ij} = [x_i, x_j]$, $(i, j = 1, 2, 3)$. To prove that it is indeed a basis we must only show that it generates $\mathfrak{ym}(3)/\mathcal{C}^2(\mathfrak{ym}(3))$ (since $\#(\mathcal{B}_2) = 6$), which is obtained using that $x_{ij} = -x_{ji}$.

The following set is a basis of $\mathfrak{ym}(3)/\mathcal{C}^3(\mathfrak{ym}(3))$:

$$\mathcal{B}_3 = \{x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{112}, x_{221}, x_{113}, x_{123}, x_{312}\},$$

where we denote $x_{ijk} = [x_i, [x_j, x_k]]$. We also define the set of triple indices of $\mathcal{B}_3$, $J_3 = \{(112), (221), (113), (123), (312)\}$.

Let us prove that $\mathcal{B}_3$ is a basis. As before, we only have to prove that it generates $\mathfrak{ym}(3)/\mathcal{C}^3(\mathfrak{ym}(3))$. This is direct, as we can see from the Yang-Mills relations

$$x_{332} = -x_{112}, \quad x_{331} = -x_{221}, \quad x_{223} = -x_{113},$$

and relations given by antisymmetry and Jacobi identity, i.e. $x_{ijk} = -x_{ikj}$, and $x_{213} = x_{123} + x_{312}$.

The case $\mathfrak{ym}(3)/\mathcal{C}^4(\mathfrak{ym}(3))$ is a little more complicated. We shall prove that

$$\begin{aligned} \mathcal{B}_4 = \{&x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{112}, x_{221}, x_{113}, x_{123}, x_{312}, \\ &x_{1112}, x_{1221}, x_{1113}, x_{1123}, x_{2221}, x_{2113}, x_{2312}, x_{3112}, x_{3221}, x_{3312}\}, \end{aligned}$$

is an ordered basis, where $x_{ijkl} = [x_i, [x_j, [x_k, x_l]]]$. Analogously, we define the set of indices of $\mathcal{B}_4$

$$\begin{aligned} J_4 = \{&(1112), (1221), (1113), (1123), (2221), \\ &(2113), (2312), (3112), (3221), (3312)\}. \end{aligned}$$

In order to prove that $\mathcal{B}_4$ is a basis it suffices again to verify that it generates $\mathfrak{ym}(3)/\mathcal{C}^4(\mathfrak{ym}(3))$. On the one hand, taking into account that

$$\begin{aligned} [[x_i, x_j], [x_k, x_l]] &= [[[x_i, x_j], x_k], x_l] + [x_k, [[x_i, x_j], x_l]] \\ &= [x_l, [x_k, [x_i, x_j]]] + [x_k, [x_l, [x_j, x_i]]] = x_{lkij} + x_{klji} \end{aligned}$$

and $[[[x_i, x_j], x_k], x_l] = [x_l, [x_k, [x_i, x_j]]] = x_{lkij}$, the set $\{x_{ijkl} : i, j, k, l = 1, 2, 3\} \cup \mathcal{B}_3$ is a system of generators. We shall prove that it is generated by $\mathcal{B}_4$. In fact, we only need to prove that it generates the set

$$\{x_{i112}, x_{i221}, x_{i113}, x_{i123}, x_{i312} : i = 1, 2, 3\},$$

because $\{x_{112}, x_{221}, x_{113}, x_{123}, x_{312}\}$ are generators of the homogeneous elements of degree 3. This last statement is direct:

$$x_{3113} = x_{1221}, \quad x_{2112} = -x_{1221}, \quad x_{2123} = x_{3221} + x_{2312} - x_{1113},$$
$$x_{1312} = \frac{x_{3112} + x_{2113} - x_{1123}}{2}, \qquad x_{3123} = \frac{x_{1112} + x_{2221} - x_{3312}}{2}.$$

We shall now briefly recall a version of the Kirillov orbit method by J. Dixmier, which we will employ. We first recall that a bilateral ideal $I \triangleleft A$ of an algebra A is called *prime* if $I \neq A$ and if $J, K \triangleleft A/I$ are two nonzero bilateral ideals of the quotient algebra A/I, then $JK \neq \{0\}$. We say that $I \triangleleft A$ is *completely prime* if A/I is a domain. Observe that every completely prime ideal is prime (cf. [Dix], 3.1.6). A bilateral ideal $I \triangleleft A$ is called *semiprime* if $I \neq A$ and every nilpotent bilateral ideal $J \triangleleft A/I$ is zero. Note that an intersection of semiprime ideals is semiprime and every prime ideal is semiprime (cf. [Dix], 3.1.6).

On the other hand, a bilateral ideal $I \triangleleft A$ is called *primitive* if it the annihilating ideal of a simple left A-module, and it called *maximal* if $I \neq A$ and if it is maximal in the lattice of bilateral ideals of A, ordered by inclusión. Every maximal ideal is primitive (cf. [Dix], 3.1.6).

If $\mathfrak{g}$ is finite dimensional Lie algebra, $\mathcal{U}(\mathfrak{g})$ is a noetherian domain, so it has a skew-field of fractions $\mathrm{Frac}(\mathcal{U}(\mathfrak{g}))$ (cf. [Dix], 3.1.16 and Thm. 3.6.12). Let $I \triangleleft \mathcal{U}(\mathfrak{g})$ be a semiprime ideal, so $\mathcal{U}(\mathfrak{g})/I$ also has a skew-field of fractions $\mathrm{Frac}(\mathcal{U}(\mathfrak{g})/I)$. We say that I is *rational* if $\mathcal{Z}(\mathrm{Frac}(\mathcal{U}(\mathfrak{g})/I)) = k$. Every rational ideal of $\mathcal{U}(\mathfrak{g})$ is primitive and, k being algebraically closed, the converse also holds (cf. [Dix], Thm. 4.5.7).

We have the following proposition.

PROPOSITION 4.1. *Let $I \triangleleft \mathcal{U}(\mathfrak{g})$ be a bilateral ideal of the universal enveloping algebra of a nilpotent Lie algebra $\mathfrak{g}$ of finite dimension. The following are equivalent:*

(i) I is primitive.

(ii) I is maximal.

(iii) There exists $r \in \mathbb{N}$ such that $\mathcal{U}(\mathfrak{g})/I \simeq A_r(k)$.

(iv) I is the kernel of a simple representation of $\mathcal{U}(\mathfrak{g})$.

Proof. Cf. [Dix], Prop. 4.7.4, Thm. 4.7.9. □

If $I \lhd \mathcal{U}(\mathfrak{g})$ is a bilateral ideal satisfying either of the previous equivalent conditions, the positive integer r (uniquely determined) such that $\mathcal{U}(\mathfrak{g})/I \simeq A_r(k)$ is called the *weight* of the ideal I (cf. [Dix], 4.7.10).

Let us suppose that $\mathfrak{g}$ is nilpotent Lie algebra of finite dimension. Given $f \in \mathfrak{g}^*$, a *polarization* of f is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that it is *subordinated* to f, i.e. $f([\mathfrak{h}, \mathfrak{h}]) = 0$, and it is maximal with respect to the previous property (cf. [Dix], Section 1.12). There exists a canonical way to construct polarizations, which are called *standard*, of a linear form $f \in \mathfrak{g}^*$ (cf. [Dix], Prop. 1.12.10). It is easily verified that the dimension of a polarization $\mathfrak{h}$ of f must be (cf. [Dix], 1.12.1)

$$\frac{\dim_k(\mathfrak{g}) + \dim_k(\mathfrak{g}^f)}{2}. \tag{4.1}$$

Furthermore, $\mathfrak{h}$ is a polarization of f on $\mathfrak{g}$ if and only if $\mathfrak{h}$ is a subalgebra subordinated to f of dimension $(\dim_k(\mathfrak{g}) + \dim_k(\mathfrak{g}^f))/2$ (cf. [Dix], 1.12.8). The weight of a primitive ideal $I(f)$ is given by $r = \dim_k(\mathfrak{g}/\mathfrak{g}^f)/2$ (cf. [Dix], Prop. 6.2.2), or using identity (4.1), $r = \dim_k(\mathfrak{g}/\mathfrak{h}_f)$, where $\mathfrak{h}_f$ is a polarization of f.

We shall now explain the connection between rational ideals and polarizations. If $f \in \mathfrak{g}^*$ be a linear functional and $\mathfrak{h}_f$ a polarization of f, we may define a representation of $\mathfrak{h}_f$ on the vector space $k.v_f$ of dimension 1 by means of $x.v_f = (f(x) + \mathrm{tr}_{\mathfrak{g}/\mathfrak{h}_f}(\mathrm{ad}_{\mathfrak{g}} x))v_f$, for $x \in \mathfrak{h}_f$ and $\mathrm{tr}_{\mathfrak{g}/\mathfrak{h}_f} = \mathrm{tr}_{\mathfrak{g}} - \mathrm{tr}_{\mathfrak{h}_f}$. Therefore, we can consider the induced $\mathcal{U}(\mathfrak{g})$-module $V_f = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h}_f)} k.v_f$. If we denote the corresponding action $\rho : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_k(V_f)$, $I(f) = \mathrm{Ker}(\rho)$ is a bilateral ideal of the enveloping algebra $\mathcal{U}(\mathfrak{g})$.

In the previous notation we have omitted the polarization in $I(f)$. This is justified by the following proposition, which states even more.

Proposition 4.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of finite dimension, let $f \in \mathfrak{g}^*$ and let $\mathfrak{h}_f$ and $\mathfrak{h}'_f$ be two polarizations of f. If we denote $\rho : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_k(V_f)$ and $\rho' : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_k(V'_f)$ the corresponding representations constructed following the previous method, then* $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho')$. *This ideal is primitive.*

On the other hand, if I is a primitive ideal of $\mathcal{U}(\mathfrak{g})$, then there exists $f \in \mathfrak{g}^*$ such that $I = I(f)$.

Proof. Cf. [Dix], Thm. 6.1.1, Thm. 6.1.4 and Thm. 6.1.7. □

The group $\mathrm{Aut}(\mathfrak{g})$ is an algebraic group whose associated Lie algebra is $\mathrm{Der}(\mathfrak{g})$. Let $\mathfrak{a}$ denote the algebraic Lie algebra generated by the ideal $\mathrm{InnDer}(\mathfrak{g})$ in $\mathrm{Der}(\mathfrak{g})$. The irreducible algebraic group G associated to $\mathfrak{a}$ is called the *adjoint algebraic group* of $\mathfrak{g}$. It is a subgroup of $\mathrm{Aut}(\mathfrak{g})$. If $\mathfrak{a} = \mathrm{InnDer}(\mathfrak{g})$, G is called the *adjoint group* of $\mathfrak{g}$. As a consequence, the group G acts on the Lie algebra $\mathfrak{g}$, so it also acts on $\mathfrak{g}^*$ with the dual action, which is called *coadjoint*.

THEOREM 4.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of finite dimension and let f and f' be two linear forms on $\mathfrak{g}$. If $I(f)$ and $I(f')$ are the corresponding bilateral ideals of $\mathcal{U}(\mathfrak{g})$, then $I(f) = I(f')$ if and only if there is $g \in G$ such that $f = g.f'$.*

Proof. Cf. [Dix], Prop. 6.2.3. □

The previous results imply that, for a nilpotent Lie algebra of finite dimension there exists a bijection

$$I : \mathfrak{g}^*/G \to \mathrm{Prim}(\mathcal{U}(\mathfrak{g}))$$

between the set of classes of linear forms on $\mathfrak{g}$ under the coadjoint action and the set of primitive ideals of $\mathcal{U}(\mathfrak{g})$.

We shall now use the previous method for the first quotients of the Yang-Mills algebra $\mathfrak{ym}(n)$ by the the ideals of the low derived sequence $\mathcal{C}^l(\mathfrak{ym}(n))$ in order to find the possible weights for each step.

Given $f \in (\mathfrak{ym}(3)/\mathcal{C}^2(\mathfrak{ym}(3)))^*$,

$$f = \sum_{i=1}^{3} c_i x_i^* + \sum_{ij} c_{ij} x_{ij}^*,$$

with $c_{12} = 0$, $c_{13} \neq 0$ and $c_{23} \neq 0$, we obtain a polarization associated to f as follows. Consider

$$\mathfrak{h}_f = k.x_1 \oplus k.x_2 \oplus k.x_{12} \oplus k.x_{13} \oplus k.x_{23}.$$

Since the weight of $I(f)$ (i.e. the positive integer n such that $\mathcal{U}(\mathfrak{g})/I(f) \simeq A_n(k)$) is equal to $\dim_k(\mathfrak{g}/\mathfrak{h}_f)$ (cf. [Dix], 1.2.1, 1.2.8 and Prop. 6.2.2), we obtain that $\mathcal{U}(\mathfrak{ym}(3)/\mathcal{C}^2(\mathfrak{ym}(3)))/I(f) \simeq A_1(k)$. It is easy to show that there are no higher weights for $\mathfrak{ym}(3)/\mathcal{C}^2(\mathfrak{ym}(3))$.

Any element of $(\mathfrak{ym}(3)/\mathcal{C}^3(\mathfrak{ym}(3)))^*$ may be written as

$$f = \sum_{i=1}^{3} c_i x_i^* + \sum_{i<j} c_{ij} x_{ij}^* + \sum_{(ijk) \in J_3} c_{ijk} x_{ijk}^*.$$

If $c_{112} = c_{123} = 1$, $c_{113} = c_{221} = c_{312} = 0$, we find a polarization associated to f as follows:

$$\mathfrak{h}_f = k.x_{12} \oplus k.x_{13} \oplus k.x_{23} \oplus k.x_{112} \oplus k.x_{221} \oplus k.x_{113} \oplus k.x_{123} \oplus k.x_{312},$$

and hence $\mathcal{U}(\mathfrak{ym}(3)/\mathcal{C}^3(\mathfrak{ym}(3)))/I(f) \simeq A_3(k)$. If $c_{112} = 1$, $c_{221} = c_{113} = c_{312} = c_{123} = 0$,

$$\mathfrak{h}_f = k.x_2 \oplus k.x_{12} \oplus k.x_{13} \oplus k.x_{23} \oplus k.x_{112} \oplus k.x_{221} \oplus k.x_{113} \oplus k.x_{123} \oplus k.x_{312},$$

and hence $\mathcal{U}(\mathfrak{ym}(3)/\mathcal{C}^3(\mathfrak{ym}(3)))/I(f) \simeq A_2(k)$.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

For $\mathfrak{ym}(3)/\mathcal{C}^4(\mathfrak{ym}(3))$, every linear functional has the form

$$f = \sum_{i=1}^{3} c_i x_i^* + \sum_{i<j} c_{ij} x_{ij}^* + \sum_{(ijk)\in J_3} c_{ijk} x_{ijk}^* + \sum_{(ijkl)\in J_4} c_{ijkl} x_{ijkl}^*.$$

If $c_{312} = c_{2312} = c_{1112} = 1$, and all other coefficients are zero, so, taking

$$\mathfrak{h}_f = k.x_{12} \oplus k.x_{13} \oplus k.x_{112} \oplus k.x_{221} \oplus k.x_{113} \oplus k.x_{123} \oplus k.x_{312} \oplus \bigoplus_{(ijkl)\in J_4} k.x_{ijkl},$$

we obtain that $\mathcal{U}(\mathfrak{ym}(3)/\mathcal{C}^4(\mathfrak{ym}(3)))/I(f) \simeq A_4(k)$.

As a consequence, we have just proved that, given n such that $1 \leq n \leq 4$, there exists an ideal I in $\mathcal{U}(\mathfrak{ym}(3))$ satisfying $A_n(k) \simeq \mathcal{U}(\mathfrak{ym}(3))/I$. In fact, a stronger statement is true:

Theorem 4.4. *Let $r \in \mathbb{N}$ be a positive integer. There exists a surjective homomorphism of algebras*

$$\mathcal{U}(\mathfrak{ym}(3)) \twoheadrightarrow A_r(k).$$

Furthermore, there exists $l \in \mathbb{N}$ such that we can choose this homomorphism satisfying that it factors through the quotient $\mathcal{U}(\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))$

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{ym}(3)) \arrow[rr, two heads] \arrow[dr, two heads] & & A_r(k) \\
& \mathcal{U}(\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))) \arrow[ur, two heads] &
\end{tikzcd}$$

Before giving the proof of this theorem, we shall state the next corollary, our main result, which follows readily from Theorem 4.4 and the fact that that every $\mathcal{U}(\mathfrak{ym}(3))$ is a quotient of every $\mathcal{U}(\mathfrak{ym}(n))$ for $n \geq 4$.

Corollary 4.5. *Let $r, n \in \mathbb{N}$ be two positive integers, satisfying $n \geq 3$. There exists a surjective homomorphism of k-algebras*

$$\mathcal{U}(\mathfrak{ym}(n)) \twoheadrightarrow A_r(k).$$

Furthermore, there exists $l \in \mathbb{N}$ such that we can choose this morphism in such a way that it factors through the quotient $\mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))$

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{ym}(n)) \arrow[rr, two heads] \arrow[dr, two heads] & & A_r(k) \\
& \mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))) \arrow[ur, two heads] &
\end{tikzcd}$$

The previous corollary shows the strong link between representations of the Yang-Mills algebra YM(n) and representations of the Weyl algebra $A_r(k)$. Concretely, using the characterization of the category of representations over the latter exhibited in [BB], we are able to give a description of a subcategory

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

of infinite dimensional modules over the Yang-Mills algebra $\mathrm{YM}(n)$, for $n \geq 3$, by means of the surjection in Corollary 4.5. In order to do so, we shall briefly recall the results presented in [BB] and we refer to it for the general definitions of weight modules, generalized weight modules and orbits. Given an orbit $\mathcal{O}$, we denote by $\mathcal{W}(\mathcal{O})$ and $\mathcal{GW}(\mathcal{O})$ the subcategories of the categories of modules over $A_r(k)$ consisting of weight modules and generalized modules with support in the orbit $\mathcal{O}$, respectively.

Also, given an orbit $\mathcal{O}$, Bavula and Bekkert construct certain categories $\mathcal{L}_{\mathcal{O},\mathrm{nil}}$ and $\mathcal{L}_{\mathcal{O},1}$, whose set of objects is $\mathcal{O}$, isomorphic to $\mathcal{GW}(\mathcal{O})$ and $\mathcal{W}(\mathcal{O})$, respectively. Furthermore, if $\mathcal{O}$ is an orbit and M is a $\mathcal{L}_{\mathcal{O},\mathrm{nil}}$-module (resp. $\mathcal{L}_{\mathcal{O},1}$-module), they define an $A_r(k)$-module $M_{\mathcal{O}}$.

Depending on the orbit, either $\mathcal{L}_{\mathcal{O},\mathrm{nil}}$ or $\mathcal{L}_{\mathcal{O},1}$ may be isomorphic to one of the tame categories exhibited in Figure 4.1 (cf. [BB], Sec. 2.5 to 2.7):

On the other hand, they define the following collection of modules over some of the previous categories, which verify the properties stated in the next proposition. For a given k-linear category $\mathcal{C}$, we denote by $\mathrm{Ind}(\mathcal{C})$ the set of isomorphism classes of indecomposable finite dimensional $\mathcal{C}$-modules.

Proposition 4.6. *(i) The indecomposable $\mathcal{B}$-modules are given by*

$$\mathrm{Ind}(\mathcal{B}) = \{\mathcal{B}_n : n \in \mathbb{N}\},$$

where $\mathcal{B}_n = k[t]/(t^n)$, for $n \in \mathbb{N}$.

(ii) The indecomposable $\mathcal{C}$-modules are given by

$$\mathrm{Ind}(\mathcal{C}) = \{\mathcal{C}_{f,n} : n \in \mathbb{N}, f \in \mathrm{Irr}(k[t]) \setminus \{t\}\}.$$

where $\mathcal{C}_{f,n} = k[t]/(f^n)$, for $n \in \mathbb{N}$ and $f \in \mathrm{Irr}(k[t]) \setminus \{t\}$.

(iii) The indecomposable $\mathcal{D}$-modules are given by

$$\mathrm{Ind}(\mathcal{D}) = \{\mathcal{D}_{n,i} : n \in \mathbb{N}, i \in \{1,2\}\},$$

where $\mathcal{D}_{n,i}$ ($n \in \mathbb{N}$, $i = 1,2$) are defined in the k-vector space generated by $e_1, \dots, e_n$, such that $\mathcal{D}_{n,1}(1)$, $\mathcal{D}_{n,2}(2)$ and $\mathcal{D}_{n,1}(2)$, $\mathcal{D}_{n,2}(1)$ contain the vector e_j with odd and even indices, respectively. The action is given by $ue_j = e_{j+1}$ and $ve_j = e_{j+1}$ ($e_{n+1} = 0$).

(iv) The description of the indecomposable $\mathcal{I}^m$-modules is more involved and we refer to [BB], Sec. 3.2. We denote by W the free monoid generated by two letters a and b and, given $m \in \mathbb{N}$, by Ω_m the set of equivalence classes of all non-periodic words.

Given $m \in \mathbb{N}$,

$$\mathrm{Ind}(\mathcal{I}^m) = \{\mathcal{I}^m_{j,w}, \mathcal{I}^m_{z,f} : z \in \Omega_m, w \in W, f \in \mathrm{Ind}(k[t]), j = 0, \dots, m-1\}.$$

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

(v) *Since the category* $\mathcal{F}$ *is a quotient of* $\mathcal{I}^4$*, we consider the following* $\mathcal{F}$*-modules* $\mathcal{F}_{1,x-1} = \mathcal{I}^4_{aabb,x-1}$*,* $\mathcal{F}_{2,x-1} = \mathcal{I}^4_{bbaa,x-1}$*,* $\mathcal{F}_{1,f} = \mathcal{I}^4_{abab,f}$*,* $\mathcal{F}_{2,f} = \mathcal{I}^4_{baba,f}$ *and* $\mathcal{F}_{j,w} = \mathcal{I}^4_{j,w}$*, for* $f \in \mathrm{Ind}(k[t])$*,* $j = 0, \dots, m-1$*,* $w = b^p(ab)^q a^r$*,* $q \in \mathbb{N}_0$*,* $p, r \in \{0, 1\}$*. In this case, the class of indecomposable* $\mathcal{F}$*-modules is*

$$\mathrm{Ind}(\mathcal{F}) = \{\mathcal{F}_{1,x-1}, \mathcal{F}_{2,x-1}, \mathcal{F}_{1,f}, \mathcal{F}_{2,f}, \mathcal{F}_{j,w} : f \in \mathrm{Ind}(k[t]), j = 0, \dots, m-1, \\ w = b^p(ab)^q a^r, q \in \mathbb{N}_0, p, r \in \{0,1\}\}.$$

(vi) *Using the functor* $\mathcal{H}^m \to \mathcal{I}^{m+2}$*, given by* $i \mapsto i$*,* $a_i \mapsto a_i$ *y* $b_i \mapsto b_i$*, we are able to define the family of* $\mathcal{H}^m$*-modules* $\mathcal{H}^m_{j,w} = \mathcal{I}^{m+2}_{j,w}$*, for* $w \in W$*,* $|w| \leq m - j$*,* $j = 0, \dots, m-1$*. The indecomposable* $\mathcal{H}^m$*-modules are given by*

$$\mathrm{Ind}(\mathcal{H}^m) = \{\mathcal{H}^m_{j,w} : w \in W, |w| \leq m - j, j = 0, \dots, m-1\}.$$

Proof. Cf. [BB], Sec. 3.1. to 3.4. □

We may now state the results describing the subcategories of weight modules and generalized weight modules over the Weyl algebras $A_r(k)$.

THEOREM 4.7. *Let us consider the first Weyl algebra* $A_1(k)$*. In this case, the subcategory* $\mathcal{W}(\mathcal{O})$ *is tame for each orbit* $\mathcal{O}$*. Moreover,*

(i) *if* $\mathcal{O} \neq \mathbb{Z}$*, then* $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{A}}\mathrm{Mod}$ *and* $\mathrm{Ind}(\mathcal{W}(\mathcal{O})) = \{\mathcal{A}_{\mathcal{O}}\}$*.*

(ii) *if* $\mathcal{O} = \mathbb{Z}$*, then* $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{H}^1}\mathrm{Mod}$ *and* $\mathrm{Ind}(\mathcal{W}(\mathcal{O})) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$*.*

On the other hand, the category $\mathcal{GW}(\mathcal{O})$ *is tame since* $\mathrm{char}(k) = 0$*. Besides, in case* $\mathcal{O} \neq \mathbb{Z}$*,* $\mathcal{GW}(\mathcal{O}) \simeq {}_{\mathcal{B}}\mathrm{Mod}$ *and* $\mathrm{Ind}(\mathcal{GW}(\mathcal{O})) = \{\mathcal{B}_{\mathcal{O},n} : n \in \mathbb{N}\}$*. Finally, if* $\mathcal{O} = \mathbb{Z}$*, then* $\mathcal{GW}(\mathcal{O}) \simeq {}_{\mathcal{D}}\mathrm{Mod}$ *and* $\mathrm{Ind}(\mathcal{GW}(\mathcal{O})) = \{\mathcal{D}_{\mathcal{O},n,i} : n \in \mathbb{N}, i = 1, 2\}$*.*

Proof. Cf. [BB], Sec. 4.1. □

THEOREM 4.8. *Let us consider the second Weyl algebra* $A_2(k)$*. Then,* $\mathcal{W}(\mathcal{O})$ *is tame for each orbit* $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$*. Moreover,*

(i) *if* $\mathcal{O}_1 \neq \mathbb{Z}$ *and* $\mathcal{O}_2 \neq \mathbb{Z}$*,* $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{A}}\mathrm{Mod}$ *and* $\mathrm{Ind}(\mathcal{W}(\mathcal{O})) = \{\mathcal{A}_{\mathcal{O}}\}$*.*

(ii) *if* $\mathcal{O}_i = \mathbb{Z}$ *and* $\mathcal{O}_j \neq \mathbb{Z}$ *(*$i, j \in \{1, 2\}$*), then* $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{H}^1}\mathrm{Mod}$ *and* $\mathrm{Ind}(\mathcal{W}(\mathcal{O})) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$*.*

(iii) *if* $\mathcal{O}_1 = \mathcal{O}_2 = \mathbb{Z}$*,* $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{F}}\mathrm{Mod}$ *and* $\mathrm{Ind}(\mathcal{W}(\mathcal{O}))$ *equals* $\{\mathcal{F}_{\mathcal{O},1,x-1}, \mathcal{F}_{\mathcal{O},2,x-1}, \mathcal{F}_{\mathcal{O},1,f}, \mathcal{F}_{\mathcal{O},2,f}, \mathcal{F}_{\mathcal{O},j,w}\}$*.*

The subcategory $\mathcal{GW}(\mathcal{O})$ *is wild for each orbit* $\mathcal{O}$*.*

Proof. Cf. [BB], Sec. 4.2. □

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

THEOREM 4.9. *Let us now consider the Weyl algebra $A_n(k)$, $n \geq 3$. We shall write each orbit $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$.*

(i) If $\mathcal{O}_i \neq \mathbb{Z}$ $(i = 1, \dots, n)$, then $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{A}}\mathrm{Mod}$ and $\mathrm{Ind}(\mathcal{W}(\mathcal{O})) = \{\mathcal{A}_{\mathcal{O}}\}$.

(ii) If the number of non-degenerate orbits $\mathcal{O}_i$ is $n-1$, then $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{H}^1}\mathrm{Mod}$ and $\mathrm{Ind}(\mathcal{W}(\mathcal{O})) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

(iii) If the number of no-degenerate orbits $\mathcal{O}_i$ is $n-2$, then $\mathcal{W}(\mathcal{O}) \simeq {}_{\mathcal{F}}\mathrm{Mod}$ and

$$\mathrm{Ind}(\mathcal{W}(\mathcal{O})) = \{\mathcal{F}_{\mathcal{O},1,x-1}, \mathcal{F}_{\mathcal{O},2,x-1}, \mathcal{F}_{\mathcal{O},1,f}, \mathcal{F}_{\mathcal{O},2,f}, \mathcal{F}_{\mathcal{O},j,w}\}.$$

(iv) If the number of non-degenerate orbits $\mathcal{O}_i$ is less than $n-2$, then $\mathcal{W}(\mathcal{O})$ es wild.

Proof. Cf. [BB], Sec. 4.3. □

Finally, we shall give the proof of Theorem 4.4.

Proof of Theorem 4.4. We have studied in detail cases $r = 1, 2, 3, 4$. We shall then restrict ourselves to the case $r \geq 5$.

We know that $\mathfrak{ym}(3) = V(3) \oplus \mathfrak{tym}(3)$ as k-modules. Also, considered as a graded Lie algebra with the special grading, the Lie ideal $\mathfrak{tym}(3)$ is a free graded Lie algebra (concentrated in even degrees) generated by a graded vector space (concentrated in even degrees) $W(3)$, that is,

$$\mathfrak{tym}(3) \simeq \mathfrak{f}_{gr}(W(3)) \quad \text{and} \quad W(3) = \bigoplus_{l \in \mathbb{N}} W(3)_{2l+2},$$

where $\dim_k(W(3)_{2l+2}) = 2l + 1$ (cf. Eq. (3.8)). In fact, we shall choose a basis for the first two homogeneous subspaces of $W(3)$ as we did previously. For $W(3)_4$ we fix the basis $\{x_{12}, x_{13}, x_{23}\}$ and for $W(3)_6$ we fix the basis $\{x_{112}, x_{113}, x_{221}, x_{123}, x_{312}\}$.

By Proposition 2.2,

$$\mathcal{U}_{gr}(\mathfrak{tym}(3)) \simeq \mathcal{U}_{gr}(\mathfrak{f}_{gr}(W(3))) \simeq T_{gr}(W(3)),$$

and hence,

$$T(\mathcal{O}(W(3))) = \mathcal{O}(T_{gr}(W(3))) \simeq \mathcal{O}(\mathcal{U}_{gr}(\mathfrak{tym}(3))) = \mathcal{U}(\mathcal{O}(\mathfrak{tym}(3))),$$

where $\mathcal{O}$ denotes the corresponding forgetful functors.

Having a morphism of k-algebras from $\mathcal{U}(\mathcal{O}(\mathfrak{tym}(3)))$ to $A_m(k)$ is the same as having a morphism of k-algebras from $T(\mathcal{O}(W(3)))$ to $A_m(k)$, which in turn is the same as having a morphism of k-vector spaces from $\mathcal{O}(W(3))$ to $A_m(k)$. The morphism of algebras will be surjective if the image of the corresponding morphism of vector spaces contains the generators as an algebra of $A_m(k)$, denoted by $p_1, \dots, p_m, q_1, \dots, q_m$ as usual.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

From now on, we shall exclusively work in the non-graded case (for algebras), so we will omit the forgetful functor $\mathcal{O}$ without confusion. However, we will keep the canonical grading of the Yang-Mills algebra $\mathfrak{ym}(3)$.

Let us suppose $m \geq 2$. We consider the morphism of k-vector spaces

$$\phi : W(3) \to A_m(k)$$

such that $\phi(W(3)_4) = \{0\}$, $\phi(x_{112}) = p_1$, $\phi(x_{221}) = p_2$, $\phi(x_{123}) = q_1$, $\phi(x_{113}) = q_2$, $\phi(x_{312}) = 0$, and such that for each generator of the Weyl algebra p_i and q_i ($i \geq 3$), there exist homogeneous elements of degree greater than 6, $w_i, w'_i \in W(3)$ such that $\phi(w_i) = p_i$ and $\phi(w'_i) = q_i$. This last condition is easily verified, taking into account that $W(3)$ is infinite dimensional. Of course this means that there are a lot of choices for this morphism.

Let d_i and d'_i be the degrees of w_i and w'_i, respectively. Let j be the maximum of the degrees d_i and d'_i, and let $l = 2j+1$. The morphism ϕ induces a unique surjective homomorphism $\Phi : \mathcal{U}(\mathfrak{tym}(3)) \twoheadrightarrow A_m(k)$, equivalent to the homomorphism of Lie algebras

$$\mathfrak{tym}(3) \to \mathrm{Lie}(A_m(k)),$$

where $\mathrm{Lie}(-) : {}_k\mathrm{Alg} \to {}_k\mathrm{LieAlg}$ denotes the functor that associates to every associative algebra the Lie algebra with the same underlying vector space and Lie bracket given by the commutator of the algebra. The latter morphism may be factorized in the following way

$$\mathfrak{tym}(3) \to \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)) \to \mathrm{Lie}(A_m(k)),$$

where the first morphism is the canonical projection. Hence, the map Φ may be factorized as

$$\mathcal{U}(\mathfrak{tym}(3)) \twoheadrightarrow \mathcal{U}(\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))) \twoheadrightarrow A_m(k).$$

We have thus obtained a surjective homomorphism of k-algebras

$$\Psi : \mathcal{U}(\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))) \twoheadrightarrow A_m(k),$$

where the Lie algebra $\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$ is nilpotent. Moreover, it is a nilpotent ideal of the (nilpotent) Lie algebra $\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$. We have, as k-modules,

$$\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)) = V(3) \oplus \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)).$$

Let I be the kernel of Ψ in $\mathcal{U}(\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))$. Taking into account that the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))$ by I is a Weyl algebra which is simple, I is a maximal two-sided ideal, and then, there exists a linear functional $f \in (\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))^*$ such that $I = I(f)$. We fix a standard polarization $\mathfrak{h}_f$ for f, i.e., a polarization constructed from a flag of ideals of $\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$. Let $\bar{f} \in (\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))^*$ be any extension of

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

f, and let $\mathfrak{h}_{\bar{f}}$ be a standard polarization for $\bar{f}$ given by extending the flag of $\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$ to $\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$, i.e., if the flag is:

$$\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)) \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3 = \mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)),$$

and we denote $\bar{f}_i$ the restriction of $\bar{f}$ to $\mathfrak{g}_i$ $(i = 1, 2, 3)$, then

$$\mathfrak{h}_{\bar{f}} = \mathfrak{h}_f + \mathfrak{g}_1^{\bar{f}_1} + \mathfrak{g}_2^{\bar{f}_2} + \mathfrak{g}_3^{\bar{f}_3}.$$

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{h}$ a subalgebra. Given $I \triangleleft \mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g})$ a two sided ideal in an enveloping algebra of $\mathfrak{h}$, consider

$$\mathfrak{st}(I, \mathfrak{g}) = \{x \in \mathfrak{g} : [x, I] \subset I\}.$$

Since $\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$ is an ideal of the finite dimensional nilpotent Lie algebra $\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$, according to Proposition 6.2.8 of [Dix]

$$\mathfrak{st}(I(f), \mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))) = \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)) + \mathfrak{g}',$$

where

$$\mathfrak{g}' = \{x \in \mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)) : f([x, \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))]) = 0\}.$$

For our ideal, we get immediately that $\bar{x}_{12}, \bar{x}_{13}, \bar{x}_{23} \in I$, but $\bar{x}_{112}, \bar{x}_{221}, \bar{x}_{123}$ do not belong to I, since $\Psi(\bar{x}_{112}) = p_1$, $\Psi(\bar{x}_{221}) = p_2$ and $\Psi(\bar{x}_{123}) = q_1$.

Let $x \in \mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$, then $x = x' + y$, where

$$x' = \sum_{i=1}^{3} c_i \bar{x}_i \in V(3),$$

and $y \in \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$. Since $[y, I(f)] \subset I(f)$, this implies that $x \in \mathfrak{st}(I(f), \mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))$ if and only if $[x', I(f)] \subset I(f)$. Explicitly,

$$[x', \bar{x}_{12}] = \sum_{i=1}^{3} c_i [\bar{x}_i, \bar{x}_{12}] = c_1 \bar{x}_{112} - c_2 \bar{x}_{221} - c_3 \bar{x}_{123}.$$

If $[x', \bar{x}_{12}] \in I$, then $\Psi([x', \bar{x}_{12}]) = 0$, or,

$$c_1 p_1 - c_2 p_2 - c_3 q_1 = 0,$$

but the generators of $A_m(k)$ are linearly independent, so $c_1 = c_2 = c_3 = 0$, which gives $x' = 0$, implying $\mathfrak{st}(I(f), \mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))) = \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$. Hence $g' \subset \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$. As a consequence, we obtain the inclusion $\mathfrak{g}_i^{\bar{f}_i} \subset \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$, whence $\mathfrak{h}_{\bar{f}} \subset \mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$. By maximality of $\mathfrak{h}_f$ in $\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3))$, we find that $\mathfrak{h}_{\bar{f}} \subset \mathfrak{h}_f$. The other inclusion is even simpler, so $\mathfrak{h}_{\bar{f}} = \mathfrak{h}_f$.

Finally, the weight of the ideal $I(\bar{f})$ is equal to

$$\begin{aligned}\dim_k((\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))/\mathfrak{h}_{\bar{f}}) &= \dim_k((\mathfrak{ym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))/\mathfrak{h}_f) \\ &= \dim_k((\mathfrak{tym}(3)/\mathcal{C}^l(\mathfrak{ym}(3)))/\mathfrak{h}_{\bar{f}}) + 3 = m + 3.\end{aligned}$$

We have then proved that $A_r(k)$ is a quotient of $\mathcal{U}(\mathfrak{ym}(3))$, for any $r \geq 5$, and, as a consequence, for any $r \in \mathbb{N}$. $\square$

Remark 4.10. *The previous proof does not work for* $\mathfrak{ym}(2)$*, since in this case* $\mathfrak{tym}(2) \simeq \mathfrak{f}_{gr}(W(2))$ *with* $\dim(W(2)) = 1$.

Definition 4.11. *Let* A *be an associative or Lie* k*-algebra, and let* $\mathcal{R} \subset {}_A\mathrm{Mod}$ *be a full subcategory of the category of (left) modules over over* A*. We shall say that* $\mathcal{R}$ separates points *of* A *if for all* $a \in A$*,* $a \neq 0$*, there exists* $M \in \mathcal{R}$ *such that* a *acts on* M *as a non null morphism.*

By Proposition 3.1.15 of [Dix], we know that an ideal I in the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ is semiprime if and only if it is the intersection of primitive ideals. Since $\mathcal{U}(\mathfrak{g})$ is a domain, the null ideal $\{0\}$ is completely prime, and hence semiprime (cf. [Dix], 3.1.6). As a consequence, $\{0\}$ is an intersection of two-sided primitive ideals. A fortiori, the intersection of all primitive two-sided ideals in $\mathcal{U}(\mathfrak{g})$ is null, that is, the Jacobson radical of the enveloping algebra of $\mathfrak{g}$ is null, i.e. $J(\mathcal{U}(\mathfrak{g})) = \{0\}$. Analogously, since every maximal two-sided ideal is primitive (cf. [Dix], 3.1.6), the intersection of all maximal two-sided ideals is null.

Let $\mathcal{W}(n)$ be the full subcategory of ${}_{\mathcal{U}(\mathfrak{ym}(n))}\mathrm{Mod}$ consisting of all modules M satisfying the following property: there exist $r, l \in \mathbb{N}$ such that the action of the Yang-Mills algebra may be factorized as follows

$$\phi : \mathcal{U}(\mathfrak{ym}(n)) \twoheadrightarrow \mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))) \to A_r(k) \to \mathrm{End}_k(M).$$

Proposition 4.12. *If* $n \geq 3$*, the category* $\mathcal{W}(n)$ *separates points of the algebra* $\mathcal{U}(\mathfrak{ym}(n))$.

Proof. Let $x \in \mathcal{U}(\mathfrak{ym}(n))$ be a non zero element. There exists then $l \in \mathbb{N}$ such that $\pi_l(x) \in \mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))$ is non zero (cf. Lemma 2.4). Since $\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n))$ is a nilpotent finite dimensional Lie algebra, the intersection of all maximal two-sided ideals is zero, so there exists a maximal two-sided ideal $J \lhd \mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))$ such that $\pi_l(x) \notin J$. On the other hand, $\mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))/J \simeq A_r(k)$, for some $r \in \mathbb{N}$ (cf. [Dix], 4.5.8 and Thm. 4.7.9). Taking into account that the inverse image of a maximal two-sided ideal by a surjective k-algebra homomorphism is maximal, $I = \pi_l^{-1}(J)$ is a maximal two-sided ideal in $\mathcal{U}(\mathfrak{ym}(n))$ and $\mathcal{U}(\mathfrak{ym}(n))/I \simeq \mathcal{U}(\mathfrak{ym}(n)/\mathcal{C}^l(\mathfrak{ym}(n)))/J \simeq A_r(k)$. If $x \in I$, then $\pi_l(x) \in \pi_l(I) \subset J$, thus $x \notin I$.

Let M be a (left) module over $A_r(k)$, such that the image of x under the previous isomorphisms is not zero (take for instance $A_r(k)$). Hence, the previous isomorphisms induce a structure of $\mathcal{U}(\mathfrak{ym}(n))$-module over M, such that x does not act as the zero endomorphism on M. The proposition is then proved. □

Remark 4.13. *Although the subcategory* $\mathcal{W}(n)$ *separates points of the Yang-Mills algebra* $\mathrm{YM}(n)$*, it does not satisfy that every element in the cate-*

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

gory ${}_{\mathrm{YM}(n)}\mathrm{Mod}$ *is isomorphic to an element in* $\mathcal{W}(n)$. *This is a consequence of the following fact: every object of* $\mathcal{W}(n)$ *has finite Gelfand-Kirillov dimension, for, if* $M \in \mathcal{W}(n)$ *is an induced module of a module over* $A_r(k)$, *then (cf. [MRS], Prop. 1.15, (ii); Prop. 3.2, (iii), (v))*

$$\text{GK-dim}_{\mathrm{YM}(n)}(M) = \text{GK-dim}_{A_r(k)}(M) \leq \text{GK-dim}_{A_r(k)}(A_r(k)) = 2r.$$

On the other hand, $\mathrm{YM}(n)$ *having infinite Gelfand-Kirillov dimension since it has exponential growth (cf. [CD]), there exist* $\mathrm{YM}(n)$*-modules (e.g. the regular module* $\mathrm{YM}(n)$ *in itself) which do not belong to* $\mathcal{W}(n)$.

$\mathcal{A} = k,$

$\mathcal{B} = k\langle t\rangle,$ $\quad$ t nilpotent,

$\mathcal{C} = k\langle t, t^{-1}\rangle,$

$\mathcal{D}$

$$\begin{tikzcd} \bullet^1 \arrow[r, shift left, "a"] & \bullet^2 \arrow[l, shift left, "b"] \end{tikzcd}, \qquad ab \text{ nilpotent},$$

$\mathcal{E}$

$$\begin{tikzcd} \bullet^1 \arrow[loop left, "a_{11}"] \arrow[r, shift left, "a_{21}"] & \bullet^2 \arrow[l, shift left, "a_{12}"] \arrow[loop right, "a_{22}"] \end{tikzcd}$$

$$a_{11}^2 = a_{12}a_{21}, \qquad a_{22}^2 = a_{21}a_{12},$$
$$a_{21}a_{11} = a_{22}a_{21} \qquad a_{12}a_{22} = a_{11}a_{12};$$
a_{11} and a_{22} nilpotent,

$\mathcal{F}$

$$\begin{tikzcd} \bullet^1 \arrow[r, shift left, "a_2"] \arrow[d, shift left, "b_1"] & \bullet^2 \arrow[l, shift left, "b_2"] \arrow[d, shift left, "a_3"] \\ \bullet^0 \arrow[u, shift left, "a_1"] \arrow[r, shift left, "b_4"] & \bullet^3 \arrow[l, shift left, "a_4"] \arrow[u, shift left, "b_3"] \end{tikzcd}$$

$a_ib_i = b_ia_i = 0$, $i = 1, \dots, 4$,
$a_ia_j = b_lb_m = 0$, if possible,

$\mathcal{G}$

$$\begin{tikzcd} \bullet^1 \arrow[r, shift left, "d"] & \bullet^0 \arrow[l, shift left, "c"] \arrow[r, shift left, "a"] & \bullet^2 \arrow[l, shift left, "b"] \end{tikzcd}$$

$ba = dc;$ $\qquad$ ab and cd nilpotent,

$\mathcal{H}^m$ $(m \in \mathbb{N})$

$$\begin{tikzcd} \bullet^0 \arrow[r, shift left, "a_1"] & \bullet^1 \arrow[l, shift left, "b_1"] \arrow[r, dotted, no head] & \bullet^{m-1} \arrow[r, shift left, "a_m"] & \bullet^m \arrow[l, shift left, "b_m"] \end{tikzcd}$$

$\mathcal{I}^m$ $(m \in \mathbb{N})$

$$\begin{tikzcd} \bullet^1 \arrow[r, shift left, "a_1"] \arrow[d, shift left, "b_m"] & \bullet^2 \arrow[l, shift left, "b_1"] \arrow[d, shift left, "a_2"] \\ \bullet^m \arrow[u, shift left, "a_m"] \arrow[r, dotted, no head] & \bullet^3 \arrow[u, shift left, "b_2"] \end{tikzcd}$$

$a_ib_i = b_ia_i = 0$, $i = 1, \dots, m$.

Figure 4.1: List of tame categories considered in [BB]. If a set of paths is said to be nilpotent, this means that we are considering the inverse limit of the categories such that this set of paths is nilpotent of finite nilpotency order.

Departamento de Matemática
Facultad de Ciencias Exctas y Naturales
Universidad de Buenos Aires
E-mail address: eherscov@dm.uba.ar

Departamento de Matemática
Facultad de Ciencias Exctas y Naturales
Universidad de Buenos Aires
E-mail address: asolotar@dm.uba.ar

References

[BB] Bavula, V.; Bekkert, V. *Indecomposable representations of generalized Weyl algebras.* Comm. Algebra **28**, (2000), no. 11, pp. 5067–5100.

[Ber1] Berger, R. *Koszulity for nonquadratic algebras.* J. Algebra **239**, (2001), no. 2, pp. 705–734. Corrigendum: *Koszulity for nonquadratic algebras II.* http://arxiv.org/abs/math/0301172.

[Ber2] Berger, R. *La catégorie des modules gradués sur une algèbre graduée* . Preprint. http://webperso.univ-st-etienne.fr/ rberger/mes-textes.html.

[CD] Connes, A.; Dubois-Violette, M. *Yang-Mills Algebra.* Lett. Math. Phys. **61**, (2002), no. 2, pp. 149–158.

[Cou] Coutinho, S. *A primer of algebraic D-modules.* London Mathematical Society Student Texts, **33**. Cambridge University Press, Cambridge, 1995.

[Dix] Dixmier, J. *Enveloping algebras.* Revised reprint of the 1977 translation. Graduate Studies in Mathematics, **11**, American Mathematical Society, Providence, RI, 1996.

[Doug] Douglas, M. *Report on the status of the Yang-Mills Millenium Prize Problem.* Preprint. http://www.claymath.org/millennium/Yang-Mills_Theory/ym2.pdf.

[HKL] Hai, P.; Kriegk, B.; Lorenz, M. *N-homogeneous superalgebras.* J. Noncommut. Geom. **2**, (2008), no. 1, pp. 1–51.

[Hum] Humphreys, J. *Introduction to Lie algebras and representation theory.* Second printing, revised. Graduate Texts in Mathematics, **9**, Springer-Verlag, New York-Berlin, 1978.

[Kel] Keller, B. *Corrections to 'Sur les A_∞-catégories'.* Preprint. http://www.math.jussieu.fr/~keller/lefevre/TheseFinale/corrainf.pdf.

[La] Lang, S. *Algebra.* Revised third edition. Graduate Texts in Mathematics, **211**, Springer-Verlag, New York, 2002.

[Lef] Lefèvre-Hasegawa, K. *Sur les A_∞-catégories.* Thèse de Doctorat, Spécialité: Mathématiques, Université Paris 7, novembre 2003. http://www.math.jussieu.fr/~keller/lefevre/TheseFinale/tel-00007761.pdf.

[MRS] McConnell, J.; Robson, J.; Small, L. *Noncommutative noetherian rings.* Graduate Studies in Mathematics, **30**, American Mathematical Society, Providence, RI, 2000.

[Mov] Movshev, M. *On deformations of Yang-Mills algebras.* http://arxiv.org/abs/hep-th/0509119.

[MS] Movshev, M.; Schwarz, A. *Algebraic structure of Yang-Mills theory.* The unity of mathematics, pp. 473–523, Progr. Math., **244**, Birkhäuser Boston, Boston, MA, 2006.

[Ne] Nekrasov, N. *Lectures on open strings and noncommutative gauge fields.* Unity from duality: gravity, gauge theory and strings (Les Houches, 2001), pp. 477–495, NATO Adv. Study Inst., EDP Sci., Les Ulis, 2003.

Feb 25 2009 18:43:25 PST
Vers. 1 - Sub. to Annals

[PP] Polishchuk, A.; Positselski, L. *Quadratic Algebras*. University Lecture Series, **37**, American Mathematical Society, Providence, RI, 2005.

[Sl] Sloane, N. *The On-Line Encyclopedia of Integer Sequences*, (2008). Published electronically at `http://www.research.att.com/~njas/sequences`.

[Wei] Weibel, C. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics, **38**. Cambridge University Press, Cambridge, 1994.

(Received)
(Revised)