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Some polynomial versions of cotype and applications ☆

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We introduce non-linear versions of the classical cotype of Banach spaces. We show that spaces with l.u.st. and cotype, and spaces having Fourier cotype enjoy our non-linear cotype. We apply these concepts to get results on convergence of vector-valued power series in infinite many variables and on ℓ_1-multipliers of vector-valued Dirichlet series. Finally we introduce cotype with respect to indexing sets, an idea that includes our previous definitions.

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1. Homogeneous cotype

Cotype, introduced in the 1970s by Maurey and Pisier, is one of the cornerstones of modern Banach space theory. We recall that a complex Banach space X has cotype $q \geq 2$ (see e.g. [10, Chapter 11]) if there exists a constant $C > 0$ such that for every finite choice of elements $x_1, \dots, x_N \in X$,

$$\Big(\sum_{k=1}^{N} \|x_k\|^q\Big)^{1/q} \leq C \Big(\int_0^1 \Big\|\sum_{k=1}^{N} r_k(t) x_k\Big\|^2 dt\Big)^{1/2}, \tag{1}$$

where r_k is the k-th Rademacher function.

It is a well known fact (see e.g. [19, Chapter 4]) that cotype can be reformulated in terms of Steinhaus variables, i.e., variables that are uniformly distributed in the torus $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then a Banach space X has cotype $q \geq 2$ if and only if there is a constant $C > 0$ such that for any choice of finitely many vectors $x_1, \dots, x_N \in X$ we have

$$\Big(\sum_{k=1}^{N} \|x_k\|^q\Big)^{1/q} \leq C \Big(\int_{\mathbb{T}^N} \Big\|\sum_{k=1}^{N} x_k z_k\Big\|^2 dz\Big)^{1/2}. \tag{2}$$

Here $\mathbb{T}^N$ is the N-dimensional torus, which is the N-fold product of $\mathbb{T}$ endowed with the N-fold product of the normalized Lebesgue measure on $\mathbb{T}$. We will later use the same notation for $N = \infty$.

If we denote by $C_q^R(X)$ and $C_q(X)$ the best constants in the inequalities (1) and (2), respectively, then we know from [19, Proposition 4.2.14] that

$$C_q(X) \leq C_q^R(X) \leq \frac{\pi}{2} C_q(X).$$

The complex approach to cotype is going to be more convenient for us.

Cotype can be seen as a property of the Banach space X in terms of linear mappings in the variables $z_1, z_2, \dots$ with values in X. Our aim in this note is to consider cotype-like properties which consider not only linear mappings, but also other algebraic combinations: polynomials (of certain classes) in the variables $z_1, z_2, \dots$ with values in X. For this, we introduce the following notation: if $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ is a multi-index (a finite sequence on $\mathbb{N}_0$ of arbitrary length), we write z^α for the monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and set $|\alpha| := \alpha_1 + \alpha_2 + \cdots$.

For each m-homogeneous polynomial on N variables

$$P(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} x_\alpha z^\alpha$$

there exists a unique symmetric m-linear form $T(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1,\dots,i_m=1}^{N} a_{i_1,\dots,i_m} z^{(1)}_{i_1} \dots z^{(m)}_{i_m}$ such that

$$P(z) = T(z, \dots, z) \text{ for every } z \in \mathbb{C}^N .$$

Then [4, Proposition 2.1] and the relation between the coefficients x_α and $a_{i_1,\dots,i_m}$ (see e.g. [5, page 544] or [2, Lemma 2.5]) immediately give that for every finite family $(x_\alpha)_{|\alpha|=m}$ (i.e., a family with only finitely many non-zero elements) we have

$$\Big(\sum_{|\alpha|=m} \|x_\alpha\|^q \Big)^{1/q} \le \big(C_q(X)\, K\big)^m \frac{m^m}{m!} (m!)^{1/q'} \Big(\int_{\mathbb{T}^\infty} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\|^2 dz \Big)^{1/2}, \tag{3}$$

where K denotes the constant in the $(2,1)$-Kahane inequality (see e.g. [10, Theorem 11.1]) and $\frac{1}{q} + \frac{1}{q'} = 1$. Let us observe that on the right-hand side we are actually integrating on some finite dimensional $\mathbb{T}^N$, where N can be taken as the maximum of those k such that there exists α with $x_\alpha \neq 0$ and $\alpha_k \neq 0$.

Note that letting $m = 1$ in this inequality we have exactly (2). Hence, (3) can be seen as a sort of homogeneous version of the classical cotype. We will then say that X has *m-homogeneous cotype q* if there exists a constant $C > 0$ such that for any finite multi-indexed sequence $(x_\alpha)_{|\alpha|=m} \subset X$ we have

$$\Big(\sum_{|\alpha|=m} \|x_\alpha\|^q \Big)^{1/q} \le C \Big(\int_{\mathbb{T}^\infty} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\|^2 dz \Big)^{1/2}. \tag{4}$$

The *constant of m-homogeneous cotype*, which we denote by $C_{q,m}(X)$, will be the best constant for which the inequality holds.

With this definition, what (3) is telling us is that if X has cotype q, then it also has m-homogeneous cotype q with $C_{q,m}(X) \le \big(C_q(X)\, K\big)^m \frac{m^m}{m!} (m!)^{1/q'}$. On the other hand, it is easy to see that if X has m-homogeneous cotype q for some m and q, then X has cotype q with $C_q(X) \le C_{q,m}(X)$.

In other words, cotype and m-homogeneous cotype are equivalent properties. This fact has interesting consequences for vector-valued power and Dirichlet series (see e.g. [4]), but for some applications (see Section 3) a better control of the behaviour of $C_{q,m}(X)$ as m grows is needed. When we do have such control, we say that the Banach space X has *hypercontractive homogeneous cotype*.

Definition 1.1. A Banach space X has hypercontractive homogeneous cotype q if there exists $C > 0$ such that for every $m \in \mathbb{N}$ and every finite family $(x_\alpha)_{|\alpha|=m}$ we have

$$\Big(\sum_{|\alpha|=m} \|x_\alpha\|^q \Big)^{1/q} \le C^m \Big(\int_{\mathbb{T}^\infty} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\|^2 dz \Big)^{1/2}.$$

Hypercontractive homogeneous cotype is clearly a local property, and it means m-homogeneous cotype for all m together with an estimate of the form $C_{q,m}(X) \leq C^m$ for some universal constant C. We consider

$$\cot(X) := \inf \left\{ 2 \leq q < \infty \,\middle|\, X \text{ has cotype } q \right\}$$

and

$$\cot_{\mathrm{Hyp}}(X) := \inf \left\{ 2 \leq q < \infty \,\middle|\, X \text{ has hypercontractive homogeneous cotype } q \right\}.$$

Although these infima are in general not attained we call them the optimal cotype and the optimal hypercontractive homogeneous cotype of X. If there is no $2 \leq q < \infty$ for which X has (hypercontractive homogeneous) cotype q, then X is said to have trivial (hypercontractive homogeneous) cotype, and we put $\cot(X) = \infty$ (or $\cot_{\mathrm{Hyp}}(X) = \infty$).

Clearly, if X has hypercontractive homogeneous cotype, then it has (classical) cotype or, in other words, $\cot(X) \leq \cot_{\mathrm{Hyp}}(X)$ for every Banach space X. We conjecture that these two concepts are actually equivalent; that is: a Banach space has hypercontractive homogeneous cotype q if and only if it has cotype q.

We are not able to prove our conjecture, but we give some positive answers. First we show that for spaces having local unconditional structure it is true (Theorem 2.1). We prove that spaces having Fourier cotype also have hypercontractive homogeneous cotype (Proposition 2.4). As a consequence we have that for Schatten classes $\mathscr{S}_r$ with $r \geq 2$ our conjecture is true, and also that for Banach spaces with type 2 the equality $\cot(X) = \cot_{\mathrm{Hyp}}(X)$ holds.

By Kahane's inequality (see e.g. [10, Theorem 11.1]), the L_2 norm on the right-hand side of inequality (2) can be changed to any other L_p-norm. Before we go into details, we give a kind of polynomial version of Kahane's inequality. This shows that in Definition 1.1 we can take any L_p-norm on the right hand side, just as in the usual definition of cotype. A recent result [8, Theorem 2.1] shows that the constant $({}^{r}/{}_{s})^{m/2}$ is almost optimal in this case.

Proposition 1.2. *For $1 \leq s \leq r < \infty$, any Banach space X and any finite sequence $(x_\alpha)_{|\alpha|=m} \subset X$ we have*

$$\left(\int\limits_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\|^r dz \right)^{1/r} \leq \left(\frac{r}{s} \right)^{\frac{m}{2}} \left(\int\limits_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\|^s dz \right)^{1/s}.$$

For the proof of Proposition 1.2, we introduce vector-valued Hardy spaces. We define them in a more general setting than needed for this proof, since we will come back to them later in Section 3.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences in $\mathbb{Z}$) the α-th Fourier coefficient $\hat{f}(\alpha)$ of $f \in L_1(\mathbb{T}^\infty, X)$ is given by

$$\hat{f}(\alpha) = \int\limits_{\mathbb{T}^\infty} f(z) z^{-\alpha} dz\,.$$

Then, given $1 \leq r \leq \infty$, the X-valued Hardy space on $\mathbb{T}^\infty$ is the subspace of $L_r(\mathbb{T}^\infty, X)$ defined as

$$H_r(\mathbb{T}^\infty, X) = \left\{ f \in L_r(\mathbb{T}^\infty, X) \mid \hat{f}(\alpha) = 0\,, \quad \forall \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \right\}.$$

The spaces $H_r(\mathbb{T}^N, X)$ with $N \in \mathbb{N}$, are defined analogously.

Given $f \in H_s(\mathbb{T}, X)$ and $0 < c < 1$, we define for $z = e^{i\theta}$ the Poisson integral

$$\mathscr{P}_c(f)(z) := \frac{1}{2\pi} P_c * f(z) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(e^{it}) P_c(\theta - t) dt,$$

where P_c denotes the Poisson kernel

$$P_c(t) = \sum_{n=-\infty}^{\infty} c^{|n|} e^{int} = \frac{1 - c^2}{1 - 2c\cos(t) + c^2}.$$

Equivalently, $\mathscr{P}_c(f)$ can be defined as the function whose Fourier coefficients are

$$\widehat{\mathscr{P}_c(f)}(n) = c^n \hat{f}(n), \quad \text{for } n \in \mathbb{N}_0.$$

As in the scalar valued case, the Poisson integral gives an 'extension' of $f \in H_s(\mathbb{T}, X)$ to a function F on the disc $\mathbb{D}$, defining for $w = \rho e^{i\theta} \in \mathbb{D}$:

$$F(w) = \mathscr{P}_\rho(f)(e^{i\theta}) = \sum_{n=0}^{\infty} \hat{f}(n) w^n. \tag{5}$$

For $s = +\infty$, we also have

$$\sup_{w \in \mathbb{D}} |F(w)| = \|f\|_{H_\infty(\mathbb{T}, X)}.$$

We refer to [3] and the references therein for details. Just for completeness, we comment that going the other way around (i.e., starting with a function on the disc and taking its boundary values to get a function on the torus) is not always possible in the vector-valued case. This is true if and only if X has the analytic Radon–Nikodým property.

The operator $\mathscr{P}_c : H_s(\mathbb{T}, X) \to H_s(\mathbb{T}, X)$ is a linear contraction, since it is given by the convolution with a function of L_1-norm one (note the normalization by 2π). Weissler in [21] proved that if $r > s \geq 1$, then $\mathscr{P}_c : H_s(\mathbb{T}, \mathbb{C}) \to H_r(\mathbb{T}, \mathbb{C})$ is again a contraction for every $c \leq \sqrt{s/r}$ and this value is optimal. We use now his result to give a vector-valued version.

Lemma 1.3. *Let $r > s \geq 1$ and set $c = \sqrt{s/r}$. Then, the mapping $\mathscr{P}_c$ is a linear contraction from $H_s(\mathbb{T}, X)$ to $H_r(\mathbb{T}, X)$.*

Proof. Take $f \in H_s(\mathbb{T}, X)$ and $\varepsilon > 0$. By classical results (as in [14, Theorem 2.7]) we can find $\varphi \in H_s$ (the scalar-valued space) with $|\varphi(z)| = \|f(z)\|_X + \varepsilon$ for all $z \in \mathbb{T}$ and $g \in H_\infty(\mathbb{T}, X)$ with $\|g\|_{H_\infty(\mathbb{T},X)} \leq 1$ such that $f = \varphi\, g$. Now, if we call F, Φ and G the extensions of f, φ and g given by (5), we have

$$\begin{aligned}
\|\mathscr{P}_c f\|_{H_r(\mathbb{T},X)} &= \Big(\int_{\mathbb{T}} \|F(cz)\|_X^r dz \Big)^{1/r} = \Big(\int_{\mathbb{T}} \|\Phi(cz) G(cz)\|_X^r dz \Big)^{1/r} \\
&\leq \|g\|_{H_\infty(\mathbb{T},X)} \Big(\int_{\mathbb{T}} |\Phi(cz)|^r dz \Big)^{1/r} \\
&= \|g\|_{H_\infty(\mathbb{T},X)} \Big(\int_{\mathbb{T}} |\mathscr{P}_c(\varphi)(z)|^r dz \Big)^{1/r} \leq \Big(\int_{\mathbb{T}} |\varphi(z)|^s dz \Big)^{1/s},
\end{aligned}$$

where the last inequality is a consequence of [21, Corollary 2.1]. Now, the last expression is not greater than

$$\Big(\int_{\mathbb{T}} (\|f(z)\| + \varepsilon)_X^s dz \Big)^{1/s} \leq \Big(\int_{\mathbb{T}} \|f(z)\|_X^s dz \Big)^{1/s} + \varepsilon = \|f\|_{H_s(\mathbb{T},X)} + \varepsilon \,.$$

Since this holds for any $\varepsilon > 0$, the proof is complete. $\square$

Polynomials with coefficients in X belong to $H_s(\mathbb{T}, X)$, so as a particular case of the lemma, for $x_0, \ldots, x_N \in X$ we have:

$$\Big(\int_{\mathbb{T}} \Big\| \sum_{k=0}^{N} x_k \,(cz)^k \Big\|_X^r dz \Big)^{1/r} \leq \Big(\int_{\mathbb{T}} \Big\| \sum_{k=0}^{N} x_k z^k \Big\|_X^s dz \Big)^{1/s}. \tag{6}$$

Iterating as in [1, Theorem 9], working with one variable at a time and applying the continuous Minkowski inequality, we can deduce from (6) that $\mathscr{P}_c$ is also a continuous contraction from $H_s(\mathbb{T}^N, X)$ to $H_r(\mathbb{T}^N, X)$. For m-homogeneous polynomials this gives:

$$\Big(\int_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha \,(cz)^\alpha \Big\|^r \Big)^{1/r} \leq \Big(\int_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\|^s \Big)^{1/s},$$

which by the homogeneity of the polynomial yields Proposition 1.2.

2. Banach spaces with hypercontractive homogeneous cotype

For $q \geq 2$, Banach lattices with nontrivial concavity q have hypercontractive homogeneous cotype q. This fact can be deduced from an analysis of the proof of [7, Theorem 5.3]. Indeed, in a first step Krivine's calculus can be used to extend [1, Theorem 9] to Banach lattices (for details on Krivine's calculus see [15, pp. 40–42]). Then, the concavity property of the Banach lattice gives the result. In this section we give other Banach spaces, different from lattices, that have hypercontractive homogeneous cotype.

2.1. Local unconditional structure and hypercontractive homogeneous cotype

Our next result shows that every Banach space with local unconditional structure (l.u.st.) and cotype q has hypercontractive homogeneous cotype q, giving the first positive answer to our conjecture. Let us recall (see e.g. [20, Definition 1.1] or [10, Chapter 17]) that a Banach space X is said to have *local unconditional structure* if there exists $\lambda > 0$ such that for every finite dimensional subspace F of X there exists a space U with unconditional basis $\{u_n\}$ and operators $T : F \to U$ and $S : U \to F$ such that $ST = \mathrm{id}_F$ and

$$\|T\| \cdot \|S\| \cdot \chi\{u_n\} \leq \lambda,$$

where $\chi\{u_n\}$ denotes the unconditional basis constant of $\{u_n\}$.

Theorem 2.1. *If X has cotype q and l.u.st., then X has hypercontractive homogeneous cotype q.*

The theorem will be a direct consequence of the next two results. Pisier in [20] introduced what is now usually called *Pisier's property* (α). The next simple lemma shows that if X has cotype q and satisfies $(\bigstar)$, which is a polynomial weaker version of property (α) with good constants, then X has hypercontractive homogeneous cotype q. Then Proposition 2.3 shows that if X has cotype q and l.u.st., then it satisfies a strong version of property $(\bigstar)$.

Lemma 2.2. *Let X be a Banach space with cotype q and suppose there exists $C > 0$ such that for every finite family $(x_\alpha)_{\alpha \in \mathbb{N}_0^N,\, |\alpha| = m} \subset X$,*

$$\left(\int_\Omega \int_{\mathbb{T}^N} \Big\| \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} x_\alpha \varepsilon_\alpha(\omega) z^\alpha \Big\|^2 dz d\omega \right)^{1/2} \leq C^m \left(\int_{\mathbb{T}^N} \Big\| \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} x_\alpha z^\alpha \Big\|^2 dz \right)^{1/2}, \qquad (\bigstar)$$

where (ε_α) are i.i.d. Rademacher random variables.

Then X has hypercontractive homogeneous cotype q.

Proof. Let $C_q = C_q^R(X)$ be the Rademacher cotype q constant of X (i.e., the best constant in (1)). For each $z \in \mathbb{T}^N$, since (x_α) is a finite family we have

$$\Big(\sum_\alpha \|x_\alpha\|^q\Big)^{2/q} = \Big(\sum_\alpha \|x_\alpha z^\alpha\|^q\Big)^{2/q} \leq C_q^2 \int_\Omega \Big\|\sum_\alpha \varepsilon_\alpha(\omega) x_\alpha z^\alpha\Big\|^2 d\omega$$

Integrating this inequality on $z \in \mathbb{T}^N$ and using (★), we obtain

$$\Big(\sum_\alpha \|x_\alpha\|^q\Big)^{2/q} \leq C_q^2 \int_{\mathbb{T}^N}\int_\Omega \Big\|\sum_\alpha \varepsilon_\alpha(\omega) x_\alpha z^\alpha\Big\|^2 d\omega\, dz \leq C_q^2 C^{2m} \int_{\mathbb{T}^N} \Big\|\sum_\alpha x_\alpha z^\alpha\Big\|^2 dz\,.$$

Therefore, X has hypercontractive homogeneous cotype q. □

In the next result we follow and adapt some of the ideas of [20]. We recall that an operator between Banach spaces $u : X \to Y$ is absolutely q-summing if there is $C > 0$ such that for every finite family $x_1, \ldots, x_n \in X$ we have

$$\Big(\sum_{j=1}^n \|ux_j\|^q\Big)^{\frac{1}{q}} \leq C \sup_{x^* \in B_{X^*}} \Big(\sum_{j=1}^n |x^*(x_j)|^q\Big)^{\frac{1}{q}}.$$

The best constant C in this inequality is called the absolutely q-summing norm of u and is denoted by $\pi_q(u)$.

Proposition 2.3. *If X has cotype q and l.u.st., then there exists $C > 0$, such that for every choice of finitely many $x_\alpha \in X$ and signs $\varepsilon_\alpha = \pm 1$*

$$\Big(\int_{\mathbb{T}^N} \Big\|\sum_{|\alpha|=m} x_\alpha \varepsilon_\alpha z^\alpha\Big\|^2 dz\Big)^{1/2} \leq C\, q^{m/2} \Big(\int_{\mathbb{T}^N} \Big\|\sum_{|\alpha|=m} x_\alpha z^\alpha\Big\|^2 dz\Big)^{1/2}.$$

In particular, X satisfies (★).

Proof. We fix $\varepsilon_\alpha = \pm 1$ for each $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = m$, and define operators $u : X^* \to L_q(\mathbb{T}^N)$ and $v : X^* \to L_1(\mathbb{T}^N)$ by

$$u(x^*)(z) = \sum_{|\alpha|=m} \varepsilon_\alpha x^*(x_\alpha) z^\alpha \quad \text{and} \quad v(x^*)(z) = \sum_{|\alpha|=m} x^*(x_\alpha) z^\alpha\,.$$

For each $x^* \in X^*$, the scalar case in Proposition 1.2 gives

$$\begin{aligned}\|u(x^*)\|_{L_q} &= \Big(\int_{\mathbb{T}^N} \Big|\sum_{|\alpha|=m} \varepsilon_\alpha x^*(x_\alpha) z^\alpha\Big|^q dz\Big)^{1/q} \leq q^{m/2} \int_{\mathbb{T}^N} \Big|\sum_{|\alpha|=m} \varepsilon_\alpha x^*(x_\alpha) z^\alpha\Big| dz \\ &= q^{m/2} \|v(x^*)\|_{L_1}\,.\end{aligned}$$

From this and the very definition of the absolutely 1-summing norm we easily deduce that $\pi_1(u) \leq q^{m/2}\pi_1(v)$. By [20, Theorem 1.1] we have

$$\pi_q({}^t u) \leq C\pi_1(u) \leq Cq^{m/2}\pi_1(v)\,.$$

Now, [20, Proposition 1.1] states that, for $1 \leq p \leq \infty$, every $\varphi_1, \ldots, \varphi_n \in L_p(\mathbb{T}^N)$ (or any other $L_p(\mu)$, μ a probability measure) and every $y_1, \ldots, y_n \in X$, the operator $S : X^* \to L_p(\mathbb{T}^N)$ given by

$$S(x^*) = \sum_{i=1}^{n} x^*(y_i)\varphi_i \tag{7}$$

satisfies

$$\pi_p(S) \leq \left(\int_{\mathbb{T}^N} \Big\| \sum_{i=1}^{n} y_i\varphi_i(z) \Big\|^p dz \right)^{1/p} \leq \pi_p({}^t S). \tag{8}$$

Note that we can write u and v as in (7), taking $\varphi_\alpha(z) = \varepsilon_\alpha z^\alpha$, and $\varphi_\alpha(z) = z^\alpha$ respectively. As a consequence, we can use the second inequality in (8) for u and the first inequality in (8) for v to obtain

$$\begin{aligned}
\left(\int_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha \varepsilon_\alpha z^\alpha \Big\|^2 dz \right)^{1/2} &\leq \left(\int_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha \varepsilon_\alpha z^\alpha \Big\|^q dz \right)^{1/q} \\
&\leq \pi_q({}^t u) \leq Cq^{m/2}\pi_1(v) \leq C\, q^{m/2} \int_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\| dz \\
&\leq C\, q^{m/2} \left(\int_{\mathbb{T}^N} \Big\| \sum_{|\alpha|=m} x_\alpha z^\alpha \Big\|^2 dz \right)^{1/2} \qquad \square
\end{aligned}$$

2.2. Fourier cotype implies hypercontractive homogeneous cotype

Now we show that Banach spaces with Fourier cotype also have hypercontractive homogeneous cotype. This is independent of our result in the previous section (Theorem 2.1), since for example the Schatten classes $\mathscr{S}_r$ have Fourier cotype but do not have l.u.st.

There are many equivalent definitions of Fourier cotype (see [11]). Let us give the one that is more akin to our framework. Given $2 \leq q < \infty$, we say that X has Fourier cotype q if there is a constant $C > 0$ such that for each choice of finitely many vectors $x_1, \dots, x_N \in X$ we have

$$\Big(\sum_{k=1}^{N} \|x_k\|^q\Big)^{1/q} \le C\Big(\int_{\mathbb{T}} \Big\|\sum_{k=1}^{N} x_k z^k\Big\|^{q'} dz\Big)^{1/q'}. \tag{9}$$

We write

$$\cot_{\mathfrak{F}}(X) := \inf\Big\{2 \le q < \infty \,\Big|\, X \text{ has Fourier cotype } q\Big\},$$

and (although this infimum in general is not attained) we call it the optimal Fourier cotype of X. In the literature (see, for example, [16]) the sums in (9) usually run from $-M$ to M or in $\mathbb{Z}$, but it is not hard to check that both definitions are equivalent: the rotation invariance of the measure dz gives

$$\begin{aligned}\Big(\int_{\mathbb{T}} \Big\|\sum_{j=-M}^{M} x_j\, z^j\Big\|^{q'} dz\Big)^{1/q'} &= \Big(\int_{\mathbb{T}} \Big\|z^M \sum_{j=-M}^{M} x_j\, z^j\Big\|^{q'} dz\Big)^{1/q'} \\ &= \Big(\int_{\mathbb{T}} \Big\|\sum_{k=0}^{2M} x_{k-M}\, z^k\Big\|^{q'} dz\Big)^{1/q'},\end{aligned}$$

from which the equivalence follows easily.

Spaces with Fourier cotype satisfy a stronger version of hypercontractive homogeneous cotype, with a uniform constant for every (homogeneous or not) polynomial of any degree. This result is basically well known in the literature on Fourier type. It can be seen as a consequence of, for example, [11, Theorem 6.14] and the equivalence between Fourier type p and Fourier cotype q when $\frac{1}{p} + \frac{1}{q} = 1$. We give the proof for the sake of completeness.

Proposition 2.4. *A Baach space X has Fourier cotype $q \ge 2$ if and only if there exists $C > 0$ such that for every finite family $(x_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ we have*

$$\Big(\sum_{\alpha} \|x_\alpha\|^q\Big)^{1/q} \le C\Big(\int_{\mathbb{T}^N} \Big\|\sum_{\alpha} x_\alpha z^\alpha\Big\|^{q'} dz\Big)^{1/q'}. \tag{10}$$

In particular, Banach spaces with Fourier cotype q have hypercontractive homogeneous cotype q.

Proof. If X satisfies (10), then it obviously has Fourier cotype q. For the reverse implication, let m be the maximum of all α_j's such that x_α is not zero. By the rotation invariance of the measures $dz_2, \dots, dz_N$, fixed $z_1 \in \mathbb{T}$ we have:

$$\begin{aligned}
&\int_{\mathbb{T}^{N-1}} \Big\| \sum_\alpha x_\alpha z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_N^{\alpha_N} \Big\|^{q'} dz_2 \cdots dz_N \\
&= \int_{\mathbb{T}^{N-1}} \Big\| \sum_\alpha x_\alpha z_1^{\alpha_1} (z_2\, z_1^{m+1})^{\alpha_2} \cdots (z_N\, z_1^{(m+1)^{N-1}})^{\alpha_N} \Big\|^{q'} dz_2 \cdots dz_N \\
&= \int_{\mathbb{T}^{N-1}} \Big\| \sum_\alpha x_\alpha z_1^{\alpha_1 + (m+1)\alpha_2 + \cdots + (m+1)^{N-1}\alpha_N} z_2^{\alpha_2} \cdots z_N^{\alpha_N} \Big\|^{q'} dz_2 \cdots dz_N.
\end{aligned}$$

As a consequence, a change in the order of integration gives

$$\int_{\mathbb{T}^N} \Big\| \sum_\alpha x_\alpha z^\alpha \Big\|^{q'} dz \tag{11}$$

$$= \int_{\mathbb{T}^{N-1}} \left(\int_{\mathbb{T}} \Big\| \sum_\alpha (x_\alpha z_2^{\alpha_2} \cdots z_N^{\alpha_N})\, z_1^{\alpha_1 + (m+1)\alpha_2 + \cdots + (m+1)^{N-1}\alpha_N} \Big\|^{q'} dz_1 \right) dz_2 \cdots dz_N. \tag{12}$$

For every α for which x_α is not zero we have $0 \le \alpha_j \le m$, $j = 1, \dots, N$. Also, if a multi-index β satisfies $0 \le \beta_j \le m$, $j = 1, \dots, N$ and

$$\alpha_1 + (m+1)\alpha_2 + \cdots + (m+1)^{N-1}\alpha_N = \beta_1 + (m+1)\beta_2 + \cdots + (m+1)^{N-1}\beta_N,$$

then we must have $\alpha = \beta$ (this is just the uniqueness of the expansion of a natural number in base $m+1$). Therefore, the powers of z_1 in the sum in (12) are all different. We can then apply (9) to the inner integral of (12) for each fixed $z_2, \dots, z_N$. This gives that the whole expression in (12) is bounded from below by

$$\frac{1}{C^{q'}} \int_{\mathbb{T}^{N-1}} \left(\sum_\alpha \| x_\alpha z_2^{\alpha_2} \cdots z_N^{\alpha_N} \|^q \right)^{q'/q} dz_2 \cdots dz_N = \frac{1}{C^{q'}} \left(\sum_\alpha \| x_\alpha \|^q \right)^{q'/q}.$$

So (11) is bounded from below by this last expression, which is the result we were looking for. □

We remark that in Proposition 2.4 we have a cotype-like inequality that holds for any polynomial of any degree and on any number of variables. Following our philosophy, we could call this *analytic cotype*.

Let us recall that a Banach space satisfying the reverse inequality in (2) for $1 \le q \le 2$ is said to have type q. It is a well known fact (which follows, for example, from [18, Section 6.1.8.6]) that if X has type 2 and cotype q_0, then it has Fourier cotype q for every $q > q_0$. Therefore, we have

$$\cot(X) = \cot_{\mathfrak{F}}(X) = \cot_{\mathrm{Hyp}}(X)$$

for every Banach space X with type 2.

2.3. Examples

By Theorem 2.1, cotype and hypercontractive homogeneous cotype coincide in $L_r(\mu)$ and, more generally, in $\mathscr{L}_r$-spaces for $1 \leq r \leq \infty$ (see Chapters 3 and 17 in [10] for the definition of $\mathscr{L}_r$-spaces and their local unconditional structure, respectively). As a consequence, an $\mathscr{L}_r$-space X has hypercontractive homogeneous cotype $\cot(X) = \max\{2, r\}$ for $1 \leq r \leq \infty$.

The Schatten classes $\mathscr{S}_r$ (as well as $\mathscr{L}_r$-spaces) have Fourier cotype $\max\{r, r'\}$ and these are the optimal values (see [12, Theorem 1.6] or [13, Theorem 6.8]). Thus by Proposition 2.4, they have hypercontractive homogeneous cotype $\max\{r, r'\}$ (in fact, they have the much stronger uniform and non-homogeneous one given in Proposition 2.4). On the other hand, these spaces have cotype $\max\{2, r\}$ and type $\min\{2, r\}$ [10, page 228]. In other words, hypercontractive homogeneous and usual cotypes coincide for Schatten classes for $r \geq 2$. Note that, since Schatten classes with $r \neq 2$ do not have l.u.st. [10, page 364], Theorem 2.1 does not apply in this case.

We summarize these positive answers to our conjecture in the following

Corollary 2.5. *Cotype and hypercontractive homogeneous cotype coincide in $\mathscr{L}_r$-spaces for $1 \leq r \leq \infty$ and in $\mathscr{S}_r$ for $2 \leq r \leq \infty$.*

3. Sets of monomial convergence for $H_p(\mathbb{T}^\infty, X)$

Each function $f \in H_p(\mathbb{T}^\infty, X)$ defines a formal power series $\sum_\alpha \hat{f}(\alpha) z^\alpha$. We address now the question of for which z's does this power series converge. Given a Banach space X and $1 \leq p \leq \infty$, we define the set of monomial convergence of $H_p(\mathbb{T}^\infty, X)$:

$$\operatorname{mon} H_p(\mathbb{T}^\infty, X) = \left\{ z \in \mathbb{C}^{\mathbb{N}} \;\middle|\; \sum_\alpha \|\hat{f}(\alpha) z^\alpha\|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X) \right\}.$$

We also define, for each $m \in \mathbb{N}$,

$$\operatorname{mon} H_p^m(\mathbb{T}^\infty, X) = \left\{ z \in \mathbb{C}^{\mathbb{N}} \;\middle|\; \sum_\alpha \|\hat{f}(\alpha) z^\alpha\|_X < \infty \text{ for all } f \in H_p^m(\mathbb{T}^\infty, X) \right\},$$

where

$$H_p^m(\mathbb{T}^\infty) = \left\{ f \in H_p(\mathbb{T}^\infty) \;\middle|\; \hat{f}(\alpha) \neq 0 \Rightarrow |\alpha| = m \right\}.$$

The problem of determining $\operatorname{mon} H_p(\mathbb{T}^\infty)$ and $\operatorname{mon} H_p^m(\mathbb{T}^\infty)$ in the scalar-valued case goes back to Bohr at the 1910s, and the so far most general result was recently given in [2] (for more information and detailed historical remarks see the references therein): For $p = \infty$ we have

$$\mathbf{B} \subset \operatorname{mon} H_\infty(\mathbb{T}^\infty) \subset \overline{\mathbf{B}},$$

where

$$\mathbf{B} = \left\{ u \in B_{c_0} \;\middle|\; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} |u_k^*|^2 < 1 \right\}$$

$$\bar{\mathbf{B}} = \left\{ u \in B_{c_0} \;\middle|\; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} |u_k^*|^2 \leq 1 \right\}$$

(u^* the decreasing rearrangement of u), and for $1 \leq p < \infty$

$$\operatorname{mon} H_p(\mathbb{T}^\infty) = \ell_2 \cap B_{c_0} \quad \text{for} \quad 1 \leq p < \infty\,.$$

In the homogeneous case we have for each m

$$\operatorname{mon} H_p^m(\mathbb{T}^\infty) = \begin{cases} \ell_{\frac{2m}{m-1},\infty} & \text{for } p = \infty \\ \ell_2 & \text{for } 1 \leq p < \infty\,. \end{cases}$$

It can be seen easily that in the preceding results scalar-valued functions can be replaced by functions with values in finite dimensional Banach spaces – but the following theorem indicates that the situation for functions with range in infinite dimensional spaces is substantially different (see also [9]).

Theorem 3.1. *Let $1 \leq p \leq \infty$, $m \in \mathbb{N}$, and X be an infinite dimensional Banach space.*

(1) *If X has trivial cotype, then*

$$\operatorname{mon} H_p(\mathbb{T}^\infty, X) = \ell_1 \cap B_{c_0} \quad \textit{and} \quad \operatorname{mon} H_p^m(\mathbb{T}^\infty, X) = \ell_1\,.$$

(2) *If X has hypercontractive homogeneous cotype $\cot(X) < \infty$, then*

$$\operatorname{mon} H_p(\mathbb{T}^\infty, X) = \ell_{\cot(X)'} \cap B_{c_0} \quad \textit{and} \quad \operatorname{mon} H_p^m(\mathbb{T}^\infty, X) = \ell_{\cot(X)'}\,.$$

To see some examples, we have mentioned in Section 2.3 that a $\mathscr{L}_r$-space X has hypercontractive homogeneous cotype $\cot(X) = \max\{2, r\}$ for $1 \leq r \leq \infty$, and that for $r \geq 2$, $\mathscr{S}_r$ has hypercontractive homogeneous cotype $\cot(\mathscr{S}_r) = r$. As a consequence, we have the following.

Corollary 3.2. *Let $1 \leq p \leq \infty$.*

(1) *If $1 \leq r \leq \infty$ and X is a $\mathscr{L}_r$-space then*

$$\operatorname{mon} H_p(\mathbb{T}^\infty, X) = \ell_{\min\{2,r'\}} \cap B_{c_0} \quad \textit{and} \quad \operatorname{mon} H_p^m(\mathbb{T}^\infty, X) = \ell_{\min\{2,r'\}}\,.$$

(2) *If* $2 \leq r \leq \infty$, *then*

$$\operatorname{mon} H_p(\mathbb{T}^\infty, \mathscr{S}_r) = \ell_{r'} \cap B_{c_0} \quad \textit{and} \quad \operatorname{mon} H_p^m(\mathbb{T}^\infty, \mathscr{S}_r) = \ell_{r'} \,.$$

We split the proof of Theorem 3.1 into two steps, that we present as separate lemmas. Before we state the first one, let us recall (see e.g. [10, Chapter 14]) that a Banach space X finitely factors $\ell_q \hookrightarrow \ell_\infty$ with $2 \leq q \leq \infty$ whenever for each N there are vectors $x_1, \dots, x_N \in X$ such that for every choice of $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ we have

$$\frac{1}{2}\|\lambda\|_\infty \leq \Big\| \sum_{n=1}^N \lambda_n x_n \Big\| \leq \|\lambda\|_q \,. \tag{13}$$

Lemma 3.3. *Let X be an infinite dimensional Banach space which finitely factors $\ell_q \hookrightarrow \ell_\infty$. Then* $\operatorname{mon} H^1_\infty(\mathbb{T}^\infty, X) \subset \ell_{q'}$.

Proof. Let us take $z \in \operatorname{mon} H^1_\infty(\mathbb{T}^\infty, X)$. By a standard closed graph argument there is a constant $c(z) > 0$ such that for each $f \in H^1_\infty(\mathbb{T}^\infty, X)$ we have

$$\sum_{n=1}^\infty \|f(e_n)\| \, |z_n| \leq c(z) \|f\|_\infty .$$

We fix some $N \in \mathbb{N}$ and choose $x_1, \dots, x_N \in X$ as in (13). Given $w_1, \dots, w_N \in \mathbb{C}$ we define $f \in H^1_\infty(\mathbb{T}^\infty, X)$ by $f(u) = \sum_{n=1}^N (x_n w_n) \, u_n$. Then we have

$$\begin{aligned} \sum_{n=1}^N |w_n z_n| &\leq 2 \sum_{n=1}^N \|(w_n x_n) z_n\| \leq 2c(z) \sup_{u \in \mathbb{T}^\infty} \Big\| \sum_{n=1}^N (w_n x_n) u_n \Big\| \\ &\leq 2c(z) \sup_{u \in \mathbb{T}^\infty} \big\| (w_n u_n)_{n=1}^N \big\|_q \leq 2c(z) \big\| (w_n)_{n=1}^N \big\|_q \,. \end{aligned}$$

Since the $w_1, \dots, w_N$ are arbitrary, this clearly proves the claim. □

Lemma 3.4. *If X has hypercontractive homogeneous cotype q, then $\ell_{q'} \cap B_{c_0} \subset$ $\operatorname{mon} H_1(\mathbb{T}^\infty, X)$.*

Proof. Assume here that $q < \infty$. We first prove that there is a constant $C > 0$ such that for each m, each $f \in H_1^m(\mathbb{T}^\infty, X)$, and each $y \in \ell_{q'} \cap B_{c_0}$ we have

$$\sum_{|\alpha|=m} \|\widehat{f}(\alpha) y^\alpha\| \leq C^m \Big(\sum_{|\alpha|=m} |y^\alpha|^{q'} \Big)^{1/q'} \|f\|_1 \,.$$

We fix such f, y and $N \in \mathbb{N}$; proceeding as in [4, page 524] we can find a function $f_N \in H_1(\mathbb{T}^N, X)$ such that $\|f_N\|_1 \leq \|f\|_1$ and $\widehat{f_N}(\alpha) = \widehat{f}(\alpha)$ for all $\alpha \in \mathbb{N}_0^N$. Using this fact, that X has hypercontractive homogeneous cotype q (with constant D, say) and Proposition 1.2 (the polynomial Kahane inequality) we have for $C = D\sqrt{2}$

$$\begin{aligned}
\sum_{\substack{\alpha\in\mathbb{N}_0^N\\|\alpha|=m}} \|\widehat{f}(\alpha)y^\alpha\| &\leq \Big(\sum_{|\alpha|=m}|y^\alpha|^{q'}\Big)^{1/q'}\Big(\sum_{|\alpha|=m}\|\widehat{f}_N(\alpha)\|^q\Big)^{1/q}\\
&\leq D^m\Big(\sum_{|\alpha|=m}|y^\alpha|^{q'}\Big)^{1/q'}\Big(\int_{\mathbb{T}^N}\|f_N(y)\|_X^2 dz\Big)^{1/2}\\
&\leq D^m\sqrt{2}^m\Big(\sum_{|\alpha|=m}|y^\alpha|^{q'}\Big)^{1/q'}\|f_N\|_1 \leq C^m\Big(\sum_{|\alpha|=m}|y^\alpha|^{q'}\Big)^{1/q'}\|f\|_1\,, \qquad (14)
\end{aligned}$$

Take now $0 < r < 1/C$, and let us check that

$$rB_{\ell_{q'}} \cap B_{c_0} \subset \operatorname{mon} H_1(\mathbb{T}^\infty, X)\,.$$

To do this we fix some $f \in H_1(\mathbb{T}^\infty, X)$ and $z = ry \in rB_{\ell_{q'}} \cap B_{c_0}$. For each N we consider f_N as above. By [4, Proposition 2.5] there is $f_N^m \in H_p(\mathbb{T}^N, X)$ such that $\widehat{f_N}(\alpha) = \widehat{f_N^m}(\alpha)$ for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = m$, $\widehat{f_N^m}(\alpha) = 0$ if $|\alpha| \neq m$, and $\|f_N^m\|_1 \leq \|f_N\|_1$. Then, applying (14) to f_N^m we get

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}_0^N}\|\widehat{f}(\alpha)z^\alpha\| &= \sum_{m=0}^\infty \sum_{\substack{\alpha\in\mathbb{N}_0^N\\|\alpha|=m}}\|\widehat{f_N^m}(\alpha)\ (ry)^\alpha\|\\
&\leq \sum_{m=0}^\infty r^m \sum_{\substack{\alpha\in\mathbb{N}_0^N\\|\alpha|=m}}\|\widehat{f_N^m}(\alpha)\ y^\alpha\|\\
&\leq \sum_{m=0}^\infty r^m C^m\Big(\sum_{|\alpha|=m}|y^\alpha|^{q'}\Big)^{1/q'}\|f_m^N\|_1\\
&\leq \sum_{m=0}^\infty r^m C^m\Big(\sum_{|\alpha|=m}|y^\alpha|^{q'}\Big)^{1/q'}\|f\|_1\\
&\leq \Big(\sum_{\alpha}|y^\alpha|^{q'}\Big)^{1/q'}\|f\|_1\sum_{m=0}^\infty r^m C^m\,.
\end{aligned}$$

Let us recall (see e.g. [6, page 24]) that

$$z \in \ell_1 \cap B_{c_0} \ \text{ if and only if } \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}|z^\alpha| < \infty\,. \qquad (15)$$

This implies that the last term is finite.

We can now complete our argument. For $z \in \ell_{q'} \cap B_{c_0}$ we choose n_0 such that

$$\tilde{z} = (0, \ldots, 0, z_{n_0}, z_{n_0+1}, \ldots) \in rB_{\ell_{q'}} \cap B_{c_0} .$$

Then $\tilde{z} \in \operatorname{mon} H_1(\mathbb{T}^\infty, X)$, and a straightforward vector-valued extension of [2, Lemma 3.7] (see also [5, Lemma 2] where the analogous result for $\operatorname{mon} H_\infty(\mathbb{T}^\infty, X)$ is shown) completes the proof. □

With this at our hand we are now ready to prove our main result in this section.

Proof of Theorem 3.1. We show parts (1) and (2) together. Take $1 \le p \le \infty$ and assume that X is an infinite dimensional Banach space with hypercontractive homogeneous cotype $\cot(X)$. By a vector-valued modification of [2, Lemma 3.3] we have

$$\operatorname{mon} H_p^m(\mathbb{T}^\infty, X) \subset \operatorname{mon} H_p^{m-1}(\mathbb{T}^\infty, X)$$

and trivially

$$\operatorname{mon} H_p^1(\mathbb{T}^\infty, X) \subset \operatorname{mon} H_\infty^1(\mathbb{T}^\infty, X) .$$

First of all, as a consequence of a deep result of Maurey and Pisier [17] (see also [10, Theorem 14.5 and page 304]) X always finitely factors $\ell_{\cot(X)} \hookrightarrow \ell_\infty$. Then Lemmas 3.3 and 3.4 give

$$\begin{aligned}\ell_{\cot(X)'} \cap B_{c_0} \subset \operatorname{mon} H_1(\mathbb{T}^\infty, X) \subset \operatorname{mon} H_p(\mathbb{T}^\infty, X) \subset \operatorname{mon} H_\infty^1(\mathbb{T}^\infty, X) \cap B_{c_0} \\ \subset \ell_{\cot(X)'} \cap B_{c_0} .\end{aligned}$$

This completes the argument. □

Let us remark that in Theorem 3.1-(2) we are assuming that X has non-trivial hypercontractive homogeneous cotype (hence also usual cotype) and both optimal values are equal and attained. If this is not the case, then our proof shows that

$$\ell_{\cot(X)'} \cap B_{c_0} \subset \operatorname{mon} H_p(\mathbb{T}^\infty, X) \subset \operatorname{mon} H_p^m(\mathbb{T}^\infty, X) \cap B_{c_0} = \ell_{\cot_{\mathrm{Hyp}}(X)'+\varepsilon} \cap B_{c_0} \tag{16}$$

for all $\varepsilon > 0$.

4. Multiplicative ℓ_1-multipliers for Hardy spaces of Dirichlet series

Power series in infinitely many variables and Dirichlet series can be identified by an ingenious idea of Bohr. For a fixed Banach space X we denote by $\mathfrak{P}(X)$ the vector space of all formal power series $\sum_\alpha c_\alpha z^\alpha$ in X and by $\mathfrak{D}(X)$ the vector space of all Dirichlet series $\sum_n a_n n^{-s}$ in X. Let $(p_n)_n$ be the sequence of prime numbers. Since each integer n

has a unique prime number decomposition $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} = p^\alpha$ with $\alpha_j \in \mathbb{N}_0$, $1 \le j \le k$, the linear mapping, that we call the Bohr transform in X,

$$\mathfrak{B}_X : \mathfrak{P}(X) \longrightarrow \mathfrak{D}(X)\,, \quad \textstyle\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \rightsquigarrow \sum_{n=1}^\infty a_n n^{-s}\,, \quad \text{where } a_{p^\alpha} = c_\alpha$$

is bijective. Given $1 \le p \le \infty$ and $m \in \mathbb{N}$, define the two linear spaces

$$\mathscr{H}_p(X) = \mathfrak{B}_X\left(H_p(\mathbb{T}^\infty, X)\right)$$

and

$$\mathscr{H}_p^m(X) = \mathfrak{B}_X\left(H_p^m(\mathbb{T}^\infty, X)\right)\,,$$

which through the norms induced by $\mathfrak{B}_X$ form (what we call) the Banach spaces of vector-valued Hardy–Dirichlet series.

A scalar sequence (b_n) is called multiplicative (or completely multiplicative) if $b_{mn} = b_n b_m$ for all m, n. Basic examples of multiplicative sequences (b_n) are the sequences $1/n^\sigma$. We call a scalar sequence (b_n) an ℓ_1-multiplier for $\mathscr{H}_p(X)$ whenever for all $\sum_n a_n n^{-s} \in \mathscr{H}_p(X)$ we have

$$\sum_{n=1}^\infty \|a_n\|_X\ |b_n| < \infty \quad \text{for all} \quad \sum_n a_n n^{-s} \in \mathscr{H}_p(X).$$

All multiplicative ℓ_1-multipliers for $\mathscr{H}_p(X)$ are denoted

$$\operatorname{mult} \mathscr{H}_p(X)\,,$$

and, given $m \in \mathbb{N}$, in the homogeneous case of course an analogous definition

$$\operatorname{mult} \mathscr{H}_p^m(X)$$

can be done. In [2, Remark 4.1] (here again the scalar case immediately transfers to the vector valued case) we have the following link between sets of monomial convergence and multiplicative ℓ_1-multipliers.

Remark 4.1. Let (b_n) be a multiplicative sequence of complex numbers, and $1 \le p \le \infty$. Then (b_n) is an ℓ_1-multiplier for $\mathscr{H}_p(X)$ if and only if $(b_{p_k}) \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$. Clearly, an analogous equivalence holds whenever we replace $\mathscr{H}_p(X)$ by $\mathscr{H}_p^m(X)$.

Remark 4.2. Suppose now that $1 \le p < \infty$. Let us observe that if $b \in \ell_p$ is multiplicative, then $|b_n| < 1$ for all n. Indeed, if some $|b_n| \ge 1$, then since the sequence is multiplicative $|b_{n^k}| \ge 1$ for every k and this contradicts the fact that b is in ℓ_p. Then, $b \in B_{c_0}$ for any multiplicative sequence $b \in \ell_p$. On the other hand, if $b \in B_{c_0}$ is multiplicative, by (15) we have that $(b_{p_k})_k \in \ell_p$ if and only if $b \in \ell_p$.

With Remarks 4.1, 4.2 and Theorem 3.1 we immediately have the following characterization of multiplicative ℓ_1-multipliers of $\mathscr{H}_p(X)$ and $\mathscr{H}_p^m(X)$, respectively.

Theorem 4.3. *Let $1 \leq p \leq \infty$, $m \in \mathbb{N}$, X be an infinite dimensional Banach space and $b = (b_n)$ be a multiplicative scalar sequence.*

(1) *If X has trivial cotype, then*

$$b \in \text{mult } \mathscr{H}_p(X) \Leftrightarrow (b_{p_k})_k \in \ell_1 \cap B_{c_0} \Leftrightarrow b \in \ell_1$$
$$b \in \text{mult } \mathscr{H}_p^m(X) \Leftrightarrow (b_{p_k})_k \in \ell_1 .$$

(2) *If X has hypercontractive homogeneous cotype $\operatorname{cot}(X) < \infty$, then*

$$b \in \text{mult } \mathscr{H}_p(X) \Leftrightarrow (b_{p_k})_k \in \ell_{\operatorname{cot}(X)'} \cap B_{c_0} \Leftrightarrow b \in \ell_{\operatorname{cot}(X)'}$$
$$b \in \text{mult } \mathscr{H}_p^m(X) \Leftrightarrow (b_{p_k})_k \in \ell_{\operatorname{cot}(X)'} .$$

If X has nontrivial cotype but does not satisfy the assumptions of (2), multiplicative multipliers are not completely characterized but we can use (16) to obtain some information about them.

To see an example, we use again the results in Section 2.3.

Corollary 4.4. *Let $1 \leq p \leq \infty$, $m \in \mathbb{N}$, and $b = (b_n)$ be a multiplicative scalar sequence.*

(1) *If $1 \leq r \leq \infty$ and X is a $\mathscr{L}_r$-space, then*

$$b \in \text{mult } \mathscr{H}_p(X) \Leftrightarrow (b_{p_k})_k \in \ell_{\min\{2,r'\}} \cap B_{c_0} \Leftrightarrow b \in \ell_{\min\{2,r'\}}$$
$$b \in \text{mult } \mathscr{H}_p^m(X) \Leftrightarrow (b_{p_k})_k \in \ell_{\min\{2,r'\}} .$$

(2) *If $2 \leq r \leq \infty$, then*

$$b \in \text{mult } \mathscr{H}_p(\mathscr{S}_r) \Leftrightarrow (b_{p_k})_k \in \ell_{r'} \cap B_{c_0} \Leftrightarrow b \in \ell_{r'}$$
$$b \in \text{mult } \mathscr{H}_p^m(\mathscr{S}_r) \Leftrightarrow (b_{p_k})_k \in \ell_{r'} .$$

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Appendix A. Cotype with respect to index sets

Throughout this note we have considered different kinds of *cotypes*: the classical (linear) cotype, homogeneous cotype, hypercontractive homogeneous cotype, Fourier cotype and analytic cotype. We end this note introducing a general setting in which all these concepts can be framed.

Let $\Lambda \subseteq \mathbb{N}_0^{(\mathbb{N})}$ be an indexing set. We say that the Banach space X has Λ-cotype q if there exists a constant $C > 0$ such that for every finite family $(x_\alpha)_{\alpha\in\Lambda} \subset X$ (i.e., a family with only finite non-zero elements) we have

$$\Big(\sum_{\alpha\in\Lambda} \|x_\alpha\|^q\Big)^{1/q} \leq C\Big(\int_{\mathbb{T}^\infty} \Big\|\sum_{\alpha\in\Lambda} x_\alpha z^\alpha\Big\|^{q'} dz\Big)^{1/q'}. \tag{17}$$

We denote by $C_{q,\Lambda}(X)$ the best constant C satisfying the previous inequality.

The usual notion of cotype turns out to be a particular case of this concept, in the sense that it corresponds to an appropriate choice of the set of multi-indices Λ. If we take

$$\Lambda_1 = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} : |\alpha| = 1\},$$

Then (17) with Λ_1 is, through Kahane's inequality, equivalent to (2). In other words, Λ_1-cotype is just cotype.

The concept of m-homogeneous cotype can also be seen as a cotype with respect to an indexing set. If we take

$$\Lambda_m = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} : |\alpha| = m\},$$

and use Proposition 1.2 (the polynomial Kahane's inequality) then m-homogeneous cotype q is Λ_m-cotype q. We can rephrase (3) and the subsequent comments: X has Λ_1-cotype if and only if X has Λ_m-cotype for some (or for all) m and

$$C_{q,\Lambda_1}(X) \leq C_{q,\Lambda_m}(X) \leq \frac{m^m}{m!}(m!)^{1/q'} K^m \sqrt{\frac{q'}{2}}^{\,m} C_{q,\Lambda_1}(X)^m .$$

Also, hypercontractive homogeneous cotype q means Λ_m-cotype for all m together with the control of the constants: $C_{q,\Lambda_m}(X) \leq C^m$. Hence our conjecture reads:

$$C_{q,\Lambda_1}(X) \leq C_{q,\Lambda_m}(X) \leq \lambda^m C_{q,\Lambda_1}(X)^m$$

for some universal $\lambda > 0$.

For Fourier cotype, let us identify $\mathbb{N}$ as a subset of $\mathbb{N}_0^{\mathbb{N}}$ in the natural way

$$\mathbb{N} \sim \{\alpha \in \mathbb{N}_0^{\mathbb{N}} : \alpha_k = 0 \text{ for } k \geq 2\}.$$

Fourier cotype is $\mathbb{N}$-cotype and analytic cotype (the inequality in Proposition 2.4) is $\mathbb{N}_0^{(\mathbb{N})}$-cotype. Finally, note that Proposition 2.4 states that $\mathbb{N}$-cotype q is equivalent to $\mathbb{N}_0^{(\mathbb{N})}$-cotype q.

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