A twisted FZZ-like dual for the 2D back hole

Gaston Giribet[1,2,3] and Matías Leoni[2]

[1] Abdus Salam International Centre for Theoretical Pysics, ICTP,
Strada Costiera 11, 34014, Trieste, Italy.

[2] Physics Department, Universidad de Buenos Aires, FCEN - UBA,
Ciudad Universitaria, pabellón 1, 1428, Buenos Aires, Argentina.

[3] Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET,
Av. Rivadavia, 1917, 1033, Buenos Aires, Argentina.

Abstract

We review and further study the duality between string theory formulated on a curved exact background (the two dimensional black hole) and string theory in flat space with a tachyon-like potential. We generalize previous results on this subject by discussing a twisted version of the Fateev-Zamolodchikov-Zamolodchikov conjecture. This duality is shown to hold at the level of N-point correlation functions on the sphere topology, and connects tree-level string amplitudes in the euclidean version of the 2D black hole ($\times$ time) to correlation functions in a non-linear σ-model in flat space but in presence of a tachyon wall potential and a linear dilaton. The dual CFT corresponds to the perturbed 2D quantum gravity coupled to $c < 1$ matter ($\times$ time), where the operator that describes the tachyon-like potential can be seen as a $n = 2$ momentum mode perturbation, while the usual sine-Liouville operator would correspond to the vortex sector $n = 1$. We show how the sine-Liouville interaction term arises through a twisting of the marginal deformation introduced here, and discuss such 'twisting' as a non-trivial realization of the symmetries of the theory. After briefly reviewing the computation of correlation functions in sine-Liouville CFT, we give a precise prescription for computing correlation functions in the twisted model. To show the new version of the correspondence we make use of a formula recently proven by S. Ribault and J. Teschner, which connects the correlation functions in the Wess-Zumino-Witten theory to correlation functions in the Liouville theory. Conversely, the duality discussed here can be thought of as a free field realization of such remarkable formula.

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1 Introduction

One of the most profound concepts in string theory is the suggestive idea that the spacetime itself could be a mere emergent notion, a sort of effective description of a more basic entity [8]. This conception relies on the existence of the duality symmetries of string theory, manifesting that concepts such as the curvature and topology of the spacetime lose their preponderant roles. This idea is particularly realized by examples that manifestly show the duality between string theory formulated on curved backgrounds (e.g. black holes) and the theory in flat space but in presence of tachyon-like potentials. This is the subject we will explore here; and we will do this by studying the worldsheet description of the 2D string theory in the black hole background (i.e. the gauged $SL(2,\mathbb{R})_k/U(1)$ Wess-Zumino-Witten (WZW) model).

1.1 The subject

The relation between string theory in the 2D black hole background and Liouville-like conformal field theories representing "tachyon wall" potentials was extensively explored in the past. One of the celebrated examples is the Mukhi-Vafa duality [9], relating a twisted version of the euclidean black hole to the $c = 1$ matter coupled to 2D gravity. The literature on the connection between the $c = 1$ CFT and the black hole CFT is actually quite rich; we should refer to the list of papers [10]-[29] and the references therein. Recently, a new relation between the 2D string theory in the euclidean black hole background and a deformation of the $c = 1$ matter CFT has received remarkable attention: This is the renowned Fateev-Zamolodchikov-Zamolodchikov conjecture (FZZ), which states the equivalence between the black hole and the often called sine-Liouville field theory [30, 32]. In the last six years this FZZ duality has been applied to study the spectrum and interactions of strings in both the black hole geometry and the Anti-de Sitter space [31, 27, 26]; and the most important application of it was so far the formulation of the matrix model for the two-dimensional black hole [32]. In fact, when one talks about the "black hole matrix model" one is actually referring to the matrix model for the sine-Liouville deformation of the $c = 1$ matter CFT, and thus the black hole description in such a framework emerges through the FZZ correspondence. This manifestly shows how useful the FZZ duality results in the context of string theory.

Although at the beginning it appeared as a conjecture, a proof of the FZZ duality was eventually given some years ago. This was done in two steps: first, by proving the equivalence of the corresponding N=2 supersymmetric extensions of both the 2D black hole σ-model and the sine-Liouville theory [33]; and, secondly, by showing that the fermionic parts of the N=2 theories eventually decouple, yielding the bosonic duality as an hereditary property [34], see also [35, 36]. This could be done because both sine-Liouville and the black hole theory admit a natural[1] embedding in N=2 theories, where the duality can be seen as a manifestation of the mirror symmetry. However, one could be also interested in seeing whether a proof of such a duality exists at the level of the bosonic theory itself. In this paper we will show how such

[1]The 2D black hole can be realized by means of the Kazama-Suzuki construction [37, 38], while the sine-Liouville theory can be seen as a sector of the N=2 Liouville theory. The bosonic version of the FZZ duality can be seen to arise by GKO quotienting the $U(1)$ R-symmetry of the N=2 version.

a duality can be actually proven (at the level of the sphere topology) without resorting to arguments based on supersymmetry but just making use of the conformal structure of the theory.

1.2 The result

Concisely, we will show that any N-point correlation functions in the $SL(2,\mathbb{R})_k/U(1)$ WZW ($\times$ *time*) on the sphere topology is equivalent to a N-point correlation functions in a two-dimensional conformal field theory that describes a linear dilaton σ-model perturbed by a tachyon-like potential. This actually resembles the FZZ correspondence; however, instead of considering a vortex perturbation with winding $|n| = 1$ here we will consider momentum modes of the sector $n = 2$. To be precise, the theory we will consider is defined by turning on the modes $\lambda_{n=2} \neq 0$ and $\lambda_{n=1} \neq 0$ in the following action

$$S = \frac{1}{4\pi} \int d^2z \left(\partial X \overline{\partial} X + \partial\varphi\overline{\partial}\varphi - \frac{1}{2\sqrt{2}} \widehat{Q} R\varphi + \sum_n \lambda_n e^{-\frac{\alpha_n}{\sqrt{2}}\varphi + in\sqrt{\frac{k}{2}}X} \right) \tag{1}$$

where $\widehat{Q} = (k-2)^{-1/2}$ and $\alpha_n = \widehat{Q}(1 + \sqrt{1 + (kn^2-4)(k-2)})$. Namely, the perturbation we will consider is given by the operator

$$\mathcal{O} = \lambda_1 e^{-\sqrt{\frac{k-2}{2}}\varphi + i\sqrt{\frac{k}{2}}X} + \lambda_2 e^{-\sqrt{\frac{2}{k-2}}(k-1)\varphi + i\sqrt{2k}X},$$

where we denoted $X = X_L(z) + X_R(\overline{z})$, which has to be distinguished from the T-dual direction $\widetilde{X} = X_L(z) - X_R(\overline{z})$. Operators $e^{-\frac{\alpha_n}{\sqrt{2}}\varphi + in\sqrt{\frac{k}{2}}X}$ are $(1,1)$-operators with respect to the stress-tensor of the free theory

$$T(z) = -\frac{1}{2} (\partial X)^2 - \frac{1}{2} (\partial\varphi)^2 - \frac{\widehat{Q}}{\sqrt{2}} \partial^2\varphi,$$

so that they represent marginal deformations of the linear dilaton theory. Coefficients λ_n in (1) must satisfy the condition $\lambda_n = \lambda_{-n}$ for the Lagrangian to be real, and thus the theory results invariant under $X \to -X$. The scaling relations between different couplings λ_n are given by standard KPZ arguments [39, 40, 41], being the scale of the theory governed by one of these constants, analogously as to how the Liouville cosmological constant introduces the scale in the $c = 1$ matter CFT. The central charge of the theory is then obtained from the operator product expansion of the stress-tensor, yielding $c = 2 + 6\widehat{Q}^2 = 2 + \frac{6}{k-2}$. Eventually, we will be interested in adding a time-like free boson to the theory in order to define a Lorentzian target space of the form $SL(2,\mathbb{R})_k/U(1) \times \mathbb{R}_{time}$, so the central charge will receive an additional contribution $+1$ coming from the time direction, yielding

$$c = 3 + 6\widehat{Q}^2 = 3 + \frac{6}{k-2}, \tag{2}$$

while the stress-tensor will result supplemented by a term $+\frac{1}{2}(\partial T)^2$. For practical purposes, this time-like direction can be thought of as an auxiliary degree of freedom, and it does not enter in the non-trivial part of the duality we want to discuss, being coupled to the other directions just by the value of the central charge[2] c.

1.3 Outline

The particular correspondence between the model (1) and the 2D black hole we will discuss turns out to be realized at the level of N-point functions on the sphere topology, and corresponds to a twisted version of the FZZ correspondence[3]. Consequently, we will discuss the latter first. Though being similar, the duality we will discuss here presents two important differences with respects to the FZZ: The first difference is that the former admits to be proven in a relatively simple way without resorting to arguments based on mirror symmetry of its supersymmetric extension; secondly, it involves higher momentum modes ($n = 2$) instead of winding modes of the sector $n = 1$. We will make the precise statement of the new correspondence in section 4, where we also address its proof. The paper is organized as follows: In section 2 we review some features of the conformal field theories that play an important role in our work. First, we review the computation of correlation functions in Liouville field theory with the purpose of emphasizing some features and refer to the analogy with the Liouville case whenever an illustrative example is needed. Secondly, we discuss some general aspects of the 2D black hole σ-model. Once these two CFTs were introduced, we discuss how correlation functions in both theories are related through a formula recently proven by S. Ribault and J. Teschner [3, 4]. Their formula connects correlation functions in both WZW and Liouville theory in a remarkably direct way [42], and it turns out to be important for proving our result. In section 3 we briefly review the FZZ dual for the 2D black hole; namely the sine-Liouville field theory. In section 4 we introduce a "twisted" version of the sine-Liouville theory, and we show that such "deformed" sine-Liouville turns out to be a dual for the 2D black hole as well. A crucial piece to show this new version of the duality is the Ribault-Teschner formula mentioned above, for which we present a realization that is eventually identified as being precisely the deformed sine-Liouville model we want to study. Section 5 contains the conclusions.

2 Conformal field theory

To begin with, let us discuss some aspects of correlation functions in Liouville field theory. The relevance of it is that Liouville theory is the prototypical example of non-compact conformal field theory [43] and thus the techniques for computing correlation functions in this model are analogous to those we will employ in the rest of the paper. Moreover, the models we will consider here are actually deformations of the Liouville theory coupled to a $c = 1 + 1$ matter field, so that it is clearly convenient to consider this model first.

[2] In the case the theory corresponds to the product $SL(2,\mathbb{R})/U(1) \times time$ the condition $c = 26$ demands $k = 52/23$. On the other hand, if the space is just the coset $SL(2,\mathbb{R})/U(1)$ the corresponding condition reads $k = 9/4$.

[3] In the sense that it involves a deformation of the sine-Liouville interaction term in the action.

2.1 Liouville theory

2.1.1 Liouville field theory coupled to $c = 1(+1)$ matter

Liouville theory naturally arises in the formulation of the two-dimensional quantum gravity and in the path integral quantization of string theory [44]. This is a non-trivial conformal field theory [45, 46] whose action reads

$$S_L[\mu] = \frac{1}{4\pi} \int d^2z \left(\partial\varphi\overline{\partial}\varphi + \frac{1}{2\sqrt{2}} QR\varphi + 4\pi\mu e^{\sqrt{2}b\varphi} \right) \tag{3}$$

where μ is a real positive parameter called "the Liouville cosmological constant". The background charge parameter takes the value $Q = b + b^{-1}$ in order to make the Liouville barrier potential $\mu e^{\sqrt{2}b\varphi}$ to be a marginal operator. In the conformal gauge, the linear dilaton term $QR\varphi$, which involves the two-dimensional Ricci scalar R, has to be understood as keeping track of the coupling with the worldsheet curvature that receives a contribution coming from the point at infinity. The theory is globally defined once one specifies the boundary conditions, and this can be done by imposing the behavior $\varphi \sim -2\sqrt{2}Q \log |z|$ for large $|z|$, that is compatible with the spherical topology. Under holomorphic transformations $z \to w$ Liouville field transforms in a way that depends on Q, namely $\varphi \to \varphi - \sqrt{2}Q \log |\frac{dw}{dz}|$. In this paper we will be interested in the coupling of Liouville theory to a $U(1)$ boson field represented by an additional $-\frac{1}{4\pi}\int d^2z\partial X\overline{\partial}X$ piece in the action (3) above. Moreover, we can also include the "time" direction $\frac{1}{4\pi}\int d^2z\partial T\overline{\partial}T$. Then, the central charge of whole theory is given by

$$c = 2 + c_L = 3 + 6Q^2,$$

where c_L refers to the Liouville central charge. Important objects of the theory are the exponential vertex operators [47]

$$V_\alpha(z) \times e^{i\sqrt{2}p_1X(z)+i\sqrt{2}p_0T(z)} = e^{\sqrt{2}\alpha\varphi(z)+i\sqrt{2}p_1X(z)+i\sqrt{2}p_0T(z)},$$

which turn out to be local operators of conformal dimension $h = \alpha(Q-\alpha)+p_1^2-p_0^2$ with respect to the stress-tensor $T(z)$ of the free theory,

$$T(z) = \frac{1}{2}(\partial T)^2 - \frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial\varphi)^2 + \frac{Q}{\sqrt{2}}\partial^2\varphi. \tag{4}$$

Now, let us discuss correlation functions.

2.1.2 Liouville correlation functions

The non-trivial part of correlation functions in the theory (4) is given by the Liouville correlation functions [46, 49, 50, 51], and these are formally defined by

$$A^L_{(\alpha_1,\ldots\alpha_N|z_1,\ldots z_N)} = \langle V_{\alpha_1}(z_1)\ldots V_{\alpha_N}(z_N)\rangle_{S_L[\mu]} = \int D\varphi e^{-S_L[\mu]} \prod_{i=1}^{N} e^{\sqrt{2}\alpha_i\varphi(z_i)}$$

and, on the spherical topology, these can be written by using that

$$\left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle_{S_L[\mu]} = b^{-1}\mu^s \Gamma(-s)\delta\left(s + b^{-1}(\alpha_1+\alpha_2+...\alpha_N) - 1 - b^{-2}\right) \times \times \prod_{r=1}^{s} \int d^2 w_r \left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{r=1}^{s} V_b(w_r) \right\rangle_{S_L[\mu=0]}, \tag{5}$$

namely,

$$A^L_{(\alpha_1,...\alpha_N|z_1,...z_N)} = b^{-1}\mu^s \Gamma(-s)\delta\left(s + b^{-1}(\alpha_1+\alpha_2+...\alpha_N) - 1 - b^{-2}\right) \times \times \prod_{r=1}^{s} \int d^2 w_r \int D\varphi e^{-S_L[\mu=0]} \prod_{i=1}^{N} e^{\sqrt{2}\alpha_i \varphi(z_i)} \prod_{r=1}^{s} e^{\sqrt{2}b\varphi(w_r)}. \tag{6}$$

This permits to compute correlation functions by employing the standard Gaussian measure and free field techniques. The overall factor $\Gamma(-s)$ and the δ-function come from the integration over the zero-mode φ_0 of the Liouville field φ, and it also yields the insertion of an specific amount, s, of screening operators $V_b(w)$ in the correlator. In obtaining this, the identity $\mu^s\Gamma(-s) = \int dx x^{-1-s} e^{-\mu x}$ and the Gauss-Bonnet theorem were used to find out the relation between s, b and the momenta α_i, which for a manifold of generic genus g and N punctures would yield

$$bs + \sum_{i=1}^{N} \alpha_i = Q(1-g). \tag{7}$$

So, the correlators can be computed through the Wick contraction of the $N+s$ operators by using the propagator $\langle \varphi(z_1)\varphi(z_2)\rangle = -2\log|z_1 - z_2|$, which corresponds to the free theory (4) and yields the operator product expansion $e^{\alpha_1\varphi(z_1)} e^{\alpha_2\varphi(z_2)} \sim |z_1 - z_2|^{-2\alpha_1\alpha_2} e^{(\alpha_1+\alpha_2)\varphi(z_2)} + ...$ In principle, this could be used to integrate the expression for $A^L_{(\alpha_1,...\alpha_N|z_1,...z_N)}$ explicitly. Nevertheless, it is worth noticing that the expression (6) can be considered just formally since, in general, s is not an integer number. Hence, in order to compute generic correlation functions one has to deal with the problem of giving a concise meaning of such integral representation. With the purpose of giving an example, let us describe here the computation of the partition function on the sphere in detail. Such case corresponds to $g = 0$ and $N = 0$, and the number of screening operators to be integrated out turns out to be $m = s - 3 = -2 + b^2$. That is, in order to compute the genus zero partition function we have to consider the correlation function of three local operators $e^{\sqrt{2}b\varphi(z)}$ inserted at the points $z_1 = 0, z_2 = 1$ and $z_3 = \infty$ to compensate the volume of the conformal Killing group, $SL(2,\mathbb{C})$. This has to be distinguished from the direct computation of the three-point function [49] of three "light" states $\alpha_1 = \alpha_2 = \alpha_3 = b$, as we will discuss below.

2.1.3 A working example: the spherical partition function

Although it is usually said that string partition function on the spherical topology vanishes, we know that this is not necessary the case when the theory is formulated on non-trivial backgrounds. A classical example of this is the two-dimensional string theory formulated in both tachyonic and gravitational non-trivial backgrounds we will be discussing along this paper. Such models admit a description in terms of the Liouville-type sigma model actions, so that the computation of the corresponding genus zero partition functions involves the computation of spherical partition function of Liouville theory or some deformation of it. Here, we will describe a remarkably simple calculation of the Liouville partition function on the spherical topology by using the free field techniques. The free field techniques to be employed here were developed so far by Dotsenko and Fateev [52, 53], and by Goulian and Li [54] (see also [56, 57, 58]). The partition function $Z_{g=0}$ is then given by

$$Z_{g=0} = \frac{\mu^{m+3}}{b}\Gamma(-m-3) \lim_{z_3 \to \infty} |z_3|^{-4} \prod_{r=1}^{m} \int d^2 w_r \int D\varphi e^{-S_L[\mu=0]} e^{\sqrt{2}b\varphi(0)} e^{\sqrt{2}b\varphi(1)} e^{\sqrt{2}b\varphi(z_3)} \prod_{r=1}^{m} e^{\sqrt{2}b\varphi(w_r)}. \tag{8}$$

with $m = -2 + b^{-2}$. According to the standard Wick rules, we can write

$$Z_{g=0} = b^{-1}\mu^{3+m}\Gamma(-m-3) \prod_{r=1}^{m} \int d^2 w_r \left(\prod_{r=1}^{m} |w_r|^{4\rho} |1-w_r|^{4\rho} \prod_{r<t}^{m-1,m} |w_t - w_r|^{4\rho} \right).$$

This can be explicitly solved for integer m by using the Dotsenko-Fateev integral formula worked out in reference [53]. Even though we are interested in the case where m is generic enough, and this can mean a negative real number, we can assume that this is an integer positive number through the integration and then try to analytically extend the final expression accordingly. Eventually, the consistency of the result strongly support this procedure. In this way, we get

$$Z_{g=0} = \frac{\mu^{3+m}}{b}\Gamma(-m-3)\Gamma(m+1)\pi^m\gamma^m(1-\rho) \prod_{r=1}^{m} \gamma(r\rho) \prod_{r=0}^{m-1} \gamma^2(1+(2+r)\rho)\gamma(-1-(3+r+m)\rho).$$

where, as usual, we denoted $\gamma(x) = \Gamma(x)/\Gamma(1-x)$; and we also denoted $\rho = -b^2$ for notational convenience. Once again, this expression only makes sense for m being a positive integer number, so that the non-trivial point here is to handle the required analytic continuation. In order to do this, we can rewrite the expression above by taking into account that $\gamma(-1-(3+r+m)\rho) = \gamma(-(r+1)\rho)$. So we can expand it as

$$Z_{g=0} = \frac{\mu^{3+m}}{b}\Gamma(-m-3)\Gamma(m+1)\pi^m\gamma^m(1-\rho) \prod_{r=1}^{m} \gamma(r\rho)\gamma(-r\rho) \prod_{r=2}^{m+1} \gamma^2(1+r\rho). \tag{9}$$

Now, some simplifications are required. First, we can use that $m = -2 + b^{-2} = -2 - \rho^{-1}$, and thus $1 + r\rho = -(m+2-r)\rho$ to arrange the last product. We can rewrite the product as

$$\gamma(1+2\rho)\gamma(1+3\rho)...\gamma(1+m\rho)\gamma(1+(m+1)\rho) = \gamma(-\rho)\gamma(-2\rho)...\gamma(-(m-1)\rho)\gamma(-m\rho),$$

that is

$$\prod_{r=2}^{m+1} \gamma(1+r\rho) = \prod_{r=1}^{m} \gamma(-r\rho),$$

and then use $\gamma(r\rho)\gamma(1-r\rho)=1$ to eventually write

$$Z_{g=0}=b^{-1}\mu^{Q/b}\Gamma(-m-3)\Gamma(m+1)\pi^m\gamma^m(1-\rho)\gamma^2(-\rho)(-1)^m\rho^{-2m}\Gamma^{-2}(m+1),$$

where the identities $\gamma(x)\gamma(-x)=\gamma(x)/\gamma(1+x)=-x^{-2}$ were also used. Again, the properties of the γ-function can be used to write $\gamma(2+\rho^{-1})=-(1+\rho^{-1})^2\gamma(1+\rho^{-1})$, $\gamma(1-\rho)=-\rho^2\gamma(-\rho)$ and $\gamma(-1-\rho)=-(1+\rho)^{-2}\gamma(-\rho)$. Then, once all is written in terms of b, the partition function reads[4]

$$Z_{g=0}=\frac{(1-b^2)\left(\pi\mu\gamma(b^2)\right)^{Q/b}}{\pi^3 Q\gamma(b^2)\gamma(b^{-2})}. \tag{10}$$

This is the exact result for the Liouville partition function on the spherical topology, which turns out to be a non trivial function of b. It oscillates with growing frequency and decreasing amplitude according b^2 approaches the values $b^2=0$ and $b^2=1$. One can use the Stirling asymptotic formula, $n!\sim n^n\sqrt{2\pi n}e^{-n}$, and the identity $\Gamma(x)\Gamma(1-x)\sin(\pi x)=\pi$ to verify that, for fixed μ, it does vanish at those points. One of the puzzling features of the expression (10) is the fact that it does not manifest the self-duality that the Liouville theory seems to present under the transformation $b\to 1/b$. In order to understand this point, it is convenient to compare the direct computation of $Z_{g=0}$ we gave above with the analogous computation of the Liouville structure constant (three-point functions) $C(\alpha_1,\alpha_2,\alpha_3)$ for the particular configuration $\alpha_1=\alpha_2=\alpha_3=b$. The difference between both calculations is merely given by the presence of the overall factor $\Gamma(-s)=\Gamma(-m-3)$ in (8). As mentioned, this factor comes from the integration over the zero-mode of the field φ, but it can be also thought of as coming from the combinatorial problem of permuting all the screening operators. Actually, for integer s this factor can be written as $\Gamma(-s)=(-1)^s\Gamma(0)/s!$, where the divergent factor $\Gamma(0)$ keeps track of a divergence due to the non-compactness of the Liouville direction. In fact, this yields the factorial $1/s!$ arising in the residue corresponding to the poles of resonant correlators. On the other hand, in the case of being computing the structure constant $C(b,b,b)$, unlike the computation of $Z_{g=0}$, such overall factor should be $\Gamma(3-s)$ instead of $\Gamma(-s)$ since one has to divide by the permutation of $s-3$ screening charges. Hence, we have $C(b,b,b)/Z_{g=0}=\Gamma(3-s)/\Gamma(-s)=-s!/(s-3)!=-(b^{-2}+1)b^{-2}(b^{-2}-1)$. This is precisely consistent with the fact that $\frac{d^3Z}{d\mu^3}=-C(b,b,b)\sim\mu^{Q/b-3}$, see Ref. [55]. Thus, this combinatorial problem appears as being the origin of the breakdown of the Liouville self-duality at the level of the partition function.

Now, let us move to study another CFT that is also a crucial piece in our discussion: the CFT that describes the 2D black hole σ-model.

2.2 String theory in the 2D black hole

2.2.1 The action and the semiclassical picture

String theory in two dimensions presents very interesting properties that make of it a fruitful ground to study features of its higher dimensional analogues. One example is given by the

[4] Notice that we have absorbed a factor $\sqrt{2}$ in the definition of the measure of the path integral.

2D black hole solution discovered in Refs. [59, 60]. This black hole solution is supported by a dilaton configuration, and it turns out to be an exact conformal background on which formulate string theory. In fact, the 2D black hole σ-model action corresponds to the gauged level-k $SL(2,\mathbb{R})_k/U(1)$ WZW theory [59]. An excellent review on this model can be found in Ref. [61].

The worldsheet action for string theory in a two dimensional metric-dilaton background, once setting $\alpha' = 2$, reads

$$S_P = \frac{1}{4\pi}\int d^2z \left(G_{\mu\nu}(X)\partial X^\mu \overline{\partial} X^\nu + R\Phi(X)\right), \tag{11}$$

where the indices $\mu, \nu = \{1, D = 2\}$ run over the two coordinates of the target space, whose metric is $G_{\mu\nu}(X)$. This action is written in the conformal gauge, so, as we discussed before, the dilaton term $R\,\Phi(X)$ has to be understood as keeping track of the coupling with the world-sheet curvature that receives a contribution coming from the point at infinity. The vanishing of the one-loop β-functions demands $R_{\mu\nu} = \nabla_\mu \nabla_\nu \Phi$, with $R_{\mu\nu}$ being now the Ricci tensor associated to the target space metric $G_{\mu\nu}$. Since the 2D black hole string theory corresponds to the $SL(2,\mathbb{R})_k/U(1)$ WZW model, it admits an exact algebraic description in terms of the current conformal algebra of the WZW theory; and we will comment on this in the following subsection. In the semiclassical limit, governed by the large k regime, the euclidean version of the background is described by the following configurations for the metric $G_{\mu\nu}$ and the dilaton Φ,

$$ds^2 = k\left(dr^2 + \tanh^2 r\; dX^2\right), \qquad \Phi(r) = \Phi_0 - 2\log\left(\cosh r\right).$$

It is well known that the geometry of the euclidean black hole is that of a semi-infinite cigar that asymptotically looks like a cylinder. The angular coordinate of such cylinder is X, while the coordinate r is the one that goes along the cigar, running from $r = 0$ (the tip of the cigar, where the string theory is strongly coupled) to $r = \infty$ (where the string coupling $e^{\Phi(r)}$ tends to zero). To get a semiclassical picture of this geometry, let us consider the large k regime and redefine the radial coordinate as $\cosh^2 r = M^{-1}e^{\sqrt{2/k}\varphi}$. Then, in the large φ approximation, and by also rescaling the angular coordinate X by a factor $\sqrt{2/k}$, the metric reads

$$ds^2 = 2\left(1 + Me^{-\sqrt{2/k}\varphi}\right)d\varphi^2 + 2\left(1 - Me^{-\sqrt{2/k}\varphi}\right)dX^2, \tag{12}$$

that asymptotically looks like the cylinder of radius $R = \sqrt{k/2}$. The parameter M is related to the mass of the black hole, and it can be fixed to any positive value by shifting φ. Considering finite-k corrections leads to a shifting in k and then the metric and the dilaton result corrected. In such case, the dilaton reads

$$\Phi(\varphi) = \Phi_0 - \log M + \sqrt{2}\widehat{Q}\varphi, \qquad \widehat{Q} = (k-2)^{-1/2}.$$

Thus, the 2D string theory in the euclidean black hole background can be semiclassically described by a deformation of the linear dilaton theory

$$S_0 = \frac{1}{4\pi}\int d^2z \left(\partial X \overline{\partial} X + \partial\varphi \overline{\partial}\varphi - \frac{1}{2\sqrt{2}}\widehat{Q}R\varphi\right); \tag{13}$$

and, according to (12) and taking into account the finite-k corrections, such "deformation" corresponds to perturbing the action (13) with the graviton-like operator [16]

$$\mathcal{O} = M\ \partial X \overline{\partial} X\ e^{-\sqrt{\frac{2}{k-2}}\varphi}; \tag{14}$$

this is true up to a BRST-trivial[5] operator of the form $\delta\mathcal{O} \sim \partial\varphi\overline{\partial}\varphi\ e^{-\sqrt{\frac{2}{k-2}}\varphi}$. In these terms, the theory can be in principle solved (e.g. its correlation functions be computed) by using the free field approach, yielding the Coulomb-like correlators $\langle\varphi(z_1)\varphi(z_2)\rangle = \langle X(z_1)X(z_2)\rangle = -2\log|z_1 - z_2|$. Operator (14) is usually called the "black hole mass operator", and it is actually a normalizable operator; so one can wonder whether its insertion is allowed since it would create non-local deformations in the worldsheet. A similar feature is exhibited by the perturbations considered in Ref. [5]. The inclusion of operator (14) in the action has to be thought of as being valid in a semiclassical picture and, for instance, can be shown to be equivalent to the free field representation of the WZW model.

Hence, in the large φ region of the space (where the theory turns out to be weakly coupled) we have that the non-linear σ-model of strings in the black hole seems to coincide with the action $S_0 + \frac{1}{4\pi}\int d^2z\ \mathcal{O}$. Furthermore, there is a way of seeing that operator (14) actually describes the dilatonic black hole σ-model beyond the semiclassical picture. To do so, it is necessary to argue that such an action unambiguously describes the full theory beyond the weak limit region [21, 5] and, for instance, reproduces the exact correlation functions. This seems to be hard to be proven in general; nevertheless, there is a nice way of showing that the perturbation (14) corresponds to the theory on the black hole background. This relies on the algebraic description of the $SL(2,\mathbb{R})_k/U(1) \times \mathbb{R}$ WZW theory and is quite direct: The point is that the action $S_0 + \frac{1}{4\pi}\int d^2z\ \mathcal{O}$, once supplemented with the BRST-trivial operator $\delta\mathcal{O}$ and a free time-like boson $-\frac{1}{4\pi}\int d^2z\ \partial T\overline{\partial}T$, can be shown to be related to the well known free field realization of the $SL(2,\mathbb{R})_k$ WZW action through a $SO(2,1)$-boost given by[6]

$$T = i\sqrt{\frac{2}{k}}u - i\sqrt{\frac{k-2}{k}}\phi, \quad X = -\sqrt{\frac{k}{2}}v + i\frac{k-2}{\sqrt{2k}}u + i\sqrt{\frac{k-2}{k}}\phi, \quad \varphi = \sqrt{\frac{k-2}{2}}(u+iv) + \phi,$$

and the standard bosonization $\gamma = e^{u+iv}$, $\beta = i\partial v e^{-u-iv}$, with $\langle\beta(z_1)\gamma(z_2)\rangle \sim (z_1 - z_2)^{-1}$, and with $\langle\phi(z_1)\phi(z_2)\rangle = -2\log|z_1 - z_2|$, [62]. In fact, this leads to the Wakimoto free field description of the $SL(2,\mathbb{R})_k$ current algebra in terms of the linear dilaton field ϕ and the β,γ ghost system [63]. In Wakimoto variables one identifies the theory as being the WZW model formulated on $SL(2,\mathbb{R})$ with the elements of the group written in the Gauss parameterization. Then, the coset theory $SL(2,\mathbb{R})_k/U(1)$ is obtained by simply taking out the time-like direction T which realizes the $U(1)$ current[7]

$$J^3 = \beta\gamma + \sqrt{\frac{k-2}{2}}\partial\phi = i\sqrt{\frac{k}{2}}\partial T;$$

[5]That means that it is pure gauge in the BRST cohomology.

[6]Please, do not mistake the time-like coordinate T for the notation used for the stress-tensor. Excuse us for this overlap in the notation.

[7]Alternatively, an additional free boson, analogous to X, can be added in order to relize the gauging, see [18, 19] and referenctes therein.

recall that this is a time-like direction so that the corresponding correlator flips its sign and thus turns out to be $\langle T(z_1)T(z_2)\rangle = +2\log|z_1 - z_2|$.

On the other hand, let us mention that the dual theory (i.e. the sine-Liouville theory) is also defined as a perturbation of (13); see (32) below. So, according to this picture, it is possible to think the FZZ duality as a relation between different marginal deformations of the same free linear dilaton background. This was the philosophy in Ref. [5], where the FZZ correspondence was seen from a generalized perspective, considering it as an example of a set of connections existing between different marginal deformations of (13). Here, we will be discussing a similar correspondence; we will consider perturbations carrying momentum modes $n = 2$ of the tachyon potential and discuss how it describes $SL(2,\mathbb{R})_k/U(1) \times \mathbb{R}$ WZW correlation functions. We will dedicate some effort to understand the relation between such $n = 2$ perturbation and the standard FZZ duality (that involves $n = 1$ modes). But, first, let us continue our description of the theory in the black hole background with appropriated detail.

2.2.2 String spectrum in the 2D black hole and its relation to AdS_3 strings

The spectrum of the 2D sting theory in the black hole background corresponds to certain sector of the Hilbert space of the gauged $SL(2,\mathbb{R})_k/U(1)$ WZW model, and is thus given in terms of certain representations of $SL(2,\mathbb{R})_k \otimes \overline{SL}(2,\mathbb{R})_k$. The string states are thus described by vectors $\left|\Phi^{\omega}_{j,m,\bar{m}}\right\rangle$ which are associated to vertex operators $\Phi^{\omega}_{j,m,\bar{m}}$, where j, m, and $\bar{m}$ are indices that label the states of the representations of the group. In order to define the string theory, it is necessary to identify which is the subset of representations that have to be taken into account. Such a subset has to satisfy several requirements[8]. In the case of the free theory these requirements are associated to the normalizability and unitarity of the string states. At the level of the interacting theory, additional properties are requested, like the closeness of the fusion rules, the factorization properties of N-point functions, etc.

The $SL(2,\mathbb{R})_k$ WZW model is behind the description of string theory in both the 2D black hole background (through the coset construction) and in AdS_3 space. These two models are closely related indeed, but still different. In the case of the black hole, the states of the spectrum are labeled by the index j of the $SL(2,\mathbb{R})$ representations with the indices m and $\overline{m}$ falling in the lattice

$$m - \overline{m} = n, \quad m + \overline{m} = -k\omega \tag{15}$$

with n and ω being integer numbers, and the conformal dimension of the vertex operators is given by

$$h = -\frac{j(j+1)}{k-2} + \frac{m^2}{k}. \tag{16}$$

On the other hand, AdS_3 string theory can be described in terms of the WZW model on the product between the coset $SL(2,\mathbb{R})_k/U(1)$ and a time-like free boson [64], so that the worldsheet turns out to be formulated in a background that is the product between the time and the euclidean black hole. This can be realized by adding the contribution[9] $-\frac{1}{4\pi}\int d^2z\, \partial T\overline{\partial}T$

[8]For an interesting discussion on non-compact conformal field theories see [43].

[9]Besides, one can represent string theory in AdS_3 space in terms of the Wakimoto free field realization mentioned above. In terms of these fields the AdS_3 metric reads $ds^2 = k\left(d\phi^2 + e^{2\phi}d\gamma d\overline{\gamma}\right)$.

to the action (13) and by supplementing the vertex operators with a factor $e^{i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)T}$ that carries the charge under the field T. Thus, the vertex operators on AdS_3 have conformal dimension given by

$$h = -\frac{j(j+1)}{k-2} - m\omega - \frac{k}{4}\omega^2, \tag{17}$$

which corresponds to adding the conformal dimension $\delta h = -\frac{(m+k\omega/2)^2}{k}$ of the time-like part to the coset contribution (16). In some sense, the string theory in the 2D black hole can be thought of as having constrained the states of the theory in AdS_3 to have vanishing bulk energy, $m + \overline{m} + k\omega = 0$. In this way, one has the theory on the background *time* $\times SL(2,\mathbb{R})_k/U(1)$ as an appropriated realization of sting theory in AdS_3 space [66, 69, 68, 70]. However, before going deeper into the string interpretation of the WZW model, some obstacles have to be overcame. In fact, even in the case of the free string theory, the fact of considering non-compact Lorentzian curved backgrounds is not trivial at all. The main obstacle in constructing the space of states is the fact that, unlike what happens in flat space, in curved space the Virasoro constraints are not enough to decouple the negative-norm string states. In the early attempts for constructing a consistent string theory in AdS_3, additional *ad hoc* constraints were imposed on the vectors of the $SL(2,\mathbb{R})_k$ representations in order to decouple the ghosts. The vectors of $SL(2,\mathbb{R})$ representations are labeled by a pair of indices j and m, and thus such additional constraints (demanded as sufficient conditions for unitarity) implied an upper bound for the index j of certain representations, and consequently an unnatural upper bound for the mass spectrum. The modern approaches to the "negative norm states problem" also include such a kind of constraint, although this fact does not imply a bound on the mass spectrum as in the old versions it did [66]. The upper bound for the index j of discrete representations, often called "unitarity bound", reads $1-k < 2j < -1$. In the case of Euclidean AdS_3, the spectrum of string theory is just given by the continuous series of $SL(2,\mathbb{C})$, parameterized by the values $j = -\frac{1}{2} + i\lambda$ with $\lambda \in \mathbb{R}$ and by real m. On its turn, the case of string theory in Lorentzian AdS_3 is richer and its spectrum is composed by states belonging to both continuous $\mathcal{C}_\lambda^{\alpha,\omega}$ and discrete $\mathcal{D}_j^{\omega,\pm}$ series. The continuous series $\mathcal{C}_\lambda^{\alpha,\omega}$ have states with $j = -\frac{1}{2} + i\lambda$ with $\lambda \in \mathbb{R}$ and $m - \alpha \in \mathbb{Z}$, with $\alpha \in [0,1) \in \mathbb{R}$ (as in $SL(2,\mathbb{C})$, obviously). On the other hand, the states of discrete representations $\mathcal{D}^{\pm,\omega}{}_j$ satisfy $j = \pm m - n$ with $n \in \mathbb{Z}_{\geq 0}$. Other important ingredient for constructing the Hilbert space is the index ω labeling the operators $\Phi^\omega_{j,m,\bar{m}}$. In the black hole background ω turns out to be given by (15), unlike what happens in AdS_3. In AdS_3 the quantum number ω is independent of the bulk kinetic energy $m + \overline{m}$ and the bulk angular momentum $m - \overline{m}$, contributing to the total energy as $m + \overline{m} + k\omega$. Then, the question arises as to how the index ω appears in the Hilbert space of the $SL(2,\mathbb{R})_k$ WZW theory. The answer is that in order to fully parameterize the spectrum in AdS_3 we have to introduce the "flowed" operators $\tilde{J}^a_n$ (with $a = 3, -, +$) which are defined through the spectral flow automorphism [66]

$$J^3_n \to \tilde{J}^3_n = J^3_n - \frac{k}{2}\omega\delta_{n,0}, \quad J^\pm_n \to \tilde{J}^\pm_n = J^\pm_{n\pm\omega} \tag{18}$$

acting of the original $\hat{sl(2)}_k$ generators J^a_n, which satisfy the Lie product that define the affine algebra

$$[J_n^-, J_m^+] = -2J_{n+m}^3 + nk\delta_{n,-m}, \qquad [J_n^3, J_m^\pm] = \pm J_{n+m}^\pm, \qquad [J_n^3, J_m^3] = -n\frac{k}{2}\delta_{n,-m}. \tag{19}$$

Then, states $\left|\Phi_{j,m,\bar{m}}^\omega\right\rangle$ belonging to the discrete representations $\mathcal{D}_j^{\pm,\omega}$ are those obeying[10]

$$\tilde{J}_0^\pm \left|\Phi_{j,m,\bar{m}}^\omega\right\rangle = (\pm j - m)\left|\Phi_{j,m\pm1,\bar{m}}^\omega\right\rangle, \quad \tilde{J}_0^3 \left|\Phi_{j,m,\bar{m}}^\omega\right\rangle = m\left|\Phi_{j,m,\bar{m}}^\omega\right\rangle \tag{20}$$

and being annihilated by the positive modes, namely

$$\tilde{J}_n^a \left|\Phi_{j,m,\bar{m}}^\omega\right\rangle = 0\ , \quad n > 0\ . \tag{21}$$

The states with $m = \pm j$ represent highest (resp. lowest) weight states, while primary states of the continuous representations $\mathcal{C}_\lambda^{\alpha,\omega}$ are annihilated by all the positive modes. On the other hand, the excited states in the spectrum are defined by acting with the negative modes J_{-n}^a ($n \in \mathbb{Z}_{>0}$) on the Kac-Moody primaries $\left|\Phi_{j,m,\bar{m}}^\omega\right\rangle$; these negative modes play the role of creation operators (*i.e.* creating the string excitation). The "flowed states" (namely those being primary vectors with respect to the $\tilde{J}_n^a$ defined with $|\omega| > 1$) are not primary with respect to the $\hat{sl(2)}_k$ algebra generated by J_n^a, and this is clear from (18). However, highest weight states in the series $\mathcal{D}_j^{+,\omega}$ are identified with lowest weight states of $\mathcal{D}_{-k/2-j}^{-,\omega}$, which means that spectral flow with $|\omega| = 1$ is closed among certain subset of Kac-Moody primaries.

The states belonging to discrete representations have a discrete energy spectrum and represent the quantum version of those string states that are confined in the centre of AdS_3 space; these are called "short strings" and are the counterpart of those states that are confined close to the tip of the cigar geometry. On the other hand, the states of the continuous representations describe massive "long strings" that can escape to the infinity, where the theory is weakly coupled. In the case of the 2D black hole the index ω of these long strings has a clear interpretation as an "asymptotically topological" degree of freedom (is not a topological one though). Because of the euclidean black hole has the geometry of a semi-infinite cigar and thus looks like a cylinder very far from the tip, the states in the asymptotic region have a winding number around such cylinder. However, this is not strictly a cylinder but has topology $\mathbb{R}^2$ instead of $\mathbb{R} \times S^1$, so that, as it happens in AdS_3, the winding number conservation can be in principle violated. Of course, this feasibility of violating ω is not evident from the background (13)-(14), which is reliable only far from the tip of the cigar, but the phenomenon can occur when string interactions take place. Instead, in the sine-Liouville theory, the violation of the winding number is understood in a clear way, as due to the explicit dependence on the T-dual direction $\tilde{X}$. We will return to this point later. Now, let us discuss the string interactions in the black hole geometry.

2.2.3 String amplitudes and correlation functions in the $SL(2,\mathbb{R})_k$ WZW theory

The string scattering amplitudes in the 2D black hole background are given by (the integration over the inserting points of) correlation functions in the $SL(2,\mathbb{R})_k$ WZW theory. The first

[10] or analogous relations for the Weyl reflected representations, namely $j \to -1 - j$.

exact computation of such WZW three and two-point functions was performed by K. Becker and M. Becker in Refs. [17, 19], and it was subsequently extended and studied in detail in Refs. [73]-[75] by J. Teschner. The interaction processes of winding string states were studied later in [67, 68], after J. Maldacena and H. Ooguri proposed the inclusion of spectral flowed states in the spectrum of the theory [66]. Moreover, several formalisms were employed to study the correlators in this non-compact CFT [79]-[87]. One of the most fruitful tools to work out the functional form of these WZW correlators was making use of the analogy existing between these and the Liouville correlators [78, 46, 77, 79]. Another useful approach to compute the exact correlation functions was the free field representation [17, 19, 84, 85, 68, 69, 70], which for the WZW model turns out to be similar to what we discussed for the Liouville theory. To be concise, let us briefly describe how this "free field computation" works for the case of the two-point function: Consider the correlation functions of exponential operators $\Phi^{\omega}_{j,m,\overline{m}} = e^{\sqrt{\frac{2}{k-2}}\widehat{j}\varphi - i\sqrt{\frac{2}{k}}mX - i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)T}$ (with $\widehat{j} = -1-j$) in the theory (13) perturbed by the operator (14), namely

$$\mathcal{O} = M\left(\sqrt{\frac{k-2}{2}}\partial\varphi + i\sqrt{\frac{k}{2}}\partial X\right)\left(\sqrt{\frac{k-2}{2}}\overline{\partial}\varphi + i\sqrt{\frac{k}{2}}\overline{\partial} X\right)e^{-\sqrt{\frac{2}{k-2}}\varphi} = M\ \beta\overline{\beta}e^{-\sqrt{\frac{2}{k-2}}\phi}. \quad (22)$$

Then, written in terms of the Wakimoto free fields[11] ϕ, γ, and β, such correlators read[12]

$$\left\langle \Phi^{\omega}_{j,m,\overline{m}}(z_1)\Phi^{-\omega}_{j,-m,-\overline{m}}(z_2)\right\rangle_{WZW} = \Gamma(-s)\delta(s+2j+1)\prod_{r=2}^{s}\int d^2\omega_r \left\langle \gamma^{-1-j-m}(z_1)\overline{\gamma}^{-1-j-\overline{m}}(z_1)\times\right.$$

$$\times\gamma^{-1-j+m}(z_2)\overline{\gamma}^{-1-j+\overline{m}}(z_2)\beta(w_1)\overline{\beta}(w_1)\prod_{r=2}^{s}\beta(w_r)\overline{\beta}(w_r)\rangle\times$$

$$\times\left\langle e^{-\sqrt{\frac{2}{k-2}}(j+1)\phi(z_1)}e^{-\sqrt{\frac{2}{k-2}}(j+1)\phi(z_2)}e^{-\sqrt{\frac{2}{k-2}}\phi(w_1)}\prod_{r=2}^{s}e^{-\sqrt{\frac{2}{k-2}}\phi(w_r)}\right\rangle, \quad (23)$$

where the screening inserted at w_1 is then taken to be fixed at infinity $w_1\to\infty$, while $z_1 = 0$ and $z_2 = 1$ as usual (this is analogous to what we did when discussed the case of Liouville partition function). It is easy to see that this can be solved by using the (analytic extension of) Dotsenko-Fateev integrals, and one eventually finds[13]

$$\left\langle \Phi^{\omega}_{j,m,\overline{m}}(0)\Phi^{-\omega}_{j,-m,-\overline{m}}(1)\right\rangle_{WZW} = -\frac{\Gamma(-j-m)\Gamma(-j+m)}{\Gamma(j+1+\overline{m})\Gamma(j+1-\overline{m})}\times .$$

$$\times\left(-\pi M\gamma\left(\frac{1}{k-2}\right)\right)^{-1-2j}\frac{\gamma(2j+2)}{k-2}\gamma\left(\frac{2j+1}{k-2}\right), \quad (24)$$

[11]Please, do not mistake the Wakimoto field γ (which is a local function on the variable z) for the Euler γ-function introduced in Eq. (9) (which is defined by $\gamma(x) = \Gamma(x)/\Gamma(1-x)$). That is, the fields γ in (23) have to be distinguished from the function γ in (24). We preferred to employ the standard notation here.

[12]In order to compare with the original computation in Ref. [17] it is necessary to consider the Weyl reflection $j\to -1-j$, which is a symmetry of the formula for the conformal dimension, actually.

[13]For instance, compare with formula (49) in Ref. [68], after the Weyl reflection.

where the m-dependent Γ-functions stand from the combinatorial problem of counting the different ways of (Wick) contracting the γ-functions with the β-functions in (23). Expression (24) is the so called $SL(2,\mathbb{R})_k$ WZW reflection coefficient $\mathcal{R}_k(j,m)$ and corresponds to the exact results for the two-point function. In articular, (24) does contain the factor $\gamma\left(\frac{2j+1}{k-2}\right)$ that keeps track of finite-k effects. Analogously, the expression of the three-point functions $\left\langle \Phi^{\omega_1}_{j_1,m_1,\overline{m}_1}(z_1)\Phi^{\omega_2}_{j_2,m_2,\overline{m}_2}(z_2)\Phi^{\omega_3}_{j_3,m_3,\overline{m}_3}(z_3)\right\rangle_{WZW}$ can be found by these means [19]. Moreover, some features of the four-point function like the physical interpretation of its divergences [67], and the crossing symmetry [46], were also studied in the last six years, and, certainly, our understanding of correlation functions in both the 2D black hole and AdS_3 backgrounds has substantially increased recently. Nevertheless, some features remain still as open questions: One of these puzzles is the factorization properties of the generic four-point function and the closeness of the operator product expansion of unitary states. Addressing these questions would require a deeper understanding of the analytic structure of the four-point function. The general expression for the N-point functions for $N > 3$ is not known; however, a new insight about its functional form appeared recently due to the discovery of a new relation between these and analogous correlators in Liouville field theory [42, 3, 4]. This relation between WZW and Liouville correlators is one of the key points for what we are going to study in this paper; so, let us give some details about it.

2.3 A connection between Liouville and WZW correlation functions

Let us comment on the particular connection that exists between the correlation functions of the two conformal theories we discussed above; namely, between Liouville and $SL(2,\mathbb{R})_k$ WZW correlation functions. This relation is a result recently obtained by S. Ribault and J. Teschner, who have found a direct way of connecting correlators in both $SL(2,\mathbb{C})_k/SU(2)$ WZW and Liouville conformal theories [3, 4]. The formula they proved is an improved version of a previous result obtained by A. Stoyanovsky some years ago [42]. The Ribault-Teschner formula (whose more general form was presented by Ribault in Ref. [4]) connects the N-point tree-level scattering amplitudes in Euclidean AdS_3 string theory to certain subset of $N+M$-point functions in Liouville field theory, where the relation between N and M turns out to be determined by the winding number of the interacting strings. Even though this formula was proven for the case of the Euclidean target space, it is likely that an analytic continuation of it also holds for the Lorentzian model. The Ribault-Teschner formula reads as follows: If $\Phi^{\omega}_{j,m,\bar{m}}$ represent the vertex operators in the WZW model, and V_α represent the vertex operators of Liouville theory, then it turns out that

$$\begin{aligned} \langle\prod_{i=1}^{N}\Phi_{j_i,m_i,\bar{m}_i}(z_i)\rangle_{WZW} &= N_k(j_1,...j_N;m_1,...m_N)\prod_{r=1}^{M}\int d^2w_r\ F_k(z_1,...z_N;w_1,...w_M)\times \\ &\quad \times\langle\prod_{t=1}^{N}V_{\alpha_t}(z_t)\prod_{r=1}^{M}V_{-\frac{1}{2b}}(w_r)\rangle_{S_L[\mu]}\ , \end{aligned} \tag{25}$$

with the normalization factor given by

$$N_k(j_1,...j_N;m_1,...m_N) = \frac{2\pi^{3-2N}b}{M!\ c_k^{M+2}}(\pi^2\mu b^{-2})^{-s}\prod_{i=1}^{N}\frac{c_k\ \Gamma(-m_i-j_i)}{\Gamma(1+j_i+\bar{m}_i)} \tag{26}$$

and the z-dependent function given by

$$\begin{aligned} F_k(z_1,...z_N;w_1,...w_M) &= \frac{\prod_{1\leq r<l}^{N}|z_r-z_l|^{k-2(m_r+m_l+\omega_r\omega_l k/2+\omega_l m_r+\omega_r m_l)}}{\prod_{1<r<l}^{M}|w_r-w_l|^{-k}\prod_{t=1}^{N}\prod_{r=1}^{M}|w_r-z_t|^{k-2m_t}}\times \\ &\quad\times\frac{\prod_{1\leq r<l}^{N}(\bar{z}_r-\bar{z}_l)^{m_r+m_l-\bar{m}_r-\bar{m}_l+\omega_l(m_r-\bar{m}_r)+\omega_r(m_l-\bar{m}_l)}}{\prod_{1<r<l}^{M}(\bar{w}_r-\bar{z}_t)^{m_t-\bar{m}_t}}, \end{aligned} \tag{27}$$

and where the parameter b of the Liouville theory is related to the Kac-Moody level k through $b^{-2}=k-2$, while the quantum numbers of the states of both conformal models are related ones to each others through the simple relation $\alpha_i = bj_i + b + b^{-2}/2$, with $i=1,2,...N$. The factor c_k in (26) is a k-dependent (j-independent) normalization; see [4]. Furthermore, the following constraints also hold $m_1+...m_N = \bar{m}_1+...\bar{m}_N = \frac{k}{2}(N-M-2)$, $\omega_1+..\omega_N = M+2-N$, $s=-b^{-1}(\alpha_1+...\alpha_N)+b^{-2}\frac{M}{2}+1+b^{-2}$, where s refers to the amount of screening operators $V_b=\mu e^{\sqrt{2}b\varphi}$ to be included in Liouville correlators in order to get a non vanishing result, as in (7). Also notice that the Liouville correlator in the r.h.s. of (25) contains M degenerate fields $V_{-1/2b}$ (i.e. states that contain null descendents in the modulo), which have conformal dimension strictly lower than zero for positive b. So that the formula (25) relates N-point functions in the WZW theory to $M+N$-point functions in Liouville field theory. Applications of (25) were discussed in [88, 89, 27, 25], and ulterior generalizations were presented in [90, 91]. The way of proving (25) was making use of the relation existing between solutions to the BPZ differential equations (satisfied by the Liouville correlation functions involved in (62), [71]) and the generalized KZ differential equation (satisfied by the WZW correlators [72, 4]). This remarkable trick allowed to demonstrate the map between correlators in both theories even though one does not know the generic form of such observables in any of the two cases.

The dictionary given by the formula (25) will play a crucial role in proving the correspondence between the 2D black hole and the flat tachyonic background we are interested in. Conversely, our result can be seen as a mere free field realization of the Ribault-Teschner formula (25). In fact, in section 4 we will describe how (25) can be thought of as an identity between the $SL(2,\mathbb{R})_k$ WZW theory and a CFT of the form *Liouville* $\times\, U(1)\times\mathbb{R}$, for which the $U(1)$ dependences of the correlators factorize out yielding the piece $F_k(z_1,...z_N;w_1,...w_M)$ in (27). In this realization, the operators $V_{-1/2b}$ are seen as M additional screening currents. The details of this can be found in subsection 4.3; but, first, let us discuss the FZZ duality.

3 The FZZ dual for the 2D black hole

In this section we will study the FZZ dual for the two-dimensional black hole; that is, the sine-Liouville field theory. We will discuss it as an example of tachyon background for the 2D string theory.

3.1 Tachyon-like backgrounds in 2D string theory

Let us consider the non-linear σ-model on a generic curved target space of metric $G_{\mu\nu}$, and in presence of both dilatonic Φ and tachyonic $\mathcal{T}$ backgrounds[14]. If we supplement the worldsheet action (11) with the tachyonic term, the σ-model takes the form

$$S_P = \frac{1}{4\pi} \int d^2z \left(G_{\mu\nu}(X)\partial X^{\mu}\overline{\partial} X^{\nu} + R\Phi(X) + \mathcal{T}(X) \right), \tag{28}$$

where, as before, $\mu, \nu = \{1, 2\}$; and where now we adopt the convention $X^1 \equiv X$ and $X^2 \equiv \varphi$ representing the two coordinates that parameterize the target space. Thus, conformal invariance at quantum level demands the vanishing of the β-functions for the action (28); and for the tachyon field, the one-loop linearized β-function reads [97]

$$\beta^{\mathcal{T}} = -\nabla_{\mu}\nabla^{\mu}\mathcal{T} + 2\nabla_{\mu}\Phi\nabla^{\mu}\mathcal{T} - 2\mathcal{T} = 0, \tag{29}$$

where higher powers of $\mathcal{T}$ were neglected. This equation, together with the one-loop β-functions for the metric and the dilaton, admit solutions of the form

$$G_{\mu\nu} = \delta_{\mu\nu}, \qquad \Phi(\varphi) = \frac{Q}{\sqrt{2}}\varphi, \qquad \mathcal{T}(X,\varphi) = \sum_n \lambda_n \, e^{\sqrt{2}a_n\varphi + i\sqrt{2}b_n X}, \tag{30}$$

with $a_n(Q - a_n) + b_n^2 = 1$, $Q = 2$. Here, the coefficient λ_n are real numbers that can be regarded as the Fourier modes of the tachyon potential. The tachyon momenta b_n are chosen to be consistent with the compactification conditions for the X direction; in particular, here we will consider $b_n = n\sqrt{k}/2$, and the tachyon potential will be of the Toda-like form

$$\mathcal{T}(X,\varphi) = \sum_{n=-\infty}^{\infty} \lambda_n e^{\sqrt{2}(1\pm\sqrt{k}|n|/2)\varphi + i\sqrt{k/2}nX}; \tag{31}$$

see (32) and (38) below. Actually, background (30) is the type of configuration we will deal with. A particular case that is of interest is the sine-Liouville theory, which we discuss below.

3.2 Sine-Liouville theory and the FZZ conjecture

3.2.1 Sine-Liouville field theory

As we said, sine-Liouville theory is a particular case of tachyon-like background, and, according to the FZZ conjecture, this is dual to the 2D string theory on the black hole spacetime. Sine-Liouville theory corresponds to perturb the free action S_0 with the operator

$$\mathcal{O}_{\widetilde{\lambda}_{-1}=\widetilde{\lambda}_{+1}=\lambda} = 4\lambda e^{-\sqrt{\frac{k-2}{2}}\varphi} \cos\left(\sqrt{k/2}\widetilde{X} \right), \tag{32}$$

[14] As it is known, in two dimensions the expression "tachyonic" has to be understood just formally, since it is well known that the tachyon is massless in $D = 2$; see [96] for an illustrative example.

which is convenient to write as

$$\mathcal{O}_{\widetilde{\lambda}_{-1}=\widetilde{\lambda}_{+1}=\lambda} = 2\lambda e^{-\sqrt{\frac{k-2}{2}}\varphi+i\sqrt{\frac{k}{2}}\widetilde{X}} + 2\lambda e^{-\sqrt{\frac{k-2}{2}}\varphi-i\sqrt{\frac{k}{2}}\widetilde{X}}, \tag{33}$$

where $\widetilde{X} = X_L(z) - X_R(\overline{z})$. The interaction term (32) resembles both the sine-Gordon and the Liouville field theories, and this gives rise the name of "sine-Liouville". Indeed, this field theory corresponds to the sine-Gordon model coupled to two-dimensional gravity.

The sine-Liouville interaction (32) can be thought of as a particular case of the action (1) if the X field is replace by its T-dual $\widetilde{X}$. It would correspond to the coupling $\widetilde{\lambda}_n = \lambda\,(\delta_{n+1} + \delta_{n-1})$. This field theory describes the phase of vortex condensation in the 2D string theory. Unlike the euclidean black hole geometry, whose topology is $\mathbb{R}^2$, sine-Liouville theory is an interacting CFT formulated on the topology $\mathbb{R} \times S^1$. The distinct topologies arise because the angular direction of the (simple connected) cigar plays a crucial role in the duality. Besides, notice that the sine-Liouville interaction term is not bounded from below, and this is ultimately related to the $\mathbb{R}^2$ topology of the cigar too.

Sine-Liouville theory and its relation to the 2D black hole have been extensively studied in the last six years, and, as we mentioned, this has led to the formulation of the matrix model for the black hole [32]. The matrix model then represented a very important tool for studying black hole physics in string theory; in particular, it permitted to address the question about the black hole formation in string theory [98]. Matrix model formulation also enabled to study the integrability of the theory from a different point of view[15], and we emphasize that all this was possible because of FZZ duality.

3.2.2 The Fateev-Zamolodchikov-Zamolodchikov conjecture

The FZZ duality is a strong-weak duality. The semiclassical limit of sine-Liouville theory corresponds to the limit $k \to 2$, where the black hole is highly curved. Conversely, the semiclassical limit of the black hole theory corresponds to the large k regime where the sine-Liouville wave function is strongly suppressed in the φ direction. Perhaps, the correct way of thinking FZZ duality is that the full theory is actually described by both the WZW and sine-Liouville models, and each of them dominates the dynamics of the theory in a different regime (where the corresponding action is reliable as a good approximation). Nevertheless, it is worth mentioning that both theories have control on the observables beyond the regime in which one would naively expect so. For instance, even though one would expect the black hole σ-model action to describe the theory only in the large k regime, it turns out that the Coulomb gas computation of correlation function using the screening operator behaving like $\sim e^{-\sqrt{\frac{2}{k-2}}\varphi}$ do reproduce the exact result, including finite-k effects[16] [17, 19, 68]. Besides, the same feature occurs for the computation in sine-Liouville theory [94]. This sourprising feature is due to the analytic extension of the Coulomb gas type expressions, which is powerful enough to reconstruct the

[15] The black hole turns out to be dual to the perturbed $c = 1$ theory, and, on the other hand, the $c = 1$ theory perturbed by the vortex or tachyon potential turns out to be integrable with the integrable structure described in terms of the Toda hierarchy [32].

[16] We have exemplified this in the previous section by computing the two-point function.

exact expression for the correlators. This is precisely what permitted to perform consistency checks of the conjecture. The interplay between perturbative poles and k-dependent poles in correlation functions of both models was first discussed in [32], where it was shown that the poles of bulk amplitudes in sine-Liouville precisely reproduce non-perturbative (finite-k effects) poles of WZW correlators[17].

The strong-weak FZZ correspondence turns out to be a very important piece for our understanding of black hole physics in string theory. So, let us briefly discuss how such correspondence works operatively. First, we present the main ingredients: The sine-Liouville vertex operators we have to consider are those of the form[18]

$$\mathcal{T}_{j,m,\overline{m}} = e^{\sqrt{\frac{2}{k-2}}j\varphi + i\sqrt{\frac{2}{k}}mX}. \tag{34}$$

The spectrum of the theory contains states obeying $m - \overline{m} = k\omega$ and $m + \overline{m} = n$, with integers n and ω. Operators (34) have conformal dimension

$$h = -\frac{j(j+1)}{k-2} + \frac{m^2}{k},$$

and the coincidence with (16) shows the convenience of this notation. A crucial observation is that the sine-Liouville theory presents symmetry under the $\widehat{sl}(2)_k$ affine algebra, and this can be realized by free field techniques by defining [86]

$$J^{\pm}(z) = \left(-i\sqrt{\frac{k}{2}}\partial X \pm \sqrt{\frac{k-2}{2}}\partial\varphi\right) e^{\mp i\sqrt{\frac{2}{k}}(T+X)}, \qquad J^3(z) = i\sqrt{\frac{k}{2}}\partial T. \tag{35}$$

These currents satisfy the OPE

$$J^3(z)J^{\pm}(w) = \pm\frac{1}{(z-w)}J^{\pm}(w) + ..., \qquad J^3(z)J^3(w) = -\frac{k/2}{(z-w)^2} + ...,$$

$$J^-(z)J^+(w) = \frac{k}{(z-w)^2} - \frac{2}{(z-w)}J^3(w) + ...,$$

and thus realize (19) by means of the usual relation $J_n^a = \frac{1}{2\pi i}\oint dz\ z^{-1-n}J^a(z)$. It is possible to verify that the sine-Liouville interaction commutes with these currents, in the sense that the OPEs yield regular terms. This matching of symmetries is an important necessary condition for stating the equivalence to the WZW theory. The next step would be that of proposing a

[17]However, again, it is important to emphasize that such finite-k poles can be directly obtained by considering the perturbative action of the WZW theory [17]. For instance, in Ref. [68] it was shown that the computation in the WZW model involving operators behaving like $\sim e^{-\sqrt{2(k-2)}\varphi}$ exactly agree with those originally computed in [17], even though the dependence on k is the opposite to the one appearing in (14).

[18]Here, we are not explicitly writing the antiholomorphic contribution $e^{i\sqrt{2/k}\bar{m}X}$ for short; it has to be understood in all the formulae below. Besides, let us notice that vertex (34) would receive an extra piece $e^{i\sqrt{\frac{2}{k}}(m+k\omega/2)T + i\sqrt{\frac{2}{k}}(\bar{m}+k\omega/2)T}$ in the case that the theory one considers is the product between the sine-Liouville action and the time direction T.

dictionary between observables. According to FZZ prescription the operators (34) turn out to be associated in one-to-one correspondence to those operators that expand $SL(2,\mathbb{R})_k$ representations in the theory on the coset [32], namely

$$\mathcal{T}_{j,m,\overline{m}} \leftrightarrow \Phi_{j,m,\overline{m}},$$

where $\Phi_{j,m,\overline{m}}$ are the vertex operators on the coset theory $SL(2,\mathbb{R})_k/U(1)$, defined through their relation to the $SL(2,\mathbb{R})_k$ vertex, namely $\Phi^{\omega}_{j,m,\overline{m}} = \Phi_{j,m,\overline{m}} \times e^{i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)T}$. Then, once the sine-Liouville operators were introduced, one can undertake the task of performing perturbative checks of the duality. To do this, one should compare the analytic structure of correlation functions in both conformal models; but, first, one has to know how to compute such quantities. So, let us review the computation of correlators for the sine-Liouville field theory.

3.2.3 Correlation functions in sine-Liouville theory

Correlation functions in the sine-Liouville theory are assumed to reproduce the analytic structure of the WZW analogues. The former can be computed by standard Coulomb gas techniques, and the precise prescription was studied in Ref. [93, 94]. In the case $N \leq 3$ these correlators were explicitly integrated, by the way. In general, N-point sine-Liouville amplitudes are expected to exhibit poles at $s_- + s_+ = \frac{2}{k-2}\left(1+\sum_{i=1}^N j_i\right)$, where the residues turn out to be expressed in terms of multiple integrals over the whole complex plane. These read

$$\begin{aligned} A^{sine-L}_{(j_1,...j_N|z_1,...z_N)} &= \frac{\lambda^{\frac{2}{k-2}(j_1+...j_N+1)}}{s_-!s_+!}\prod_{r=1}^{s_+}\int d^2v_r\prod_{t=1}^{s_-}\int d^2v_t\,\langle\mathcal{T}_{j_1,m_1,\overline{m}_1}(z_1)\mathcal{T}_{j_2,m_2,\overline{m}_2}(z_2)... \\ &\quad ...\mathcal{T}_{j_N,m_N,\overline{m}_N}(z_N)\prod_{r=1}^{s_+}\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(u_r)\prod_{t=1}^{s_-}\mathcal{T}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}(v_t)\Big\rangle_{S_{[\lambda=0]}} \end{aligned} \tag{36}$$

with $S_{[\lambda=0]} = S_0$, yielding

$$A^{sine-L}_{(j_1,...j_N|z_1,...z_N)} = \frac{\lambda^{\frac{2}{k-2}(j_1+...j_N+1)}}{\Gamma(s_-+1)\Gamma(s_++1)}\prod_{a<b}^{N-1,N}|z_a-z_b|^{-\frac{4j_aj_b}{k-2}}(z_a-z_b)^{\frac{2}{k}m_am_b}(\bar{z}_a-\bar{z}_b)^{\frac{2}{k}\bar{m}_a\bar{m}_b}\times$$

$$\begin{aligned} &\times\prod_{r=1}^{s_+}\int d^2u_r\prod_{l=1}^{s_-}\int d^2v_l\prod_{r<t}^{s_+-1,s_+}|u_r-u_t|^2\prod_{l<t}^{s_--1,s_-}|v_t-v_s|^2\prod_{l=1}^{s_-}\prod_{r=1}^{s_+}|v_l-u_r|^{2-2k}\times \\ &\times\prod_{a=1}^{N}\prod_{r=1}^{s_+}|z_a-u_r|^{2(j_a+m_a)}(\bar{z}_a-\bar{u}_r)^{m_a-\overline{m}_a}\prod_{b=1}^{N}\prod_{l=1}^{s_-}|z_b-v_l|^{2(j_b-m_b)}(\bar{z}_b-\bar{v}_l)^{m_b-\overline{m}_b}, \end{aligned} \tag{37}$$

that follows from the Wick contraction of operators $\mathcal{T}_{j,m,\overline{m}}$ and $\mathcal{T}_{1-\frac{k}{2},\pm\frac{k}{2},\pm\frac{k}{2}}$. The poles that correspond to bulk amplitudes in sine-Liouville theory can be shown to arise though the integration over the zero-mode of the field φ [65]. In the case $N = 3$ the pole structure of (37) was shown to agree with that of the black hole theory, for which the finite-k poles represent non-perturbative worldsheet effects. This non-trivial matching between analytic structures was one of the strongest evidence in favor of the FZZ conjecture at perturbative level [32, 94, 70]. An important piece of information is encoded in the fact that the sine-Liouville correlators scale as $\lambda^{\frac{2}{k-2}(j_1+j_2+...j_N+1)}$ while the black hole correlators scale as $M^{1+\widehat{j}_1+\widehat{j}_2+..\widehat{j}_N}$. In particular, it tells us something about how the sine-Liouville correlators behave in the large k limit.

In Ref. [94] the authors translated the integrals $\prod_{r,l}\int d^2u_r \int d^2v_l$ in (36) into the product of contour integrals. In this way, the integral representation above results described by standard techniques developed in the context of rational conformal field theory. Such techniques were used to evaluate the correlators to give a formula for the contour integrals. The first step in the calculation is to decompose the u_r complex variables (resp. v_l) into two independent real parameters (*i.e.* the real and imaginary part of u_r) which take values in the whole real line. Secondly, a Wick rotation for the imaginary part of (u_r) has to be performed in order to introduce a shifting parameter ε which is subsequently used to elude the poles in z_a. Then, the contours are taken in such a way that the poles at $v_r \to z_a$ are avoided by considering the alternative order with respect to this inserting points. A detailed description for the prescription can be found in the section 3 of Ref. [94]; see also Ref. [95].

3.2.4 On the violation of the winding number conservation

Now, let us return to the feature of the violation of winding conservation. From the point of view of the sine-Liouville field theory the violation of the total winding number in (36) is given by $\sum_{a=1}^{N=3}\omega_a = k^{-1}\sum_{a=1}^{N=3}(m_a+\overline{m}_a) = s_- - s_+$ and comes from the insertion of a different amount of screening operators s_- and s_+. It can be proven that for the three-point functions, the winding can be violated up to $|\sum_{a=1}^{N=3}\omega_a| \leq N-2 = 1$ and, presumably, this is the same for generic N. The key point for obtaining such a constraint is noticing that the integrand that arises in the Coulomb gas-like prescription contains contributions of the form

$$\int d^2v_r d^2v_t |v_r - v_t|^2 ...$$

that come from the product expansion of two operators $\mathcal{T}_{1-\frac{k}{2},\pm\frac{k}{2},\pm\frac{k}{2}}$ inserted at the points v_r and v_t for $0 \leq r,t \leq s_-$ (and the same for the points u_l with $0 \leq l \leq s_+$), and where the dots "..." stand for "other dependences" on v_r and u_l. As explained in [94], the integral vanishes for certain alignments of contours due to the fact that the exponent of $|v_r - v_t|$ is $+2$. Conversely, in the case where such exponent is generic enough (let us say 2ρ, following the notation of [94]), the integral has a phase ambiguity due to the multi-valuedness of $|v_r - v_t|^{2\rho}$ in the integrand. Then, those integrals containing two contours of v_r and v_t just next to each other vanish, and this precisely happens for all the contributions of those correlators satisfying $|s_+ - s_-| = |\sum_{a=1}^{N}\omega_a| > N-2$. This led Fukuda and Hosomichi to prove that, for the three-point function, there are only three terms that contribute: one with $\sum_{a=1}^{3}\omega_a = 1$, a second with $\sum_{a=1}^{3}\omega_a = -1$, and the conserving one, $\sum_{a=1}^{3}\omega_a = 0$. A similar feature is exhibited in the "twisted" sine-Liouville model we will consider in the next section, see [1].

On the other hand, one can wonder about how the violation of the winding number conservation is seen from the point of view of the black hole theory, where, unlike what happens in sine-Liouville theory, the action does not seem to break the winding conservation. That it, even though the geometric reason why the winding is not conserved in the cigar is quite clear, it is not obvious how to understand such non-conservation in the calculation of correlators. The answer to this puzzle was first given in Ref. [30], and subsequently reviewed in [67]. In fact, the computation of the winding violating correlators in the WZW theory is far from being as simple as in the case of sine-Liouville theory. In the WZW theory such computation requires the insertion of one additional operator for each unit in which the winding number is being violated. This additional operator is the often called "spectral flow operator" $\Phi^1_{-\frac{k}{2},\pm\frac{k}{2},\pm\frac{k}{2}}$, and this is an auxiliary operator that plays the role of changing (in one unit) the winding number ω of a given $SL(2,\mathbb{R})$-state involved in the correlator. The spectral flow operator corresponds to a conjugate representation of the identity operator, so it has conformal dimension zero. For instance, the three-point scattering amplitudes (violating winding in one unit) in the 2D black hole would be actually given in terms of a four-point correlation functions involving a fourth dimension-zero operator $\Phi^1_{-\frac{k}{2},\pm\frac{k}{2},\pm\frac{k}{2}}$, before extracting the appropriate divergent factor coming from the coincidence limit of spectral flow operator and the evaluation at $m = \bar{m} = \pm k/2$; see [67, 27] for the details.

Regarding the computation of correlation functions where the winding number conservation is violated, let us mention that the most simple way of computing such observables is that of making use of the twisted dual model we will introduce in the next section. Perhaps this is the most useful application it has, and we will comment on it later. Now, let us to introduce the new dual model for the 2D black hole; which we will call the "twisted model" because it involves momentum modes of the higher sector $n = 2$.

4 A twisted dual for the 2D black hole

Now, we will discuss an alternative dual description of string theory in the 2D black hole ($\times$ *time*). First, we will introduce a family of perturbations of the linear dilaton background (13) and, in particular, we will introduce the perturbation that corresponds to the twisted version of the sine-Liouville model which we want to relate to the black hole σ-model. After doing this, we will make the precise statement of such duality and show how to prove it by using the formula (25).

4.1 Perturbations of higher winding and momentum modes

4.1.1 Momentum mode perturbations

Let us begin by considering a rather general deformation of the theory (13), including higher modes of momentum and winding. The interaction term in (1) is given by the operator

$$\mathcal{O}_{\lambda_n} = \sum_{n=-\infty}^{\infty} \lambda_n e^{-\frac{\alpha_n}{\sqrt{2}}\varphi + in\sqrt{\frac{k}{2}}X}, \tag{38}$$

for which the condition $\lambda_n = \lambda_{-n}$ is required to be real. Each term in this sum represents a marginal deformation of the linear dilaton theory (13), and if the T-dual direction $\widetilde{X}$ is considered instead of X then this operator describes the sine-Liouville field theory in the particular case $\widetilde{\lambda}_{n=1} = \widetilde{\lambda}_{n=-1} \neq 0$. The case $n = 0$ is also included in the sum. In that case the exponent is given by $\alpha_{n=0} = \frac{1+\sqrt{9-4k}}{\sqrt{k-2}}$, so that it is real (represents a "Liouville-like wall potential") only for values $k \leq 9/4$. The value that saturates this bound, $k = 9/4$, precisely corresponds to the black hole background, i.e. for which the central charge of the coset $SL(2,\mathbb{R})_k/U(1)$ itself turns out to be 26. At $k = 9/4$ the interaction term for $n = 0$ turns out to be $e^{-\sqrt{2}\varphi}$, i.e. the cosmological constant. We have to point out that for $k = 9/4$ the interaction (38) agrees with the two-dimensional string theory in an arbitrary winding background studied by V. Kazakov, I. Kostov and D. Kutasov in[19] Ref. [32]. That is, for $k = 9/4$ operator (38) reads[20]

$$\mathcal{O}_{\lambda_n} = \lambda_0 \varphi e^{-\sqrt{2}\varphi} + \sum_{n\neq 0} \lambda_n e^{(|n|R-\sqrt{2})\varphi + inRX}. \tag{39}$$

with $R = \sqrt{k/2} = 3/2\sqrt{2}$. The matrix model incorporating these perturbations is constructed by implementing a deformed version of the Haar measure on the $U(N)$ group manifold. The details of the matrix model construction can be found in [32]; here we will not discuss the subject beyond the scope of the continuous limit.

4.1.2 Adding vortex type perturbations

It is instructive to explore other deformations. For instance, let us consider the more general family

$$\mathcal{O}_{\lambda_n,\widetilde{\lambda}_n} = \sum_{n\neq 0} e^{-\frac{\alpha_n^{(-)}}{\sqrt{2}}\varphi}\left(\lambda_n^{(-)} e^{in\sqrt{\frac{k}{2}}X} + \widetilde{\lambda}_n^{(-)} e^{in\sqrt{\frac{k}{2}}\widetilde{X}}\right) + \sum_{n\neq 0} e^{-\frac{\alpha_n^{(+)}}{\sqrt{2}}\varphi}\left(\lambda_n^{(+)} e^{in\sqrt{\frac{k}{2}}X} + \widetilde{\lambda}_n^{(+)} e^{in\sqrt{\frac{k}{2}}\widetilde{X}}\right), \tag{40}$$

with $\alpha_n^{(\pm)} = \widehat{Q}(1 \mp \sqrt{1 + (kn^2-4)(k-2)})$, so that (38) corresponds to the branch $\alpha_n^{(-)}$. In the 2D black hole, couplings λ_n turn the momentum modes on, while $\widetilde{\lambda}_n$ are the couplings of vortex operators turning winding modes on, instead. Notice that perturbation (40) not only includes the usual sine-Liouville interaction $\alpha_{\pm 1}^{(-)}$, but also includes the dual sine-Liouville interaction introduced by A. Mukherjee, M. Mukhi and A. Pakman in Ref. [5] when the modes $\alpha_{\pm 1}^{(+)}$ are considered[21]. Operators of the branches $\alpha_n^{(\pm)}$ have a large k behavior $\sim e^{\pm\sqrt{\frac{k}{2}}|n|\varphi}$, so that only those of the branch $\alpha_n^{(-)}$ decrease for large φ (where the theory is weakly coupled) in the black

[19] See formula (3.19) in Ref. [32] and notice that the notation there relates to the one employed here by $\varphi = \sqrt{2}\phi$.

[20] Actually, the contribution $n = 0$ at the point $k = 9/4$ leads to the operator $\varphi e^{-\sqrt{2}\varphi}$ instead of $e^{-\sqrt{2}\varphi}$. This comes from the fact that there are two possible values for $\alpha_{n=0} = (1 \pm \sqrt{9-4k})/\sqrt{k-2}$ which coincide (a resonance) in the limit $k \to 9/4$ producing a degenerangy analogous to the case of the Liouville cosmological term in the $b \to 1$ limit [92]. Also notice the difference between the signs of the exponents of (31) and (39); which is due to the sign of the background charge in each case.

[21] Notice that the notation in [5] relates with ours here by $\varphi = -\sqrt{2}\phi$.

hole semiclassical limit $k \to \infty$. For our purpose, the interesting operators are those having momentum $n = 2$. In particular, here we are mainly interested in the case $\alpha_2^{(-)} = \frac{2}{\sqrt{k-2}}(k-1)$; this is the one that will enable us to present an alternative dual description for the 2D black hole. For notational convenience, let us point out that operators with momentum $\alpha_n^{(\pm)}$ can be written as

$$\mathcal{T}_{j_n^{\pm},m_n,m_n} = e^{-\frac{\alpha_n^{(\pm)}}{\sqrt{2}}\varphi + in\sqrt{\frac{k}{2}}X}, \quad \mathcal{T}_{j_n^{\pm},m_n,-m_n} = e^{-\frac{\alpha_n^{(\pm)}}{\sqrt{2}}\varphi + in\sqrt{\frac{k}{2}}\widetilde{X}},$$

so carrying momentum $j_n^{\pm} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+(k-2)(kn^2-4)}$ and momentum or winding number $m_n \pm \bar{m}_n = kn$. Here we will discuss how certain correlation functions of the model defined by the action (1), when the momentum modes $n = 2$ (represented by operators $\mathcal{T}_{1-k,k,k}$) and $n = 1$ (respectively represented by $\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}$) are turned on, precisely agree with the correlation functions of the $SL(2,\mathbb{R})_k$ WZW model. So, now we are ready to make the main statement about the correspondence and describe the precise prescription for computing the correlators.

4.2 Statement of the correspondence

4.2.1 Preliminary: some definitions

We will discuss how the particular deformation of the family (38) given by $\lambda_n = \mu\delta_{n-2} + \lambda\delta_{n-1}$ is dual to the 2D black hole, in a similar way as the sine-Liouville model is so. That is, by "dual" we mean that there exists a direct correspondence between correlation functions of both CFTs at the level of the sphere topology. Then, the interaction operator we will consider is

$$\mathcal{O}_{\lambda_1=\lambda,\lambda_2=\mu} = \lambda e^{-\sqrt{\frac{k-2}{2}}\varphi + i\sqrt{\frac{k}{2}}X} + \mu e^{-\sqrt{\frac{2}{k-2}}(k-1)\varphi + i\sqrt{2k}X}, \tag{41}$$

where the scaling relation between the coupling constants μ and λ goes like $\lambda^2 = a_k\mu$ with a k-dependent proportionality factor a_k that will be specified below. In the large k limit this operator behaves as

$$\mathcal{O}_{\lambda_1=\lambda,\lambda_2=\mu} \sim \lambda e^{-\sqrt{k/2}(\varphi-iX)} + \frac{1}{a_k}(\lambda e^{-\sqrt{k/2}(\varphi-iX)})^2 \sim \lambda e^{-\sqrt{k/2}(\varphi-iX)} + \frac{1}{a_k}\left(\mathcal{O}_{\lambda_1=\lambda,\lambda_2=0}\right)^2.$$

Also notice that

$$\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}} = e^{-\sqrt{\frac{k-2}{2}}\varphi + i\sqrt{\frac{k}{2}}X}, \qquad \mathcal{T}_{1-k,k,k} = e^{-\sqrt{\frac{2}{k-2}}(k-1)\varphi + i\sqrt{2k}X}, \tag{42}$$

so that

$$\mathcal{O}_{\lambda_1=\lambda,\lambda_2=\mu} = \lambda\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}} + \frac{\lambda^2}{a_k}\mathcal{T}_{1-k,k,k}. \tag{43}$$

Taking into account (34), we notice that the perturbation $\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}$ corresponds to an operator that satisfies the unitarity bound $1-k < 2j < -1$ only for $k > 3$, while the operator $\mathcal{T}_{1-k,k,k}$ does not satisfy that bound for any value of k grater than 2. With operators (42), we define the following correlationfunctions

$$\left\langle \widetilde{\mathcal{T}}_{j_1,m_1,\overline{m}_1}(z_1)...\widetilde{\mathcal{T}}_{j_N,m_N,\overline{m}_N}(z_N)\right\rangle_{S_{[\lambda]}} = \frac{\Gamma(-s)}{b}\delta\left(s+1+j_1+...j_N+M+(N-2-M)k/2\right)\times$$

$$\times \frac{1}{M! c_k^M} \delta\left(m_1 + \overline{m}_1 + ... m_N + \overline{m}_N + k(M+2-N)\right) \delta_{m_1 - \overline{m}_1 + ... + m_N - \overline{m}_N} \times$$
$$\times \prod_{r=1}^{M} \int d^2 w_r \prod_{t=1}^{s} \int d^2 v_t \left\langle \widetilde{\mathcal{T}}_{j_1, m_1, \overline{m}_1}(z_1) ... \widetilde{\mathcal{T}}_{j_N, m_N, \overline{m}_N}(z_N) \prod_{r=1}^{M} \mathcal{T}_{1-\frac{k}{2}, \frac{k}{2}, \frac{k}{2}}(w_r) \prod_{t=1}^{s} \mathcal{T}_{1-k,k,k}(v_t) \right\rangle_{S_{[\lambda=0]}} \quad (44)$$

where $b^{-2} = k-2$, and where we fixed $a_k = c_k^{-2}$. The value of λ was also fixed to a specific value. The vertex operators $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ appearing in this expression are related to those introduced in (34) through

$$\widetilde{\mathcal{T}}_{j,m,\overline{m}} = \frac{c_k \Gamma(-m-j)}{\pi^2 \Gamma(1+j+\overline{m})} \mathcal{T}_{\widetilde{j}, \widetilde{m}, \widetilde{\overline{m}}}, \quad (45)$$

with

$$\widetilde{j} = -j(k-1) - m(k-2) - k/2, \quad \widetilde{m} = jk + m(k-1) + k/2, \quad (46)$$

and analogously for $\widetilde{\overline{m}}$. Again, notice that in (44) we already fixed the value of λ to a specific value c_k, which is a k-dependent numerical factor that is ultimately related to the one appearing in (25). Realization (44) is similar to (5) in Liouville field theory and defines the correlators we will consider here. The overall factor $\frac{\Gamma(-s)}{bM! c_k^M}$ and the δ-functions are understood once the prescription for inserting the screenings when computing the correlators is specified. These factors come from the integration over the zero-modes of the fields φ and X. Besides, the condition $\sum_{i=1}^{N}(m_i - \overline{m}_i) = 0$ also holds. The conditions imposed by these δ-functions are equivalent to demand $\sum_{i=1}^{N}(\widetilde{m}_i - \widetilde{\overline{m}}_i) = 0$ and $\sum_{i=1}^{N}(\widetilde{m}_i + \widetilde{\overline{m}}_i) + k(M+2s) = 0$, being $M+s$ the total amount of screenings to be inserted. We discuss our prescription for the insertion of the screening charges below.

4.2.2 A Coulomb gas-like prescription

In this realization, the interaction operators $\mathcal{T}_{1-\frac{k}{2}, \frac{k}{2}, \frac{k}{2}}$ and $\mathcal{T}_{1-k,k,k}$ act in (44) as screening operators, analogously to the computation of Liouville correlation functions. Because of the δ-functions appearing in (44), the amount of these screening operators to be inserted turns out to be given by

$$s = 1 - N - \sum_{i=1}^{N} j_i + \frac{k-2}{2}(M+2-N), \quad M = N - 2 + \sum_{i=1}^{N} \omega_i. \quad (47)$$

However, the statement is not complete unless one specifies how conditions (47) are to be satisfied. This is because in principle there is no a unique way of choosing s and M in order to obey the first of the charge symmetry conditions in (47). Thus, let us be precise about the prescription to compute the r.h.s. of (44): The prescription adopted here is that M represents a positive integer number of operators $\mathcal{T}_{1-\frac{k}{2}, \frac{k}{2}, \frac{k}{2}}$ to be inserted, and M is actually fixed by the winding numbers ω_i of the N interacting states. On the other hand, the amount s of operators $\mathcal{T}_{1-k,k,k}$ is then appropriately chosen to make the r.h.s. of (44) to be nonzero; and this is going to be the case even if (44) has to be analytically extended to non-integer values of s

(we already discussed correlation functions with a non-integer amount of screening operators in section 2). This is the set of correlation functions we will consider here; and we emphasize that the equivalence between CFTs we will state in the following subsection 4.2.3 has to be understood as holding only if the prescription employed to compute the observables is the one we just gave in this subsection. Now, once we precisely defined the correlators (44), let us present the main assertion.

4.2.3 Correspondence between correlation functions

The statement is that the following identity between correlation functions holds

$$\left\langle \Phi^{\omega_1}_{j_1,m_1,\overline{m}_1}(z_1)...\Phi^{\omega_N}_{j_N,m_N,\overline{m}_N}(z_N)\right\rangle_{WZW} = \widehat{c}_k^{\,-2} \left\langle \widetilde{\mathcal{T}}_{j_1,m_1,\overline{m}_1}(z_1)...\widetilde{\mathcal{T}}_{j_N,m_N,\overline{m}_N}(z_N)\right\rangle_S \tag{48}$$

where $\widehat{c}_k^2$ is a numerical factor (independent of N) that will be specified below, and where the correlators in the r.h.s. are given by (44) computed with the prescription specified above. This relation reads

$$\left\langle \Phi^{\omega_1}_{j_1,m_1,\overline{m}_1}(z_1)...\Phi^{\omega_N}_{j_N,m_N,\overline{m}_N}(z_N)\right\rangle_{WZW} = \frac{\Gamma(-s)}{\widehat{c}_k^2 bM! c_k^{M-N}} \prod_{i=1}^{N} \frac{\Gamma(-m_i - j_i)}{\Gamma(1+j_i+\overline{m}_i)} \delta_{m_1-\bar{m}_1+...+m_N-\bar{m}_N} \times$$
$$\times \delta\left(\sum_{i=1}^{N}(m_i+\overline{m}_i) + (M+2-N)k\right) \delta\left(s+1+\sum_{i=1}^{N} j_i + M + (N-2-M)k/2\right) \times$$
$$\times \prod_{r=1}^{M}\int d^2w_r \prod_{t=1}^{s}\int d^2v_t \left\langle \prod_{i=1}^{N} \mathcal{T}_{\widetilde{j}_i,\widetilde{m}_i,\widetilde{\overline{m}}_i}(z_i) \prod_{r=1}^{M} \mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(w_r) \prod_{t=1}^{s} \mathcal{T}_{1-k,k,k}(v_t)\right\rangle_{S_{[\lambda=0]}} . \tag{49}$$

This is the main result here. Equation (48) gives a realization of any N-point function of the $SL(2,\mathbb{R})_k/U(1)$ WZW correlators in terms of the analogous observables in the theory (1) if the perturbation is taken to be $\lambda_n = \mu\delta_{n-2} + \lambda\delta_{n-1}$. The perturbation involved in this realization corresponds to the operators $\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}$ and $\mathcal{T}_{1-k,k,k}$, having momentum modes $n=1$ and $n=2$, respectively. This is different from the standard FZZ duality, which corresponds to $\lambda_n = \widetilde{\lambda}\delta_{n-1} + \widetilde{\lambda}\delta_{n+1}$, instead. The perturbation of the linear dilaton theory (13) with operators of different winding numbers was also considered in Ref. [5], where it was suggested that the multiply-wound tachyon operators are linked to the called higher-spin black holes. It would be very interesting to understand the relation with the realization of [5] better and confirm such picture.

4.2.4 Conjugate representations and spectral flow

An interesting feature of the statement made above is that the r.h.s. of (48) involves "conjugate operators" instead of the ones introduced in (34). Ones are related to each others by (46), which represents a symmetry of the formula for the conformal dimension (17) (and not only for it, actually). Notice that, in particular, we have

$$\widetilde{\mathcal{T}}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}} \sim \mathcal{T}_{1-k,k,k}, \tag{50}$$

and operators $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ and $\mathcal{T}_{j,m,\overline{m}}$ have exactly the same conformal dimension. Moreover, the automorphism can be extended in order to be valid for the theory formulated on the product $SL(2,\mathbb{R})_k/U(1)\times\mathbb{R}$ by including the new winding number

$$\widetilde{\omega} = -\omega - 1 - 2(j+m). \tag{51}$$

In such case, the operators $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ and $\mathcal{T}_{j,m,\overline{m}}$, when both are extended by including the time-like factor $e^{i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)T}$, satisfy

$$-\frac{j(j+1)}{k-2} - m\omega - \frac{k}{4}\omega^2 = -\frac{\widetilde{j}(\widetilde{j}+1)}{k-2} - \widetilde{m}\widetilde{\omega} - \frac{k}{4}\widetilde{\omega}^2$$

and also have the same momentum under the J^3 current of the WZW model, namely

$$m + \frac{k}{2}\omega = \widetilde{m} + \frac{k}{2}\widetilde{\omega}.$$

To understand the relation between $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ and $\mathcal{T}_{j,m,\overline{m}}$ in the algebraic framework, let us comment on the $SL(2,\mathbb{R})_k$ representations again: As we said, the principal continuous series $\mathcal{C}_\lambda^{\alpha,\omega}$ correspond to $j=-\frac{1}{2}+i\lambda$ with $\lambda\in\mathbb{R}$ and thus, through (46), this results in the new values $j=-\frac{1}{2}+i\tilde{\lambda}-m(k-2)$ with $\tilde{\lambda}\in\mathbb{R}$, which only belong to the continuous series if $m=0$. Besides, if we perform the change (46) for generic $\tilde{\lambda}$, then m turns out to be a non-real number after of that. Then, the relation between j,m and $\widetilde{j},\widetilde{m}$ can not be thought of as a simple identification between states of different continuous representations but it does correspond to different free field realizations (at least in what respects to the continuous series $\mathcal{C}_\lambda^{\alpha,\omega}$). On the other hand, concerning the discrete representations, it is worth mentioning that the quantity $j+m$ remains invariant under the involution (46); though it is not the case for the difference $j-m$ that, instead, remains invariant under a $\mathbb{Z}_2$ reflected version of (46). Then, unlike the states of continuous representations, the transformation defined by (46) and (51) is closed among certain subset of states of discrete representations. This is because such transformation maps states of the discrete series with $2(j+m)\in\mathbb{Z}$ among themselves. In particular, the case $m+j=0$ corresponds to the well known identification between discrete series $\mathcal{D}_j^{\pm,\omega=0}$ and $\mathcal{D}_{-k/2-j}^{\mp,\omega=\pm1}$ since in that case (46) and (51) reduce to $j\to -k/2-j,\ m\to k/2-m=k/2+j,\ \omega\to -1-\omega$ (i.e. it includes[22] such spectral flow transformation as a particular case). Also notice that the condition $m-\bar{m}\in\mathbb{Z}$ is not preserved for generic values of k. The fixed points of (46) describe a line in the space of representations, parameterized by $j+1/2=-m(k-2)/k$; in particular, a fixed point for generic k corresponds to $j=-1/2$ and $m=0$, for which (51) reduces to $\omega\to-\omega$. Also, in the tensionless limit $k\to 2$ transformation (46) agrees with the Weyl reflection $j\to -1-j$. The relation between quantum numbers manifested by (46) permits to visualize the relation between the vertex considered in our construction and those of reference [62], and we emphasize that these correspond to two different (alterative) representations of

[22]More precisely, the identification between Kac-Moody primary highest-weight (lowets-weight) states that is induced by the sector $\omega=1$ of the spectral flow coincides with a particular case of the identification given by the symmetry (46), (51).

the vertex operators. The relation between both is a kind of "twisting" and is presumably related to the representations studied in [70] for the WZW theory. Certainly, the conjugate representations of the $SL(2,\mathbb{R})_k$ vertex algebra do resemble the twisting (45), and it seems to be a nice connection between correlators (44) and the free field representation studied in [69, 68, 70, 62]. Conjugate representations transform in a particular way under the Kac-Moody affine $\widehat{sl}(2)_k$ algebra, and are analogous to those introduced by Dotsenko for the case of $\widehat{su}(2)_k$ in Refs. [56, 57]. For the non-compact WZW these were first introduced in Ref. [30] to describe winding violating amplitudes. Here, through Eq. (45), these appear again (although we are referring to them as "twisted") within a similar context.

4.2.5 Remark on the $\hat{sl}(2)_k$ affine symmetry

To understand these twisted sectors better, let us make some remarks on the $sl(2)_k$ symmetry of the action (1) when it is perturbed as we did so. We are claiming (and we will prove in the following subsection) that correlators (44) do transform appropriately under the $sl(2)_k$ symmetry in order to describe the WZW correlators. However, even though one can prove this a posteriori, the question arises as to why does it happen if the operators $\mathcal{T}_{1-k,k,k}$ do not seem to commute with the $\widehat{sl}(2)_k$ currents though. To be precise, even though one eventually proves that the free field representation employed here transforms properly by construction (*e.g.* it reproduces solutions of the KZ equation), it is also true that this is not obvious because the screening operators do not seem to commute with the free field representation of the $sl(2)_k$ current algebra (35) as one could naively expect. The explanation of this puzzling feature is that the vertex operators $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ do not satisfy the usual OPE with the $sl(2)_k$ currents either, and thus this restores the symmetry. To see this explicitly, one has to consider the generators of the affine algebra (35), and then verify that those currents do not have regular OPE with the operators $\mathcal{T}_{1-k,k,k}$. The remarkable point is that this is precisely what makes the $SL(2,\mathbb{R})_k$ to be restored: While these currents do not present regular OPE with the operator $\mathcal{T}_{1-k,k,k}$, these do not satisfy the usual OPE with the twisted vertex operators $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ either; and both facts seem to combine in such a way that render the set of observables (44) $SL(2,\mathbb{R})_k$ invariant[23]. This depends on the presence of the normalization factor $\frac{\Gamma(-j-m)}{\Gamma(j+\overline{m}+1)}$ in (45), since its presence is not innocuous for the transformation properties under the generators $J^{\pm}_n$. This feature makes out of the correspondence (48) a non-trivial assertion.

4.3 Proving the correspondence

Here we will show that the formula (48) immediately follows from the relation (25) between WZW and Liouville correlators. With the aim of being clear, here we address the proof in two steps: First, we rewrite the correlators (44) and the operators involved there in a convenient way[24]. The second step will be using the formula (25) to make contact with the WZW correlators.

[23]G.G. thanks Yu Nakayama for addressing his attention to this remarkable point.

[24]In Ref. [7] similar techniques were used to prove a different (though related) correspondence: the one between Liouville and sine-Liouvile correlation functions.

4.3.1 Step 1: Rewriting the correlators

As we said, the proof of formula (48) directly follows from the Stoyanovsky-Ribault-Teschner map (25) we discussed in section 2. In order to make the proof simpler, let us begin by redefining fields as follows

$$\varphi(z) = (1-k)\widehat{\varphi}(z) + i\sqrt{k(k-2)}\widehat{X}(z), \quad X(z) = i\sqrt{k(k-2)}\widehat{\varphi}(z) + (k-1)\widehat{X}(z). \tag{52}$$

That is

$$-\sqrt{\frac{k-2}{2}}\widehat{\varphi} + i\sqrt{\frac{k}{2}}\widehat{X} = -\sqrt{\frac{k-2}{2}}\varphi + i\sqrt{\frac{k}{2}}X \tag{53}$$

and

$$-\frac{1}{\sqrt{k-2}}\varphi = \frac{k-1}{\sqrt{k-2}}\widehat{\varphi} - i\sqrt{k}\widehat{X}. \tag{54}$$

Also notice that it implies

$$\partial\varphi\overline{\partial}\varphi + \partial X\overline{\partial}X = \partial\widehat{\varphi}\overline{\partial}\widehat{\varphi} + \partial\widehat{X}\overline{\partial}\widehat{X}; \tag{55}$$

so that the free field correlators are $\langle\widehat{\varphi}(z_1)\widehat{\varphi}(z_2)\rangle = \left\langle\widehat{X}(z_1)\widehat{X}(z_2)\right\rangle = -2\log|z_1 - z_2|$. One can wonder whether the field redefinition (52) is well defined or not since it is a complex transformation and then both $\widehat{\varphi}$ and $\widehat{X}$ would acquire a non-real part. However, the correct way of thinking this transformation is first considering a Wick rotation of the X direction and then, after the transformation, Wick rotate $\widehat{X}$ back. It turns out to be a perfectly defined transformation for the Wick rotated fields iX and $i\widehat{X}$, which can be seen as real time-like bosons. Transformation (52) is a $U(1,1)$ transformation, with determinant -1. In fact, one can also turn it into a $SU(2)$-rotation by supplementing (52) with a reflection $X \to -X$ (that is also a symmetry of the theory). In that case, it is clear that (54) and (55) remain invariant, while (53) changes its sign in the second term of the r.h.s.. So, in principle, it would be possible to consider a $U(1,1) \times \overline{SU}(2)$ chiral transformation (for the holomorphic part and the anti-holomorphic part, respectively) in order to transform dependences on X into dependences on $\widetilde{X}$.

In terms of these new fields $\widehat{\varphi}$ and $\widehat{X}$ one finds that the linear dilaton theory defined by the action $S_0 - \frac{1}{4\pi}\int d^2z\, \partial T\bar{\partial}T$ takes the form

$$S = \frac{1}{4\pi}\int d^2z \left(-\partial T\bar{\partial}T + \partial\widehat{X}\bar{\partial}\widehat{X} + \partial\widehat{\varphi}\bar{\partial}\widehat{\varphi} + \frac{1}{2\sqrt{2}}QR\widehat{\varphi} - i\sqrt{2k}R\widehat{X}\right) \tag{56}$$

with $Q = b + b^{-1}$, and $b^{-2} = k-2$, so that $Q = \frac{k-1}{\sqrt{(k-2)}} = b + b^{-1}$ (cf. Eq. (4) in section 2). That is, the background charge operator $e^{-\sqrt{\frac{2}{k-2}}\varphi}$ transform through (46) into a new background charge operator $e^{\sqrt{2}Q\varphi - i\sqrt{2k}}$, where $\tilde{j} = -1$, $\tilde{m} = 0$ while $j = k-1$, $m = -k$. Consequently, the stress-tensor reads [1]

$$T(z) = \frac{1}{2}(\partial T)^2 - \frac{1}{2}(\partial\widehat{X})^2 - i\sqrt{\frac{k}{2}}\partial^2\widehat{X} - \frac{1}{2}(\partial\widehat{\varphi})^2 + \frac{k-1}{\sqrt{2(k-2)}}\partial^2\widehat{\varphi}, \tag{57}$$

and the dilaton now acquires a linear dependence on both directions $\widehat{X}$ and $\widehat{\varphi}$. This kind of CFT, representing $c<1$ matter coupled to perturbed 2D gravity, was recently discussed in Refs. [99, 100, 101, 102, 55]. It is possible to verify that this stress-tensor leads to the appropriated central charge $c = 3 + \frac{6}{k-2}$, as it is of course expected. On the other hand, in terms of the new fields the interaction (perturbation) term $\lambda \mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(w_r) + \mu \mathcal{T}_{1-k,k,k}(v_t)$ takes the form

$$\mathcal{O}_{\lambda_1,\lambda_2} = c_k^{-1} e^{-\sqrt{\frac{k-2}{2}}\widehat{\varphi}+i\sqrt{\frac{k}{2}}\widehat{X}} + e^{\sqrt{\frac{2}{k-2}}\widehat{\varphi}}, \tag{58}$$

where we already fixed the scale μ to a specific value by shifting the zero-mode of the Liouville field $\widehat{\varphi}$, and we also specified the numerical factor c_k as being the ratio between the couplings μ and λ^2 in (41). Notice that in these coordinates the second term in the perturbation $\mathcal{O}_{\lambda_2,\lambda_1}$ turns to be diagonalized (no dependence on $\widehat{X}$ arise there) and agrees with the Liouville cosmological constant $\mu e^{\sqrt{2}b\widehat{\varphi}}$. On the other hand, the first term in (58) still has the form of one of the two exponentials that form the cosine interaction (33) in sine-Liouville theory; this is due to (53). On the other hand, the vertex operators in terms of $\widehat{X}$ and $\widehat{\varphi}$ take the form[25]

$$\widetilde{\mathcal{T}}_{j,m,\overline{m}} = \frac{c_k \Gamma(-m-j)}{\pi^2 \Gamma(1+\overline{m}+j)} V_\alpha \times e^{i\sqrt{\frac{2}{k}}(m-\frac{k}{2})\widehat{X}+i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)T}, \tag{59}$$

with the Liouville field $V_\alpha = e^{\sqrt{2}\alpha\widehat{\varphi}}$, with $\alpha = bj + b + b^{-2}/2 = b(j+k/2)$. Expanding the correlators we get

$$\frac{1}{\hat{c}_k^2}\left\langle \widetilde{\mathcal{T}}_{j_1,m_1,\overline{m}_1}(z_1)...\widetilde{\mathcal{T}}_{j_N,m_N,\overline{m}_N}(z_N)\right\rangle_{S_{[\lambda,\mu]}} = \frac{\Gamma(-s)}{bM!c_k^M\hat{c}_k^2}\prod_{r=1}^{M}\int d^2w_r \prod_{t=1}^{s}\int d^2v_t \left\langle \prod_{i=1}^{N}\widetilde{\mathcal{T}}_{j_i,m_i,\overline{m}_i}(z_i) \times \right.$$
$$\left. \times \prod_{r=1}^{M}\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(w_r)\prod_{t=1}^{s}\mathcal{T}_{1-k,k,k}(v_t)\right\rangle_{S_{[\lambda=0,\mu=0]}}$$

and this can be written as

$$= \frac{1}{k\hat{c}_k^2}\delta\left(\omega_1+...\omega_N+N-2-M\right)\prod_{a=1}^{N}\frac{c_k\ \Gamma(m_a-j_a)}{\pi^2\Gamma(1+j_a-\bar{m}_a)}\left\langle\prod_{t=1}^{N}e^{i\sqrt{\frac{2}{k}}(m_t+\frac{k}{2}\omega_t)T(z_t)}\right\rangle_{S_{[\lambda=0]}} \times$$
$$\times\frac{1}{M!c_k^M}\delta_{m_1-\overline{m}_1+...m_N-\overline{m}_N}\prod_{r=1}^{M}\int d^2w_r \left\langle\prod_{t=1}^{N}e^{i\sqrt{\frac{2}{k}}(m_t-\frac{k}{2})\widehat{X}(z_t)}\prod_{r=1}^{M}e^{i\sqrt{\frac{k}{2}}\widehat{X}(w_r)}\right\rangle_{S_{[\lambda=0]}} \times$$
$$\times\frac{\Gamma(-s)}{b}\delta\left(s-1-\frac{2+M}{2b^2}+\frac{\alpha_1+...\alpha_N}{b}\right)\prod_{t=1}^{s}\int d^2v_t \left\langle\prod_{t=1}^{N}V_{\alpha_t}(z_t)\prod_{r=1}^{M}V_{-\frac{1}{2b}}(w_r)\prod_{t=1}^{s}V_b(v_t)\right\rangle_{S_{L[\mu=0]}}, \tag{60}$$

[25] Again, we are not explicitly writing the antiholomorphic contribution $e^{i\sqrt{\frac{2}{k}}\overline{m}\widehat{X}}$ for short. It has to be understood in what follows.

where $S_{[\lambda=0]}$ here refers to the unperturbed action

$$S_{[\lambda=0]} = \frac{1}{4\pi}\int d^2z\left(-\partial T\overline{\partial}T + \partial\widehat{X}\overline{\partial}\widehat{X} - \frac{i}{2}\sqrt{\frac{k}{2}}R\widehat{X}\right). \tag{61}$$

The third line in (60) turns out to be a $N+M$-point correlation function in Liouville field theory (see Eq. (5) in section 2), defined by the Liouville action

$$S_L[\mu] = \frac{1}{4\pi}\int d^2z\left(\partial\widehat{\varphi}\overline{\partial}\widehat{\varphi} + \frac{1}{2\sqrt{2}}QR\widehat{\varphi} + 2\pi\mu e^{\sqrt{2}b\widehat{\varphi}}\right),$$

with $Q = b + b^{-1}$. Recall that the parameter b of the Liouville theory is related to the Kac-Moody level k through $b^{-2} = k-2$, while the quantum numbers α_i are defined in terms of j_i by $\alpha_i = bj_i + b + b^{-2}/2$, for $i = 1, 2, ...N$. After the Wick contraction, we find

$$\frac{\Gamma(-s)}{bM!c_k^M\widehat{c}_k^2}\prod_{r=1}^{M}\int d^2w_r\prod_{t=1}^{s}\int d^2v_t\left\langle\widetilde{\mathcal{T}}_{j_1,m_1,\overline{m}_1}(z_1)...\widetilde{\mathcal{T}}_{j_N,m_N,\overline{m}_N}(z_N)\prod_{r=1}^{M}\mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(w_r)\prod_{t=1}^{s}\mathcal{T}_{1-k,k,k}(v_t)\right\rangle_{S[\lambda=0]} =$$

$$= N_k(j_1,...j_N;m_1,...m_N)\prod_{r=1}^{M}\int d^2w_r\ F_k(z_1,...z_N;w_1,...w_M)\langle\prod_{t=1}^{N}V_{\alpha_t}(z_t)\prod_{r=1}^{M}V_{-\frac{1}{2b}}(w_r)\rangle_{S_L[\mu]}\ , \tag{62}$$

where $\mu = b^2/\pi^2$ and where, after fixing the value $\hat{c}_k^2 = 2c_k^2/b\pi^3$, the normalization factor is

$$N_k(j_1,...j_N;m_1,...m_N) = \frac{2\pi^{3-2N}b}{M!\ c_k^{M+2-N}}\prod_{i=1}^{N}\frac{\Gamma(-m_i-j_i)}{\Gamma(1+j_i+\bar{m}_i)} \tag{63}$$

and the function $F_k(z_1,...z_N;w_1,...w_M)$ is given by

$$\begin{aligned} F_k(z_1,...z_N;w_1,...w_M) \quad = \quad & \frac{\prod_{1\leq r<l}^{N}|z_r-z_l|^{k-2(m_r+m_l+\omega_r\omega_l k/2+\omega_l m_r+\omega_r m_l)}}{\prod_{1<r<l}^{M}|w_r-w_l|^{-k}\prod_{t=1}^{N}\prod_{r=1}^{M}|w_r-z_t|^{k-2m_t}}\times \\ & \times\frac{\prod_{1\leq r<l}^{N}(\bar{z}_r-\bar{z}_l)^{m_r+m_l-\bar{m}_r-\bar{m}_l+\omega_l(m_r-\bar{m}_r)+\omega_r(m_l-\bar{m}_l)}}{\prod_{1<r<l}^{M}(\bar{w}_r-\bar{z}_t)^{m_t-\bar{m}_t}}. \end{aligned} \tag{64}$$

Remarkably, this has reproduced the r.h.s. of formula (25); cf. Eq. (27). Notice that the exponents of the differences $|z_r - z_l|$ in (64) do depend on whether the theory is being formulated on the coset $SL(2,\mathbb{R})_k/U(1)$ or on its product with the time T. The vertex operators (59) are the only fields that carry the T-dependences, so that the rest of the OPEs are not affected.

According to (47), the amount of perturbations involved in (62) is constrained by the following conditions

$$\sum_{i=1}^{N}m_i = \sum_{i=1}^{N}\bar{m}_i = \frac{k}{2}(N-M-2), \quad s = -b^{-1}\sum_{i=1}^{N}\alpha_i + b^{-2}\frac{M}{2} + 1 + b^{-2}\ , \tag{65}$$

where the number s corresponds to the amount of screening operators $V_b = \mu e^{\sqrt{2}b\widehat{\varphi}}$ to be included in Liouville correlators. The whole amount of vertex operators involved in the r.h.s. of (62) is then $N+M+s$, and is related to the winding numbers of the strings through

$$\sum_{i=1}^{N} \omega_i = M + 2 - N \geq -|N-2|. \tag{66}$$

Notice that the value of $\sum_{i=1}^{N}\omega_i$ can not be lower than $2-N$ if M represents a positive integer number. Allowing negative values of winding numbers requires the insertion of screening operators with $n=-1$ in addition to those of $n=+1$. M runs between 0 and $N-2$, which implies that, according to the prescription given in subsection 4.2.2, the absolute value of the violation of winding number conservation could not exceed $N-2$. This is an interesting feature, and it is not trivial at all to fully understand this bound. We can say here that it is closely related to the $\widehat{sl}(2)_k$ symmetry of the theory, and we refer to the appendix D of Ref. [67] for a nice explanation. It is worth mentioning that the selection rule for winding number violation (66) was already part of the original FZZ conjecture [30]. A short note about this rule can be also found in Ref. [103].

The formula (62)-(64), with the conditions (65), is the main ingredient for proving (48). It only remains to argue that the r.h.s. of (62) actually represents a WZW correlator; and, actually, it can be already observed since it directly follows from the formula (25). Indeed, the r.h.s. of (62) agrees with the l.h.s. of (25) and this would complete the proof of (48). Let us conclude the job by further commenting on it.

4.3.2 Step 2: Realizing the Stoyanovsky-Ribault-Teschner map

As we just mentioned, the last step in proving (48) is showing that the r.h.s. of Eq. (62) precisely describes a WZW N-point function, and, actually, this immediately follows from the main result of Ref. [4] (see formula (3.29) there, which we wrote in the Eq. (25) in section 2). Hence, we have managed to rewrote our result (48) in such a way that its proof turns out to be a direct consequence of the observation made by S. Ribault in his paper [4], where he showed that the l.h.s. of Eq. (62) is precisely equal to a correlation function in the $SL(2,\mathbb{R})_k$ WZW model. Our achievement was to prove that the auxiliary overall function $F_k(z_1,...z_N;w_1,...w_M)$ standing in the Ribault-Teschner formula can be also thought of as coming from the correlation functions of the linear dilaton CFT realized by the field $\widehat{X}$; namely

$$F_k(z_1,...z_N;w_1,...w_M) = \left\langle \prod_{t=1}^{N} e^{i\sqrt{\frac{2}{k}}\left((m_t-\frac{k}{2})\widehat{X}(z_t)+(m_t+\frac{k}{2}\omega_t)T(z_t)\right)} \prod_{r=1}^{M} e^{i\sqrt{\frac{k}{2}}\widehat{X}(w_r)} \right\rangle_{S_{[\lambda=0]}}. \tag{67}$$

That is, we showed how the Ribault-Teschner formula can be seen as an identity between correlators of two different two-dimensional σ-models with three-dimensional target space each. While one of these is the $SL(2,\mathbb{R})_k$ WZW, the other is of the form

$$Liouville \times \mathcal{M}_k \otimes \mathbb{R} \tag{68}$$

of which Liouville theory is just a part. The $\mathbb{R}$ factor corresponds to the time-like direction, parameterized by T. On the other hand, the $\mathcal{M}_k$ factor is a $U(1)$ direction, parameterized by the field $\widehat{X}$, and describes a linear dilaton theory with central charge $c = 1 - 6k < 1$. In fact, notice that the contribution of $\widehat{X}$ to the central charge is actually negative because $k > 2$. The field $\widehat{X}$ interacts with the Liouville field $\widehat{\varphi}$ through the tachyon-like potential, so that the first product in (68), unlike the second, is not a direct product. The time direction, instead, does not interact with the other fields, and it only contributes to the total central charge and the conformal dimension of the vertex operators.

The fact that a construction like (68) is possible is not a minor detail: Realizing that the Ribault-Teschner formula (25) admits to be interpreted as the equivalence between these two CFTs demanded not only the existence of a realization like (67), but also demanded the contribution of the central charge coming from the $U(1) \otimes time$ part to agree with the difference between the Liouville central charge $c_L = 1 + 6Q^2$ and the $SL(2,\mathbb{R})$ WZW central charge $c_{SL(2)} = 3 + 6\widehat{Q}^2 = 3k/(k-2)$, being reminded of $b^{-2} = k - 2$. Moreover, such value for the central charge of the CFT defined by fields $\widehat{X}$ and T had to be consistent with the conformal dimension of the fields in (67), leading to reproduce the formula for the conformal dimension of the WZW vertex operators. Besides, another feature that had to be explained was the presence of the M additional fields $V_{-1/2b}$ arising in the r.h.s. of (62). Their presence is now understood as follows: Since we know that the (the M-multiple integral of the) product between the function $F_k(z_1, ...z_N; w_1, ...w_M)$ and the $N+M$-point Liouville correlation function does satisfy the KZ equation and so represents a correlation function in the WZW theory, it is then expected that the Liouville degenerate fields $V_{-1/2b}$ arising there admit to be expressed as a $(1,1)$-operator in a "bigger" theory with the form $Liouville \times CFT$ (i.e. the screening charge $V_{-1/2b} \times e^{i\sqrt{k/2}\widehat{X}}$ standing as the first term of the r.h.s. of (58)). That is, even though $V_{-1/2b}$ has dimension $h = -\frac{1}{2} - \frac{3}{4b^2} \neq 1$ with respect to the Liouville stress-tensor, it does correspond to a $(1,1)$-operator[26] $V_{-1/2b} \otimes V_{CFT}$ with respect to the stress-tensor of the bigger model $Liouville \times CFT$. Of course, the theory also admits as a screening operator the one that was already the screening for the "Liouville part of the theory", namely $V_b \otimes I$; so (58) can be written as the sum of both, $\mathcal{O}_{\lambda_1,\lambda_2} = c_k^{-1} V_{-1/2b} e^{i\sqrt{k/2}\widehat{X}} + V_b$.

Notice that all the requirements mentioned above are actually obeyed by the theory defined by the action (61) perturbed by the operator (58). Hence, we have given a free field representation of the Ribault-Teschner formula (25). Related to this, in Ref. [4] it was commented that a parafermionic realization of (25) is also known, and the unpublished work by V. Fateev was referred. The parafermion representation leads (see Eq. (3.31) in [4]) to a formula similar to (25) provided the replacement of the factor $\prod_{1\leq r<l}^{N}(z_r - z_l)^{\frac{k}{2} - (m_r + m_l + \omega_r\omega_l\frac{k}{2} + \omega_l m_r + \omega_r m_l)}(\overline{z}_r - \overline{z}_l)^{\frac{k}{2} - (\overline{m}_r + \overline{m}_l + \omega_r\omega_l\frac{k}{2} + \omega_l\overline{m}_r + \omega_r\overline{m}_l)}$ in (27) by a factor $\prod_{1\leq r<l}^{N}(z_r - z_l)^{\frac{k}{2} + \frac{2}{k}m_r m_l - m_r - m_l}(\overline{z}_r - \overline{z}_l)^{\frac{k}{2} + \frac{2}{k}\overline{m}_r\overline{m}_l - \overline{m}_r - \overline{m}_l}$, and notice that this is exactly what we find in our language (67) if we exclude the T dependence in the vertex operators. This realizes a correspondence like (25) but for the case of the coset $SL(2,\mathbb{R})_k/U(1)$. See the "notes" at the end of Ref. [1] where the similarities with Fateev's

[26] Even though the operator $V_{-1/2b} \times e^{i\sqrt{k/2}\widehat{X}}$ has dimension 1, it is not strictly correct to refer to it as a "screening" operator due to the remark on the $\widehat{sl}(2)_k$ transformation properties made in section 4.2.5.

work were already mentioned. Besides, a realization of the Ribault-Teschner formula in terms of Liouville times a $c < 1$ matter CFT was independently presented by S. Nakamura and V. Niarchos in Ref. [6]. We would like to explore the similarities between our realization and the one in that paper; we just realized that the realization in [6] does closely parallels ours.

Summarizing: because of Ribault-Teschner formula, it turns out that the correlation function in the r.h.s. of (62) does correspond to the string amplitude in the black hole ($\times time$) background, where the winding number conservation is being violated in an amount $|N-2-M|$. Consequently, this implies that the l.h.s. of (62) do correspond to WZW correlators as well, and this completes the proof of (48). However, it has to be emphasized that the correspondence between BPZ and KZ equations was proven for the Lorentzian theory, namely holding for continuous representations. Thus, considering its validity beyond such regime assumes a sort of analytic continuation. The convergence of integrals in (62) is the subtle point here.

4.4 A consistency check of the correspondence

We have proven formula (48); this was first done in Ref. [1], but the order of the presentation was rather different there. Formula (48) turns out to be a useful tool for computing correlators in the WZW theory. A concise example was given in Ref. [2], where the free field representation in terms of the $Liouville \times U(1) \otimes \mathbb{R}$ conformal field theory (56)-(58) was employed to compute WZW three-point functions for the particular case where the total winding number is violated in one unit. This quantity turns out to be proportional to the Liouville correlator

$$\left\langle \Phi^{\omega_1}_{j_1,m_1,\overline{m}_1}(0)\Phi^{\omega_2}_{j_2,m_2,\overline{m}_2}(1)\Phi^{\omega_3}_{j3,m_3,\overline{m}_3}(\infty)\right\rangle_{WZW} \sim \prod_{i=1}^{3}\frac{\Gamma(-m_i-j_i)}{\Gamma(j_i+1+\bar{m}_i)}\prod_{t=1}^{s}\int d^2v_t \left\langle e^{\sqrt{\frac{2}{k-2}}(j_1+1)\widehat{\varphi}(0)} \times \right.$$

$$\left. \times e^{\sqrt{\frac{2}{k-2}}(j_2+1)\widehat{\varphi}(1)} e^{\sqrt{\frac{2}{k-2}}(j_3+1)\widehat{\varphi}(\infty)} \prod_{t=1}^{s} e^{\sqrt{\frac{2}{k-2}}\widehat{\varphi}(v_t)} \right\rangle_{S_L[\mu=0]} \delta\left(s+j_1+j_2+j_3+1+\frac{k}{2}\right),$$

and, up to an irrelevant k-dependent (j-m-independent) factor and having fixed the value of the black hole mass, the final result reads

$$\left\langle \Phi^{\omega_1}_{j_1,m_1,\overline{m}_1}(0)\Phi^{\omega_2}_{j_2,m_2,\overline{m}_2}(1)\Phi^{\omega_3}_{j3,m_3,\overline{m}_3}(\infty)\right\rangle_{WZW} = \left(\pi\gamma\left(\frac{1}{k-2}\right)\right)^{-j_1-j_2-j_3-\frac{k}{2}-1}\prod_{i=1}^{3}\frac{\Gamma(-m_i-j_i)}{\Gamma(j_i+1+\bar{m}_i)}\times$$

$$\times\frac{G_k(j_1+j_2+j_3+\frac{k}{2})G_k(-j_1-j_2+j_3-\frac{k}{2})G_k(j_1-j_2-j_3-\frac{k}{2})G_k(1+j_1-j_2+j_3-\frac{k}{2})}{\gamma\left(-j_1-j_2-j_3-\frac{k}{2}\right)\gamma\left(-\frac{2j_2+1}{k-2}\right)G_k(-1)G_k(2j_1+1)G_k(1-k-2j_2)G_k(2j_3+1)}\times$$
$$\times\delta(m_1+m_2+m_3-k/2)\delta(\bar{m}_1+\bar{m}_2+\bar{m}_3-k/2)\delta(s+j_1+j_2+j_3+1+k/2). \qquad (69)$$

where the special function $G_k(x)$ is defined through

$$G_k(x) = (k-2)^{\frac{x(k-1-x)}{2(k-2)}}\Gamma_2(-x|1,k-2)\Gamma_2(k-1+x|1,k-2),$$

in terms of the Barnes function $\Gamma_2(x|1,y)$

$$\ln \Gamma_2(x|1,y) = \lim_{\varepsilon\to 0} \frac{d}{d\varepsilon} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left((x+n+my)^{-\varepsilon} - (1-\delta_{n,0}\delta_{m,0})(n+my)^{-\varepsilon} \right),$$

where the presence of the factor $(1-\delta_{n,0}\delta_{m,0})$ in the r.h.s. means that the sum in the second term does not take into account the step $m = n = 0$. Expression (69) does reproduce the exact result, so that agrees with the result obtained in Refs. [30, 67, 81]. The details of the computation can be found in [2]. Rather than an application, the calculation of (69) can be considered as a consistency check of the representation (68) proposed here (and in [1]) to represent WZW correlators. Besides, it also represents an operative advantage since (unlike other free field realizations for which the computation of violating winding three-point function involves the additional spectral flow operator) this turns out to be integrable in terms of the Dotsenko-Fateev type integrals (cf. the calculations in Refs. [68, 67]). Nevertheless, it is worth pointing out that the consistency check discussed here is the most simple (non-trivial) computation one can do within this framework; this is because it did not involve the degenerate Liouville fields $V_{-1/2b}$. Unlike, the screening that we did use to realize (69) was the new one we introduced; namely, the operator $\mathcal{T}_{1-k,k,k} = e^{\sqrt{\frac{2}{k-2}}\widehat{\varphi}} = e^{-\sqrt{\frac{2}{k-2}}(k-1)\varphi + i\sqrt{2k}X}$, which represents a $n = 2$ perturbation. A less trivial consistency check would be that of trying to reproduce the winding-conservative WZW three-point function in the often called m-basis, which would require to make use of a non-trivial integral representation of the (hypergeometric) special function of the kind studied in Refs. [82, 83, 104]. Related to this point, let us mention that explicit expressions for Liouville four-point functions involving one degenerate state $V_{-\frac{1}{2b}}$ were recently obtained [52, 102]. According to the relation (25), these four-point functions are the ones representing three-point functions that conservate the winding conservation in the WZW side.

Other applications of the Stoyanovsky-Ribault-Teschner correspondence (25), (48), were early discussed in Ref. [88, 89, 27, 25]. In subsection 4.5.3, we will review one of the observations made in [27].

4.5 Remarks

4.5.1 A comment on generalized minimal gravity

Now, we would like to make a brief comment on the theory defined by the action (56) and the perturbation (58); and let us focus our attention on the two-dimensional sector corresponding to the fields $\widehat{\varphi}$ and $\widehat{X}$. Because of the field redefinitions (52), it turns out that the theory could be written as the Liouville theory coupled to a $c < 1$ CFT. Then, the natural question arises as to whether such a $c < 1$ model can be identified with one of the quoted minimal models. As it is well known, the CFT minimal models are characterized by two integers p and q which yield the value of the central charge, being $c = 1 - 6(\beta^{-1} - \beta)^2$ with rational[27] $\beta^2 = p/q$ satisfying $q > p$ (so that $\beta < 1$). In our case, the value of the central charge of the $c < 1$ theory (corresponding

[27]Besides, a generalized version of these CFTs can be considered, being valid for generic values of β, [50].

to the part of the theory governed by the field $\widehat{X}$) turns out to be $c = 1 - 6k$ and, then, in order to identify this with one of the minimal models we should demand $k = (p-q)^2/pq$ (that is $k = (\beta^{-1} - \beta)^2$) [71]. However, since we are interested in the whole range $k > 2$, it turns out that the condition $c = 1-6(p-q)^2/pq$ is only consistent with particular values of k. One example is precisely the model ($p = 1$, $q = 4$) which does correspond to $k = 9/4$, which is the value of k for the 2D theory on the coset. In such case, and taking into account that k also satisfies $k = 2+b^{-2}$, we would have $\beta = b$ so that the theory corresponds to the often called 2D minimal gravity (model that is supposed to be exactly solved). For more general case, the 2D theory defined by the fields $\widehat{\varphi}$ and $\widehat{X}$ can be regarded as the Liouville theory coupled to a generalized minimal model (with non-necessarily rational β^2) perturbed by (58). Such perturbation would then correspond to a Liouville-dressed operator in the minimal model too. The operators of the minimal models admit a representation in terms of the exponential form $\Phi_{mn} = e^{i\alpha_{mn}\widehat{X}}$, having conformal dimension $h_{mn} = \frac{1}{4}(m\beta^{-1} - n\beta)^2 - \frac{1}{4}(\beta^{-1} - \beta)^2 = \alpha_{mn}(\alpha_{mn} + \beta - \beta^{-1})$ for two positive integers m and n; that is, the momenta can take values $\alpha_{mn} = \frac{1}{2}(n-1)\beta - \frac{1}{2}(m-1)\beta^{-1}$ or $\alpha_{mn} = \frac{1}{2}(m+1)\beta^{-1} - \frac{1}{2}(n+1)\beta$. So, a perturbation operator with the form $e^{i\sqrt{k/2}n\widehat{X} + \sqrt{2}a_n\widehat{\varphi}}$ can be regarded as a dressed field $\Phi_{n-1,n-1}$ of the minimal model (p, q) with $k = (p-q)^2/pq$. In these terms, what we have proven is a correspondence between N-point functions in the WZW theory and a subset of correlation functions of perturbed Liouville gravity coupled to generalized minimal models.

4.5.2 Duality between tachyon-like backgrounds

By using the relation between Liouville correlators and WZW correlators we wrote down identity (48). This gives a dual description for the 2D string theory in the black hole background. One of the questions that arise is about the relation between (48) and the standard FZZ correspondence. In fact, both models appear as alternative dual descriptions of the WZW theory, so that we can use WZW correlators as an intermediate step to eventually write the following seemly self-duality relation

$$\prod_{r=1}^{s_+} \int d^2u_r \prod_{t=1}^{s_-} \int d^2v_t \left\langle \prod_{i=1}^{N} \mathcal{T}_{j_i,m_i,\overline{m}_i}(z_i) \prod_{r=1}^{s_+} \mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(u_r) \prod_{t=1}^{s_-} \mathcal{T}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}(v_t) \right\rangle_{S_{[\lambda=0]}} \sim$$

$$\sim \prod_{r=1}^{\widetilde{s}_+} \int d^2u_r \prod_{t=1}^{\widetilde{s}_{++}} \int d^2\omega_t \left\langle \prod_{i=1}^{N} \widetilde{\mathcal{T}}_{j_i,m_i,\overline{m}_i}(z_i) \prod_{r=1}^{\widetilde{s}_+} \mathcal{T}_{1-\frac{k}{2},\frac{k}{2},\frac{k}{2}}(u_r) \prod_{t=1}^{\widetilde{s}_{++}} \mathcal{T}_{1-k,k,k}(\omega_t) \right\rangle_{S_{[\lambda=0]}}, \qquad (70)$$

where the fearful symbol $\sim$ stands to make explicit the fact that this identity depends on the details of how the FZZ conjecture relates the WZW correlators to those of sine-Liouville theory[28]. This relation between correlators, realized by means of the Coulomb gas realization, yields a non-trivial integral identity. On the other hand, one can wonder whether (70) has to be referred as a self-duality of sine-Liouville field theory or not. In fact, it merely looks like

[28]As far as we know, the checks of FZZ duality were performed by comparing the analytic structures of both theories rather than verifying exact numerical matching.

a duality between two different deformations of the linear dilaton theory (13) rather than a "self-duality". However, one can see that both sides in the identity above are in some sense connected to sine-Liouville theory, and not only the left hand side. Actually, the perturbation $\mathcal{T}_{1-k,k,k}$, that represents the momentum $n=2$ operator, is connected to that of $n=1$ by the conjugation relation (45). That is, while $j=1-m=1-k$ for the $n=2$ operator $\mathcal{T}_{1-k,k,k}$, the dual momenta (dual according to (46)) are[29] $\widetilde{j}=1+\widetilde{m}=1-k/2$, and correspond to the momenta of the $n=-1$ operator $\mathcal{T}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}$. Thus, we could relate the correlators in the l.h.s. of (70) to the following one[30]

$$\sim \prod_{r=1}^{\widetilde{s}_+}\int d^2u_r \prod_{t=1}^{\widetilde{s}_-}\int d^2\omega_t \left\langle \prod_{i=1}^{N}\widetilde{\mathcal{T}}_{j_i,m_i,\overline{m}_i}(z_i)\prod_{r=1}^{\widetilde{s}_+}\mathcal{T}_{1-\frac{k}{2},+\frac{k}{2},+\frac{k}{2}}(u_r)\prod_{t=1}^{\widetilde{s}_-}\widetilde{\mathcal{T}}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}(\omega_t)\right\rangle_{S_{[\lambda=0]}} \tag{71}$$

with $\frac{k-2}{2}\widetilde{s}_+-\widetilde{s}_--\frac{k}{2}(N-2)=\frac{k-2}{2}(s_++s_-)$ and $\widetilde{s}_+-(N-2)=s_+-s_-$. Hence, (70) turns out to be a twisted version of the sine-Liouville model, i.e. can be written as in (71). The presence of the tildes $\sim$ on the operators in gives rise to the expression "twisted"; twisted in the sense that (46) is applied to the operators $\mathcal{T}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}$ but is not applied to the operators $\mathcal{T}_{1-\frac{k}{2},+\frac{k}{2},+\frac{k}{2}}$. This kind of relation between correlators (70) and (71) is reminiscent of what happens in the WZW theory, where standard and conjugate representations stand as alternative realizations of the same correlation functions. Thus, this suggests that (70) could be manifesting some kind of self-duality relating two different realization of the same conformal theory[31]. Morally, the price to be paid to twist (namely, to conjugate) the N vertex operators $\mathcal{T}_{j_i,m_i,\overline{m}_i}$ in (70) is that of twisting the lelf-handed screening operators $\mathcal{T}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}\to\widetilde{\mathcal{T}}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}\propto\mathcal{T}_{1-k,k,k}$, while keeping the right-handed $\mathcal{T}_{1-\frac{k}{2},+\frac{k}{2},+\frac{k}{2}}$ unchanged. Consequently, the number of insertions changes from s_- to $\widetilde{s}_{++}=\widetilde{s}_-$ (and also from s_+ to $\widetilde{s}_+$) by keeping the formal relation $N-2=\widetilde{s}_+-s_++s_-$ fixed. Roughly speaking, the right hand side of (70) looks like a "half" of a sine-Liouville theory, because just one of the two exponential operators $\mathcal{T}_{1-\frac{k}{2},\pm\frac{k}{2},\pm\frac{k}{2}}$ that form the cosine interaction (33) is present, while the operators $\mathcal{T}_{1-k,k,k}$ seem to arise there for compensating the conservation laws that make the correlator to be nonzero. In the WZW theory, the analogue to the "twisting" that connects the operators $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ to operators $\mathcal{T}_{j,m,\overline{m}}$ would be the relation existing between conjugate and standard representations of the $\widehat{sl}(2)_k$ vertex algebra [18, 20, 69, 68, 70]. The relation between representations $\widetilde{\mathcal{T}}_{j,m,\overline{m}}$ and $\mathcal{T}_{j,m,\overline{m}}$ connects operators of the winding sector n to those of the sector $n+1$. Presumably, the twisted version of the FZZ duality we presented in (48) can be extended in order to include higher momentum and winding modes $n>2$. This would rise the obvious question as to what would these twisted sectors be describing in terms of the black hole picture. As it was pointed out in [32], if the $c=1$ theory is perturbed by operators of the sector n, then it behaves equivalently to the theory compactified in a different

[29]Strictly speaking, one has to consider the automorphism $m\to\widetilde{m}=-jk-m(k-1)-k/2$ instead of (46), which is a composition with the reflection $m\to -m$.

[30]up to a k-dependent factor of the form $(b_k)^{\widetilde{s}_-}$, with b_k being independent on j_i, m_i and $\overline{m}_i$.

[31]Let us also mention that another realization of the same correlators is possible if one replace $\widetilde{\mathcal{T}}_{1-\frac{k}{2},-\frac{k}{2},-\frac{k}{2}}\propto e^{-\sqrt{\frac{2}{k-2}}(k-1)\widehat{\varphi}+i\sqrt{2k}\widehat{X}}$ by its $k-2$ power $e^{-\sqrt{2(k-2)}(k-1)\widehat{\varphi}+i\sqrt{2k}(k-2)\widehat{X}}$. This is because of the Liouville self-duality under $b\leftrightarrow b^{-1}$.

radius R/n and perturbed by the sine-Liouville operators. In some sense, this is related to what was early studied in Ref. [21]. Nevertheless, the perturbation we considered here presents operators of both sectors $n = 1$ and $n = 2$, so being a sort of chirally twisted case. We would like to understand this deformations better. Our hope is to make contact to the results of Refs. [5] and [21] in trying to answer this question, but this certainly requires further study.

4.5.3 The $c \to 0$ limit of the *Liouville* $\times U(1) \times \mathbb{R}$ model

To conclude, we would like to discuss the particular limit where the central charge of the model (68) vanishes. This was first studied in Refs. [26] and [27] (see also [105]-[108]). This limit corresponds to $k \to 0$, which, in fact, is far from being well understood. Actually, one can rise several question concerning whether in such a limit the CFT is well defined or not. However, let us avoid these questions here and merely assume that such an extension is admissible. In the limit $k \to 0$, the Liouville central charge becomes $c_L = -2$ while the background charge for the field $\widehat{X}$ vanishes, so that the central charge for the $U(1) \times \mathbb{R}$ theory (i.e. the fields $\widehat{X}$ and T) turns out to be $+2$. The functional form of the correlation functions in the $k \to 0$ limit requires a careful analysis because of subtle features arising through the analytic continuation in the b complex plane [109, 50]. However, we can further speculate and assume for a while that an extension of the correspondence (25) between WZW and Liouville theory still holds at $k = 0$. At this point, the sine-Liouville action actually coincides with the Liouville action supplemented with that of a $c = 1$ field $\widehat{X}$. This is because of the identification $b^{-2} = k - 2$ and the fact that $Q = -\widehat{Q}$ at the point $k = 0$. Besides, at $k = 0$ the sine-Liouville interaction (32) does correspond to the Liouville cosmological constant $e^{i\widehat{\varphi}/\sqrt{2}}$. This suggestive matching between both actions can be tested at the level of correlation functions as well. In fact, with the authors of [26], we could assume that the FZZ conjecture is still valid in the limit $k \to 0$ and, then, by invoking the Ribault-Teschner formula (25), eventually conclude that the sine-Liouville correlators model coincide with the correlators of the Liouville theory (times the free boson $\widehat{X}$) at $k = 0$. To see this, let us point out the following remarkable facts: First, notice that, because we are taking a limit $R = \sqrt{k/2}$ going to zero (*i.e.* the asymptotic radius of the cigar), it is just enough to observe what happens with the modes $m = \bar{m} = 0$ on the cigar. From the point of view of the T-dual model, the dual radius $\tilde{R} \sim 1/\sqrt{k}$ of the cylinder goes to infinity and the states with finite momentum $p = \frac{m}{\sqrt{k}}$ (keeping p fixed) decouple generating a $U(1)$ factor $\sim e^{i\sqrt{2}p\widehat{X}}$ in the correlation functions. Secondly, one can show (see [27]) that for $k = 0$ the formula (25) reads

$$\left\langle \Phi^{\omega_1}_{j_1,m_1,\overline{m}_1}(z_1) ... \Phi^{\omega_N}_{j_N,m_N,\overline{m}_N}(z_N) \right\rangle_{WZW} \sim \prod_{i=1}^{N} \mathcal{R}_0(j_i, 0) \ \langle V_{-\frac{i}{\sqrt{2}}j_1}(z_1) ... V_{-\frac{i}{\sqrt{2}}j_N}(z_N) \rangle_{S_L[\mu]}; \qquad (72)$$

with $p_1 + p_2 + ... p_N = \omega_1 + \omega_2 + ... \omega_N = M - N + 2 = 0$. The function $\mathcal{R}_k(j, m)$ is the reflection coefficient of WZW model, which is given by the two-point function (24). The arising of these reflection coefficients (one for each vertex operator) is ultimately attributed to the fact that the momenta of the WZW vertex operators were the Weyl reflected $\widehat{j}_i = -1 - j_i$ instead of j_i (notice that the Liouville correlator in (72) scales like $\mu^{\hat{j}_1 + .. \hat{j}_N + 1}$). We also observe in (72) that, besides the s integrals over the screening insertions required in the Liouville correlators, we implicitly have $M = N - 2$ additional integrals over the variables v_t where M operators $V_{-1/2b}(v_t)$ are inserted. This is consistent with what one would expect since $k = 0$ implies $b^2 = -1/2$ and then the degenerate fields $V_{-1/2b}$ turn out to agree with the screening operators V_b. Hence, at $k = 0$ the integrals over such variables v_t are nothing more than screening insertions in Liouville correlation functions[32], and this is the reason why we did not explicitly write them in (72). This shows that the Ribault-Teschner formula turns out to be consistent with the FZZ conjecture. That is, at $k = 0$ sine-Liouville agrees with the product between Liouville theory and a free $c = 1$ boson, so that for the particular case $k = 0$ equation (25) actually states the identity between N-point correlation functions in sine-Liouville theory and N-point correlation functions in the 2D black hole. Nevertheless, we should emphasize that all these digressions are strongly based on the assumption that the CFT is still well defined in the regime $k < 2$ and, as far as we know, this is still far from being clear.

5 Conclusions

It is usually accepted that, probably, the FZZ duality is just an example of a more general phenomenon which should be interesting to understand in a deeper way [32, 5]. The purpose of this paper was precisely to discuss an example of such kind of generalization. We studied a correspondence between two-dimensional string theory in the euclidean black hole ($\times time$) and a (higher mode) tachyon perturbation of a linear dilaton background. Our main result is presented in Eq. (48).

The tachyon perturbation we considered here corresponds to momentum modes $n = 1$ and $n = 2$, and so it can be considered as a kind of deformation of the standard FZZ sine-Liouville theory. We argued that such a "deformation" (or "twisting" in the sense of (46)) can be thought of as a conjugate representation of the sine-Liouville interaction term, presumably related to the conjugate representations of operators in the WZW model [70].

In section 4 we have given a dictionary that permits to express any N-point correlation function in the $SL(2,\mathbb{R})_k$ WZW model on the sphere topology in terms of a correlation function in the tachyon perturbed linear dilaton background, and we have given a precise prescription for computing those correlators in the Coulomb gas approach. This correspondence between correlators was proven by rewriting a nice formula worked out by S. Ribault and J. Teschner in Refs. [4, 3], which directly follows from the relation between the solutions of the KZ and the BPZ equations. Our result (44) realizes the general version of the formula proven in [4]. In fact, following [1], we showed that the auxiliary overall function $F_k(z_1, ...z_N; w_1, ...w_M)$ standing in the Ribault-Teschner formula (25) can be also seen as coming from the correlation functions of a linear dilaton CFT perturbed by a tachyon-like operator of higher ($n \geq 1$) momentum modes. Thus, the twisted dual we discussed here turns out to be a free field realization of the Ribault-Teschner formula. A remarkable feature of such realization is that the $n = 2$ mode perturbation $\mathcal{T}_{1-k,k,k}$ turns out to be related to the sine-Liouville potential in the same way as

[32] G.G. specially thanks Yu Nakayama for collaboration in this particular computation. See Ref. [27].

to how the twisted tachyon-like vertex operators $\widetilde{\mathcal{T}}_{j_i,m_i,\overline{m}_i}$ are related to the operators $\mathcal{T}_{j_i,m_i,\overline{m}_i}$ of the standard FZZ prescription. Both representations $\widetilde{\mathcal{T}}_{j_i,m_i,\overline{m}_i}$ and $\mathcal{T}_{j_i,m_i,\overline{m}_i}$ have the same eigenvalues under the Cartan $U(1)$ generator J_0^3 and the Virasoro-Casimir operator L_0. Besides, it turns out that the fact that $\mathcal{T}_{j_i,m_i,\overline{m}_i}$ and $\widetilde{\mathcal{T}}_{j_i,m_i,\overline{m}_i}$ transform distinctly under the action of the $\widehat{sl}(2)_k$ generators $J_n^{\pm}$ combines with the fact that the $n=2$ operator $\mathcal{T}_{1-k,k,k}$ transforms non-trivially under those generators either, and this makes the correlation functions (44) behave properly under the $sl(2)_k$ algebra.

Tachyon-like perturbations of the linear dilaton background involving higher winding modes were also studied recently by Mukherjee, Mukhi and Pakman in Ref. [5], where they presented a generalized perspective of the FZZ correspondence. One of the task for the future is to understand the relation to [5] better. Besides, the understanding of the connection of our result to the standard FZZ correspondence also deserves more analysis. Regarding this, we would like to conclude by mentioning that the idea of the proof of (48) given in section 4 here could be actually adapted to prove the standard FZZ duality (on the sphere) if one considers the appropriated pieces in the literature. A key point in doing this would be a result obtained some time ago by V. Fateev, who has found a very direct way of showing the relation between correlation functions in both Liouville and sine-Liouville theories [7]. Such connection, once combined with the Ribault-Teschner formula [3, 4], would yield a proof of the FZZ duality at the level of correlation functions on the sphere topology without resorting to arguments based on supersymmetry.

This paper is an extended version of our contribution to the XVIth International Colloquium on Integrable System and Quantum Symmetries, to be held in Prague, in June 2007. Besides, these notes are based on Refs. [1, 2] and summarize the contents of the seminars that one of the authors has delivered at several institutions in the last year and a half. Gaston Giribet would like to thank S. Murthy and K. Narain for conversations and for very important comments. He is also grateful to V. Fateev for sharing his unpublished work [7], and to Yu Nakayama for previous collaboration in related subjects. The partial support of Universidad de Buenos Aires, Agencia ANPCyT and CONICET through grants UBACyT X861, PICT 34557, PIP6160 is also acknowledged. G.G. is member of CIC, CONICET, Argentina.

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