

GREEDY WALK ON THE REAL LINE

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We consider a self-interacting process described in terms of a single-server system with service stations at each point of the real line. The customer arrivals are given by a Poisson point processes on the space–time half plane. The server adopts a greedy routing mechanism, traveling toward the nearest customer, and ignoring new arrivals while in transit. We study the trajectories of the server and show that its asymptotic position diverges logarithmically in time.

1. Introduction. We consider a self-interacting process described in terms of a single-server system with service stations at each point of the real line. The system is described as follows. Initially, there is a Poisson field of customers in \mathbb{R} with unit intensity and the server starts at $x = 0$. Customers arrive as a Poisson point process in the space–time $\mathbb{R} \times \mathbb{R}_+$ with intensity $\lambda > 0$. When not serving, the server chooses the nearest customer and travels toward it at speed $0 < v \leq \infty$, ignoring new arrivals. The service then takes T units of time with $\mathbb{E}T = 1$, after which the customer leaves the system. This is a common example of a routing mechanism that depends on the system state, and targeting the nearest customer is known as a *greedy strategy*.

The particular interest in customer-server systems in continuous space stems from their transparent description of large systems with spacial structure, in contrast with finite systems where phenomenological properties are often obscured by combinatorial aspects of the model. However, systems with greedy routing strategies in the continuum are extremely *sensitive to microscopic perturbations*, and their rigorous study represents a challenging problem; a topic that has been active for almost three decades [2, 7, 8, 12, 15, 18, 19, 24–26, 32, 33].

The system described above arises naturally in the question of stability of a greedy server on the circle \mathbb{R}/\mathbb{Z} . It was conjectured in [12] that the greedy server on \mathbb{R}/\mathbb{Z} is stable when $\lambda < 1$, regardless of the speed v . This was verified only under light-traffic assumptions, that is, for large enough v given λ [19], and for the greedy server on a discrete ring $\mathbb{Z}/n\mathbb{Z}$ [15, 16, 26, 35].¹ Yet, discrete models have not been

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¹Stability was also shown for a number of other finite graphs and a broader class of service strategies [35], as well as several nongreedy policies [18], a gated-greedy variant on convex spaces [2] and random nongreedy servers on general spaces [1]. See [33] for a recent review.

able to grasp the microscopic nature of the greedy mechanism in continuous space, and there are major obstacles in extrapolating any approach based on a discrete approximation.

On the other hand, stability under the same conditions is known to hold for the *polling server* on \mathbb{R}/\mathbb{Z} , that is, the server whose strategy is to always travel in the same direction [17].² Simulations indicate not only that the greedy server is stable, but also that under heavy traffic conditions its dynamics resembles that of the polling server [12].

This prompts a detailed study of its *local behavior*, and it is natural to describe it with a model on an *infinite line*. The model on the line is clearly different from that on the circle as a queueing system, for its total arrival rate is infinite. The study of the former is rather intended to give *mathematical insight* on the server's local motion, as is confirmed in [34]. It was shown in [21] that the position of the greedy server on \mathbb{Z} is *transient*. But again, the behavior in this case is governed by averaging effects inside each discrete cell overcrowded by waiting customers, and its understanding is of little help for the continuous-space system. The goal of this paper is to study the greedy server on \mathbb{R} .

The main difficulty in studying this model is due to the interplay between the server's motion and the environment of waiting customers that surround it. This interplay is given by the *interaction* at the *microscopic level* resulting from the greedy choice of the next customer and the removal of those who have been served. The server's path is locally *self-repelling*, since the removal of already served customers makes it less likely for the greedy server to take the next step back into the recently visited regions.

There are several deep studies of processes which in different ways are self-repelling. This includes examples of self-interacting walks such as the random walk avoiding its past convex hull [3, 42], the prudent walk [4, 9], the "true" self-avoiding walk [39, 40] and excited random walks [6]. In the continuum setup, one has the self-interacting diffusion with repulsion [28], the perturbed Brownian motions [10, 11, 13, 14, 30], the excited Brownian motions [31] and random paths with bounded local time [5].³ It was clear since these models were introduced that they could not be treated via standard methods and tools. Despite the existence of a few disconnected techniques that have proved useful in specific situations, this rich research field still lacks a systematic basis of study.⁴ A lot remains to be understood

²Several other state-independent strategies have been analyzed and, in particular, the following two: after each service, the server decides to move next in a direction chosen at random; the server follows the path of a Brownian motion. Stability under the same conditions holds in both cases, see [17].

³See the introductions of [28, 31] for concise reviews on these models, [27] for a review on reinforced walks, and [29] for a comprehensive survey on the field up to 2007.

⁴Except for the family of universality classes given by the Schramm–Löwner Evolutions [36], which include 2-dimensional loop-erased random walk [22] and several other models [23, 37, 38].

even in dimension $d = 1$, and in particular none of the known techniques seems to be applicable in our case.

The future evolution of the greedy server's position is of course influenced by its previous path. But unlike the above models, here there is no direct prescription of such influence in terms of occupation times. A similar situation occurs while defining the "true self-repelling motion" in $d = 1$ [41], although the authors show that its evolution depends on the occupation times, and moreover that such dependency is local.

Notice that, for the greedy server, "self-repulsion" does not imply immediately "repulsion toward ∞ ," since the server is allowed to backtrack, in which case it starts being repelled back toward the origin.⁵ Another particular feature of this model is an inverse relation between the strength of self-repulsion (measured by the bias in the probability that the server takes the next step backward) and the average speed of the server. The attraction felt by the server upon reaching unexplored regions is increasing in time, due to the accumulation of customers that keep arriving throughout the whole evolution, but at the same time this high concentration of customers causes the subsequent traveled distances to become shorter at the same proportion.

In this paper, we introduce a framework based on a randomized representation of the customers environment as viewed from the server (namely, it learns only the information that is necessary and sufficient to determine the next movement, and the positions of further waiting customers remain unknown). This allows a fine description of the system behavior. As a consequence of this approach, we show transience and describe the server's asymptotics, setting up an old question in the field (stated, e.g., as Open Problem 4 in [33]).

THEOREM 1. *Let S_t denote the server position at time t . Assume that $\mathbb{E}e^{\alpha T} < \infty$ for some $\alpha > 0$. Then for any $v > 0$ and $\lambda > 0$ the greedy server on the real line is transient. Moreover,*

$$\frac{S_t}{\lambda^{-1} \log t} \rightarrow \pm 1$$

with probability 1/2 each.

REMARK 1. In our approach, it is important that the arrivals form a Poisson process in space–time, and that they are independent of the service times.

REMARK 2. Assume that at time 0– the set of waiting customers is distributed as a Poisson point process with intensity $\mu(x) dx$, for some nonnegative bounded measurable function μ with $\int \mu = \infty$, and with an additional deterministic finite set of points. Then Theorem 1 remains true (with essentially the same

⁵A difficulty similar in spirit was faced in [28], where it was proved that a certain diffusion with self-repelling potential has a power-law asymptotic behavior.

proof), except for the lack of symmetry in the probabilities of S_t diverging to $+\infty$ or $-\infty$.

REMARK 3. There is a dynamic version of the greedy server, where new arrivals are not ignored while the server is traveling. This variation might be studied by similar arguments, but the dynamic mechanism introduces some extra complications that will not be considered here.

REMARK 4. The assumption of constant speed is natural in several contexts where the terminal speed is quickly achieved, but not crucial in our construction. In fact, for a server moving with constant acceleration, or any other mechanical constraints (which restarts after each service), mild modifications of our method yield the same results. Notice that the value of v plays no role in Theorem 1.

Heuristically, the asymptotics described by Theorem 1 is what one should expect to happen, assuming that the server will indeed move most of the times in the same direction. Suppose that all of the first N customers were found to the right of the server. The typical distance between the server and the next customer to the right is about $\frac{1}{N}$, because customers have been arriving to this region for about N time units. To the left of the server, there are regions of size about $\frac{1}{N-1}$, $\frac{1}{N-2}$, $\frac{1}{N-3}$, etc., where the arrival of customers is rather recent: they must have happened during the last 1, 2, 3, etc., units of time. If the server is eventually moving only to the right (or the excursions to the left are very sparse in time), the server position S_N should therefore diverge as $\log N$.⁶

However, the probability that the next customer is found to the left of the server is about $\frac{c}{N}$, which implies that it will happen some time in the future. In fact, the server will make an excursion of length $\frac{c}{N}$ to the left for infinitely many N , for any constant c , in contrast with its discrete variant. Nevertheless, the probability that the two next customers are both to the left is about $\frac{c}{N^2}$. One may thus push this argument and show that indeed, with positive probability, the system will never produce microscopic scenarios capable of causing important changes in the server's course.

To make the above observation rigorous, we introduce a dynamic block construction, where the block sizes are increasing at each step, and combine it with a renewal argument. The size ℓ_k of the blocks (groups of sequentially served customers) should increase slow enough so that the cleared region left by a block is wide enough to support the next one, but fast enough so that the probability of atypical gaps inside the blocks is summable in k . It turns out that a growth $\ell_k \sim k^\eta$, with $0 < \eta < \frac{1}{2}$, works well for this purpose.

⁶This heuristics is confirmed by the asymptotic behavior of a continuous model in \mathbb{R} where there is no greedy mechanism and the server is always moving to the right [20], as well as for the greedy server on \mathbb{Z} [21].

This paper is divided as follows. In Section 2, we present the evolution of the customers environment as viewed from the server, and study its properties. We state Proposition 1 about the behavior of the greedy server on the real line at specific times (a block argument) and show how it implies Theorem 1, and in Section 3 we prove Proposition 1. In these sections, we consider the case where T is deterministic and $v = \infty$. This case contains the most important features of the construction and the block argument, but is simpler to present. The general case is considered in Section 4.

2. The process viewed from the server. We consider a particular construction of the initial state by assuming that there are arrivals during $t \in [-1, 0]$, before the service starts at $t = 0$. This is of course equivalent to simply starting at $t = 0$ with a Poisson field of points. Let ν denote the random set of arrivals in $\{(x, t) : x \in \mathbb{R}, t > -1\}$.

We want to construct the process by following a progressive exploration of the space–time until finding the mark $(x^*, t^*) \in \nu$ corresponding to the nearest waiting customer, getting as little information as possible about ν . The server is thus unaware of existing customers further than the nearest one, and keeps record of the last time when each point in space was explored in the seek of waiting customers.

For the reasons mentioned above, here we consider the case $T = 1$ and $v = \infty$, and postpone the general case to Section 4. Thus, the server's position S_t remains constant on intervals $t \in [n - 1, n)$. By rescaling space, we can assume $\lambda = 1$.

Starting at $t = 0$, each region on the space has potentially witnessed the arrival of customers during 1 unit of time. The first customer is then found at an exponentially-distributed distance, to the left or to the right with equal probabilities. Discovering its position reveals the presence of a point in ν , as well as a region where ν has no points. For the second customer, there is a region in space that has potentially witnessed the arrival of customers during 1 unit of time (namely, the region explored on the previous step), and the complementary region has not been queried during the last 2 units of time. The position of the third customer is already more involved, and the positions of both of the previous customers are important in determining the regions where ν is still unexplored. Yet there is a general description which is amenable to study, which motivates the construction described hereafter and depicted in Figure 1.

A *potential* is a piecewise continuous function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that there is a unique point $x_* = S(u)$ where it attains its maximum $\mathcal{M} = \mathcal{M}(u) = u(x_*)$.

Given a pair of positive numbers $w = (E, U)$, where $0 < E < \infty$ and $0 < U < 1$, we define the operator \mathcal{H}_w as follows. Let u be given and take $z > 0$ as the unique number such that

$$\int_{x_* - z}^{x_* + z} (\mathcal{M} - u) dx = E.$$

Let

$$a = \mathcal{M} - u(x_* - z), \quad b = \mathcal{M} - u(x_* + z),$$

choose

$$x^* = \begin{cases} x_* - z, & \text{if } U \in \left(0, \frac{a}{a+b}\right], \\ x_* + z, & \text{if } U \in \left(\frac{a}{a+b}, 1\right), \end{cases}$$

and finally

$$(1) \quad (\mathcal{H}_w(u))(x) = \begin{cases} \mathcal{M} + 1, & x = x^*, \\ \mathcal{M}, & x \in [x_* - z, x_* + z], x \neq x^*, \\ u(x), & \text{otherwise.} \end{cases}$$

Notice that $\mathcal{M}(\mathcal{H}_w(u)) = \mathcal{M}(u) + 1$, $S(\mathcal{H}_w(u)) = x^*$, and $\int_{\mathbb{R}} [\mathcal{H}_w(u) - u] dx = E$. Moreover, if u is unimodal, then $\mathcal{H}_w(u)$ is also unimodal.

Start with $u_0^0: \mathbb{R} \rightarrow \mathbb{R}$ given by $u_0^0(x) = \delta_0(x) - 1$, let $(E_n)_n$ and $(U_n)_n$ be independent i.i.d. sequences of exponential and uniform random variables, and write $w_n = (E_n, U_n)$. Let $u_n^0 = \mathcal{H}_{w_n}(u_{n-1}^0)$ and write $S_n = S(u_n^0)$.

LEMMA 1. *The sequence $(S_n)_{n=1,2,\dots}$ defined above has the same distribution as the sequence $(S_{t-})_{t=1,2,\dots}$ given by the positions of the greedy server at integer times.*

The lemma follows from the properties of the Poisson point process ν on

$$\Gamma_u := \{(x, t) : x \in \mathbb{R}, u(x) \leq t < \infty\} \subseteq \mathbb{R}^2.$$

Indeed, consider a progressive exploration at the left and right vertical boundaries of the continuously-expanding region $\{(x, t) : x_* - z \leq x \leq x_* + z, u(x) \leq t \leq \mathcal{M}(u)\}$ as z increases, starting from 0 until finding the first point (x^*, t^*) of ν . The variable E is given by the area of the explored region. The variable U is related to the position of (x^*, t^*) on the union of the two disjoint vertical intervals where this region is growing, and is given by $t^* = \mathcal{M}(u) - |(a+b)U - a|$, $x^* = x_* + z \cdot \text{sgn}[(a+b)U - a]$. By the properties of a Poisson point process, E and U are independent of each other, distributed as standard exponential and uniform variables, regardless of how ν had been explored outside Γ_u .

We now consider some properties of the operators \mathcal{H} . Let $\theta_z u = u(z + \cdot)$. For any potential u , any number c , and any point z , $\mathcal{M}(\theta_z u + c) = \mathcal{M}(u) + c$ and $S(\theta_z u + c) = S(u) - z$. It follows from the definition of \mathcal{H} that

$$(2) \quad \mathcal{H}_w(\theta_z u + c) = \theta_z \mathcal{H}_w(u) + c.$$

A potential u is said to be *centered* if $S(u) = 0$ and $\mathcal{M}(u) = 0$. Define the operator $\Theta^u(\cdot) = \theta_{S(u)}(\cdot) - \mathcal{M}(u)$, so that $\Theta^u(u)$ is centered. For given potentials u and \tilde{u} ,

$$(3) \quad \Theta^{\Theta^u(\tilde{u})} \circ \Theta^u = \Theta^{\tilde{u}}.$$

The natural *shifts* in this evolving sequence of potentials $(u_n^0)_{n \geq 0}$ is given for each k by $(u_n^k)_{n \geq 0}$ defined as $u_n^k := \Theta^{u_1^{k-1}}(u_{n+1}^{k-1})$. Expanding this recursion and using (3) yields

$$(4) \quad u_n^k = \Theta^{u_1^{k-1}}(\Theta^{u_1^{k-2}}(u_{n+2}^{k-2})) = \Theta^{u_2^{k-2}}(u_{n+2}^{k-2}) = \dots = \Theta^{u_k^0}(u_{n+k}^0).$$

In particular, $u_0^k = \Theta^{u_k^0}(u_k^0)$. Writing

$$\mathcal{H}_n^k = \mathcal{H}_{w_{k+n}} \circ \dots \circ \mathcal{H}_{w_{k+2}} \circ \mathcal{H}_{w_{k+1}},$$

it follows from (2) that $u_n^k = \mathcal{H}_n^k(u_0^k)$. Therefore, u_0^k is determined by w_1, w_2, \dots, w_k , whereas $(u_n^k)_{n \geq 0}$ is determined by w_{k+1}, w_{k+2}, \dots and u_0^k itself.

The above properties imply that the evolution of $(u_n^0)_n$ is a homogeneous, translation-invariant, height-invariant, Markov chain in the space of potentials. At any moment k , we can take u_k^0 and move the axes so that the origin is placed on its maximum (i.e., apply $\Theta^{u_k^0}$), obtaining u_0^k , and from this point on the evolution of $(u_n^k)_n$ is independent of (u_1^0, \dots, u_k^0) , and obeys the same transition rules. Moreover, $(u_n^k)_n$ is related to $(u_{k+n}^0)_n$ by $u_n^k = \Theta^{u_k^0}(u_{k+n}^0)$. An example depicting this construction is shown in Figure 1.

This motivates us to define the evolution of the greedy server model starting from any centered potential u as the initial u_0^0 , not necessarily given by $\delta_0 - 1$. Namely, the system starts at $t = 0$, with customer arrivals in space–time given by a Poisson point process ν on Γ_u . We denote its law by \mathbb{P}^u .

In the proof of Theorem 1, we only use two properties of $u_0^0(x) = \delta_0(x) - 1$. We say that a potential u is *unimodal* if u is nondecreasing on $(-\infty, S(u))$ and nonincreasing on $(S(u), +\infty)$. We say that a potential u is *bounded* if $m(u) := \mathcal{M}(u) - \inf_{x \in \mathbb{R}} u(x)$ is finite. Each of these conditions is preserved by the operators of the form \mathcal{H}_ω . Since they are also preserved by θ_z and $u \mapsto u + c$, starting from $u_0^0(x) = \delta_0(x) - 1$ the potentials u_n^k are unimodal and bounded for any k and n . Proposition 1 below, though, uses only unimodality.

Grouping customers. Theorem 1 is proved by grouping customers as in the following proposition. Write

$$\ell_j = \lceil 12j^{1/4} + 1 \rceil, \quad j = 1, 2, 3, \dots$$

PROPOSITION 1. *Let (S_n) be a greedy random walk generated by a centered unimodal initial potential u . For any $\varepsilon > 0$, there exists $\delta > 0$ depending on ε but not on the potential u , and a sequence of stopping times $L_0, L_1, L_2, L_3, \dots$ with the following properties.*

Let $\sigma = \text{sgn } S_1$, $Z_j = \sigma S_{L_j}$, $N_j = L_j - u(S_{L_j})$, $Q_j = L_{j+1} - L_j$ and $X_j = Z_{j+1} - Z_j$. Then, with probability at least δ , we have, for all $j = 1, 2, 3, \dots$,

$$(5) \quad \begin{cases} Q_j = \ell_j \text{ or } \ell_j + 1, \\ X_j^- \leq X_j \leq X_j^+, \\ Z_{j-1} < \sigma S_n < Z_{j+1}, \end{cases} \quad \text{for } L_j \leq n < L_{j+1},$$

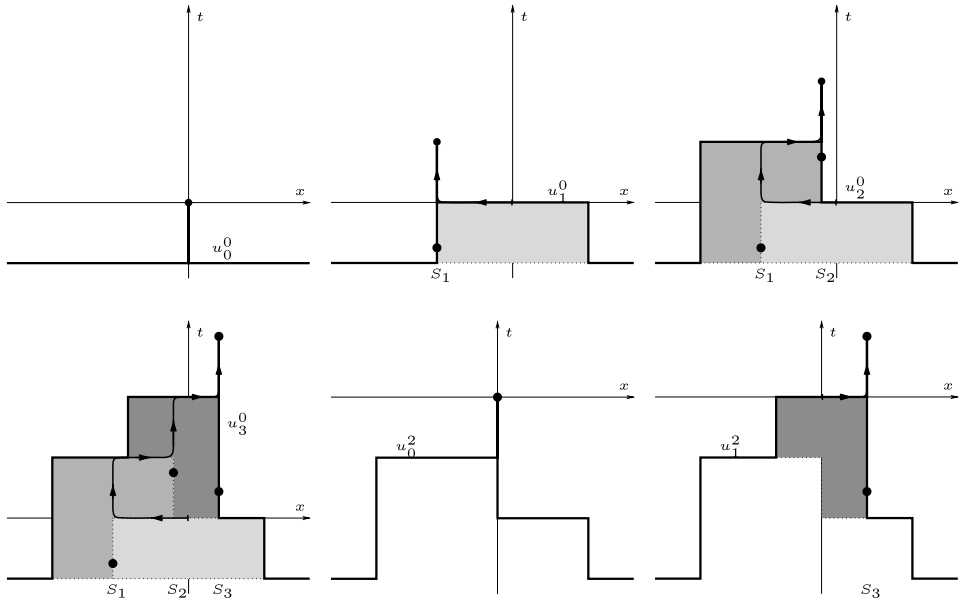


FIG. 1. Revealing three points of $v \subseteq \mathbb{R} \times (-1, \infty)$ to determine the greedy server's first steps. Before starting, the configuration is unknown on the whole $v \subseteq \mathbb{R} \times (-1, \infty)$, represented by the graph $u_0^0(x) = \delta_0(x) - 1$. The nearest customer found at time 0 corresponds to the bold point (x^*, t^*) in the second plot (middle above), where the graph of u_1^0 covers the region that had to be explored in order to find (x^*, t^*) . After serving this customer, the point in v corresponding to the nearest customer corresponds to a new bold point appearing in the third plot, where the graph of u_2^0 covers the total region explored in these two steps. The server's trajectory is depicted by the arrowed, curly path, and consists of unit service times alternated with instantaneous space displacements. The fourth plot (below, left) shows the three points of v determining the construction of u_3^0 , the region of $\Gamma_{u_0^0}$ explored and the path performed by the server during the interval $[0, 3)$. The fifth and sixth plots (below, center and right) depict the Markovian nature of this procedure. At the second customer's departure time, we place the axes on the maximum of u_2^0 , obtaining u_2^2 . Notice that in this picture there is no record of the past trajectory and the location of the other two points also called (x^*, t^*) . It turns out that the potential is enough in order to determine the future evolution, and we find the same point (x^*, t^*) corresponding to the next customer.

where

$$X_j^- = (1 - \varepsilon) \frac{\ell_j - 1}{N_{j+1}} \quad \text{and} \quad X_j^+ = (1 + \varepsilon) \frac{\ell_j}{N_j}.$$

In words, Z_j is the server position after serving L_j customers (in case $\sigma = +1$, otherwise the picture is mirrored), N_j is the discontinuity of the potential at the server's position at this moment, and finally X_j measures the displacement in space after serving the next Q_j customers.

The above proposition is proved in the next section. Let us show how it implies the main result.

PROOF OF THEOREM 1 FOR $v = \infty$ AND $T = 1$. Let ε be any positive number. The system starts at time $n_0 = 0$ from the potential u_0^0 , and by Proposition 1, with probability at least δ the events (5) hold for all j , for some sequence of stopping times L_j . If it does not hold for all j , let j_* be the first j for which condition (5) is violated, and call $n_1 = L_{j_*+1}$. Whether (5) occurs or not is determined by $(u_n^0)_{n=0,1,\dots,L_{j_*+1}}$. Since L_{j_*+1} is a stopping time, defining $n_1 = \infty$ on the event that (5) is satisfied for all j , we have that n_1 is also a stopping time. Therefore, at time n_1 the system restarts from some unimodal bounded potential $u_0^{n_1}$, ignoring the past history, that is, conditioned on n_1 and $u_{n_1}^0$, $(u_n^{n_1})_{n \geq 0}$ is distributed as $\mathbb{P}^{u_0^{n_1}}$. Again, starting from such potential there is probability at least δ that (5) holds for all j , with $(u_n^0)_n$ replaced by $(u_n^{n_1})_n$. It thus takes at most a geometric number of restarts (with parameter δ) to get a success, so there is an a.s. finite time n_* such that condition (5) holds for all j , with $(u_n^0)_n$ replaced by $(u_n^{n_*})_n$. Notice that σ takes a possibly new value at each attempt.

We write $a \sim_\varepsilon b$ if $\limsup |\frac{a}{b} - 1| \leq \varepsilon$ and $a \sim b$ if $\frac{a}{b} \rightarrow 1$. By definition of ℓ and L , we have $L_j \sim \frac{48j^{5/4}}{5}$ and $\ell_j/L_j \sim \frac{5}{4j}$. Now, by construction of N , $L_j \leq N_j \leq L_j + m(u_0^{n_*})$ and, therefore, $N_{j+1} \sim N_j \sim L_j$. Finally, assuming that (5) holds for all j , $X_j \sim_\varepsilon \ell_j/L_j$.

But $Z_{j+1} = Z_0 + \sum_{i=1}^j X_i$, and putting these all together gives

$$Z_{j+1} \sim_\varepsilon \frac{5}{4} \log j.$$

Finally, the position S_n is given by $S_n = S_{n_*} + \sigma Z_j$ at times n satisfying $n = n_* + L_j$ and, therefore,

$$S_n \sim_\varepsilon \sigma \log n \quad \text{a.s.}$$

Since ε was arbitrary,

$$\frac{S_n}{\sigma \log n} \rightarrow 1 \quad \text{a.s.}$$

and using Lemma 1 this completes the proof of Theorem 1 for $v = \infty$ and $T = 1$. □

3. Block argument. In this section, we prove Proposition 1. Let $0 < \varepsilon < \frac{1}{2}$. Here and in the next section, each time C or c (resp., c_ε or C_ε) appears, it denotes a different constant (resp., function of ε) that is positive, finite and universal. We write $a \vee b$ for $\max\{a, b\}$.

We are going to define the event A_j that step j is successful. For each j , the occurrence of A_j implies (5), and we will show that there exists a sequence p_0, p_1, p_2, \dots , depending only on ε , such that

$$(6) \quad \mathbb{P}^u(A_j | A_{j-1}, A_{j-2}, \dots, A_0) \geq p_j$$

and

$$(7) \quad \prod_{j=0}^{\infty} p_j > 0.$$

For the latter, we show that p_j increases fast enough so that $1 - p_j$ is summable, and that $p_j > 0$ for all j . Let us drop the superscript 0 in the potentials u_n^0 .

We start with $j = 0$, omitted in the statement of Proposition 1. Define $\ell_0 = 1$, and take $L_0 = 0$, $Z_0 = S_0 = 0$, and $L_1 = Q_0 = 1$. We choose $\sigma = \text{sgn}(S_1)$. Let $Z_1 = X_0 = \sigma S_1 = |S_1|$, and $N_1 = L_1 - u(\sigma Z_1)$. We say that step 0 is *successful* if

$$(8) \quad X_0 \geq X_0^- := \frac{4}{N_1},$$

otherwise we declare step 0 to have *failed* and stop. The next steps $j = 1, 2, 3, \dots$ are described assuming for simplicity that $\sigma = +1$.

Suppose that steps $0, 1, 2, \dots, j - 1$ have been successful and start from u_{L_j} . Step j may be successful in two situations. First, if each of the next ℓ_j customers $S_{L_j+1}, S_{L_j+2}, \dots, S_{L_j+\ell_j}$ satisfy $S_n > S_{n-1}$, in which case we take $Q_j = \ell_j$. Second, if there is one $\tilde{n} \in \{L_j + 1, \dots, L_j + \ell_j\}$ such that $S_{\tilde{n}} < S_{\tilde{n}-1}$, and $S_n > S_{n-1}$ for all $n \in \{L_j + 1, \dots, L_j + \ell_j, L_j + \ell_j + 1\}$ except \tilde{n} , in which case we take $Q_j = \ell_j + 1$. If none of these two happen, we declare step j to have *failed* and stop. Otherwise, in either of the above two cases we say that step j is *successful* if (5) is satisfied.⁷

Notice that, for $j \geq 1$, if step $j - 1$ is successful we have

$$(9) \quad \begin{cases} \mathcal{M}(u_{L_j}) = L_j, \\ u_{L_j}(x) = u_0(x) \leq L_j - N_j, & \text{for } x > Z_j, \\ u_{L_j}(x) \geq L_j - Q_{j-1}, & \text{for } Z_j - X_{j-1}^- < x < Z_j. \end{cases}$$

Having described the grouping steps, it remains to show (6) and (7).

Recall from the previous section that, once u_n is fixed, the position of the next customer S_{n+1} is determined by a pair E_{n+1}, U_{n+1} of exponentially- and uniformly-distributed random variables, or alternatively by the Poisson point process ν restricted to the region $\{(x, t) : u_n(x) < t \leq \mathcal{M}(u_n)\}$.

We start with $j = 0$. In this step, we pay a *finite price* p_0 to produce a potential which exhibits a *plateau* with convenient shape, namely a potential satisfying (8). Recall that E_1 and U_1 are the exponential and uniform random variables used in order to produce u_1 from u_0 . Consider the event that E_1 and U_1 satisfy the

⁷We could have taken Q_j always equal $\ell_j + 1$ and have a simpler proposition with nonrandom times L_j . In this case, we would define step j to be successful if $S_n > S_{n-1}$ for all $j = L_j + 1, \dots, L_j + \ell_j + 1$ except for possibly one \tilde{n} in $L_j + 1, \dots, L_j + \ell_j$. This would result in a simpler statement but less robust proof. More precisely, the simple estimate (10) below would not suffice, and a special treatment would be needed for the last point $S_{L_j+\ell_j+1}$.

following two requirements. The first requirement is that $E_1 > 8$. The second one is that, given E_1 , the variable U_1 lies on the largest interval among $[0, \frac{a}{a+b}]$ and $[\frac{a}{a+b}, 1]$; see (1). This is when σ is determined. In the worst case, this interval has length $\frac{1}{2}$, whence the probability that both conditions are satisfied is at least $p_0 = \frac{1}{2}e^{-8} > 0$. The requirement for U_1 implies that $u(S_1) \leq u(-S_1)$. Hence, by monotonicity of u_0 , the occurrence of the above event implies that

$$\begin{aligned} 8 < \int_{-X_0}^{+X_0} -u(x) \, dx &\leq \int_{-X_0}^{+X_0} \max_{[-X_0, +X_0]} (-u) \, dx = -2X_0u(\sigma X_0) = -2X_0u(S_1) \\ &\leq 2X_0N_1. \end{aligned}$$

The above inequality implies A_0 and, therefore, $\mathbb{P}^u(A_0) \geq p_0 > 0$.

Fix some $j = 1, 2, 3, \dots$. We will describe events B_1, B_2, B_3 , omitting the dependency on j , such that $B_1 \cap B_2 \cap B_3$ implies A_j . The conditional probability of $B_1 \cap B_2 \cap B_3$ given u_{L_j} can be bounded from below by some number p_j that does not depend on the potential u_{L_j} as long as it satisfies (9). This in turn implies (6).

We stress that, even though the knowledge about these events inconveniently provides more information about ν than needed in determining $u_{L_{j+1}}$, we only study them with the purpose of estimating the probability of A_j . The occurrence of the latter is entirely determined by $u_{L_j}, u_{L_{j+1}}, u_{L_{j+2}}, \dots, u_{L_{j+1}}$.

We consider the evolution given by the point process ν itself rather than the construction specified in (1). We write $\nu_i = \nu \cap R_i$, where

$$\begin{aligned} R_1 &= \{(x, t) : x > Z_j, u(x) < t \leq L_j\}, \\ R_2 &= \{(x, t) : Z_j < x < Z_j + X_j^+, L_j < t \leq L_j + \ell_j + 1\} \\ &\cup \{(x, t) : Z_j - X_{j-1}^- < x < Z_j, u_{L_j}(x) < t \leq L_j + \ell_j + 1\}. \end{aligned}$$

The first event considered is

$$B_1 := [|\nu_2| \leq 1].$$

Notice that, conditioned on u_{L_j} , the number of points $|\nu_2|$ is distributed as a Poisson random variable with mean given by the area $|R_2|$. Now, on the event that u_{L_j} satisfies (9),

$$\begin{aligned} |R_2| &\leq (\ell_j + 1)X_j^+ + (\ell_{j-1} + 1)X_{j-1}^- + (\ell_j + 1)X_{j-1}^- \leq 3(\ell_j + 1)X_j^+ \\ &\leq C \frac{(\ell_j + 1)^2}{N_j} \leq C \frac{1}{j^{3/4}} \end{aligned}$$

since

$$\ell_j \leq Cj^{1/4} \quad \text{and} \quad N_j \geq L_j \geq Q_0 + \dots + Q_{j-1} \geq Cj^{5/4},$$

and therefore

$$(10) \quad \mathbb{P}(B_1 | u_{L_j}) \geq 1 - C|R_2|^2 \geq 1 - C \frac{1}{j^{3/2}}.$$

We also need the estimate to be positive for all j , which follows from

$$\mathbb{P}(B_1|u_{L_j}) \geq \mathbb{P}(v_2 = \emptyset|u_{L_j}) = e^{-|R_2|} \geq e^{-c} > 0.$$

We now consider the events B_2 and B_3 , which depend on v_1 . Define

$$(11) \quad A(x) = \int_{Z_j}^x [L_j - u(z)] dz, \quad x \geq Z_j,$$

and write $v_1 = \{(x_1, t_1), (x_2, t_2), (x_3, t_3), \dots\}$ with $x_0 = Z_j < x_1 < x_2 < x_3 < \dots$. By definition of v_1 , we have that $(A(x_n) - A(x_{n-1}))_{n=1,2,3,\dots}$ are i.i.d. exponential random variables with mean 1, independent of u_{L_j} . The events B_2 and B_3 are defined in terms of $A(x_n)$, $n = 1, 2, 3, \dots$, whence the estimates on their probabilities are always uniform on u_{L_j} .

Consider the event

$$(12) \quad B_2 := [(1 - \varepsilon)(\ell_j - 1) < A(x_{\ell_j-1}) < A(x_{\ell_j}) < (1 + \varepsilon)\ell_j].$$

By Chernoff's exponential bounds,

$$(13) \quad \mathbb{P}(B_2) \geq 1 - e^{-c_\varepsilon \ell_j}.$$

Consider the event

$$(14) \quad B_3 := \left[A(x_n) - A(x_{n-1}) \leq \frac{\ell_j}{12} \text{ for } n = 1, 2, \dots, \ell_j \right].$$

By a simple union bound, we have

$$(15) \quad \mathbb{P}(B_3) \geq 1 - \ell_j e^{-\ell_j/12} \geq 1 - C e^{-c \ell_j}.$$

Using (13) and (15), we get

$$\mathbb{P}(B_2 \cap B_3) \geq 1 - C_\varepsilon e^{-c_\varepsilon \ell_j}.$$

Now, since $\ell_j \geq 12$, we have

$$\mathbb{P}(B_2 \cap B_3) \geq \mathbb{P}(1 - \varepsilon < A(x_n) - A(x_{n-1}) < 1 \text{ for } n = 1, 2, \dots, \ell_j) > e^{-c_\varepsilon \ell_j} > 0$$

and thus adjusting c_ε we get

$$\mathbb{P}(B_2 \cap B_3) \geq 1 - e^{-c_\varepsilon \ell_j}.$$

Since v_1 is conditionally independent of $v_2 \cup v_3$ given u_{L_j} , we have that

$$\mathbb{P}(B_1 \cap B_2 \cap B_3|u_{L_j}) \geq p_j$$

for

$$p_j = (1 - e^{-c_\varepsilon \ell_j})(e^{-c} \vee (1 - Cj^{-3/2})).$$

Notice that the sequence $(p_j)_{j=0,1,2,\dots}$ satisfies (7), thus it only remains to show that $B_1 \cap B_2 \cap B_3$ implies A_j .

Suppose B_1, B_2 and B_3 happen. By (9) and monotonicity of u , we have

$$N_j[x_n - x_{n-1}] \leq A(x_n) - A(x_{n-1}) \leq [L_j - u(x_n)][x_n - x_{n-1}],$$

whence by (12)

$$(16) \quad x_{\ell_j-1} - Z_j \leq x_{\ell_j} - Z_j \leq (1 + \varepsilon) \frac{\ell_j}{N_j} = X_j^+,$$

and by (14)

$$(17) \quad x_n - x_{n-1} \leq \frac{\ell_j}{12N_j} \leq \frac{X_{j-1}^-}{3}.$$

Moreover, for $n = 1, 2, \dots, \ell_j - 1$,

$$A(x_n) - A(x_{n-1}) \leq [L_j - u(x_{\ell_j-1})][x_n - x_{n-1}]$$

and, by (12),

$$x_{\ell_j} - Z_j \geq x_{\ell_j-1} - Z_j \geq (1 - \varepsilon) \frac{\ell_j - 1}{L_j - u(x_{\ell_j-1})} \geq (1 - \varepsilon) \frac{\ell_j - 1}{N_{j+1}} = X_j^-$$

as long as $Z_{j+1} = S_{L_j+Q_j} \geq x_{\ell_j-1}$.

Therefore, to prove (5) it suffices to show that

$$(18) \quad \begin{cases} x_{\ell_j-1} \leq S_{L_j+Q_j} \leq x_{\ell_j}, \\ x_0 - X_{j-1}^- \leq S_{L_j+n} < S_{L_j+Q_j}, \quad n = 1, 2, \dots, Q_j - 1. \end{cases}$$

The remainder of the proof is dedicated to proving (18) assuming (16), (17) and that B_1 occurs.

We first recall that the points in $(x, t) \in \nu$ that correspond to customers $(S_{L_j+1}, S_{L_j+2}, \dots, S_{L_j+Q_j})$ are such that $u_{L_j}(x) < t \leq L_j + \ell_j + 1$. When these points are neither in R_1 nor in R_2 , they must be in R_3 given by $t \in (u_{L_j}(x), L_j + \ell_j + 1]$ and

$$(19) \quad x < Z_j - X_{j-1}^- \quad \text{or} \quad x > Z_j + X_j^+.$$

The points in R_1 are given by $(x_n, t_n)_{n=1,2,\dots}$, and R_2 is either empty or contains one point, denoted by (x', t') .

Let n' be the maximal index between 0 and ℓ_j such that

$$(S_{L_j}, S_{L_j+1}, S_{L_j+2}, \dots, S_{L_j+n'-1}, S_{L_j+n'}) = (x_0, x_1, x_2, \dots, x_{n'-1}, x_{n'}).$$

If $n' = \ell_j$, we have $Q_j = \ell_j$, thus (18) is satisfied. So suppose $n' \leq \ell_j - 1$. We claim that

$$S_{L_j+n'+1} = x'$$

with x' satisfying

$$x_{n'} - \frac{\ell_j}{12N_j} \leq x' < x_{n'+1},$$

and moreover

$$(20) \quad S_{L_j+n+1} = x_n \quad \text{for } n = n' + 1, n' + 2, \dots, \ell_j,$$

that is, the points in R_3 cannot participate in the construction of $S_{L_j+Q_j}$.

In the case $x' < x_{n'}$, we will have $Q_j = \ell_j + 1$ and $S_{L_j+Q_j} = x_{\ell_j}$. Otherwise, $x_{n'} < x' < x_{n'+1}$, we will have $Q_j = \ell_j$, and in this case $S_{L_j+Q_j} = x_{\ell_j-1}$ if $n' \leq \ell_j - 2$ or $S_{L_j+Q_j} = x' \in (x_{\ell_j-1}, x_{\ell_j})$ if $n' = \ell_j - 1$. Therefore, (18) is always satisfied.

It thus remains to prove the above claim. By definition of n' , the point $(x', t') \in \nu$ corresponding to $S_{L_j+n'+1}$ cannot be in R_1 . But it cannot be in R_3 either. Indeed, since $S_{L_j+n'} = x_{n'}$ and

$$x_{n'} < x_{n'+1} \leq x_{n'} + \frac{\ell_j}{12N_j},$$

we must have

$$x_0 - \frac{\ell_j}{12N_j} \leq x_{n'} - \frac{\ell_j}{12N_j} \leq x' < x_{n'+1},$$

thus x' cannot satisfy (19). Therefore, (x', t') is the only point in ν_2 .

We finally show (20). Start with $n = n' + 1$. Write $\tilde{x} = S_{L_j+n'+2}$, corresponding to a point $(\tilde{x}, \tilde{t}) \in \nu$. This point cannot be in R_2 , since (x', t') was the only such point. As before,

$$|x' - x_{n'+1}| \leq |x' - x_{n'}| + |x_{n'} - x_{n'+1}| \leq \frac{\ell_j}{6N_j},$$

thus we must have

$$\tilde{x} < x_{n'+1} \leq x_{\ell_j} \leq Z_j + X_j^+$$

and

$$|\tilde{x} - x'| < \frac{\ell_j}{6N_j},$$

whence

$$\tilde{x} > x' - 2\frac{\ell_j}{N_j} \geq x_0 - \frac{\ell_j}{4N_j}$$

and again \tilde{x} cannot satisfy (19) either. Therefore, $(\tilde{x}, \tilde{t}) \in \nu_1$ which implies $\tilde{x} = x_{n'+1}$. For $n = n' + 2, \dots, \ell_j$, the argument is the same.

4. Finite speed and random service times. In this section, we show how the proof of Theorem 1 for the particular case $T = 1, v = \infty$ can be adapted to more broad conditions as stated in Section 1. We start describing the analogous construction for the stochastic evolution of potentials. Assume that at time $t = 0$ the server starts serving a customer at $x = 0$ (for convenience, we consider here the potentials corresponding to times when service starts). Assume also that the set of waiting customers is given by a Poisson Point Process on \mathbb{R} with intensity $-u_0(x) dx$ for a unimodal potential u_0 with maximal value $u_0(0) = 0$. In analogy with (1), given $w = (T, E, U)$ we define the operator \mathcal{H}_w by

$$\int_{x_*-z}^{x_*+z} (\mathcal{M} + T - u) dx = E, \quad \begin{aligned} a &= \mathcal{M} + T - u(x_* - z), \\ b &= \mathcal{M} + T - u(x_* + z), \end{aligned}$$

$$x^* = \begin{cases} x_* - z, & \text{if } U \in \left(0, \frac{a}{a+b}\right), \\ x_* + z, & \text{if } U \in \left(\frac{a}{a+b}, 1\right), \end{cases}$$

and

$$(\mathcal{H}_w(u))(x) = \begin{cases} \mathcal{M} + T + \frac{z}{v}, & x = x^*, \\ \mathcal{M} + T, & x \in [x_* - z, x_* + z], x \neq x^*, \\ u(x), & \text{otherwise.} \end{cases}$$

Notice that $\mathcal{M}(\mathcal{H}_w(u)) = \mathcal{M}(u) + T + \frac{z}{v}$, $S(\mathcal{H}_w(u)) = x^*$, and $\int_{\mathbb{R}} [\mathcal{H}_w(u) - u] dx = E$.

We take an i.i.d. sequence $(\omega_n)_{n=1,2,\dots}$, where each $\omega_n = (T_n, E_n, U_n)$ has independent coordinates, distributed respectively as the service time, a standard exponential, and a uniform on $[0, 1]$. We define u_n^k by (4) and let $u_n = u_n^0$ and $u = u_0$.

Define $t_n = \mathcal{M}(u_n^0)$ and $S_n = S(u_n^0)$. In analogy with Lemma 1, we have

LEMMA 2. *The pair sequence $(t_n, S_n)_{n=1,2,\dots}$ described above has the same distribution as $(t_n, S_n)_{n=1,2,\dots}$ given by the beginning of service times and the corresponding positions.*

For the evolution $(u_n)_{n=0,1,2,\dots}$ we will define a sequence of stopping times $0 = \mathcal{N}_0 < \mathcal{N}_1 < \mathcal{N}_2 < \dots$ in \mathbb{N}_0 , as well as the corresponding events of success A_j defined in terms of $u_{\mathcal{N}_j}$ and whose occurrence is determined by $u_0, u_1, \dots, u_{\mathcal{N}_{j+1}}$.

The construction will have the following properties. For some sequence p_j and any u that is centered and unimodal,

$$(21) \quad \mathbb{P}^u(A_j | u_{\mathcal{N}_j}) \geq p_j \quad \text{on } A_0 \cap \dots \cap A_{j-1} \quad \text{and} \quad \prod_j p_j > 0.$$

Moreover, the event $\bigcap_{j=0}^{\infty} A_j$ implies $S_n \sim_{\varepsilon} \sigma \log n$ just as in the proof of Theorem 1 in the end of Section 2.

Step 0 provides $\sigma = \pm 1$ which indicates the direction in which subsequent blocks are supposed to grow. In the steps described below we let $\mathcal{N}_j = Q_0 + \dots + Q_{j-1}$, where Q_j is the number of customers served in each block, $L_j = \mathcal{M}(u_{\mathcal{N}_j})$ the physical time, $N_j = L_j - u(S_{\mathcal{N}_j})$ the height of discontinuity in the potential, $Z_j = \sigma S_{\mathcal{N}_j}$, $X_j = Z_{j+1} - Z_j$ physical displacement during each block, and $M_j = L_{j+1} - L_j$ is the time elapsed within each block. In this setting, the time L_{j+1} is given by the instant when the server reaches the customer located at σZ_{j+1} , and the next block starts.

Let ℓ_j be given as above, and write $m_j = \ell_1 + \dots + \ell_j$. Let j_* be such that

$$\frac{1}{m_{j_*}} \cdot \frac{1}{v} < \frac{1}{16} \quad \text{and} \quad \mathbb{P}\left(\frac{n}{2} < T_1 + \dots + T_n < 2n\right) > 0 \quad \text{for all } n > \ell_{j_*}.$$

Fix $m = 1 + m_{j_*}$. In steps $1, \dots, j_*$ we will relax the lower bound on time and take $M_j^- = 0$. This is compensated by finding a big number of customers at step 0, namely $Q_0 = m$. So the triggering step will take care of however small the speed v is, as well as complications arising from the distribution of T .

The event A_0 is defined by the following conditions. First, that S_m is an unexplored point, that is, $\sigma S_m > \sigma S_n$ for all $n = 0, 1, 2, \dots, m - 1$ and some $\sigma = \pm 1$. Second, $M_0 \geq M_0^- = m_{j_*}$. Finally,

$$T_m \leq 1 \quad \text{and} \quad X_0^- \leq |S_m - S_{m-1}| \leq X_0^+,$$

where $X_0^- = \frac{4}{N_1}$ and $X_0^+ = v$.

We claim that $\mathbb{P}^u(A_0) \geq p_0$ for some $p_0 > 0$ that does not depend on u . To prove the claim, consider the following events. First, suppose $T_1 \geq 1$. Suppose also that $E_1 \geq 16$, and U_1 is such that S_1 lies on the bigger side of $-u$, as in Section 3. Assuming that all these happen, σ is determined by which direction S_1 was found, that is, the higher side of the plateau in u_1 . In the sequel we assume for simplicity that $\sigma = +1$, otherwise mirror the system around $x = 0$. Now suppose that, for all $n = 2, \dots, m - 1$, $T_n \geq 1$, $E_n \in [0, 1]$, and $U_n > \frac{1}{2}$. Finally, suppose that $T_m \leq 1$, $8 \leq E_m < 16$ and $U_m > \frac{1}{2}$.

Let us show that these events imply A_0 , which proves our claim. By assumption $S_1 > 0$ and

$$16 \leq E_1 = \int_{-S_1}^{S_1} [T_1 - u(x)] dx \leq -2S_1 \cdot u(S_1)$$

and thus

$$-u(S_1) \geq \frac{8}{S_1}.$$

Writing $z_2 = |S_2 - S_1|$, we have

$$1 \geq E_2 = \int_{S_1 - z_2}^{S_1 + z_2} [\mathcal{M}(u_1) + T_2 - u_1(x)] dx \geq \int_{S_1}^{S_1 + z_2} [-u_1(x)] dx \geq -u(S_1) \cdot z_2$$

and thus $z_2 \leq \frac{S_1}{8}$ and $0 < S_1 - z_2 < S_1$. Since $u_1(x) \leq 0$ for $x > S_1$ and $u_1(x) = T_1 > 0$ for $-S_1 < x < S_1$, the choice of $U_2 > \frac{1}{2}$ implies that $S_2 = S_1 + z_2 > S_1$. By the same argument, $E_3 \leq 1$ implies $|S_3 - S_2| \leq \frac{S_1}{8}$, and thus $U_3 > \frac{1}{2}$ implies $S_3 > S_2$, and so on. Therefore, $S_{m-1} > S_1$ and $u_{m-1}(x) = u(x)$ for $x > S_{m-1}$. As before, writing $z_m = |S_m - S_{m-1}|$ we have

$$16 > E_m = \int_{S_{m-1}-z_m}^{S_{m-1}+z_m} [\mathcal{M}(u_{m-1}) + T_m - u_{m-1}(x)] dx \geq z_m \cdot [T_1 + \dots + T_m - u(S_{m-1})],$$

thus $z_m < 2S_1$, and since $U_m > \frac{1}{2}$ we have $S_m > S_{m-1}$. Moreover, $T_1 + \dots + T_m \geq m_{j^*}$ and thus $S_m - S_{m-1} = z_m < v = X_0^+$. Finally,

$$8 \leq E_m = \int_{S_{m-1}-z_m}^{S_{m-1}+z_m} [\mathcal{M}(u_{m-1}) + T_m - u_{m-1}(x)] dx \leq 2z_m \cdot [\mathcal{M}(u_m) - u(S_m)] = 2z \cdot N_1,$$

and thus $S_m - S_{m-1} = z_m \geq X_0^-$. This proves the above claim.

For $j \geq 1$, we define Q_j and the event A_j as in Section 3, with condition (5) replaced by

$$\begin{cases} Q_j = \ell_j \text{ or } \ell_j + 1, \\ X_j^- \leq X_j \leq X_j^+, \\ M_j^- \leq M_j \leq M_j^+, \\ Z_{j-1} < S_n < Z_{j+1}, \end{cases} \quad \text{for } \mathcal{N}_j \leq n < \mathcal{N}_{j+1},$$

where $X_j^- = (1 - \varepsilon) \frac{\ell_{j-1}}{N_{j+1}}$, $X_j^+ = (1 + \varepsilon) \frac{\ell_j}{N_j}$, $M_j^+ = 3\ell_j + 3$ and $M_j^- = \frac{1}{2}\ell_j \mathbb{1}_{j > j^*}$.

Assuming that $(A_0 \cap \dots \cap A_{j-1})$ occurs, since $u_{\mathcal{N}_j}$ is unimodal it must satisfy

$$\begin{cases} \mathcal{M}(u_{\mathcal{N}_j}) = L_j, \\ u_{\mathcal{N}_j}(x) = u_0(x) \leq L_j - N_j, & \text{for } x > Z_j, \\ u_{\mathcal{N}_j}(x) \geq L_j - M_{j-1}, & \text{for } Z_j - X_{j-1}^- < x < Z_j. \end{cases}$$

For $j = 1$ the last condition is replaced by $u_{\mathcal{N}_1}(x) \geq L_1 - T_m - \frac{X_0^+}{v} \geq L_1 - 2$. Moreover,

$$N_j = -u(\sigma Z_j) + L_j \geq L_j \geq M_0^- + \dots + M_{j-1}^- \geq m_j \geq cj^{5/4}$$

and thus $X_j^+ \leq Cj^{-1}$ and $X_{j-1}^- \leq Cj^{-1}$. Therefore, $M_j^+ \leq Cj^{1/4}$.

To estimate $\mathbb{P}^u(A_j | u_{\mathcal{N}_j})$ on $(A_0 \cap \dots \cap A_{j-1})$, we consider the events B_1, B_2 , and B_3 as in Section 4.

The region analogous to $R_2 \subseteq \mathbb{R} \times \mathbb{R}$ is contained in the union of the rectangles $[Z_j - X_{j-1}^-, Z_j] \times [L_j - M_{j-1}^+, L_j + M_j^+]$ and $[Z_j, Z_j + X_j^+] \times [L_j, L_j + M_j^+]$

with M_0 replaced by $T_m + \frac{X_0^+}{v} \leq 2$ for $j = 1$. The above inequalities imply that $|R_2| \leq (X_{j-1}^- + X_j^+)(M_{j-1}^+ + M_j^+) \leq Cj^{-3/4}$. Therefore,

$$\mathbb{P}^\mu(B_1 | u_{\mathcal{N}_j}) \geq \left(1 - \frac{C}{j^{3/2}} \vee e^{-c}\right).$$

Events B_2 and B_3 are defined by (12) and (14). Therefore, occurrence of $B_2 \cap B_3$ implies inequalities (16) and (17), and its probability satisfies (21). The desired bounds for S_n for $\mathcal{N}_j \leq n < \mathcal{N}_{j+1}$ and for X_j thus follow exactly as in Section 3.

It remains to control M_j , which was not necessary in the case $T = 1, v = \infty$ because $M_j = Q_j$ in that setup. But M_j is composed of Q_j service times plus traveling time. The latter is nonnegative and bounded by

$$2\frac{X_j}{v} \leq 2\frac{X_j^+}{v} \leq 2\frac{\ell_j/N_j}{v} \leq 2\frac{\ell_j/m}{v} \leq \ell_j.$$

Therefore, the inequality $M_j^- \leq M_j \leq M_j^+$ holds whenever the sum of Q_j service times is bigger than $\frac{1}{2}Q_j \mathbb{1}_{j > j_*}$ and less than $2Q_j$. The probability of this event is exponentially high in ℓ_j , and positive by the choice of j_* . This completes the proof of Theorem 1.

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