Permutation group S_N and large N_c excited baryons

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We study the excited baryon states for an arbitrary number of colors N_c from the perspective of the permutation group S_N of N objects. Classifying the transformation properties of states and quark-quark interaction operators under S_N allows a general analysis of the spin-flavor structure of the mass operator of these states, in terms of a few unknown constants parametrizing the unknown spatial structure. We explain how to perform the matching calculation of a general two-body quark-quark interaction onto the operators of the $1/N_c$ expansion. The inclusion of core and excited quark operators is shown to be necessary. Considering the case of the negative parity $L = 1$ states transforming in the MS of S_N, we discuss the matching of the one-gluon and the Goldstone-boson exchange interactions.

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I. INTRODUCTION

The nonrelativistic quark model (NRQM) [1,2] has been used with a varying degree of success to describe the baryonic states, both for ground state and excited [3] baryons. Recently, the NRQM gained additional significance in the baryon sector, where it provides an explicit realization of the contracted $SU(4)_c$ symmetry, which becomes manifest in the large N_c limit [4]. This symmetry connects states with different spins and isospins, and provides an important organizing principle for the construction of a systematic expansion in $1/N_c$ for physical quantities such as masses and couplings. Many detailed applications to the physics of ground state baryons, both heavy and light, have been presented.

The $1/N_c$ expansion has been applied also to orbitally excited baryons, up to subleading order in $1/N_c$ [5–11]. The structure of the operators contributing to the physical properties of these states is more complicated. The construction of these operators makes use of the decomposition of the spin-flavor states into ''core'' and ''excited'' quark subsystems. The operators appearing in the $1/N_c$ expansion are the most general structures which can be constructed from $SU(4)$ generators acting on the core and excited quarks.

The construction of the states and operators for excited states presented in Ref. [7] has been inspired by the quark model picture of the excited states. To our knowledge a formal justification in terms of an underlying symmetry is still lacking. Such a symmetry argument would be desirable for several reasons. First, it would help in order to establish the completeness of the set of operators contributing to any given order in $1/N_c$. Second, such a symmetry argument could point the way for the treatment of higher excitations with a more complicated spin-flavor structure. Finally, in several recent papers [12,13] the validity of the usual approach based on the core-excited quark decomposition has been questioned, and a symmetry argument should settle the objections raised in these works.

In this paper we use the simple observation that the spin-flavor states of a system of N quarks can be classified into irreps of $SU(4) \times S_N^{\rm sf}$, with S_N the permutation group of N objects. Including also the orbital degrees of freedom, the complete permutation symmetry is $S_N \subset S_N^{\rm orb} \times S_N^{\rm sf}$, the diagonal subgroup of the permutations acting on both orbital and spin-flavor degrees of freedom. Of course, although S_N is a good symmetry of the quark model Hamiltonian, $S_N^{\rm sf}$ is not, and mixing between different irreps can occur in general (configuration mixing).

The permutation symmetry $S_N^{\rm sf}$ has nontrivial implications. It constrains the form of the allowed operators which can contribute to any given matrix element. The spin-flavor structure of the mass operator for these states contains only those spin-flavor operators $\mathcal{O}^{\rm sf}$ which appear in the decomposition of the interaction Hamiltonian $\mathcal{H} = \sum_R \mathcal{R}_R^{\rm orb} \mathcal{O}_R^{\rm sf}$ under $S_N^{\rm orb} \times S_N^{\rm sf}$.

The ground state baryons transform in the singlet irrep of $S_N^{\rm sf}$, and thus the only operators with nonzero matrix elements are also singlets of $S_N^{\rm sf}$. More interesting predictions are obtained for the orbitally excited baryons, which transform nontrivially under $S_N^{\rm sf}$. We consider in some detail the states transforming in the $[N-1, 1]$ (MS) irrep of $S_N^{\rm sf}$.

We perform the explicit decomposition of the most general two-body quark-quark interaction into irreps of $S_N^{\rm orb} \times S_N^{\rm sf}$, which gives a general expression for the mass operator of the excited states. We present explicit results for the one-gluon exchange, and one Goldstone-boson exchange mediated quark-quark interaction. This generalizes the $N_c = 3$ results of Collins and Georgi [14] to arbitrary N_c and also allows to extract the otherwise im-

plicit N_c dependence of the operators and their matrix elements.

The analysis based on S_N makes the connection to the $1/N_c$ expansion particularly simple, allowing the matching of any microscopic quark-quark interaction onto the operators of the $1/N_c$ expansion. In addition to the class of operators transforming as $[N]$ and $[N-1,1]$ of $S_N^{\rm sf}$, previously considered in [14] for $N_c = 3$, we also discuss a new class of operators transforming in the $[N-2,2]$, $[N-2,1,1]$ irreps, which appear only for $N_c > 3$. The matching calculation for two different models confirms the scalings and the completeness of the set of effective operators considered in [7]. We find that the criticism of Matagne and Stancu [12,13] of the usual $1/N_c$ expansion for excited baryons [5–11] is unfounded, and the disagreement with existing work is due to ignoring operators transforming like nonsinglets under $S_N^{\rm sf}$.

As an application of our results, we point out that the models for quark dynamics discussed here make definite predictions for the phenomenology of the excited baryons, in the form of a hierarchy of the coefficients of operators in the $1/N_c$ expansion. Comparing with the results of a fit we conclude that at leading order in $1/N_c$ the Goldstone-boson exchange model is favored.

II. THE $S_N^{\rm sf}$ PERMUTATION SYMMETRY

We describe in this section the S_N transformation properties of the spin-flavor wave function of a system of $N = N_c$ quarks with spin and isospin. For simplicity we consider only two light quark flavors u, d and assume isospin symmetry. In the nonrelativistic quark model, the spin-flavor wave functions appear as building blocks for the total baryon wave function, along with the orbital wave function. More important, they are relevant in the context of the $1/N_c$ expansion, where the quark representation provides a realization of the large N_c spin-flavor symmetry [4].

The spin-flavor baryon states are constructed by taking products of one-body states with spin and isospin

$$|u_\uparrow\rangle, \qquad |u_\downarrow\rangle, \qquad |d_\uparrow\rangle, \qquad |d_\downarrow\rangle. \tag{1}$$

We keep track of the identity of the states of the N_c quarks, such that in general, a state will not be left invariant under a permutation of the quarks. For any finite N_c, such a state will be completely specified by giving its transformation under the symmetry group $SU(4) \times S_N$.

For any spin-flavor state, the irreps of the spin-flavor $SU(4)$ and permutation group are related such that they correspond to the same Young tableaux. For example, the multiplicity of a state with given spin, isospin, and their projections $|S, I; S_3, I_3\rangle$ is given by the dimension of the S_N irrep with the same Young diagram as the $SU(4)$ irrep. For this reason it may appear that the spin-flavor irrep is sufficient to describe the state. However, it will be seen that the S_N information can give constraints on the possible form of the operators which do not follow from the spin-flavor structure alone.

We start by considering the ground state baryons. They transform in the completely symmetric irrep S of $S_N^{\rm sf}$. The only operators which have nonvanishing matrix elements between these states transform also in the S irrep of the permutation group.

Next consider states with nontrivial transformation under the permutation group. As a first example of such states consider the nonstrange states in the MS irrep of $S_N^{\rm sf}$. An explicit analysis of the spin-flavor structure in the nonrelativistic quark model performed below in Sec. III shows that these states can be identified with the $L = 1$ orbitally excited baryons.

Consider the mass operator of these states, and the constraints imposed on it from the permutation group. It must transform like R under the permutation group, where R is any irrep such that $MS \times R$ contains MS. The allowed irreps must satisfy the following condition (see p. 257 of [15]): consider the Young diagrams of R and of MS (partition $[N-1,1]$). By removing one box from the R diagram, and the addition of one box to the resulting diagram, the Young diagram of MS must be obtained. A simple examination of the respective Young diagrams shows that this condition is satisfied by the following irreps of S_N (see Appendix A for notations):

$$R = S, MS, E, A'. \tag{2}$$

Thus, the most general operator contributing to the mass operator of these states will be a linear combination of operators transforming in each of the irreps of $S_N^{\rm sf}$ given in Eq. (2). The S operators contain only building blocks S^i, T^a, G^{ia} for the entire hadron: total hadron spin, isospin, and the G^{ia} operator. The MS operators are constructed by combining operators acting on the excited and core subsystems of the hadron consisting of 1 and $N_c - 1$ quarks, respectively. Finally, the E and A' operators have a more complicated structure, and have not been considered previously.

In the next section we will show by explicit calculation in the nonrelativistic quark model how each of these operators is generated from specific models of quark-quark interactions. We consider in detail two such models, the one-gluon exchange and the Goldstone-boson exchange mediated interactions.

III. S_N AND THE NONRELATIVISTIC QUARK MODEL

We consider in this section the spin-flavor structure of the orbitally excited baryons transforming in the MS irrep of $S_N^{\rm sf}$ in the nonrelativistic quark model with N_c quarks. For $N_c = 3$, these states correspond to the negative parity baryons with $L = 1$ and transform in the 70^- of $SU(6)$ (or the 20^- of $SU(4)$). This contains several spin-isospin

multiplets of $SU(2) \times SU(3) \subset SU(6)$. Denoting them as ${}^{(2S+1)}F$, these multiplets are ${}^4 8 \oplus^2 8 \oplus^2 10 \oplus^2 1$.

We would like to construct these states for arbitrary N_c quarks, with special attention to their transformation properties under the permutation group. For generality we will keep L arbitrary. Fermi statistics constrains the spin-flavor-orbital wave function to be completely symmetric. Thus this wave function must transform under the completely symmetric irrep of S_N, the permutation group acting on all degrees of freedom of the N quarks: orbital, spin, and flavor. This group is the diagonal subgroup of $S_N^{\rm orb} \times S_N^{\rm sf}$, the product of the permutation groups acting independently on the orbital and spin-flavor degrees of freedom, respectively. Any symmetric state in orbital-spin-flavor space is decomposed into a sum of inner products of irreps of $S_N^{\rm orb} \times S_N^{\rm sf}$ as

$$S = \sum_{R=S,MS,\cdots} R^{\rm orb} \otimes R^{\rm sf}. \tag{3}$$

The sum is over states which transform in the same irrep R under independent permutations of the respective degrees of freedom. We use here and in the following the notations of Appendix A for the irreps of the permutation group.

The first term in the sum, with $R = S$, corresponds to states with orbital wave functions symmetric under any exchange of two quarks. These states include the ground state baryons, and some of their radial excitations. According to Eq. (3), their spin-flavor wave function is also completely symmetric under permutations. For two light quark flavors, this implies that they transform in the symmetric irrep of $SU(4)$.

A. The MS states and relation to CCGL

Consider next the second term in the sum Eq. (3), corresponding to $R = MS$. They correspond to orbitally excited baryons, for example, the $L = 1$ baryons in the 70^-. A basis for the spin-flavor wave function can be constructed using the method of the Young operators (see Appendix A). This can be chosen as the set of $N-1$ wave functions, with $k = 2, 3, \cdots, N$

$$|\phi_k\rangle = |q_k\rangle \otimes |i_c\rangle_{N-1} - |q_1\rangle \otimes |i_c\rangle_{N-1}, \tag{4}$$

where $|q_k\rangle$ denotes the spin-flavor state of the quark k, and $|i_c\rangle_{N-1}$ denotes the spin-flavor state of the subset of $N-1$ quarks ("core") obtained by removing quark k from the N quarks. The latter states are symmetric under any permutation of the $N-1$ quarks. The states ϕ_i are not orthogonal, and have the scalar products

$$\langle \phi_i | \phi_j \rangle = S_{ij}, \qquad S_{ij} = \begin{cases} 2, & i = j \\ 1, & i \neq j \end{cases}. \tag{5}$$

The basis states ϕ_k have the following transformations under the action of the transpositions P_{ij} (exchange of the quarks i, j)

$$P_{1j}\phi_k = \begin{cases} -\phi_k & , j = k \\ \phi_k - \phi_j & , j \neq k \end{cases} \tag{6}$$

$$P_{ij}\phi_k = \phi_k \quad \text{if } (i,j) \neq 1, k \tag{7}$$

$$P_{ik}\phi_k = P_{ki}\phi_k = \phi_i \quad \text{if } i \neq 1. \tag{8}$$

We will adopt these transformation relations as defining a basis for the MS irrep of S_N, both for states and operators.

A similar construction can be used also for the orbital wave functions with MS permutation symmetry. They form a set of $N-1$ functions and will be denoted as $\chi_k^m(\vec{r}_1, \cdots, \vec{r}_N)$ with $k = 2, 3, \cdots, N$. The index $m = \pm 1, 0$ denotes the projection of the orbital angular momentum $\vec{L}$ along the z axis.

For definiteness we adopt a Hartree representation for the orbital wave functions, in terms of one-body wave functions $\varphi_s(r)$ for the ground state orbitals, and $\varphi_p^m(r)$ for the orbitally excited quark. The Young operator basis for the orbital wave functions is given by

$$\chi_k^m(\vec{r}_1, \cdots, \vec{r}_N) = \varphi_p^m(\vec{r}_k)\Pi_{i=1, i\neq k}^N \varphi_s(\vec{r}_i) - \varphi_p^m(\vec{r}_1)\Pi_{i=2}^N \varphi_s(\vec{r}_i). \tag{9}$$

These basis wave functions have the same transformation properties under transpositions as the spin-flavor basis functions, Eqs. (6)–(8). We emphasize that the main results of our paper do not depend on the use of the Hartree representation, as explained in more detail below.

The complete spin-flavor-orbital wave function of a baryon B with mixed-symmetric spin-flavor symmetry is written as the $MS \times MS \to S$ inner product of the two basis wave functions, for the orbital and spin-flavor components, respectively. Making explicit the spin-isospin degrees of freedom, the wave function is given by

$$|B; (LS)JJ_3, II_3\rangle = \sum_{m, S_3} \left(\begin{matrix} L & S \\ m & S_3 \end{matrix} \middle| \begin{matrix} J \\ J_3 \end{matrix} \right) \sum_{k,l=2}^N \phi_k(SS_3, II_3)\chi_l^m M_{kl}. \tag{10}$$

For simplicity of notation, we will drop the explicit dependence of the basis states on spin-isospin, and the $SU(2)$ Clebsch-Gordan coefficient coupling the orbital angular momentum with the quark spin $L + S = J$. They will be implicitly understood in all expressions written below. In Appendix B we give an explicit example of this construction for a baryon with $N_c = 3$, and compare it with the baryon states constructed in Ref. [7].

The matrix of coefficients M_{ij} are the Clebsch-Gordan coefficients for the $MS \times MS \to S$ inner product of two irreps of S_N. They can be determined by requiring that the state Eq. (10) is left invariant under the action of transpositions acting simultaneously on the spin-flavor and orbital components. In the MS basis defined by the transformations Eqs. (6) the matrix M_{ij} is

$$\hat{M} = \begin{pmatrix} 1 & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} \\ -\frac{1}{N-1} & 1 & -\frac{1}{N-1} & \cdots & -\frac{1}{N-1} \\ -\frac{1}{N-1} & -\frac{1}{N-1} & 1 & \cdots & -\frac{1}{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{N-1} & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & 1 \end{pmatrix}. \quad (11)$$

Since any permutation can be represented as a product of transpositions, the state given in Eq. (10) transforms indeed in the S irrep of the overall S_N group.

The form of the baryon state Eq. (10) simplifies greatly with an alternative choice for the basis of MS spin-flavor and orbital states. This is the so-called Yamanouchi basis [15], which is constructed such that i) the basis vectors are orthogonal and ii) each of them is invariant under permutations of mutually overlapping subsets of the N quarks. Considering the spin-flavor states, the Yamanouchi basis ψ_k, $k = 2, \cdots, N$ is given by

$$\begin{aligned} \psi_2 &= \frac{1}{\sqrt{N(N-1)}}(\phi_2 + \phi_3 + \cdots + \phi_N) \\ \psi_j &= \frac{1}{\sqrt{(N-j+2)(N-j+1)}}(-(N-j+1)\phi_{j-1} \\ &\quad + \phi_j + \cdots + \phi_N), \\ j &= 3, \cdots, N. \end{aligned} \quad (12)$$

These states satisfy the normalization conditions $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. Also, using Eqs. (6) one can see that $\psi_2, \psi_3, \psi_4, \cdots, \psi_N$ are invariant under permutations of the subsets $(2, 3, \cdots, N) \supset (3, \cdots, N) \supset (4, \cdots, N) \supset \cdots \supset (N-1, N)$ of the N quarks, respectively.

Expressed in terms of the Yamanouchi basis ψ_k, the matrix M, giving the $MS \times MS \to S$ coupling, is proportional to the unit matrix

$$M_{ij} = \frac{N}{N-1}\delta_{ij}, \quad (13)$$

such that the baryon state Eq. (10) is given simply (with ξ_k the Yamanouchi basis of orbital states) by

$$|B; (LS)JJ_3, II_3\rangle = \frac{N}{N-1}\sum_{m,S_3} \left(\begin{array}{cc|c} L & S & J \\ m & S_3 & J_3 \end{array}\right) \times \sum_{k=2}^{N} \psi_k(SS_3, II_3)\xi_k^m. \quad (14)$$

We pause at this point to compare the state Eq. (10) with the MS states constructed in Ref. [7], and commonly used in the literature in the context of the $1/N_c$ expansion. These states are constructed as tensor products of an "excited" quark whose identity is fixed as quark 1, with a symmetric "core" of $N-1$ quarks. We will refer to these states as CCGL states. With this convention, the wave function used in Ref. [7] has the form

$$|CCGL\rangle = \Phi(SI)\varphi_p^m(\vec{r}_1)\Pi_{i=2}^N\varphi_s(\vec{r}_i), \quad (15)$$

where $\Phi(SI)$ denotes the spin-flavor component, and the remainder is the orbital wave function in Hartree form. By construction, the spin-flavor wave function $\Phi(SI)$ transforms in the MS irrep of $SU(4)$, and thus also like MS under S_N. It is symmetric under any exchange of the core quarks, $P_{ij}\Phi(SI) = \Phi(SI)$ for $i, j \neq 1$. These two properties identify it uniquely in terms of the MS basis functions defined in Eq. (4) as $\Phi(SI) = \frac{1}{\sqrt{N(N-1)}}\sum_{k=2}^N \phi_k$. The normalization factor is chosen such that the state is normalized as $\langle\Phi(SI)|\Phi(SI)\rangle = 1$.

Symmetrizing under the "excited" quark index $i = 1, 2, \cdots, N$, one finds for the properly normalized symmetric state

$$\begin{aligned} |CCGL\rangle &\to \frac{1}{\sqrt{N}}\Sigma_{i=1}^N P_{1i}|CCGL\rangle \\ &= \frac{1}{N\sqrt{N-1}}\Sigma_{i=2}^N\Sigma_{j=2}^N(P_{1i}\phi_j)\chi_i \\ &= -\frac{\sqrt{N-1}}{N}|B\rangle, \end{aligned} \quad (16)$$

where the terms with different i in the first sum are orthogonal states. This gives the relation between the CCGL state and the symmetric state constructed above in Eq. (10).

B. $S_N \to S_N^{\rm orb} \times S_N^{\rm sf}$ decomposition of the interaction Hamiltonian

In this section we study the S_N transformation properties of the interaction Hamiltonian in the nonrelativistic quark model. For definiteness we adopt the one-gluon-exchange potential, which follows from a perturbative expansion in $\alpha_s(m_Q)$ in the heavy quark limit $m_Q \gg \Lambda_{QCD}$. We formulate our discussion in sufficiently general terms to allow the treatment of any other Hamiltonian containing only two-body interactions.

We consider a Hamiltonian containing a spin-flavor symmetric term H_0 (the confining potential and kinetic terms), plus spin-isospin dependent two-body interaction terms V_{ij}

$$H = H_0 + g_s^2\sum_{i<j}\frac{\lambda_i^a}{2}\frac{\lambda_j^a}{2}V_{ij} \to H_0 - g_s^2\frac{N_c+1}{2N_c}\sum_{i<j}V_{ij}, \quad (17)$$

where λ^a are the generators of $SU(3)$ color in the fundamental representation, and g_s is the strong gluon coupling to the quarks, scaling like $g_s \sim O(N_c^{-1/2})$. The second equality holds on color-singlet hadronic states, on which the color interaction evaluates to the color factor $(N_c+1)/(2N_c)$. We will restrict ourselves only to color neutral hadronic states in this paper.

In the nonrelativistic limit, the two-body interaction V_{ij} contains three terms: the spin-spin interaction (V_{ss}), the

quadrupole interaction V_q, and the spin-orbit terms $V_{\rm so}$. We write these interaction terms in a slightly more general form as [14], where $f_{0,1,2}(r_{ij})$ are unspecified functions of the interquark distances

$$V_{\rm ss} = \sum_{i<j=1}^{N} f_0(r_{ij})\vec{s}_i \cdot \vec{s}_j, \tag{18}$$

$$V_q = \sum_{i<j=1}^{N} f_2(r_{ij})[3(\hat{r}_{ij} \cdot \vec{s}_i)(\hat{r}_{ij} \cdot \vec{s}_j) - (\vec{s}_i \cdot \vec{s}_j)], \tag{19}$$

$$V_{\rm so} = \sum_{i<j=1}^{N} f_1(r_{ij})[(\vec{r}_{ij} \times \vec{p}_i) \cdot \vec{s}_i - (\vec{r}_{ij} \times \vec{p}_j) \cdot \vec{s}_j + 2(\vec{r}_{ij} \times \vec{p}_i) \cdot \vec{s}_j - 2(\vec{r}_{ij} \times \vec{p}_j) \cdot \vec{s}_i]. \tag{20}$$

We would like to decompose the Hamiltonian H into a sum of terms transforming according to irreps of the permutation group acting on the spin-flavor degrees of freedom $S_N^{\rm sf}$. The operators in Eq. (17) are two-body interactions, of the generic form

$$V = \sum_{1\le i<j\le N} \mathcal{R}_{ij}\mathcal{O}_{ij}, \tag{21}$$

where $\mathcal{R}_{ij}$ acts only on the orbital coordinates of the quarks i, j, and $\mathcal{O}_{ij}$ acts only on their spin-flavor degrees of freedom. For example, the spin-spin interaction $V_{\rm ss}$ has $\mathcal{R}_{ij} = f_0(r_{ij})$ and $\mathcal{O}_{ij} = \vec{s}_i \cdot \vec{s}_j$. Of course, V must be symmetric under any permutation of the N quarks, but the transformation of the spin-flavor and orbital factors separately can be more complicated. We distinguish two possibilities for the transformation of these operators under a transposition of the quarks i, j, corresponding to the two irreps of S_2:

(i) symmetric two-body operators: $P_{ij}\mathcal{R}^s_{ij}P^{-1}_{ij} = \mathcal{R}^s_{ij}$ and $P_{ij}\mathcal{O}^s_{ij}P^{-1}_{ij} = \mathcal{O}^s_{ij}$.

(ii) antisymmetric two-body operators: $P_{ij}\mathcal{R}^a_{ij}P^{-1}_{ij} = -\mathcal{R}^a_{ij}$ and $P_{ij}\mathcal{O}^a_{ij}P^{-1}_{ij} = -\mathcal{O}^a_{ij}$.

The spin-spin and quadrupole interactions $V_{\rm ss}$, V_q are composed of symmetric two-body operators, while the spin-orbit interaction $V_{\rm so}$ contains both symmetric and antisymmetric components.

In general, the k-body operators can be classified into irreps of the permutation group of k objects S_k. For example, there are three classes of three-body operators, corresponding to the S, MS, A irreps of S_3.

We start by considering the symmetric two-body operators. The set of all spin-flavor operators $\mathcal{O}^s_{ij}$ (and analogous for the orbital operators $\mathcal{R}^s_{ij}$) with $1 \le i < j \le N$ form a $\frac{1}{2}N(N-1)$ dimensional reducible representation of the S_N group, which contains the following irreps:

$$\{\mathcal{O}^s_{ij}\} = S \oplus MS \oplus E. \tag{22}$$

We will use as a basis for the operators on the right-hand side the Young operator basis supplemented by the phase convention Eq. (6) as explained in Appendix A. The projection of the operators onto irreps of S_N are as follows.

The S projection.

$$\mathcal{O}^S = \sum_{i<j} \mathcal{O}^s_{ij} \tag{23}$$

The MS projection.

$$\mathcal{O}^{MS}_k = \sum_{2\le j\ne k\le N} (\mathcal{O}^s_{1j} - \mathcal{O}^s_{kj}), \qquad k = 2, 3, \cdots, N \tag{24}$$

The E projection. These operators can be labeled by the three integers i, j, k appearing in the standard Young tableau $[1i\cdots][jk]$ corresponding to the respective operator, and are given by

$$\mathcal{O}^E_{ijk} = \mathcal{O}^s_{1i} - \mathcal{O}^s_{1k} - \mathcal{O}^s_{ij} + \mathcal{O}^s_{jk}, \quad k > i,\ j = 2, \cdots, N-1. \tag{25}$$

In Appendix A we give as an illustration the complete basis of the S, MS, E operators for $N_c = 5$.

The interaction $V^{\rm symm}$ constructed with symmetric two-body operators is symmetric under S_N, and its decomposition under $S_N \subset S_N^{\rm orb} \times S_N^{\rm sf}$ has the form

$$V^{\rm symm} = \frac{2}{N(N-1)}\mathcal{R}^S\mathcal{O}^S + \frac{N-1}{N(N-2)}\sum_{j,k=2}^{N}\mathcal{R}^{MS}_j\mathcal{O}^{MS}_k M_{jk} + c_E^{\rm symm}\sum_{j,k=1}^{(1/2)N(N-3)}\mathcal{R}^E_j\mathcal{O}^E_k N_{jk}. \tag{26}$$

The matrices M_{jk} and N_{jk} are Clebsch-Gordan coefficients for the $MS \times MS \to S$ and $E \times E \to S$ reductions for the irreps of the S_N group. The matrix M_{jk} is given in Eq. (11) in explicit form. We do not have explicit results for c_E and the matrix N_{jk} for arbitrary N, although for any given N they can be found as explained in Appendix A.

We consider next the case of the antisymmetric two-body operators $\mathcal{O}^a_{ij}$. They form a $\frac{1}{2}N(N-1)$ dimensional reducible representation of S_N, which is decomposed into irreps as

$$\{\mathcal{O}^a_{ij}\} = MS \oplus A'. \tag{27}$$

This can be verified by noting that the sum of the dimensions of the irreps on the right-hand side is indeed $\frac{1}{2}N(N-1)$. We list in Appendix A the explicit basis of operators for the A' irrep with $N = 5$. The projection of these operators onto irreps of S_N gives the following basis operators.

The MS projection.

$$\mathcal{O}^{MS}_k = \sum_{j=1}^{N} (\mathcal{O}^a_{1j} - \mathcal{O}^a_{kj}), \qquad k = 2, 3, \cdots, N \tag{28}$$

The A' projection.

These operators can be labeled by the two integers j, k appearing in the first column of the standard Young tableaux $[1\ldots][j][k]$ corresponding to the given operator. They can be chosen as

$$\mathcal{O}^{A'}_{jk} = 2\mathcal{O}^a_{1j} + 2\mathcal{O}^a_{jk} - 2\mathcal{O}^a_{1k}, \qquad 2 \le j < k \le N. \tag{29}$$

The decomposition of the interaction $V^{\rm anti} = \sum_{i<j} \mathcal{R}^a_{ij}\mathcal{O}^a_{ij}$ consisting of products of antisymmetric two-body interactions, into operators transforming as irreps of $S^{\rm orb}_N \times S^{\rm sf}_N$, has a similar form to Eq. (26),

$$V^{\rm anti} = \frac{N-1}{N^2} \sum_{j,k=2}^{N} \mathcal{R}^{MS}_j \mathcal{O}^{MS}_k M_{jk} + c^{anti}_E \sum_{j,k=1}^{(1/2)(N-1)(N-2)} \mathcal{R}^{A'}_j \mathcal{O}^{A'}_k Q_{jk}, \tag{30}$$

where M_{jk} is given in Eq. (11), and Q_{jk} is the Clebsch-Gordan coefficient for the reduction $A' \times A' \to S$.

C. Mass operator—One-gluon exchange interaction

The discussion of the interaction Hamiltonian in the previous section was completely general, and did not assume anything about the hadronic states. In this section we consider its matrix elements on the $|B\rangle$ states constructed in Sec. III A, with special regard to their spin-flavor structure. The permutation symmetry reduces the matrix elements of the orbital operators $\mathcal{R}_{ij}$ to a small number of unknown reduced matrix elements, which can be expressed as overlap integrals. Counting the number of the contributing irreps gives that there are three undetermined reduced matrix elements for each symmetric two-body operator, and two reduced matrix elements for each antisymmetric two-body operator.

We compute the matrix elements of the two-body interaction Hamiltonian on the $|B\rangle$ states constructed in Eq. (10). The matrix elements of the spin-flavor operators on the basis functions ϕ_i are given by

$$\langle \phi_i | \mathcal{O}^S | \phi_j \rangle = \langle \mathcal{O}^S \rangle (1 + \delta_{ij}), \tag{31}$$

$$\langle \phi_i | \mathcal{O}^{MS}_k | \phi_j \rangle = \langle \mathcal{O}^{MS} \rangle (1 - \delta_{ik}\delta_{ij}), \tag{32}$$

$$\langle \phi_i | \mathcal{O}^E_{klm} | \phi_j \rangle = \langle \mathcal{O}^E \rangle \frac{1}{2} [(-\delta_{ik} + \delta_{im})(1 + \delta_{jl}) + (-\delta_{jk} + \delta_{jm})(1 + \delta_{il})], \tag{33}$$

where $\langle \mathcal{O}^S \rangle, \langle \mathcal{O}^{MS} \rangle, \langle \mathcal{O}^E \rangle$ are reduced matrix elements. The proportionality of the matrix elements to just one reduced matrix element follows from the Wigner-Eckart theorem for the S_N group. The form of the Clebsch-Gordan coefficients is specific to the MS basis used in this paper, and can be derived by repeated application of Eqs. (6) to the states and operators.

For the orbital operators $\mathcal{R}$, an additional complexity is introduced by the presence of the magnetic quantum numbers of the initial and final state orbital basis functions χ^m_p. The dependence on m, m' is given by the Lorentz structure of the orbital operator. The simplest case corresponds to a Lorentz scalar, for which the matrix elements are given by

$$\langle \chi^{m'}_i | \mathcal{R}^S | \chi^m_j \rangle = \langle \mathcal{R}^S \rangle (1 + \delta_{ij}) \delta_{mm'}, \tag{34}$$

$$\langle \chi^{m'}_i | \mathcal{R}^{MS}_k | \chi^m_j \rangle = \langle \mathcal{R}^{MS} \rangle (1 - \delta_{ik}\delta_{ij}) \delta_{mm'}, \tag{35}$$

$$\langle \chi^{m'}_i | \mathcal{R}^E_{kln} | \chi^m_j \rangle = \langle \mathcal{R}^E \rangle \frac{1}{2} [(-\delta_{ik} + \delta_{in})(1 + \delta_{jl}) + (-\delta_{jk} + \delta_{jn})(1 + \delta_{il})] \delta_{mm'}. \tag{36}$$

Inserting these expressions into Eq. (26) and combining all factors we find the following result for the matrix element of a symmetric two-body operator $V^{\rm symm}$, expressed as a sum over irreps of $S^{\rm orb}_N \times S^{\rm sp\text{-}fl}_N$:

$$\langle V^{\rm symm} \rangle \equiv \frac{\langle B | V^{\rm symm} | B \rangle}{\langle B | B \rangle} = \frac{2}{N(N-1)} \langle \mathcal{R}^S \rangle \langle \mathcal{O}^S \rangle + \frac{1}{N} \langle \mathcal{R}^{MS} \rangle \langle \mathcal{O}^{MS} \rangle + \frac{N(N-3)}{4(N-1)} \langle \mathcal{R}^E \rangle \langle \mathcal{O}^E \rangle. \tag{37}$$

The contribution of the E operators in the last term is obtained using an alternative method, sketched in Appendix C, which does not require the knowledge of the Clebsch-Gordan coefficients N_{jk}.

The reduced matrix elements depend on the precise form of the interaction. We consider for definiteness the spin-spin interaction $V_{\rm ss}$ in some detail. For this case the reduced matrix elements of the symmetric operators are given by

$$\langle \mathcal{R}^S \rangle = \frac{1}{2}(N-1)(N-2) I_s + (N-1) I_{\rm dir} - I_{\rm exc}, \tag{38}$$

$$\langle \mathcal{O}^S \rangle = \langle \Phi(SI) | \frac{1}{2} \vec{S}^2 - \frac{3}{8} N | \Phi(SI) \rangle. \tag{39}$$

For ease of comparison with the literature on the $1/N_c$ expansion for excited baryons, we expressed the reduced matrix element of the spin-flavor operator as a matrix element on the state $|\Phi(SI)\rangle$ where the excited quark is quark no. 1. This state is given by $\Phi(SI) = \frac{1}{\sqrt{N(N-1)}} \times \sum_{k=2}^{N} \phi_k$. The reduced matrix elements of the orbital operator are expressed in terms of the three overlap integrals over the one-body wave functions

$$I_s = \int d\vec{r}_1 d\vec{r}_2 f_0(r_{12}) |\varphi_s(\vec{r}_1)|^2 |\varphi_s(\vec{r}_2)|^2,$$
$$I_{\rm dir} = \int d\vec{r}_1 d\vec{r}_2 f_0(r_{12}) |\varphi_s(\vec{r}_1)|^2 |\varphi_p^m(\vec{r}_2)|^2,$$
$$I_{\rm exc} = \int d\vec{r}_1 d\vec{r}_2 f_0(r_{12}) \varphi_s^*(\vec{r}_1) \varphi_p^{m*}(\vec{r}_2) \varphi_p^m(\vec{r}_1) \varphi_s(\vec{r}_2). \tag{40}$$

The spin-flavor operator transforming in the MS irrep is $\mathcal{O}_k^{MS} = (\vec{s}_1 - \vec{s}_k) \cdot \vec{S}$, and the corresponding orbital operator is $\mathcal{R}_k^{MS} = \sum_{j=2, j\neq k}^N [f_0(r_{1j}) - f_0(r_{kj})]$. Their reduced matrix elements are

$$\langle \mathcal{R}^{MS} \rangle = (N-2)(I_{\rm dir} - I_s) - 2 I_{\rm exc}, \tag{41}$$

$$\langle \mathcal{O}^{MS} \rangle = \frac{1}{N-2} \langle \Phi(SI) | -\vec{S}^2 + N \vec{s}_1 \cdot \vec{S}_c + \frac{3}{4} N | \Phi(SI) \rangle, \tag{42}$$

where the overlap integrals are the same as defined in Eqs. (40). We denoted $\vec{S}_c$ the "core" spin, defined as $\vec{S}_c = \vec{S} - \vec{s}_1$. The reduced matrix element of the orbital operator $\langle \mathcal{R}^{MS} \rangle$ is computed by taking representative values of i, j, k in Eq. (35) and evaluating the integrals. The derivation of the spin-flavor reduced matrix element $\langle \mathcal{O}^{MS} \rangle$ is given in Appendix D. We postpone the discussion of the E operators for a later section, as they require a separate treatment.

The quadrupole interaction is treated in a similar way. This is written as

$$V_q = \frac{1}{2} \sum_{i<j} f_2(r_{ij}) (3 \hat{r}_{ij}^a \hat{r}_{ij}^b - \delta^{ab}) (s_i^a s_j^b + s_i^b s_j^a) \equiv \sum_{i<j} \mathcal{Q}_{ij}^{ab} \mathcal{O}_{ij}^{ab}, \tag{43}$$

where $\mathcal{Q}_{ij}^{ab}$ acts only on the orbital degrees of freedom, $\mathcal{O}_{ij}^{ab} = s_i^a s_j^b + s_i^b s_j^a$ on spin-flavor, and a, b are spatial indices. As mentioned, this is a symmetric two-body operator. The projections onto S and MS of each of the factors are obtained as explained above. The symmetric projections are

$$\mathcal{Q}_S^{ab} = \sum_{i<j} \mathcal{Q}_{ij}^{ab} = \frac{1}{2} \sum_{i<j} f_2(r_{ij}) (3 \hat{r}_{ij}^a \hat{r}_{ij}^b - \delta^{ab}), \tag{44}$$

$$\mathcal{O}_S^{ab} = \sum_{i<j} \mathcal{O}_{ij}^{ab} = \frac{1}{2} \left(\{S^a, S^b\} - \frac{1}{2} N \delta^{ab} \right), \tag{45}$$

where the braces denote symmetrization with respect to the spatial indices a, b, $\{X^a, Y^b\} = X^a Y^b + X^b Y^a$. Their matrix elements between the MS basis for orbital and spin-flavor states are given by the Wigner-Eckart theorem for the permutation group

$$\langle \chi_i^{m'} | \mathcal{Q}_S^{ab} | \chi_j^m \rangle = S_{ij} \left(\frac{1}{2} \{L^a, L^b\} - \frac{1}{3} L(L+1) \delta^{ab} \right)_{m'm} \langle \mathcal{Q}_S \rangle, \tag{46}$$

$$\langle \phi_i | \mathcal{O}_S^{ab} | \phi_j \rangle = S_{ij} \langle \mathcal{O}_S^{ab} \rangle. \tag{47}$$

The dependence on the magnetic quantum numbers m, m' in the orbital matrix element is given by the most general symmetric and traceless tensor which can be formed from the angular momentum. We will denote it as $L_2^{ab} = \frac{1}{2} \times \{L^a, L^b\} - \frac{1}{3} L(L+1) \delta^{ab}$.

The reduced matrix elements are given by

$$\langle \mathcal{Q}_S \rangle = (N-1) \mathcal{K}_{\rm dir} - \mathcal{K}_{\rm exc}, \tag{48}$$

$$\langle \mathcal{O}_S^{ab} \rangle = \frac{1}{2} \langle \Phi(SI) | \{S^a, S^b\} - \frac{1}{2} N \delta^{ab} | \Phi(SI) \rangle. \tag{49}$$

The convolution integrals $\mathcal{K}_{\rm dir,exc}$ appearing in the reduced matrix element of the orbital operator $\mathcal{Q}_S$ are similar to those introduced for the spin-spin interaction.

$$\mathcal{K}_{\rm dir}(L_2^{ab})_{m'm} \equiv \frac{1}{2} \int d\vec{r}_1 d\vec{r}_2 f_2(r_{12}) (3 \hat{r}_{12}^a \hat{r}_{12}^b - \delta^{ab}) \times |\varphi_s(\vec{r}_1)|^2 \varphi_p^{m'*}(\vec{r}_2) \varphi_p^m(\vec{r}_2), \tag{50}$$

$$\mathcal{K}_{\rm exc}(L_2^{ab})_{m'm} \equiv \frac{1}{2} \int d\vec{r}_1 d\vec{r}_2 f_2(r_{12}) (3 \hat{r}_{12}^a \hat{r}_{12}^b - \delta^{ab}) \times \varphi_s^*(\vec{r}_1) \varphi_p^{m'*}(\vec{r}_2) \varphi_p^m(\vec{r}_1) \varphi_s(\vec{r}_2). \tag{51}$$

The MS projections of the operators are

$$[\mathcal{Q}^{ab}]_k^{MS} = \frac{1}{2} \sum_{j=2\neq k}^N f_2(r_{1j}) (3 \hat{r}_{1j}^a \hat{r}_{1j}^b - \delta^{ab}) - \frac{1}{2} \sum_{j=2\neq k}^N f_2(r_{kj}) (3 \hat{r}_{kj}^a \hat{r}_{kj}^b - \delta^{ab}), \tag{52}$$

$$[\mathcal{O}^{ab}]_k^{MS} = \{s_1^a - s_k^a, S^b\}. \tag{53}$$

Their reduced matrix elements are given by

$$\langle \mathcal{Q}_{MS} \rangle = (N-2) \mathcal{K}_{\rm dir} - 2 \mathcal{K}_{\rm exc}, \tag{54}$$

$$\langle \mathcal{O}_{MS}^{ab} \rangle = \frac{1}{N-2} \langle \Phi(SI) | N \{S_c^a, s_1^b\} - \{S^a, S^b\} + \frac{1}{2} N \delta^{ab} | \Phi(SI) \rangle. \tag{55}$$

Finally we discuss the computation of the matrix element of the spin-orbit interaction $V_{\rm so}$, which is more involved. As mentioned, this operator can be written as a sum of symmetric and antisymmetric two-body operators

$$V_{\rm so} = V_{\rm so}^{\rm symm} + V_{\rm so}^{\rm anti} = \frac{3}{2}\sum_{i<j}\vec{\mathcal{L}}^s_{ij}\cdot(\vec{s}_i+\vec{s}_j) - \frac{1}{2}\sum_{i<j}\vec{\mathcal{L}}^a_{ij}\cdot(\vec{s}_i-\vec{s}_j), \quad (56)$$

where we defined the symmetric and antisymmetric orbital operators

$$\vec{\mathcal{L}}^s_{ij} = f_1(r_{ij})\vec{r}_{ij}\times(\vec{p}_i-\vec{p}_j), \qquad \vec{\mathcal{L}}^a_{ij} = f_1(r_{ij})\vec{r}_{ij}\times(\vec{p}_i+\vec{p}_j). \quad (57)$$

They are decomposed as $S\oplus MS\oplus E$ and $MS\oplus A'$, respectively, and thus their matrix elements taken between MS states of S_N acting on spin-flavor depend on five reduced matrix elements. At $N_c=3$ the E operators do not exist, and thus this number reduces to four, which agrees with the counting of Ref. [14]. Table I summarizes the counting of the independent reduced matrix elements arising from each of the terms in the quark interaction Hamiltonian.

Consider first the symmetric piece of the spin-orbit interaction $V_{\rm so}^{\rm symm}$. The projections of the orbital and spin-flavor factors onto S and MS irreps of the S_N group are: the symmetric S components

$$[\vec{\mathcal{L}}^s]_S = \sum_{i<j}\vec{\mathcal{L}}^s_{ij}, \qquad [\vec{s}_i+\vec{s}_j]_S = (N-1)\vec{S} \quad (58)$$

and the MS components

$$[\vec{\mathcal{L}}^s]^{MS}_k = \sum_{j=2\neq k}^{N}\vec{\mathcal{L}}^s_{1j} - \sum_{j=2\neq k}^{N}\vec{\mathcal{L}}^s_{kj}, \qquad [\vec{s}_i+\vec{s}_j]^{MS}_k = (N-2)(\vec{s}_1-\vec{s}_k). \quad (59)$$

The matrix element of the $V_{\rm so}^{\rm symm}$ operator is given by the same relation as Eq. (37) except that the reduced matrix elements are vectors. Taking into account that the only available vector is the angular momentum $\vec{L}$, the most general form for the matrix elements of the orbital operators $\vec{\mathcal{L}}^s_{ij}$ are given as

TABLE I. Summary of the spin-flavor contributions to the mass operator of the $L=1$ baryons arising from each of the three terms in the interaction Hamiltonian (for both gluon-exchange and Goldstone-boson exchange interactions). The table counts the independent structures, corresponding to each irrep of S_N. The dashes represent terms forbidden by S_N, and the 0's show operators absent for other reasons. The E operators contribute only for $N_c>3$.

	Symmetric two-body			Antisymmetric two-body	
	S	MS	E	MS	A′
$V_{\rm ss}$	1	1	1	-	-
V_q	1	1	1	-	-
$V_{\rm so}$	1	1	0	1	0

$$\langle\chi_i^{m'}|[\vec{\mathcal{L}}^s]_S|\chi_j^m\rangle = S_{ij}(\vec{L})_{m'm}\langle\mathcal{L}^s_S\rangle, \quad (60)$$

$$\langle\chi_i^{m'}|[\vec{\mathcal{L}}^s]^{MS}_k|\chi_j^m\rangle = (1-\delta_{ik}\delta_{jk})(\vec{L})_{m'm}\langle\mathcal{L}^s_{MS}\rangle. \quad (61)$$

Collecting together all factors, the matrix element of the $V_{\rm so}^{\rm symm}$ interaction is given by (we neglect here the contribution of the $E\otimes E$ operators to the matrix element, which does not contribute for $N_c=3$)

$$\frac{\langle B|V_{\rm so}^{\rm symm}|B\rangle}{\langle B|B\rangle} = \frac{2}{N}\vec{L}\cdot\vec{S}\langle\mathcal{L}^s_S\rangle + \frac{1}{N}(N\vec{s}_1\cdot\vec{L}-\vec{S}\cdot\vec{L})\langle\mathcal{L}^s_{MS}\rangle. \quad (62)$$

As usual, the expression on the right-hand side is understood as an operator acting on the CCGL-type spin-flavor states $\Phi(SI)$, for which the excited quark is quark no. 1.

The scalar coefficients appearing as reduced matrix elements of the S and MS orbital operators are given by overlap integrals over the two-body interaction

$$\langle\mathcal{L}^s_S\rangle = (N-1)\mathcal{J}^s_{\rm dir} - \mathcal{J}^s_{\rm exc}, \qquad \langle\mathcal{L}^s_{MS}\rangle = (N-2)\mathcal{J}^s_{\rm dir} - 2\mathcal{J}^s_{\rm exc}, \quad (63)$$

where the coefficients $\mathcal{J}^s_{\rm dir,exc}$ are defined by the integrals

$$(\vec{L})_{m'm}\mathcal{J}^s_{\rm dir} = \int d\vec{r}_1 d\vec{r}_2 f_1(r_{12})\vec{r}_{12}\times(\vec{p}_1-\vec{p}_2) \times|\varphi_s(\vec{r}_1)|^2\varphi_p^{m'*}(\vec{r}_2)\varphi_p^m(\vec{r}_2), \quad (64)$$

$$(\vec{L})_{m'm}\mathcal{J}^s_{\rm exc} = \int d\vec{r}_1 d\vec{r}_2 f_1(r_{12})\vec{r}_{12}\times(\vec{p}_1-\vec{p}_2)\varphi_s^*(\vec{r}_1) \times\varphi_p^{m'*}(\vec{r}_2)\varphi_p^m(\vec{r}_1)\varphi_s(\vec{r}_2). \quad (65)$$

Their scaling is $\mathcal{J}_i\sim O(N_c^0)$ since they depend only on the wave functions of the pair of interacting quarks, but not on the total number of quarks.

The reduced matrix elements $\langle\mathcal{L}^s_S\rangle$, $\langle\mathcal{L}^s_{MS}\rangle$ scale like $O(N_c)$. Using this in Eq. (62), one can see that the contribution of the first term to the mass operator scales like $O(\alpha_s N_c^0)\sim O(N_c^{-1})$, while the second term gives a contribution of leading order $O(\alpha_s N_c)\sim O(N_c^0)$ proportional to $\vec{s}_1\cdot\vec{L}$, together with another power-suppressed contribution.

Finally, we present also the contribution of the antisymmetric piece of the spin-orbit interaction to the mass operator. According to Eq. (30), this is decomposed into $MS\otimes MS + A'\otimes A'$ under $S_N^{\rm orb}\times S_N^{\rm sf}$.

We start by computing the $MS\otimes MS$ term, which contributes

$$\langle V_{\rm so}^{\rm anti}\rangle \equiv \frac{\langle B|V_{\rm so}^{\rm anti}|B\rangle}{\langle B|B\rangle} = \frac{N-2}{N^2}\langle\vec{\mathcal{L}}^a_{MS}\rangle\langle\vec{\mathcal{O}}^a_{MS}\rangle. \quad (66)$$

The spin-flavor reduced matrix element can be obtained as explained in Appendix D, with the result

$$\langle \vec{\mathcal{O}}^a_{MS} \rangle = \frac{N}{N-2} \langle \Phi(SI) | N\vec{s}_1 - \vec{S} | \Phi(SI) \rangle, \tag{67}$$

and the orbital reduced matrix element is

$$\langle \vec{\mathcal{L}}^a_{MS} \rangle = N \mathcal{J}^a_{\rm dir} (\vec{L})_{m'm}. \tag{68}$$

The overlap integral is given explicitly by

$$(\vec{L})_{m'm} \mathcal{J}^a_{\rm dir} = \int d\vec{r}_1 d\vec{r}_2 \vec{\mathcal{L}}^a_{12}(r_{12}) |\varphi_s(\vec{r}_1)|^2 \varphi^{m'*}_p(\vec{r}_2) \varphi^m_p(\vec{r}_2). \tag{69}$$

This gives the total result for the $MS \times MS$ projection of the antisymmetric piece of the spin-orbit interaction

$$\langle V^{\rm anti}_{\rm so} \rangle = \mathcal{J}^a_{\rm dir} (N\vec{s}_1 \cdot \vec{L} - \vec{S} \cdot \vec{L}). \tag{70}$$

Finally, we consider also the A' projection of the antisymmetric spin-orbit interaction. According to Eq. (29), the A' projection of the spin-flavor operator $\mathcal{O}^a_{ij} = \vec{s}_i - \vec{s}_j$ vanishes

$$\mathcal{O}^{A'}_{jk} = 2(\vec{s}_1 - \vec{s}_j) + 2(\vec{s}_j - \vec{s}_k) - 2(\vec{s}_1 - \vec{s}_k) = 0, \tag{71}$$

such that the A' operators do not contribute. In addition, for $N = 3$ the orbital A' contribution vanishes because of T-reversal invariance. We give the detailed argument in the following, since it is independent on the spin-flavor structure of the operator.

The A' projection of $\vec{\mathcal{L}}^a_{ij}$ is given by Eq. (29) and can be taken as $[\vec{\mathcal{L}}]^{A'} = \mathcal{L}^a_{12} + \mathcal{L}^a_{23} - \mathcal{L}^a_{13}$. On the two-dimensional space of the MS orbital states χ^m_i, its matrix elements have the form

$$\langle \chi^{m'}_i | [\vec{\mathcal{L}}^a_{ij}]^{A'} | \chi^m_j \rangle = \begin{pmatrix} 0 & -3\mathcal{J}^a_{\rm exc} \\ 3\mathcal{J}^a_{\rm exc} & 0 \end{pmatrix}_{ij} (\vec{L})_{m'm}, \tag{72}$$

where the exchange integral is defined as

$$(\vec{L})_{m'm} \mathcal{J}^a_{\rm exc} = \int d\vec{r}_1 d\vec{r}_2 \vec{\mathcal{L}}^a_{12}(r_{12}) \varphi^*_s(\vec{r}_1) \varphi^{m'*}_p(\vec{r}_2) \varphi^m_p(\vec{r}_1) \varphi_s(\vec{r}_2). \tag{73}$$

Hermiticity of the $\vec{\mathcal{L}}^a_{12}$ operator implies that the overlap integral is purely imaginary, $\mathrm{Re}\mathcal{J}^a_{\rm exc} = 0$. However, T-reversal invariance of the Hamiltonian forbids imaginary terms in the mass operator, such that the overlap integral must vanish. For $N > 3$, a second overlap integral can contribute, which can be nonvanishing.

In summary, the mass operator of the orbitally excited baryons in the nonrelativistic quark model with the gluon-exchange potential Eq. (17) is given by

$$M = c_0 \mathbf{1} - g_s^2 \frac{N_c + 1}{2N_c} (a\vec{S}^2 + b\vec{s}_1 \cdot \vec{S}_c + cL^{ab}_2 S^a S^b + dL^{ab}_2 S^a_c s^b_1 + e\vec{L} \cdot \vec{S} + f\vec{L} \cdot \vec{s}_1) + \cdots. \tag{74}$$

The spin-flavor operators are understood to act on the CCGL-type state $\Phi(SI)$, for which the excited quark is quark no. 1. The ellipses denote operators transforming in the E irrep, which appear only for $N > 3$, and are considered in the next section.

The coefficients $a - f$ are given by linear combinations of the orbital overlap integrals introduced above

$$a = \frac{1}{N(N-1)} \left[\frac{(N-1)(N-2)}{2} \mathcal{I}_s + (N-1)\mathcal{I}_{\rm dir} - \mathcal{I}_{\rm exc} \right] - \frac{1}{N(N-2)} [(N-2)(\mathcal{I}_{\rm dir} - \mathcal{I}_s) - 2\mathcal{I}_{\rm exc}], \tag{75}$$

$$b = \frac{1}{N-2} [(N-2)(\mathcal{I}_{\rm dir} - \mathcal{I}_s) - 2\mathcal{I}_{\rm exc}], \tag{76}$$

$$c = \frac{2}{N(N-1)} [(N-1)\mathcal{K}_{\rm dir} - \mathcal{K}_{\rm exc}] - \frac{2}{N(N-2)} [(N-2)\mathcal{K}_{\rm dir} - 2\mathcal{K}_{\rm exc}], \tag{77}$$

$$d = \frac{2}{N-2} [(N-2)\mathcal{K}_{\rm dir} - 2\mathcal{K}_{\rm exc}], \tag{78}$$

$$e = \frac{3}{N} [(N-1)\mathcal{J}^s_{\rm dir} - \mathcal{J}^s_{\rm exc}] - \frac{3}{2N} [(N-2)\mathcal{J}^s_{\rm dir} - 2\mathcal{J}^s_{\rm exc}] + \frac{1}{2} \mathcal{J}^a_{\rm dir}, \tag{79}$$

$$f = \frac{3}{2} [(N-2)\mathcal{J}^s_{\rm dir} - 2\mathcal{J}^s_{\rm exc}] - \frac{N}{2} \mathcal{J}^a_{\rm dir}. \tag{80}$$

The matrix elements of the operators on quark model states can be found in the Appendix of Ref. [7] for any values of L and N_c.

The expression Eq. (74) holds regardless of the orbital wave functions, and summarizes in a compact operator form the most general result for the masses of the orbitally excited states with MS spin-flavor symmetry, in the presence of one-gluon exchange quark-quark interactions. Although we used in deriving these results the Hartree approximation, they have a more general validity. The Hartree approximation is an useful device for computing the leading N_c dependence of the orbital-reduced matrix elements, but is not essential for the arguments leading to the spin-flavor structure of the mass operator. Expressing the latter directly in terms of the orbital-reduced matrix elements will produce an expression similar to Eq. (74), for which the N dependence is implicit in the reduced matrix elements of the orbital operators.

D. Goldstone-boson exchange interaction

We consider in this section the mass operator of the orbitally excited baryons in a second model for the quark-quark interaction. In Ref. [16] it was suggested that pion-exchange mediated quark-quark interactions can reproduce better the observed mass spectrum of these

states. The physical idea is that at the energy scales of quarks inside a hadron, the appropriate degrees of freedom are quarks, gluons and the Goldstone bosons of the broken chiral group $SU(2)_L \times SU(2)_R \to SU(2)$ [17]. The exchange of Goldstone bosons changes the short distance form of the quark-quark interactions, and introduces a different spin-flavor structure.

The interaction Hamiltonian of this model has the form

$$H = H_0 + \frac{g_A^2}{f_\pi^2} \sum_{i<j} \tilde{V}_{ij} \tag{81}$$

and has the same spin structure as the one-gluon interaction, but with additional explicit isospin dependence. g_A is a quark-pion coupling which scales like $O(N_c^0)$ with the number of colors N_c, and $f_\pi^2 \sim O(N_c)$. The two-body interactions $\tilde{V}_{ij}$ include the following contributions: spin-spin interaction $\tilde{V}_{\rm ss}$, a tensor interaction $\tilde{V}_q$, and the spin-orbit interaction $\tilde{V}_{\rm so}$. We write these interaction terms as [14],

$$\tilde{V}_{\rm ss} = \sum_{i<j=1}^{N} g_0(r_{ij}) \vec{s}_i \cdot \vec{s}_j t_i^a t_j^a, \tag{82}$$

$$\tilde{V}_q = \sum_{i<j=1}^{N} g_2(r_{ij}) [3(\hat{r}_{ij} \cdot \vec{s}_i)(\hat{r}_{ij} \cdot \vec{s}_j) - (\vec{s}_i \cdot \vec{s}_j)] t_i^a t_j^a, \tag{83}$$

$$\tilde{V}_{\rm so} = \sum_{i<j=1}^{N} g_1(r_{ij}) [(\vec{r}_{ij} \times \vec{p}_i) \cdot \vec{s}_i - (\vec{r}_{ij} \times \vec{p}_j) \cdot \vec{s}_j + 2(\vec{r}_{ij} \times \vec{p}_i) \cdot \vec{s}_j - 2(\vec{r}_{ij} \times \vec{p}_j) \cdot \vec{s}_i] t_i^a t_j^a, \tag{84}$$

where the isospin generators are $t^a = \frac{1}{2}\tau^a$, and $g_i(r_{ij})$ are unspecified functions.

The mass operator of the orbitally excited baryons with this interaction potential can be computed using the approach presented for the one-gluon exchange interaction. The orbital matrix elements have the same form (although with different overlap integrals), while the spin-flavor operators are different.

We give in some detail the projection of the spin-spin interaction $\tilde{V}_{ij} = g_0(r_{ij}) \vec{s}_i \cdot \vec{s}_j t_i^a t_j^a$ onto operators transforming as irreps of $S_N^{\rm sf}$. The symmetric projection of the spin-flavor operator $\mathcal{O}_{ij} = \vec{s}_i \cdot \vec{s}_j t_i^a t_j^a$ is

$$\mathcal{O}_S = \sum_{i<j} \mathcal{O}_{ij} = \frac{1}{2}\left(G^{ka} G^{ka} - \frac{9}{16} N\right), \tag{85}$$

where we denoted $G^{ka} = \sum_{i=1}^{N} s_i^k t_i^a$. The G^2 operator can be expressed in terms of the $SU(4)$ Casimir using the identity (for $F = 2$ light quark flavors)

$$G^{ka} G^{ka} = \frac{1}{16} N(3N + 4) - \frac{1}{4} \vec{S}^2 - \frac{1}{4} T^a T^a. \tag{86}$$

The reduced matrix element of the MS projection can be found as explained in Appendix D, and is given by

$$\langle \mathcal{O}_{MS} \rangle = \frac{1}{N-2}\left(N g_1^{ka} G_c^{ka} - G^{ka} G^{ka} + \frac{9}{16} N\right). \tag{87}$$

The $g_1 G_c$ terms can be reduced to simpler operators using the reduction rule Eq. (4.5) in Ref. [7], which for $F = 2$ gives

$$g_1^{ka} G_c^{ka} = -\frac{1}{16}(N + 3) - \frac{1}{4} \vec{s}_1 \cdot \vec{S}_c - \frac{1}{4} t_1^a T_c^a. \tag{88}$$

Finally, the operators have to be expressed as sums of terms acting on the core and excited quark.

Collecting the contributions of all terms in the Hamiltonian we find the following general result for the mass operator in the Goldstone-boson exchange model:

$$M = c_0 \mathbf{1} + \frac{g_A^2}{f_\pi^2}(a \vec{S}_c^2 + b \vec{s}_1 \cdot \vec{S}_c + c t_1^a T_c^a + d L_2^{ij} g_1^{ia} G_c^{ja} + e L_2^{ij} \{S_c^i, S_c^j\} + f L^i S_c^i + g L^i t_1^a G_c^{ia} + h L^i g_1^{ia} T_c^a). \tag{89}$$

The coefficients $a - h$ are given by linear combinations of the orbital overlap integrals

$$a = -\frac{1}{2N(N-1)}\left[\frac{1}{2}(N-1)(N-2)\mathcal{I}_s + (N-1)\mathcal{I}_{\rm dir} - \mathcal{I}_{\rm exc}\right] + \frac{1}{2N(N-2)}[(N-2)(\mathcal{I}_{\rm dir} - \mathcal{I}_s) - 2\mathcal{I}_{\rm exc}], \tag{90}$$

$$b = c = -\frac{1}{2N(N-1)}\left[\frac{1}{2}(N-1)(N-2)\mathcal{I}_s + (N-1)\mathcal{I}_{\rm dir} - \mathcal{I}_{\rm exc}\right] - \frac{1}{4N}[(N-2)(\mathcal{I}_{\rm dir} - \mathcal{I}_s) - 2\mathcal{I}_{\rm exc}], \tag{91}$$

$$d = \frac{4}{N(N-1)}[(N-1)\mathcal{K}_{\rm dir} - \mathcal{K}_{\rm exc}] + \frac{2}{N}[(N-2)\mathcal{K}_{\rm dir} - 2\mathcal{K}_{\rm exc}], \tag{92}$$

$$e = \frac{1}{4N(N-1)}[(N-1)\mathcal{K}_{\rm dir} - \mathcal{K}_{\rm exc}] - \frac{1}{4N(N-2)}[(N-2)\mathcal{K}_{\rm dir} - 2\mathcal{K}_{\rm exc}], \tag{93}$$

$$f = \frac{3(N-2)}{4N(N-1)}[(N-1)\mathcal{J}_{\rm dir}^s - \mathcal{J}_{\rm exc}^s] - \frac{3}{4N}[(N-2)\mathcal{J}_{\rm dir}^s - 2\mathcal{J}_{\rm exc}^s], \tag{94}$$

$$g = \frac{3}{N(N-1)}[(N-1)\mathcal{J}^s_{\rm dir} - \mathcal{J}^s_{\rm exc}] + \frac{3}{2N}[(N-2)\mathcal{J}^s_{\rm dir} - 2\mathcal{J}^s_{\rm exc}] + \frac{1}{2}\mathcal{J}^a_{\rm dir}, \quad (95)$$

$$h = \frac{3}{N(N-1)}[(N-1)\mathcal{J}^s_{\rm dir} - \mathcal{J}^s_{\rm exc}] + \frac{3}{2N}[(N-2)\mathcal{J}^s_{\rm dir} - 2\mathcal{J}^s_{\rm exc}] - \frac{1}{2}\mathcal{J}^a_{\rm dir}. \quad (96)$$

Comparing with the mass operator formula for the one-gluon exchange interaction Eq. (74), we find that more effective operators are present for this case (eight vs six). Note that two of the coefficients are equal ($b = c$). The total number of unknown constants is the same in both cases and is given by the seven reduced matrix elements of the orbital operators. An additional simplification occurs for the one-gluon interaction case, where the MS projections of the symmetric and antisymmetric spin-orbit interaction give the same operator (compare the second term in Eq. (62) and (70)). This reduces the number of independent spin-flavor structures for this case from seven to six.

E. The *E* operators

In addition to the operators considered so far, transforming in the S, MS, and A' irreps, there are additional operators transforming in the E irrep of S_N which appear only for $N_c > 3$. They are introduced only by the spin-spin $V_{\rm ss}$ and tensor V_q interactions in the Hamiltonian. In this section we study their spin-flavor structure.

We start by considering the spin-spin interaction $\mathcal{O}_{ij} = \vec{s}_i \cdot \vec{s}_j$. The E projection of this operator is given by

$$\mathcal{O}^E_{abc} = (\vec{s}_1 - \vec{s}_b) \cdot (\vec{s}_a - \vec{s}_c), \quad (97)$$

where a, b, c are the integers appearing in the standard Young tableaux in the order $[1c\cdots][ab]$. It turns out that it is impossible to express the reduced matrix element of E operators as matrix elements on the CCGL-type state $\Phi(SI)$. The reason is that the diagonal matrix elements of the $\mathcal{O}^E_{abc}$ operators on the state $\Phi(SI)$ vanish. This can be seen by symmetrizing the operator $\mathcal{O}^E_{abc}$ under permutations of the $N-1$ quarks different from quark 1, which gives a vanishing result.[1]

On the other hand, the off-diagonal matrix elements of $\mathcal{O}^E_{abc}$ between $CCGL$-type states with different excited quarks, e.g., quarks 1 and 2, are nonvanishing. We will denote them as $\Phi_1(SI) \equiv \Phi(SI)$ and $\Phi_2(SI)$; they are expressed in terms of the basis states ϕ_i as

$$|\Phi_1(SI)\rangle = \frac{1}{\sqrt{N(N-1)}}(\phi_2 + \phi_3 + \cdots + \phi_N),$$
$$|\Phi_2(SI)\rangle = \frac{1}{\sqrt{N(N-1)}}(-(N-1)\phi_2 + \phi_3 + \cdots + \phi_N). \quad (98)$$

Consider for definiteness the matrix element of $\mathcal{O}^E_{234} = (\vec{s}_1 - \vec{s}_3) \cdot (\vec{s}_2 - \vec{s}_4)$, which can be written using the Wigner-Eckart theorem as

$$\langle \Phi_1(SI)|\mathcal{O}^E_{234}|\Phi_2(SI)\rangle = \frac{N}{2(N-1)}\langle \mathcal{O}_E\rangle \quad (99)$$

where the Clebsch coefficient has been computed using Eq. (33). This shows that the reduced matrix element of the E operators $\langle \mathcal{O}_E\rangle$ can be expressed as the off-diagonal matrix elements between $CCGL$-type states with different excited quarks.

The operator on the left-hand side of Eq. (99) can be put in a simpler form by noting that both states $\Phi_{1,2}(SI)$ are symmetric under any permutation of the "core" quarks $(3, 4, \cdots, N)$. This core contains all $N-2$ quarks which are different from quarks 1,2. Therefore the only nonvanishing contribution to the matrix element in (99) will come from the component of $\mathcal{O}^E_{234}$ which is symmetric under the core quarks. This component is given as a sum over all permutations Π_c of the core quarks $(3, 4, \cdots, N)$

$$\mathcal{O}^E_{234} \to \frac{1}{(N-2)!}\Sigma_{\Pi_c}\Pi_c\mathcal{O}^E_{234}. \quad (100)$$

The sum over permutations can be computed explicitly for the spin-spin interaction with the result, for arbitrary N,

$$\frac{1}{(N-2)!}\Sigma_{\Pi_c}\Pi_c\mathcal{O}^E_{234} = (\vec{s}_1 \cdot \vec{s}_2) - \frac{1}{N-2}(\vec{s}_1 + \vec{s}_2) \cdot \vec{S}_{cc} + \frac{2}{(N-2)(N-3)}\sum_{i<j=3}^{N} \vec{s}_i \cdot \vec{s}_j \quad (101)$$

where we denoted $S_{cc} = s_3 + s_4 + \cdots + s_N$ the $(N-2)$-quarks core spin. The sum in the last term runs over all quark pairs in the core.

Combining this with Eq. (99) one finds the reduced matrix element of the E operator as an off-diagonal matrix element on CCGL states with different excited quarks

$$\frac{N}{2(N-1)}\langle \mathcal{O}_E\rangle = \langle \Phi_1(SI)|\mathcal{O}^E_{234}|\Phi_2(SI)\rangle = \langle \Phi_1(SI)|(\vec{s}_1 \cdot \vec{s}_2) - \frac{1}{N-2}(\vec{s}_1 + \vec{s}_2) \cdot \vec{S}_{cc} + \frac{1}{(N-2)(N-3)}\left(\vec{S}^2_{cc} - \frac{3}{4}(N-2)\right) \times |\Phi_2(SI)\rangle. \quad (102)$$

[1] In group theoretic language, this is due to the fact that any E operator transforms as $MS_{N-1} + E_{N-1}$ under the S_{N-1} subgroup of S_N which leaves quark 1 unchanged. Since $\Phi_1(SI)$ is a singlet under this subgroup, the matrix elements of E_N on this state must vanish.

This specifies the reduced matrix of the operator $\langle \mathcal{O}_E \rangle$ as an off-diagonal matrix element on CCGL-type states.

The contribution of the E component of the tensor interaction is computed in an analogous way. Combining everything we find the total contribution of the E operators to the mass operator Eq. (74) in the one-gluon exchange model as $-g_s^2 \frac{N_c+1}{2N_c} \delta M_E$ with

$$\begin{aligned}\delta M_E = g\Big((\vec{s}_1 \cdot \vec{s}_2) - \frac{1}{N-2}(\vec{s}_1 + \vec{s}_2)\cdot \vec{S}_{cc} \\ + \frac{1}{(N-2)(N-3)}\Big[S_{cc}^2 - \frac{3}{4}(N-2)\Big]\Big) \\ + hL_2^{ab}\Big(s_1^a s_2^b - \frac{1}{N-2}(s_1^a + s_2^a)S_{cc}^b \\ + \frac{1}{2(N-2)(N-3)}\{S_{cc}^a, S_{cc}^b\}\Big) \end{aligned} \tag{103}$$

with

$$g = (N-3)\mathcal{I}_{\text{exc}}, \qquad h = (N-3)\mathcal{K}_{\text{exc}}. \tag{104}$$

The matrix element of the operators in (103) is understood to be taken between the off-diagonal CCGL-type spin-flavor states $\langle \Phi_1(SI)|$ and $|\Phi_2(SI)\rangle$ states defined as in Eq. (98).

For the purposes of N power counting, it is convenient to express the operators in Eq. (103) as diagonal matrix elements on the $\Phi_1(SI)$ state. This can be done using the relation

$$|\Phi_2(SI)\rangle = P_{12}|\Phi_1(SI)\rangle \tag{105}$$

together with the explicit representation of the transposition operator P_{12} on a system of spin-isospin $1/2$ quarks

$$P_{12} = \frac{1}{4}(1 + 4\vec{s}_1 \cdot \vec{s}_2)(1 + 4t_1^a t_2^a). \tag{106}$$

We illustrate the method of calculation by giving the details of the off-diagonal matrix element of the first term in Eq. (103). This can be written as

$$\begin{aligned}\langle \Phi_1(SI)|\vec{s}_1 \cdot \vec{s}_2|\Phi_2(SI)\rangle &= \langle \Phi_1(SI)|\vec{s}_1 \cdot \vec{s}_2 P_{12}|\Phi_1(SI)\rangle \\ &= \frac{1}{N-1}\langle \Phi_1(SI)|\sum_{k=2}^{N} \vec{s}_1 \cdot \vec{s}_k P_{1k}| \\ &\quad \times \Phi_1(SI)\rangle. \end{aligned} \tag{107}$$

In the last step we used the symmetry of the wave function $\Phi_1(SI)$ under the $S_{N-1}^{\text{sp-fl}}$ subgroup of $S_N^{\text{sp-fl}}$ acting only on the quarks $2, 3, \cdots, N$, and projected the operator $\vec{s}_1 \cdot \vec{s}_2 P_{12}$ onto its symmetric component under this subgroup. The sum over k can be performed in closed form with the result

$$\begin{aligned}\langle \Phi_1(SI)|\vec{s}_1 \cdot \vec{s}_2|\Phi_2(SI)\rangle &= \frac{1}{4(N-1)}\langle \Phi_1(SI)| - \vec{s}_1 \cdot \vec{S}_c \\ &\quad + \frac{3}{4}(N-1) - 4g_1^{ia}G_c^{ia} \\ &\quad + 3t_1^a T_c^a|\Phi_1(SI)\rangle \\ &= \langle \Phi_1(SI)|\frac{1}{N-1}t_1^a T_c^a \\ &\quad + \frac{N}{4(N-1)}|\Phi_1(SI)\rangle, \end{aligned} \tag{108}$$

where we used the reduction rule Eq. (88) in the last step to eliminate $\vec{s}_1 \cdot \vec{S}_c$, $g_1^{ia}G_c^{ia}$ in favor of $t_1^a T_c^a$.

The remaining terms in Eq. (103) can be computed in a similar way. After a lengthy calculation one finds for the off-diagonal matrix element of the first term, arising from the spin-spin interaction,

$$\begin{aligned}&\langle \Phi_1(SI)|(\vec{s}_1 \cdot \vec{s}_2) - \frac{1}{N-2}(\vec{s}_1 + \vec{s}_2)\cdot \vec{S}_{cc} + \frac{1}{(N-2)(N-3)}\Big(\vec{S}_{cc}^2 - \frac{3}{4}(N-2)\Big)|\Phi_2(SI)\rangle \\ &= \langle \Phi_1(SI)|\frac{3N-13}{4(N-2)(N-3)}t_1 T_c - \frac{N^2+4N+11}{4(N-1)(N-2)(N-3)}s_1 S_c - \frac{2}{(N-2)(N-3)}g_1 G_c \\ &\quad - \frac{N+3}{4(N-1)(N-2)(N-3)}S_c^2 - \frac{1}{(N-2)(N-3)}g_1 S_c T_c + \frac{N}{4(N-2)}|\Phi_1(SI)\rangle. \end{aligned} \tag{109}$$

In writing the final result we assumed two light flavors $F = 2$, which allows the use of the identity $S_c^i G_c^{ia} = \frac{1}{4}(N+1)T_c^a$.

The matrix element of the second term in Eq. (103), representing the contribution of the E projection of the quadrupole interaction V_q, can be expressed as a diagonal matrix element on $\Phi_1(SI)$ in a similar way, with the result

$$\begin{aligned}&L_2^{ij}\langle \Phi_1(SI)|\{s_1^i, s_2^j\} - \frac{1}{N-2}\{(s_1^i + s_2^i), S_{cc}^j\} + \frac{1}{(N-2)(N-3)}\{S_{cc}^i, S_{cc}^j\}|\Phi_2(SI)\rangle \\ &= L_2^{ij}\langle \Phi_1(SI)|\frac{2N-5}{(N-2)(N-3)}\{g_1^{ia}, G_c^{ja}\} + \frac{N+1}{4(N-1)(N-2)}\{s_1^i, S_c^j\} - \frac{N+3}{4(N-1)(N-2)(N-3)}\{S_c^i, S_c^j\} \\ &\quad - \frac{1}{(N-2)(N-3)}t_1^a\{S_c^i, G_c^{ja}\} - \frac{1}{(N-2)(N-3)}\{g_1^{ia}, S_c^j\}T_c^a|\Phi_1(SI)\rangle. \end{aligned} \tag{110}$$

A few comments are in order about the form of these results. Note that the E operators arising from the spin-spin and tensor interactions introduce a nontrivial flavor dependence (manifested through the operators $t_1 T_c$, $g_1 G_c$) which is not observed in the S, MS projections of these interactions. Furthermore, in addition to the operators encountered so far, three new structures are introduced by the E operators, given by $g_1^{ia} S_c^i T_c^a$ and $L_2^{ij} t_1^a \{S_c^i, G_c^{ja}\}$, $L_2^{ij}\{g_1^{ia}, S_c^j\} T_c^a$. The operator $L_2^{ij}\{g_1^{ia}, G_c^{ja}\}$ contributes at $O(N_c^0)$, and the remaining ones start at order $O(N_c^{-1})$.

IV. MATCHING ONTO THE $1/N_c$ EXPANSION

The spin-flavor structure of the matrix elements of the one-gluon interaction Eq. (74) matches a subset of the operators appearing in the $1/N_c$ expansion of the mass operator of the orbitally excited states. We adopt here the basis of operators of Ref. [7]. Working to order $1/N_c$, the most general set of operators for the mass of these states is

$$\hat{M} = c_1 N_c \mathbf{1} + c_2 L^i s^i + c_3 \frac{3}{N_c} L_2^{ij} g^{ia} G_c^{ja} + \sum_{i=4}^{8} c_i \mathcal{O}_i. \tag{111}$$

The terms proportional to $c_{2,3}$ contribute at order $O(N_c^0)$, and the remaining operators proportional to $c_{4\text{–}8}$ are of order $1/N_c$. A complete basis of subleading operators can be chosen as [7]

$$\mathcal{O}_4 = L^i s^i + \frac{4}{N_c+1} L^i t^a G_c^{ia}, \qquad \mathcal{O}_5 = \frac{1}{N_c} L^i S_c^i,$$
$$\mathcal{O}_6 = \frac{1}{N_c} S_c^2, \qquad \mathcal{O}_7 = \frac{1}{N_c} s^i S_c^i, \qquad \mathcal{O}_8 = \frac{1}{N_c} L_2^{ij}\{s^i, S_c^j\},$$
$$\mathcal{O}_9 = \frac{1}{N_c} L^i g^{ia} T_c^a, \qquad \mathcal{O}_{10} = \frac{1}{N_c} t^a T_c^a,$$
$$\mathcal{O}_{11} = \frac{1}{N_c^2} L_2^{ij} t^a \{S_c^i, G_c^{ja}\}. \tag{112}$$

Matching the one-gluon exchange quark-quark interaction we find a leading-order $O(N_c^0)$ contribution to the mass coming from the spin-orbit interaction

$$c_2 = -\frac{g_s^2 N_c}{4}(3\mathcal{J}^s_{\rm dir} - \mathcal{J}^a_{\rm dir}), \qquad c_3 = 0. \tag{113}$$

This confirms in a direct way the prediction obtained from the $1/N_c$ expansion of the breaking of the $SU(4)$ spin-flavor symmetry at leading order in N_c [5,7]. The nonrelativistic quark model with gluon-mediated quark interactions displays the same breaking phenomenon.

The coefficients of the $O(N_c^{-1})$ operators are given by

$$c_4 = 0, \quad c_5 = -\frac{g_s^2 N_c}{4}(3\mathcal{J}^s_{\rm dir} + \mathcal{J}^a_{\rm dir}), \quad c_6 = -\frac{g_s^2 N_c}{4}\mathcal{I}_s,$$
$$c_7 = -\frac{g_s^2 N_c}{2}\mathcal{I}_{\rm dir}, \quad c_8 = -\frac{g_s^2 N_c}{2}\mathcal{K}_{\rm dir} \tag{114}$$

and $c_{9,10,11} = 0$. In addition, the E operators contribute to the coefficients $c_{6,7,10}$, as shown in Eq. (109), and to $c_{3,8,11}$ as seen in Eq. (110). They also introduce two operators not present for $N = 3$, $g_1^{ia} S_c^i T_c^a$, and $L_2^{ij}\{g_1^{ia}, S_c^j\} T_c^a$, contributing at order $O(N_c^{-2})$, in agreement with Ref. [7].

Several general comments can be made about these results.

The explicit calculation confirms the N_c power counting rules given in Ref. [5], giving the leading contribution of each operator. The two-body quark interactions considered here produce one-, two-, and three-body ($\mathcal{O}_{17} = \frac{1}{N_c^2} L_2^{ij}\{S_c^i, S_c^j\}$ of Ref. [7], which correctly appears at order $O(1/N_c^2)$ in Eqs. (74) and (110)) effective operators in the $1/N_c$ expansion. Higher-order iterations of the interaction Hamiltonian will also generate more higher-order operators.

An important conclusion following from this calculation is that operators with nontrivial permutation symmetry are indeed required by a correct implementation of the $1/N_c$ expansion. This disagrees with the $1/N_c$ expansion recently proposed by Matagne and Stancu [12], which does not allow for such operators.

A distinctive prediction of the one-gluon exchange potential is the vanishing of the coefficient c_3 of one of the leading-order operators (up to contributions from the E operators, which appear only for $N > 3$). Expressed in terms of the tower masses M_0, M_1, M_2, (see Ref. [9]) this relation is equivalent to a mass relation at leading order in $1/N_c$

$$M_0 - \frac{3}{2} M_1 + \frac{1}{2} M_2 = O(N_c^{-1}). \tag{115}$$

We consider next the matching of the mass operator Eq. (89) in the model with quark-quark interactions mediated by Goldstone-boson exchange. For this case the coefficients of the leading operators are

$$c_2 = -\frac{1}{8}\tilde{g}_A^2(3\mathcal{J}^s_{\rm dir} + \mathcal{J}^a_{\rm dir}), \qquad c_3 = \frac{2}{3}\tilde{g}_A^2 \mathcal{K}_{\rm dir} \tag{116}$$

where we defined $\tilde{g}_A^2 = N_c g_A^2/f_\pi^2$ a coefficient of order $\sim O(N_c^0)$.

The coefficients of the subleading $O(N_c^{-1})$ operators are

$$c_4 = \frac{1}{8}\tilde{g}_A^2(3\mathcal{J}^s_{\rm dir} + \mathcal{J}^a_{\rm dir}), \qquad c_5 = 0,$$
$$c_6 = -\frac{1}{4}\tilde{g}_A^2 \mathcal{I}_s, \qquad c_7 = c_{10} = -\frac{1}{4}\tilde{g}_A^2 \mathcal{I}_{\rm dir}, \tag{117}$$
$$c_8 = c_{11} = 0, \qquad c_9 = \frac{1}{2}\tilde{g}_A^2(3\mathcal{J}^s_{\rm dir} - \mathcal{J}^a_{\rm dir}).$$

For this case the coefficients of both leading operators can be of natural size. There are predictions for the vanishing of the coefficients of some subleading operators, and also one relation between the coefficients of the leading and subleading operators $c_2 = -c_4$. The Goldstone-boson interaction also generates a three-body operator, $\mathcal{O}_{17}$ of

TABLE II. Irreducible representations of S_N contained in $MS \times MS$.

Name	Partition	Dim	Character $\chi^{(R)}_{1^\alpha,2^\beta,3^\gamma,\ldots}$
S	$[N]$	1	1
MS	$[N-1,1]$	$N-1$	$\alpha-1$
E	$[N-2,2]$	$\frac{1}{2}N(N-3)$	$\frac{1}{2}(\alpha-1)(\alpha-2)+\beta-1$
A'	$[N-2,1,1]$	$\frac{1}{2}(N-1)(N-2)$	$\frac{1}{2}(\alpha-1)(\alpha-2)-\beta$

Ref. [7], which correctly appears at order $O(1/N_c^2)$ in Eq. (89).

The coefficients of the leading-order operators $c_{1,2,3}$ have been determined in [9] from a fit to the masses of the nonstrange $L=1$ baryons, working at leading order in $1/N_c$. The results depend on the assignment of the observed baryons into the irreps of the contracted symmetry (towers). There are four possible assignments, but only two of them are favored by data. These two assignments give the coefficients

$$\text{assignment 1: } c_2^{(0)} = 83 \pm 14 \text{ MeV}, \quad c_3^{(0)} = -188 \pm 28 \text{ MeV} \tag{118}$$

$$\text{assignment 3: } c_2^{(0)} = -12 \pm 16 \text{ MeV}, \quad c_3^{(0)} = 142 \pm 38 \text{ MeV}. \tag{119}$$

Comparing with the predictions Eq. (113) of the one-gluon quark-quark potential model, we see that there is no evidence in the data for a suppression of the coefficient c_3 relative to c_2. For both assignments, the coefficient c_3 is sizeable, a situation which favors the Goldstone-boson exchange model, or at least indicates that some kind of flavor-dependent effective interactions cannot be neglected. A more detailed analysis, including also the predictions for the subleading coefficients, will be presented elsewhere.

V. CONCLUSIONS

In this paper we analyzed the spin-flavor structure of excited baryons containing N_c quarks, from the perspective of the permutation group S_N of $N=N_c$ objects. The group S_N is the diagonal subgroup of the product $S_N^{\rm orb} \times S_N^{\rm sf}$ of permutations acting separately on the orbital and spin-flavor wave functions, respectively.

The permutation group imposes restrictive constraints on the form of the allowed spin-flavor operators contributing to any physical quantity, such as masses and couplings. In this paper we discussed in detail the mass operator of the orbitally excited baryons with MS spin-flavor symmetry.

In the quark model with a quark-quark interaction Hamiltonian V, the hadronic mass operator contains only those spin-flavor operators which appear in the decomposition of the Hamiltonian under $S_N \to S_N^{\rm orb} \times S_N^{\rm sf}$. This decomposition has the generic form $V = \sum_R V_R^{\rm orb} V_R^{\rm sf}$, with R irreps of the S_N group, which depend on V. Taking the matrix elements of the orbital operators $V_R^{\rm orb}$ on hadronic states replaces them with reduced matrix elements, parametrizing in a general way our ignorance about the orbital wave functions. The remaining dependence on the spin-flavor degrees of freedom is displayed explicitly in operator form, regardless of the unknown orbital wave functions.

This approach is similar to the method used for $N_c=3$ in Ref. [14] to obtain the predictions of the nonrelativistic quark model in a form independent of the orbital wave functions. In addition to extending this result to arbitrary N_c, we focus here on the transformation of the operators under the permutation group $S_N^{\rm sf}$ acting on the spin-flavor degrees of freedom, which is crucial for the connection of these results with the $1/N_c$ expansion.

We constructed explicitly the $S_N \to S_N^{\rm orb} \times S_N^{\rm sf}$ decomposition of the most general two-body operator V, considering, in particular, the case of the hyperfine gluon-exchange and Goldstone-boson exchange mediated quark-quark interactions. The main results of this analysis are Eqs. (74) and (89), which summarize in a compact operator form the most general structure of the mass operator for the excited baryons allowed by the considered $q-q$ interactions.

The results for the mass operator match precisely the effective operators appearing in the $1/N_c$ expansion for these states [5,7], We find that the decomposition into core and excited quark operators used previously in the literature on the subject is both necessary, and a consequence of the S_N symmetry. We confirm by explicit calculation the N_c counting rules of Refs. [5,7]. In particular we confirm that the effective spin-orbit interaction is of leading order $O(N_c^0)$, as was correctly stated in Ref. [5].

Comparing with existing fits to the masses of nonstrange $L=1$ excited baryons, we find that flavor-dependent interactions cannot be neglected, and may be necessary to supplement the gluon-exchange model. This is in line with the chiral quark picture proposed in Ref. [17]. The approach discussed here allows further tests of the nature of the $q-q$ forces, as manifested through the hadronic properties of these states. We hope to report progress on this issue in the near future.

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APPENDIX A: THE PERMUTATION GROUP

In this appendix we give a few details about the permutation group of N objects S_N, and its irreducible representations which are used in the main text.

The irreps of S_N are identified by a partition $[n_1, n_2, \cdots, n_j]$ with $\sum_{i=1}^{j} n_i = N$. Each partition can be represented as a Young diagram with n_1 boxes on the first row, n_2 on the second row, etc.

The irreps of S_N which are relevant for the present work are given in Table II. We list also the characters for the conjugacy class $1^\alpha 2^\beta 3^\gamma \cdots$ containing α 1-cycles, β 2-cycles, etc. Using these expressions and the orthogonality theorem for characters, the projection of any operator onto these irreps of S_N can be computed explicitly.

The choice of a basis for a given irrep is not unique and several possible choices are adopted in the literature. The most elegant choice is the orthogonal Yamanouchi-Katani basis, which is constructed recursively in terms of the chain $S_{N-1} \subset S_N \subset S_{N+1} \cdots$. However, this basis is not convenient for arbitrary N, since the matrices of the irreducible representations of S_N cannot be given in a simple closed form.

In this paper we use the Young operator basis [15]. Consider a particular standard Young tableaux (SYT) of the shape corresponding to a given irrep of S_N. This is a Young tableaux with the numbers 1, 2, $\cdots$, N inserted such that the numbers increase from the left to the right in each row, and increase as one moves down in each column. We will denote a SYT by giving the sequence of indices in each row, e.g., [125][3][4] for a Young diagram with three rows in the A' of S_5. Denote with P the sum of all horizontal permutations, which exchange only elements in the same row, and with Q the sum of all vertical permutations, which interchange only elements in the same column, multiplied with the parity of each permutation.

The Young operator Y corresponding to a given Young tableau is defined as

$$Y = QP. \tag{A1}$$

This can be applied to any state or operator to project out the component with the desired transformation under S_N.

As an example we give the explicit construction of the basis of symmetric two-body operators $\mathcal{O}_{ij}$ transforming under irreps of S_N, for $N = 5$. As shown in Sec. III, these operators are decomposed as $\{\mathcal{O}_{ij}\} = S + MS + E$ under the permutation group S_N.

The symmetric component is given by the sum over all components

$$\mathcal{O}^S = \mathcal{O}_{12} + \mathcal{O}_{13} + \mathcal{O}_{14} + \mathcal{O}_{15} + \mathcal{O}_{23} + \mathcal{O}_{24} + \mathcal{O}_{25} + \mathcal{O}_{34} + \mathcal{O}_{35} + \mathcal{O}_{45}. \tag{A2}$$

The MS operators are constructed by applying the Young projection operators to a certain operator $O_{ij}^{(0)}$. This is chosen such that the resulting basis of operators satisfies the relations Eqs. (6). Considering the Young tableaux $[1ab\cdots][k]$ corresponding to O_k^{MS}, the starting operator must be chosen as $O_{ij}^{(0)} = O_{1a}$. Thus, for O_2^{MS} the Young projection operator is applied to O_{13}, and for the remaining $O_{k\geq 3}^{MS}$, to O_{12}. As an illustration, we give below the explicit MS operator basis for $N = 5$, where we show also the standard Young diagram corresponding to each projector:

$$\mathcal{O}_2^{MS} = \mathcal{O}_{13} + \mathcal{O}_{14} + \mathcal{O}_{15} - \mathcal{O}_{23} - \mathcal{O}_{24} - \mathcal{O}_{25}, \quad [1345][2] \tag{A3}$$

$$\mathcal{O}_3^{MS} = \mathcal{O}_{12} + \mathcal{O}_{14} + \mathcal{O}_{15} - \mathcal{O}_{23} - \mathcal{O}_{34} - \mathcal{O}_{35}, \quad [1245][3] \tag{A4}$$

$$\mathcal{O}_4^{MS} = \mathcal{O}_{12} + \mathcal{O}_{13} + \mathcal{O}_{15} - \mathcal{O}_{24} - \mathcal{O}_{34} - \mathcal{O}_{45}, \quad [1235][4] \tag{A5}$$

$$\mathcal{O}_5^{MS} = \mathcal{O}_{12} + \mathcal{O}_{13} + \mathcal{O}_{14} - \mathcal{O}_{25} - \mathcal{O}_{35} - \mathcal{O}_{45}, \quad [1234][5]. \tag{A6}$$

The E projection contains five operators. They can be obtained by acting with the Young projectors $Y(E)$ of the standard Young diagram $[1a\cdots][bc]$ onto O_{1a}, and are given by

$$\mathcal{O}_1^E = \mathcal{O}_{12} - \mathcal{O}_{15} - \mathcal{O}_{24} + \mathcal{O}_{45}, \quad [123][45] \tag{A7}$$

$$\mathcal{O}_2^E = \mathcal{O}_{12} - \mathcal{O}_{15} - \mathcal{O}_{32} + \mathcal{O}_{35}, \quad [124][35] \tag{A8}$$

$$\mathcal{O}_3^E = \mathcal{O}_{12} - \mathcal{O}_{14} - \mathcal{O}_{23} + \mathcal{O}_{34}, \quad [125][34] \tag{A9}$$

$$\mathcal{O}_4^E = \mathcal{O}_{13} - \mathcal{O}_{15} - \mathcal{O}_{23} + \mathcal{O}_{25}, \quad [134][25] \tag{A10}$$

$$\mathcal{O}_5^E = \mathcal{O}_{13} - \mathcal{O}_{14} - \mathcal{O}_{23} + \mathcal{O}_{24}, \quad [135][24]. \tag{A11}$$

These results can be used to express any interaction of $N = 5$ particles interacting through symmetric two-body potentials which are factorized into spin-isospin and orbital operators. Expressed in terms of the operators transforming in the irreps of S_N, $\tilde{O}_k^T = (O_S, O_2^{MS}, O_3^{MS}, O_4^{MS}, O_5^{MS}, O_1^E, O_2^E, O_3^E, O_4^E, O_5^E)$, the interaction Hamiltonian takes a block diagonal form

$$\sum_{i<j}\mathcal{O}_{ij}\mathcal{R}_{ij}=\sum_{k,l}\tilde{\mathcal{O}}_k\begin{pmatrix}\frac{1}{10}&0&0&0&0&0&0&0&0&0\\0&\frac{4}{15}&-\frac{1}{15}&-\frac{1}{15}&-\frac{1}{15}&0&0&0&0&0\\0&-\frac{1}{15}&\frac{4}{15}&-\frac{1}{15}&-\frac{1}{15}&0&0&0&0&0\\0&-\frac{1}{15}&-\frac{1}{15}&\frac{4}{15}&-\frac{1}{15}&0&0&0&0&0\\0&-\frac{1}{15}&-\frac{1}{15}&-\frac{1}{15}&\frac{4}{15}&0&0&0&0&0\\0&0&0&0&0&\frac{1}{2}&-\frac{1}{6}&-\frac{1}{6}&-\frac{1}{6}&\frac{1}{3}\\0&0&0&0&0&-\frac{1}{6}&\frac{1}{2}&-\frac{1}{6}&-\frac{1}{6}&0\\0&0&0&0&0&-\frac{1}{6}&-\frac{1}{6}&\frac{1}{2}&\frac{1}{6}&-\frac{1}{3}\\0&0&0&0&0&-\frac{1}{6}&-\frac{1}{6}&\frac{1}{6}&\frac{1}{2}&-\frac{1}{3}\\0&0&0&0&0&\frac{1}{3}&0&-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix}_{kl}\tilde{\mathcal{R}}_l. \tag{A12}$$

This is a particular case for $N=5$ of the general relation Eq. (26) which is valid for any N.

We also present the construction of the operator basis for the antisymmetric two-body operators. As mentioned, these operators transform like the $MS+A'$ irreps.

The MS operators are given by

$$O_2^{MS}=2O_{12}-O_{23}-O_{24}-O_{25}+O_{13}+O_{14}+O_{15},\quad [1345][2] \tag{A13}$$

$$O_3^{MS}=2O_{13}+O_{23}-O_{34}-O_{35}+O_{12}+O_{14}+O_{15},\quad [1245][3] \tag{A14}$$

$$O_4^{MS}=2O_{14}+O_{24}+O_{34}-O_{45}+O_{12}+O_{13}+O_{15},\quad [1235][4] \tag{A15}$$

$$O_5^{MS}=2O_{15}+O_{25}+O_{35}+O_{45}+O_{12}+O_{13}+O_{14},\quad [1234][5]. \tag{A16}$$

The A' operators depend only on the indices in the first column of the SYT. Considering the SYT $[1ab][c][d]$, the Young projector $Y(A')$ reduces to the antisymmetrizer in the indices $[1cd]$, $A_{1cd}=1-P_{1c}-P_{1d}-P_{cd}+P_{cd1}+P_{c1d}$, when applied to O_{1c}. The basis of operators is

$$O_1^{A'}=2\mathcal{O}_{12}-2\mathcal{O}_{13}+2\mathcal{O}_{23},\quad [145][2][3] \tag{A17}$$

$$O_2^{A'}=2\mathcal{O}_{12}-2\mathcal{O}_{14}+2\mathcal{O}_{24},\quad [135][2][4] \tag{A18}$$

$$O_3^{A'}=2\mathcal{O}_{12}-2\mathcal{O}_{15}+2\mathcal{O}_{25},\quad [134][2][5] \tag{A19}$$

$$O_4^{A'}=2\mathcal{O}_{13}-2\mathcal{O}_{14}+2\mathcal{O}_{34},\quad [125][3][4] \tag{A20}$$

$$O_5^{A'}=2\mathcal{O}_{13}-2\mathcal{O}_{15}+2\mathcal{O}_{35},\quad [124][3][5] \tag{A21}$$

$$O_6^{A'}=2\mathcal{O}_{14}-2\mathcal{O}_{15}+2\mathcal{O}_{45},\quad [123][4][5]. \tag{A22}$$

The decomposition of a symmetric Hamiltonian containing antisymmetric two-body operators, expressed in the basis $\tilde{O}_k^T=(O_2^{MS},O_3^{MS},O_4^{MS},O_5^{MS},O_1^{A'},O_2^{A'},O_3^{A'},O_4^{A'},O_5^{A'},O_6^{A'})$, has again a block diagonal form

$$\sum_{i<j}\mathcal{O}_{ij}\mathcal{R}_{ij}=\sum_{k,l}\tilde{\mathcal{O}}_k\begin{pmatrix}\frac{4}{25}&-\frac{1}{25}&-\frac{1}{25}&-\frac{1}{25}&0&0&0&0&0&0\\-\frac{1}{25}&\frac{4}{25}&-\frac{1}{25}&-\frac{1}{25}&0&0&0&0&0&0\\-\frac{1}{25}&-\frac{1}{25}&\frac{4}{25}&-\frac{1}{25}&0&0&0&0&0&0\\-\frac{1}{25}&-\frac{1}{25}&-\frac{1}{25}&\frac{4}{25}&0&0&0&0&0&0\\0&0&0&0&\frac{3}{20}&-\frac{1}{20}&-\frac{1}{20}&\frac{1}{20}&\frac{1}{20}&0\\0&0&0&0&-\frac{1}{20}&\frac{3}{20}&-\frac{1}{20}&-\frac{1}{20}&0&\frac{1}{20}\\0&0&0&0&-\frac{1}{20}&-\frac{1}{20}&\frac{3}{20}&0&-\frac{1}{20}&-\frac{1}{20}\\0&0&0&0&\frac{1}{20}&-\frac{1}{20}&0&\frac{3}{20}&-\frac{1}{20}&\frac{1}{20}\\0&0&0&0&\frac{1}{20}&0&-\frac{1}{20}&-\frac{1}{20}&\frac{3}{20}&-\frac{1}{20}\\0&0&0&0&0&\frac{1}{20}&-\frac{1}{20}&\frac{1}{20}&-\frac{1}{20}&\frac{3}{20}\end{pmatrix}_{kl}\tilde{\mathcal{R}}_l. \tag{A23}$$

This is the particular case for $N=5$ of the general relation Eq. (30), valid for any N.

APPENDIX B: AN EXPLICIT EXAMPLE FOR $N_c=3$

We illustrate here the construction of the baryon states used in this paper, given in Eq. (10), on the example of the $N_{5/2}$ state. This is a member of the 70^- multiplet for $N_c=3$ with quantum numbers $I=1/2$, $J=5/2$.

We start by constructing the MS basis of spin-flavor states given in Eq. (4). They are defined as $|\phi_k\rangle=(P_{1k}-1)|\phi\rangle$, where the reference state $|\phi\rangle$ is chosen as an eigenstate of S^2, S_3, I^2, I_3

$$|\phi\rangle = (|q_1\rangle \otimes |i_c = 1\rangle)^{I=(1/2),S=(3/2)}_{I_3=+(1/2),S_3=+(3/2)}$$
$$= \frac{1}{3}(2d_\uparrow u_\uparrow u_\uparrow - u_\uparrow u_\uparrow d_\uparrow - u_\uparrow d_\uparrow u_\uparrow). \quad \text{(B1)}$$

We indicate explicitly the spin and isospin couplings, which were dropped in the main text. The normalization of $|\phi\rangle$ is chosen so that the basis states satisfy Eq. (5). We obtain

$$|\phi_2\rangle = u_\uparrow d_\uparrow u_\uparrow - d_\uparrow u_\uparrow u_\uparrow, \quad \text{(B2)}$$

$$|\phi_3\rangle = u_\uparrow u_\uparrow d_\uparrow - d_\uparrow u_\uparrow u_\uparrow. \quad \text{(B3)}$$

Applying the total isospin operator

$$I^2 = \frac{9}{4} + \frac{1}{2}\sum_{i<j}(\tau_i^3\tau_j^3 + 2\tau_i^+\tau_j^- + 2\tau_i^-\tau_j^+) \quad \text{(B4)}$$

with $\tau^\pm = (\tau^1 \pm i\tau^2)/2$, where $\vec{\tau}$ are the Pauli matrices, it is easy to check that these states have the same quantum numbers as the reference state $|\phi\rangle$. Combining the spin-flavor and orbital components, our symmetrized state Eq. (10) is given by

$$|B\rangle = \sum_{k,l=2}^{3} \phi_k\chi_l M_{kl}$$
$$= \phi_2\chi_2 + \phi_3\chi_3 - \frac{1}{2}(\phi_2\chi_3 + \phi_3\chi_2) \quad \text{(B5)}$$

with the MS orbital basis wave functions $\chi_2 = sp^+s - p^+ss$, $\chi_3 = ssp^+ - p^+ss$, and reads explicitly

$$|B\rangle = \tilde{d}_\uparrow^+ u_\uparrow u_\uparrow + u_\uparrow\tilde{d}_\uparrow^+ u_\uparrow + u_\uparrow u_\uparrow\tilde{d}_\uparrow^+ - \frac{1}{2}(\tilde{u}_\uparrow^+ u_\uparrow d_\uparrow + \tilde{u}_\uparrow^+ d_\uparrow u_\uparrow$$
$$+ u_\uparrow\tilde{u}_\uparrow^+ d_\uparrow + d_\uparrow\tilde{u}_\uparrow^+ u_\uparrow + u_\uparrow d_\uparrow\tilde{u}_\uparrow^+ + d_\uparrow u_\uparrow\tilde{u}_\uparrow^+). \quad \text{(B6)}$$

This state is symmetric under all three quarks, and is normalized as $\langle B|B\rangle = 9/2$.

Next we compare this with the state constructed in Ref. [7] (denoted as the CCGL state) which is given in Eq. (15). Here we always choose to couple the excited quark in the first position, which leads to the explicit form

$$|(LS)J, J_3; I, I_3\rangle_{CCGL} = |\left(1, \frac{3}{2}\right)\frac{5}{2}, +\frac{5}{2}; \frac{1}{2}, +\frac{1}{2}\rangle$$
$$= \frac{1}{\sqrt{6}}(\tilde{u}_\uparrow^+ u_\uparrow d_\uparrow + \tilde{u}_\uparrow^+ d_\uparrow u_\uparrow - 2\tilde{d}_\uparrow^+ u_\uparrow u_\uparrow) \quad \text{(B7)}$$

where $\tilde{q}^+$ denotes the quark no. 1 with orbital angular momentum $|L = 1, L_3 = +1\rangle = p^+$.

The CCGL state Eq. (B7) can be written as $|CCGL\rangle = \Phi(SI) \otimes |p^+ss\rangle$ with $\Phi(S=\frac{3}{2}, I=\frac{1}{2}) = \frac{1}{\sqrt{2\cdot 3}}(|\phi_2\rangle + |\phi_3\rangle)$. The relation of the symmetrized state Eq. (B6) with the CCGL state given in Eq. (B7) is given by

$$\frac{1}{\sqrt{3}}\sum_{i=1}^{3} P_{1i}|CCGL\rangle = -\frac{\sqrt{2}}{3}|B\rangle \quad \text{(B8)}$$

and agrees with Eq. (16).

APPENDIX C: THE PROOF OF EQ (37)

The matrix elements of a two-body symmetric operator on the ϕ_i basis can be given in closed form, for any N. Using the representation of the two-body operator $(kl) = O_{kl}$ on the basis of Fock states, these matrix elements are

$$\langle\phi_i|(1k)|\phi_j\rangle = (I_d - I_e)(\delta_{ik} + \delta_{jk}) + I_d(1 - \delta_{ik})(1 - \delta_{jk}),$$
$$k = 2, \cdots, N \quad \text{(C1)}$$

$$\langle\phi_i|(kl)|\phi_j\rangle = I_d\delta_{ij}(\delta_{ik} + \delta_{jl}) + I_e(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$
$$k \neq l = 2, \cdots, N. \quad \text{(C2)}$$

We denoted here the two overlap integrals as $I_d = \langle sp|(12)|sp\rangle$, $I_e = \langle sp|(12)|ps\rangle$.

Using these expressions, the matrix element of operators transforming into irreps of S_N can be obtained by projecting onto the appropriate irrep.

$$\langle\phi_i|O_S|\phi_j\rangle = \langle\phi_i|\sum_{1\le k<l\le N}(kl)|\phi_j\rangle = \langle O_S\rangle(1 + \delta_{ij}) \quad \text{(C3)}$$

$$\langle\phi_i|O_k^{MS}|\phi_j\rangle = \langle O_{MS}\rangle(1 - \delta_{ij}\delta_{ik}) \quad \text{(C4)}$$

$$\langle\phi_i|O_{klm}^E|\phi_j\rangle = \langle O_E\rangle\frac{1}{2}[(-\delta_{ik} + \delta_{im})(1 + \delta_{jl})$$
$$+ (-\delta_{jk} + \delta_{jm})(1 + \delta_{il})] \quad \text{(C5)}$$

where the reduced matrix elements are

$$\langle O_S\rangle = (N - 1)I_d - I_e \quad \text{(C6)}$$

$$\langle O_{MS}\rangle = (N - 2)I_d - 2I_e \quad \text{(C7)}$$

$$\langle O_E\rangle = 2I_e. \quad \text{(C8)}$$

The matrix element of the interaction Hamiltonian in Eq. (37) can be computed as

$$\langle B|V_{\text{symm}}|B\rangle = \sum_{1\le k<l<N}\text{Tr}[M(kl)M(kl)] = \frac{1}{2}N(N - 1)T \quad \text{(C9)}$$

where M is the matrix in Eq. (11) of the paper, and the matrices $(kl)_{ij}$ are given above in Eqs. (C1). In the last step we used the fact that each term in the sum is the same T, such that the sum is simply the number of terms times T.

The result for the matrix element of V_{symm} must be given equivalently by Eq. (37) as a sum over the irreps S, MS, E

$$\langle B|V_{\rm symm}|B\rangle = \frac{N^2}{N-1}(c_S\langle O_S\rangle^2 + c_{MS}\langle O_{MS}\rangle^2 + c_E\langle O_E\rangle^2) \tag{C10}$$

where the reduced matrix elements are given above in Eqs. (C6) in terms of the overlap integrals I_d, I_e. Comparing the two expressions for this matrix element allows the determination of the coefficients $c_{S,MS,E}$.

We start by computing the matrix element Eq. (C9). As mentioned, each term in the sum is the same, and can be computed as

$$\begin{aligned} T &\equiv {\rm Tr}[M(1k)M(1k)] \\ &= 2\left(1+\frac{1}{(N-1)^2}\right)(I_d^2+I_e^2) - \frac{8}{N-1}I_dI_e. \end{aligned} \tag{C11}$$

This must be matched by the expression Eq. (C10)

$$\begin{aligned} \langle B|V_{\rm symm}|B\rangle &= \frac{1}{2}N(N-1)T \\ &= \frac{2N}{(N-1)^2}[(N-1)I_d - I_e]^2 \\ &\quad + \frac{N}{N-1}[(N-2)I_d - 2I_e]^2 \\ &\quad + c_E\frac{N^2}{N-1}[2I_e]^2. \end{aligned} \tag{C12}$$

Matching the coefficients of I_d^2, I_dI_e and I_e^2 on both sides of this relation gives the coefficients of the three terms in Eq. (C10) corresponding to the three irreps of S_N:

$$c_S = \frac{2}{N(N-1)}, \qquad c_{MS} = \frac{1}{N}, \qquad c_E = \frac{N(N-3)}{4(N-1)}. \tag{C13}$$

This completes the proof of Eq. (37).

APPENDIX D: THE REDUCED MATRIX ELEMENT $\langle O_{MS}\rangle$

We present here the details of the derivation of the reduced matrix element of an MS spin-flavor operator as a matrix element on the CCGL state $\Phi(SI)$ with fixed identity of the "excited" quark (such as, e.g., Eq. (42)). As explained in Sec. III A, this state is given by

$$\Phi(SI) = \frac{1}{\sqrt{N(N-1)}}\sum_{k=2}^{N}\phi_k. \tag{D1}$$

The matrix element of an MS operator on the MS basis states is given by the S_N Wigner-Eckart theorem

$$\langle\phi_i|\mathcal{O}_k^{MS}|\phi_j\rangle = \langle\mathcal{O}^{MS}\rangle(1-\delta_{ik}\delta_{ij}). \tag{D2}$$

Summing over the index k of the operator, the matrix element on the $\Phi(SI)$ state is

$$\langle\Phi(SI)|\sum_{k=2}^{N}\mathcal{O}_k^{MS}|\Phi(SI)\rangle = (N-2)\langle\mathcal{O}^{MS}\rangle, \tag{D3}$$

which can be used to express $\langle\mathcal{O}^{MS}\rangle$ as a matrix element on the CCGL-type spin-flavor state $\Phi(SI)$.

The advantage of taking the sum $\sum_{k=2}^{N}\mathcal{O}_k^{MS}$ is that it singles out the quark no. 1, just as in the state $\Phi(SI)$. An explicit calculation gives

$$\sum_{k=2}^{N}\mathcal{O}_k^{MS} = \begin{cases} -2\sum_{i<j}\mathcal{O}_{ij}^s + N\sum_{i=2}^{N}\mathcal{O}_{1i}^s & \text{for symmetric two-body operators} \\ N\sum_{i=2}^{N}\mathcal{O}_{1i}^a & \text{for antisymmetric two-body operators} \end{cases}. \tag{D4}$$

Using this relation, one finds, for example, for the spin-spin interaction $\mathcal{O}_{ij} = \vec{s}_i\cdot\vec{s}_j$

$$\sum_{k=2}^{N}\mathcal{O}_k^{MS} = -\vec{S}^2 + N\vec{s}_1\cdot\vec{S}_c + \frac{3}{4}N, \tag{D5}$$

which gives directly Eq. (42) after combining it with Eq. (D3).

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