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OPTIMIZATION PROBLEM FOR EXTREMALS OF THE TRACE INEQUALITY IN DOMAINS WITH HOLES

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We study the Sobolev trace constant for functions defined in a bounded domain Ω that vanish in the subset A. We find a formula for the first variation of the Sobolev trace with respect to the hole. As a consequence of this formula, we prove that when Ω is a centered ball, the symmetric hole is critical when we consider deformation that preserve volume but is not optimal for some case.

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1. Introduction and Main Results

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$ and $1 < p < \infty$. We denote by p^* the critical exponent for the Sobolev trace immersion given by $p^* = p(N-1)/(N-p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$.

For any $A \subset \overline{\Omega}$, which is a smooth open subset, we define the space

$$W_A^{1,p}(\Omega) = \overline{C_0^\infty(\overline{\Omega}\backslash A)},$$

where the closure is taken in $W^{1,p}$-norm. By the Sobolev Trace Theorem, there is a compact embedding

$$W_A^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega), \tag{1.1}$$

for all $1 \leq q < p^*$. Thus, given $1 \leq q < p^*$, there exists a constant $C = C(q,p)$ such that

$$C\left\{\int_{\partial\Omega} |u|^q \,\mathrm{d}S\right\}^{\frac{p}{q}} \leq \int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x.$$

The best (largest) constant in the above inequality is given by

$$S_q(A) := \inf_{u \in W_A^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x}{\left\{\displaystyle\int_{\partial\Omega} |u|^q \,\mathrm{d}S\right\}^{\frac{p}{q}}}. \tag{1.2}$$

By (1.1), there exist an extremal for $S_q(A)$. Moreover, an extremal for $S_q(A)$ is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega\setminus\overline{A}, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega\setminus\overline{A}, \\ u = 0 & \text{on } \partial A, \end{cases} \tag{1.3}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian, $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative and λ depends on the normalization of u. When $\|u\|_{L^q(\partial\Omega)} = 1$ we have that $\lambda = S_q(A)$. Moreover, when $p = q$, the problem (1.3) becomes homogeneous and therefore is a nonlinear eigenvalue problem. In this case, the first eigenvalue of (1.3) coincides with the best Sobolev trace constant $S_q(A) = \lambda_1(A)$ and it is shown in [9] that it is simple (see also [3]). Therefore, if $p = q$, the extremal for $S_p(A)$ is unique up to constant factor. In the linear setting, i.e., when $p = q = 2$, this eigenvalue problem is known as the Steklov eigenvalue problem, see [11].

The aim of this paper is to analyze the dependence of the Sobolev trace constant $S_q(A)$ with respect to variations on the set A. To this end, we compute the so-called *shape derivative* of $S_q(A)$ with respect to regular perturbations of the *hole* A.

Let $V : \mathbb{R}^N \to \mathbb{R}^N$ be a regular (smooth) vector filed, globally Lipschitz, with support in Ω and let $\psi_t : \mathbb{R}^N \to \mathbb{R}^N$ be defined as the unique solution to

$$\begin{cases} \dfrac{d}{dt}\psi_t(x) = V(\psi_t(x)) & t > 0 \\ \psi_0(x) = x & x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

We have

$$\psi_t(x) = x + tV(x) + o(t) \quad \forall\, x \in \mathbb{R}^N.$$

Now, we define $A_t := \psi_t(A) \subset \Omega$ for all $t > 0$ and

$$S_q(t) = \inf_{u \in W_{A_t}^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x}{\left\{\displaystyle\int_{\partial\Omega} |u|^q \,\mathrm{d}S\right\}^{\frac{p}{q}}}. \tag{1.5}$$

Observe that $A_0 = A$ and therefore $S_q(0) = S_q(A)$.

In the linear case $p = q = 2$, Rossi studies the best constant of the Sobolev trace embedding in a domain without holes, see [10]. He finds a formula for the first variation of the best constant with respect to the domain. As a consequence he proves that the ball is a critical domain when we consider deformations that preserve volume.

In [2], Fernández Bonder, Groisman and Rossi analyze this problem in domain with holes and prove that $S_2(t)$ is differentiable with respect to t at $t = 0$ and it holds

$$\frac{d}{dt} S_2(t)\,|_{t=0} = -\int_{\partial A} \left(\frac{\partial u}{\partial \nu}\right)^2 \langle V, \nu\rangle \, \mathrm{d}S,$$

where u is a normalized eigenfunction for $S_2(A)$ and ν is the exterior normal vector to $\Omega\backslash\overline{A}$. Furthermore, in the case that Ω is the ball B_R with center 0 and radius $R > 0$ the authors show that a centered ball $A = B_r, r < R$, is critical in the sense that $S_2'(A) = 0$ when considering deformations that preserves volume and that this configuration is not optimal.

We say that a hole A^* is optimal for the parameter $\alpha, 0 < \alpha < |\Omega|$, if $|A^*| = \alpha$ and

$$S_q(A^*) = \inf_{\substack{A\subset\overline{\Omega}\\ |A|=\alpha}} S_q(A).$$

Therefore, in the case $p = q = 2$, there is a lack of symmetry in the optimal configuration.

Here we extend these results to the more general case $1 \le p < \infty$ and $1 \le q < p^*$. Our method differs from the one in [2] in order to deal with the nonlinear character of the problem.

Our first result states

Theorem 1.1. *Suppose $A \subset \overline{\Omega}$ is a smooth open subset and let $1 \le q < p^*$. Then, with the previous notation, we have that $S_q(t)$ is differentiable at $t = 0$ and*

$$S_q'(0) = \frac{d}{dt} S_q(t)\bigg|_{t=0} = (1-p)\int_{\partial A} \left|\frac{\partial u}{\partial \nu}\right|^p \langle V, \nu\rangle \, \mathrm{d}S,$$

where u is a normalized extremal (according to $\|u\|_{L^q(\partial\Omega)} = 1$) for $S_q(A)$ and ν is the exterior normal vector to $\Omega\backslash\overline{A}$.

Remark 1.2. If u is an extremal for $S_q(A)$ we have that $|u|$ is also an extremal associated to $S_q(A)$. Then in the previous theorem we can suppose that $u \ge 0$ in Ω. Moreover, by [8], we have that for all $U \subset\subset \Omega$ open subset such that $U \cap \partial A \neq \emptyset$ is a smooth open set there exists $\delta \in (0, 1)$ such that $u \in C^{1,\delta}(\overline{U\backslash\overline{A}})$ and $u > 0$ on $\partial\Omega\backslash\partial A$ if $\Omega\backslash\overline{A}$ satisfies the interior ball condition for all $x \in \partial\Omega\backslash\partial A$, see [12].

In the case that $\Omega = B_R$, we have the next result

Theorem 1.3. *Let $\Omega = B_R$ and let the hole be a centered ball $A = B_r$. Then, if $1 \le q \le p$, this configuration is critical in the sense that $S_q'(B_r) = 0$ for all deformations V that preserve the volume of B_r.*

But, if q is sufficiently large, the symmetric hole with a radial extremal is not an optimal configuration. In fact, we prove

Theorem 1.4. *Let $r > 0$ and $1 < p < \infty$ be fixed. Let $R > r$ and*

$$Q(R) = \frac{1}{S_p(B_r)^{\frac{p}{p-1}}}\left(1 - \frac{N-1}{R} S_p(B_r)\right) + 1. \tag{1.6}$$

If $q > Q(R)$ then the centered hole B_r is not optimal.

Finally, to study the asymptotic behavior of $Q(R)$

Proposition 1.5. *The function $Q(R)$ has the following asymptotic behavior*

$$\lim_{R\to r} Q(R) = 1^- \quad \textit{and} \quad \lim_{R\to+\infty} Q(R) = p.$$

Observe that $Q(R) < 1$ for R close to r and therefore the symmetric hole with a radial extremal is not an optimal configuration for R close to r.

2. Proof of Theorem 1.1

2.1. *Preliminary results*

The proof of Theorem 1.1 require some technical results. In this subsection we use some ideas from [4].

Given $u \in W^{1,p}_{A_t}(\Omega)\backslash W^{1,p}_0(\Omega)$ we consider $v = u \circ \psi_t$, so $v \in W^{1,p}_A(\Omega)\backslash W^{1,p}_0(\Omega)$ and $\nabla v^T = {}^T\psi_t' \nabla(u\circ\psi_t)^T$, where ψ_t' denotes the differential matrix of ψ_t and ${}^TA$ is the transpose of matrix A. Thus, by the change of variables formula, we have that

$$\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x = \int_\Omega \{|{}^T[\psi_t']^{-1}\nabla v^T|^p + |v|^p\} J(\psi_t)\,\mathrm{d}x,$$

here $J(\psi_t)$ is the usual Jacobian of ψ_t. Moreover, since $\mathrm{supp}(V) \subset \Omega$, we have that

$$\int_{\partial\Omega} |u|^q \,\mathrm{d}S = \int_{\partial\Omega} |v|^q \,\mathrm{d}S.$$

In [5], are proved the following asymptotic formulas

$$[\psi_t']^{-1}(x) = Id - tV'(x) + o(t), \tag{2.1}$$

$$J(\psi_t)(x) = 1 + t\,\mathrm{div}\; V(x) + o(t). \tag{2.2}$$

Then, by (2.1) and (2.2), we have

$$\begin{aligned}\int_\Omega |v|^p J(\psi_t)\,\mathrm{d}x &= \int_\Omega |v|^p\{1+t\,\mathrm{div}\ V+o(t)\}\,\mathrm{d}x\\ &= \int_\Omega |v|^p\,\mathrm{d}x + t\int_\Omega |v|^p\,\mathrm{div}\ V\,\mathrm{d}x + o(t)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |{}^T[\psi_t']^{-1}\nabla v^T|^p J(\psi_t)\,\mathrm{d}x &= \int_\Omega |[Id - t^T V' + o(t)]\nabla v^T|^p\{1+t\,\mathrm{div}\ V+o(t)\}\,\mathrm{d}x\\ &= \int_\Omega |\nabla v - t^T V'\nabla v^T + o(t)|^p\{1+t\,\mathrm{div}\ V+o(t)\}\,\mathrm{d}x,\end{aligned}$$

since

$$|\nabla v - t^T V'\nabla v^T + o(t)|^p = |\nabla v|^p - pt|\nabla v|^{p-2}\langle \nabla v, {}^T V'\nabla v^T\rangle + o(t)$$

we obtain that

$$\begin{aligned}\int_\Omega |{}^T[\psi_t']^{-1}\nabla v^T|^p J(\psi_t)\,\mathrm{d}x &= \int_\Omega |\nabla v|^p\,\mathrm{d}x + t\int_\Omega |\nabla v|^p\,\mathrm{div}\ V\,\mathrm{d}x\\ &\quad - pt\int_\Omega |\nabla v|^{p-2}\langle \nabla v, {}^T V'\nabla v^T\rangle\,\mathrm{d}x + o(t).\end{aligned}$$

Thus, we conclude

$$\begin{aligned}\int_\Omega |\nabla u|^p + |u|^p\,\mathrm{d}x &= \int_\Omega \{|{}^T[\psi_t']^{-1}\nabla v^T|^p + |v|^p\}J(\psi_t)\,\mathrm{d}x\\ &= \int_\Omega |v|^p\,\mathrm{d}x + \int_\Omega |\nabla v|^p\,\mathrm{d}x + t\int_\Omega \{|\nabla v|^p + |v|^p\}\mathrm{div}\ V\,\mathrm{d}x\\ &\quad - pt\int_\Omega |\nabla v|^{p-2}\langle \nabla v, {}^T V'\nabla v^T\rangle\,\mathrm{d}x + o(t).\end{aligned}$$

Therefore, we can rewrite (1.5) as

$$S_q(t) = \inf_{v\in W_A^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \{\rho(v) + t\gamma(v)\}, \tag{2.3}$$

where

$$\rho(v) = \frac{\displaystyle\int_\Omega |\nabla v|^p + |v|^p\,\mathrm{d}x}{\left\{\displaystyle\int_{\partial\Omega} |v|^q\,\mathrm{d}S\right\}^{p/q}},$$

and

$$\gamma(v) = \frac{\displaystyle\int_\Omega \{|\nabla v|^p + |v|^p\}\mathrm{div}\ V\,\mathrm{d}x - p\int_\Omega |\nabla v|^{p-2}\langle \nabla v, {}^T V'\nabla v^T\rangle\,\mathrm{d}x}{\left\{\displaystyle\int_{\partial\Omega} |v|^q\,\mathrm{d}S\right\}^{p/q}} + o(1).$$

Given $t \geq 0$, let $v_t \in W^{1,p}_A(\Omega) \backslash W^{1,p}_0(\Omega)$ such that $\|v_t\|_{L^q(\partial\Omega)} = 1$ and

$$S_q(t) = \varphi(t) + t\phi(t),$$

where

$$\varphi(t) = \rho(v_t) \quad \text{and} \quad \phi(t) = \gamma(v_t) \quad \forall\, t \geq 0.$$

We observe that $\varphi, \phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and

Lemma 2.1. *The function ϕ is nonincreasing.*

Proof. Let $0 \leq t_1 \leq t_2$. By (2.3), we have that

$$\varphi(t_2) + t_1\phi(t_2) \geq S_q(t_1) = \varphi(t_1) + t_1\phi(t_1) \tag{2.4}$$

$$\varphi(t_1) + t_2\phi(t_1) \geq S_q(t_2) = \varphi(t_2) + t_2\phi(t_2). \tag{2.5}$$

Subtracting (2.4) from (2.5), we get

$$(t_2 - t_1)\phi(t_1) \geq (t_2 - t_1)\phi(t_2).$$

Since $t_2 - t_1 \geq 0$, we obtain

$$\phi(t_1) \geq \phi(t_2).$$

This ends the proof. □

Remark 2.2. Since ϕ is nonincreasing, we have

$$\phi(t) \leq \phi(0) \quad \forall\, t \geq 0,$$

and there exists

$$\phi(0^+) = \lim_{t \to 0^+} \phi(t).$$

Corollary 2.3. *The function φ is nondecreasing.*

Proof. Let $0 \leq t_1 \leq t_2$. Again, by (2.3), we have that

$$\varphi(t_2) + t_1\phi(t_2) \geq S_q(t_1) = \varphi(t_1) + t_1\phi(t_1) \tag{2.6}$$

so

$$\varphi(t_2) - \varphi(t_1) \geq t_1(\phi(t_1) - \phi(t_2)).$$

Since $0 \leq t_1 \leq t_2$, by Lemma 2.1, we have that $\phi(t_1) - \phi(t_2) \geq 0$. Then

$$\varphi(t_2) - \varphi(t_1) \geq 0$$

that is what we wished to prove. □

Now we can prove that $S_q(t)$ is continuous at $t = 0$.

Theorem 2.4. *The function $S_q(t)$ is continuous at $t = 0$, i.e.*

$$\lim_{t \to 0^+} S_q(t) = S_q(0).$$

Proof. Given $t \geq 0$ so, by Corollary 2.3,

$$S_q(t) - Sq(0) = \varphi(t) + t\phi(t) - \varphi(0) \geq t\phi(t).$$

On the other hand, by (2.3), we have that

$$S_q(t) \leq \varphi(0) + t\phi(0) = S_q(0) + t\phi(0).$$

Then

$$t\phi(t) \leq S_q(t) - Sq(0) \leq t\phi(0).$$

Thus, by Remark 2.2,

$$\lim_{t \to 0^+} S_q(t) - S_q(0) = 0.$$

This finishes the proof. □

Thus, from Remark 2.2 and Theorem 2.4, we obtain the following corollary:

Corollary 2.5. *The function φ is continuous at $t = 0$, i.e.*

$$\lim_{t \to 0^+} \varphi(t) = \varphi(0).$$

Proof. We observe that

$$\varphi(t) - \varphi(0) = S_q(t) - S_q(0) - t\phi(t).$$

Then, by Remark 2.2 and Theorem 2.4,

$$\lim_{t \to 0^+} \varphi(t) - \varphi(0) = 0.$$

That proves the result. □

Finally, we prove the following:

Theorem 2.6. *The function φ is differentiable at $t = 0$ and*

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(0) = 0.$$

Proof. Let $0 < r < t$. By (2.3), we get

$$S_q(r) = \varphi(r) + r\phi(r) \le \varphi(t) + r\phi(t),$$

and

$$S_q(t) = \varphi(t) + t\phi(t) \le \varphi(r) + t\phi(r).$$

So

$$\frac{r}{t}(\phi(r) - \phi(t)) \le \frac{\varphi(t) - \varphi(r)}{t} \le \phi(r) - \phi(t).$$

Hence, taking limits when $r \to 0^+$, by Remark 2.2 and Corollary 2.1, we have that

$$0 \le \frac{\varphi(t) - \varphi(0)}{t} \le \phi(0^+) - \phi(t).$$

Now, taking limits when $t \to 0^+$, and again by Remark 2.2, we get

$$\lim_{t\to 0^+} \frac{\varphi(t) - \varphi(0)}{t} = 0,$$

as we wanted to show. □

2.2. *Proof of Theorem* 1.1

We proceed in three steps.
Step 1. We show that $S_q(t)$ is differentiable at $t = 0$ and

$$S_q'(0) = \phi(0^+).$$

We have that

$$\frac{S_q(t) - S_q(0)}{t} = \frac{\varphi(t) - \varphi(0)}{t} + \phi(t).$$

Then, by Remark 2.2 and Theorem 2.6,

$$S_q'(0) = \lim_{t\to 0^+} \frac{S_q(t) - S_q(0)}{t} = \phi(0^+).$$

Step 2. We show that there exists u extremal for $S_q(A)$ such that $\|u\|_{L^q(\partial\Omega)} = 1$ and

$$\phi(0^+) = \int_\Omega (|\nabla u|^p + |u|^p)\mathrm{div}\, V \,\mathrm{d}x - p\int_\Omega |\nabla u|^{p-2}\langle \nabla u, {}^T V' \nabla u\rangle \,\mathrm{d}x.$$

By Theorem 2.1,

$$\|v_t\|^p_{W^{1,p}(\Omega)} = \varphi(t) \to \varphi(0) = S_q(0) \quad \text{when } t \to 0^+. \tag{2.7}$$

Then there exists $u \in W^{1,p}(\Omega)$ and $t_n \to 0^+$ when $n \to \infty$ such that

$$v_{t_n} \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \tag{2.8}$$

$$v_{t_n} \to u \quad \text{strongly in } L^q(\partial\Omega), \tag{2.9}$$

$$v_{t_n} \to u \quad \text{a.e. in } \Omega. \tag{2.10}$$

By (2.9) and (2.10), $u \in W_A^{1,p}(\Omega)$ and $\|u\|_{L^q(\partial\Omega)} = 1$ and by (2.8)

$$S_q(0) = \lim_{n\to\infty} \|v_{t_n}\|^p_{W^{1,p}(\Omega)} \geq \|u\|^p_{W^{1,p}(\Omega)} \geq S_q(0),$$

then

$$S_q(0) = \|u\|^p_{W^{1,p}(\Omega)}. \tag{2.11}$$

Moreover, by (2.7), (2.8) and (2.11), we have that

$$v_{t_n} \to u \quad \text{strongly in } W^{1,p}(\Omega).$$

Therefore

$$\begin{aligned}\phi(0^+) &= \lim_{n\to\infty} \phi(v_{t_n})\\ &= \int_\Omega (|\nabla u|^p + |u|^p)\text{div } V \,\mathrm{d}x - p\int_\Omega |\nabla u|^{p-2}\langle \nabla u,\ {}^T V' \nabla u^T\rangle \,\mathrm{d}x.\end{aligned}$$

Step 3. Finally, we show that

$$\begin{aligned}S_q'(0) &= \int_\Omega (|\nabla u|^p + |u|^p)\text{div } V \,\mathrm{d}x - p\int_\Omega |\nabla u|^{p-2}\langle \nabla u,\ {}^T V' \nabla u^T\rangle \,\mathrm{d}x\\ &= (1-p)\int_{\partial A} \left|\frac{\partial u}{\partial \nu}\right|^p \langle V, \nu\rangle \,\mathrm{d}S.\end{aligned}$$

To show this we require that $u \in C^2$. However, this is not true in general. Since u is an extremal for $S_q(A)$ and $\|u\|_{L^q(\partial\Omega)} = 1$, we know that u is weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega\backslash\overline{A},\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = S_q(A)|u|^{q-2}u & \text{on } \partial\Omega\backslash\overline{A},\\ u = 0 & \text{on } \partial A,\end{cases}$$

and by [8] we get that u belongs to the class $C^{1,\delta}$ for some $0 < \delta < 1$.

Now, in order to overcome our difficulty, we proceed as follows. We consider the following problem, let $\varepsilon > 0$

$$S_\varepsilon := \inf_{v \in W_A^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega (|\nabla v|^2 + \varepsilon^2)^{\frac{p-2}{2}} |\nabla v|^2 + |v|^p \, \mathrm{d}x}{\left\{ \displaystyle\int_{\partial\Omega} |v|^q \, \mathrm{d}S \right\}^{\frac{p}{q}}}. \tag{2.12}$$

Let u_ε be the normalized positive eigenvalue associated to S_ε. Observe that the eigenfunction is weak solution to

$$\begin{cases} -\mathrm{div}(|\nabla u_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2} \nabla u_\varepsilon) + |u_\varepsilon|^{p-2} u_\varepsilon = 0 & \text{in } \Omega \setminus \overline{A}, \\ (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{(p-2)/2} \dfrac{\partial u_\varepsilon}{\partial \nu} = S_\varepsilon |u_\varepsilon|^{q-2} u_\varepsilon & \text{on } \partial\Omega, \\ u_\varepsilon = 0 & \text{on } \partial A. \end{cases} \tag{2.13}$$

It is well known that the solution u_ε to (2.13) is of class $C^{2,\rho}(\Omega \setminus \overline{A})$ for some $0 < \rho < 1$ (see [6]).

Thus, since $u_\varepsilon \in W_A^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)$ and $\|u_\varepsilon\|_{L^q(\partial\Omega)} = 1$ for all $\varepsilon > 0$, we have that

$$\begin{aligned} S_q(A) &\le S_\varepsilon \\ &\le \int_\Omega (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 + |u_\varepsilon|^p \, \mathrm{d}x \\ &\le \int_\Omega (|\nabla u|^2 + \varepsilon^2)^{\frac{p-2}{2}} |\nabla u|^2 + |u|^p \, \mathrm{d}x. \end{aligned}$$

Then $\lambda_\varepsilon \to S_q(0)$ as $\varepsilon \to 0^+$ and the normalized eigenfunction u_ε associated to λ_ε are bounded in $W^{1,p}(\Omega)$ uniformly in $\varepsilon > 0$. Therefore, there exists a sequence, that we still call $\{u_\varepsilon\}$, and a function $w \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup w \quad \text{weakly in } W^{1,p}(\Omega), \\ u_\varepsilon &\to w \quad \text{strongly in } L^q(\partial\Omega), \\ u_\varepsilon &\to w \quad \text{a.e. in } \Omega. \end{aligned}$$

Hence, $w \in W_A^{1,p}(\Omega)$, $\|w\|_{L^q(\partial\Omega)} = 1$ and

$$\begin{aligned} S_q(A) &= \lim_{\varepsilon \to 0^+} S_\varepsilon \\ &= \lim_{\varepsilon \to 0^+} \int_\Omega (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2 + |u_\varepsilon|^p \, \mathrm{d}x \\ &\ge \int_\Omega |\nabla w|^p + |w|^p \, \mathrm{d}x \\ &\ge S_q(A). \end{aligned}$$

These imply that w is a normalized positive extremal for $S_q(A)$ and $\|u_\varepsilon\|_{W^{1,p}(\Omega)} \to \|w\|_{W^{1,p}(\Omega)}$ as $\varepsilon \to 0^+$, and therefore

$$u_\varepsilon \to w \quad \text{strongly in } W^{1,p}(\Omega).$$

Let $U \subset\subset \Omega$ be a smooth open subset such that $U \backslash \overline{A}$ is a smooth open set and the support of V is contained in U. By [8], there exists $\delta \in (0,1)$ such that $w, u_\varepsilon \in C^{1,\delta}(\overline{U\backslash \overline{A}})$. Moreover, there exists a constant C independent of $\varepsilon > 0$ such that

$$\|u_\varepsilon\|_{C^{1,\delta}(\overline{U\backslash \overline{A}})} \le C.$$

Then, we have that $u_\varepsilon \to w$ and $\nabla u_\varepsilon \to \nabla w$ uniformly in $\overline{U\backslash \overline{A}}$ as $\varepsilon \to 0^+$.

Hence,

$$\begin{aligned} S_q'(0) &= \int_\Omega (|\nabla w|^p + |w|^p)\text{div } V \, \mathrm{d}x - p\int_\Omega |\nabla w|^{p-2}\langle \nabla w, \ {}^T V' \nabla w^T\rangle \, \mathrm{d}x \\ &= \lim_{\varepsilon\to 0^+} \int_\Omega [(|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p}{2}}| + |u_\varepsilon|^p]\text{div } V \, \mathrm{d}x \\ &\quad - p\int_\Omega |\nabla w|^{p-2}\langle \nabla w, {}^T V' \nabla w^T\rangle \, \mathrm{d}x, \end{aligned}$$

and since

$$\begin{aligned} \text{div}(|u_\varepsilon|^p V) &= |u_\varepsilon|^p \text{div } V + p|u_\varepsilon|^{p-2} u_\varepsilon \langle \nabla u_\varepsilon, V\rangle, \\ \text{div}((|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p}{2}} V) &= (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p}{2}} \text{div } V + p(|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon D^2 u_\varepsilon, V\rangle \\ &= (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p}{2}} \text{div } V + p(|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, \nabla\langle \nabla u_\varepsilon, V\rangle\rangle \\ &\quad - \ p(|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, {}^T V' \nabla u_\varepsilon^T\rangle, \end{aligned}$$

we have that

$$S_q'(0) = \lim_{\varepsilon \to 0^+} a_\varepsilon - p b_\varepsilon$$

where

$$\begin{aligned} a_\varepsilon &= \int_\Omega \text{div}((|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p}{2}} V + |u_\varepsilon|^p V) \, \mathrm{d}x, \\ b_\varepsilon &= \int_\Omega \{(|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, \nabla\langle \nabla u_\varepsilon, V\rangle\rangle + |u_\varepsilon|^{p-2} u_\varepsilon \langle \nabla u_\varepsilon, V\rangle\} \, \mathrm{d}x. \end{aligned}$$

Now, integrating by parts and using that $\text{supp}(V) \subset \Omega$ and $u_\varepsilon = 0$ on $\partial\Omega$, we obtain that

$$a_\varepsilon = \int_{\partial A} (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p}{2}} \langle V, \nu\rangle \, \mathrm{d}S,$$

and since u_ε is solution of (2.13), we have

$$b_\varepsilon = \int_{\partial A} (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{\frac{p-2}{2}} \langle \nabla u_\varepsilon, V\rangle \langle \nabla u_\varepsilon, \nu\rangle \, \mathrm{d}S,$$

where ν is the exteriror normal vector to $\Omega\backslash\overline{A}$. Then using that $\nabla w_\varepsilon \to \nabla w$ uniformly in $\overline{U\backslash\overline{A}}$ as $\varepsilon \to 0^+$, we get that

$$S_q'(0) = \int_{\partial A} |\nabla w|^p \langle V, \nu\rangle \, \mathrm{d}S - p\int_{\partial A} |\nabla w|^{p-2} \langle \nabla w, \nu\rangle \langle \nabla w, V\rangle \, \mathrm{d}S.$$

Hence, since $\nabla w = \frac{\partial w}{\partial \nu}\nu$ on ∂A,

$$S_q'(0) = (1-p)\int_{\partial A} \left|\frac{\partial w}{\partial \nu}\right|^p \langle V, \nu\rangle \, \mathrm{d}S,$$

as we wanted to show. □

3. Lack of Symmetry in the Ball

In this section, we consider the case where $\Omega = B_R$ and $A = B_r$ with $r < R$ and show Theorems 1.3 and 1.4 and Proposition 1.5. The proofs are based on the arguments of [2, 7] adapted to our problem. In order to simplify notations, we write $S_q(r)$ instead $S_q(B_r)$.

First, we proof Theorem 1.3, for this we need the following proposition

Proposition 3.1. *Let* $1 < q < p$. *The non-negative solution of* (1.3) *is unique.*

Proof. Suppose that there exist two non-negative solutions u and v of (1.3). By Remark 1.2 it follows that $u, v > 0$ on $\partial\Omega$. Let $v_n = v + \frac{1}{n}$ with $n \in \mathbb{N}$, using first Picone's identity (see [1]) and the weak formulation of (1.3) we have

$$\begin{aligned}
0 &\leq \int_{B_R} |\nabla u|^p \, \mathrm{d}x - \int_{B_R} |\nabla v_n|^{p-2} \nabla v_n \nabla\left(\frac{u^p}{v_n^{p-1}}\right) \mathrm{d}x \\
&= \int_{B_R} |\nabla u|^p \, \mathrm{d}x - \int_{B_R} |\nabla v|^{p-2} \nabla v \nabla\left(\frac{u^p}{v_n^{p-1}}\right) \mathrm{d}x \\
&= -\int_{B_R} u^p \, \mathrm{d}x + \lambda\int_{\partial B_R} u^q \, \mathrm{d}S + \int_{B_R} v^{p-1}\frac{u^p}{v_n^{p-1}} \, \mathrm{d}x - \lambda\int_{\partial B_R} v^{q-1}\frac{u^p}{v_n^{p-1}} \, \mathrm{d}S \\
&\leq \lambda\int_{\partial B_R} u^q \, \mathrm{d}S - \lambda\int_{\partial B_R} v^{q-1}\frac{u^p}{v_n^{p-1}} \, \mathrm{d}S.
\end{aligned}$$

Thus, by the Monotone Convergence Theorem,

$$\begin{aligned}
0 &\leq \int_{\partial B_R} u^q \, \mathrm{d}S - \int_{\partial B_R} v^{q-1}\frac{u^p}{v^{p-1}} \, \mathrm{d}S \\
&= \int_{\partial B_R} u^p(u^{q-p} - v^{q-p}) \, \mathrm{d}S.
\end{aligned}$$

Note that the role of u and v in the above equation are exchangeable. Therefore, adding we get

$$0 \leq \int_{\partial B_R} (u^p - v^p)(u^{q-p} - v^{q-p})\, \mathrm{d}S.$$

Since $q < p$ we have that $u \equiv v$ on ∂B_R. Then, by uniqueness of solution to the Dirichlet problem, we get $u \equiv v$ in B_R. □

Remark 3.2. As the problem (1.3) is rotationally invariant, by uniqueness we obtain that the non-negative solution of (1.3) must be radial. Therefore, if $\Omega = B_R, A = B_r$ and $1 \leq q \leq p$ we can suppose that the extremal for $S_q(r)$ found in the Theorem 1.1 is non-negative and radial.

Now we can prove the Theorem 1.3,

Proof of Theorem 1.3. We consider $\Omega = B_R, A = B_r$ and $1 \leq q \leq p$. By Theorem 1.3 and Remark 3.2 there exist a non-negative and radial normalized extremal for $S_q(r)$ such that

$$S_q'(0) = (1-p) \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu} \right|^p \langle V, \nu \rangle \, \mathrm{d}S.$$

Since u is radial

$$\frac{\partial u}{\partial \nu} \equiv c \quad \text{on } \partial B_r,$$

where c is a constant.

Thus, using that we are dealing with deformations V that preserves the volume of the B_r, we have that

$$S_q'(0) = (1-p)c^p \int_{\partial B_r} \langle V, \nu \rangle \, \mathrm{d}S = (p-1)c^p \int_{B_r} \mathrm{div}(V)\, \mathrm{d}x = 0.$$

□

To prove Theorem 1.4, we need two previous results.

Proposition 3.3. *Let $r > 0$ fixed. Then, there exists a positive radial function u_0 such that*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \mathbb{R}^N \backslash B_r, \\ u = 0 & \text{on } \partial B_r. \end{cases} \tag{3.1}$$

This u_0 is unique up to a constant factor and for any $R > r$ the restriction of u_0 to B_R is the first eigenfunction of (1.3) with $q = p$.

Proof. For $R > r$, let u_R be the unique solution of the Dirichlet problem

$$\begin{cases} \Delta_p u_R = |u_R|^{p-2} u_R & \text{in } B_R \backslash \overline{B_r}, \\ u(R) = 1, \\ u(r) = 0. \end{cases}$$

Then, by uniqueness, u_R is a non-negative and radial function. Moreover, by the regularity theory and maximum principle we have $\frac{\partial u_R}{\partial \nu}(r) \neq 0$ (see [8, 12]). Thus, for any $R > r$, we define the restriction of u_0 by

$$u_0 = \frac{u_R}{\dfrac{\partial u_R}{\partial \nu}(r)}.$$

By uniqueness of the Dirichlet problem, it is easy to check that u_0 is well defined and is a non-negative radial solution of (3.1). Furthermore, by the simplicity of $S_p(r), u_0$ is the eigenfunction associated to $S_p(r)$ for every $R > r$. □

Proposition 3.4. *Let v be a radial solution of* (1.3)*. Then v is a multiple of u_0. In particular any radial minimizer of* (1.2) *is a multiple of u_0.*

Proof. Let $a > 0$ be such that $v = au_0$ on $\partial B(0, R)$. Then v and au_0 are two solutions to the Dirichlet problem $\Delta_p w = w^{p-1}$ and $w = v$ on $\partial(B_R \backslash \overline{B_r})$. Hence, by uniqueness, we have that $v = au_0$ in B_R. □

Remark 3.5. If $1 < q < p$ then the solution of (1.3), by Remark 3.2 and Proposition 3.4, is a multiple of u_0.

Now we can deal with the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $R > r$ be fixed and consider u_0 to be the non-negative radial function given by Proposition 3.3 such that that $u_0 = 1$ on ∂B_R. Then, by Proposition 3.4, it is enough to prove that u_0 is not a minimizer for $S_q(r)$ when $q > Q(R)$.

First let us move this symmetric configuration in the x_1 direction. For any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ we denote $x_t = (x_1 - t, x_2, \dots, x_N)$ and define

$$U(t)(x) = u_0(x_t).$$

Observe that U vanishes in $A_t := B_r(te_1)$ (the ball with center te_1 and radius r) a subset of B_R of the same measure of B_r for all t small.

Consider the function

$$h(t) = \frac{f(t)}{g(t)}$$

where

$$f(t) = \int_{B_R} |\nabla U|^p + U^p \, \mathrm{d}x \quad \text{and} \quad g(t) = \left(\int_{\partial B_R} U^q \, \mathrm{d}S \right)^{\frac{p}{q}}.$$

We observe that $h(0) = 0$ and since h is an even function, we have $h'(0) = 0$. Now,

$$h''(0) = \left. \frac{f''g^2 - fgg'' - 2f'gg' - 2fgg'}{g^3} \right|_{t=0}.$$

Next we compute these terms. First, since u_0 is the first eigenfunction of (1.3) with $q = p$ and $u_0 = 1$ on ∂B_R, we get

$$f(0) = S_p(r)|\partial B_R| \quad \text{and} \quad g(0) = |\partial B_R|^{\frac{p}{q}}.$$

Thus, by Gauss–Green's theorem and using the fact that u_0 is radial, we get

$$f'(0) = -\int_{B_R} \frac{\partial}{\partial x_1}(|\nabla u_0|^p + u_0^p)\,\mathrm{d}x = \int_{\partial B_R} (|\nabla u_0|^p + u_0^p)\nu_1\,\mathrm{d}S = 0.$$

Again, since u_0 is radial,

$$g'(0) = \frac{p}{q}\left(\int_{\partial B_R} u^q\,\mathrm{d}S\right)^{\frac{p}{q}-1}\left(\int_{\partial B_R} \frac{\partial u^q}{\partial x_1}\,\mathrm{d}S\right) = 0.$$

Finally, using that $u_0 = 1$ on ∂B_R, we obtain

$$g''(0) = p|\partial B_R|^{\frac{p}{q}-1}\int_{\partial B_R} (q-1)\left(\frac{\partial u_0}{\partial x_1}\right)^2 + \frac{\partial^2 u_0}{\partial x_1^2}\,\mathrm{d}S$$

and, by the Gauss–Green's theorem

$$\begin{aligned} f''(0) &= p\int_{B_R} \frac{\partial}{\partial x_1}\left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial x_1} + \frac{1}{p}\frac{\partial u_0^p}{\partial x_1}\right)\mathrm{d}x \\ &= p\int_{\partial B_R} \left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial x_1} + \frac{1}{p}\frac{\partial u_0^p}{\partial x_1}\right)\nu_1\,\mathrm{d}S. \end{aligned}$$

Then

$$\begin{aligned} h''(0) = \frac{p}{|\partial B_R(0)|^{p/q}}\Bigg[&\int_{\partial B_R} \left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial x_1} + \frac{1}{p}\frac{\partial u_0^p}{\partial x_1}\right)\nu_1\,\mathrm{d}S \\ &- S_p(r)\int_{\partial B_R} (q-1)\left(\frac{\partial u_0}{\partial x_1}\right)^2 + \frac{\partial^2 u_0}{\partial x_1^2}\,\mathrm{d}S\Bigg]. \end{aligned}$$

Thus, since u_0 is radial, we get

$$\begin{aligned} h''(0) = \frac{p}{N|\partial B_R(0)|^{p/q}}\Bigg[&\int_{\partial B_R} \left(\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial \nu} + \frac{1}{p}\frac{\partial u_0^p}{\partial \nu}\right)\mathrm{d}S \\ &- S_p(r)\int_{\partial B_R} (q-1)|\nabla u_0|^2 + \Delta u_0\,\mathrm{d}S\Bigg]. \end{aligned}$$

Now, by definition, $u_0(x) = u_0(|x|)$ and α satisfies

$$(s^{N-1}|u_0'|^{p-1}u_0')' = s^{N-1}u_0^{p-1} \quad \forall\, s > r$$

with $u_0(R) = 0$ and $u_0(r) = 0$, moreover, by Proposition 3.3, we have

$$u_0'(s)^{p-1} = S_p(r)u_0(s)^{p-1} \quad \forall\, s > r.$$

Then

$$\frac{1}{2}|\nabla u_0|^{p-2}\frac{\partial|\nabla u_0|^2}{\partial\nu}+\frac{1}{p}\frac{\partial u_0^p}{\partial\nu}=\frac{S_p(r)^{\frac{1}{p-1}}}{p-1}\left(1-\frac{N-1}{R}S_p(r)\right)+S_p(r)^{\frac{1}{p-1}}$$

and

$$S_p(r)[(q-1)|\nabla u_0|^2+\Delta u_0]=(q-1)S_p(r)^{\frac{p+1}{p-1}}+\frac{S_p(r)^{\frac{1}{p-1}}}{p-1}\left(1-\frac{N-1}{R}S_p(r)\right)$$
$$+\frac{N-1}{R}S_p(r)^{\frac{p}{p-1}}.$$

Therefore

$$h''(0)=\frac{pS_p^{\frac{1}{p-1}}}{N|\partial B_R|^{\frac{p}{q}-1}}\left[1-(q-1)S_p(r)^{\frac{p}{p-1}}-\frac{N-1}{R}S_p(r)\right].$$

Thus, if $q>Q(R)$ we get that $h''(0)<0$ and so 0 is a strict local maxima of ψ. So we have proved that

$$S_q(r)=h(0)>h(t)\geq S_q(B_r(te_1))$$

for all t small. Therefore a symmetric configuration is not optimal. □

To finish the paper we prove Proposition 1.5.

Proof of Proposition 1.5. We proceed in two steps.
Step 1. First we show that, for $R>r$, $S_p(R,r)=S_p(r)$ verifies the differential equation

$$\frac{\partial S_p}{\partial R}=-\frac{N-1}{R}S_p+1-(p-1)S_p^{\frac{p}{p-1}} \tag{3.2}$$

with the condition

$$S_p|_{R=r}=+\infty.$$

Again we consider $u_0(x)=u_0(|x|)$ the non-negative radial function given by Proposition 3.3. Thus, for all $R>r$, we get

$$\begin{cases}(p-1)(u_0')^{p-2}u_0''+\dfrac{N-1}{R}(u_0')^{p-1}=u_0^{p-1},\\ u_0'(R)^{p-1}=S_pu_0(R)^{p-1},\\ u_0(r)=0.\end{cases}$$

Then

$$S_p=\left(\frac{u_0'(R)}{u_0(R)}\right)^{p-1}.$$

Thus

$$\frac{\partial S_p}{\partial R} = (p-1)\left(\frac{u_0'(R)}{u_0(R)}\right)^{p-2} \frac{u_0''(R)u_0(R) - u_0'(R)^2}{u_0(R)^2}$$
$$= (p-1)\left(\frac{u_0'(R)}{u_0(R)}\right)^{p-2} \frac{u_0''(R)}{u_0(R)} - (p-1)S_p^{\frac{p}{p-1}}$$
$$= (p-1)\frac{u_0'(R)^{p-2}u_0''(R)}{u_0(R)^{p-1}} - (p-1)S_p^{\frac{p}{p-1}}$$
$$= 1 - \frac{N-1}{R}S_p - (p-1)S_p^{\frac{p}{p-1}}.$$

On the other hand, since (by definition) $\frac{\partial u_0}{\partial \nu} \equiv 1$ on ∂B_r, we get that $u'(r) = 1$. Then

$$\lim_{R\to r} S_p = \lim_{R\to r}\left(\frac{u_0'(R)}{u_0(R)}\right)^{p-1} = +\infty.$$

Now, it is easy to check that $\lim_{R\to r} Q(R) = 1^-$.

Step 2. Finally, we prove that

$$\lim_{R\to+\infty} Q(R) = p.$$

We begin differentiating (3.2) to obtain

$$\frac{\partial^2 S_p}{\partial R^2} = \frac{N-1}{R^2}S_p - \frac{N-1}{R}\frac{\partial S_p}{\partial R} - pS_p^{\frac{1}{p-1}}\frac{\partial S_p}{\partial R}.$$

Then, since $S_p > 0$, at any critical point ($S_p' = 0$) we have that $S_p'' > 0$. Thus, S_p has at most one critical point, which is a minimum. If S_p has a minimum, then there exist $R_0 > r$ such that $S_p'(R_0) = 0$. Moreover, since $S_p'(R) \neq 0$ for any $R \neq R_0$ and $S_p \to +\infty$ as $R \to r$ and by (3.2), we get that $S_p' < 0$ for all $r < R < R_0$ and $S_p' > 0$ for all $R > R_0$. Thus, using again (3.2) we have that $S_p^{\frac{p}{p-1}} < \frac{1}{p-1}$ for all $R > R_0$. Then S_p is strictly increasing as a function of R and bonded for all $R > R_0$. Consequently $S_p' \to 0$ as $R \to +\infty$. It follows, by (3.2), that $S_p^{\frac{p}{p-1}} \to \frac{1}{p-1}$ as $R \to +\infty$. On the other hand using (1.6) and (3.2) we see that

$$S_p = (Q(R) - p)S_p^{\frac{p}{p-1}}. \tag{3.3}$$

So, if S_p has a minimum, we get that $Q(R) > p$ for all $R > R_0$ and $Q(R) \to p^+$ as $R \to +\infty$. Now, If S_p has not critical points so $S_p' \neq 0$ for all $R > r$ and using that $S_p \to +\infty$ as $R \to r$ and (3.2) we get that $S_p' < 0$ for all $R > r$. Consequently, in this case, S_p is strictly decreasing and therefore $S_p' \to 0$ as $R \to +\infty$ and by (3.2) we have that $S_p \to \frac{1}{p-1}$ as $R \to +\infty$. Then, if S_p has not critical points, we get $Q(R) < p$ and $Q(R) \to p^-$ as $R \to +\infty$. □

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References

[1] W. Allegretto and Y. X. Huang, A Picone's identity for the p-Laplacian and applications, *Nonlinear Anal.* **32**(7) (1998) 819–830.

[2] J. Fernández Bonder, P. Groisman and J. D. Rossi, Optimization of the first Steklov eigenvalue in domains with holes: A shape derivative approach, *Ann. Mat. Pura Appl. (4)* **186**(2) (2007) 341–358.

[3] J. Fernández Bonder and J. D. Rossi, A nonlinear eigenvalue problem with indefinite weights related to the Sobolev trace embedding, *Publ. Mat.* **46**(1) (2002) 221–235.

[4] J. García Melián and J. Sabina de Lis, On the perturbation of eigenvalues for the p-Laplacian, *C. R. Acad. Sci. Paris Sér. I Math.* **332**(10) (2001) 893–898.

[5] A. Henrot and M. Pierre, *Optimization de Forme: Un Analyse Géométric*, Mathematics and Applications, Vol. 48 (Springer-Verlag, 2005).

[6] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural′ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translated from the Russian by S. Smith, Translations of Mathematical Monographs, Vol. 23 (American Mathematical Society, Providence, R.I., 1967).

[7] E. J. Lami Dozo and O. Torné, Symmetry and symmetry breaking for minimizers in the trace inequality, *Commun. Contemp. Math.* **7**(6) (2005) 727–746.

[8] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12**(11) (1988) 1203–1219.

[9] S. Martínez and J. D. Rossi, Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition, *Abstr. Appl. Anal.* **7**(5) (2002) 287–293.

[10] J. D. Roossi, First variations of the best Sobolev trace constant with respect to the domain, *Math. Bull. Canad.* **51**(1) (2008) 140–145.

[11] M. W. Steklov, Sur les problémes fondamentaux en physique mathématique, *Ann. Sci. Ecole Norm. Sup.* **19** (1902) 445–490.

[12] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**(3) (1984) 191–202.