UNIFORM LECH'S INEQUALITY

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ABSTRACT. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. We prove that if $\mathrm{e}(\widehat{R}_{\mathrm{red}}) > 1$, then the classical Lech's inequality can be improved uniformly for all $\mathfrak{m}$-primary ideals, that is, there exists $\varepsilon > 0$ such that $\mathrm{e}(I) \leq d!(\mathrm{e}(R) - \varepsilon)\ell(R/I)$ for all $\mathfrak{m}$-primary ideals $I \subseteq R$. This answers a question raised in [3]. We also obtain partial results towards improvements of Lech's inequality when we fix the number of generators of I.

1. Introduction

The origin of this paper is a simple inequality of Lech, proved in [5], that connects the colength and the multiplicity of an $\mathfrak{m}$-primary ideal in a Noetherian local ring $(R, \mathfrak{m})$.

Theorem 1 (Lech's inequality)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let I be an $\mathfrak{m}$-primary ideal of R. Then we have*

$$\mathrm{e}(I) \leq d! \, \mathrm{e}(R)\ell(R/I),$$

where $\mathrm{e}(I)$ denotes the Hilbert–Samuel multiplicity of I and $\mathrm{e}(R) := \mathrm{e}(\mathfrak{m})$.

Lech observed that his inequality is never sharp if $d \geq 2$ (see [5, Page 74, after (4.1)]): that is, when $d \geq 2$ we always have a strict inequality in Theorem 1. In [3], we raised the problem of improving Lech's inequality by replacing $\mathrm{e}(R)$ with a smaller constant. This problem is partially motivated by [6], where Mumford considered the quantity

$$\sup_{\sqrt{I}=\mathfrak{m}} \left\{ \frac{\mathrm{e}(I)}{d!\ell(R/I)} \right\}$$

and showed that this has close connections with singularities on the compactification of the moduli spaces of smooth varieties constructed via Geometric Invariant Theory. Our proposed refinement of Lech's inequality is the following (see [3, Conjecture 1.2]):

Conjecture 2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$.*

(a) *If $\widehat{R}$ has an isolated singularity, i.e., $\widehat{R}_P$ is regular for all $P \in \operatorname{Spec} \widehat{R} - \{\mathfrak{m}\}$, then*

$$\lim_{N \to \infty} \sup_{\substack{\sqrt{I}=\mathfrak{m} \\ \ell(R/I)>N}} \left\{ \frac{\mathrm{e}(I)}{d!\ell(R/I)} \right\} = 1.$$

(b) *We have $\mathrm{e}(\widehat{R}_{\mathrm{red}}) > 1$ if and only if*

$$\lim_{N \to \infty} \sup_{\substack{\sqrt{I}=\mathfrak{m} \\ \ell(R/I)>N}} \left\{ \frac{\mathrm{e}(I)}{d!\ell(R/I)} \right\} < \mathrm{e}(R).$$

Roughly speaking, we expect that the constant $\mathrm{e}(R)$ on the right hand side of Lech's inequality can usually be replaced by a smaller number as long as the colength of the ideal is large. In [3], we established the first part of Conjecture 2 when R has positive characteristic with perfect residue field and the second part of Conjecture 2 when R has equal characteristic. Our main goal in this article is to settle the second part of Conjecture 2 in full generality by proving the following, which can be viewed as a uniform version of Lech's inequality:

Theorem 3 (=Theorem 14)**.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Suppose $\mathrm{e}(\widehat{R}_{\mathrm{red}}) > 1$. Then there exists $\varepsilon > 0$ such that for any $\mathfrak{m}$-primary ideal I, we have*

$$\mathrm{e}(I) \leq d!(\mathrm{e}(R) - \varepsilon)\ell(R/I).$$

The main case of Conjecture 2 (b) follows immediately from Theorem 3, see Corollary 15. Our approach to Theorem 3 is similar to the strategy in the equal characteristic case proved in [3, Theorem 5.8]. However, the main reason that the argument in [3] does not carry to mixed characteristic is because it crucially relies on a refined version of Lech's inequality for ideals with a fixed number of generators in equal characteristic (see [3, Proposition 5.7], recalled in Theorem 11) which essentially follows from work of Hanes [2, Theorem 2.4] on Hilbert–Kunz multiplicity. We do not know whether such a version of Lech's inequality holds in mixed charactersitic (though we expect it holds, see Conjecture 24). Due to the absence of this ingredient in mixed characteristic, we prove Theorem 3 by carefully passing to certain associated graded rings to reduce to an equal characteristic setting so that [3, Proposition 5.7] can be applied.

On the other hand, our strategy in the proof of Theorem 3 does allow us to obtain a weaker version of [3, Proposition 5.7] valid in all characteristics for integrally closed ideals. A value of this result is not just in mixed characteristic, it also removes the need of a reduction modulo p argument used in characteristic 0 to deduce [3, Theorem 5.8] from the result of Hanes.

Theorem 4 (=Corollary 23)**.** *Let $d \geq 2$ and $N \geq d$ be two positive integers. Then there exists a constant $c = c(N,d) \in (0,1)$ such that for any Noetherian local ring $(R,\mathfrak{m})$ of dimension d and any $\mathfrak{m}$-primary integrally closed ideal I which can be generated by N elements we have*

$$\mathrm{e}(I) \leq d!c\,\mathrm{e}(R)\ell(R/I).$$

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2. Preliminaries

Throughout this article, all rings are commutative, Noetherian, with multiplicative identity 1. We use $\ell(M)$ to denote the length of a finite R-module M and $\mu(M)$ to denote the minimal number of generators of M.

Definition 5. Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d and I be an $\mathfrak{m}$-primary ideal. The *Hilbert–Samuel multiplicity* of I is defined as

$$\mathrm{e}(I) = \lim_{n\to\infty} \frac{d!\ell(R/I^n)}{n^d}.$$

It is well-known that $\mathrm{e}(I)$ is always a positive integer. The Hilbert–Samuel multiplicity is closely related to integral closure. Recall that an element $x \in R$ is integral over an ideal I if it satisfies an equation of the form $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ where $a_k \in I^k$. The set of all elements x integral over I is an ideal and is denoted by $\overline{I}$, called the integral closure of I. The Hilbert–Samuel multiplicity is an invariant of the integral closure, i.e., $\mathrm{e}(I) = \mathrm{e}(\overline{I})$. Thus, we always have an inequality $\mathrm{e}(I)/\ell(R/I) \leq \mathrm{e}(\overline{I})/\ell(R/\overline{I})$. In particular, Conjecture 2 can be restricted to integrally closed ideals. Another related concept is $\mathfrak{m}$-full ideals. We briefly recall the definition following [7].

Definition 6. Let $(R,\mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{m} = (x_1,\ldots,x_n)$ and let $\widetilde{R} = R(t_1,\ldots,t_n)$, and consider the general linear form $z = t_1x_1 + \cdots + t_nx_n$. An ideal I of R is called $\mathfrak{m}$-*full* if $\mathfrak{m}I\widetilde{R} : z = I\widetilde{R}$.

The following remark summarizes some useful properties of $\mathfrak{m}$-full ideals

Remark 7. With notation as in Definition 6, we have

(1) If I is $\mathfrak{m}$-full, then $I\widetilde{R} : z = I\widetilde{R} : \mathfrak{m}$ ([7, Lemma 1]).
(2) If I is integrally closed, then I is $\mathfrak{m}$-full or $I = \sqrt{(0)}$ ([1, Theorem 2.4]).
(3) If I is $\mathfrak{m}$-primary and $\mathfrak{m}$-full, then $\mu(I) \geq \mu(J)$ for any ideal $J \supseteq I$ ([7, Theorem 3]).
(4) If I is $\mathfrak{m}$-primary and $\mathfrak{m}$-full, then $\mu(I) = \ell(\widetilde{R}/(z,I)\widetilde{R}) + \mu(I(\widetilde{R}/z\widetilde{R}))$ ([7, Theorem 2]).

The associated graded ring. Our key argument relies on passage to certain associated graded rings in order to transfer to the equal characteristic setting. We record some notations and simple facts about initial (form) ideals in associated graded rings. Let $J \subseteq R$ be an ideal and let $\mathrm{gr}_J(R) = \bigoplus_n J^n/J^{n+1}$ be the associated graded ring of R with respect to J. If $I \subseteq R$ is another ideal then we will use

$$\mathrm{in}_J(I) := \bigoplus_n \frac{I \cap J^n + J^{n+1}}{J^{n+1}} \subseteq \mathrm{gr}_J(R)$$

to denote the initial ideal of I (or form ideal in the notation of [5]) in the associated graded ring. Now let $(R,\mathfrak{m})$ be a Noetherian local ring and $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. It is well-known and easy to check that $\ell(\mathrm{gr}_J(R)/\mathrm{in}_J(I)) = \ell(R/I)$. Furthermore, since $\mathrm{in}_J(I)^n \subseteq \mathrm{in}_J(I^n)$ and $\dim(R) = \dim(\mathrm{gr}_J(R))$, we have $\mathrm{e}(I) \leq \mathrm{e}(\mathrm{in}_J(I))$.

Lemma 8. *Let $(R,\mathfrak{m})$ be a Noetherian local ring. Then $\mathrm{gr}_{(\mathfrak{m},T)}(R[T]) = \mathrm{gr}_{\mathfrak{m}}(R)[T]$. Moreover, via this identification, the initial ideal of a T-homogeneous ideal $I = \sum_k I_kT^k$ is $\sum_k \mathrm{in}_{\mathfrak{m}}(I_k)T^k$.*

Proof. The first claim follows from the second by considering the unit ideal (so that $I_k = R$ for all k). We know that the image of I on the left hand side is

$$\mathrm{in}_{(\mathfrak{m},T)}(I) = \bigoplus_{n\geq 0} \frac{I \cap (\mathfrak{m},T)^n}{I \cap (\mathfrak{m},T)^{n+1}}.$$

Note that $I \cap (\mathfrak{m}, T)^n = \sum_{k=0}^{\infty}(I_k \cap \mathfrak{m}^{n-k})T^k$, where we set $\mathfrak{m}^{n-k} := R$ for $k \geq n$ for convenience. By restricting to fixed T-degree components and noting that $I_k \cap \mathfrak{m}^{n-k} = I_k \cap \mathfrak{m}^{n+1-k} = I_k$ for $n > k$, we may further decompose

$$\mathrm{in}_{(\mathfrak{m},T)}(I) = \bigoplus_{n\geq 0}\bigoplus_{k=0}^{n} \frac{I_k \cap \mathfrak{m}^{n-k}}{I_k \cap \mathfrak{m}^{n+1-k}}T^k = \bigoplus_{k\geq 0}\left(\bigoplus_{n\geq k} \frac{I_k \cap \mathfrak{m}^{n-k}}{I_k \cap \mathfrak{m}^{n+1-k}}\right)T^k = \bigoplus_{k\geq 0}\mathrm{in}_{\mathfrak{m}}(I_k)T^k. \quad \square$$

3. Main Results

In this section we prove our main results. We begin with a few lemmas.

Lemma 9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I = I_0 + I_1T + I_2T^2 + \cdots$ be a T-homogeneous ideal of finite colength in $R[T]$. Then we have*

1. *$\mu(I) \leq 1 + \ell(R/I_0)(\mu(I_0) + \ell(R/I_0)/2 - 1/2)$.*
2. *If I_0 is $\mathfrak{m}$-full, then $\mu(I) \leq 1 + \ell(R/I_0)\mu(I_0)$.*
3. *If $\dim(R) = 1$, then $\mu(I) \leq 1 + \ell(R/I_0)(\mathrm{e}(R) + \ell(\mathrm{H}^0_{\mathfrak{m}}(R))$.*

Proof. We have containments $I_0 \subseteq I_1 \subseteq \cdots$ and this sequence will eventually include the unit ideal R. Let $n_0 = 0$ and define the sequence $\{n_k\}$ inductively so that $n_{k+1} = \min\{n \mid I_n \neq I_{n_k}\}$. Let $C = \ell(R/I_0)$ and note that $\ell(R/I_{n_{k+1}}) \leq \ell(R/I_{n_k}) - 1 \leq C - k$, so the length of this sequence is at most C. Without loss of generality, we may assume there are exactly $C + 1$ distinct ideals $I_{n_0}, I_{n_1}, \dots, I_{n_C} = R$ (having fewer distinct ideals in this sequence will result in smaller and thus better bound on $\mu(I)$).

Let $x_{k,1}, \dots, x_{k,D_k}$ be a minimal generating set for I_{n_k}. Then it is easy to see that I can be generated by $\{x_{k,i}T^{n_k}, T^{n_C}\}_{k=0,i=1}^{k=C-1,i=D_k}$. Now in case (2), we have $\mu(I_0) \geq \mu(J)$ for any $I_0 \subset J$ by Remark 7 and thus $D_k \leq \mu(I_0)$ for each k, and in case (3), we have $D_k \leq \mathrm{e}(R) + \ell(\mathrm{H}^0_{\mathfrak{m}}(R))$ for each k (for example, see [3, Lemma 5.5]). Hence the assertion follows. Finally, in case (1), note that $D_k = \mu(I_{n_k}) \leq \mu(I_{n_0}) + \ell(I_{n_k}/I_{n_0})$ for each k. Thus we have

$$\mu(I) \leq 1 + \sum_{k=1}^{C-1}(\mu(I_0) + \ell(I_{n_k}/I_{n_0})) \leq 1 + \sum_{i=0}^{C-1}(\mu(I_0) + i) = 1 + C(\mu(I_0) + C/2 - 1/2).$$

This completes the proof. $\square$

We next prove a local Bertini-type result, this should be well-known to experts and we thank Bernd Ulrich for suggesting the argument.

Lemma 10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring which satisfies Serre's condition (R_s) and has dimension at least $s + 2$. If $\mathfrak{m} = (x_1, \dots, x_n)$, then $R(t_1, \dots, t_n)/(t_1x_1 + \cdots + t_nx_n)$ still satisfies (R_s).*

Proof. Let P be a height $s+1$ prime in $S = R[t_1, \dots, t_n]$ that contains $z = t_1x_1 + \cdots + t_nx_n$. It is enough to show that S_P/zS_P is regular.

Let $Q = P \cap R$. We first claim that $\mathrm{ht}(Q) \leq s$. For if $\mathrm{ht}(Q) = s + 1$, then we must have $P = Q[t_1, \dots, t_n]$, but then P cannot contain z because $Q \neq \mathfrak{m}$ (since $\mathrm{ht}(\mathfrak{m}) = \dim(R) \geq s + 2 > \mathrm{ht}(Q)$), which is a contradiction. Thus, without loss of generality, we assume that $x_1 \notin Q$, so $R_Q[t_1, \dots, t_n]/(z) \cong R_Q[t_2, \dots, t_n]$ is regular because R_Q is regular. Therefore, $(S/zS)_P \cong (R_Q[t_2, \dots, t_n])_P$ is also regular. $\square$

We will need the following version of Lech's inequality, which is proved in [3] using Hanes' work on Hilbert–Kunz multiplicity [2, Theorem 2.4] and reduction mod $p > 0$.

Theorem 11 ([3, Proposition 5.7])**.** *Let $d \geq 2$ and $N \geq d$ be two positive integers. Then there exists a constant $c = c(N, d) \in (0, 1)$ such that for any equal characteristic Noetherian local ring $(R, \mathfrak{m})$ of dimension d and any $\mathfrak{m}$-primary ideal I with $\mu(I) \leq N$, we have*

$$\mathrm{e}(I) \leq d! c\, \mathrm{e}(R)\ell(R/I).$$

In fact, one can take $c = (1 - \frac{1}{N^{1/(d-1)}})^{d-1}$.

We now prove our main technical result.

Theorem 12. *Let $(R, \mathfrak{m})$ be a two-dimensional Noetherian complete local ring which satisfies Serre's condition (R_0). If $\mathrm{e}(R) > 1$, then there exists $\epsilon > 0$ such that $\mathrm{e}(I) \leq 2(\mathrm{e}(R) - \epsilon)\ell(R/I)$ for all $\mathfrak{m}$-primary ideals I.*

Proof. Let $P_1, \ldots, P_n$ be the minimal primes of R such that $\dim(R/P_i) = 2$. Since R is (R_0), we know that 0 has a primary decomposition

$$0 = P_1 \cap P_2 \cap \cdots \cap P_n \cap P_{n+1} \cap \cdots \cap P_m \cap Q_1 \cap \cdots \cap Q_k$$

where $P_{n+1}, \ldots, P_m$ are (possibly) minimal primes of R whose dimensions are less than 2 and $Q_1, \ldots, Q_k$ are (possibly) embedded components. If we replace R by $\widetilde{R} = R/(P_1 \cap \cdots \cap P_n)$, then it follows by the additivity formula for multiplicities that for all $\mathfrak{m}$-primary ideals $I \subseteq R$, we have $\mathrm{e}(I, R) = \mathrm{e}(I, \widetilde{R})$ while $\ell(R/I) \geq \ell(\widetilde{R}/I\widetilde{R})$. It follows that

$$\frac{\mathrm{e}(I)}{2 \cdot \ell(R/I)} \leq \frac{\mathrm{e}(I\widetilde{R})}{2 \cdot \ell(\widetilde{R}/I\widetilde{R})}.$$

Therefore to prove the result for R, it is enough to establish it for $\widetilde{R}$ (note that $\mathrm{e}(R) = \mathrm{e}(\widetilde{R})$). Thus we may replace R by $\widetilde{R}$ to assume that R is reduced and equidimensional.

Let $x_1, \ldots, x_n$ be a generating set of $\mathfrak{m}$. Note that by Lemma 10, $R(t_1, \ldots, t_n)/(t_1x_1 + \cdots + t_nx_n)$ satisfies (R_0). Since it is excellent, its completion still satisfies (R_0). Note that the depth of the completion of $R(t_1, \ldots, t_n)$ is at least one (since this is true for R). Therefore, after replacing R by the completion of $R(t_1, \ldots, t_n)$, we may assume that there exists a nonzerodivisor $z \in \mathfrak{m}$ such that z is a part of a minimal reduction of $\mathfrak{m}$ and $S = R/zR$ is (R_0), and that the residue field of R is infinite. Note that $\mathrm{e}(S) = \mathrm{e}(R)$ since z is part of a minimal reduction of $\mathfrak{m}$. By [3, Proposition 4.10], there exists C such that for all ideals $J \subseteq S$ with $\ell(S/J) > C$, we have that $\mathrm{e}(J) \leq \frac{3}{2} \cdot \ell(S/J)$. Let us fix this C.

We first consider an arbitrary $\mathfrak{m}$-primary ideal $I \subseteq R$ such that $\ell(S/IS) \leq C$. We take the associated graded ring of R with respect to the ideal (z), and we use $\mathrm{in}_z(I)$ to denote the initial ideal of I in $\mathrm{gr}_z(R)$. Since z is a nonzerodivisor, we have $\mathrm{gr}_z(R) \cong S[T]$. It follows that

$$\mathrm{in}_z(I) = I_0 + I_1T + \cdots + I_{N-1}T^{N-1} + T^N$$

where $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{N-1}$ are $\mathfrak{m}_S$-primary ideals in S. Note that our assumption on I says that $\ell(S/I_0) \leq C$, which is a constant that does not depend on I, N, or any of the I_i. Since

$\mathrm{e}(\mathrm{in}_z(I)) \geq \mathrm{e}(I)$ and $\ell(R/I) = \ell(S[T]/\mathrm{in}_z(I))$, we have

$$\frac{\mathrm{e}(I)}{2\cdot\ell(R/I)} \leq \frac{\mathrm{e}(\mathrm{in}_z(I))}{2\cdot\ell(S[T]/\mathrm{in}_z(I))}. \tag{1}$$

We next take the associated graded ring of $S[T]$ with respect to $(\mathfrak{m}, T)$. By Lemma 8 the initial ideal of $\mathrm{in}_z(I)$ is

$$J := \mathrm{in}_{\mathfrak{m}}(I_0) + \mathrm{in}_{\mathfrak{m}}(I_1)T + \cdots + \mathrm{in}_{\mathfrak{m}}(I_{N-1})T^{N-1} + T^N \subseteq \mathrm{gr}_{\mathfrak{m}}(S)[T].$$

Note that $\mathrm{e}(J) \geq \mathrm{e}(\mathrm{in}_z(I))$ and $\ell(S[T]/\mathrm{in}_z(I)) = \ell((\mathrm{gr}_{\mathfrak{m}} S)[T]/J)$, thus we have

$$\frac{\mathrm{e}(\mathrm{in}_z(I))}{2\cdot\ell(S[T]/\mathrm{in}_z(I))} \leq \frac{\mathrm{e}(J)}{2\cdot\ell(R/J)}. \tag{2}$$

By Lemma 9, $\mu(J) \leq 1 + C(\mathrm{e}(S) + \ell(\mathrm{H}^0_{\mathrm{in}_{\mathfrak{m}}(\mathfrak{m})}(\mathrm{gr}_{\mathfrak{m}}(S)))$. Since $\mathrm{gr}_{\mathfrak{m}}(S)[T]$ contains a field, $S/\mathfrak{m}S$, and has dimension two, by Theorem 11, there exists a constant $0 < \epsilon \ll 1$ (which depends on C, R and S, but not on I!) such that

$$\frac{\mathrm{e}(J)}{2\cdot\ell(R/J)} \leq (1-\epsilon)\,\mathrm{e}(\mathrm{gr}_{\mathfrak{m}}(S)[T]) = (1-\epsilon)\,\mathrm{e}(R). \tag{3}$$

Putting (1), (2), (3) toegther, we have proved the theorem for all I such that $\ell(S/IS) \leq C$. We can further shrink ϵ to guarantee that $\frac{3}{2} < (1-\epsilon)\,\mathrm{e}(R)$ since $\mathrm{e}(R) > 1$.

Finally, we use induction on $\ell(R/I)$ to show that ϵ works for all $\mathfrak{m}$-primary ideal $I \subseteq R$. We may assume that $\ell(S/IS) > C$, then by [3, Lemma 5.1] and the second paragraph of the proof, we have

$$\frac{\mathrm{e}(I)}{2\cdot\ell(R/I)} \leq \max\left\{\frac{\mathrm{e}(I:z)}{2\cdot\ell(R/(I:z))}, \frac{\mathrm{e}(IS)}{\ell(S/IS)}\right\} \leq \max\left\{(1-\epsilon)\,\mathrm{e}(R), \frac{3}{2}\right\} = (1-\epsilon)\,\mathrm{e}(R),$$

where for the second inequality we are using induction on the colength. □

We next deduce the higher dimensional case from the two-dimensional case via induction on dimension, this is similar to the strategy in [3, Theorem 5.8], the only difference is that here we use Lemma 10 instead of Flenner's result [3, Lemma 5.4] (in equal characteristic).

Corollary 13. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d such that $\widehat{R}$ satisfies Serre's condition (R_0). If $d \geq 2$ and $\mathrm{e}(R) > 1$, then there exists $\epsilon > 0$ such that $\mathrm{e}(I) \leq d!(\mathrm{e}(R) - \epsilon)\ell(R/I)$ for all $\mathfrak{m}$-primary ideals I.*

Proof. We may assume that R is complete. We use induction on $d \geq 2$. Theorem 12 provides the base case. Suppose $d \geq 3$ and that $x_1, \dots, x_n$ is a generating set for $\mathfrak{m}$. We can replace R by $R(t_1, \dots, t_n)$. Then we consider $R' = R(t_1, \dots, t_n)/(t_1x_1 + \cdots + t_nx_n)$. By Lemma 10, R' (and hence $\widehat{R'}$) still satisfies (R_0), $\dim(R') = d-1$, and $\mathrm{e}(R') = \mathrm{e}(R)$. By induction the assertion holds for R'. That is, there exists ϵ such that $\mathrm{e}(J) \leq (d-1)!(\mathrm{e}(R') - \epsilon)\ell(R'/J)$ for any $\mathfrak{m}$-primary ideal $J \subseteq R'$. We use induction on $\ell(R/I)$ to show that the same ϵ works for R (the initial case $I = \mathfrak{m}$ is obvious). By [3, Lemma 5.1] we have

$$\frac{\mathrm{e}(I)}{d!\ell(R/I)} \leq \max\left\{\frac{\mathrm{e}(I:z)}{d!\ell(R/(I:z))}, \frac{\mathrm{e}(IR')}{(d-1)!\ell(R'/IR')}\right\} \leq \mathrm{e}(R') - \epsilon = \mathrm{e}(R) - \epsilon.$$

where $z = t_1x_1 + \cdots + t_nx_n$. This completes the proof. □

Here is our uniform Lech's inequality, now valid in all characteristic.

Theorem 14 (Uniform Lech's inequality)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Suppose $\mathrm{e}(\widehat{R}_{\mathrm{red}}) > 1$. Then there exists $\varepsilon > 0$ such that for any $\mathfrak{m}$-primary ideal I, we have*

$$\mathrm{e}(I) \leq d!(\mathrm{e}(R) - \varepsilon)\ell(R/I).$$

Proof. This follows by the same argument as in [3, Proof of Corollary 5.9], we just replace the citation of [3, Theorem 5.8] by Corollary 13 above. □

Now we can prove Conjecture 2.

Corollary 15. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. Then we have $\mathrm{e}(\widehat{R}_{\mathrm{red}}) > 1$ if and only if*

$$\lim_{N\to\infty} \sup_{\substack{\sqrt{I}=\mathfrak{m} \\ \ell(R/I)>N}} \left\{ \frac{\mathrm{e}(I)}{d!\ell(R/I)} \right\} < \mathrm{e}(R).$$

Proof. The "if" direction was proved in [3, Proposition 5.3]. For the "only if" direction, the one-dimensional case was proved in [3, Proposition 5.11], and when $d \geq 2$, the result follows immediately from Theorem 14. □

3.1. **Uniform Lech's inequality for ideals with fixed number of generators.** In this subsection we prove partial results towards a characteristic-free version of Theorem 11. We first treat the two-dimensional case, we will need the following corollary of Lemma 9.

Corollary 16. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an $\mathfrak{m}$-full $\mathfrak{m}$-primary ideal. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ and define $S = R(t_1, \ldots, t_n)/\operatorname{H}^0_{\mathfrak{m}}(R)R(t_1, \ldots, t_n)$ with the general linear form $z = t_1x_1 + \cdots + t_nx_n$. Then $\mathrm{in}_z(IS)$, the initial ideal of IS in $\mathrm{gr}_z(S)$, can be generated by at most $1 + \mu(I)(\mu(I) - 1)/2$ homogeneous elements.*

Proof. Since S has positive depth, z is a nonzerodivisor on S. By Lemma 9, we know that $\mathrm{in}_z(IS)$ can be generated by at most

$$1 + \ell(S/(I, z))(\mu(I(S/zS)) + \ell(S/(I, z))/2 - 1/2)$$

(homogeneous) elements. Both the minimal number of generators and the colength do not increase when passing to a quotient ring. Hence if we let $\widetilde{R} = R(t_1, \ldots, t_n)$ then the above bound is no greater than $1 + \ell(\widetilde{R}/(I, z)\widetilde{R})(\mu(I(\widetilde{R}/z\widetilde{R})) + \ell(\widetilde{R}/(I, z)\widetilde{R})/2 - 1/2)$.

Because I is $\mathfrak{m}$-full, we know that $\mu(I) = \mu(I(\widetilde{R}/z\widetilde{R})) + \ell(\widetilde{R}/(I, z)\widetilde{R})$ by Remark 7. Let $\ell(\widetilde{R}/(I, z)\widetilde{R}) = t$. Then our upper bound becomes $1 + t(\mu(I) - t/2 - 1/2)$ which clearly has the maximum at $t = \mu(I) - 1/2$. Since t is an integer, we can use $t = \mu(I)$ instead. The result follows. □

Theorem 17. *There exists a constant $c = c(N) \in (0, 1)$ such that for any Noetherian local ring $(R, \mathfrak{m})$ of dimension two and any $\mathfrak{m}$-primary $\mathfrak{m}$-full ideal I which can be generated by N elements we have*

$$\mathrm{e}(I) \leq 2c\,\mathrm{e}(R)\ell(R/I).$$

Proof. We use the notation of Corollary 16. Observe that passing from R to S does not affect multiplicity and does not increase the colength. Let $J := \mathrm{in}_z(IS)$ be the initial ideal of IS in $\mathrm{gr}_z(S) \cong (S/zS)[T]$. We know that

$$\frac{\mathrm{e}(I)}{2\cdot \ell(R/I)} \le \frac{\mathrm{e}(IS)}{2\cdot \ell(S/IS)} \le \frac{\mathrm{e}(J)}{2\cdot \ell(\mathrm{gr}_z(S)/J)}. \tag{4}$$

We next write $J = J_0 + J_1T + \cdots + J_{K-1}T^{K-1} + T^K$ as a T-homogenous ideal of $(S/zS)[T]$. By Corollary 16, we know that J can be generated by at most $1 + N(N-1)/2$ homogeneous elements. Following the proof of Lemma 9, we define the sequence $\{n_k\}$ that labels the distinct J_i by setting $n_0 = 0$ and $n_{k+1} = \min\{n \mid J_n \neq J_{n_k}\}$. We now choose appropriately $\le 1 + N(N-1)/2$ generators $\{a_{i,j}T^j\}$ of J, so that $a_{1,0}, a_{2,0}, \ldots$ generate $J_0 = J_{n_0}$ and J_{n_k} is generated by $a_{i,j}$ with $j \le k$. By adjoining one more generator to each J_{n_k} if necessary, we may assume that the chosen generating set contain a minimal reduction of each J_{n_k} as one of $a_{i,k}$ (note that each J_{n_k} is an ideal in the one-dimensional ring S/zS). The total number of adjoined generators is at most $N(N-1)/2$, because there are at most $1 + N(N-1)/2$ distinct J_{n_k}s and we do not need to adjoin a new generator to J_{n_0} (since we can let $a_{1,0}$ be a minimal reduction of J_0).

Inspired by positive characteristic methods, for each positive integer q we define $J^{[q]}$ as the ideal generated by $\{a_{i,j}^q T^{jq}\}$ and define $J_i^{[q]}$ accordingly. Note that $J^{[q]}$ and each $J_i^{[q]}$ in principle might depend on the chosen generating set $\{a_{i,j}T^j\}$ but this will not be a problem. By definition, we have

$$J^{[q]} = J_0^{[q]} + J_0^{[q]}T + \cdots J_0^{[q]}T^{q-1} + J_1^{[q]}T^q + \cdots + J_1^{[q]}T^{2q-1} + J_2^{[q]}T^{2q} + \cdots.$$

It follows that $\ell((S/zS)[T]/J^{[q]}) = q\sum_{i=0}^{K-1} \ell(S/(z, J_i^{[q]}))$. Note that since $\dim(S/zS) = 1$ and our selected list of generators contains a minimal reduction a_i of every J_i, we have that

$$\mathrm{e}(J_i) = \lim_{q\to\infty} \frac{\ell(S/(z, a_i^q))}{q} \ge \lim_{q\to\infty} \frac{\ell(S/(z, J_i^{[q]}))}{q} \ge \lim_{q\to\infty} \frac{\ell(S/(z, J_i^q))}{q} = \mathrm{e}(J_i).$$

Therefore we have

$$\lim_{q\to\infty} \frac{\ell((S/zS)[T]/J^{[q]})}{q^2} = \sum_{i=0}^{K-1} \frac{\ell(S/(z, J_i^{[q]}))}{q} = \sum_{i=0}^{K-1} \mathrm{e}(J_i).$$

Applying Lech's inequality (Theorem 1) to each $J_i \subseteq S/zS$, we then have

$$\lim_{q\to\infty} \frac{\ell((S/zS)[T]/J^{[q]})}{q^2} \le \mathrm{e}(S/zS) \sum_{i=1}^{K-1} \ell(S/(z, J_i)) = \mathrm{e}(S/zS)\ell((S/zS)[T]/J). \tag{5}$$

At this point, we follow the argument in [2, Theorem 2.4]. First, since $J^{[q]}$ is clearly contained in J^q and is generated by at most $1 + N(N-1)$ elements, for any integer s we can surject $1 + N(N-1)$ copies of $(S/zS)[T]/J^s$ onto $J^{[q]}/(J^{[q]} \cap J^{q+s})$. Thus

$$\ell((S/zS)[T]/J^{[q]}) \ge \ell((S/zS)[T]/J^{q+s}) - (1 + N(N-1))\ell((S/zS)[T]/J^s).$$

After setting $s = q/(N(N-1))$ this gives

$$\lim_{q\to\infty} \frac{\ell((S/zS)[T]/J^{[q]})}{q^2} \geq \frac{\mathrm{e}(J)}{2}\left(\left(1+\frac{1}{N(N-1)}\right)^2 - \frac{1+N(N-1)}{N^2(N-1)^2}\right) = \frac{\mathrm{e}(J)}{2}\left(1+\frac{1}{N(N-1)}\right).$$

Combining with (5) we obtain that

$$\frac{\mathrm{e}(J)}{2\cdot \ell((S/zS)[T]/J)} \leq \mathrm{e}(S/zS)\left(1+\frac{1}{N(N-1)}\right)^{-1} = \mathrm{e}(R)\left(1+\frac{1}{N(N-1)}\right)^{-1}.$$

This together with (4) completes the proof, and we can take $c = (1+\frac{1}{N(N-1)})^{-1}$. □

Corollary 18. *Let $(R,\mathfrak{m})$ be a two-dimensional Noetherian local ring. Let $\mathfrak{m} = (x_1,\dots,x_n)$ and define $R' = R(t_1,\dots,t_n)/(t_1x_1+\cdots+t_nx_n)$. Then for any positive integer N there exists $\varepsilon > 0$ such that for any $\mathfrak{m}$-primary ideal I with $\ell(R'/IR') \leq N$, we have*

$$\mathrm{e}(I) \leq 2(1-\varepsilon)\,\mathrm{e}(R)\ell(R/I).$$

Proof. We may assume I is integrally closed since replacing I by its integral closure $\overline{I}$ will not affect $\mathrm{e}(I)$ and will not increase $\ell(R/I)$. By Remark 7 and [3, Lemma 5.5], $\mu(I) \leq N+\mu(IR')$ is then bounded by a constant independent of I, so we may apply Theorem 17. □

Remark 19. One can give an alternative proof of Theorem 12 (and thus the uniform Lech's inequality Theorem 14) via Corollary 18: in fact, Corollary 18 handles exactly the case $\ell(S/IS) \leq C$ in the proof of Theorem 12. This alternative approach avoids the use of Theorem 11 (i.e., [3, Proposition 5.7]), which benefits in equal characteristic 0 as it avoids the reduction mod p argument needed to prove Theorem 11.

Finally, we treat the higher dimensional case, the proof turns out to be easier. We need a couple of lemmas. The first one is due to Mumford.

Lemma 20 ([6, Proof of Lemma 3.6])**.** *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d and let $I = I_0 + I_1T + I_2T^2 + \cdots + I_{N-1}T^{N-1} + T^N \subseteq R[T]$ be a T-homogeneous ideal of finite colength. Then we have*

$$\frac{\mathrm{e}(I)}{(d+1)!\ell(R[T]/I)} \leq \frac{\sum_{i=0}^{N-1}\mathrm{e}(I_i)}{d!\sum_{i=0}^{N-1}\ell(R/I_i)} \leq \max_i\left\{\frac{\mathrm{e}(I_i)}{d!\ell(R/I_i)}\right\}.$$

Lemma 21. *For any Noetherian local ring $(R,\mathfrak{m})$ of dimension $d \geq 2$ and any $\mathfrak{m}$-primary ideal I of colength at most N we have*

$$\mathrm{e}(I) \leq d!\left(1-\frac{1}{d!N}\right)\mathrm{e}(R)\ell(R/I).$$

Proof. The proof of Lech's inequality via Noether's normalization given in [4] shows that it is enough to prove the statement in an equicharacteristic regular local ring of dimension d. Since it was shown by Lech [5, Page 74, after (4.1)] that in dimension at least two, we always have strict inequality in Theorem 1 (so that each rational number appeared on the left hand side below must be strictly less than 1), it follows that

$$\max_{\ell(R/I)\leq N}\left\{\frac{\mathrm{e}(I)}{d!\ell(R/I)}\right\} \leq 1-\frac{1}{d!N}.$$ □

Theorem 22. *Let $d > 2$ and $N \geq d$ be two positive integers. Then there exists a constant $c = c(N, d) \in (0, 1)$ such that for any Noetherian local ring $(R, \mathfrak{m})$ of dimension d and any $\mathfrak{m}$-primary $\mathfrak{m}$-full ideal I which can be generated by N elements we have*

$$\mathrm{e}(I) \leq d! c \, \mathrm{e}(R) \ell(R/I).$$

Proof. Let us write $\mathfrak{m} = (x_1, \ldots, x_n)$ and define $\widetilde{R} = R(t_1, \ldots, t_n)$ with the general linear form $z = t_1 x_1 + \cdots + t_n x_n$. By Remark 7, for any $\mathfrak{m}$-primary $\mathfrak{m}$-full ideal I we have

$$\mu(I(\widetilde{R}/z\widetilde{R})) + \ell(\widetilde{R}/(I, z)\widetilde{R}) = \mu(I) \leq N.$$

We apply the same reduction as in the first paragraph of the proof of Theorem 17: we can pass from R to $S := (R/\mathrm{H}^0_{\mathfrak{m}}(R)) \otimes \widetilde{R}$ because this will not affect multiplicity and will not increase the colength, now z is a nonzerodivisor on S and we can further pass from S to $\mathrm{gr}_z(S) \cong (S/zS)[T]$ and note that if c works for all T-homogeneous ideals in $\mathrm{gr}_z(S)$, then it will work for S.

We now write $\mathrm{in}_z(IS)$, the initial ideal of IS in $\mathrm{gr}_z(S) \cong (S/zS)[T]$, as $I_0 + I_1 T + I_2 T^2 + \cdots + I_{K-1} T^{K-1} + T^K$. By Lemma 20, it is enough to show that there exists $c > 0$ such that $\mathrm{e}(I_i) \leq (d-1)!(1-c) \, \mathrm{e}(S/zS) \ell(S/(z, I_i))$. But now we have $\ell(S/(z, I_i)) \leq N$ for each I_i and $\dim(S/zS) = d - 1 \geq 2$. Thus the assertion follows from Lemma 21 applied to S/zS (and we can actually take $c = 1 - \frac{1}{(d-1)!N}$). □

Corollary 23. *Let $d \geq 2$ and $N \geq d$ be two positive integers. Then there exists a constant $c = c(N, d) \in (0, 1)$ such that for any Noetherian local ring $(R, \mathfrak{m})$ of dimension d and any $\mathfrak{m}$-primary integrally closed ideal I which can be generated by N elements we have*

$$\mathrm{e}(I) \leq d! c \, \mathrm{e}(R) \ell(R/I).$$

Proof. Since we are in dimension at least two, any $\mathfrak{m}$-primary integrally closed ideal is $\mathfrak{m}$-full, see Remark 7. The conclusion follows from Theorem 17 when $d = 2$ and Theorem 22 when $d > 2$. □

As we mentioned in the introduction, we expect Corollary 23 holds without assuming I is integrally closed (and recall that this is true in equal characteristic by Theorem 11).

Conjecture 24. *Let $d \geq 2$ and $N \geq d$ be two positive integers. Then there exists a constant $c = c(N, d) \in (0, 1)$ such that for any Noetherian local ring $(R, \mathfrak{m})$ of dimension d and any $\mathfrak{m}$-primary ideal I which can be generated by N elements we have*

$$\mathrm{e}(I) \leq d! c \, \mathrm{e}(R) \ell(R/I).$$

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