Algebraic equations for constant width curves and Zindler curves

David Rochera

BCAM – Basque Center for Applied Mathematics, Mazarredo 14, E-48009 Bilbao (Basque Country), Spain

Abstract

An explicit method to compute algebraic equations of curves of constant width and Zindler curves generated by a family of middle hedgehogs is given thanks to a property of Chebyshev polynomials. This extends the methodology used by Rabinowitz and Martinez-Maure in particular constant width curves to generate a full family of algebraic equations, both of curves of constant width and Zindler curves, defined by trigonometric polynomials as support functions.

Keywords: Constant width curves, Zindler curves, Planar algebraic curves, Hedgehogs, Resultant
2020 MSC: 52A10, 53A04, 14H50

1. Introduction

A planar convex body K is called of constant width if its width, defined as the distance between any pair of parallel supporting lines to K, is constant regardless the direction in which we measure it. The boundary of K is a called a curve of constant width. The circle is the most trivial example of this kind of curves, but there are infinitely many more (see [10] for an introduction to the topic). It is probably the Reuleaux triangle the most famous non-circular example of a constant width curve, but there are several known methods to construct smooth curves of constant width different from the circle, see e.g. [16], [11] or [1], among others. Other classical approaches include their construction as the involutes of some specific curves [6] or their generation by some support functions [7, 15].

There is another kind of curves, maybe not as famous as constant width curves but that has a very strong relationship with them. These are Zindler curves [17]. The property that defines a Zindler curve is that all chords which

Email address: `drochera@bcamath.org` (David Rochera)

This is a preprint of an article published in Journal of Symbolic Computation. The final revised and authenticated version is available online at: `https://doi.org/10.1016/j.jsc.2022.03.001`.

cuts the curve perimeter (or area) into halves, have the same length. Zindler curves are also the boundaries of figures of constant density that float in water in equilibrium in any position [5].

It is known that there is a "duality" between curves of constant width and Zindler curves (see [10] or [13]), in the sense that, under some convexity assumptions, a right angle rotation of the constant length chord that describes the width in the first case or that cuts the perimeter in a half in the second case yields the other figure. The locus of all these midpoints also describes another curve called the *middle hedgehog*, which is a projective hedgehog by construction (see Figure 1).

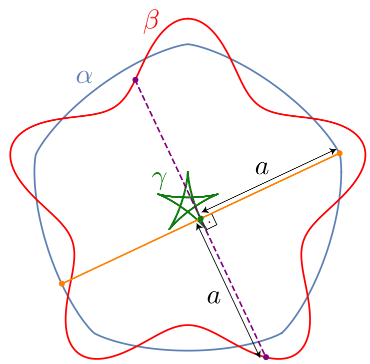

Figure 1: A curve of constant width α and its "dual" Zindler curve β. The midpoint curve γ is the middle hedgehog.

Rabinowitz asked in [15] for algebraic equations describing non-circular constant width curves. He found a quite complicated expression and he asked if simpler expressions could be given. Recently, Martinez-Maure in [9] gave an algebraic equation of a non-circular constant width curve which is simpler than the one of Rabinowitz. This was done from a parameterization of the constant width curve by a support function $p(t) = 8 - \sin(3\,t)$. However, in any case, it seems that the complexity of general algebraic equations is unavoidable for trigonometric polynomial support functions, because Bardet and Bayen showed in [2] that the minimum degree of an implicit equation defining a non-circular constant width curve of this kind is 8. The interested reader can also see [4], [12] and [14] for related works.

The objective of this short paper is to provide an explicit method to compute the algebraic equation of any constant width curve or any Zindler curve constructed from a middle hedgehog γ parameterized by a support function of the kind

$$p(t) = \frac{1}{b}\sin\big(n\,t\big), \tag{1}$$

where $b \in \mathbb{R}$ and $n = 2\,k+1$, for $k \in \mathbb{N}$. Note that the hedgehog γ is projective because $p(t) + p(t+\pi) = 0$ and that it will exhibit n cusps.

The method we propose for finding the algebraic equation is based on the same technique used by Rabinowitz in [15] and by Martinez-Maure in [9]. The main contribution is to notice that thanks to the properties of Chebyshev polynomials, a generalized version of the same procedure can be given explicitly to get an algebraic equation for any $k \in \mathbb{N}$. In addition, we provide an analogous method to get similar conclusions in the case of Zindler curves.

The method is reduced to compute the resultant of two polynomials of degrees $2\,n+2$ and $n+1$ (Theorems 1 and 2), so that symbolic computation is usually needed to compute the algebraic equation. Moreover, as we increase n, the degree of the algebraic equation increases too and it gets a wider and a more complicated form.

Finally, we observe that the polynomials we use to compute the resultant are very similar for pairs of "dual" curves (constant width curves and Zindler curves), as well as the resulting algebraic equations. This seems particularly clear in some examples, such as the one we consider in Section 5.

2. Definition of the curves and Chebyshev polynomials

2.1. Duality between curves of constant width and Zindler curves

Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be a projective hedgehog defined by a support function p:

$$\gamma(t) = p(t)\,(\cos t,\, \sin t) + p'(t)\,(-\sin t,\, \cos t),$$

where $p(t) + p(t+\pi) = 0$. From it we can construct a curve α of constant width $2\,a$ simply by taking the support function $h(t) = a + p(t)$. This is equivalent to constructing a continuous branch of the double offset to the hedgehog γ at a distance a. This is, we go from each point $\gamma(t)$ in a normal direction to γ a distance a. The resulting parameterization of α can be written as

$$\alpha(t) = \big(a + p(t)\big)\,(\cos t,\, \sin t) + p'(t)\,(-\sin t,\, \cos t).$$

Analogously, given the same hedgehog we can construct a Zindler curve β for a chord length $2\,d$ simply by going from each point $\gamma(t)$ in a tangent direction to γ a distance d. This construction is related to tire-tracks of bicycles, where the hedgehog γ represents the rear wheel of a bicycle tire-track and the Zindler curve its front wheel (see e.g. [3]). However, the Zindler curve is not parameterized by a support function. Instead, its parameterization can be written as

$$\beta(t) = p(t)\,(\cos t,\, \sin t) + \big(d + p'(t)\big)\,(-\sin t,\, \cos t).$$

From now on, γ, α and β will be the curves defined above.

We must remark that there are some restrictions on the figures obtained by these curves. A curve of constant width is usually assumed to be convex, although some non-convex cases also exhibit a special kind of "constant width", as pointed out in [8], but this width does not correspond to the actual width of the full figure. Non-convex figures can be avoided by setting a sufficiently high value for a. Analogously, a Zindler curve must be such that the chords from which we construct the figure and that divides its perimeter into two halves cut the curve at precisely two points (and not more). Similarly, this can be achieved by setting a sufficiently high value for d.

2.2. A property of Chebyshev polynomials

We look for a method to compute algebraic equations for constant width curves and Zindler curves generated by a middle hedgehog parameterized by a support function (1). The core of our method is based on a property of Chebyshev polynomials. Denote by T_n the Chebyshev polynomial of degree n. These polynomials can be defined recursively as

$$\begin{aligned}
&T_0(x) = 1,\\
&T_1(x) = x,\\
&T_n(x) = 2\,x\,T_{n-1}(x) - T_{n-2}(x), \quad n \geq 2.
\end{aligned}$$

If $n = 2\,k+1$, for $k \in \mathbb{N}$, recall that these polynomials satisfy that

$$\cos(n\,t) = T_n\big(\cos(t)\big) \tag{2}$$

and

$$\sin(n\,t) = (-1)^k\,T_n\big(\sin(t)\big). \tag{3}$$

The property we are referring to above is the following.

Lemma 1. *If n is an odd natural number and $x \in [0,1]$, then*

$$T_n\big(\sqrt{1-x^2}\big) = \sqrt{1-x^2}\,p_{n-1}(x),$$

where

$$p_{n-1}(x) = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2\,k} (-1)^k\,x^{2k}\,(1-x^2)^{\frac{n-2k-1}{2}} \tag{4}$$

is a polynomial of degree $n-1$ in x.

Proof. An explicit expression of T_n is given by

$$T_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2\,k} \big(x^2-1\big)^k\,x^{n-2k},$$

where $[\,\cdot\,]$ denotes the integer part. In our case, we have that

$$\left[\frac{n}{2}\right] = \frac{n-1}{2}.$$

Thus,

$$T_n\big(\sqrt{1-x^2}\big) = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2\,k} (-1)^k\,x^{2\,k}\,\big(\sqrt{1-x^2}\big)^{n-2k}.$$

Since $n-2\,k$ is odd, we get

$$
\begin{aligned}
T_n\big(\sqrt{1-x^2}\big) &= \sqrt{1-x^2}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{2\,k}(-1)^k\,x^{2k}\,\big(\sqrt{1-x^2}\big)^{n-2k-1}\\
&= \sqrt{1-x^2}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{2\,k}(-1)^k\,x^{2k}\,(1-x^2)^{\frac{n-2k-1}{2}}\\
&= \sqrt{1-x^2}\,p_{n-1}(x),
\end{aligned}
$$

which is what we wanted to prove. □

3. Method for constant width curves

Let $\alpha:[0,2\pi]\to\mathbb{R}^2$ be a parametric curve defined by a support function of the kind

$$
h(t)=a+p(t)=a+\frac{1}{b}\sin\big(n\,t\big),
$$

where $n=2\,k+1$, for $k\in\mathbb{N}$. Since $h(t)+h(t+\pi)=2\,a$, we have that α is a curve of constant width $2\,a$.

The coefficient $\frac{1}{b}$ is just a scale factor. As higher is n, a higher value of a is needed to get a convex curve. By setting a sufficiently low scale factor $1/b$, the full figure is scaled and, therefore, a smaller value of a is needed.

The next result describes the method to compute the algebraic equation for this kind of constant width curves.

Theorem 1. *The algebraic equation of the constant width curve α can be obtained by computing the resultant of the polynomials*

$$
(1-s^2)\,\big(a\,b+(-1)^k\,T_n(s)-n\,s\,p_{n-1}(s)\big)^2-b^2\,x^2
$$

and

$$
s\,\big(a\,b+(-1)^k\,T_n(s)\big)+n\,\big(1-s^2\big)\,p_{n-1}(s)-b\,y,
$$

which are of degrees $2\,n+2$ and $n+1$, respectively, and where p_{n-1} is the polynomial defined by (4).

Proof. The explicit expression of the curve $\alpha(t)=\big(x(t),\,y(t)\big)$ is

$$
\begin{aligned}
x(t) &= \frac{1}{b}\Big[\cos(t)\,\big(a\,b+\sin(n\,t)\big)-n\sin(t)\cos(n\,t)\Big],\\
y(t) &= \frac{1}{b}\Big[\sin(t)\,\big(a\,b+\sin(n\,t)\big)+n\cos(t)\cos(n\,t)\Big],
\end{aligned}
$$

where remember that $n=2\,k+1$, for $k\in\mathbb{N}$.

We can expand the expression of the curve in terms of $\sin(t)$ and $\cos(t)$ using (2) and (3) as follows:

$$x(t) = \frac{1}{b}\left[\cos(t)\left(a\,b + (-1)^k\,T_n\big(\sin(t)\big)\right) - n\sin(t)\,T_n\big(\cos(t)\big)\right],$$
$$y(t) = \frac{1}{b}\left[\sin(t)\left(a\,b + (-1)^k\,T_n\big(\sin(t)\big)\right) + n\cos(t)T_n\big(\cos(t)\big)\right].$$

Let's write now both equations only in terms of $s = \sin(t)$. For that, a substitution of $\cos(t)$ by $\sqrt{1-s^2}$ is also performed:

$$x(t) = \frac{1}{b}\left[\sqrt{1-s^2}\left(a\,b + (-1)^k\,T_n(s)\right) - n\,s\,T_n\left(\sqrt{1-s^2}\right)\right],$$
$$y(t) = \frac{1}{b}\left[s\left(a\,b + (-1)^k\,T_n(s)\right) + n\,\sqrt{1-s^2}\,T_n\left(\sqrt{1-s^2}\right)\right].$$

Now, by Lemma 1, we can write

$$x(t) = \frac{1}{b}\,\sqrt{1-s^2}\left[a\,b + (-1)^k\,T_n(s) - n\,s\,p_{n-1}(s)\right],$$
$$y(t) = \frac{1}{b}\left[s\left(a\,b + (-1)^k\,T_n(s)\right) + n\left(1-s^2\right)p_{n-1}(s)\right],$$

where p_{n-1} is the polynomial of degree $n-1$ given by (4).

On the one hand, observe that $x(t)$ is the product of $\sqrt{1-s^2}$ by a polynomial of degree n in s. On the other hand, $y(t)$ is itself a polynomial of degree $n+1$ in s. We can square the first equation to get

$$b^2\,x^2 = (1-s^2)\left(a\,b + (-1)^k\,T_n(s) - n\,s\,p_{n-1}(s)\right)^2.$$

The second equation can be written as

$$b\,y = s\left(a\,b + (-1)^k\,T_n(s)\right) + n\left(1-s^2\right)p_{n-1}(s).$$

From this, the statement follows, where the resultant is used to eliminate the variable s. □

4. Method for Zindler curves

A very similar method to the one described in the previous section also works to obtain the algebraic equation of Zindler curves constructed from the same middle hedgehog. The main difference from the previous case is that now we must write the expressions in terms of $c = \cos(t)$ to get equations easy to handle.

Theorem 2. *The algebraic equation of the Zindler curve β can be obtained by computing the resultant of the polynomials*

$$(1-c^2)\left(-b\,d - n\,T_n(c) + (-1)^k\,c\,p_{n-1}(c)\right)^2 - b^2\,x^2$$

and

$$c\left(b\,d+n\,T_n(c)\right)+(-1)^k\,(1-c^2)\,p_{n-1}(c)-b\,y,$$

which are of degrees $2\,n+2$ and $n+1$, respectively, and where p_{n-1} is the polynomial defined by (4).

Proof. The explicit parametric expression of the curve β is

$$\begin{aligned} x(t) &= \frac{1}{b}\Big[-\sin(t)\left(b\,d+n\cos(n\,t)\right)+\cos(t)\sin(n\,t)\Big],\\ y(t) &= \frac{1}{b}\Big[\cos(t)\left(b\,d+n\cos(n\,t)\right)+\sin(t)\sin(n\,t)\Big]. \end{aligned}$$

For $c=\cos(t)$, we have

$$\begin{aligned} x(t) &= \frac{1}{b}\Big[-\sqrt{1-c^2}\left(b\,d+n\,T_n(c)\right)+(-1)^k\,c\,T_n\big(\sqrt{1-c^2}\big)\Big],\\ y(t) &= \frac{1}{b}\Big[c\left(b\,d+n\,T_n(c)\right)+(-1)^k\,\sqrt{1-c^2}\,T_n\big(\sqrt{1-c^2}\big)\Big]. \end{aligned}$$

Now, by Lemma 1:

$$\begin{aligned} x(t) &= \frac{1}{b}\,\sqrt{1-c^2}\,\Big[-\big(b\,d+n\,T_n(c)\big)+(-1)^k\,c\,p_{n-1}(c)\Big],\\ y(t) &= \frac{1}{b}\Big[c\left(b\,d+n\,T_n(c)\right)+(-1)^k\,(1-c^2)\,p_{n-1}(c)\Big], \end{aligned}$$

where p_{n-1} is the polynomial of degree $n-1$ given by (4). Again, $x(t)$ is the product of $\sqrt{1-c^2}$ by a polynomial of degree n in c and $y(t)$ is itself a polynomial of degree $n+1$ in c.

Finally, the desired algebraic equation is found by eliminating c from the equations

$$b^2\,x^2=(1-c^2)\left(-b\,d-n\,T_n(c)+(-1)^k\,c\,p_{n-1}(c)\right)^2$$

and

$$b\,y=c\left(b\,d+n\,T_n(c)\right)+(-1)^k\,(1-c^2)\,p_{n-1}(c)$$

using a resultant. □

5. Examples of algebraic equations

Consider now an example. Let $\gamma:[0,2\pi]\to\mathbb{R}^2$ be the middle hedgehog parameterized by the support function

$$p(t)=-\sin(3\,t).$$

Here, $b=-1$ and $k=1$.

Let's compute the algebraic equation of the associated curve of constant width α and the Zindler curve β for $a=d=8$ using Theorems 1 and 2.

In this case, the polynomial (4) has the form

$$p_2(x) = 1 - 4\,x^2.$$

By Theorem 1, to compute the algebraic equation of α, we must compute the resultant of the polynomials

$$-64\,s^8 + 64\,s^6 + 128\,s^5 - 128\,s^3 - 64\,s^2 - x^2 + 64$$

and

$$8\,s^4 - 12\,s^2 - 8\,s + y + 3.$$

This produces the algebraic equation (3) given by Martinez-Maure in [9]:

$$\begin{aligned}&\left(\left(x^2+y^2\right)^2 + 8\,y\left(y^2-3\,x^2\right)\right)^2 + 432\,y\left(y^2-3\,x^2\right)\left(351 - 10\left(x^2+y^2\right)\right)\\ &= 567^3 + 28\left(x^2+y^2\right)^3 + 486\left(x^2+y^2\right)\left(67\left(x^2+y^2\right) - 567\cdot 18\right).\end{aligned}$$

We can also apply Theorem 2 to get the associated algebraic equation for the Zindler curve β: we must compute the resultant of the polynomials

$$-64\,c^8 + 192\,c^6 + 128\,c^5 - 192\,c^4 - 256\,c^3 + 128\,c - x^2 + 64$$

and

$$8\,c^4 - 4\,c^2 - 8\,c + y - 1.$$

As we can see, there is a strong similarity with the polynomials above to compute the algebraic equation of the constant width curve α. In fact, the resulting algebraic equation for β is also very similar to that of α:

$$\begin{aligned}&\left(\left(x^2+y^2\right)^2 + 8\,y\left(y^2-3\,x^2\right)\right)^2 - 16\,y\left(y^2-3x^2\right)\left(251 + 14\left(x^2+y^2\right)\right)\\ &= 55^3 + 28\left(x^2+y^2\right)^3 + 18\left(x^2+y^2\right)\left(17\left(x^2+y^2\right) + 55\cdot 26\right).\end{aligned}$$

The curves α and β defined by the algebraic equations above can be visualized in Figure 2.

6. Conclusions

We have provided an explicit method, very easy to implement in a Computer Algebra System (CAS), to compute algebraic equations of a wide class of curves of constant width and Zindler curves. The use of symbolic computation is justified because of the complexity of the corresponding algebraic equations, which can be obtained from a resultant.

Chebyshev polynomials constitute an elegant tool to implement the explicit expressions which are needed to compute the resultant in a CAS. We must mention that, by construction, the same method works for any hedgehog of constant width and any generalized Zindler curve which is obtained from the

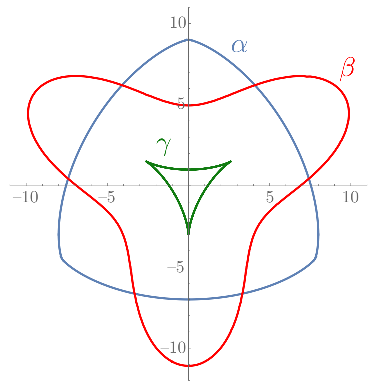

Figure 2: The curve α of constant width $2\,a = 16$ and the Zindler curve β for a chord length $2\,d = 16$. The curve γ is the middle hedgehog.

same middle hedgehog. This is, there is actually no constraint on the real values a and d to get the corresponding algebraic equation with the same method.

It would be interesting to see if similar methods are possible to get algebraic equations in higher scenarios, such as in the case of surfaces of constant width or the corresponding floating bodies in equilibrium in three dimensions. However, as far as the author knows, there are still very few references regarding the theoretical background for the latter case.

Acknowledgments

The author has been partially funded by the BCAM Severo Ochoa accreditation of excellence, Spain (SEV-2017-0718).

References

[1] Rachid Ait-Haddou, Walter Herzog, and Luc Biard, *Pythagorean-hodograph ovals of constant width*, Comput. Aided Geom. Design **25** (2008), no. 4-5, 258–273.

[2] Magali Bardet and Térence Bayen, *On the degree of the polynomial defining a planar algebraic curves of constant width*, 2013, https://arxiv.org/abs/1312.4358.

[3] Gil Bor, Mark Levi, Ron Perline, and Sergei Tabachnikov, *Tire tracks and integrable curve evolution*, Int. Math. Res. Not. IMRN (2020), no. 9, 2698–2768.

[4] Javier Bracho, Luis Montejano, and Déborah Oliveros, *A classification theorem for Zindler carrousels*, J. Dynam. Control Systems **7** (2001), no. 3, 367–384.

[5] ______, *Carousels, Zindler curves and the floating body problem*, Period. Math. Hungar. **49** (2004), no. 2, 9–23.

[6] John F. Burke, *A curve of constant diameter*, Math. Mag. **39** (1966), 84–85.

[7] Andrew D. Irving, *Curves of Constant Width & Centre Symmetry Sets*, 2006, MSc mini-dissertation.

[8] Paul J. Kelly, *Curves with a kind of constant width*, Amer. Math. Monthly **64** (1957), 333–336.

[9] Yves Martinez-Maure, *Non-circular algebraic curves of constant width: an answer to Rabinowitz*, Canadian Mathematical Bulletin (2021), 1–5.

[10] Horst Martini, Luis Montejano, and Déborah Oliveros, *Bodies of constant width*, Birkhäuser/Springer, Cham, 2019, An introduction to convex geometry with applications.

[11] Horst Martini and Zokhrab Mustafaev, *A new construction of curves of constant width*, Comput. Aided Geom. Design **25** (2008), no. 9, 751–755.

[12] Horst Martini and Senlin Wu, *On Zindler curves in normed planes*, Canad. Math. Bull. **55** (2012), no. 4, 767–773.

[13] Déborah Oliveros, *Los volantines: sistemas dinámicos asociados al problema de la flotación de los cuerpos*, Ph.D. thesis, Faculty of Science, National University of Mexico, 1997.

[14] Chatchawan Panraksa and Lawrence C. Washington, *Real algebraic curves of constant width*, Period. Math. Hungar. **74** (2017), no. 2, 235–244.

[15] Stanley Rabinowitz, *A polynomial curve of constant width*, Missouri J. Math. Sci. **9** (1997), no. 1, 23–27.

[16] Isaak M. Yaglom and Vladimir G. Boltyanskiĭ, *Convex figures*, Translated by Paul J. Kelly and Lewis F. Walton, Holt, Rinehart and Winston, New York, 1960.

[17] Konrad Zindler, *Über konvexe Gebilde II*, Monatsh. Math. Phys. **31** (1921), 25–56.