


## Codimensions of algebras with additional structures

Dedicated to our dear colleague Vesselin Drensky on the occasion of his seventieth anniversary.

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Received: 30.12.2021

Accepted/Published Online: 23.02.2022

Final Version: ..2022

**Abstract:** Let  $A$  be an associative algebra endowed with an automorphism or an antiautomorphism  $\varphi$  of order  $\leq 2$ . One associates to  $A$ , in a natural way, a numerical sequence  $c_n^\varphi(A)$ ,  $n = 1, 2, \dots$ , called the sequence of  $\varphi$ -codimensions of  $A$  which is the main tool for the quantitative investigation of the polynomial identities satisfied by  $A$ . In [13] it was proved that such a sequence is eventually nondecreasing in case  $\varphi$  is an antiautomorphism. Here we prove that it still holds in case  $\varphi$  is an automorphism and present some recent results about the asymptotics of  $c_n^\varphi(A)$ .

**Key words:** Polynomial identity,  $\varphi$ -identity, growth.

### 1. Introduction

Let  $A$  be an algebra over a field  $F$  of characteristic zero. It is well-known that the study of the polynomial identities satisfied by  $A$  is equivalent to the study of the multilinear ones and in this setting an effective way to measure such identities is through the sequence of codimensions  $c_n(A)$ ,  $n = 1, 2, \dots$ , of  $A$ . Recall that if  $P_n$  is the space of multilinear polynomials in the noncommuting variables  $x_1, \dots, x_n$  and  $\text{Id}(A)$  is the ideal of identities of  $A$ , then  $c_n(A) = \dim P_n / (P_n \cap \text{Id}(A))$ .

In case  $A$  is a PI-algebra, i.e. it satisfies a nontrivial polynomial identity,  $c_n(A)$ ,  $n = 1, 2, \dots$ , is exponentially bounded [37].

In the 1980's two main conjectures about the asymptotic behaviour of the codimensions of a PI-algebra were made:  $\lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}$  exists and is a nonnegative integer (the Amitsur's conjecture) and there exist a constant  $C$ , a semi-integer  $q$  and an integer  $d \geq 0$  such that  $c_n(A) \simeq Cn^q d^n$  (the Regev's conjecture). The first conjecture was proved by Giambruno and Zaicev in [20, 21]. They proved that for any PI-algebra  $A$ , there exist constants  $C_1 > 0, C_2, t, s, d$  such that

$$C_1 n^t d^n \leq c_n(A) \leq C_2 n^s d^n,$$

for all  $n \geq 1$ , and  $d$  is an integer called the PI-exponent  $\text{exp}(A)$  of  $A$ . Later Berele and Regev in [8, 10] solved affirmatively the conjecture of Regev for algebras with 1. Moreover, since the sequence of codimensions is eventually nondecreasing ([25]) then by [8, 10] it follows that if  $A$  is any arbitrary PI-algebra

$$C_1 n^t \text{exp}(A)^n \leq c_n(A) \leq C_2 n^t \text{exp}(A)^n \quad (1.1)$$

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2010 AMS Mathematics Subject Classification: 16R10, 16R50, 16W55, 16P90

holds where  $C_1 > 0, C_2, t$  are constants and  $t \in \frac{1}{2}\mathbb{Z}$ .

This result gives a second invariant of a T-ideal, after the PI-exponent, namely

$$t = \lim_{n \rightarrow \infty} \log_n \frac{c_n(A)}{\exp(A)^n}.$$

This last invariant was explicitly computed in [3] for the so-called fundamental algebras.

Now we turn our attention to algebras with additional structures.

Let  $F$  be a field of characteristic zero,  $A$  an associative algebra over  $F$  endowed with an automorphism or antiautomorphism  $\varphi$  of order  $\leq 2$  and  $c_n^\varphi(A)$ ,  $n = 1, 2, \dots$ , the sequence of  $\varphi$ -codimensions of  $A$ . It was proved that if  $A$  satisfies a nontrivial identity then  $c_n^\varphi(A)$ ,  $n = 1, 2, \dots$ , is exponentially bounded ([19]). Moreover, an explicit bound related to the ordinary identities of the algebra  $A$  was found in [6]. Some of the questions arising in this setting are the following: can one prove the Amitsur and Regev's conjectures for  $\varphi$ -algebras? The exponential rate of growth of the  $\varphi$ -codimensions was computed for finite dimensional algebras in [2, 7, 22] and for general PI-algebras in [1, 12, 18] and it turned out to be a nonnegative integer called the  $\varphi$ -exponent  $\exp^\varphi(A)$  of the algebra.

The Regev's conjecture was verified for finite dimensional  $\varphi$ -simple algebras in [9, 16, 26]: If  $A$  is a  $\varphi$ -simple algebra over an algebraically closed field  $F$  of characteristic zero then

$$c_n^\varphi(A) \simeq Cn^t(\dim A)^n,$$

for some constant  $C$ , where  $t \in \frac{1}{2}\mathbb{Z}$  is explicitly computed.

Now starting with the well-known inequalities for PI-algebras given in [12, 18]:

$$C_1 n^t \exp^\varphi(A)^n \leq c_n^\varphi(A) \leq C_2 n^s \exp^\varphi(A)^n \tag{1.2}$$

with  $C_1 > 0, C_2, t, s$  constants, we shall see that, for finite dimensional algebras ([13, 16]) and, as a consequence for finitely generated algebras,  $t = s \in \frac{1}{2}\mathbb{Z}$ . In this way we get a second invariant  $\lim_{n \rightarrow \infty} \log_n \frac{c_n^\varphi(A)}{\exp^\varphi(A)^n}$  of a  $T^\varphi$ -ideal, after the  $\varphi$ -exponent.

Such result is accomplished by studying a special class of algebras, the so-called  $\varphi$ -fundamental algebras. These are finite dimensional algebras that can be defined in terms of some multialternating polynomials and for such algebras the polynomial factor  $t$  in (1.2) is related to the structure of the algebra and can be determined explicitly.

Finally we shall prove that if  $A$  is any  $\varphi$ -algebra satisfying a nontrivial polynomial identity, then its sequence of  $\varphi$ -codimensions is eventually nondecreasing. For  $\varphi$ -algebras endowed with an antiautomorphism of order 2 the proof was given in [13].

## 2. On $\varphi$ -codimensions and $\varphi$ -fundamental algebras

Throughout this paper we shall denote by  $F$  a field of characteristic zero and by  $A$  an associative algebra over  $F$  endowed with an automorphism or antiautomorphism  $\varphi$  of order  $\leq 2$ ; such an algebra  $A$  will be called  $\varphi$ -algebra. Let us write  $A = A_0^\varphi + A_1^\varphi$ , where  $A_0^\varphi = \{a \in A \mid \varphi(a) = a\}$  and  $A_1^\varphi = \{a \in A \mid \varphi(a) = -a\}$ . In case  $\varphi$  is an involution (antiautomorphism)  $A_0^\varphi = A^+$  and  $A_1^\varphi = A^-$  denote the subspaces of symmetric and skew elements, respectively. If  $\varphi$  is an automorphism then  $A$  is a  $\mathbb{Z}_2$ -graded algebra (superalgebra) with grading

$(A^0, A^1)$ , where  $A^0 = A_0^\varphi$  and  $A^1 = A_1^\varphi$ . Conversely, any  $\mathbb{Z}_2$ -graded algebra  $A$ , can be viewed as an algebra with  $\varphi$ -action, where  $\varphi$  is an automorphism of  $A$  of order  $\leq 2$ . In fact, if  $(A^0, A^1)$  is the given  $\mathbb{Z}_2$ -grading, then  $\varphi : A \rightarrow A$  such that  $\varphi(a^0 + a^1) = a^0 - a^1$ , for all  $a^0 \in A^0, a^1 \in A^1$  is an automorphism of order  $\leq 2$ . Recall that the elements of  $A^0$  and  $A^1$  are called homogeneous of degree zero (or even elements) and degree one (or odd elements), respectively.

Let  $X = \{x_1, x_2, \dots\}$  be a countable set and let

$$F\langle X, \varphi \rangle = F\langle x_1, x_1^\varphi, x_2, x_2^\varphi, \dots \rangle$$

be the free associative algebra endowed with an automorphism or antiautomorphism  $\varphi$  of order  $\leq 2$ . In order to simplify the notation we shall simply write  $f = f(x_1, \dots, x_n)$  to indicate a  $\varphi$ -polynomial of  $F\langle X, \varphi \rangle$  in which the variables  $x_1, \dots, x_n$  or their image by  $\varphi$  appear. Recall that  $f = f(x_1, \dots, x_n) \in F\langle X, \varphi \rangle$  is a  $\varphi$ -identity of  $A$  and we write  $f \equiv 0$  if  $f(a_1, \dots, a_n) = 0$  for all  $a_1, \dots, a_n \in A$ .

We denote by  $\text{Id}^\varphi(A) = \{f \in F\langle X, \varphi \rangle \mid f \equiv 0 \text{ on } A\}$  the  $\Gamma^\varphi$ -ideal of  $\varphi$ -identities of  $A$ , i.e.  $\text{Id}^\varphi(A)$  is an ideal of  $F\langle X, \varphi \rangle$  invariant under all endomorphisms of the free algebra commuting with the  $\varphi$ -action.

It is well known that in characteristic zero  $\text{Id}^\varphi(A)$  is completely determined by its multilinear polynomials. We denote by  $P_n^\varphi$  the space of multilinear  $\varphi$ -polynomials of degree  $n$  in  $x_1, \dots, x_n$ , i.e. for every  $i = 1, \dots, n$ , either  $x_i$  or  $x_i^\varphi$  appears in every monomial of  $P_n^\varphi$  at degree 1 (but not both).

There is a natural action of the symmetric group  $S_n$  on the left on  $P_n^\varphi$ : if  $\sigma \in S_n$  and  $f = f(x_1, \dots, x_n) \in P_n^\varphi$ , then  $\sigma f = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ . The outcome is that

$$P_n^\varphi(A) = \frac{P_n^\varphi}{P_n^\varphi \cap \text{Id}^\varphi(A)}$$

has a structure of  $S_n$ -module and its dimension,  $c_n^\varphi(A)$ , is called the  $n$ th  $\varphi$ -codimension of  $A$ .

Despite its importance the exact computation of the  $\varphi$ -codimensions of an algebra is extremely difficult, and it has been done for very few algebras (see [17, 29–31, 33]). That is why one is led to study the asymptotic behaviour of the sequence of  $\varphi$ -codimensions. Such a sequence is bounded from above by the dimension of  $P_n^\varphi$  which is  $2^n n!$  but, in case  $A$  is a PI-algebra, it was proved in [19] that, as in the ordinary case,  $c_n^\varphi(A), n = 1, 2, \dots$ , is exponentially bounded. Actually, by a well-known theorem of Amitsur ([5]), this is still true if  $A$  satisfies a  $\varphi$ -identity in case  $\varphi$  is an antiautomorphism. The exponential rate of growth of  $c_n^\varphi(A), n = 1, 2, \dots$  was computed and shown to be an integer (see [1, 2, 7, 12, 18, 22]).

**Theorem 2.1** *Let  $A$  be a  $\varphi$ -algebra over a field of characteristic zero satisfying a nontrivial polynomial identity. Then there exist constants  $d \geq 0, C_1 > 0, C_2, t_1, t_2$  such that*

$$C_1 n^{t_1} d^n \leq c_n^\varphi(A) \leq C_2 n^{t_2} d^n. \tag{2.1}$$

Hence  $\lim_{n \rightarrow \infty} \sqrt[n]{c_n^\varphi(A)} = \exp^\varphi(A)$ , the  $\varphi$ -exponent of  $A$ , exists and is a nonnegative integer.

As a consequence of the above theorem we have that the sequence of  $\varphi$ -codimensions  $c_n^\varphi(A), n = 1, 2, \dots$ , is either polynomially bounded or grows as an exponential function  $d^n$  with  $d \geq 2$ .

In case of polynomial growth, if  $A$  is an algebra with 1, in [28, 32] it was proved that

$$c_n^\varphi(A) = qn^k + O(n^{k-1})$$

is a polynomial with rational coefficients. Moreover its leading term satisfies the inequalities

$$\frac{1}{k!} \leq q \leq \sum_{i=0}^k 2^{k-i} \frac{(-1)^i}{i!}.$$

Let us write down the inequalities given in (2.1) keeping in mind that  $d = \exp^\varphi(A)$ :

$$C_1 n^{t_1} \exp^\varphi(A)^n \leq c_n^\varphi(A) \leq C_2 n^{t_2} \exp^\varphi(A)^n. \tag{2.2}$$

Now one can ask if the polynomial factor in (2.2) is uniquely determined, i.e.  $t_1 = t_2$ , giving in this way a second invariant of a  $T^\varphi$ -ideal, after the  $\varphi$ -exponent. The answer is positive for finite dimensional  $\varphi$ -algebras ([15, 16]) and, as a consequence, by the main result in [4, 34, 35], for finitely generated algebras.

**Theorem 2.2** [15, 16] *Let  $A$  be a finitely generated  $\varphi$ -algebra over a field  $F$  of characteristic zero. If  $A$  satisfies a nontrivial polynomial identity then*

$$C_1 n^t \exp^\varphi(A)^n \leq c_n^\varphi(A) \leq C_2 n^t \exp^\varphi(A)^n,$$

where  $t \in \frac{1}{2}\mathbb{Z}$ , for some constants  $C_1 > 0, C_2$ . Hence  $\lim_{n \rightarrow \infty} \log_n \frac{c_n^\varphi(A)}{\exp^\varphi(A)^n}$  exists and is a half integer.

Now, a more concrete question would be the following: can one compute such polynomial factor for a certain class of algebras relating it to the structure of the algebra itself? The answer is positive for the class of  $\varphi$ -fundamental algebras defined in [14–16].

Let us recall the definition of  $\varphi$ -fundamental algebra.

We recall that a  $\varphi$ -polynomial  $f(x_1, \dots, x_n, Y)$  linear in the variables  $x_1, \dots, x_n$  (and in some other set of variables  $Y$ ) is alternating in  $x_1, \dots, x_n$  if  $f$  vanishes whenever we identify any two of these variables. This is equivalent to say that the polynomial changes sign whenever we exchange any two of these variables (here we exchange the indices of the two variables).

Now assume that  $A$  is a finite dimensional  $\varphi$ -algebra over an algebraically closed field  $F$  of characteristic zero.

By the Wedderburn–Malcev theorem [24, Theorem 3.4.4] for  $\varphi$ -algebras we can write

$$A = \bar{A} \oplus J$$

where  $\bar{A}$  is a semisimple subalgebra of  $A$ ,  $J = J(A)$  is the Jacobson radical and both  $\bar{A}$  and  $J$  are stable under the  $\varphi$ -action. Moreover

$$\bar{A} = A_1 \oplus \dots \oplus A_q, \tag{2.3}$$

where  $A_1, \dots, A_q$  are  $\varphi$ -simple algebras.

We recall that the  $(t, s)$ -index of  $A$  is  $Ind_{t,s}(A) = (\dim \bar{A}, s)$  where  $s \geq 0$  is the smallest integer such that  $J^{s+1} = 0$ .

We start with the following construction. Let  $J^s \neq 0$ ,  $J^{s+1} = 0$  and let  $n = \dim J$ . Then define

$$A' = \bar{A} * F\langle x_1, \dots, x_n, \varphi \rangle,$$

the free product of  $\bar{A}$  and the free algebra  $F\langle x_1, \dots, x_n, \varphi \rangle$ . If  $I_1$  is the  $\varphi$ -ideal generated by  $\{f(A') \mid f \in \text{Id}^\varphi(A)\}$ , then since  $f(\bar{A}) = 0$ , for  $f \in \text{Id}^\varphi(A)$ , we have that  $I_1 \subseteq I$ , the  $\varphi$ -ideal of  $A'$  generated by  $x_1, \dots, x_n$ .

We define

$$\mathcal{A}_s = A' / (I^{s+1} + I_1),$$

a finite dimensional algebra with  $\text{Id}^\varphi(\mathcal{A}_s) = \text{Id}^\varphi(A)$ . Also, if  $I' = I / (I^{s+1} + I_1)$ , then  $\mathcal{A}_s \cong \bar{A} + I'$ ,  $(I')^s \neq 0$ ,  $(I')^{s+1} = 0$  and  $\text{Ind}_{t,s}(\mathcal{A}_s) = (\dim \bar{A}, s) = \text{Ind}_{t,s}(A)$ .

Then we define

$$\mathcal{B}_0 = \mathcal{A}_s / (I')^s.$$

Hence  $\text{Id}^\varphi(A) = \text{Id}^\varphi(\mathcal{A}_s) \subseteq \text{Id}^\varphi(\mathcal{B}_0)$ , and  $\text{Ind}_{t,s}(\mathcal{B}_0) = (\dim \bar{A}, s - 1)$ .

Now, for any  $1 \leq i \leq q$  we denote

$$B_i = A_1 \oplus \dots \hat{A}_i \dots \oplus A_q + J,$$

where the symbol  $\hat{A}_i$  means that the algebra  $A_i$  is omitted in the direct sum.

**Definition 2.3** *The algebra  $A$  is  $\varphi$ -fundamental if either  $A$  is  $\varphi$ -simple or  $s > 0$  and*

$$\text{Id}^\varphi(A) \not\subseteq \bigcap_{i=1}^q \text{Id}^\varphi(B_i) \cap \text{Id}^\varphi(\mathcal{B}_0).$$

We remark that in case  $s > 0$ , the algebras  $B_i$  have a lower  $(t, s)$ -index. In fact the first index is lower. Also the algebra  $\mathcal{B}_0$  has a lower  $(t, s)$ -index, since  $\text{Ind}_{t,s}(\mathcal{B}_0) = (\dim \bar{A}, s - 1)$ . The main feature of these algebras is that any finite dimensional algebras satisfies the same  $\varphi$ -identities as a finite direct sum of  $\varphi$ -fundamental algebras.

**Proposition 2.4** *Every finite dimensional  $\varphi$ -algebra satisfies the same  $\varphi$ -identities as a finite direct sum of  $\varphi$ -fundamental algebras.*

Next we define the Kemer  $\varphi$ -index of  $A$ . Let  $\Gamma \subseteq F\langle X, \varphi \rangle$  be the ideal of  $\varphi$ -identities of  $A$ . Then  $\beta(\Gamma)$  is defined as the greatest integer  $t$  such that for every  $\mu \geq 1$ , there exists a multilinear  $\varphi$ -polynomial  $f(X_1, \dots, X_\mu, Y) \notin \Gamma$  alternating in the  $\mu$  sets  $X_i$  with  $|X_i| = t$ . Moreover  $\gamma(\Gamma)$  is defined as the greatest integer  $s$  for which there exists for all  $\mu \geq 1$ , a multilinear  $\varphi$ -polynomial  $f(X_1, \dots, X_\mu, Z_1, \dots, Z_s, Y) \notin \Gamma$  alternating in the  $\mu$  sets  $X_i$  with  $|X_i| = \beta(\Gamma)$  and in the  $s$  sets  $Z_j$  with  $|Z_j| = \beta(\Gamma) + 1$ . Then  $\text{Ind}_K^\varphi(\Gamma) = (\beta(\Gamma), \gamma(\Gamma))$  is called the Kemer  $\varphi$ -index of  $\Gamma$ . Since  $\Gamma = \text{Id}^\varphi(A)$ , we also say that  $(\beta(\Gamma), \gamma(\Gamma)) = (\beta(A), \gamma(A)) = \text{Ind}_K^\varphi(A)$  is the Kemer  $\varphi$ -index of  $A$ .

In general we have that  $\text{Ind}_K^\varphi(A) \leq \text{Ind}_{t,s}(A)$  in the left lexicographic order and in case  $A$  is  $\varphi$ -fundamental  $\text{Ind}_K^\varphi(A) = \text{Ind}_{t,s}(A)$  by the following.

**Theorem 2.5** ([15, 16]) *A finite dimensional  $\varphi$ -algebra  $A$  is  $\varphi$ -fundamental if and only if  $\text{Ind}_K^\varphi(A) = \text{Ind}_{t,s}(A)$ .*

In the following theorems the polynomial factor in (2.2) is explicitly computed for  $\varphi$ -fundamental algebras.

**Theorem 2.6** [16] *Let  $A = \bar{A} + J$  be a  $\varphi$ -fundamental algebra over an algebraically closed field  $F$  of characteristic zero, where  $\varphi$  is an antiautomorphism of order 2, and  $s \geq 0$  the least integer such that  $J^{s+1} = 0$ . Write  $\bar{A} = A_1 \oplus \cdots \oplus A_r \oplus A_{r+1} \oplus \cdots \oplus A_q$ , a direct sum of  $\varphi$ -simple algebras where  $A_1, \dots, A_r$  are not simple algebras, then*

$$C_1 n^{-\frac{1}{2}(\dim(\bar{A})^- - r) + s} (\dim \bar{A})^n \leq c_n^\varphi(A) \leq C_2 n^{-\frac{1}{2}(\dim(\bar{A})^- - r) + s} (\dim \bar{A})^n,$$

for some constants  $C_1 > 0, C_2$ , where  $(\bar{A})^- = \{a \in \bar{A} \mid \varphi(a) = -a\}$  is the Lie algebra of skew elements of  $\bar{A}$ . Hence

$$\lim_{n \rightarrow \infty} \log_n \frac{c_n^\varphi(A)}{\exp^\varphi(A)^n} = -\frac{1}{2}(\dim(\bar{A})^- - r) + s.$$

**Theorem 2.7** *Let  $A = \bar{A} + J$  be a  $\varphi$ -fundamental algebra over an algebraically closed field  $F$  of characteristic zero, where  $\varphi$  is an automorphism of order  $\leq 2$ , and let  $\bar{A} = A_1 \oplus \cdots \oplus A_q$  be a direct sum of  $\varphi$ -simple algebras,  $J^{s+1} = 0, s \geq 0$ . Then*

$$C_1 n^{-\frac{1}{2}(\dim(\bar{A})^0 - q) + s} (\dim \bar{A})^n \leq c_n^\varphi(A) \leq C_2 n^{-\frac{1}{2}(\dim(\bar{A})^0 - q) + s} (\dim \bar{A})^n,$$

for some constants  $C_1 > 0, C_2$ , where  $(\bar{A})^0 = \{a \in \bar{A} \mid \varphi(a) = a\}$ . Hence

$$\lim_{n \rightarrow \infty} \log_n \frac{c_n^\varphi(A)}{\exp^\varphi(A)^n} = -\frac{1}{2}(\dim(\bar{A})^0 - q) + s.$$

In case  $\varphi$  is an automorphism of order 1 we rediscover the result given in [3] in the setting of ordinary fundamental algebras.

### 3. Nondecreasing sequences of $\varphi$ -codimensions

In this section we shall prove that if  $A$  is an associative  $\varphi$ -algebra then the sequence of  $\varphi$ -codimensions  $c_n^\varphi(A)$ ,  $n = 1, 2, \dots$ , is eventually nondecreasing. In case  $\varphi$  is an antiautomorphism of order 2 the result was proved in [13].

We start by recalling some basic definitions in case  $\varphi$  is an automorphism. Let  $B = \bigoplus_{(g,i) \in \mathbb{Z}_2 \times \mathbb{Z}_2} B^{(g,i)}$  be a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra and write  $B = B^0 \oplus B^1$ , where  $B^0 = \bigoplus_{g \in \mathbb{Z}_2} B^{(g,0)}$  and  $B^1 = \bigoplus_{g \in \mathbb{Z}_2} B^{(g,1)}$ .

Let  $E = \langle e_1, e_2, \dots \mid e_i e_j = -e_j e_i \rangle$  be the infinite dimensional Grassmann algebra over  $F$  and let  $E = E^0 \oplus E^1$  be its standard  $\mathbb{Z}_2$ -grading. Here  $E^0$  (resp  $E^1$ ) is the span of all monomials in the  $e_i$ 's of even (resp. odd) length. Then, the Grassmann envelope of  $B$ ,

$$E(B) = (B^0 \otimes E^0) \oplus (B^1 \otimes E^1)$$

has a natural  $\mathbb{Z}_2$ -grading (induced from one of the  $\mathbb{Z}_2$ -gradings of  $B$ ) given by  $E(B) = \bigoplus_{g \in \mathbb{Z}_2} E(B)^g$ , where  $E(B)^0 = (B^{(0,0)} \otimes E^0) \oplus (B^{(0,1)} \otimes E^1)$  and  $E(B)^1 = (B^{(1,0)} \otimes E^0) \oplus (B^{(1,1)} \otimes E^1)$ .

In order to compute the exponential rate of growth of the  $\varphi$ -codimensions of an algebra one applies a result, proved independently in [4] and [34], which extends an important theorem of Kemer ([27, Theorem

2.3]) to the graded case. We state here the result for  $\varphi$ -algebras (superalgebras): let  $A$  be a  $\varphi$ -algebra over a field of characteristic zero satisfying a nontrivial polynomial identity. Then there exists a finite dimensional  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra  $B$  such that  $\text{Id}^\varphi(A) = \text{Id}^\varphi(E(B))$ .

Now if the field  $F$  is algebraically closed by the Wedderburn–Malcev theorem ([11]), we can write

$$B = C + J$$

where  $C$  is a maximal semisimple subalgebra of  $B$  and  $J = J(B)$  is its Jacobson radical. It is well-known that  $J$  is a graded ideal, moreover by [36] we assume, as we may, that  $C$  is a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded subalgebra of  $B$ . Hence we can write

$$C = C_1 \oplus \cdots \oplus C_k$$

where  $C_1, \dots, C_k$  are  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded simple algebras.

In [1, 2, 12] the authors proved that  $\exp^\varphi(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^\varphi(E(B))}$  coincides with the maximal dimension of an admissible subalgebra  $D_1 \oplus \cdots \oplus D_h$  of  $C$ , i.e. an algebra such that  $D_1 J D_2 J \cdots J D_h \neq 0$  with  $D_1, \dots, D_h \in \{C_1, \dots, C_k\}$  distinct.

In the following theorem we prove that the  $\varphi$ -codimensions  $c_n^\varphi(A)$ ,  $n = 1, 2, \dots$ , are eventually nondecreasing.

**Theorem 3.1** *Let  $A$  be a  $\varphi$ -algebra over a field of characteristic zero satisfying a nontrivial polynomial identity. Then the sequence of  $\varphi$ -codimensions  $c_n^\varphi(A)$ ,  $n = 1, 2, \dots$ , is eventually nondecreasing, that is,  $c_{n+1}^\varphi(A) \geq c_n^\varphi(A)$ , for  $n$  large enough.*

**Proof** If  $\varphi$  is an antiautomorphism the proof was given in [13]. So we assume that  $\varphi$  is an automorphism and let  $B = C + J$  be a finite dimensional  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra with  $J^t = 0$ , for some  $t$ , such that  $\text{Id}^\varphi(A) = \text{Id}^\varphi(E(B))$ . We shall prove that if  $n \geq t$ ,  $c_n^\varphi(E(B)) \leq c_{n+1}^\varphi(E(B))$ .

If  $B$  is a nilpotent algebra, i.e.  $C = 0$ , then  $c_n^\varphi(E(B)) = 0$  for any  $n \geq t$  and we are done.

Now assume that  $C \neq 0$ .

Given  $n \geq t$  let  $c_n^\varphi(E(B)) = r$  and let  $f_1, \dots, f_r$  be  $\varphi$ -polynomials of  $P_n^\varphi$  in the variables  $x_1, x_1^\varphi, \dots, x_n, x_n^\varphi$  that are linearly independent modulo  $P_n^\varphi \cap \text{Id}^\varphi(E(B))$ . For any  $1 \leq i \leq r$ , we construct the following  $\varphi$ -polynomials:

$$h_i = h_i(x_1, \dots, x_{n+1}) = \sum_{j=1}^n f_i(x_1, \dots, x_{n+1}x_j + x_jx_{n+1}, \dots, x_n) \in P_{n+1}^\varphi,$$

where for any  $j = 1, \dots, n$ , we have substituted in  $f_i$  the variable  $x_j$  with  $x_{n+1}x_j + x_jx_{n+1}$ .

We shall prove that  $h_1, \dots, h_r$  are linearly independent modulo  $P_{n+1}^\varphi \cap \text{Id}^\varphi(E(B))$ .

Suppose by contradiction that  $h = \sum_i \alpha_i h_i \equiv 0$  is a  $\varphi$ -identity of  $E(B)$  with some  $\alpha_i \neq 0$ . Since  $f_1, \dots, f_r$  are linearly independent modulo  $P_n^\varphi \cap \text{Id}^\varphi(E(B))$ , we have that  $f = \sum_i \alpha_i f_i$  is not a  $\varphi$ -identity of  $E(B)$ .

Recall that  $E(B) = B^0 \otimes E^0 + B^1 \otimes E^1$ . Hence we can choose homogeneous elements  $a_1, \dots, a_n$  in a basis  $\mathcal{B} = \mathcal{B}^{(0,0)} \cup \mathcal{B}^{(0,1)} \cup \mathcal{B}^{(1,0)} \cup \mathcal{B}^{(1,1)}$  of  $B$ , where  $\mathcal{B}^{(i,j)} \subseteq C^{(i,j)} \cup J^{(i,j)}$  and suitable  $g_1, \dots, g_n \in E^0 \cup E^1$  such that

$$f(a_1 \otimes g_1, \dots, a_n \otimes g_n) \neq 0 \tag{3.1}$$

in  $E(B)$ .

Notice that, for any  $i = 1, \dots, r$ , there exists a  $\varphi$ -polynomial  $p_i(x_1, \dots, x_n)$  such that

$$f_i(a_1 \otimes g_1, \dots, a_n \otimes g_n) = p_i(a_1, \dots, a_n) \otimes g_1 \cdots g_n.$$

Hence, since the nonzero evaluation of  $f$  in (3.1) is equal to

$$\sum_{i=1}^r \alpha_i f_i(a_1 \otimes g_1, \dots, a_n \otimes g_n) = \left( \sum_{i=1}^r \alpha_i p_i(a_1, \dots, a_n) \right) \otimes g_1 \cdots g_n,$$

we must have that  $\sum_{i=1}^r \alpha_i p_i(a_1, \dots, a_n) \neq 0$ .

By using left and right multiplication by the unit element  $e$  of  $C$  we can decompose the Jacobson radical  $J$  of  $B$  into the direct sum of graded  $C$ -bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for  $i \in \{0, 1\}$ ,  $J_{ik}$  is a left faithful module or a 0-left module according as  $i = 1$  or  $i = 0$ , respectively. Similarly,  $J_{ik}$  is a right faithful module or a 0-right module according as  $k = 1$  or  $k = 0$ , respectively. Moreover, for  $i, k, l, m \in \{0, 1\}$ ,  $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$  where  $\delta_{kl}$  is the Kronecker delta ([23, Lemma 2]).

Now, without loss of generality we may assume that if  $a_i \in J$  then  $a_i \in J_{kl}$ , for some  $k, l \in \{0, 1\}$ . Take  $g_0 \in E^0$  such that  $g_0 g_1 \cdots g_n \neq 0$ ; then if  $b \in C \cup J_{00} \cup J_{01} \cup J_{10} \cup J_{11}$ ,  $g \in E$  we have:

$$(e \otimes g_0)(b \otimes g) + (b \otimes g)(e \otimes g_0) = \begin{cases} 2b \otimes g_0 g, & \text{if } b \in C \cup J_{11} \\ b \otimes g_0 g, & \text{if } b \in J_{10} \cup J_{01}, \\ 0, & \text{if } b \in J_{00}. \end{cases}$$

Hence since  $n \geq t$ , by (3.1) some  $a_j$  must lie in  $C$  and we have:

$$h_i(a_1 \otimes g_1, \dots, a_n \otimes g_n, e \otimes g_0) = \alpha p_i(a_1, \dots, a_n) \otimes g_0 g_1 \cdots g_n,$$

where  $\alpha$  is a positive integer.

Thus

$$\sum_{i=1}^r \alpha_i h_i(a_1 \otimes g_1, \dots, a_n \otimes g_n, e \otimes g_0) = \alpha \left( \sum_{i=1}^r \alpha_i p_i(a_1, \dots, a_n) \right) \otimes g_0 g_1 \cdots g_n \neq 0,$$

contrary to our assumption. In conclusion the  $\varphi$ -polynomials  $h_1, \dots, h_r$  are linearly independent modulo  $P_{n+1}^\varphi \cap \text{Id}^\varphi(E(B))$  and the proof is complete.  $\square$

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