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Research Article

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Nonlinear Thermal Transport with Inertia in Thin Wires: Thermal Fronts and Steady States

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Abstract: In a series of papers we have obtained results for nonlinear heat transport when thin wires exchange heat non-linearly with the surroundings, with particular attention to propagating solitons. Here we obtain and discuss new results related to the propagation of nonlinear heat fronts and some conceptual aspects referring to the application of the second principle of thermodynamics to some nonlinear steady states related to non-propagating solitons.

Keywords: nonlinear thermal fronts, Maxwell-Cattaneo law, nonlinear radiative transfer, auxiliary equation method, nonlinear steady states

1 Introduction

The interest in the transmission of information by means of soliton lightwaves [1] has stimulated an analogous interest in the transmission of information by means of thermal solitons [2]. Solitons are localized perturbations of the system, propagating or not, keeping their form along time. Their main peculiarity is the particle-like behavior, which explains the suffix "on," as in "photon" or "phonon." The two main kinds of solitons are the pulse-like solitons ("sech"-type solitons, also named bright solitons) and the front-like solitons ("tanh"-type solitons, also named dark solitons).

These waves are the outcome of the combination of dispersive terms and nonlinear terms. This has fostered the interest in nonlinear versions of generalized heat transport equations such as Maxwell-Cattaneo and Guyer–Krumhansl equations [3–8], keeping in mind the restrictions required by the extended formulations of thermodynamics [9–15].

The propagation of solitons for transmitting bits of information has recently suggested to consider thermal waves along thin wires when the heat exchange between the wire and the surroundings is nonlinear [2, 16, 17]. In [2] the energetic cost for transmission of one bit was especially outlined, because the energy cost on information transmission and processing is a relevant practical topic. Other ways of having heat solitons is by means of exothermic chemical reactions or phase transitions with latent heat [18–20], but these situations are not so useful for the transmission of many bits.

We consider that heat exchange between the wires and the environment has nonlinear contributions in the difference of their respective temperatures [21, 22]. In [2] we proposed a mathematical model for the thermal exchange between the wire and the environments using the Stefan–Boltzmann equation for the lateral heat exchange. We used the same mathematical model but with a different kind of non-linearity (a flux-limiter) in [23]. A comparison between the two models of non-linearity was made in [24] assuming two relaxation times for the longitudinal heat flux and for the transversal heat exchange. In these papers we obtained solutions of the kind "sech^2," because we were interested in the use of localized propagating solitons to transmit bits of information. There, we were interested in the minimum amount of energy necessary to transmit

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a bit of information. Instead, in propagating fronts the transmission of thermal energy, rather than of information, is the main interest. We will be interested in how fast a thermal front propagates along the wire, transmitting with it an amount of energy. We will deal with two aspects: (a) propagating fronts related to the function tanh and (b) non-propagating solitons of the form sech^2, which raise interesting questions regarding the second law.

The paper is organized as follows. Section 2 introduces the mathematical model; Section 3 explains the mathematical method applied to the model for searching traveling wave solutions; Sections 4 and 5 deal with some nonlinear wave solutions in the radiative heat exchange regime; finally, Section 6 is devoted to the discussions and conclusions.

2 The mathematical model

The mathematical model proposed in [2] for heat propagation along a heat-conducting wire of radius r with lateral radiative heat exchange with the environment is

$$\begin{cases} \rho c \dfrac{\partial T}{\partial t} = -\nabla q - 2\pi r q_t, \\ \tau \dfrac{\partial q}{\partial t} = -q - \lambda \nabla T, \\ \tau \dfrac{\partial q_t}{\partial t} + q_t = r g(T) = 2\pi r^2 \sigma_{SB}(T^4 - T_0^4), \end{cases} \tag{1}$$

namely, the energy balance equation (first equation) and two constitutive equations for the longitudinal (the second equation, known as Maxwell-Cattaneo equation [19, 20]) and lateral heat flux (third equation). As in [2, 23, 24], the temperature T depends only on z (the distance along the axis), ρ is the mass density, and c is the specific heat per unit mass; moreover, $\mathbf{q} = q(z)\,\hat{\mathbf{z}}$ is the longitudinal heat transfer along the cylinder and q_t is the transverse heat per unit area which the cylinder exchanges with the environment. We also assume that both the second and the third equations in (1) have the same relaxation time τ. The model with two distinct relaxation times has been considered in [24]. In the second equation λ stands for the thermal conductivity along the wire. The term $g(T)$ in the third equation is the Stefan–Boltzmann law for radiative transfer across the lateral walls, with σ_{SB} being the Stefan–Boltzmann constant.

Here T is assumed to be the local-equilibrium temperature; more general versions of temperature, including nonlinear contributions in the fluxes, could be considered, in the line pointed out in [21]. This would be an additional source of non-linearity, worthy of future examination and physical discussion.

By differentiating the first equation in (1) with respect to time and using the second equation in (1), we find

$$\tau \rho c \frac{\partial^2 T}{\partial t^2} = \lambda \frac{\partial^2 T}{\partial z^2} - \rho c \frac{\partial T}{\partial t} - g(T). \tag{2}$$

Expression (2) is still valid for $T \to T - T_0$, with T_0 being the homogeneous temperature of the environment.

If we write $T = T_0 + \Delta T$, expanding the fourth power law coming from the Stefan–Boltzmann law we obtain a polynomial of fourth order in ΔT for $g(T)$. The dimensionless form of the equation is

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - \frac{\partial u}{\partial t_1} - b\left(4u + 6u^2 + 4u^3 + u^4\right), \tag{3}$$

where

$$t_1 = t/\tau, \quad z_1 = z\sqrt{\frac{\rho c}{\lambda \tau}}, \quad u = \frac{(T - T_0)}{T_0}, \quad \tilde{g}(u) = \frac{\tau}{\rho c T_0} g(T - T_0), \tag{4}$$

with $b = 2\pi r \dfrac{\tau \sigma_{SB} T_0^3}{\rho c}$. If there is no lateral heat exchange, the coefficient b will be zero.

If $\lambda \to \infty$ and $\tau \to \infty$ but λ/τ is finite, (2) leads to

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - b\left(4u + 6u^2 + 4u^3 + u^4\right). \tag{5}$$

Both (3) and (5) can be summarized by

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - a\frac{\partial u}{\partial t_1} - b\left(4u + 6u^2 + 4u^3 + u^4\right), \tag{6}$$

where $a = 1$ stands for equation (3) and $a = 0$ for equation (5).

Here, we will explore the possibility of propagation fronts, related to "tanh," and of steady states related to "sech^2." However, applying the mathematical procedure described in the subsequent section we have seen that these kinds of solitons are not solutions of equation (6), but the situation changes if we consider the truncation $\tilde{g}(u) = b(4u + 6u^2)$, which is valid for small values of the field u, namely $\Delta T < T_0$.

3 Auxiliary method for traveling waves

In this section we recall, for the sake of clarity to the reader, the main steps of the auxiliary equation method used in [2, 23, 24], which have been introduced in [25], [26], [27]. The reader familiar with this method may pass over this section.

The main peculiarity of this method is to allow to find some exact traveling wave solutions of the 1 + 1 nonlinear equation $E(z, t, u, u_z, u_t, ...) = 0$. The first step is transforming the partial differential equation in the ordinary equation, $E(\xi, u, u_\xi, u_{\xi\xi}, ...) = 0$, by means of $\xi = kz - \omega t$.

The second step is to choose for $u(\xi)$ a polynomial form

$$u(\xi) = \sum_{i=0}^{n} u_i y(\xi)^i, \tag{7}$$

where u_i are constants to be determined and the functions $y(\xi)$ are solutions of an auxiliary equation, such as [25–27]

$$y(\xi)' = 1 - y(\xi)^2, \tag{8}$$

whose solution is $y(\xi) = \tanh(\xi)$, mimicking a propagating front, or the equation

$$y(\xi)'^2 = y(\xi)^2(1 - y(\xi)^2), \tag{9}$$

whose solution is $y(\xi) = \text{sech}(\xi)$, mimicking a propagating pulse.

The third step is determining the coefficients u_i in (7), after introducing (7) in $E(z, t, u, u_z, u_t, ...) = 0$ and taking into account the auxiliary equation $y(\xi)$. The integer n (the exponent of $y(\xi)$ in (7)) is found by balancing the higher-order linear term with the higher nonlinear term of the equation. Finally one obtains an algebraic system of equations for the coefficients u_i which may be solved.

4 Traveling fronts associated to the auxiliary equation (8)

According to the first step of the method, we need to consider the moving frame of reference $\xi = kz - \omega t = k(z - vt)$, where v is the speed given by $v = \omega/k$. Then, the second-order truncated form of equation (6) is

$$(\omega^2 - k^2)\frac{\partial^2 u}{\partial \xi^2} - a\omega\frac{\partial u}{\partial \xi} + b\left(4u + 6u^2\right) = 0. \tag{10}$$

In this subsection we use the auxiliary equation $y(\xi)' = 1 - y(\xi)^2$, namely (8), with the solution $\tanh(\xi)$, and we follow the procedure described in the previous section. Indeed, by balancing the highest nonlinear term u^2 with the highest linear term $\frac{\partial^2 u}{\partial \xi^2}$ keeping in mind (8), from which we find the value $n = 2$, the sum (7) becomes $u(\xi) = u_0 + u_1 y(\xi) + u_2 y(\xi)^2$. After substituting the latter expression of $u(\xi)$ in (10) we find the following algebraic system, which allows to find the coefficients u_0, u_1, and u_2

$$\begin{cases} -a\omega u_1 + 2bu_0(3u_0 + 2) + 2u_2\left(\omega^2 - k^2\right) = 0, \\ 2u_1\left(6bu_0 + 2b + k^2 - \omega^2\right) - 2a\omega u_2 = 0, \\ a\omega u_1 + 4u_2(3bu_0 + b + 2(k-\omega)(k+\omega)) + 6bu_1^2 = 0, \\ 2\left(a\omega u_2 + 6bu_1 u_2 + k^2(-u_1) + \omega^2 u_1\right) = 0, \\ 6u_2\left(bu_2 - k^2 + \omega^2\right) = 0. \end{cases} \tag{11}$$

Case $a = 0$

Setting $a = 0$ in the above system (11), we find solutions obtained in [2] and reported in Section 5, namely (18) and (20).

Case $a = 1$

In this case we find $u_2 = 1/6$, $u_1 = 1/3$, and $u_0 = -1/2$, so we have the traveling front solution

$$u(z_1, t_1) = \frac{1}{6}\left(\tanh^2(kz_1 - \omega t) + 2\tanh(kz_1 - \omega t) - 3\right), \tag{12}$$

with $\omega = -\frac{5}{3}b$ and $k = \pm\frac{\sqrt{b}\sqrt{50b+3}}{3\sqrt{2}}$, as plotted in Fig. 1.

Solution (12) can also be written in dimensional form:

$$\frac{\Delta T(z,t)}{T_0} = \frac{1}{6}\left(\tanh^2(k_d z - \omega_d t) + 2\tanh(k_d z - \omega_d t) - 3\right), \tag{13}$$

with $\omega_d = -\frac{5}{3}\frac{2\pi r\sigma_{SB}T_0^3}{\rho c}$ and $k_d = \pm\frac{\sqrt{\rho c\omega_d(10\tau\omega_d + 1)}}{\sqrt{10\lambda}}$.

The propagation velocity v_d is given by

$$v_d = \frac{\omega_d}{k_d} = \pm\frac{\omega_d\sqrt{10\lambda}}{\sqrt{\rho c\omega_d(10\tau\omega_d + 1)}}. \tag{14}$$

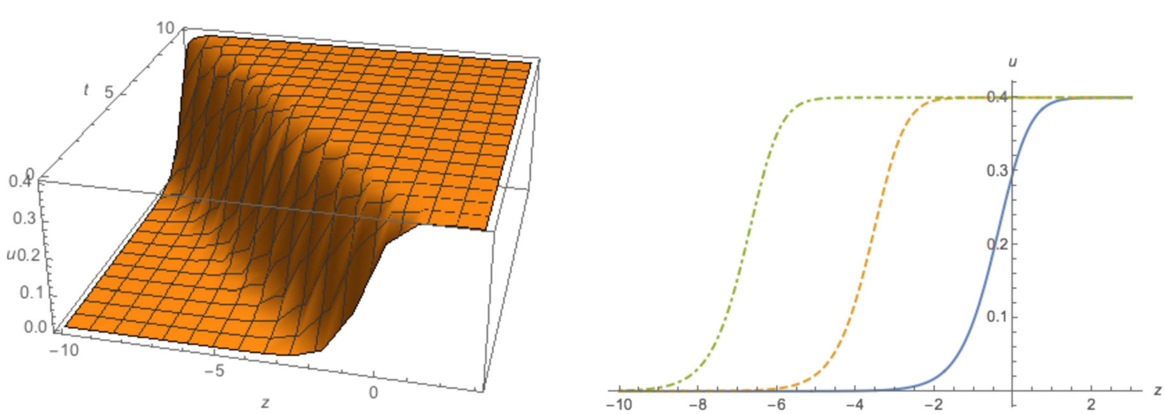

Figure 1: 3D plot of (12) for $\omega = 1$ (left) and 2D plot of (12) for three different values of the dimensionless time $t = 0$, $t = 3$, and $t = 6$ (right).

Another solution found for $u_2 > 0$ is $u_2 = 1/6$, $u_1 = -1/3$, and $u_0 = -1/2$

$$u(z_1, t_1) = \frac{1}{6}\left(\tanh^2(kz_1 - \omega t) - 2\tanh(kz_1 - \omega t) - 3\right), \tag{15}$$

with $\omega = \frac{5}{3}b$ and $k = \mp\frac{\sqrt{b}\sqrt{50b+3}}{3\sqrt{2}}$.

Two other soliton solutions, corresponding to $u_2 = -1/6$, $u_1 = -1/3$, and $u_0 = -1/6$ or to $u_2 = -1/6$, $u_1 = 1/3$, and $u_0 = -1/6$, are

$$u(z_1, t_1) = -\frac{1}{6}(\tanh(kz_1 - \omega t_1) + 1)^2, \tag{16}$$

with $\omega = \frac{5}{3}b$ and $k = \mp\frac{\sqrt{b}\sqrt{50b-3}}{3\sqrt{2}}$, and

$$u(z_1, t_1) = -\frac{1}{6}(\tanh(kz_1 - \omega t) - 1)^2, \tag{17}$$

with $\omega = -\frac{5}{3}b$ and $k = \mp\frac{\sqrt{b}\sqrt{50b-3}}{3\sqrt{2}}$.

Note that in (15) k is always real, whereas in (12) (respectively (17)) it is real for $b > 3/50$, which in view of the meaning of b (below (3)) refers to $\tau > \frac{3}{2\pi r 50}\frac{\rho c}{\sigma_{SB}T_0^3}$. This comment illustrates that the value of the relaxation time will be one of the physical factors whose concrete numerical value allows or forbids some concrete kinds of solutions.

The particular kind of front which will be observed will depend on the initial temperature profile imposed along the wire.

To sustain such a propagation of the energy front, energy must be injected to the system at $z_1 = -\infty$ at a rate $\dot{e} = \rho c \Delta T v_d \pi r^2$. Indeed, the advance of the front makes that the volume of the hot part of the system increases with $v\pi r^2$ per unit time; since the difference in internal energy between the cold part and the hot part is $\rho c \Delta T$, the mentioned result for $\dot{e}$ follows in a direct way. Thus, an operational way of controlling which kind of solution is chosen is related to boundary conditions and to the imposed initial temperature profile. For instance, if the energy is supplied to the system at a rate different than the one calculated here, the solution (12) will not be physically realizable.

5 Traveling pulses and stationary solutions associated to the auxiliary equation (9)

In this section we recall the results obtained in [2], where we have applied the auxiliary equation method by equation (9), namely $y(\xi)'^2 = y(\xi)^2(1 - y(\xi)^2)$. Following the previous procedure we find the following solutions.

Case $a = 0$

The first solution is [2]

$$u(z_1, t_1) = \mathrm{sech}^2(k(z_1 - vt_1)) - \frac{2}{3}, \tag{18}$$

with $\omega^2 = k^2 + b$ and $v = \omega/k$, which in dimensional form are $\omega_d^2 = \frac{2\pi r\sigma_{SB}T_0^3 + \lambda k_d^2}{\tau\rho c}$ and

$$v = \omega_d/k_d = \sqrt{\frac{\lambda}{\tau\rho c}}\sqrt{1 + \frac{2\pi r\sigma_{SB}T_0^3}{\lambda k_d^2}}. \tag{19}$$

Another solution is [2]

$$u(z_1, t_1) = -\text{sech}^2(k(z_1 - vt)), \tag{20}$$

with $\omega^2 = k^2 - b$ and $v = \omega/k$, which in dimensional form are $\omega_d^2 = \dfrac{\lambda k_d^2 - 2\pi r \sigma_{SB} T_0^3}{\tau \rho c}$ and

$$v = \omega_d/k_g = \sqrt{\frac{\lambda}{\tau \rho c}} \sqrt{1 - \frac{2\pi r \sigma_{SB} T_0^3}{\lambda k_d^2}}. \tag{21}$$

We point out that the speed in (18) is higher than for the high-frequency linear waves, namely $\sqrt{\frac{\lambda}{\tau \rho c}}$. The opposite occurs instead in (20). As already stressed in [2], this is of conceptual interest because the presence of the relaxation term in the transport equation allows to avoid infinite speed of propagation for thermal pulses.

In this case we also find stationary solutions. The first stationary solution is [2]

$$u(z_1, t_1) = -\frac{1}{3}(2 - 3\,\text{sech}^2(kz_1)), \tag{22}$$

with $\omega = 0$ and $k^2 = -b$, which in dimensional form is

$$\frac{\Delta T(z,t)}{T_0} = -\frac{1}{3}(2 - 3\,\text{sech}^2(k_d z)), \tag{23}$$

with $\omega_d = 0$ and $\lambda k_d^2 = -2\pi r \sigma_{SB} T_0^3$. Note that from the latter expression it follows that k_d is imaginary and that "sech" becomes "sec," which is plotted in Fig. 2. Since $\sec(h\pi/2) = \infty$ ($h \in \mathbb{Z}$), the solution may have a physical meaning in a restricted interval of ξ.

Another stationary and localized solution is [2]

$$u(z_1, t_1) = -\text{sech}^2(kz_1), \tag{24}$$

with $\omega = 0$ and $k^2 = b$, or in dimensional form

$$\frac{\Delta T(z,t)}{T_0} = -\text{sech}^2(k_d z), \tag{25}$$

with $\omega_d = 0$ and $\lambda k_d^2 = 2\pi r \sigma_{SB} T_0^3$. Solution (25) is plotted in Fig. 2.

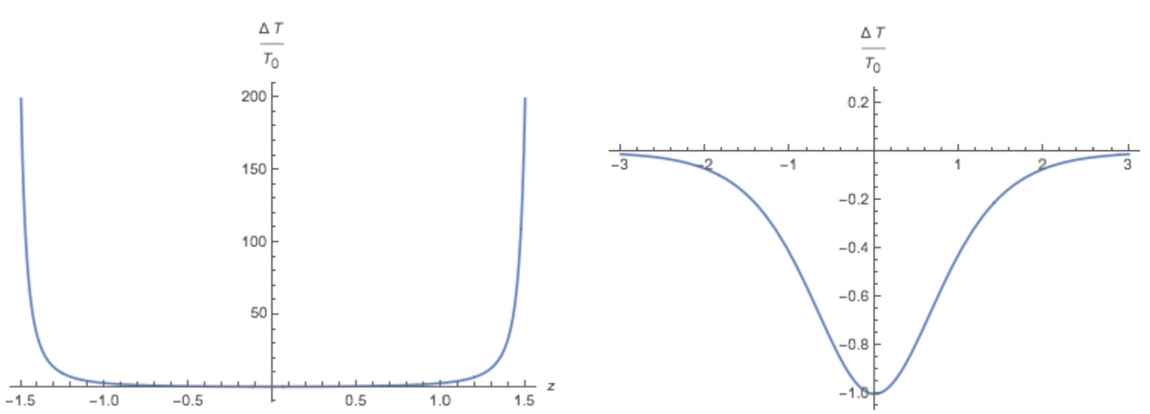

Figure 2: Plot of the stationary solution (23) for $2\pi r \sigma_{SB} T_0^3/\lambda = 1$ (left) and plot of (25) for $k_d = 1$, namely for $2\pi r \sigma_{SB} T_0^3/\lambda = 1$ (right).

Case $a = 1$

In the case $a = 1$ we find again the same stationary and localized solutions just written. Note that the system is not expected to spontaneously reach the states (23) or (25) starting from the homogeneous state, but these profiles will be the result of imposed initial conditions. Stability of solutions (23) and (25) should be explored in the future. They are consistent with the first and the second law but they are strongly non-intuitive.

6 Discussion

We have considered the mathematical model proposed in [2], which, apart from the propagation of bright solitons ("sech"-type solitons), allows the propagation of nonlinear fronts; see (12), (15), (16), and (17), considered in Section 4. Note that the solutions are not simply proportional to $\tanh(k_d z - \omega_d t)$, but are second-order polynomials in $\tanh(k_d z - \omega_d t)$. We have pointed out the relevance of the value of the relaxation time τ in allowing or forbidding some kinds of solitons, as commented below (17).

The entropy production is given by

$$\sigma = -q_l \lambda \nabla T + \left(2\frac{\pi}{r}\right) q_t (T - T_0) > 0 \tag{26}$$

according to the second law. The first term may be negative on the condition that the second term is sufficiently positive or viceversa, or both terms may be positive. Note indeed that if the lateral heat exchange is taken to be zero (this would correspond to formally taking $\sigma_{SB} = 0$ in Stefan–Boltzmann's law), the k_d in (22) or (24) would be zero, and the profile would be homogeneous.

The entropy production (26) is the local-equilibrium entropy production, concerning longitudinal and transversal heat transfer. We have used it because the discussion here refers to a steady-state situation. Had we considered a non-local equilibrium temperature with nonlinear contribution in the fluxes, a more general expression for the entropy and the entropy production incorporating nonlinear terms in $\mathbf{q}$ and q_t should be taken into account [9, 10].

A more general view should also include the heat flow outside the system, because this is a forced state; in the environment, there will be a heat flow in a direction opposite to the heat flow along the wire, but if the environment is at a uniform temperature this will not contribute to an entropy production.

Solutions (23) and (25) are far from being intuitively acceptable. We focus on the situation corresponding to (25), whose temperature profile is plotted in Fig. 2. In Fig. 3(a) we sketch the behavior of the axial and the lateral heat flows according to temperature profile (25). When this profile is introduced into the Fourier law and the Stefan–Boltzmann law, the behavior of the heat flux is into the wire (lateral heat flow) and towards $z = 0$ (axial heat flux). This makes that heat would accumulate at $z = 0$, in such a way that this solution cannot correspond to a steady state.

A possible way of physically interpreting solution (25) as a stationary and localized solution would be that either longitudinal heat flows in a way opposite to Fourier's law, as in Fig. 3(b), or that the lateral heat flows in a way opposite to Stefan–Boltzmann, as in Fig. 3(c). These two situations would be compatible with a steady state, in contrast to Fig. 3(a). They could also be compatible with the second law, provided that the

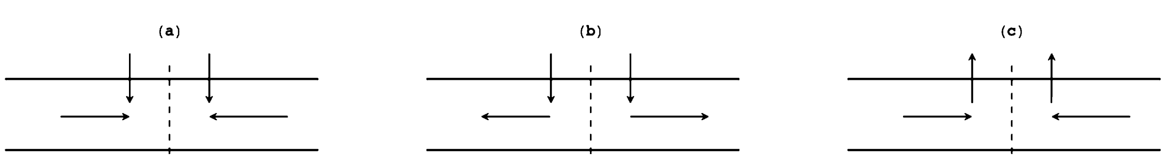

Figure 3: Applying the temperature profile (25) along the wire to the Fourier law and the Stefan–Boltzmann law leads to heat behavior sketched in (a), where heat flows into the system and towards $z = 0$ along the system. Panels (b) and (c) yield possible alternatives consistent with a steady state, commented in the text.

positive entropy production of lateral heat flowing into the wire in Fig. 3(b) compensates the negative entropy production related to the behavior of longitudinal heat flow in the same figure (and vice versa, in Fig. 3(c)).

Solitons have been observed in many physical systems. In principle, the solitons and fronts considered here should be observable. In order for lateral effects to be especially relevant, thin wires should be considered; as a fast and non-invasive way of measuring temperature, observations based on radiation would be especially convenient.

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