SLOPE INEQUALITIES FOR FIBRATIONS OF NON-MAXIMAL ALBANESE DIMENSION

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Dedicated to Professor Fabrizio Catanese for his seventieth birthday

ABSTRACT. We study and obtain Slope inequalities for fibred irregular varieties of non-maximal Albanese dimension. We give a comparison theorem between Clifford-Severi and Slope inequalities for this type of fibrations. We also obtain a set of Slope inequalities considering the geometry of the Albanese map and the associated eventual maps.

Contents

1. INTRODUCTION

We consider triplets (X, L, a) where X is an irregular, complex, projective variety of dimension n, L is a line bundle on X and $a : X \longrightarrow A$ is a non trivial generating map to an abelian variety of dimension q. Se will have a fibration $f : X \longrightarrow B$ onto a smooth curve. We will assume that the fibration is *irregular*, i.e. $\dim a(F) > 0$, where F is a general fibre of f. The main example of an irregular fibration is the case of a fibred variety $f : X \longrightarrow B$, with $a = \mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$ and irregularity $q = q(X) > b = g(B)$.

In this situation, several invariants associated to the triplet can be defined: the *continuous rank* $h_a^0(X, L)$, the *continuous positive degree* $\deg_a^+ f_* L$ and the *eventual map* ϕ_L. If $h_a^0(X, L) \neq 0$ we define the Clifford-Severi slope of (X, L, a) as $\lambda(L, a) = \mathrm{vol}(L)/h_a^0(X, L)$ and the Slope of L with respect to f as $s(f, L, a) = \mathrm{vol}(L)/\deg_a^+ f_* L$.

When we consider a *good* class of triplets $\mathcal{F}$ we denote $\lambda_{\mathcal{F}}(n)$ and $s_{\mathcal{F}}(n)$ to be the minimum values of such slopes when $(X, L, a) \in \mathcal{F}$ or $(F, L_{|F}, a_{|F}) \in \mathcal{F}$, respectively, and $n = \dim(X)$.

When varieties in $\mathcal{F}$ are of maximal a-dimension, in [2] we prove that

(1.1) $$\lambda_{\mathcal{F}}(n) \geq s_{\mathcal{F}}(n) \geq n\,\lambda_{\mathcal{F}}(n-1).$$

We also characterize fibrations with minimal slope in the family. Observe that this provides a way to inductively give higher dimensional Clifford-Severi and Slope inequalities just giving an inequality in low dimension. Moreover, we deduce a huge set of Slope inequalities just using all the existing Clifford-Severi inequalities for varieties of maximal a-dimension. In order to obtain these results, in [2] we develop a version of the method of Xiao adapted to the irregular setting, the so called *continuous Xiao's method*.

The aim of this work is to obtain similar results for classes of varieties of non-maximal a-dimension. Even if the main interesting case is the one of varieties X of non-maximal Albanese dimension and $a = \mathrm{alb}_X$, we need to consider general maps $a : X \longrightarrow A$ due to our methods of proof. Under this assumption it is easy to see that $\lambda_{\mathcal{F}}(n) = s_{\mathcal{F}}(n) = 0$, since in this case $h^0_a(X, L) > 0$ does not implies bigness. We define new meaningful invariants $\overline{\lambda}_{\mathcal{F}}(n)$ and $\overline{s}_{\mathcal{F}}(n)$ restricting the class $\mathcal{F}$ to line bundles L with *continuous moving part L_c big*.

Adapting the arguments given in [2], our first result is an analogous to (1.1) (see Proposition 3.3, Theorem 3.4 and Remark 3.7), and allows to obtain Slope inequalities for varieties of non maximal a-dimension from Clifford-Severi inequalities of maximal a-dimension ones. More concretely, given a good class of triplets $\mathcal{F}$, we define the subclass $\mathcal{F}_p$ imposing the extra condition that $c(X) = \dim(X) - \dim a(X) \leq p$. Then

Theorem A.

(i) $\overline{\lambda}_{\mathcal{F}_p}(n) \geq \overline{s}_{\mathcal{F}_p}(n) \geq (n-p)\,\lambda_{\mathcal{F}_0}(n-p-1)$.

(ii) *If equality* $\overline{s}_{\mathcal{F}_p}(n) = (n-p)\,\lambda_{\mathcal{F}_0}(n-p-1)$ *holds, then* $f_*(L \otimes a^*(\alpha))^+$ *is semistable for general* $\alpha \in \widehat{A}$ *and F is covered by $(n-p-1)$-dimensional varieties V of maximal $a_{|V}$-dimension such that $\lambda(R, a_{|V}) = \lambda_{\mathcal{F}_0}(n-p-1)$, for some $R \in \mathrm{Pic}(V)$.*

The technique used here is again the continuous Xiao's method.

The second part of the paper is devoted to obtaining new Slope inequalities considering the geometry of T and G, the connected components of the general fibres of a and ϕ_L, respectively. We refer to Section 2 and Remark 3.9 for definitions. Again we use continuous Xiao's method, adapting the arguments of [1] and [12]. Our main result is

Theorem B. *Let $f : X \longrightarrow B$ an irregular fibration with general fibre F and $k = \dim a(F)$. Then:*

(i) $s(f, L, a) \geq \mathrm{vol}_{F|G}(L)\,(k+1)!$

(ii) *If $(L_{|F})_c$ is $a_{|F}$-big, then $s(f, L, a) \geq \beta(L_{|T}, n, k+1)\,(k+1)!$*
In particular, if F is not uniruled, then $s(f, L, a) \geq 2\,(k+1)!$

(iii) *If equality holds in (i) or (ii), then $f_*(L \otimes a^*(\alpha))^+$ is semistable for general $\alpha \in \widehat{A}$ and $a(F)$ is (the translate of) an abelian subvariety K of A.*

where β is a numerical invariant introduced by Jiang, related to the covering gonality of the variety (see Remark 3.9 for definition).

In Section 2 we survey all the techniques we will use, known results on this topic and the involved definitions. Section 3 is devoted to proving theorems A and B.

<u>Notations and Conventions.</u> We work over $\mathbb{C}$. Varieties are projective and smooth unless otherwise stated. We will use notation of divisor or line bundles indistinctly. Given a triplet (X, L, a) we will write $L \otimes \alpha$ instead of $L \otimes a^*(\alpha)$, for $\alpha \in \widehat{A}$.

2. Preliminaries and technical results

For benefit of the reader, we collet here a series of definitions and known results we will use in next section, each of them stated with the corresponding reference.

We consider triplets (X, L, a) where X is a smooth irregular variety of dimension n with a nontrivial generating map $a : X \longrightarrow A$ to an abelian variety of dimension q and L is a line bundle on X such that $h_a^0(X, L) = \min\{h^0(X, L \otimes \alpha) \,|\, \alpha \in \widehat{A}\} \neq 0$.

Multiplication maps ([6], Section 2). We will often consider situations *up to a multiplication map*, meaning that we will consider base changes via a multiplication map on A by some d, which is étale of degree d^{2q}:

$$(2.1) \qquad \begin{array}{ccc} X^{(d)} & \xrightarrow{\nu_d} & X \\ a_d \downarrow & & \downarrow a \\ A & \xrightarrow{\nu_d} & A \end{array}$$

We will denote $L^{(d)} := \nu_d^*(L)$. Continuous rank and volume are multiplicative through a multiplication map.

Continuous moving divisor L_c ([6], Section 2). Up to a blow-up, there is a decomposition $L = P + W$ such that, for any $d >> 0$ and divisible enough and any general $\alpha \in \widehat{A}$, $P^{(d)}$ is base point free and is the moving divisor of $|L^{(d)} \otimes \alpha|$ and $W^{(d)}$ is its fixed divisor. Following [1], P and W are called the *continuous moving divisor* and *continuous fixed divisor* of L, respectively. According to the notation of [12], we will set $L_c := P$ for the continuous moving part.

Eventual map and eventual degree. Up to a blow-up, there is a factorization of the map a, $X \to X_L \to A$ such that the map $\phi_L : X \longrightarrow X_L$ verifies the following properties:

- $L_c = \phi_L^*(R_L)$ for some line bundle R_L on X_L which induces a base point free generically finite morphism on X_L ([6]).
- Up to a multiplication map, the linear system $|L_c^{(d)} \otimes \alpha|$, for α general, is base point free and induces the map $\phi_L^{(d)} : X^{(d)} \longrightarrow X_L^{(d)}$ ([1], [6]).
- Since the map ϕ_L factorizes a, it is generically finite provided X is of maximal a-dimension. It is birational if $\deg a = 1$.
- The map ϕ_L is generically finite provided L_c is a-big ([1]).
- A birational model of the map ϕ_L is given by the natural map $\rho : X \dashrightarrow \mathbb{P}_A(a_*L)$, where $X_L := \rho(X)$ ([12]).
- When the eventual map ϕ_L is generically finite, we define the *eventual degree* of L to be $e(L) = \deg(\phi_L)$ (cf. [6], Section 3). We can extend the definition to any L just considering the degree of the finite part in the Stein factorization of ϕ_L.
- $\kappa(L_c) = \dim \phi_L(X) \geq \dim a(X)$ ([12]).

Good classes of triplets. ([2], Section 4). Given a family $\mathcal{F}$ of triplets with $h_a^0(X, L) \neq 0$, we say that the family is *good* if it is stable via the following four operations:

(1) If $(X, L, a) \in \mathcal{F}$, then $(\overline{X}, \sigma^*L, a \circ \sigma) \in \mathcal{F}$, where $\sigma : \overline{X} \longrightarrow X$ is a birational morphism.
(2) If $(X, L, a) \in \mathcal{F}$, then $(X^{(d)}, L^{(d)}, a_d) \in \mathcal{F}$.
(3) If $(X, L, a) \in \mathcal{F}$, then $(X, L', a) \in \mathcal{F}$ for $L' \leq L$ such that $h_a^0(X, L') > 0$.
(4) If $(X, L, a) \in \mathcal{F}$, then $(M, L_{|M}, a_{|M}) \in \mathcal{F}$, for a general smooth M, moving in a base point free linear system on X.

Clifford-Severi inequalities and equalities (maximal a-dimension). Given a triplet (X, L, a) with $h_a^0(X, L) > 0$, we define its Clifford-Severi slope as

$$\lambda(L, a) = \frac{\mathrm{vol}(L)}{h_a^0(X, L)}.$$

Observe that this slope remains constant under multiplication maps (cf. [6], section 2). Given a good class $\mathcal{F}$ we define $\lambda_{\mathcal{F}}(n)$ to be the infimum of the Clifford-Severi slopes of triplets in $\mathcal{F}$, of dimension n. Clifford Severi-Inequalities for a given good class $\mathcal{F}$ are inequalities of type

$$\lambda_{\mathcal{F}}(n) \geq \lambda_n.$$

There are many known Clifford-Severi inequalities for different good classes of maximal a-dimension varieties: see Remark 4.9 in [2] for a (almost) complete list. For example:

- In general, for any line bundle L we have $\lambda_{\mathcal{F}}(n) \geq e(L)\, n!$ ([5]).

- Higher dimensional Severi inequality states that $\lambda_{\mathcal{F}}(n) \geq 2\,n!$ if $\mathcal{F}$ is defined by the property that L is *numerically subcanonical*.

Moreover, we can characterize some limit cases ([7]):

- (*First Clifford-Severi line*) If $\lambda(L, a) = n!$, then, (up to a birational modification of X and an étale map on A), we have that $\dim A = n$ and a is a birational map.
- (*Second Clifford-Severi line*) If $L \leq K_X$ and $\lambda(L, a) = 2\,n!$, then, (up to a birational modification of X and an étale map on A), we have that $\dim A = n$ and a is a double cover.

Clifford-Severi inequalities (non-maximal a-dimension). In the case of triplets of non-maximal a-dimension, situation is not so clear and depends heavily on conditions of bigness of L or L_c and the geometry of the fibre of the map a or ϕ_L. The case of irregular threefolds is well understood by results of Zhang ([14]).

Here you can find a (non complete) list of known results for arbitrary dimension n. We set $k = \dim a(X)$, and G and T for a connected component of the general fibre of ϕ_L and a, respectively.

In [1], Main Theorem and Remark 5.8, the author proves

- If L_c is a-big and L is numerically r-subcanonical, then $\lambda(L, a) \geq \delta(r)\,k!$.
- If L is nef and a-big then $\lambda(L, a) \geq (L_{|G})^{n-k}\,k! \geq\,k!$.

When $k = n - 1$, Zhang gives a better bound ([15]):

- If g is the genus of the curve T, then $\lambda(L, a) \geq 2\,\frac{g-1}{g+n-2}\,n!$.

Finally Jiang ([12]) gives a set of inequalities depending on the geometry of G or T. The simplest ones are the following (see Proposition 3.6 and Theorem 3.1 in [12] and Remark 3.9 for a more detailed result):

- If L is big, then $\lambda(L, a) \geq \mathrm{vol}_{X|G}(L)\,k!$.
- If L_c is big and T is not uniruled, then $\lambda(L, a) \geq 2\,k!$.

Remark 2.1. The proof of Main Theorem (iii) in [1] uses implicitly the volume of $L_{|G}$ (see Remark 5.8 in the cited paper). In Corollary B (iii), loc. cit., extending this inequality to K_X in the singular setting, it is erroneously assumed that $\mathrm{vol}_{X|G}(K_X) \geq 1$, assuming that the minimal variety X is Gorenstein.

Irregularly fibred triplets. ([2], Section 2) Given a triplet (X, L, a), we will say that it is *irregularly fibred* (with respect to a) if moreover we have a fibration $f : X \longrightarrow B$ onto a smooth curve, such that $\dim a(F) > 0$, where F is a general smooth fibre of f.

In this case, $a(F)$ generates a translate of a positive dimensional abelian subvariety K of A. We denote the relative irregularity of f with respect to a to be $q_f = \dim K$. We have that $q_f = \dim A - \bar{b}$, where $\bar{b}$ is the dimension of the intersection $f^*\mathrm{Pic}^0(B) \cap a^*\widehat{A}$. The fibration is irregular iff $q_f \geq 1$.

If f is an irregular fibration as above, the family of vector bundles $\{f_*(L \otimes \alpha)\}_{\alpha \in \widehat{A}}$ has constant type of Harder-Narasimhann filtration for $\alpha \in U_0$, for some nonempty open set U_0. We set $\{(r_i, \mu_i)\}$ for their ranks and slopes.

For $\alpha \in U_0$ we will write $f_*(L \otimes \alpha)^+$ for the biggest nef subbundle of $f_*(L \otimes \alpha)$ and we denote

$$\deg_a^+ f_*L = \deg f_*(L \otimes \alpha)^+ \geq 0.$$

We will define the (continuous) Slope of L w.r.t. f to be:

$$s(f, L, a) = \frac{\mathrm{vol}(L)}{\deg_a^+ f_*L} \in (0 + \infty].$$

which is also constant under multiplication maps.

Slope inequalities. Given a good class $\mathcal{F}$, we will say that an irregular fibration f, with general fibre F, is of type $\mathcal{F}$ if $(F, L_{|F}, a_{|F}) \in \mathcal{F}$ (the triplet (X, L, a) is not necessarily in $\mathcal{F}$). We define $s_{\mathcal{F}}(n)$ to be the infimum of slopes of fibrations f of type $\mathcal{F}$, where $n = \dim X$.

Slope inequalities for the family $\mathcal{F}$ is a set of inequalities for any n:

$$s_{\mathcal{F}}(n) \geq \lambda_n.$$

In [11] Hu and Zhang give slope inequalities for $L = K_f$ and X of maximal Albanese dimension, by direct computation, giving properties of the limit cases.

In [2], Theorem 4.11, a broad generalization is given, for any L, establishing an equivalence between Clifford-Severi inequalities and Slope inequalities for a given good class $\mathcal{F}$ of varieties of maximal a-dimension. Moreover, this result allows to produce automatically a whole set of Clifford-Severi and Slope inequalities for any dimension, just given one inequality in low dimension, typically 1 or 2 (see Remark 4.12 and Corollary 1.1 in [2]). The main result can be stated as:

$$\lambda_{\mathcal{F}}(n) \geq s_{\mathcal{F}}(n) \geq n \, \lambda_{\mathcal{F}}(n - 1).$$

Continuous Xiao's method and derived inequalities. ([2], Section 3) Take a general $\alpha_0 \in \widehat{A}$ and let $L_0 = L \otimes \alpha_0$. After a suitable blow-up and multiplication map there is a filtration by nef line bundles:

$$T_1 \leq T_2 \leq ... \leq T_m \leq L_0$$

such that, for all i $N_i := T_i - \mu_i F$ is nef and, if $P_i := N_{i|F}$, then

- P_i is a base point free linear system on F such that $h^0(F, P_i) = h^0_{a_{|F}}(F, P_i) \geq r_i$.
- $\deg_a^+ f_*(L) = \deg(\mathcal{E}_m^{\alpha_0}) = \sum_{i=1}^m r_i(\mu_i - \mu_{i+1})$

By convention we can take, coherently, $(N_{m+1}, \mu_{m+1}) = (T_m, 0)$ or, in case L is nef, $(N_{m+1}, \mu_{m+1}) = (L, 0)$.

Fix $r \leq n$. Consider an ordered, increasing partition of the set $I = \{1, ..., m\}$ given by subsets I_s, $s = 1, ..., r-1$ (some of the sets I_s may be empty), with $I_{r-1} \neq \emptyset$. Define, decreasingly, for $s = 1, ..., r-1$

$$b_s = \left\{ \begin{array}{ll} \min I_s & \text{if } I_s \neq \emptyset \\ b_{s+1} & \text{otherwise} \end{array} \right.$$

Then we have that, for any $Q_1, ..., Q_{n-r}$ nef $\mathbb{Q}$-Cartier divisors the following inequality holds:

$$(2.2) \qquad Q_1 ... Q_{n-r} \left[N_{m+1}^r - \left(\sum_{s=1}^{r-1} (\prod_{k>s} P_{b_k}) \sum_{i \in I_s} (\sum_{l=0}^{s} P_i^{s-l} P_{i+1}^l)(\mu_i - \mu_{i+1}) \right) \right] \geq 0.$$

We will use the following particular cases.

Taking $r = n$, and any partition:

$$(2.3) \qquad \text{vol}(L) = \text{vol}(L_0) \geq N_{m+1}^n \geq \sum_{s=1}^{n-1} (\prod_{k>s} P_{b_k}) \sum_{i \in I_s} (\sum_{l=0}^{s} P_i^{s-l} P_{i+1}^l)(\mu_i - \mu_{i+1}).$$

Taking any r, the trivial partition $I_{r-1} = I$ ($I_s = \emptyset$ for $s < r-1$) and $Q_i = N_{m+1}$:

$$(2.4) \qquad \text{vol}(L) = \text{vol}(L_0) \geq N_{m+1}^n \geq \sum_{i=1}^{m} P_m^{n-r} \left[P_{i+1}^{r-1} + P_{i+1}^{r-2} P_i + ... + P_i^{r-1} \right] (\mu_i - \mu_{i+1}).$$

3. SLOPE INEQUALITIES FOR NON MAXIMAL a-DIMENSION FIBRATIONS

In [2] we study the equivalence of Slope and Clifford-Severi inequalities for general irregular varieties and fibrations in a good class $\mathcal{F}$, which mostly applies to maximal a-dimension varieties. Our aim is to study whether this equivalence can be extended to classes of varieties of non-maximal a-dimension.

Parts of the main result in [2] given in Theorem 4.11 hold for families of non maximal a-dimension as well, but they are not interesting since in these cases $\lambda_{\mathcal{F}}(n)$ and $s_{\mathcal{F}}(n)$ vanish. The reason is that condition $h_a^0(X, L) > 0$ does not imply bigness if the variety is not of maximal a-dimension. Indeed, if $a : X \longrightarrow A$ verifies that $k = \dim a(X) < n = \dim X$, take $L = a^*H$ for any H very ample on A. Then clearly $\text{vol}(L) = 0$ and $h_a^0(X, L) > 0$. The same phenomena occur if X is fibred: we can construct examples with $\deg_a^+ f_*L \neq 0$ and $\text{vol}(L) = 0$. So we need to impose extra hypotheses on the triplets to obtain nontrivial inequalities. Natural conditions are a-bigness of L or its continuous moving part L_c. There

are several strategies (see [1] and [12]), according to whether L or L_c are a-big. Observe that bigness of L_c implies bigness of L but the viceversa does not hold (see Remarks 3.7 and 3.8 in [1]). Observe also that in [1] it is shown that a-bigness of L_c implies bigness of L. In particular, for L_c, bigness and a-bigness are equivalent, provided $h_a^0(X, L) \neq 0$.

If we restrict our good classes $\mathcal{F}$ adding the condition of bigness of L (or L_c), the resulting subclass is not *good* since bigness is not stable by subbundles (and so condition (3) fails).

Nevertheless, we can obtain some closely related Slope inequalities for fibrations of non maximal a-dimension with adapted arguments.

Let us first introduce some extra notation.

Definition 3.1. Given a class $\mathcal{F}$ of triplets we define:
  (i) $\mathcal{F}_p = \{(X, L, a) \in \mathcal{F} \,|\, c(X) \leq p \}$.
  (ii) $\overline{\lambda}_{\mathcal{F}_p}(n) = \inf\{\lambda(L, a) \,|\, (X, L, a) \in \mathcal{F}_p,\ n = \dim X,\ L_c \text{ big}\}$.
  (ii') $\widehat{\lambda}_{\mathcal{F}_p}(n) = \inf\{\lambda(L, a) \,|\, (X, L, a) \in \mathcal{F}_p,\ n = \dim X,\ L \text{ big}\}$.
  (iii) $\overline{s}_{\mathcal{F}_p}(n) = \inf\{s(f, L, a) \,|\, f \text{ of type } \mathcal{F}_p,\ n = \dim X,\ (L_{|F})_c \text{ big}\}$.
  (iii') $\widehat{s}_{\mathcal{F}_p}(n) = \inf\{s(f, L, a) \,|\, f \text{ of type } \mathcal{F}_p,\ n = \dim X,\ (L_{|F}) \text{ big}\}$.

Remark 3.2.   • If the class $\mathcal{F}$ is good, so is $\mathcal{F}_p$.
  • If we consider classes of maximal a-dimension as in [2], then $\mathcal{F} = \mathcal{F}_0$. In this case, since $h_a^0(X, L) \neq 0$ implies bigness, we have that $\lambda_{\mathcal{F}}(n) = \overline{\lambda}_{\mathcal{F}_0}(n) = \widehat{\lambda}_{\mathcal{F}_0}(n)$ and $s_{\mathcal{F}}(n) = \overline{s}_{\mathcal{F}_0}(n) = \widehat{s}_{\mathcal{F}_0}(n)$.

One of the two inequalities between Clifford-Severi and Slope inequalities given in Theorem 4.11 in [2] holds without change in this new setting:

Proposition 3.3. *Let $\mathcal{F}$ be a good class of triplets of irregular varieties. Then, for all n and $p \leq n - 2$ we have*

$$\overline{\lambda}_{\mathcal{F}_p}(n) \geq \overline{s}_{\mathcal{F}_p}(n) \quad \text{and} \quad \widehat{\lambda}_{\mathcal{F}_p}(n) \geq \widehat{s}_{\mathcal{F}_p}(n).$$

Proof. We refer to the proof of the maximal a-dimension case using Pardini's trick given in Theorem 4.11, (ii) of [2]. As pointed out in Remark 4.13 (loc.cit.), only properties (1), (2) and (3) of a good class are used, and bigness of L or L_c are maintained in all the process.

The condition $p \leq n - 2$ ensures that $\dim a(X) \geq 2$. In this case, the sections of $\nu_d^*(H)$ are irreducible and f_d has connected fibres (and so it is an irregular fibration).

The rest of the proof holds without changes.

$\square$

Unfortunately, the second inequality $s_{\mathcal{F}}(n) \geq n\,\lambda_{\mathcal{F}}(n-1)$ given in Theorem 4.11 in [2] is more subtle and can not be extended directly to the cases of non maximal a-dimension. When L_c is a-big two possible strategies are possible. The first option, following [1], is by

hyperplane section argument since the eventual map allows us to maintain the process inside the good class $\mathcal{F}$. This is the content of Theorem 3.4 and Remark 3.7.

The second option is to consider the geometry of T, the connected component of the general fibre of $a_{|F}$. This approach uses Theorem 3.1 in [12], adapted to the irregularly fibred case via a suitable use of the continuous Xiao's method. This is the content of Theorem 3.10 (ii).

In general $(L_{|F})_c$ may not be $a_{|F}$-big. In this case, we can also obtain a good lower estimation of $s(f, L, a)$, but we need to consider the geometry of G, a connected component of the general fibre of the eventual map $\phi_{L_{|F}}$. This approach adapts the argument in Proposition 3.6 in [12] to the relative setting via continuous Xiao's method and is the content of Theorem 3.10 (i).

Observe that bounds in Theorem 3.10 in general are sharper than those in Theorem 3.4 when considering a single fibred triplet (X, L, a) but strongly depend on properties not well behaved in a good class $\mathcal{F}$.

As a by product of the use of Xiao's method, in theorems 3.4 and 3.10 we can give properties in the limit cases, being those in Theorem 3.4 analogous to those obtained in the cases of maximal a-dimension varieties.

Theorem 3.4. *Let $\mathcal{F}$ be a good class of triplets. Let (X, L, a) be a fibred triplet $f : X \longrightarrow B$ of type $\mathcal{F}$ such that f is of a-dimension k (i.e., $k = \dim a(F)$). Assume that L_c is f-big. Then:*

    (i) $s(f, L, a) \geq (k + 1) \lambda_{\mathcal{F}_0}(k)$.

    (ii) If equality holds, then:

        – There is a family of varieties V of dimension k covering F, and line bundles $R \leq L_{|V}$ such that $(V, R, a_{|V})$ are of maximal $a_{|V}$-dimension and verify the Clifford-Severi equality $\mathrm{vol}(R) = \lambda_{\mathcal{F}_0}(k) h^0_{a_{|V}}(V, R)$.

        – $f_*(L \otimes \alpha)^+$ is semistable for general $\alpha \in \widehat{A}$. If, moreover, L is nef, then $f_*(L \otimes \alpha)$ is nef and semistable.

Remark 3.5. If $\mathcal{F}$ is the class of all the triplets (X, L, a), then $\lambda_{\mathcal{F}_0}(k) = k!$ and equality in Theorem 3.4 implies that $(V, R, a_{|V})$ is in the *first Clifford-Severi line* and so we can conclude that $q_f = k$.

Analogously, if $\mathcal{F}$ is the class of all the triplets (X, L, a) with $L \leq K_X$, then $\lambda_{\mathcal{F}_0}(k) = 2k!$ and again we can conclude that $q_f = k$ since then $(V, R, a_{|V})$ is in the *second Clifford-Severi line*.

Remark 3.6. Statement (i) of the above theorem, combined with Clifford-Severi inequalities for maximal a-dimension varieties as given in Remark 4.9 in [2] gives a broad set of Slope inequalities.

For example, in general we have $s(f, L, a) \geq (k+1)!$ and, if L is numerically r-subcanonical, then $s(f, L, a) \geq \delta(r)(k + 1)!$

Remark 3.7. Theorem 3.4 together with Proposition 3.3 can be rephrased as, for $p \leq n - 2$:

$$\overline{\lambda}_{\mathcal{F}_p}(n) \geq \overline{s}_{\mathcal{F}_p}(n) \geq (n - p)\,\lambda_{\mathcal{F}_0}(n - p - 1).$$

Proof. (i) Let (X, L, a) be a triplet with a fibration $f : X \longrightarrow B$ of type $\mathcal{F}$.
Let us apply inequality (2.4) with $r = k + 1 = \dim a(F) + 1$:

$$(3.1) \qquad \mathrm{vol}(L) = \mathrm{vol}(L_0) \geq N_{m+1}^n \geq \sum_{i=1}^{m} P_m^{n-k-1} \left[P_{i+1}^k + P_{i+1}^{k-1} P_i + \ldots + P_i^k \right] (\mu_i - \mu_{i+1}).$$

Since $L_c \leq T_m = N_{m+1}$ (see Lemma 3.1 in [2]), we have by hypothesis that $P_m = T_{m|F}$ is big and hence $a_{|F}$-big. Hence, its eventual map is generically finite. Moreover, since it is continuously globally generated by construction, up to a multiplication map we can assume that the linear system $|P_m|$ is base point free. Take $V_1, \ldots, V_{n-k-1} \in |P_m|$ general sections and let $V = V_1 \cap \ldots \cap V_{n-k-1}$. Hence V is a smooth variety of maximal $a_{|V}$-dimension k. Let $R_i = P_{i|V}$. By the properties of a good class, we have that the triplet $(V, R_i, a_{|V}) \in \mathcal{F}_0$. Hence we have that

$$(3.2) \qquad\qquad\qquad\qquad R_i^k \geq \lambda_{\mathcal{F}_0}(k)\, h^0_{a_{|V}}(V, R_i).$$

Observe that

$$P_m^{n-k-1} P_i^k = R_i^k.$$

Finally, we use that $P_i - P_m \leq 0$ and then $h^0_{a_{|F}}(F, P_i - P_m) = 0$, and the same holds by cutting by successive V_i. Then we can conclude that

$$(3.3) \qquad\qquad\qquad\qquad h^0_{a_{|V}}(V, R_i) \geq h^0_{a_{|F}}(F, P_i) \geq r_i.$$

Finally, observe that using general Clifford-Severi inequality for irregular varieties (Main Theorem in [1])

$$(3.4) \quad \delta_i := R_{i+1}^k + R_{i+1}^{k-1} R_i + \ldots + R_i^k \geq R_{i+1}^k + k\, R_i^k \geq \lambda_{\mathcal{F}_0}(k)\,(r_{i+1} + k r_i) \geq (k + 1)\, \lambda_{\mathcal{F}_0}(k)\, r_i.$$

Then we can conclude

$$(3.5) \quad \mathrm{vol}(L) = \mathrm{vol}(L_0) \geq N_{m+1}^n \geq \sum_{i=1}^{m} (k + 1)\, \lambda_{\mathcal{F}_0}(k)\, r_i (\mu_i - \mu_{i+1}) = (k + 1)\, \lambda_{\mathcal{F}_0}(k)\, \deg_a^+ f_* L.$$

(ii) Assume that equality holds. Then we have equality in (3.2), (3.3), (3.4) and (3.5), which imply, for all $i = 1, \ldots, m$:

- $r_{i+1} = r_i,$

- $h^0_{a_{|V}}(V, R_i) = h^0_{a_{|F}}(F, P_i) = r_i$,
- $R_i^k = \lambda_{\mathcal{F}}(k)_0\, h^0_{a_{|V}}(V, R_i)$,
- $\mathrm{vol}(L_0) = T_m^n$.

Hence we have that $m = 1$ (so $f_*(L \otimes \alpha)^+$ is semistable), $h^0_{a_{|V}}(V, R_m) = h^0_{a_{|F}}(F, P_m) = r_m$, and (V, R_m) verifies the Clifford-Severi equality $R_m^k = \lambda_{\mathcal{F}_0}(k)\, h^0_{a_{|V}}(V, R_m)$.

Observe that if equality holds then L is big since $s(f, L, a) > 0$. We have that $L_0 = T_m + Z_m$ and $\mathrm{vol}(L_0) = T_m^n$. If L is moreover nef, then we have that $Z_m = 0$ (cf. Theorem A in [10]). Hence:

$$\mathrm{rank} f_*(L_0)^+ = r_1 = h^0_{a_{|F}}(F, P_1) = h^0_{a_{|F}}(F, L_{0|F}) = \mathrm{rank} f_*(L_0)$$

and hence $f_*(L_0)$ is semistable and nef. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.8. *Let X be a smooth, projective variety and L a big line bundle on X. Let $\phi : X \longrightarrow Y$ be a fibred space, G a general fibre of ϕ and $R \in \mathrm{Pic}(Y)$ a line bundle such that $L' = \phi^*(R) \leq L$. Then:*

$$\mathrm{vol}_X(L) \geq \mathrm{vol}_{X|G}(L)\, \mathrm{vol}_Y(R).$$

Proof. We have that $L = \phi^*(R) + Z$ with $Z \geq 0$. The result is obvious if L is nef. We will reduce to this case via Fujita Approximation theorem. In the general case, assume R is big, otherwise the result is trivial. Following Theorem 3.5 in [8], there is an extension of the volume function given by the moving intersection numbers which is non decreasing in each factor, superadditive and coincides with the intersection product for nef line bundles. Let $e = \dim(Y)$. For any birational compatible modifications of X and Y and any decompositions $L = W_1 + E_1$ and $R = W_2 + E_2$ such that W_i are big and nef $\mathbb{Q}$-divisors and E_i are effective $\mathbb{Q}$-divisors, we have:

$$\mathrm{vol}(L) = <L, ..., L> \geq <W_1, ..., W_1, \phi^*(W_2), ..., \phi^*(W_2)> = W_1^{n-e}(\phi^*(W_2))^e = (W_{1|G})^{n-e}W_2^e$$

since the moving intersection numbers are nondecreasing and W_i are nef.

To conclude, we apply Fujita Approximation theorem for the volume and the relative volume (see, for example, Proposition 2.11 and Theorem 2.13 in [9]). For any $\epsilon > 0$, there are birational modifications of X and Y (that we can make compatible with the above hypotheses) and decompositions $L = W_1 + E_1$ and $R = W_2 + E_2$ as above and such that

- $\mathrm{vol}_{X|G}(L) \geq W_{1|G}^{n-e} \geq \mathrm{vol}_{X|G}(L) - \epsilon$ and
- $\mathrm{vol}_Y(R) \geq W_2^e \geq \mathrm{vol}_Y(R) - \epsilon$.

We apply the above inequality for any such decomposition and we conclude that

$$\mathrm{vol}_X(L) \geq \mathrm{vol}_{X|G}(L)\, \mathrm{vol}_Y(R).$$

$\square$

Remark 3.9. Let Z be a smooth, projective variety, and L a line bundle on Z. In [12], two invariants to compute the positivity of L are defined: $\delta(L)$ and $\delta_1(L)$, the second being the minimum of volumes of subline bundles of L inducing generically finite maps, when restricted to general positive dimensional subvarieties V covering Z. We clearly have that $\delta_1(L) \geq cov.gon(Y)$. For any $k \leq s \leq \dim(Z)$, define

$$\beta(L, s, k) = \min\{\binom{s}{k}\delta(L), \delta_1(L)\}.$$

We have that $\beta(L, s, k) \geq 1$ and that $\beta(L, s, k) \geq 2$, provided Y is not uniruled.

In [12] Theorem 3.1, the following Clifford-Severi inequality is proved. Consider a triplet (X, L, a) of dimension n. Let $k = \dim a(X)$ and T be a connected component of the general fibre of a. Assume that L_c is a-big. Then:

$$\lambda(L, a) \geq \beta(L_{|T}, n, k)\, k!$$

Now we can give an extension of the Clifford-Severi inequalities for non-maximal Albanese dimension varieties given in [12] to the fibred cases.

Theorem 3.10. Let (X, L, a) be a fibred triplet. Assume that $f : X \longrightarrow B$ is of a-dimension k (i.e., $k = \dim a(F)$). Let G be a connected component of the general fibre of the eventual map $\phi_{L_{|F}}$ and let T be a connected component of the general fibre of $a_{|F}$. Then

    (i) If $L_{|F}$ is $a_{|F}$-big, then $s(f, L, a) \geq \mathrm{vol}_{F|G}(L)\,(k+1)!$

    (ii) If $(L_{|F})_c$ is $a_{|F}$-big, then $s(f, L, a) \geq \beta(L_{|T}, n, k+1)\,(k+1)!$
        In particular, if F is not uniruled, then $s(f, L, a) \geq 2\,(k+1)!$

   (iii) If equality holds in (i) or (ii), then $f_*(L \otimes \alpha)^+$ is semistable for general $\alpha \in \widehat{A}$ and $q_f = k$.

Proof. (i) Since $(T_m)_{|F} = P_m$ is continuously globally generated and coincides by construction with $(L_{|F})_c$, up to a multiplication map and a birational modification, we can consider that $|P_m|$ induces the eventual map of $L_{|F}$ and $a_{|F}$ factorizes through this map. Up to a further birational modification and multiplication map, we can consider the relative map induced by the quotient $f_*(L_0)^+ = f_*(T_m)^+ = \mathcal{E}_m \longrightarrow T_m$. The Stein factorization of such map gives a relative fibration $\phi_m : X \longrightarrow X_m$ over B, with general fibre G and a factorization as $a = \bar{a} \circ \phi_m$ for some $\bar{a} : X_m \longrightarrow A$. We can assume X_m to be smooth. Then $T_m = \phi_m^*(L_m)$ for some line bundle L_m on X_m such that $g_*(L_m)^+ = f_*(T_m)^+ = f_*(L_0)^+ = \mathcal{E}_m^{\alpha_0}$, where $g : X_m \longrightarrow B$ is the induced fibration over B.

When restricted to a general fibre $\overline{F}$, this is just the Stein factorization of the eventual map $\phi_{L_{|F}}$.

Then we apply Lemma 3.8:

$$\mathrm{vol}_X(L) \geq \mathrm{vol}_{X|G}(L)\,\mathrm{vol}_{X_m}(L_m).$$

Observe that L_m restricted to the fibre of g is $\overline{a}$-big. Hence, we can apply Remark 3.6 to $(X_m, L_m, \overline{a})$ and obtain

$$\mathrm{vol}_{X_m}(L_m) \geq (k+1)!\,\deg_{\overline{a}}^+ g_*(L_m) = (k+1)!\,\deg_a^+ f_*(L_0).$$

(ii) Consider the set of indexes $I = \{1, ..., m\}$ and let us construct an increasing ordered partition as follows. For $s = 1, ..., n-1$, consider $I_s = \{i \in I \,|\, \kappa(P_i) = s\}$. Recall that $\kappa(P_i) = \dim \phi_{P_i}(F)$. Since $(L_{|F})_c = P_m$ is $a_{|F}$-big then its eventual map is generically finite, and so $I_{n-1} \neq \emptyset$. On the other hand, since $a_{|F}$ factorizes through any eventual map, we have that $\dim \phi_i(F) \geq k$, and hence $I_s = \emptyset$ if $s < k$.

Consider now Xiao's inequality (2.2):

$$(3.6) \qquad N_{m+1}^n \geq \sum_{s=1}^{n-1} (\prod_{k>s} P_{b_k}) \sum_{i \in I_s} (\sum_{l=0}^{s} P_i^{s-l} P_{i+1}^l)(\mu_i - \mu_{i+1}).$$

Consider the Stein factorization of the eventual maps induced by $|P_i|$, $\phi_i : F \longrightarrow F_i$, and let $R_i \in \mathrm{Pic}(F_i)$ be such that $P_i = \phi_i^*(R_i)$. We can assume that all the F_i are smooth and that ϕ_i factorizes through ϕ_j if $i < j$. Let G_i the generic fibre of ϕ_i. Observe that if $i, i' \in I_s$, then $G_i = G_{i'}$ and so we denote it by G_s. Observe that $\dim G_s = n - 1 - s$. Since eventual maps factorizes $a_{|F}$, we also have maps $a_i : F_i \longrightarrow A$ such that $a_{|F} = a_i \circ \phi_i$.

Then we have that, for $i \in I_s$:

$$\sum_{l=0}^{s} P_i^{s-l} P_{i+1}^l \geq (s+1)P_i^s \geq [(k+1)R_i^s]\,G_s.$$

Since (F_i, R_i, a_i) is a triplet such that $(R_i)_c = R_i$ is big and of a_i-dimension k, we can apply general Clifford-Severi inequaliy to obtain $R_i^s \geq k! h_{a_i}^0(F_i, R_i) = k! h_{a_{|F}}^0(F, P_i) \geq k!\, r_i$.

Summing up we have

$$(3.7) \qquad \sum_{l=0}^{s} P_i^{s-l} P_{i+1}^l \geq [k!\,(r_{i+1} + k r_i)]\,G_s \geq [(k+1)!\,r_i]\,G_s.$$

When $s \leq n-2$, we have that $(\prod_{k>s} P_{b_k})G_s$ is the volume of $P_{b_{n-1}}$ restricted to a curve in T. Since $|P_{b_{n-1}}|$ induces a generically finite map, we have that $(\prod_{k>s} P_{b_k})G_s \geq \delta_1(L_{|T})$. Hence, for any $i \in I_s$ we have

$$(\prod_{k>s} P_{b_k})(\sum_{l=0}^{s} P_i^{s-l} P_{i+1}^l)(\mu_i - \mu_{i+1})) \geq (s+1)\delta_1(L_{|T})k!r_i \geq (k+1)!\delta_1(L_{|T})r_i.$$

When $s = n - 1$, and $i \in I_{n-1}$ we can use Theorem 3.1 in [12] to obtain

$$P_i^{n-1-l} P_{i+1}^l \geq P_i^{n-1} \geq \beta(L_{|T}, n-1, k)k!r_i,$$

and so

$$(\sum_{l=0}^{n-1} P_i^{n-1-l} P_{i+1}^l)(\mu_i - \mu_{i+1}) \geq n\beta(L_{|T}, n-1, k)\, k!\, r_i.$$

Now we take the minimal lower bound for $s \leq n - 2$ and $s = n - 1$ and observe that

$$\min\{n\,\beta(L_{|T}, n-1, k)\, k!, (k+1)!\,\delta_1(L_{|T}) = \beta(L_{|T}, n, k+1)(k+1)!$$

We finally obtain

$$\mathrm{vol}(L) \geq N_{m+1}^n \geq \beta(L_{|T}, n, k+1)(k+1)! \sum_{i=1}^{m} r_i(\mu_i - \mu_{i+1}) = \beta(L_{|T}, n, k+1)(k+1)! \deg_a^+ f_* L.$$

(iii) If equality holds in (i), then equality holds for $s(g, L_m, \overline{a})$ and so Theorem 3.4 applies. If equality holds in (ii), then equality holds in any step and in particular in formula (3.7). If $r_{i+1} = r_i$ for all i then $m = 1$ and $f_*(L_0)^+$ is semistable.

In the case (i), equality implies directly $q_f = k$ just applying Remark 3.5 to $(X_m, L_m, \overline{a})$. In the case (ii), equality implies that the triplet (F_i, R_i, a_i) (where $a_i(F_i) = a(F) = K$) is in the *first Clifford-Severi line* and so $q_f = \dim K = \dim F_i = k$ (see Section 2).   □

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