Robust Zonotopic Prognostics Approaches for LPV Systems Based on Set-Membership and Extended Kalman Filter

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***Abstract*—This paper proposes robust model-based prognostics approaches based on zonotopic Joint Estimation of States and Parameters (JESP) for Linear Parameter-Varying (LPV) systems. Zonotopes are employed due to their simple computations with a reduced number of vertices. Thus, Zonotopic Set-Membership (ZSM) and Zonotopic Extended Kalman Filter (ZEKF) approaches are investigated for the JESP which plays a crucial role in the proposed Prognostics and Health Management (PHM) approach. The zonotopic estimators are optimally-tuned using a specially formulated Linear Matrix Inequality (LMI) framework to guarantee a high estimation accuracy and less conservative results. Furthermore, a Recursive ZSM (RZSM) approach is derived from a conventional Recursive Least Squares (RLS) filter for the sake of Remaining Useful Life (RUL) forecasting of exponentially-decayed parameters. Additionally, a polynomial RUL forecasting approach has been also proposed based on the ZEKF approach. Finally, a degraded DC-DC converter is modelled as an LPV system and examined with the proposed approaches, and the obtained results show their efficiency.**

***Index Terms*—Model-based prognostics, zonotopes, set-membership, extended Kalman filter, linear parameter-varying, linear matrix inequality, joint state-parameter estimation**

I. INTRODUCTION

Model-based PHM approaches are mainly based on state observations in various engineering applications [1]. The more the system is critical, the more the stability and accuracy constraints should be guaranteed. From the perspective of the high complexity level of the prognostics techniques, the PHM is directed towards complex uncertain systems with unknown behaviors. The latter can be described as nonlinear dynamical systems that require advanced JESP techniques to deal with such uncertainties [2]. On one hand, random Gaussian perturbations and noises characterize the stochastic approaches which are employed in the EKF for nonlinear systems. On the other hand, the remarkable feature of bounding the unknown uncertainties in the deterministic set-membership approach is very interesting for the reliability assessment, especially in the prognostics framework. Furthermore, the set-membership approach makes use of the several type of sets for bounding uncertainties. Polytopes such as boxes and parallelotopes are efficient with linear models, ellipsoids require simple implementation, yet they lack flexibility and could lead to pessimistic estimation, and zonotopes provides a trade-off between simple arithmetic computations and high flexibility when compared to other sets [1], [3], [4]. On account of feasible application of zonotopes, the ZSM approach is employed for JESP based on the prediction, measurement and intersection steps. Moreover, the ZSM-based JESP carries out the burden of multi-output systems and guarantees the estimation process by repeating the intersection between the measurement strip of each output and the predicted zonotope [5]. Hence, the JESP is the reliable base of the proposed PHM approach that aims to forecast the RUL on a macro level. Wherefore, a RZSM approach has been derived based on the conventional RLS filter in a zonotopic scheme to indirectly predict the bounded RUL of the system based on the estimated zonotopic parameter from ZSM and a general exponential decay for degradation.

Moreover, based on the broad application of the EKF, in addition to the provided flexibility of zonotopes, a ZEKF approach is also provided in this paper for the JESP [6], [7]. Hence, we also propose a zonotopic RUL prediction based on

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polynomial parameter degradation which predicts the End of Life (EoL) of the system and provides a bounded RUL [8].

A nonlinear embedding has been applied to the dynamical system with varying parameters which transforms the nonlinear system to an LPV that performs well with both proposed approaches [9]. Additionally, an LMI-based optimization is formulated to guarantee optimal, fast and accurate tuning of the ZSM and ZEKF estimators. Beside the existing online tuning techniques, the proposed LMI-based problem can be applied to nonlinear systems in addition to the fact that is designed to be solved offline and only simple heuristic operations are solved online. Hence, these benefits provide a faster estimation and forecasting due to less computational operations and time.

This paper is structured as follows. Section II provides a background material regarding zonotopes. Section III highlights the problem formulation of the PHM approaches for LPV systems. Thus, Section IV addresses the proposed estimation and forecasting approaches. Moreover, both prognostics approaches have been examined with a degraded DC-DC converter in Section V that shows their effectiveness in the results. Finally, the conclusions are drawn in Section VI.

II. Preliminaries

Definition II.1. *A zonotope $\mathcal{Z} = \langle \mathbf{c}, \mathbf{G} \rangle \subset \mathbb{R}^n$ with center $c \in \mathbb{R}^n$ and the generator matrix $\mathbf{G} \in \mathbb{R}^{n \times m}$ is a polytopic set defined as a linear image of the unit hypercube $[-1, 1]^m$, as follows:*

$$\langle \mathbf{c}, \mathbf{G} \rangle = \{\mathbf{c} + \mathbf{G}s, \|s\|_\infty \leq 1\}. \tag{1}$$

Definition II.2 (Sum of Zonotopes)**.** *The Minkowski sum of two zonotopes results into a zonotope. Given two zonotopes $\mathcal{Z}_1 = \mathbf{c}_1 \oplus \mathbf{G}_1 \mathcal{B}^{m_1} \in \mathbb{R}^n$, and $\mathcal{Z}_2 = \mathbf{c}_2 \oplus \mathbf{G}_2 \mathcal{B}^{m_2} \in \mathbb{R}^n$, their sum is defined as follows:*

$$\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2 = (\mathbf{c}_1 + \mathbf{c}_2) \oplus [\mathbf{G}_1 \ \mathbf{G}_2] \mathcal{B}^{m_1 + m_2}. \tag{2}$$

Definition II.3 (Reduction operator)**.** *The weighted reduction operator $\downarrow_{q,w}$ is used to reduce and limit the size of the vertices of the generator matrix $\mathbf{G}$ to q columns, while preserving the inclusion property of zonotopes.*

Definition II.4 (F_W-radius)**.** *The weighted Frobenius radius of a given zonotope $\mathcal{Z} = \langle \mathbf{c}, \mathbf{G}_{\mathcal{Z}} \rangle \in \mathbb{R}^n$ is the weighted Frobenius norm of the same zonotope:*

$$\|\mathcal{Z}\|_{F,W} = \|\mathbf{G}\|_{F,W}. \tag{3}$$

The $F-$radius is similar to the F_W-radius with W is an identity matrix.

III. Problem Formulation for Prognosis Framework

A. Problem set-up

Consider the average model of a switched discrete-time linear parameter-varying system:

$$\mathbf{x}_{k+1} = \mathbf{A}(\rho_k)\mathbf{x}_k + \mathbf{B}(\rho_k)\mathbf{u}_k + \mathbf{E}_\omega \omega_k, \tag{4}$$

$$\mathbf{y}_k = \mathbf{C}(\rho_k)\mathbf{x}_k + \mathbf{D}(\rho_k)\mathbf{u}_k + \mathbf{E}_\upsilon \upsilon_k, \tag{5}$$

where ρ_k is a vector that contains all the varying parameters (i.e. $\rho_{k,1}, \ldots, \rho_{k,n_\rho}$). $\mathbf{x}_k \in \mathbb{R}^{n_x}$, $\mathbf{y}_k \in \mathbb{R}^{n_y}$ and $u_k \in \mathbb{R}^{n_u}$ are the states, outputs and inputs of the system respectively. For the sake of notation simplicity, in the following we will use $\mathbf{A}_k = \mathbf{A}(\rho_k) \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{B}_k = \mathbf{B}(\rho_k) \in \mathbb{R}^{n_x \times n_y}$, $\mathbf{C}_k = \mathbf{C}(\rho_k) \in \mathbb{R}^{n_u \times n_y}$ and $\mathbf{D}_k = \mathbf{D}(\rho_k) \in \mathbb{R}^{n_u \times n_u}$ for denoting the state matrix, input matrix, output matrix and feedthrough matrix, respectively. $\mathbf{E}_\omega \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{E}_\upsilon \in \mathbb{R}^{n_y \times n_y}$ are the direction matrices for the process and measurement noises, uncertainties and perturbations, $\omega_k \in \mathbb{R}^{n_x}$ and $\upsilon_k \in \mathbb{R}^{n_y}$, respectively. The uncertainties are assumed to be bounded by unitary hypercube zonotope centered at 0

$$\omega_k = \langle \omega_c = 0, \mathbf{I}_{n_\omega} \rangle = [-1, 1], \quad \forall k \geq 0, \tag{6a}$$

$$\upsilon_k = \langle \upsilon_c = 0, \mathbf{I}_{n_\upsilon} \rangle = [-1, 1], \quad \forall k \geq 0, \tag{6b}$$

In this paper, the varying parameters ρ of the LPV model (4) describe the nonlinear degradation equations of power electronic components that are augmented to the state vector, and embedded in the state-space matrices such that $\mathbf{x}_k \in \mathbb{R}^{n_{m+p}}$ contains m states and p parameters:

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{x}_k & \mathbf{x}_{\rho_k} \end{bmatrix}^\intercal = \begin{bmatrix} x_1 & \ldots & x_m & \rho_1 & \ldots & \rho_p \end{bmatrix}^\intercal. \tag{7}$$

Moreover, $\rho_1, \ldots, \rho_p$ are considered as unknown but bounded as $[\underline{\rho_k}, \overline{\rho_k}]$. Furthermore, the small unknown variations $\delta\rho_k$ are included as uncertainties in the direction matrix $\mathbf{E}_\omega$. Henceforth, $\mathbf{x}$ denotes the joint states and parameters vector of the zonotope $\mathcal{X}$, and $\mathbf{x}_\rho$ represents the augmented parameter vector only of the zonotope $\mathcal{X}_\rho$. $\rho_{0,i}$ denotes the initial value of the parameter ρ_i.

B. Proposed PHM approaches

The proposed PHM approaches in Figure 1 are mainly based on the following three steps:

1) *System Modeling:* The dynamical model with augmented nonlinear parameters which are characterized as degradation models, is represented in LPV form. Thus, the latter contains the varying joint states and parameters that will be estimated afterwards using zonotopes for the sake of RUL forecasting.
2) *Zonotopic-based JESP:* The joint estimation plays a crucial role in the PHM approach. Since the LPV is modelled to contain all the joint variations, we have proposed two approaches for the JESP. The ZSM and the ZEKF are optimally tuned based on an LMI solver. Hence, their investigation in the JESP framework can guarantee a robust base for the following step of RUL forecasting.
3) *RUL Forecasting:* The aim of this whole study is to forecast the RUL of degraded systems such as power converters. Thus, based on the ZSM and the ZEKF estimators, two RUL forecasting approaches are assigned to them accordingly. The target of the RZSM is to retrieve the zonotopic parameter estimation from the ZSM estimator and recursively predict the exponentially decayed degradation models that will result into an

online forecasting of bounded RUL. Unlike the RZSM predictor, the polynomial EoL approach is applied to the ZEKF and forecast the bounded RUL based on a polynomial EoL model solving.

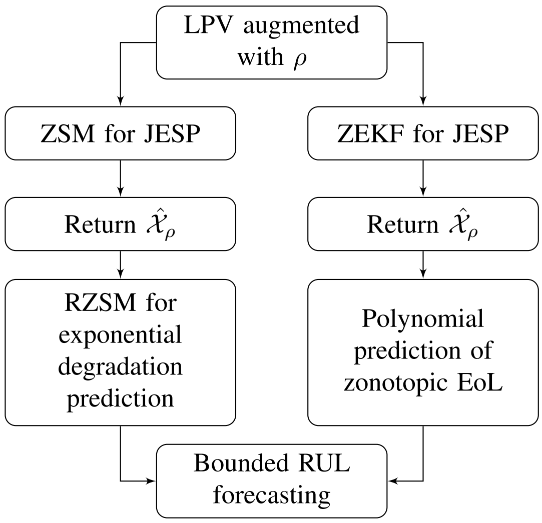

Fig. 1. Diagram of the two proposed approaches for JESP and RUL forecasting

IV. Proposed Approaches for JESP and RUL Forecasting

This section is dedicated to firstly explain the ZSM and the RZSM approaches. Secondly, the ZEKF and the polynomial EoL approaches are assigned for zonotopic JESP and RUL forecasting. Thirdly, the LMI-based optimal tuning for both approaches is addressed. It should be noted that for the sake of notation simplification, the vectors and matrices of the ZEKF approach will be superscripted by (ZEKF), and similarly for the ZSM. ($\hat{}$) and ($\tilde{}$) denote estimated and predicted elements, respectively.

A. ZSM-based JESP and RUL forecasting

The consistent state-bounding zonotope $\hat{\mathcal{X}}_{k,i}$ at each component of the measured output $y_{k,i}$ can be obtained based on a zonotopic *Luenberger observer* as follows [3], [5], [10]:

$$\hat{\mathcal{X}}_{k,i}^{\text{ZSM}} = \hat{\mathbf{x}}_k^{\text{ZSM}} \oplus \hat{\mathbf{G}}_k^{\text{ZSM}} \mathcal{B}^{m+n_x+n_y}, \tag{8}$$

where,

$$\hat{\mathbf{x}}_k^{\text{ZSM}} = \tilde{\mathbf{x}}_k^{\text{ZSM}} + \lambda_{k,i}(y_k - (\mathbf{C}_k\tilde{\mathbf{x}}_k^{\text{ZSM}} + \mathbf{D}_k\mathbf{u}_k)), \tag{9a}$$

$$\hat{\mathbf{G}}_k^{\text{ZSM}} = [(\mathbf{I} - \lambda_k\mathbf{C}_k)\tilde{\mathbf{G}}_{k,\downarrow_{q,W}}^{\text{ZSM}} \qquad -\lambda_k\mathbf{E}_\upsilon], \tag{9b}$$

with,

$$\tilde{\mathbf{x}}_k^{\text{ZSM}} = \mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1}^{\text{ZSM}} + \mathbf{B}_{k-1}\mathbf{u}_{k-1}, \tag{10a}$$

$$\tilde{\mathbf{G}}_k^{\text{ZSM}} = [\mathbf{A}_{k-1}\hat{\mathbf{G}}_{k-1,\downarrow_{q,W}}^{\text{ZSM}} \quad \mathbf{E}_\omega], \tag{10b}$$

where $\hat{\mathbf{x}}_k^{\text{ZSM}}$ is the center of the consistent state-bounding zonotope, $\hat{\mathbf{G}}_k^{\text{ZSM}}$ denotes its generator matrix, $\tilde{\mathbf{x}}_k^{\text{ZSM}}$ denotes the center of the uncertain predicted zonotope $\tilde{\mathcal{X}}_k^{\text{ZSM}}$, and λ is the tuning matrix that is extensively explained throughout this section. Additionally, i refers to the i^{th} output component at each time sample k.

Given each measurement $y_{k,i}$ of the measurements vector $\mathbf{y}_k$ at time instant k of the same model, the consistent measurement set $\mathcal{X}_{y_{k,i}}$, denoted by the strip is computed as follows:

$$\mathcal{X}_{y_{k,i}} = \{\mathbf{x} \in \mathbb{R}_{n_x} : |y_{k,i} - (\mathbf{C}_{k,i}^{\intercal}\mathbf{x}_k^{\text{ZSM}} + \mathbf{D}_k\mathbf{u}_k)| \leq \mathbf{E}_{\upsilon_{k,i}}\}, \tag{11}$$

where $\mathbf{C}_{k,i}^{\intercal}$ is a vector of the i^{th} row of the matrix $\mathbf{C}_k$ at time instant k. The estimated consistent zonotope (8) is obtained by intersecting the uncertain predicted zonotope (10) and the strip (11) as:

$$\hat{\mathcal{X}}_{k,i}^{\text{ZSM}} = \tilde{\mathcal{X}}_k^{\text{ZSM}} \cap \mathcal{X}_{\mathbf{y}_{k,i}} = \langle \hat{\mathbf{x}}_{k,i}^{\text{ZSM}}, \hat{\mathbf{G}}_{k,i}^{\text{ZSM}} \rangle, \tag{12}$$

Remark. *The computation of the predicted state set for multi-output systems is implemented as in the single-output case by repeating the intersection per each output.*

Proposition IV.1. *For a guaranteed zonotopic inclusion, the intersection between the measurement strip $\mathcal{X}_{y_{k,i}}$ of each of the output components, and the predicted state-bounding set $\tilde{\mathcal{X}}_k$ is successively repeated for $i = 1, \ldots, n_y$ at each time sample k [3], [5]. Thus, this correction step is repeated with every output component $y_{k,i}$ of $\mathbf{y}_k$ until n_y in order to obtain the consistent estimated state zonotopic set with guaranteed inclusion, at each time instant k as:*

$$\begin{aligned} \hat{\mathcal{X}}_{k,n_y}^{ZSM}(\lambda_1, \ldots, \lambda_{n_y}) = \hat{\mathbf{x}}_{k,n_y}^{ZSM}(\lambda_1, \ldots, \lambda_{n_y}) \\ \oplus \hat{\mathbf{G}}_{k,n_y}^{ZSM}(\lambda_1, \ldots, \lambda_{n_y})\mathcal{B}^{m+n_x+n_y}, \end{aligned} \tag{13}$$

1) RZSM approach for RUL forecasting: The continuous-time degradation variation is proposed as follows:

$$\Delta\rho_{t,i} \equiv x_{\rho_{t,i}}^{\text{ZSM}} - \rho_{0,i} = \alpha_{1,i}e^{\alpha_{2,i}t}, \tag{14}$$

Then, it is discretized and rewritten in logarithmic form, to be able to deal with the previously estimated zonotopic parameter $\hat{\mathcal{X}}_{\rho_{k,i}}^{\text{ZSM}}$ of the augmented parameter $\rho_{k,i}$ as:

$$\ln(\hat{x}_{\rho_{k,i}}^{\text{ZSM}} - \rho_{0,i}) = \ln(\alpha_{1,i}) + \alpha_{2,i} \cdot k \cdot T_s, \tag{15}$$

where $k \in \mathbb{N}$ and T_s is the sampling time, ρ_0 is known and $\hat{x}_{\rho_{k,i}}^{\text{ZSM}}$ is the center of the consistent estimated zonotope $\hat{\mathcal{X}}_{\rho_{k,i}}^{\text{ZSM}}$. The degradation equation (15) is reformulated as follows:

$$\underbrace{\ln(\hat{x}_{\rho_{k,i}}^{\text{ZSM}} - \rho_{0,i})}_{\mathbf{y}_{\rho_{k,i}}} = \underbrace{\begin{bmatrix} 1 & k \cdot T_s \end{bmatrix}}_{\mathbf{C}_\alpha} \underbrace{\begin{bmatrix} \ln(\alpha_{1,k,i}) & \alpha_{2,k,i} \end{bmatrix}^{\intercal}}_{\alpha_i^{\intercal}}. \tag{16}$$

Next, the same ZSM approach equations (9) and (10) are utilized for the α parameters estimation and prediction as:

$$\hat{\alpha}_{k,i} = \mathbf{A}_\alpha\tilde{\alpha}_{k,i} + \mathbf{L}_{\alpha_{k,i}}(y_{\rho_{k,i}} - \mathbf{C}_{\alpha_{k,i}}\tilde{\alpha}_{k,i}) \tag{17a}$$

$$\hat{\mathbf{G}}_{\alpha_{k,i}} = \begin{bmatrix} (\mathbf{I} - \mathbf{L}_{\alpha_{k,i}}\mathbf{C}_{\alpha_{k,i}})\tilde{\mathbf{G}}_{\alpha_{k,i}} & -\mathbf{L}_{\alpha_{k,i}}E_{\alpha_\upsilon} \end{bmatrix} \tag{17b}$$

where $\mathbf{A}_\alpha = \mathbf{I}$ and $\mathbf{L}_\alpha$ is the tuning matrix of the RZSM for degradation parameters estimation.

Moreover, the bounded degradation models are predicted online from k measurement sample to N as:

$$\underline{\tilde{\mathbf{x}}}_{\rho_{[k\to N]},i} = \rho_{0,i} + \underline{\hat{\alpha}}_{1,i} e^{\underline{\hat{\alpha}}_{2,i}[k+1\to N]}, \tag{18a}$$

$$\overline{\tilde{\mathbf{x}}}_{\rho_{[k\to N]},i} = \rho_{0,i} + \overline{\hat{\alpha}}_{1,i} e^{\overline{\hat{\alpha}}_{2,i}[k+1\to N]}. \tag{18b}$$

Finally, given the experimentally defined threshold (TH) ρ^{TH} of the power electronic components in the literature, the RUL forecasting is computed such that:

$$\underline{\tilde{\mathrm{RUL}}}_{k,i} \;:\; \overline{\tilde{\mathbf{x}}}_{\rho_{[k\to N]},i} \;\cap\; \rho_i^{\mathrm{TH}}, \quad k \le t_{k,i}^{\mathrm{EoL}} < N \tag{19a}$$

$$\overline{\tilde{\mathrm{RUL}}}_{k,i} \;:\; \underline{\tilde{\mathbf{x}}}_{\rho_{[k\to N]},i} \;\cap\; \rho_i^{\mathrm{TH}}, \quad k \le t_{k,i}^{\mathrm{EoL}} < N \tag{19b}$$

B. ZEKF-based JESP and RUL forecasting

The ZEKF is designed to follow the same structure of the EKF for the estimation process with zonotopic bounding. Unlike the ZSM approach, the zonotopic state estimation $\hat{\mathcal{X}}^{\mathrm{ZEKF}} = \langle \hat{\mathbf{x}}^{\mathrm{ZEKF}}, \hat{\mathbf{G}}^{\mathrm{ZEKF}} \rangle$ is based on the current time measurement y_k instead of the previous y_{k-1}. Thus, by introducing the tuning matrix Λ to the model (4), the center and the generator matrix of $\hat{\mathcal{X}}^{\mathrm{ZEKF}}$ are given as [1], [6], [11]:

$$\hat{\mathbf{x}}_{k+1}^{\mathrm{ZEKF}} = \mathbf{A}_k \hat{\mathbf{x}}_k^{\mathrm{ZEKF}} + \mathbf{B}_k \mathbf{u}_k + \Lambda_k(\mathbf{y}_k - \mathbf{C}_k \hat{\mathbf{x}}_k^{\mathrm{ZEKF}} - \mathbf{D}_k \mathbf{u}_k), \tag{20a}$$

$$\hat{\mathbf{G}}_{k+1}^{\mathrm{ZEKF}} = \Big[(\mathbf{A}_k - \Lambda_k \mathbf{C}_k) \hat{\mathbf{G}}_{k,\downarrow_{q,W}}^{\mathrm{ZEKF}}, \quad \mathbf{E}_\omega, \quad -\Lambda_k \mathbf{E}_\upsilon \Big], \tag{20b}$$

where $\mathbf{y}_k$ is the measured output vector and $\hat{\mathbf{y}}_k$ is its estimation at each time step.

Remark. *There is no strip intersection process needed in the ZEKF approach where the structure follows the conventional EKF prediction and estimation process. Moreover, is is worth noting that Λ is introduced to the ZEKF approach instead of the observer gain in a conventional filter. All the tuning matrices of the ZSM and ZEKF are optimally computed instead of employing the conventional online gains.*

1) ZEKF-based polynomial RUL forecasting: Consider two general polynomial equations that describe the variations of an unknown degradation model in an interval form as:

$$\Delta\underline{\rho}_{k,i} \equiv \underline{\hat{x}}_{\rho_{k,i}}^{\mathrm{ZEKF}} - \rho_{0,i} = \underline{b}_{p,i}\underline{\beta}^p + \underline{b}_{p-1,i}\underline{\beta}^{p-1} + \cdots + \underline{b}_{1,i}\underline{\beta}, \tag{21a}$$

$$\Delta\overline{\rho}_{k,i} \equiv \overline{\hat{x}}_{\rho_{k,i}}^{\mathrm{ZEKF}} - \rho_{0,i} = \overline{b}_{p,i}\overline{\beta}^p + \overline{b}_{p-1,i}\overline{\beta}^{p-1} + \cdots + \overline{b}_{1,i}\overline{\beta}. \tag{21b}$$

It is worth noting that a linear EoL-RUL model has been considered in this paper. Thus, the bounded EoL equations are obtained as follows:

1) At $t = 0$, the system is assumed to be 100% healthy with 0% degradation
2) Solve (21) with the previously estimated $\hat{\mathbf{x}}_{\rho_k}^{\mathrm{ZEKF}}$ and return the coefficients b.
3) Resolve the same equation with the calculated coefficients b for $\Delta\rho_k = \max(\mathrm{TH})\%$, where the max(TH) is the experimentally obtained accepted threshold of each power electronic components, and return β^{EoL}.
4) Forecast the interval RUL equations as follows:

$$\underline{\mathrm{RUL}}_{k,i} = \underline{\beta}_{k,i}^{\mathrm{EoL}} - k, \quad \overline{\mathrm{RUL}}_{k,i} = \overline{\beta}_{k,i}^{\mathrm{EoL}} - k, \tag{22}$$

C. Optimal computation of $\lambda^*, \mathbf{L}_\alpha^*$ and Λ^*

1) LMI-based optimization of the tuning matrices: $\lambda, \mathbf{L}_\alpha$ and Λ play a significant role in the correction step of the prediction of states and parameters. The optimal tuning serves the whole prognosis framework and not limited to the JESP only. In particular, the goal of tuning is to minimize the effect of the uncertainties and increase the accuracy of the estimation due to robustness constraints. Therefore, there exist few methods for the computations of the tuning matrices that can guarantee an optimal enclosure of zonotopes. The online technique is based on the reduction of the F_W-radius of the generator matrices of zonotopes. It has shown remarkable performances, yet it is limited to linear models and can consume more computational time and process than the offline techniques. Hence, an LMI-based optimization problem has been proposed to solve a reduced and limited number of tuning matrices offline, where only a simple arithmetic operation is computed online to obtain the optimal tuning matrix. $\lambda^*, \mathbf{L}_\alpha^*$ and Λ^* are considered the optimal tuning matrices and are all tuned using the LMI-based technique. For notations simplification purposes, only λ^* is addressed and the same techniques apply to $\mathbf{L}_\alpha^*$ and Λ^* as well. Given the LPV model in (4) and following the structure of the zonotopic JESP in ZSM, RZSM and ZEKF, the optimal tuning matrix λ^* is computed offline if there exists a positive scalar γ such that:

$$\lambda^\star = \mathbf{\Psi}^{-1}\mathbf{W}. \tag{23}$$

Hence, by introducing the matrices $\mathbf{Q}$ and $\mathbf{R}$ as tuning parameters, the proposed optimization problem is solved following this form:

$$\begin{aligned}
&\underset{\mathbf{W},\mathbf{\Psi}}{\text{minimize}} \quad \gamma \\
&\text{subject to} \\
&\begin{bmatrix} \gamma\mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{\Psi} \end{bmatrix} \succeq 0, \\
&\begin{bmatrix} -\mathbf{\Psi} & \mathbf{\Psi}\mathbf{A} - \mathbf{W}\mathbf{C} & \mathbf{\Psi}\sqrt{Q}^{\mathsf{T}} & \mathbf{W} \\ \star & -\mathbf{\Psi} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mathbf{I} & \mathbf{0} \\ \star & \star & \star & -R^{-1} \end{bmatrix} \preceq 0
\end{aligned} \tag{24}$$

where $\star$ denotes symmetrical elements. Furthermore, the LMI-based framework (24) reduces and limits the maximum number of solving the optimization problem to N computations. Consequently, only the interpolation of the offline-obtained tuning matrices is computed online at each k as follows:
An approach is proposed to solve the LMI optimization problem offline with reduced arithmetic operations. Based on the augmented vector (7) and the model (4), $\rho_{k,i}$ denotes the varying parameters in zonotopic form as $\mathcal{X}_{\rho_{k,i}} = \langle \rho_{0,i}, \mathbf{G}_{\rho_{k,i}} \rangle$. Where $\rho_{0,i}$ is the center of the parameter zonotope $\mathcal{X}_{\rho_{k,i}}$ and denotes the nominal value of ρ_i.

$$\mathcal{X}_{\rho_{k,i}} \subset \mathbb{R}_+^{n_\rho} : |x_{\rho_{k,i}} - \rho_{0,i}| \le \rho_i^{\mathrm{TH}}, \quad \forall k > 0, \tag{25}$$

where ρ_i^{TH} is the value of the statistically and experimentally obtained TH of the physical components [12]. Thus, the polytopic representation of (4) is obtained using the bounding box approach and considering the range of variation of the varying parameters.

$$\mathbf{x}_{k+1} = \sum_{i=1}^{N} \mu_i(\rho_k)\left(\mathbf{A}_i\mathbf{x}_k + \mathbf{B}_i\mathbf{u}_k\right), \tag{26a}$$

$$\mathbf{y}_k = \sum_{i=1}^{N} \mu_i(\rho_k)\left(\mathbf{C}_i\mathbf{x}_k + \mathbf{D}_i\mathbf{u}_k\right), \tag{26b}$$

where μ_i are the coefficients of the polytopic decomposition such that:

$$\sum_{i=1}^{N} \mu_i(\rho_k) = 1, \;\; \mu_i(\rho_k) \geq 0, \quad \forall i = 1, \ldots, N. \tag{27}$$

Finally, a varying value for the tuning matrix (23) is obtained as follows:

$$\lambda^{\star}(\rho_k) = \sum_{i=1}^{N} \mu_i(\rho_k)\,\lambda_i^{\star}, \tag{28}$$

where $\lambda_i^{\star}$ are obtained by solving (24) at the vertices of the polytopic model (26a)-(26b).

2) Online computation of the tuning matrices: There exists a classical online method to compute the tuning parameter, based on the minimization of the Frobenius norm of the generator vertices using. Additionally, the proof of the computation of the optimal λ^*, $\mathbf{L}_\alpha$ and Λ^* has shown its independence of the weighting matrix [1], [4]. λ^*_{on}, $\mathbf{L}_{\alpha_{\text{on}}}$ and Λ_{on^*} denote the tuning matrices of the online classical approach, and are determined (by only featuring λ^*_{on} and applies for all) as:

$$\lambda^*_{k_{\text{on}}} = \frac{\tilde{\mathbf{G}}_k\tilde{\mathbf{G}}_k^{\intercal}\mathbf{C}}{\mathbf{C}^{\intercal}\tilde{\mathbf{G}}_k\tilde{\mathbf{G}}_k^{\intercal}\mathbf{C} + \mathbf{E}_v\mathbf{E}_v^{\intercal}}. \tag{29}$$

Remark. *It is worth mentioning that the same tuning matrices are obtained if the LMI-based optimization was solved online or offline by interpolation. Nevertheless, it is preferable to adopt the polytopic approach for the sake of less computational time and memory consumption, In addition to its application to nonlinear systems unlike the online approach.*

V. Case Study

A 30 kW Boost converter is examined as an application of the proposed prognostics approaches in terms of degraded MOSFET. The cascading damage affects various parameters in the system. Briefly, the interconnection of the components complicates the estimation process of the states due to the very slow degradation behaviors. Moreover, from the technical perspective, the degradation models are nonlinear models that are included into the system model. Thus, to reduce the computational complexity of the online Jacobian linearization, the system has been transformed into an LPV. See [2] for detailed analysis about the degradation behaviors of the critical components, their effects and their modeling with the full numerical application which are not included in this paper due to the size limitations. The following state-space matrices characterize the average model of the boost converter with MOSFET degradation:

$$\mathbf{A}_k = \begin{bmatrix} -1.136\times 10^5 & -1363 & 0 & 0 \\ 622.6 & -0.00727 - 0.0385\hat{x}_{\rho_k} & -4452 & -2397.26\hat{\mathbf{x}}_{\mathbf{2,k}} \\ 0 & 130 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{30a}$$

$$\mathbf{B}_k = \begin{bmatrix} 113636.6 & 0 \\ 6226 & 356.1 \\ 0 & 200 \\ 0 & 0 \end{bmatrix}, \tag{30b}$$

$$\mathbf{C}_k = \begin{bmatrix} -9.0909 & 0.909 & 0 & 0 \\ 0 & 0.052 & 1 & 0 \end{bmatrix}, \tag{30c}$$

$$\mathbf{D}_k = \begin{bmatrix} 9.0909 & 0 \\ 0 & -0.08 \end{bmatrix}, \tag{30d}$$

where x_ρ denotes the fault precursor of the MOSFET R_{ON}, the switching frequency is 15 kHz and the duty cycle is $d = 0.35$. Figure 2 illustrates the estimated zonotopic parameter $\hat{\mathcal{X}}_\rho$ and shows an increment in the value of the on-resistance R_{ON} of the MOSFET throughout the whole degradation process. The other states such as input capacitor voltage, inductor current, and output capacitor voltage are also affected by the cascading damage. However, they are not illustrated in this paper due the size limitations. The centers of the ZSM and the ZEKF approaches are estimated with high accuracy when compared to the empirical degradation model of MOSFET. Additionally, the estimated bounds by the ZSM show a tighter enclosure than the estimated bounds of the ZEKF.

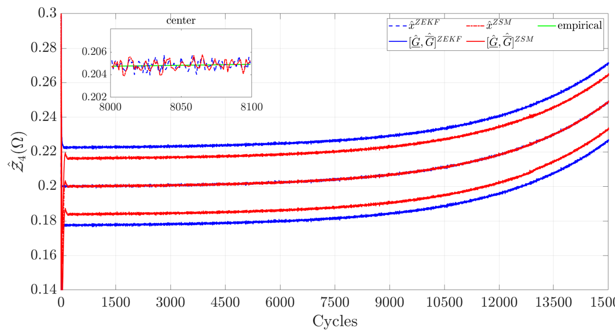

Fig. 2. Zonotopic estimation of the varying parameter $\mathcal{X}_\rho$ by ZSM and ZEKF approaches

Moreover, based on the aforementioned explanation, both RUL forecasting approaches require a zonotopic estimation of the degraded parameter. Thus, Figure 3 illustrates the forecasted bounded RUL by each approach. The empirical RUL of the system is shown in green in function of cycles. Hence, the ZSM provides a bounded RUL forecasting with full enclosure around the empirical RUL. On the other hand, the results of the polynomial RUL prediction of the ZEKF do not exceed the safe zone of the empirical model. However, the bounds converge towards the real EoL in a very conservative way

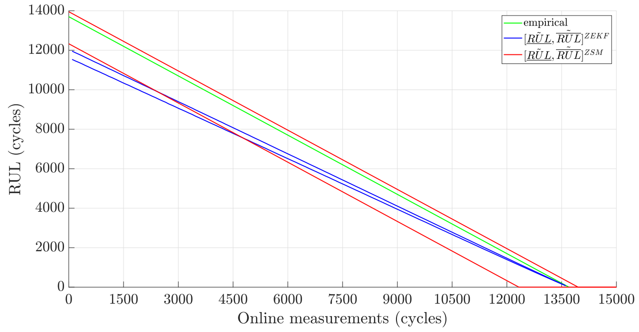

Fig. 3. Bounded RUL forecasting using the RZSM and the polynomial ZEKF approaches

with narrow bounds.
As an example, for an online measurement at cycle = 6000, the empirical RUL, the predicted zonotopic RUL $[\underline{\tilde{RUL}}, \overline{\tilde{RUL}}]$ of the ZEKF, and the RZSM approaches are approximated at 7700 cycles, $[6510, 6750]$ and $[6330, 7953]$ respectively. Consequently, the RZSM provides the most consistent and optimistic RUL bounds.

VI. Conclusions

A complete investigation about the performance of the two proposed prognostics approaches has been addressed in this paper. Zonotopes are employed for the sake of their known flexibility and simple computations with high accuracy. In broad, the proposed PHM is designed for dynamical systems with unknown degraded parameters which is transformed into an LPV model. Moreover, the ZSM and the ZEKF have shown efficient estimation results with very close bounds, due to the formulated LMI problem that guarantees an optimal tuning for both estimators. Furthermore, an online bounded RUL forecasting has been provided based on the estimated zonotopic varying parameter in each approach. Thus, the specially-designed RZSM for parameter estimation, predicts the degradation models in a general exponential decay and indirectly forecasts the RUL. On the other hand, the polynomial prediction approach also provides a bounded RUL forecasting based on polynomial models. Finally, the results have shown that the RZSM is perfectly bounding the empirical RUL, and too conservative RUL enclosures have been obtained by the polynomial ZEKF approach. Current and future studies will consider more degraded parameters in the system in order to upgrade from micro-level to macro-level prognosis.

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