On the Degree of Transitivity of a Fuzzy Relation

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Abstract

Considering a family $\{T_c\}_{c\in[0,1]}$ generated from a t-norm T, the degree of T-transitivity of a fuzzy relation R is revisited and proved to coincide with the greatest c for which R is T_c-transitive.

This fact gives rise to the study of new families of t-norms to generate different degrees of transitivity with respect to them.

The mappings transforming fuzzy relations into transitive fuzzy relations smaller than or equal to the given ones are studied.

Keywords: t-norm, Archimedean t-norm, fuzzy relation, T-transitivity, degree of transitivity.

1 Introduction

Transitivity with respect to a t-norm T (T-transitivity) is probably the most important property a fuzzy relation can fulfill. This is mainly because fuzzy equivalence relations and fuzzy preorders- fuzzifying the concepts of equivalence relation and preorder respectively- are T-transitive fuzzy relations.

It may happen though that a fuzzy relation R which we require to be T-transitive for theoretical or practical reasons do not satisfy this property.

In fact, when R comes from applied domains, where data are dependent on the accuracy of empirical measurements or to the subjectivity of human assessments, it cannot be expected that R is neatly transitive with respect to a given t-norm. In these cases it is proposed to consider the degree of transitivity of R and then the property of being T-transitive turns out to be fuzzy.

For a given t-norm T, a fuzzy relation R on a universe X is T-transitive when for all $x, y, z \in X$, $T(R(x,y), R(y,z)) \leq R(x,z)$. Considering the residuation $\overrightarrow{T}$ of the t-norm (see Section 2) this is equivalent to

$$\overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z)) = 1$$

for all $x, y, z \in X$. The first author that fuzzified this concept and defined and studied the degree of T-transitivity of a fuzzy relation was Gottwald [13, 14, 15] (see also [4, 16]). For a left continuous t-norm T the degree of transitivity of a fuzzy relation R on a universe X was defined by

$$\inf_{x,y,z\in X} \overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z)).$$

In [3] L. Běhounek generalized this definition by considering a fuzzy equality E on X and imposing compatibility between R and E:

$$\inf_{x,x',y,y',z,z'\in X} \overrightarrow{T}(T(E(x,x'), E(y,y'), E(z,z'), R(x,y), R(y',z))|R(x',z')).$$

Later on, in [6, 7] the degree of transitivity of R was related to the degree of equivalence or similarity $E_T(R,S)$ (Definition 2.19) between R and a T-transitive fuzzy relation S taken as referential: If $E_T(R,S) \geq \alpha$, then the degree of transitivity of R is greater than or equal to $\alpha^{(3)} = T(\alpha, T(\alpha,\alpha))$. In [11, 12] different ways of obtaining a transitive relation from a reflexive and symmetric fuzzy relation (i.e., a proximity relation) were provided.

Gottwald introduced the degree of transitivity of a fuzzy relation in the context of formal fuzzy logic. It is written as a logic sentence and depends on the conjunction (t-norm) (and the implication) associated to the corresponding Logic. In the present manuscript, following the motto that good definitions deserve to be generalized, the definition has been re-interpreted by considering a family of t-norms, called $\{T_c\}$ in the paper. The degree of transitivity coincides with the greatest c for which the relation is T_c-transitive.

All the previous considerations assume that the t-norm T is given, and based on this assumption a measure on how far R is from being transitive is provided by means of the transitivity degree.

The reason why a fuzzy relation R that "should" be T-transitive does not satisfy completely this property can be analyzed from two perspectives. We can think that we are asking too much in the sense that

a) the t-norm is too big or

b) R has to be modified.

The transitivity with respect to a t-norm T indicates the level of structuralization of the data represented by the relation R. The smaller the t-norm T, the less structuralised. If T is the drastic t-norm (i.e. $T(x, y) = 0$ for all $x, y < 1$), then virtually every relation R is T-transitive. At the other end of the spectrum, if $T = \min$ the condition of being T-transitive becomes very restrictive. Therefore, if we choose a) (i.e. to change the t-norm T), a procedure must be provided in order to determine a new t-norm small enough to accommodate R, but still optimal in some sense.

If we follow b) instead, and we consider that is R that should be modified, a reasonable way to replace R by a new relation has to be provided.

These two points of view turn out to be equivalent in the sense that will be explained in Sections 3 and 4.

Going back to the problem of finding a suitable t-norm T to fit the relation R, it is worth recalling that choosing optimal candidates from parametric families of objects is standard mathematical practice. This is the case, for example, of Parametric Statistics, function approximation by means of orthogonal families of polynomials, Computer Aided Design or Neural Networks, among many others. In the present situation it happens sometimes that we are bounded to a certain family $\{T_a\}$ of t-norms because they have to satisfy a certain condition, property or functional equation. In these cases we are constrained to this family and it is reasonable to define a degree of transitivity with respect to it. In Section 4 we recall a couple of ways to obtain a family of t-norms from a given one that include some of the most used ones, namely, Yager's, Dombi's, Aczél-Alsina's t-norms.

For a fuzzy relation R its T-transitive closure is known to be the smallest fuzzy relation greater than or equal to R. This provides the best upper approximation of R by a T-transitive fuzzy relation. More complex is finding good lower approximations of R. In [9] good lower approximations for reflexive and symmetric fuzzy relations are obtained for universes of finite cardinality. Later on in [8] optimal lower approximations, called openings, of these relations were obtained again for finite cardinality but with a non-efficient algorithm. The problem of finding lower approximations for arbitrary fuzzy relations in universes of non necessarily finite cardinality was still open and Section 5 provides an easy way to obtain them.

The paper is structured as follows: In the next section, the basic definitions and results concerning t-norms including the definitions of T-transitivity, degree of T-transitivity and power with respect to a t-norm will be recalled as well as some results on fuzzy relations. In Section 3, a family $A = \{T_c\}_{c\in[0,1]}$ of t-norms will be generated from a given t-norm T and the degree of T-transitivity of a fuzzy relation R will coincide with the index c_0 of the greatest t-norm of A for which R is T_{c_0}-transitive. Inspired by this result, in Section 4 two different ways of generating families from a given continuous Archimedean t-norm will be recalled and corresponding new degrees of T-transitivity associated to these families will be defined and studied. Moreover as a generalization of Section 3, the mappings transforming a fuzzy relation R into a transitive one smaller than or equal to R will be studied in Section 5. The last section of the paper contains some concluding remarks.

2 Preliminaries

This section contains some definitions and properties related to t-norms and fuzzy relations that will be needed in the article. For more details on t-norms we recommend [17].

Definition 2.1. [17] *A t-norm is a binary operation* $T : [0,1] \times [0,1] \to [0,1]$ *which satisfies the following properties for all* $x, y, z, t \in [0,1]$

- $T(x,y) = T(y,x)$
- $T(x,y) \leq T(z,t)$ *if* $x \leq z$ *and* $y \leq t$
- $T(x, T(y,z)) = T(T(x,y), z)$
- $T(x,1) = x$

Example 2.2.

- *The minimum t-norm* min *is defined by* $T(x,y) = \min(x,y)$ *for all* $x, y \in [0,1]$.
- *The Łukasiewicz t-norm is defined by* $T(x,y) = \max(0, x+y-1)$.
- *The product t-norm is defined by* $T(x,y) = x \cdot y$.

Remark. The minimum t-norm is the greatest t-norm.

Definition 2.3. *Let* T *be a t-norm and* $x_1, x_2, ..., x_n \in [0,1]$. $T(x_1, x_2, ..., x_n)$ *is defined recursively.*

- $T(x_1) = x_1$.
- $T(x_1, x_2, ..., x_n) = T(T(x_1, x_2, ..., x_{n-1}), x_n)$.

In particular, for $x \in [0,1]$ *and* $n \in \mathbb{N}$, $x_T^{(n)}$ *will denote* $T(\overbrace{x, x, ..., x}^{n\ times})$ *and* $x_T^{(0)} = 1$.

Definition 2.4. *For a t-norm* T, *an* $x \in [0,1]$ *is an idempotent element of* T *if and only if* $T(x,x) = x$. $E(T)$ *will be the set of idempotent elements of* T.

Definition 2.5. *A t-norm* T *is Archimedean if and only if* $E(T) = \{0,1\}$.

Example 2.6. *The product and Łukasiewicz t-norms are Archimedean t-norms, while the minimum t-norm is not.*

Definition 2.7. *For a t-norm* T, *an* $x \in [0,1]$ *is nilpotent if and only if there exists an* $n \in N$ *such that* $x_T^{(n)} = 0$. $Nil(T)$ *will be the set of nilpotent elements of* T.

Proposition 2.8. *If a t-norm* T *is continuous Archimedean, then* $Nil(T)$ *is* $[0,1)$ *or* $\{0\}$. *In the first case,* T *is called nilpotent and strict in the second case.*

Proposition 2.9 (Ling's Theorem)**.** *A continuous t-norm* T *is Archimedean if and only if there exists a continuous and strictly decreasing function* $t : [0,1] \to [0,\infty)$ *with* $t(1) = 0$ *such that*

$$T(x,y) = t^{[-1]}(t(x) + t(y))$$

where $t^{[-1]}$ *is the pseudo inverse of* t, *defined by*

$$t^{[-1]}(x) = \begin{cases} t^{-1}(x) & \text{if } x \in [0, t(0)] \\ 0 & \text{otherwise.} \end{cases}$$

T is strict if $t(0) = \infty$ and nilpotent otherwise.

t is called an additive generator of T and two generators of the same t-norm differ only by a positive multiplicative constant.

If T is nilpotent (strict), its additive generator with $t(0) = 1$ ($t(\frac{1}{2}) = 1$) is called its normalized additive generator.

Example 2.10.

1. *$t(x) = 1 - x$ is an additive generator of the Łukasiewicz t-norm.*
2. *$t(x) = -\log(x)$ is an additive generator of the product t-norm.*

Lemma 2.11. *Let t be an additive generator of a continuous Archimedean t-norm and $t^{[-1]}$ its pseudoinverse. Then*

$$(t \circ t^{[-1]})(x) = \begin{cases} x & \text{if } x \leq t(0) \\ t(0) & \text{otherwise} \end{cases}$$

for all $x \geq 0$.

Lemma 2.12. *Let T be a continuous Archimedean t-norm and t its normalized additive generator. Then*

$$(t \circ t^{[-1]})(x) = x$$

for all $x \in [0, 1]$.

Definition 2.13. *Let T be a t-norm. Its residuation $\overrightarrow{T}$ is the mapping $\overrightarrow{T} : [0,1] \times [0,1] \to [0,1]$ defined for all $x, y \in [0,1]$ by*

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0,1] \mid T(x, \alpha) \leq y\}.$$

The residuation is usually defined only if T is a left continuous t-norm. In this case $\left\langle [0,1], \wedge, \vee, T, \overrightarrow{T}, 0, 1 \right\rangle$ is a complete commutative residuated lattice [4] where T and $\overrightarrow{T}$ satisfy the adjoiness property

$$T(x, y) \leq z \Longleftrightarrow y \leq \overrightarrow{T}(x|z).$$

In this case, $\overrightarrow{T}$ is also called the residual implication associated to T.

Definition 2.14. *Let T be a t-norm. Its biresiduation $\overleftrightarrow{T}$ is defined for all $x, y \in [0,1]$ by*

$$\overleftrightarrow{T}(x,y) = \min(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

If T is left continuous, then $\overrightarrow{T}$ and $\overleftrightarrow{T}$ play the role of implication and biimplication respectively.

Definition 2.15. *A fuzzy relation R on a universe X is a mapping $R : X \times X \to [0,1]$.*

In other words, R is a fuzzy set of $X \times X$. For two elements $x, y \in X$ $R(x,y)$ is the degree in which x and y are related by R.

Definition 2.16. *Let T be a t-norm. A fuzzy relation R on a set X is T-transitive if for all $x, y, z \in X$,*

$$T(R(x,y), R(y,z)) \leq R(x,z).$$

Definition 2.17. *Let T be a t-norm. A T-transitive fuzzy relation R on a set X is a Tindistinguishability operator if it satisfies for all $x, y \in X$,*

- *$R(x,x) = 1$ (Reflexivity)*
- *$R(x,y) = R(y,x)$ (Symmetry)*

Definition 2.18. *[13] For a fuzzy relation R on a set X and a left continuous t-norm T, the degree of T-transitivity $\alpha_T(R)$ of R with respect to T is*

$$\alpha_T(R) = \inf_{x,y,z \in X} \overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z)).$$

$\alpha_T(R) = 1$ if and only if R is T-transitive.

Definition 2.19. *[18] The degrees of similarity or equivalence $E_T(\mu, \nu)$, $E_T(R, S)$ between two fuzzy subsets μ and ν of a set X and between two fuzzy relations R and S on a set X are defined, respectively, by*

$$\begin{aligned} E_T(\mu,\nu) &= \inf_{x \in X} \overleftrightarrow{T}(\mu(x), \nu(x)) \\ E_T(R,S) &= \inf_{x,y \in X} \overleftrightarrow{T}(R(x,y), S(x,y)). \end{aligned}$$

where $\overleftrightarrow{T}$ is the biresiduation associated to T.

In Section 4 we will need the concept of power with respect to a t-norm [1, 20, 18, 17].

Definition 2.20. *Given a continuous t-norm T and $m, n \in \mathbb{N}$, the n-th root $x_T^{(\frac{1}{n})}$ of x with respect to T is defined by*

$$x_T^{(\frac{1}{n})} = \sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

and for $m, n \in \mathbb{N}$, $x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)}$.

Lemma 2.21. [1] *Let T be a continuous t-norm. If k, m, n are non-negative integers and $k, n \neq 0$, then $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$.*

The powers $x_T^{(\frac{m}{n})}$ can be extended to irrational exponents in a straightforward way.

Definition 2.22. [18] *If $r \in \mathbb{R}^+$ is a positive real number, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers with $\lim_{n\to\infty} a_n = r$. For any $x \in [0,1]$, the power $x_T^{(r)}$ is*

$$x_T^{(r)} = \lim_{n\to\infty} x_T^{(a_n)}.$$

Continuity assures the existence of the last limit and independence of the sequence $\{a_n\}_{n\in\mathbb{N}}$.

Proposition 2.23. [18] *Let T be a continuous Archimedean t-norm with additive generator t, $x \in [0,1]$ and $r \in \mathbb{R}^+$. Then*

$$x_T^{(r)} = t^{[-1]}(rt(x)).$$

Proposition 2.24. [18] *Let T be a continuous t-norm, R a T-transitive relation on X and $r > 0$. Then the fuzzy relation $R_T^{(r)}$ defined for all $x, y \in X$ by $R_T^{(r)}(x,y) = (R(x,y))_T^{(r)}$ is a T-transitive relation on X.*

Example 2.25.

- *If T is a continuous Archimedean t-norm with additive generator t and R a T-transitive fuzzy relation on X, then $t^{[-1]}(r \cdot t(R))$ is a T-transitive fuzzy relation on X.*

- *If T is the Łukasiewicz t-norm and R a T-transitive fuzzy relation on X, then $\max(0, 1 - r + r \cdot R)$ is a T-transitive fuzzy relation on X.*

- *If T is the product t-norm and R a T-transitive fuzzy relation on X, then R^r is a T-transitive fuzzy relation on X.*

3 Different Interpretations of α_T

In this section we will provide different characterizations of the degree α_T of T-transitivity of a fuzzy relation that are related to its T_c-transitivity with respect to the members of the family $\{T_c\}_{c\in[0,1]}$ of t-norms defined below.

Definition 3.1. *Given a t-norm T and $c \in [0,1]$ let T_c be the t-norm defined for all $x, y \in [0,1]$ by*

$$T_c(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ T(x,y,c) & \text{otherwise} \end{cases}$$

if $c \in (0,1]$ and $T_0 = T_D$, where T_D stands for the smallest t-norm.

Proposition 3.2.

a) T_c is a t-norm for every $c \in [0,1]$.

b) T_c is continuous (left-continuous) if and only if T is continuous (left-continuous) and $c = 1$.

c) The family $\{T_c\}_{c\in[0,1]}$ satisfies

1. $T_1 = T$
2. $c \leq c'$ *implies* $T_c \leq T_{c'}$
3. $\lim_{c\to 0} T_c = T_D$.

Proof. The results are straightforward and we will only prove b).

If $c = 1$, then $T_c = T$.

If $c < 1$, we will prove that T_c is not continuous (left-continuous) at the point $(1,1)$. $T_c(1,1) = 1$ but for $(x,y) \neq (1,1)$, $T_c(x,y) = T(x,y,c) \leq c < 1$. □

Definition 3.3. *Let R be a fuzzy relation on a set X, T a t-norm and $c \in [0,1]$. Then $R_{(T,c)}$ is defined for all $x, y \in X$ by*

$$R_{(T,c)}(x,y) = T(R(x,y),c).$$

Proposition 3.4. *Let T be a left-continuous t-norm. R a fuzzy relation on a set X and $c \geq 0$.*

a) *If* $c \leq \alpha_T(R)$*, then* $R_{(T,c)}$ *is a* T*-transitive fuzzy relation.*

b) *If* T *is strictly monotone on* $]0,1]^2$*, then* $R_{(T,c)}$ *is a* T*-transitive fuzzy relation on* X *if and only if* $c \leq \alpha_T(R)$*.*

c) R *is* T_c*-transitive if and only if* $c \leq \alpha_T(R)$*.*

Proof.

a) For all $x, y, z \in X$,

$$c \leq \overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z))$$

which is equivalent to

$$T(R(x,y), R(y,z), c) \leq R(x,z)$$

which implies

$$T(R(x,y), R(y,z), c, c) \leq T(R(x,z), c)$$

and this inequality is equivalent to

$$T(R_{(T,c)}(x,y), R_{(T,c)}(y,z)) \leq R_{(T,c)}(x,z).$$

b) $R_{(T,c)}$ is T-transitive if and only if for all $x, y, z \in X$,

$$T(R_{(T,c)}(x,y), R_{(T,c)}(y,z)) \leq R_{(T,c)}(x,z)$$

or, equivalently,

$$T(R(x,y), R(y,z), c, c) \leq T(R(x,z), c).$$

Since T is strictly monotone it satisfies the cancellation law.

$$c = 0 \text{ or } T(R(x,y), R(y,z), c) \leq R(x,z).$$

This is equivalent to

$$c \leq \overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z))$$

and since this is satisfied for all $x, y, z \in X$, $c \leq \alpha_T(R)$.

The sufficiency follows from a).

c) R is T_c-transitive if and only if for all $x, y, z \in X$,

$$T_c(R(x, y), R(y, z)) \leq R(x, z)$$

or, equivalently,

$$T(R(x, y), R(y, z), c) \leq R(x, z).$$

This is equivalent to

$$c \leq \overrightarrow{T}(T(R(x, y), R(y, z))|R(x, z))$$

and since this is satisfied for all $x, y, z \in X$, $c \leq \alpha_T(R)$.

The sufficiency follows from a).

□

Proposition 3.4a) and 3.4b) provide a characterization of $\alpha_T(R)$ as the greatest c for which $R_{(T,c)}$ is T-transitive.

Proposition 3.4c) provides the second characterization of $\alpha_T(R)$.

The third characterization is Corollary 3.7:

Lemma 3.5. [5] *Let T be a left-continuous t-norm. For $x, y \in [0, 1]$, $\overrightarrow{T}(x|T(x, y)) \geq y$.*

Proposition 3.6. *Let T be a left-continuous t-norm, R a fuzzy relation on a universe X and $c \in [0, 1]$. Then*

1. *$E_T(R, R_{(T,c)}) \geq c$*
2. *If T is strictly monotone on $]0, 1]^2$ and there exist $x, y \in X$ with $R(x, y) \neq 0$, then $E_T(R, R_{(T,c)}) = c$.*

Proof.

1.

$$\begin{aligned} E_T(R, R_{(T,c)}) &= \inf_{x,y \in X} \overleftrightarrow{T}(R(x, y), R_{(T,c)}(x, y)) \\ &= \inf_{x,y \in X} \overleftrightarrow{T}(R(x, y), T(R(x, y), c)) \\ &= \inf_{x,y \in X} \overrightarrow{T}(R(x, y) \mid T(R(x, y), c)) \geq c. \end{aligned}$$

2. $\overrightarrow{T}(R(x,y) \mid T(R(x,y),c)) = \sup\{\alpha \mid T(\alpha, R(x,y)) \leq T(R(x,y),c)\}$. T satisfies the cancellation law and therefore from}

$$T(\alpha, R(x,y)) \leq T(R(x,y),c)$$

it follows

$$\alpha \leq c$$

and from here

$$\overrightarrow{T}(R(x,y) \mid T(R(x,y),c)) \leq c.$$

Therefore,

$$\inf_{x,y\in X} \overrightarrow{T}(R(x,y) \mid T(R(x,y),c)) \leq c.$$

□

Corollary 3.7. *Let T be a left-continuous t-norm, R a fuzzy relation on a universe X and $c \in [0,1]$. Then*

1. $E_T(R, R_{(T,\alpha_T(R))}) \geq \alpha_T(R)$
2. *If T is strictly monotone on $]0,1]^2$, then $E_T(R, R_{(T,\alpha_T(R))}) = \alpha_T(R)$.*

4 Alternative degrees for continuous Archimedean t-norms

Inspired by Section 3 we will consider alternative degrees of transitivity of a fuzzy relation R with respect to a continuous Archimedean t-norm T. For this, we will build two families $\{T_\lambda\}_{\lambda>0}$ from T, one in each of the following subsections, and study the transitivity of R with respect to the t-norms of these families.

4.1 Powers of Additive Generators

Definition 4.1. [10, 17] *Let T be a continuous Archimedean t-norm, t an additive generator of T and $\lambda > 0$. T_λ is the continuous Archimedean t-norm generated by an additive generator t_λ given for all $x \in [0,1]$ by $t_\lambda(x) = (t(x))^\lambda$.*

The pseudo-inverse $t_\lambda^{[-1]}$ of t_λ is $t_\lambda^{[-1]}(x) = t^{[-1]}(x^{\frac{1}{\lambda}})$ for all $x \in [0,\infty]$ and for every $\lambda > 0$ T_λ is explicitly given by

$$T_\lambda(x,y) = \begin{cases} t^{-1}(((t(x))^\lambda + (t(y))^\lambda)^{\frac{1}{\lambda}}) & \text{if } ((t(x))^\lambda + (t(y))^\lambda)^{\frac{1}{\lambda}} \leq t(0) \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in [0,1]$.

Example 4.2.

- *From the Łukasiewicz t-norm, we obtain the family of Yager's t-norms [22].*
- *From the t-norm with additive generator $t(x) = \frac{1-x}{x}$ for all $x \in [0,1]$, we obtain the family of Dombi t-norms [10].*
- *From the product t-norm, we obtain the family of Aczél-Alsina t-norms [2].*

Proposition 4.3. [17] *The family $\{T_\lambda\}_{\lambda>0}$ satisfies*

1. $T_1 = T$
2. $\lambda \leq \lambda'$ *implies* $T_\lambda \leq T_{\lambda'}$
3. $\lim_{\lambda\to\infty} T_\lambda = \min$
4. $\lim_{\lambda\to 0} T_\lambda = T_D$.

Definition 4.4. *Let R be a fuzzy relation on a set X, T a continuous Archimedean t-norm, t an additive generator of T and $\lambda > 0$. R_λ^t is the fuzzy relation defined for all $x, y \in X$ by $R_\lambda^t(x,y) = t^{[-1]}((t(R(x,y)))^\lambda)$.*

Proposition 4.5. *Let R be a fuzzy relation on a set X, T a continuous Archimedean t-norm, t and $u = \mu t$ ($\mu > 0$) two additive generators of T and $\lambda > 0$. Then*

$$R_\lambda^u(x,y) = t^{[-1]}(\mu^{\lambda-1}(t(R(x,y)))^\lambda)$$

for all $x, y \in X$.

Proposition 4.5 shows that R_λ^t is not canonical in the sense that it depends on the additive generator. It is not true in general that R_λ^t is T-transitive if and only if R_λ^u is. The next proposition gives necessary conditions for this equivalence.

Proposition 4.6. *Let R be a fuzzy relation on a set X, T a continuous Archimedean t-norm, t and $u = \mu t$ ($\mu > 0$) two additive generators of T and $\lambda > 0$. If T is nilpotent, t is its normalized additive generator and $\mu^{\lambda-1} \leq 1$. Then*

a) R^t_λ is T-transitive if and only if R^u_λ is T-transitive.

b) R^t_λ is T-transitive if and only if R is T_λ-transitive.

Proof.

a) R^u_λ is T-transitive if and only if for all $x, y, z \in X$,

$$T(R^u_\lambda(x,y), R^u_\lambda(y,z)) \leq R^u_\lambda(x,z) \;\Leftrightarrow$$

$$t(R^u_\lambda(x,y)) + t(R^u_\lambda(y,z)) \geq t(R^u_\lambda(x,z)) \;\Leftrightarrow$$

$$t(t^{[-1]}(\mu^{\lambda-1}(t(R(x,y)))^\lambda)) + t(t^{[-1]}(\mu^{\lambda-1}(t(R(y,z)))^\lambda)) \geq t(t^{[-1]}(\mu^{\lambda-1}(t(R(x,z)))^\lambda)) \;\overset{*}{\Leftrightarrow}$$

$$\mu^{\lambda-1}(t(R(x,y)))^\lambda + \mu^{\lambda-1}(t(R(y,z)))^\lambda \geq \mu^{\lambda-1}(t(R(x,z)))^\lambda \;\Leftrightarrow$$

$$(t(R(x,y)))^\lambda + (t(R(y,z)))^\lambda \geq (t(R(x,z)))^\lambda \;\Leftrightarrow$$

$$t(t^{[-1]}((t(R(x,y)))^\lambda) + t(t^{[-1]}((t(R(y,z)))^\lambda) \geq t(t^{[-1]}((t(R(x,z)))^\lambda) \;\overset{*}{\Leftrightarrow}$$

$$t(R^t_\lambda(x,y)) + t(R^t_\lambda(y,z)) \geq t(R^t_\lambda(x,z)) \;\Leftrightarrow$$

$$T(R^t_\lambda(x,y), R^t_\lambda(y,z)) \leq R^t_\lambda(x,z)$$

which means that R^t_λ is T-transitive.

The equivalences with a star follow from Lemma 2.11.

b) R^t_λ is T-transitive if and only if for all $x, y, z \in X$,

$$T(R^t_\lambda(x,y), R^t_\lambda(y,z)) \leq R^t_\lambda(x,z) \;\Leftrightarrow$$

$$t(R^t_\lambda(x,y)) + t(R^t_\lambda(y,z)) \geq t(R^t_\lambda(x,z)) \;\Leftrightarrow$$

$$t(t^{[-1]}((t(R(x,y)))^\lambda)) + t(t^{[-1]}((t(R(y,z)))^\lambda)) \geq t(t^{[-1]}((t(R(x,z)))^\lambda)) \;\overset{*}{\Leftrightarrow}$$

$$t(R(x,y))^\lambda + t(R(y,z))^\lambda \geq t(R(x,z))^\lambda \;\Leftrightarrow$$

$$t_\lambda(R(x,y)) + t_\lambda(R(y,z)) \geq t_\lambda(R(x,z)) \;\Leftrightarrow$$

$$T_\lambda(R(x,y), R(y,z)) \leq R(x,z)$$

where the equivalence with a star follows from Lemma 2.11.

$\square$

Definition 4.7. *Let T be a continuous Archimedean t-norm and R a fuzzy relation on a set X. $\lambda_0(R) = \sup\{\lambda \geq 0 \mid R \text{ is } T_\lambda-transitive\}$ is the degree of transitivity of R with respect to the family $\{T_\lambda\}$.*

The next result is a corollary of Proposition 4.6.

Corollary 4.8. *Let T be a continuous Archimedean t-norm and R a fuzzy relation on a set X. Then $\lambda_0(R) = \min(\sup\{\lambda \geq 0 \mid R^t_\lambda \text{ is } T-transitive\}, 1)$ for any additive generator t of T if T is strict and for t the normalized additive generator is T is nilpotent.*

Remark. It can happen that $\lambda_0 = 0$ even in sets of finite cardinality. As an example, it is well known that for a fuzzy relation R it is necessary, in order to be transitive with respect to a given t-norm T, that $R(x,z) = R(y,z)$ if $R(x,y) = 1$ [1]. So, for a fuzzy relation S with $S(x,y) = 1$ and $S(x,z) \neq S(y,z)$, S^t_λ will not be T-transitive for any $\lambda \geq 0$.

Example 4.9. *Let R be the fuzzy relation with matrix*

$$\begin{pmatrix} 1 & 0.3 & 0.8 \\ 0.3 & 1 & 0.6 \\ 0.8 & 0.6 & 1 \end{pmatrix}$$

Then $\lambda_0(R) = 0.564$ with respect to the Łukasiewicz t-norm while $\alpha_T(R) = 0.9$.

The rest of this subsection studies the similarity between the elements of the family of t-norms and between a fuzzy relation R and the corresponding fuzzy relations R^t_λ. The study of the mentioned similarities shows how similar is the original t-norm and fuzzy relation to the modified one and can be used for estimation of an error in applications.

Proposition 4.10. *Let $\lambda_1 \geq \lambda_2$ and T a nilpotent t-norm. Then $E_T(T_{\lambda_1}, T_{\lambda_2}) = t^{-1}((1 - 2^{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}})t(0))$.*

[1] If R is T-transitive and $R(x,y) = 1$, then $T(R(x,y), R(y,z)) = R(y,z) \leq R(x,z)$ and similarly $R(x,z) \leq R(y,z)$.

Proof.

If $\lambda_1 = \lambda_2$, then $t^{-1}((1 - 2^{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}})t(0)) = t^{-1}(0) = 1$.

If $\lambda_1 > \lambda_2$, then

$$E_T(T_{\lambda_1}, T_{\lambda_2}) = \inf_{x,y\in[0,1]} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y)) = \inf_{x,y\in[0,1]} \overrightarrow{T}(T_{\lambda_1}(x,y)|T_{\lambda_2}(x,y)).$$

Let us partition $[0,1]^2$ into three subsets A, B, C:

a) A is the set of pairs (x,y) such that $T_{\lambda_1}(x,y) = 0$ (and therefore $T_{\lambda_2}(x,y) = 0$),

b) B is the set of pairs (x,y) such that $T_{\lambda_1}(x,y) \neq 0$ and $T_{\lambda_2}(x,y) = 0$ and

c) C is the set of pairs (x,y) such that $T_{\lambda_1}(x,y) \neq 0$ and $T_{\lambda_2}(x,y) \neq 0$.

a)

$$\inf_{(x,y)\in A} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y)) = 1.$$

b)

$$\begin{aligned}
\inf_{(x,y)\in B} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y)) &= \inf_{(x,y)\in B} \overleftrightarrow{T}(T_{\lambda_1}(x,y), 0) \\
&= \inf_{(x,y)\in B} (t^{-1}(t(0) - t(t^{[-1]}((((t(x))^{\lambda_1} + (t(y))^{\lambda_1})^{\frac{1}{\lambda_1}}))))) \\
&= t^{-1}(\sup_{(x,y)\in B} (t(0) - t(t^{[-1]}((((t(x))^{\lambda_1} + (t(y))^{\lambda_1})^{\frac{1}{\lambda_1}}))))) \\
&= t^{-1}(\sup_{(x,y)\in B} (t(0) - ((t(x))^{\lambda_1} + (t(y))^{\lambda_1})^{\frac{1}{\lambda_1}})).
\end{aligned}$$

Putting $t(x) = X$ and $t(y) = Y$ and $B' = \{(t(x), t(y)) \mid (x,y) \in B\}$, we consider the function $F : B' \to [0, t(0)]$

$$F(X,Y) = t(0) - (X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}}.$$

We want to find where F attains its maximum.

$F(0,Y) = t(0) - Y$ and this would be maximum if $Y = 0$ but the corresponding point (x,y) with $t(x) = 0$ and $t(y) = 0$ is $(1,1)$ that does not belong to B. Similarly with $F(X,0)$. We can consider therefore the points (X,Y) with $X \neq 0$ and $Y \neq 0$.

The critical points of F would be the solutions of

$$\begin{cases} \frac{\partial F}{\partial x} = -(X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}-1} X^{\lambda_1 - 1} = 0 \\ \frac{\partial F}{\partial x} = -(X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}-1} Y^{\lambda_1 - 1} = 0 \end{cases}$$

But this system does not have solutions in B'.

The maximum will belong to the border of the domain of F which is composed by the curves $C_{\lambda_1} : (X_1^{\lambda} + Y_1^{\lambda})^{\frac{1}{\lambda_1}} = t(0)$ and $C_{\lambda_2} : (X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}} = t(0)$. The points (X,Y) of the curve C_{λ_1}, correspond to points (x,y) with $T_{\lambda_1}(x,y) = T_{\lambda_2}(x,y) = 0$ and there cannot be a maximum in C_{λ_1}. We can apply Lagrange multipliers to find the maximum of $F(X,Y)$ subject to the constraint $(X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}} = t(0)$:

$$\begin{cases} -(X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}-1} X^{\lambda_1 - 1} = k(X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}-1} X^{\lambda_2 - 1} \\ -(X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}-1} Y^{\lambda_1 - 1} = k(X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}-1} Y^{\lambda_2 - 1} \\ (X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}} = t(0) \end{cases}$$

By dividing both members of the first equation by the corresponding members of the second one, we get

$$\frac{X^{\lambda_1 - 1}}{Y^{\lambda_1 - 1}} = \frac{X^{\lambda_2 - 1}}{Y^{\lambda_2 - 1}}.$$

From that,

$$X^{\lambda_1 - \lambda_2} = Y^{\lambda_1 - \lambda_2}$$

and finally, $X = Y$ and using the third equality, the only maximum is attained at the point (X_0, Y_0) with $X_0 = Y_0 = 2^{-\frac{1}{\lambda_2}} t(0)$. If $(X_0, X_0) = (t(x_0), t(x_0))$, then $t(x_0) = 2^{-\frac{1}{\lambda_2}} t(0)$. Therefore

$$\begin{aligned} \inf_{(x,y)\in B} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y)) &= t^{-1}(t(0) - ((t(x_0))^{\lambda_1} + (t(x_0))^{\lambda_1}))^{\frac{1}{\lambda_1}}) \\ &= t^{-1}(t(0) - 2^{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}} t(0)). \end{aligned}$$

c)

$$\begin{aligned}
\inf_{(x,y)\in C} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y)) &= \inf_{(x,y)\in C} \overrightarrow{T}(T_{\lambda_1}(x,y)|T_{\lambda_2}(x,y)) \\
&= \inf_{(x,y)\in C} t^{-1}(t(T_{\lambda_2}(x,y)) - t(T_{\lambda_1}(x,y))) \\
&= \inf_{(x,y)\in C} t^{-1}(t(t^{-1}(((t(x))^{\lambda_2} + (t(y))^{\lambda_2})^{\frac{1}{\lambda_2}})) \\
&- t(t^{-1}(((t(x))^{\lambda_1} + (t(y))^{\lambda_1})^{\frac{1}{\lambda_1}}))) \\
&= t^{-1}(\sup_{(x,y)\in C} (((t(x))^{\lambda_2} + (t(y))^{\lambda_2})^{\frac{1}{\lambda_2}} \\
&- ((t(x))^{\lambda_1} + (t(y))^{\lambda_1})^{\frac{1}{\lambda_1}})).
\end{aligned}$$

Putting $t(x) = X$ and $t(y) = Y$ and $C' = \{(t(x), t(y)) \mid (x, y) \in C\}$, we consider the function $F : C' \to [0, t(0)]$

$$F(X,Y) = (X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}} - (X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}}.$$

We want to find where F attains its maximum.

If $X = 0$ or $Y = 0$, then $F(X,Y) = 0$ and at the points of the form $(X, 0)$ or $(0, Y)$ F does not attain its maximim.

Let us find its critical points:

$$\begin{cases}
\frac{\partial F}{\partial x} = (X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}-1} X^{\lambda_2 - 1} - (X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}-1} X^{\lambda_1 - 1} = 0 \\
\frac{\partial F}{\partial x} = (X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}-1} Y^{\lambda_2 - 1} - (X^{\lambda_1} + Y^{\lambda_1})^{\frac{1}{\lambda_1}-1} Y^{\lambda_1 - 1} = 0
\end{cases}$$

If $X \neq 0$, then $Y \neq 0$ and

$$\frac{Y^{\lambda_2 - 1}}{X^{\lambda_2 - 1}} = \frac{Y^{\lambda_1 - 1}}{X^{\lambda_1 - 1}}.$$

From this,

$$X^{\lambda_1 - \lambda_2} = Y^{\lambda_1 - \lambda_2}$$

and $X = Y$. However then

$$(2X^{\lambda_2})^{\frac{1}{\lambda_2}-1} X^{\lambda_2 - 1} = (2X^{\lambda_1})^{\frac{1}{\lambda_1}-1} X^{\lambda_1 - 1}$$

and hence

$$2^{\frac{1}{\lambda_2}-1} = 2^{\frac{1}{\lambda_1}-1}$$

and this implies $\lambda_1 = \lambda_2$, which is a contradiction.

Therefore the maximum belongs to the border of the domain of F. This border is composed by the segments L_1 joining the points $(0, t(0))$ with $(0,0)$ and L_2 joining the points $(t(0), 0)$ and $(0,0)$ and the curve $C_{\lambda_2} : (X^{\lambda_2} + Y^{\lambda_2})^{\frac{1}{\lambda_2}} = t(0)$. The maximum on this curve has already been found in b).

Finally,

$$\begin{aligned}E_T(T_{\lambda_1}, T_{\lambda_2}) = \min(&\inf_{(x,y)\in A} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y)), \inf_{(x,y)\in B} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y)),\\ &\inf_{(x,y)\in C} \overleftrightarrow{T}(T_{\lambda_1}(x,y), T_{\lambda_2}(x,y))) = \min(1, t^{-1}(t(0) - 2^{\frac{1}{\lambda_1}-\frac{1}{\lambda_2}} t(0)))\\ &= t^{-1}((1 - 2^{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}) t(0)).\end{aligned}$$

□

In particular,

$$E_T(T, T_\lambda) = \begin{cases} t^{-1}((1 - 2^{1-\frac{1}{\lambda}}) t(0)) & \text{if } \lambda < 1 \\ t^{-1}((1 - 2^{\frac{1}{\lambda}-1}) t(0)) & \text{if } \lambda \geq 1. \end{cases}$$

Example 4.11. *If T is the Łukasiewicz t-norm, then*

$$E_T(T, T_\lambda) = \begin{cases} 2^{1-\frac{1}{\lambda}} & \text{if } \lambda < 1 \\ 2^{\frac{1}{\lambda}-1} & \text{if } \lambda \geq 1. \end{cases}$$

Lemma 4.12. *Let T be a continuous Archimedean t-norm and t an additive generator of T normalized if T is nilpotent. Defining $x^t_\lambda = t^{[-1]}((t(x))^\lambda)$ for all $x \in [0,1]$ and $\lambda > 0$, $\overleftrightarrow{T}(x, x^t_\lambda) = t^{-1}(|t(x) - (t(x))^\lambda|)$.*

The next proposition provides a lower bound for $E_T(R, R^t_\lambda)$.

Proposition 4.13. *Let T be a continuous Archimedean t-norm and t an additive generator of T, normalized if T is nilpotent. Given a fuzzy relation R on a set X, $E_T(R, R^t_\lambda) = \inf_{x,y\in X} t^{-1}(|t(R(x,y)) - (t(R(x,y)))^\lambda|) \geq \inf_{z\in[0,1]} t^{-1}(|t(z) - (t(z))^\lambda|)$.*

Proof.

$$\begin{aligned}
E_T(R, R^t_\lambda) &= \inf_{x,y \in X} \overleftrightarrow{T}(R(x,y), R^t_\lambda(x,y)) \\
&\geq \inf_{z \in [0,1]} \overleftrightarrow{T}(z, z^t_\lambda) \\
&= \inf_{z \in [0,1]} t^{-1}(|t(z) - (t(z))^\lambda|).
\end{aligned}$$

□

Example 4.14. *If in the last proposition T is the Łukasiewicz t-norm and $t(x) = 1 - x$ for all $x \in [0,1]$ its normalized additive generator, then the lower bound $\inf_{z \in [0,1]} t^{-1}(|t(z) - (t(z))^\lambda|)$ is*

$$\begin{cases} 1 + \lambda^{\frac{1}{1-\lambda}} - \lambda^{\frac{\lambda}{1-\lambda}} & \text{if } \lambda < 1 \\ 1 - \lambda^{\frac{1}{1-\lambda}} + \lambda^{\frac{\lambda}{1-\lambda}} & \text{if } \lambda \geq 1. \end{cases}$$

Figure 1 plots the lower bound for the corresponding λ.

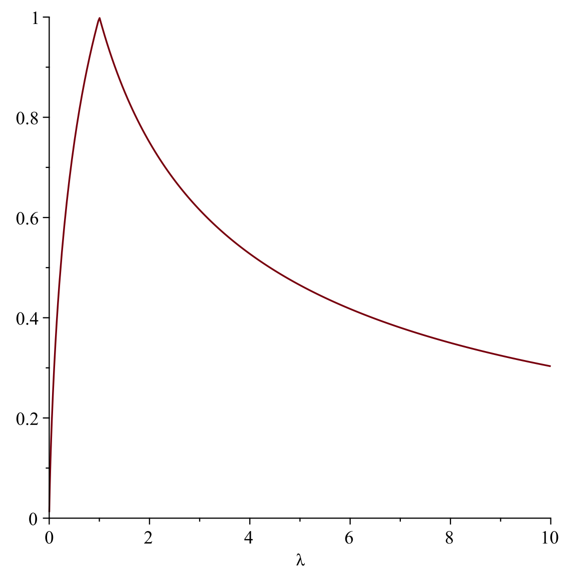

Figure 1: Lower bound for λ.

4.2 Powers with respect to a Strict t-norm

Another idea for obtaining new relations from the given fuzzy relation R is calculating their powers $R_T^{(r)}$ but, as we know from Proposition 2.24, if R is not T-transitive, none of its powers $R_T^{(r)}$ can be T-transitive either. Nevertheless, we can consider a new strict continuous Archimedean t-norm T^* and the powers $R_{T^*}^{(r)}$ of R.

In this section we will assume that T is a continuous Archimedean t-norm, t an additive generator of T, t^* an additive generator of a fixed strict continuous Archimedean t-norm and $\lambda > 0$.

Definition 4.15. *${}_\lambda T$ is the continuous Archimedean t-norm generated by an additive generator ${}_\lambda t$ given for all $x \in [0,1]$ by ${}_\lambda t(x) = t(x_{T^*}^{(\lambda)})$.*

This definition coincides with Example 3.32(ii) of [17] as is shown in the next lemma.

Lemma 4.16. *${}_\lambda t(x) = t(t^{*-1}(\lambda t^*(x)))$ for all $x \in [0,1]$.*

Lemma 4.17. *The pseudo-inverse ${}_\lambda t^{[-1]}$ of t^λ is ${}_\lambda t^{[-1]}(x) = (t^{[-1]}(x))_{T^*}^{(\frac{1}{\lambda})} = t^{*-1}(\frac{1}{\lambda}t^*(t^{[-1]}(x))$ for all $x \in [0,\infty]$.*

Lemma 4.18. *If $T^* = T$, then ${}_\lambda T = T$ for all $\lambda > 0$.*

Proof. ${}_\lambda t(x) = t(t^{-1}(\lambda t(x))) = \lambda t(x)$. $\square$

Lemma 4.19.

$$ {}_\lambda T(x,y) = (t^{[-1]}(t(x_{T^*}^{(\lambda)}) + t(y_{T^*}^{(\lambda)})))_{T^*}^{(\frac{1}{\lambda})}. $$

In general, the members of the family $\{{}_\lambda T\}_{\lambda>0}$ are not comparable. A counterexample is given in Example 6.4(ii) of [17].

Proposition 4.20. *${}_\lambda T \leq_\mu T$ if and only if $f : [0,_\mu t(0)] \to [0,\infty]$ defined for all $x \in [0,_\mu t(0)]$ by $t((t^{[-1]}(x))_{T^*}^{(\frac{\lambda}{\mu})})$ is subadditive.*

Proof. ${}_\lambda T \leq_\mu T$ if and only if ${}_\lambda t \circ_\mu t^{[-1]} : [0,_\mu t(0)] \to [0,\infty]$ is subadditive (Theorem 6.2 of [17].)

$$ {}_\lambda t \circ_\mu t^{[-1]}(x) =_\lambda t(t^{[-1]}(x))_{T^*}^{(\frac{1}{\mu})}) = t(((t^{[-1]}(x))_{T^*}^{(\frac{1}{\mu})})_{T^*}^{\lambda}) = t((t^{[-1]}(x))_{T^*}^{(\frac{\lambda}{\mu})}). $$

$\square$

Example 4.21. *Let T is the Łukasiewicz t-norm and T^* the product t-norm. If $\lambda \geq \mu > 0$, then ${}_\lambda T \leq_\mu T$.*

Proof. In this case, $x_{T^*}^{(r)} = x^r$ for all $r \geq 0$ and $t((t^{[-1]}(x))_{T^*}^{(\frac{\lambda}{\mu})}) = 1-(1-x)^{\frac{\lambda}{\mu}}$ which is a subadditive function when $\lambda \geq \mu > 0$. $\square$

Definition 4.22. *Let R be a fuzzy relation on a set X, T a continuous Archimedean t-norm with additive generator t, $\lambda > 0$ and T^* a strict continuous Archimedean t-norm. R_λ^* is the fuzzy relation defined for all $x, y \in X$ by $R_\lambda^*(x,y) = (R(x,y))_{T^*}^{(\lambda)}$.*

Proposition 4.23. *R_λ^* is T-transitive if and only if R is ${}_\lambda T$-transitive.*

Proof. R_λ^* is T-transitive if and only if for all $x, y, z \in X$,

$$T(R_\lambda^*(x,y), R_\lambda^*(y,z)) \leq R_\lambda^*(x,z) \iff$$

$$t(R_\lambda^*(x,y)) + t(R_\lambda^*(y,z)) \geq t(R_\lambda^*(x,z)) \iff$$
$$t((R(x,y))_{T^*}^{(\lambda)}) + t((R(y,z))_{T^*}^{(\lambda)}) \geq t((R(x,z))_{T^*}^{(\lambda)})$$

if and only if R is ${}_\lambda T$-transitive. $\square$

Definition 4.24. *Let R be a fuzzy relation on a set X, T a continuous Archimedean t-norm with additive generator t, $\lambda > 0$ and T^* a strict continuous Archimedean t-norm. $\lambda_0^*(R) = \sup\{\frac{1}{\lambda} \mid R \text{ is } {}_\lambda T - transitive\}$ is the degree of transitivity of R with respect to the family $\{{}_\lambda T\}_{\lambda > 0}$.*

Corollary 4.25. *$\lambda_0^*(R) = \sup\{\frac{1}{\lambda} \mid R_\lambda^* \text{ is } T - transitive\}$.*

Example 4.26. *We have seen in Example 4.9 that for the fuzzy relation R with matrix*

$$\begin{pmatrix} 1 & 0.3 & 0.8 \\ 0.3 & 1 & 0.6 \\ 0.8 & 0.6 & 1 \end{pmatrix},$$

if T is the Łukasiewicz t-norm, then $\alpha_T(R) = 0.9$ and $\lambda_0(R) = 0.564$. In this case, considering T^* the product t-norm, $\lambda_0^*(R) = \frac{1}{1.579} = 0.6331$.

5 How to make a Fuzzy Relation Transitive

For a given fuzzy relation R, in this section, a characterization of the mappings that provide a transitive relation smaller than or equal to R will be obtained when the t-norm is continuous Archimedean. More specifically, given a continuous Archimedean t-norm T and a fuzzy relation R with degree of transitivity $\alpha_T(R)$, we will characterize in Proposition 5.7 the mappings $\sigma : [0,1] \to [0,1]$ for which

- $\sigma \circ R$ is T-transitive
- $\sigma \circ R \leq R$.

Moreover, Proposition 5.9 will provide a way to obtain t-norms satisfying that R is transitive with respect to them.

Definition 5.1. *A fuzzy relation R on a set X is a proximity relation if for all $x, y \in X$*

- $R(x,x) = 1$
- $R(x,y) = R(y,x)$.

Recall that if moreover R is T-transitive for a given t-norm T, then R is called a T-indistinguishability operator (Definition 2.17).

Definition 5.2. *Let X be a set. A mapping $m : X \times X \to [0,\infty]$ is δ-triangular if $\delta = \max(0, \sup_{x,y,z \in X}\{m(x,z) - m(x,y) - m(y,z)\})$.*

$\delta = 0$ if and only if m satisfies the triangular inequality ($m(x,z) \leq m(x,y) + m(y,z)$ for all $x, y, z \in X$).

Proposition 5.3. *Let X be a set, $\delta > 0$, $m : X \times X \to [0,\infty]$ a δ-triangular mapping, $m_0 = \inf_{(x,y)\in X\times X} m(x,y)$, $\varepsilon : [0,\infty] \to [0,\infty]$ a mapping, $\varepsilon_0 = \inf_{a \geq m_0} \varepsilon(a)$ and $\varepsilon_1 = \sup_{a \geq m_0} \varepsilon(a)$. If ε satisfies the condition $2 \cdot \varepsilon_0 - \varepsilon_1 \geq \delta$, then*

1. *$m' = (i + \varepsilon) \circ m$ (where i is the identity map) satisfies the triangular inequality*
2. *$m' \geq m$.*

NOTE: If $\varepsilon_0 = \varepsilon_1 = \infty$ then we assume $2 \cdot \varepsilon_0 - \varepsilon_1 \geq \delta$ for all $\delta \geq 0$.

Proof.

a) If $\varepsilon_1 = \infty$, then $\varepsilon_0 = \infty$ and ε restricted to $[m_0, \infty]$ is the constant mapping $\varepsilon(a) = \infty$. Hence, $m'(x, y) = \infty$ for all $x, y \in X$.

b) If $\varepsilon_1 < \infty$,

1. $$(i + \varepsilon)(m(x, y)) + (i + \varepsilon)(m(y, z)) \geq (i + \varepsilon)(m(x, z))$$

   is equivalent to

   $$\varepsilon(m(x, y)) + \varepsilon(m(y, z)) - \varepsilon(m(x, z)) \geq m(x, z) - m(x, y) - m(y, z).$$

   This equality holds since

   $$\varepsilon(m(x, y)) + \varepsilon(m(y, z)) - \varepsilon(m(x, z)) \geq 2 \cdot \varepsilon_0 - \varepsilon_1 \geq \delta \geq m(x, z) - m(x, y) - m(y, z).$$

2. $m' = m + \varepsilon(m) \geq m$.

□

In order to deal with pseudometrics and proximity relations it is necessary to split $m(x, y) = 0$ and $\varepsilon(0)$ from the rest of possible values of m and ε. To this end, we define a new condition on m.

If a mapping $m : X \times X \to [0, \infty]$ satisfies that if $m(x, y) = 0$, then $m(x, z) = m(y, z)$ for all $x, y, z \in X$, then we will say that m satisfies Condition (*).

Proposition 5.4. *Let X be a set, $\delta > 0$, $m : X \times X \to [0, \infty]$ a δ-triangular mapping satisfying condition (*), with $m(x, x) = 0$ and $m(x, y) = m(y, x)$ for all $x, y \in X$. Let $M_0 = \{(x, y) \text{ s.t. } 0 < m(x, y)\}$, $m_0 = \inf_{(x,y) \in M_0} m(x, y)$ and suppose $m_0 > 0$. Let $\varepsilon : [0, \infty] \to [0, \infty]$ be a mapping such that $\varepsilon(0) = 0$, $\varepsilon_0 = \inf_{a \geq m_0} \varepsilon(a)$ and $\varepsilon_1 = \sup_{a \geq m_0} \varepsilon(a)$. If ε satisfies the condition $2 \cdot \varepsilon_0 - \varepsilon_1 \geq \delta$ for all $a \in [m_0, \infty]$ then*

1. *$m' = (i + \varepsilon) \circ m$ (where i is the identity map) is a pseudometric on X*

2. *$m' \geq m$.*

Proof. 1. We need to proof the inequality

$$(i+\varepsilon)(m(x,y)) + (i+\varepsilon)(m(y,z)) \geq (i+\varepsilon)(m(x,z)).$$

If $m(x,y) > 0$, $m(y,z) > 0$ and $m(x,z) > 0$ then proceed as in proof of Proposition 5.3.

If $m(x,y) = 0$ (or $m(y,z) = 0$) then Condition (*) ensures that $m(y,z) = m(x,z)$, (resp. m(x,y)=m(x,z)), and therefore the inequality holds.

If $m(x,z) = 0$ then $(i+\varepsilon)(m(x,z)) = 0$ and therefore the inequality holds too.

□

Lemma 5.5. [19] *Let T be a continuous Archimedean t-norm and t an additive generator of T. Then*

- *If R is a T-transitive fuzzy relation on a set X, then $t \circ R$ satisfies the triangular inequality.*
- *If $m : X \times X \to [0,\infty]$ satisfies the triangular inequality, then $t^{[-1]}(m)$ is a T-transitive fuzzy relation on X.*

Lemma 5.6. *Let T be a continuous Archimedean t-norm, t an additive generator of T, R a fuzzy relation on a set X and $\delta > 0$. Then R has degree of transitivity $\alpha_T(R)$ if and only if $m = t \circ R$ is a δ-triangular mapping with $\delta = t(\alpha_T(R))$.*

Proof.

a) Case $\alpha_T(R) = 1$.

   If $\alpha_T(R) = 1$ then R is T-transitive. Due to Lemma 5.5 this is equivalent to m satisfying the triangular inequality, which means $t(\alpha_T(R)) = 0$.

b) Case $\alpha_T(R) < 1$.

   Necessity

If $\alpha_T(R) < 1$, then, in order to calculate $\alpha_T(R)$ we can restrict ourselves to the subset $A \subseteq X^3$ of elements $(x, y, z) \in X^3$ for which $\overrightarrow{T}(T(R(x, y), R(y, z))|R(x, z)) < 1$, or, equivalently, $T(R(x, y), R(y, z)) > R(x, z)$.

$$\begin{aligned}
\alpha_T(R) &= \inf_{(x,y,z)\in A} \overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z)) \\
&= \inf_{(x,y,z)\in A} t^{-1}(t(R(x,z)) - t(t^{-1}(t(R(x,y)) + t(R(y,z))))) \\
&= \inf_{(x,y,z)\in A} t^{-1}(m(x,z) - m(x,y) - m(y,z))
\end{aligned}$$

and

$$\begin{aligned}
t(\alpha_T(R)) &= \sup_{(x,y,z)\in A} \{m(x,z) - m(x,y) - m(y,z)\} \\
&= \sup_{x,y,z\in X} \{m(x,z) - m(x,y) - m(y,z)\} = \delta.
\end{aligned}$$

Sufficiency

$\delta = t(\alpha_T(R)) > 0$ and in order to evaluate δ we can restrict ourselves to the subset $B \subseteq X^3$ of elements $(x, y, z) \in X^3$ for which $m(x, z) - m(x, y) - m(y, z) \geq 0$.

From here,

$$\begin{aligned}
\alpha_T(R) &= \inf_{(x,y,z)\in X} \overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z)) \\
&= \inf_{(x,y,z)\in B} \overrightarrow{T}(T(R(x,y), R(y,z))|R(x,z)) \\
&= \inf_{(x,y,z)\in B} t^{-1}(\{t(R(x,z)) - t(R(x,y)) - t(R(y,z))\}) \\
&= \inf_{(x,y,z)\in B} t^{-1}(\{m(x,z) - m(x,y) - m(y,z)\}) \\
&= t^{-1}(\sup_{(x,y,z)\in B} \{m(x,z) - m(x,y) - m(y,z)\}).
\end{aligned}$$

□

Proposition 5.7. *Let R be a fuzzy relation on a set X, T a continuous Archimedean t-norm, t an additive generator of T and $\alpha_T(R)$ the degree of transitivity of R. Let $m_0 = \inf_{(x,y)\in X\times X} t \circ R(x, y)$, $\varepsilon : [0, \infty] \to [0, \infty]$ a mapping, $\varepsilon_0 = \inf_{a\geq m_0} \varepsilon(a)$ and $\varepsilon_1 = \sup_{a\geq m_0} \varepsilon(a)$. If ε satisfies the condition $2 \cdot \varepsilon_0 - \varepsilon_1 \geq t(\alpha_T(R))$ (where i is the identity map) then $\sigma = t^{[-1]} \circ (i + \varepsilon) \circ t$ (where i is the identity map) satisfies*

1. $\sigma \circ R$ is T-transitive

2. $\sigma \circ R \leq R$.

Proof.

1. By the previous lemma, $t \circ R$ is $t(\alpha_T(R))$-triangular and, due to Proposition 5.3, $(i+\varepsilon) \circ t \circ R$ is a mapping satisfying the triangular inequality. Therefore, $t^{[-1]} \circ (i+\varepsilon) \circ t \circ R$ is a T-transitive fuzzy relation.

2. $(i+\varepsilon) \circ t \circ R \geq t \circ R$ and

$$t^{[-1]} \circ (i+\varepsilon) \circ t \circ R \leq t^{[-1]} \circ t \circ R = R.$$

$\square$

Proposition 5.8. *Let R be a proximity relation on a set X, T a continuous Archimedean t-norm, t an additive generator of T and $\alpha_T(R)$ the degree of transitivity of R. Suppose $t \circ R$ satisfies condition (*). Let $M_0 = \{(x,y) \text{ s.t. } 0 < t \circ R(x,y)\}$, $m_0 = \inf_{(x,y)\in M_0} t \circ R(x,y)$ with $m_0 > 0$, $\varepsilon : [0,\infty] \to [0,\infty]$ a mapping such that $\varepsilon(0) = 0$, $\varepsilon_0 = \inf_{a \geq m_0} \varepsilon(a)$ and $\varepsilon_1 = \sup_{a \geq m_0} \varepsilon(a)$. If ε satisfies the condition $2 \cdot \varepsilon_0 - \varepsilon_1 \geq t(\alpha_T(R)$ then $\sigma = t^{[-1]} \circ (i+\varepsilon) \circ t$ (where i is the identity map) satisfies*

1. $\sigma \circ R$ is a T-indistinguishability operator

2. $\sigma \circ R \leq R$.

Proof. Proceed as in the previous proof but apply Proposition 5.4 instead. $\square$

It seems that $s = (i+\varepsilon) \circ t$ should be an additive generator of a continuous t-norm for which R would be transitive. This is not necessarily the case because s need not to be continuous. If $m_0 \neq 0$, ε is continuous on $[m_0, \infty]$ and we replace ε by $\varepsilon' : [0,\infty] \to [0,\infty]$ defined by

$$\varepsilon'(a) = \begin{cases} \varepsilon(a) & \text{if } a > m_0 \\ \frac{\varepsilon(m_0)}{m_0} \cdot a & \text{if } 0 \leq a \leq m_0, \end{cases}$$

then ε' is continuous and the next result follows.

Proposition 5.9. *With the previous notations, if R is a fuzzy relation on X, $m_0 \neq 0$ and ε is continuous on $[m(0), \infty]$, then $s = (i + \varepsilon') \circ t$ is the additive generator of a t-norm T and R is T-transitive.*

Remarks

- $\varepsilon'_{|[0,m_0]}$ can be replaced by any strictly increasing continuous mapping f with $f(0) = 0$ and $f(m_0) = \varepsilon(m_0)$.
- If, moreover, R is reflexive and symmetric, then R is a T-indistinguishabilty operator.

We would like to find a T-transitive relation for which the degree of similarity with the original relation is maximal. If R is T-transitive then evidently the maximal such similarity degree is 1. Therefore we take the corresponding similarity degree as a degree of the transitivity.

Definition 5.10. *Let T be a nilpotent t-norm, t an additive generator of T, R a fuzzy relation on a set X with degree of transitivity $\alpha_T(R)$, $m_0 = \inf_{(x,y)\in X\times X}(t \circ R)(x, y)$, $\varepsilon : [0, \infty] \to [0, \infty]$ a mapping, $\varepsilon_0 = \inf_{a\geq m_0} \varepsilon(a)$ and $\varepsilon_1 = \sup_{a\geq m_0} \varepsilon(a)$. If ε satisfies $2 \cdot \varepsilon_0 - \varepsilon_1 \geq t(\alpha_T(R))$ and $\sigma = t^{[-1]} \circ (i + \varepsilon) \circ t$, then the degree of transitivity $\alpha_\sigma(R)$ of R with respect to σ is $\alpha_\sigma(R) = E_T(\sigma(R), R)$.*

Example 5.11. *Considering again the fuzzy relation R from Example 4.9 with matrix*

$$\begin{pmatrix} 1 & 0.3 & 0.8 \\ 0.3 & 1 & 0.6 \\ 0.8 & 0.6 & 1 \end{pmatrix}$$

and $\alpha_T(R) = 0.9$, the additive generator $t(x) = 1 - x$ of the Łukasiewicz t-norm and ε defined by

$$\varepsilon(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{2+x}{10} & \text{if } 0 < x \leq 1 \\ 0.3 & \text{if } x > 1 \end{cases} ,$$

$\sigma(R)$ is the transitive relation with matrix

$$\begin{pmatrix} 1 & 0.03 & 0.58 \\ 0.03 & 1 & 0.36 \\ 0.58 & 0.36 & 1 \end{pmatrix}.$$

and the degree of transitivity with respect to σ is $\alpha_\sigma(R) = E_T(\sigma(R), R) = 0.73$.

As it can be guessed from this example, the degree of transitivity of a fuzzy relation R with respect to a mapping σ is always smaller than or equal to $\alpha_T(R)$.

Proposition 5.12. *Let $\sigma : [0,1] \to [0,1]$ be a mapping defined in Definition 5.10 and R a fuzzy relation on a set X with degree of transitivity $\alpha_T(R)$. If $\alpha_\sigma(R)$ is the degree of transitivity of R with respect to σ, then $\alpha_\sigma(R) \leq \alpha_T(R)$.*

Proof. From $2\cdot\varepsilon_0 - \varepsilon_1 \geq t(\alpha_T(R))$, it follows that $\varepsilon_0 \geq t(\alpha_T(R))$ and a fortiori $\varepsilon(x) \geq t(\alpha_T(R))$ for all $x \in [m_0, \infty)$. From this, $\sigma(R) \leq T(R, \alpha_T(R))$ or, equivalently, $E_T(\sigma(R), R) = \inf_{x,y\in X} \overleftrightarrow{T}(\sigma(R(x,y))|R(x,y)) \leq \alpha_T(R)$. □

As a corollary of Proposition 5.7, we will give an alternative proof of Proposition 3.4 for continuous Archimedean t-norms.

Proposition 5.13. *Let T be a continuous Archimedean t-norm. R a fuzzy relation on a set X and $c \leq \alpha_T(R)$. Then $R_{(T,c)}$ is a T-transitive fuzzy relation.*

Proof. With the previous notation, take as ε the mapping $\varepsilon(x) = t(c)$ for $x > 0$ and $\varepsilon(0) = 0$. ε fulfills the conditions of the last proposition and

$$(t^{[-1]} \circ (i + \varepsilon) \circ t)(R(x,y)) = t^{[-1]}(t(R(x,y)) + t(c)) = R_{(T,c)}(x,y).$$

□

Finally, it T is a continuous strict Archimedean t-norm, the only mapping σ providing a T-transitive fuzzy relation $\sigma(R)$ to every fuzzy relation R on a given set X is the constant mapping $\sigma(x) = 0$ for all $x \in [0,1]$ which means that the obtained T-transitive relation from every fuzzy relation R is the smallest fuzzy relation assigning the value 0 to any pair $(x,y) \in X^2$. If T is nilpotent then there are more mappings σ satisfying the former condition.

Proposition 5.14. *Let X be a set and T a continuous strict Archimedean t-norm. Then the only mapping $\sigma : [0,1] \to [0,1]$ satisfying for any fuzzy relation R on X*

1. $\sigma \circ R$ is T-transitive

2. $\sigma \circ R \leq R$

is the constant mapping $\sigma(x) = 0$ for all $x \in [0, 1]$.

Proof. Given $a \in [0, 1]$, consider a fuzzy relation R such that $R(x, y) = R(y, z) = a$ and $R(x, z) = 0$ for some $x, y, z \in X$.

Since $\sigma \circ R$ is T-transitive and $\sigma \circ R \leq R$ we have $T(\sigma(a), \sigma(a)) = T(\sigma \circ R(x, y), \sigma \circ R(y, z)) \leq \sigma \circ R(x, z) \leq R(x, z) = 0$. Therefore $\sigma(a) = 0$ because T is strict.

□

Proposition 5.15. *Let X be a set and T a nilpotent t-norm, and $\alpha = \sup\{a \in [0, 1]$ s.t. $T(a, a) = 0\}$. Then the mapping $\sigma : [0, 1] \to [0, 1]$ defined by*

$$\sigma(a) = \begin{cases} \alpha & \text{if } a > \alpha \\ a & \text{if } a \leq \alpha \end{cases}$$

is the biggest mapping satisfying for any fuzzy relation R on X

1. *$\sigma \circ R$ is T-transitive*

2. *$\sigma \circ R \leq R$.*

Proof. $\sigma(a) \leq a$ for all $a \in [0, 1]$. Therefore, $\sigma \circ R \leq R$.

As for the transitivity, $T(\sigma \circ R(x, y), \sigma \circ R(y, z)) \leq T(\alpha, \alpha) = 0 \leq \sigma \circ R(x, z)$.

Let σ' be a mapping satisfying 1 and 2, and $b \in [0, 1]$.We need to proof that $\sigma'(b) \leq \sigma(b)$. $\sigma' \circ R \leq R$ for all R implies that $\sigma' \leq i$ (the identity map).

If $b \leq \alpha$, then $\sigma'(b) \leq i(b) = b = \sigma(b)$. If $b > \alpha$ consider a fuzzy relation R such that $R(x, y) = R(y, z) = b$ and $R(x, z) = 0$ for some $x, y, z \in X$. Then $\sigma'(b) \leq \alpha$ follows from: $T(\sigma'(b), \sigma'(b)) = T(\sigma' \circ R(x, y), \sigma' \circ R(y, z)) \leq \sigma' \circ R(x, z) \leq i \circ R(x, z) = 0$.

□

The defined degrees $\alpha_\sigma(R)$ of a fuzzy relation R depend on σ. As we would like to find a T-transitive relation for which the degree of similarity with the original relation is maximal we can consider the supremum of all degrees $\alpha_\sigma(R)$ as its degree of transitivity from below.

Definition 5.16. *Let T be a nilpotent t-norm, t an additive generator of T and R a fuzzy relation on a set X with degree of transitivity $\alpha_T(R)$.Consider* *the family F of all mappings σ satisfying the conditions of Definition 5.10. Then the degree of transitivity from below $\alpha_b(R)$ of R is $\alpha_b(R) = \sup\{\alpha_\sigma(R) \mid \sigma \in F\}$.*

As an immediate consequence of Propostion 5.12, we obtain that for every fuzzy relation R, $\alpha_b(R) \leq \alpha_T(R)$.

6 Concluding Remarks

The degree of transitivity of a fuzzy relation R has been revisited. From the original t-norm T a family $(T_c)_{c\in[0,1]}$ and a family of fuzzy relations $(R_{(T,c)})_{c\in[0,1]}$ have been generated in such a way that the degree of T-transitivity $\alpha_T(R)$ of R coincides with the greatest $c \in [0,1]$ for which R is T_c-transitive and for which $R_{(T,c)}$ is T-transitive.

Inspired by these relationships, different degrees of transitivity related to new families of t-norms have been provided in Section 4 following two directions.

In the first one, from a continuous Archimedean t-norm with additive generator t, the family $(T_\lambda)_{\lambda>0}$ of continuous Archimedean t-norms with additive generators t^λ is considered and the degree of transitivity of a fuzzy relation R with respect to $(T_\lambda)_{\lambda>0}$ is defined as the supremum of λ such that R is T_λ-transitive.

In the second one, a new strict t-norm T^* is involved in the generation of a new family $({}_\lambda T)_{\lambda>0}$ and the degree of transitivity of a fuzzy relation with respect to this family is analyzed.

In Section 5 the mappings σ transforming a fuzzy relation R into a T-transitive fuzzy relation $\sigma(R)$ smaller than or equal to R have been studied and characterized.

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