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## The six books of Diophantus' Arithmetic increased and reduced to specious. Lost manuscript of Jacques Ozanam (1640-1718)

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| Abstract: | The introduction of a new analytical method, thanks fundamentally to François Viète (1540-1603) and René Descartes (1596-1650) and the later dissemination of their works, resulted in a profound change in the way of thinking and doing mathematics. This change, known as process of algebrization, occurred during the seventeenth and early eighteenth centuries and allowed a great transformation in mathematics. Among many other consequences, this process gave rise to the treatment of the results with the new analytical method in the classic treatises, which allowed new visions of such treatises and the obtention of new results. Among those treatises is the Arithmetic of Diophantus of Alexandria (approx. 200-284) which was written, using the new algebraic language, by the French mathematician Jacques Ozanam (1640-1718), who in addition to profusely increasing the original problems of Diophant, solved them in a general way, thus obtaining multiple geometric consequences. The work is handwritten, it has never been published, it has been lost for almost 300 years and the known references show its importance. We will show that Ozanam's manuscript was quoted as an important work on several occasions by others mathematicians of the time, among which stands out G. W. Leibniz (1646-1716). Once the manuscript has been located, in this article, our aim is to show and analyze this work of Ozanam, its content, its notation and its structure and how, through the new algebraic method, he not only solved and expanded the questions proposed by Diofante, but also introduced a connection between the algebraic solutions and what he called geometric determinations by obtaining loci from the solutions. |
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# *The six books of Diophantus' Arithmetic increased and reduced to specious.* Lost manuscript of Jacques Ozanam (1640-1718)

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**Abstract** The introduction of a new analytical method, thanks fundamentally to François Viète (1540-1603) and René Descartes (1596-1650) and the later dissemination of their works, resulted in a profound change in the way of thinking and doing mathematics. This change, known as process of algebrization, occurred during the seventeenth and early eighteenth centuries and allowed a great transformation in mathematics. Among many other consequences, this process gave rise to the treatment of the results with the new analytical method in the classic treatises, which allowed new visions of such treatises and the obtention of new results. Among those treatises is the Arithmetic of Diophantus of Alexandria (approx. 200-284) which was written, using the new algebraic language, by the French mathematician Jacques Ozanam (1640-1718), who in addition to profusely increasing the original problems of Diophant, solved them in a general way, thus obtaining multiple geometric consequences. The work is handwritten, it has never been published, it has been lost for almost 300 years and the known references show its importance. We will show that Ozanam's manuscript was quoted as an important work on

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several occasions by others mathematicians of the time, among which stands out G. W. Leibniz (1646-1716). Once the manuscript has been located, in this article, our aim is to show and analyze this work of Ozanam, its content, its notation and its structure and how, through the new algebraic method, he not only solved and expanded the questions proposed by Diofante, but also introduced a connection between the algebraic solutions and what he called geometric determinations by obtaining loci from the solutions.

**Keywords** Jacques Ozanam · Arithmetic · 17th century · Symbolic language · Algebrization · Diophantus · Loci.

**Mathematics Subject Classification (2000)** 01A45 · 01A99

## 1 Introduction

Throughout the seventeenth century, new algebraic methods were introduced, bringing a new way of thinking and doing mathematics, and facilitating two fundamental advances: the appearance of analytical geometry and, a few decades later, the birth of the infinitesimal calculus. An evolution from an eminently geometric thought to a more algebraic form of mathematical thought took place as shown by Mahoney[1].

This paradigm shift starts in 1591 with the publication of *In Artem Analyticen Isagoge* [55], written by the French mathematician François Viète (1540-1603), who established connections between the new algebra he was starting and the resolution of geometric problems. Viète introduced in the resolution of problems the use of symbols, letters, to represent both the unknown and the known magnitudes, which allowed him to search for solutions to equations in a general way.

Viète's new algebraic procedures were progressively adopted to solve mathematical problems in general and geometric problems in particular, as happened in the outstanding cases of René Descartes (1596-1650) and Pierre de Fermat (1601-1665). Descartes published in 1637 *La Géométrie* as an appendix to his *Discours de la méthode* [14], which became the most influential work in the evolution of the relationship between geometry and the new algebra. In the years following the publication of *La Géométrie*, this new algebraic way of thinking was imposed, allowing the rapid evolution of mathematics throughout the 17th century[2].

[1] Mahoney [31] has studied the impact, not free of tensions, of this new way of algebraic thought: "Close examination of the works of leading mathematicians of the seventeenth century often reveals a certain tension between two modes of mathematical thought: an old, traditional, geometric mode and a new, in many ways revolutionary, algebraic mode", and he gives it a great importance in the process of change of mathematics during the seventeenth century: "In the light of the brilliant mathematical achievements of the later seventeenth century, in particular the infinitesimal calculus, there is a risk of overlooking the most important and basic achievement of mathematics at the time, to wit, the transition from the geometric mode of thought to the algebraic".

[2] For further information about this process of algebrization see, among others, Bos [7], Mancosu [32] or Massa Esteve [33], [34].

After the diffusion of the works of Viète and Descartes, many mathematicians were adopting the new algebraic method. Among them was the French mathematician Jacques Ozanam (1640-1718) who was born in the 17th century, which means that he was trained, studied and taught in years when mathematics were changing profoundly. Ozanam was a prolific author of mathematical texts in which questions of both pure and applied mathematics were addressed. He devoted efforts to the development of texts for the teaching of mathematics; among others stand out *Cours de mathématique* (1693) [41], work that had two reissues and was translated into English, or the *Dictionaire mathématique* (1641) [44] which was also reissued and translated into English. Ozanam, perhaps due to financial necessity, spent much of his time teaching mathematics, hence the publication of some of his books devoted to the teaching of mathematics. However, he also wrote several treatises that were not specifically intended for teaching but for deepening studies of mathematics and applications to other branches such as physics or engineering. Thus, he published trigonometric tables, tables of logarithms, or treatises on instruments and mathematics applied to mechanics[3].

Ozanam had an extraordinary love of mathematics as his works show, particularly, *Récréations Mathématiques et Physiques* (1694) [45][4], which was republished at least ten times until 1844[5]. This is, perhaps, Ozanam's best known work and probably the one that produced the most benefits for his needy economy[6]. It is a collection of recreational problems that were intended to illustrate and facilitate a liking for mathematics.

Another of Ozanam's outstanding works, and yet only known by references from other mathematicians, was *Les six livres de l'arithmétique de Diophante d'Alexandrie augmentez & reduits à la specieuse*,[7] written in French. In Figure 1 the original title of the manuscript is shown and you can perceive its careful calligraphy. This work was never published and was only known by references to its handwritten text. Its whereabouts have been unknown since the late

[3] C. Càndito [9] collects the reference to 25 works by Jacques Ozanam and reviews the content of his main published works. She also offers a possible explanation of Ozanam's mathematical motivations and interests.

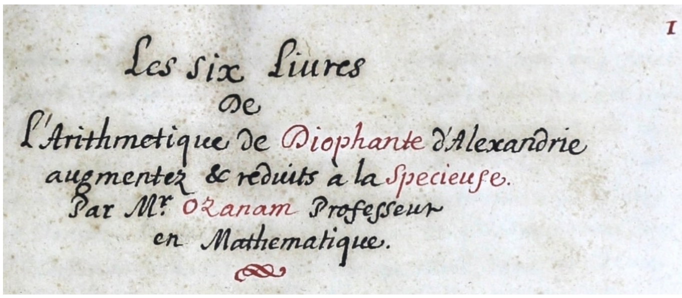
[4] In the preface, Ozanam writes: "[...] to defend pleasures which are engaging by their usefulness, & which are so common, so easy, & so appropriate to all those who have reason, that they cannot be taken away from men without depriving them of what is more pleasant to the life". Our translation of "[...] défendre des plaisirs qui sont engageans par leur utilité, & qui sont si communs, si faciles, & si propres à tous ceux qui ont de la raison, qu'on ne peut pas les ôter aux hommes, sans les priver de ce qu'il y a de plus agréable dans la vie."

[5] Also translated into English in 1814 [25] by Charles Hutton (1737-1823).

[6] C. Càndito [9] explains how Ozanam, despite being born into a wealthy family, did not obtain any part of the family inheritance that, at that time, passed in its entirety to the eldest of the children. Ozanam was the second and his father guided him through ecclesiastical studies that he, after the death of his father, left without finishing, to dedicate himself to the study of mathematics. Despite abandoning his ecclesiastical career, Ozanam's mathematical training was greatly influenced by Jesuits such as C.F.M. Deschales (1621-1678) or J. de Billy (1602-1679), as we will see later. For more details on Ozanam's biography, see [50] and B. Fontenelle [18].

[7] *The Six Books of Diophantus of Alexandria's arithmetic increased and reduced to the specious [new algebra].*

18th century, when the last reference to Ozanam's handwritten text appears in the sales catalog [2] of the Chancelier D'Aguesseau[8] library, until a few years ago.



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Les six Liures
de
l'Arithmetique de Diophante d'Alexandrie
augmentez & reduits a la Specieuse.
Par Mr. Ozanam Professeur
en Mathematique.

**Fig. 1** Image of the title of the manuscript p. 1 of [42].

We have been able to access this work by Ozanam thanks to an article [10] of the French professor Jean Cassinet[9] (1928-1999), who found the manuscript. He pointed out that he had been lucky to find the manuscript which, he claimed, matched the one in the Chancelier D'Aguesseau library sales catalog. Statement which we agree with. The details given on pages 165 and 166 of the sales catalog of the library, as we will show later in the Section 8.3, coincide with the description of the manuscript made by Cassinet, which is the one we have studied. He did not explain what his "stroke" of luck consisted of and only said that he found the manuscript in the Library of the Academy of Sciences in Turin; that he had access to it thanks to Professor Tulio Viola and that he obtained permission from Professor Romano, a member of the Academy, to publish all or part of the manuscript. Furthermore, Cassinet stated that the manuscript was to be studied by a group of science historians from Toulouse, but we are not aware of any other publication apart from the aforementioned article [10].

Cassinet, with some inaccuracies, wrote about the references to this work of which he was aware. He made a schematic index of the work and stopped to describe in more detail only the first 45 pages of the second volume of the manuscript, which appear with the title *Traité des simples, des doubles et des triples égalitèz*. This part, however, had been published by Ozanam in 1702 in chapter II of his *Nouveaux éléments d'Algèbre* [43], but Cassinet did not dive into the fundamental part of the text: the 1085 pages that occupy "The six books of the Arithmetic of Diophantus" which is precisely the one that had received numerous praises, as we will see later.

[8] Henri François D'Aguesseau (1668-1751) was Chancelier of France from 1717 to 1722, high position appointed directly by the king whose task was the administration of justice in France.

[9] Jean Cassinet was a French mathematician and historian of mathematics, professor at the University of Toulouse.

Having access to Cassinet's work, and after verifying that nothing else had been published about the manuscript, we contacted the Turin Academy of Sciences, which made it easier for us to pay for the digitization of Ozanam's text. We have such a digitization and we have studied the manuscript, especially in the part that had made it famous: the "Diophantus". The present work is the result of this study.

The aims of this article are to study and analyze the contents of Ozanam's manuscript; how he solved the questions originally posed by Diophantus of Alexandria using the new algebra and how the power of this new method, handled with elegance and expertise, enabled him to obtain general results and expand Diophantus' questions with numerous conclusions, applications and geometric determinations. To this end, we will analyze, by way of examples, his solutions and generalizations of some of these Diophantus' questions, showing the novelties with respect to what Viète had written about some of the Diophantus questions and with respect to the Diophantus of Claude Gaspar Bachet de Méziriac (1584-1638), the most important reference at that time. We will show what Ozanam himself claimed to pursue with his work as well as with his notation. We will make a chronological list of the references found about the Ozanam's manuscript on the Arithmetic of Diophantus and we will explain the importance that some notable mathematicians of the time gave to this unpublished work by Ozanam, which continued to be referred by several authors even about years after his death, as we will see in detail in Section 8.

## 2 The Arithmetic of Diophantus until the 17th century

Before continuing with the content and characteristics of Ozanam's manuscript, we will present a brief overview of what was known and what had been published about Diophantus' Arithmetic at the time Ozanam wrote his work. This will allow us to compare what Ozanam did with what was being done at that time.

The Arithmetic of Diophantus (or the Arithmetics as Rashed and Houzel prefer to call it [49]) was a mathematical treatise written by Diophantus of Alexandria on the third century of our era[10]. The Arithmetic consisted of 13 books, six of which have arrived so far in some Greek versions and in some Arabic translations, the latter made about the 9th century. In 1971 four more books were found, translated into Arabic also about the first millennium. Three of them basically coincided with books 4, 5 and 6 already known; and the last would be a seventh book of which there is no known Greek version[11].

Arab mathematicians already knew the Arithmetic of Diophantus about the ninth century and had translations from Greek to Arabic. However, in

[10] Although very little is known about Diophantus and neither the period of time in which he lived nor when he wrote the Arithmetic is known exactly, there is agreement among historians in placing him around the third century A.D. See Meskens [35], Heath [24] or Rashed [48].

[11] R. Rashed and C. Houzel do in [49] an interesting investigation and reflection on the different known versions in classical Greek and Arabic of the Arithmetic of Diophantus.

Western Europe the work of Diophantus was not known until the 15th century, except for the works of Leonardo of Pisa (1170-1250), better known as Fibonacci, in which arithmetic problems of the Diophantine type appear, although it is not clear that Fibonacci knew, at least in depth, the work of Diophantus[12]. From this century on, a growing interest in the Greek classics, particularly in mathematics, began spreading firstly around Italy, and throughout the rest of Europe especially from the impulse that the Council of Florence supposed in the 15th century with the attempt to reunify the Latin and Byzantine churches.

The first news of the Arithmetic of Diophantus in the West came from the hand of the German Johann Müller (1436-1476) known by the nickname of Regiomontanus who, on the occasion of some lectures, visited Padua (Italy) in 1464. In the introduction of one of them, having the title "Oratio introductoria in omnes scientias mathematicas"[Introductory discourse in all mathematical sciences], he referred to the need to look back at the Greek classics and specifically at the Arithmetic of Diophantus, of which he had found in Venice, short time ago, some Greek manuscripts and that nobody had translated from Greek to Latin. He said that Diophantus' work contained the essence of arithmetic, the "ars rei census", which the Arabs call algebra. Regiomontanus realized that of the thirteen books named in the preface, he had only found the first six and showed his aim to undertake the translation, but Regiomontanus did not translate any of the books[13].

The Italian Rafael Bombelli (1526-1572) was the first to introduce the Arithmetic of Diophantus in the West. Bombelli published in 1572, in his treatise *L'Algebra* [5], an Italian translation of a good part of the problems in Diophantus's work. Bombelli had made a first manuscript of his Algebra and, before publishing it, he had access through Antonio Maria Pazzi (? -1585), professor of mathematics in Rome, to the manuscripts on Diophantus's work located at the Vatican Library. Both of them were working on the translation of the Diophantine manuscripts. Bombelli's contact with Diophantus' work caused him to change much of the content of Book III of his Algebra before publishing it, introducing a large part of Diophantus' problems, specifically 143, and also incorporating Diophantus' method[14].

Three years later, the German Wilhelm Holtzman (1532-1576), better known for the latinization of his name Guilielmus Xylander, translated the Diophantus into Latin in 1575. He added some comments and introduced some symbology but, unlike Bombelli, in certain parts of the original text that were not understandable, he limited himself to making a literal translation that still

[12] Meskens states in [35] that Fibonacci received his mathematical training in the Arab world where he traveled until 1200. He adds that it is not clear that he knew the Diophantus' work since the problems of a Diophantine type that he collects do not correspond to the treatment of Diophantus of Alexandria.

[13] Meskens offers in [35] more details about Regiomontanus and the Diophantus.

[14] For more information on Bombelli's work and Diophantine problems, see Agostini [1], Bortolotti [6], Jayawardene [26], [27], Ver Eecke [54] or Wagner [57].

did not clarify the mathematical content[15]. In 1577 the French mathematician Guillaume Gosselin (? -1590) published his work "De arte magna seu de occulta parte numerorum quae et Algebra et Almucabala vulgo dicitur"[22] where, from Xylander's translation, includes Diophantus' problems.[16]

The Belgian Simon Stevin (1548-1620) published in 1585 a French translation of the first four books of Diophantus as an appendix to his work "L'Arithmétique" [52]. Stevin relied on Xylander's translation. The last two missing books were incorporated by Albert Girard (1595-1632) in a new edition of Stevin's work in 1625[17].

The first substantial change in the treatment of Diophantus took place when Viète published in 1593 "Zeteticorum libri quinque" [56] which, although it was not a translation or a compilation of Diophantus' Arithmetic, addressed numerous Diophantine problems. Viète's approach was a novelty, he used the methods that he had incorporated with his new algebra and he wanted to solve in a general way the problems posed by Diophantus that, despite its original general formulation, in its resolution concrete numerical cases were taken. Furthermore, Viète took a geometric approach to problems, frequently using right triangles. However, Viète's algebraic language was logically very primitive, but it helped him to adopt a general and abstract approach. Later, in Section 7, we will see an example of how Viète solved one of Diophantus' questions, in order to be able to compare it with Ozanam's treatment.

Finally, the Diophantus published in 1621 [3] by the jesuit Bachet in a bilingual edition in Greek-Latin, was considered from the beginning a great work. It became the reference book for anyone who, at that time, wanted to study the Arithmetic of Diophantus and, of course, it was for Ozanam as well. Bachet's text is written in two columns, on the right is the original Greek version and on the left is the Latin translation. Bachet's treatment was very rigorous and also innovative and not limited to a mere translation of the Greek text; he made a reflection on the problems of Diophantus and obtained general solutions as Viète had already done but, in addition, he appended numerous comments and new problems from those of Diophantus himself[18], what justifies and explains the success of his Diophantus.

However, Bachet did not incorporate the new algebraic language that was progressively imposing in mathematics, but rather retained a rhetorical and cosist language[19]. Obviously, Viète went further in this regard, incorporating his new algebraic language.

[15] See Meskens [35].

[16] For more information on Gosselin's work see Kouteynikoff [28].

[17] On the evolution and knowledge of the Arithmetic of Diophantus of Alexandria, and in particular between the 16th and 17th centuries, detailed information can be found in the works of Paul Tannery (1843-1904) [53], Thomas L. Heath (1861-1940) [24], Paul Ver Eecke (1867-1959) [54] or more recently Freguglia [19], Rashed [48], [49], Christianidis [11], Christianidis and Oaks [12] and Meskens [35].

[18] Meskens in [35] comments on the influence of the text of Bachet in his time and later, as well as the characteristics of its content.

[19] Meskens states in [35] that "[...] Bachet remained faithful to the classical style, but evidently went much further in content, so that we can safely say he was a research mathe-

In this context and with these references, Ozanam wrote his Diophantus. Especially with the reference to Bachet's text, as Ozanam himself quotes in his work. Ozanam, as we will show, would go further in generalization, in the use of the new algebraic language, in numerous extensions with new questions and geometric determinations.

## 3 The lost manuscript

Ozanam's manuscript on the six books of Diophantus' Arithmetic was quoted and referenced by some mathematicians and historians of mathematics, and even by Ozanam himself in some of his published works. In all cases, the quality of the work stands out, thus the innovativeness of the mathematical treatment of the classic content of the Arithmetic of Diophantus, and the original contributions that enriched, expanded and generalized the Diophantine text. However, as we have already said, this work was never published.

In this section, we are going to make a chronological tour through the different texts for which we know that there exist references to this work by Ozanam. Cassinet in [10] made a list of the references that he had found on the manuscript and that we collect and specify here. We add the references that Ozanam himself made to "his Diophantus" in some of his works; references that were not cited by Cassinet; another letter from Leibniz and another mention of Montucla.

### 3.1 Leibniz

Chronologically, the first written reference to Ozanam's work on Diophantus is found in a letter that Gottfried Wilhelm Leibniz (1646-1716)[20] addressed to Henry Oldenburg (1619-1677), secretary of the Royal Society of London, dated February 26, 1672 (February 16 in the current Gregorian calendar) in which he says that, in Paris, Ozanam lives, a very talented young man who is applying algebraic methods to solve Diophantus' problems and who is managing to solve problems whose solution was not known:

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matician of great stature. His preconditions for working with numbers made him shy away from Viète's algebra of kinds".

[20] The correspondence between Leibniz and Ozanam on mathematical questions was fluid as can be seen, for example, in [30]. Leibniz's good opinion of Ozanam's work is shown in a note on Ozanam's Algebra, published by Leibniz in "Le Journal des Savans" [29]: "The Algebra of M. Ozanam, which I just received, seems to me much better than most of the ones we have seen in a long time, which do nothing but copy Descartes and his commentators. I am glad that he recovers some of the precepts of Viète, inventor of La Specieuse, which do not deserve to be forgotten"; our translation of "L'Algebre de M. Ozanam, que je viens de recevoir, me paroist bien meilleure que la pluspart de celles qu'on a vûës depuis quelque temps, qui ne font que copier Descartes & ses Commentateurs. Je suis bien aise qu'il fasse revivre une partie des preceptes de Viète, inventeur de la Specieuse, qui meritoient de n'estre point oubliez".

> At Paris there is Mr. Ozanam, a young man highly skilled in algebra, who is to give us something of that sort (the same as was developed by Diophantus), having found a way of solving problems which could not be solved by the Diophantine or any algebraic method known hitherto.[21]

The following reference is found in another letter also from Leibniz to Oldenburg dated October 16, 1674 (October 6 in the current Gregorian calendar). In this letter Leibniz already spoke of the manuscript explicitly and, among other things, he told Oldenburg that he had been able to see a manuscript by Ozanam on the Arithmetic of Diophantus. Leibniz gave a brief idea that Ozanam had written the manuscript in the symbolic language of the new algebra and that Father De Billy[22] (1602-1679) praised Ozanam's work. Leibniz affirmed that Ozanam had added numerous questions that had not been collected neither by Diophantus nor by Bachet; and given the novel, broad and rigorous character of Bachet's Diophantus, this statement is a clear example of the quality that Leibniz observed in Ozanam's work. Although Leibniz claimed that the text would soon be published, it was never edited:

> Jacques Ozanam, about whom I at some time spoke to you, and whom Père Billy has mentioned with praise in his writings, showed me his work on Diophantus, soon to be committed to the press, reduced into symbolism. He has added throughout questions omitted by Diophantus and Bachet and he has added a seventh book filled with supplementary questions[...][23]

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[21] Translation of Hall and Hall in [23] from the original Latin text: "Parisiis est dominus Osannam juvenis in Algebra versatissimus, qui nobis aliquid in eo genere Idem Diophantum promotum dabit, reperta ratione solvendi problemata, quae neque ex Diophanto, neque ex cognita hactenus Algebra poterant solvi".

[22] Jacques De Billy, a French Jesuit and mathematician, considered an expert in the work of Diophantus, had Ozanam as a disciple and was related to Bachet. He also maintained a frequent correspondence with Fermat. Part of the comments and mathematical observations of De Billy to Fermat, were collected under the title of "Doctrinae analyticae inventum novum" in the edition of the Arithmetic of Diophantus published by Fermat's son. Jean Bertet (1622-1692), Jesuit and mathematician, stated in a letter to Wallis, dated 1671 [4] p.534, that Ozanam was a disciple of De Billy, although the editors of [4] say that this statement is not verified. Also in "Mémoires pour l'histoire des sciences & des beaux-arts" [20], p. 372, in 1721, it is stated that Ozanam was a disciple of De Billy. However, in "Mémoires pour servir à l'histoire des hommes illustres" [37] it is stated that he was not a disciple of De Billy, but of Claude F. M. Deschales (1621-1678), also French Jesuit and mathematician. This hypothesis is consistent with the fact that Ozanam made a in 1720 new edition "revûë, corrigée & augmentée" [revised, corrected and augmented] [15] of Les Elements D'Euclide of Deschales which had been published in 1672 [16]. Ozanam also quotes Deschales on p. 2 of the introduction to his Elements of Euclid that are part of volume I of his Cours de mathématique [41].
In any case, the relationship between De Billy and Ozanam was fluid, as shown by the previous references or by the excerpts of letters between the two of them, that were collected by Tannery and Henry in the "Oeuvres de Fermat"[17] pp. 138-140.

[23] Translation of Hall and Hall in [23] of the original Latin text collected by Wallis in [58]: "Jacobus Osanna de quo TIBI aliquando locutus sum, et cuius P. Billy in scriptis suis cum elogio meminit, monstravit mihi super Diophantum suum mox praelo committendum; ad

Next, Leibniz told Oldenburg how Ozanam had proposed a problem that he considered minor and gave him the reasons why he had agreed to solve it[24]. However, he returned to the manuscript of the Arithmetic of Diophantus written with the new symbolic algebra, which he said was what he was interested in telling him. In his reflection, Leibniz stressed that the algebraic language allowed Ozanam to generalize the problems originally proposed by Diophanthus of Alexandria, without being subjected to the purely numerical:

> I think that the Diophantus of the same Ozanam will be worth reading; for he gives the work in such a way that he gets rid of all the lemmas sought from the nature of numbers, and shows they can always be found out by his analytical method. These are the matters which I judged worth writing, indeed, because I understand that a symbolic [version of] Diophantus is also to be, or perhaps is, published with you. [in reference to the Royal Society].[25]

This means that the Diophantus was handwritten prior to the date of Leibniz's letter to Oldenburg and that, furthermore, the text must have been fairly refined since Leibniz himself said that it would soon be printed. Therefore, we can deduce that Ozanam wrote his treatise before October 1674.

### 3.2 The "Diophantus" in other works by Ozanam

Ozanam himself made reference in his publications to "his Diophantus", understanding by "Diophantus" his version on the six books of the Arithmetic of Diophantus of Alexandria. Thus, in 1687 he published three treatises in a single volume [38], [39], [40] and in them he referred to his Diophantus in precise terms as we will see below. In view of these references that Ozanam made himself, it could give the impression that its publication was close. This means that as many as 13 years had passed since Leibniz had read that manuscript that he described as interesting and innovative, without the text seeing the light of day. Despite the fact that Leibniz himself, as we have seen, announced to Oldenburg its prompt publication.

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symbola revocatum. Adjicit passim Quaestiones a Diophanto et Bacheto praetermissas; sed et librum septimum addet refertum quaestionibus Paralipomenis". The seventh book with complementary questions referred to by Leibniz is not to be found in the manuscript we have studied. [42].

[24] Cassinet, in the aforementioned paper [10], erroneously states that the author of this letter was John Wallis (1616-1703) and quotes his work *Operum mathematicorum* [58]. This error may be due to the fact that Wallis included a section dedicated to gathering a collection of letters (pp. 617-708) and among them, on pages 617 and 618 was the aforementioned letter from Leibniz to Oldenburg in which Ozanam's Diophantus is mentioned. However, Wallis picked up this letter correctly attributing its authorship to Leibniz.

[25] Translation of Hall and Hall in [23] of the original Latin text collected by Wallis in [58]: "Diophantum ipsius Osannae puto fore lectu dignum; dat enim operam, ut lemmata omnia ex numerorum natura petita, expungat, et ut semper ostendat ipsum inveniendi modum analyticum. Sed haec quidem vel ideo scriptu digna putavi, quia Diophantum symbolicum apud Vos quoque edi editumve esse intelligo".

The three treatises published in a single volume could be viewed separately; in fact, each treatise is numbered independently. The first was the *Treatise on the lines of the first genre* (1687) [38][26] and Ozanam referred to his Diophantus on two occasions. Although this treatise is devoted to conic sections, Ozanam included, from p. 137 until the end (p. 151), a chapter with the title of "Problems" that looks like an appendix in which a collection of problems is gathered. Among them, the one that appears with the number four is a problem that can be called "diophantic" since it deals with relationships between numbers. This problem ask:

> To find four numbers, such that their sum is equal to a given number and the difference between any two of them is a square number[27].

In the solution of the problem, Ozanam wrote about his Diophantus while waiting for its publication:

> I will not stop here to show you how to find the three indefinite preceding numbers, [...] and you will be able to find them in our Diophantus when I have the good fortune that it appears [...][28]

And later, on p. 149 and following the resolution of this same problem IV, he again referred to his Diophantus:

> You see from the different solutions that this problem is easy for one who knows how to find four numbers, such that the difference of any two of them is a square number; what we have shown in various ways in our Diophantus[29].

The following treatise is entitled *Treatise on the Construction of Equations to Solve Indeterminate Problems* (1687) [39][30]. In the introduction, *Au lecteur*, of this treatise, Ozanam said that it was necessary to know how to solve some problems by geometry and added the following, referring to his Diophantus:

> [...]as you will see in the Questions that we add at the beginning of this treatise, some of which have been taken from our Diophantus, as in

[26] *Traité des lignes du premier genre explained par une méthode nouvelle & facile* [38]. It is a treatise on conic sections, their properties, and how to build them.

[27] Our translation of [38], p. 144. "Trouver quatre Nombres, tels que leur somme soit egale a Nombre donné & que la difference de deux quelconques soit un Nombre quarré". This problem appears in the second volume of the manuscript [42], p. 259 as the first problem added to Question IX of Book III, although Ozanam, in the manuscript, does not impose the condition that the sum of the four numbers be given.

[28] Our translation of [38], p. 144. "Je ne m'arresteray pas icy à vous enseigner la maniere de trouver les trois Nombres precedens indefinis,[...] et que vous la pourrez trouver dans nostre Diophante, lorsqu'il aura le bonheur de paroistre[...]".

[29] Our translation of [38], p. 149. "Vous voyez par ces solutions differentes, que ce Probleme est aisé à celuy qui sçait trouver quatre Nombres, tels que la difference de deux quelconques soit un Nombre quarré; ce que nous avons enseigné en plusieurs manieres dans nostre Diophante".

[30] *Traité de la construction des equations pour la solution des problèmes indéterminés* [39].

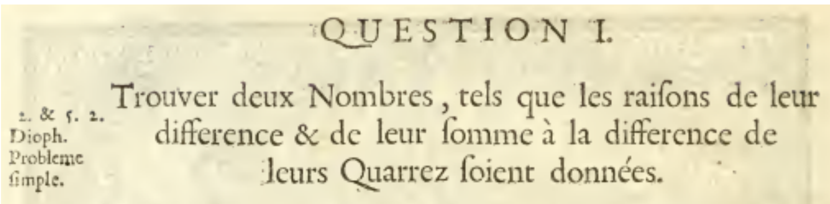
> the preceding treatise, to make you better understand the way to solve arithmetic problems by geometry,[...][31]

Ozanam justified the introduction of certain questions on numbers (Diophantine questions), by the need for knowing them to solve geometric problems and he said again that they were extracted from his Diophantus. The "preceding treatise" to which he referred is the Treatise on the Lines of First Genre [38] that we have quoted before.

Furthermore, in this treatise on the construction of equations, some notes were printed in the margins that made reference to certain texts or certain results. Thus on page 2 of this treatise, Question I was a Diophantine problem that said:

> To find two numbers, such that the ratios of their difference and their sum to the difference of their squares are given[32].

In the margin of this question appeared the reference: "2.& 5.2. Dioph. Probleme simple", see Figure 2.


QUESTION I.

2. & 5. 2. Dioph. Probleme simple.

Trouver deux Nombres, tels que les raiſons de leur difference & de leur ſomme à la difference de leurs Quarrez ſoient données.

**Fig. 2** Question I of the Treatise of the construction of equations, p.2 of [39].

This reference coincided with the numbering of the same question in the manuscript, that is, 2.& 5.2 meant Book Two of the Arithmetic of Diophantus, question 5, sub-question 2. See Figure 3.

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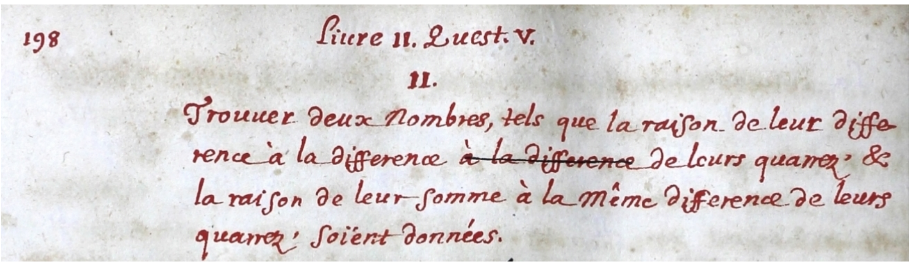
[31] Our translation of the first page, unnumbered, of the introduction *Au lecteur* [39] "[...]comme vous verrez dans les Questions que nous ajoûtons au commencement de ce Traité, dont quelques-unes ont été tirées de nôtre Diophante, comme dans le Traité precedent, pour vous mieux faire comprendre la maniere de resoudre par Géométrie les Problèmes d'Arithmétique,[...]".

[32] Proper translation of [39] p. 2. "Trouver deux Nombres, tels que les raisons de leur difference & de leur somme à la difference de leurs Quarrez soient données".



198 Livre II. Quest. V.

II.

Trouver deux Nombres, tels que la raison de leur difference à la difference ~~à la difference~~ de leurs quarrez & la raison de leur somme à la même difference de leurs quarrez soient données.

**Fig. 3** Question 5 of the second book of the manuscript, p. 198 of [42].

The formulation of this question and the algebraic notation used also coincide: it is a matter of finding two numbers $x$ and $y$ such that

$$\begin{array}{c} lx - ly\ ,\ xx - yy\ ::\ r, s \\ lx + ly\ ,\ xx - yy\ ::\ a, b. \end{array}$$

Here $l$ denoted the unit and was introduced to respect the law of homogeneous and the symbol :: indicated proportion. In our current notation it would be:

$$\frac{x-y}{x^2-y^2} = \frac{r}{s};\quad \frac{x+y}{x^2-y^2} = \frac{a}{b}.$$

In the Treatise on the construction of equations there are three more references like the previous one and, although there is agreement with identical questions collected in the manuscript, in those cases the numbering doesn't match[33].

In the last of the treatises, entitled *Treatise on loci explained by a short and easy method* (1687) [40][34], again Ozanam referred to his Diophantus in similar terms to those of the two previous treatises, this time on three occasions. In two of them, questions II, p. 11 and III, p. 13, the coincidence is exact in writing and numbering with the same questions of the manuscript pages 176 and 169 respectively. In the third, the reference in the Treatise is to Book IV of Diophantus, question IV and there is no coincidence with the manuscript. Finally, at the end of question V, on p. 19 of the Treatise, Ozanam refers again to his Diophantus:

> We have given in our Diophantus several solutions to this Question [...][35]

[33] On p. 12 of the Treatise, question III carries the reference to book II of Diophantus, question II, subquestion II that, in content but not in numbering, coincides with book II, question V, subquestion II on p. 198 of the manuscript. The same is true for two other questions on pp. 14 and 16 of the Treatise that coincide in content and not in numbering with questions on pp. 201 and 210 of the manuscript, respectively.

[34] *Traité des lieux geometriques, expliquez par une methode courte & facile* [40].

[35] Our translation of [39], p. 19. "Nous avons donné dans nôtre Diophante plusieurs solutions de cette Question[...]".

## 4 The content of the manuscript

The manuscript consists of two separately bound volumes. The first one consists of 752 pages, including the covers, of which 746 correspond to the contents of the book; written in exquisite penmanship in two colours, red and black. The parts in red are usually those corresponding to the statements of the questions, the algebraic expressions, the titles and what Ozanam called "canon", which we will comment on later and which was nothing else but an explanation of the needed operations to obtain the solution, as can be seen in Figures 4, 5, 6 and 9. This first volume contains the first two books of the Arithmetic of Diophantus. Book I, which goes from page 7 to page 168, consists of 43 questions with 58 other additions to many of them. It even adds problems to those extensions, for example question XXXIII has 24 extensions numbered and the last one has 7 added problems; moreover, two questions are accompanied by geometrical determinations, namely, geometrical versions of the problem. Book II goes from page 169 to page 745 with 36 questions and 213 extensions. For example, question V is accompanied by 41 added questions, or question VII which incorporates 50 additional questions. Many of them are accompanied by geometrical determinations.

The second volume has 441 pages devoted to Diophantus, including the covers and some unbound folios that appear between some pages and begins with the Treatise on Simple, Double and Triple Equalities which is written in a single color, black, and with a somewhat less careful penmanship than that of the first volume. This treatise occupies the first 45 pages of the second volume and, as we have said, was published separately in 1702 as part of the treatise [43]. This volume continues with Book III of Diophantus, which returns to the careful calligraphy of the first volume and the writing in red and black. After question I of Book III, which occupies 19 pages, there are 16 loose leaves with annotations, with a much more sloppy handwriting, even in some of them there are stains and deletions, which indicates that they are drafts. After two unnumbered blank pages, Book III continues but from question IV with a different writing only in black, similar to that of the treatise on simple, double and triple equalities. As indicated in the sales catalog of the Chancelier D'Aguessau library [2], questions II and III are missing, as we will see later in Section 8.3. Furthermore, the numbering of the pages is different and starts at number 149[36]. Book III concludes on page 221 with question XXII and incorporates 46 additional questions throughout the book. This book does not contain any geometric determination.

From page 221, Book IV continues with a total of 46 questions plus 70 additional ones without geometric determinations, until page 332 where Book V begins, consisting of 33 questions with 11 added questions, again without

[36] Although we cannot be sure, it seems quite clear that it was an earlier version of Diophantus' treatise and it gives the impression that the work was being refined and, perhaps, preparing the final version prior to publication; note, for example, that in the heading of each page, it only says "Book III" without indicating the question that is addressed on that page, while in the two previous books it was indicated.

geometric determinations. On page 385, Book V ends and Book VI begins with 26 questions; all of them begin with the phrase "To find a right triangle such that ...", with the exception of Question XVI. Book VI does incorporate numerous geometric illustrations with 14 additional questions and concludes on page 441.

Regarding the questions contained in the books and the order in which they are written, Ozanam collects all the questions in the same order as those that appear in Bachet's Diophantus, including the added questions, except for questions XXIII and XXIV of Book III, which Ozanam collects as additional questions. As we have said, at that time the main reference for the Diophantus' Arithmetic was the Bachet's version and, as it could not be otherwise, Ozanam knew him in depth and names Bachet at several times, such as on p. 409 of volume II, where he refers to some questions added by Bachet saying that he has collected them and that he improves and develops them.

However, the differences between Bachet and Ozanam are considerable, as Leibniz had already pointed out. Ozanam fully introduced the new algebraic language, which Bachet did not use, and his effective handling of that language allowed him to obtain general results and new questions; furthermore, the large number of geometric determinations that Ozanam incorporates into the thread of many of the questions, especially in Books I and II, underlines the connection between the new algebra and geometry, between algebraic problems and geometric determinations, as we will see throughout this paper.

## 5 The notation in the manuscript

The algebraic notation used by Ozanam in the manuscript, and throughout all his work, follows that introduced by Descartes[37][14] and already had many similarities to the notation used today, although with some differences. Keep in mind that, as we have already shown in Section 1, the new algebraic language began at the beginning of the 17th century and the notation and use of symbols were by no means unified, especially at the beginning. The process of evolution until the use and meaning of symbols became common was relatively short, taking into account the profound change that it entailed in the evolution of mathematics (see, among others, Massa Esteve [34] or Serfati [51]).

Ozanam, on page 5 of the first volume of the manuscript, in the last paragraphs of the preface, established the notation that he would use following the symbols that Descartes had already used and said:

> We have employed the letters $x$, $y$, $z$, $w$ for the unknown quantities, and the other letters without distinction for the known ones, and for the indeterminate ones, except the letter $l$, which will always be used

[37] Although it is not cited explicitly in the manuscript, it seems clear that Ozanam follows Descartes' notation as he had already done in 1693 in his Cours de mathématique [41] p. 41 and 46 of the preface of volume I or p. 5 of the preface to volume III which does quote Descartes. He also quotes Viète in volume I of the Cours de mathématique pp. 15 and 39; on this last page because of the law of homogeneous.

> for unity, when the matter be about comparing together, by addition or by subtraction, two quantities of different types, as occurs in several Diophantine problems. In this case, it is necessary to multiply the lower of these two quantities by the unit as many times as necessary to make it as large as the highest, or to divide the highest by the unit as many times as necessary to make it homogeneous to the lowest. This can always be done without changing the problem, because the unit when multiplying or dividing does not produce any change[38].

Ozanam, as Descartes already did, reserved the letter $l$ to designate the unit that he would use to equalize "the size" of the quantities. This "size" was concerned with the geometric dimension, that is, Viète's law of homogeneous: it was not allowed to establish an equality between quantities of different dimensions. This Ozanam's quotation is very relevant because he justified that with this procedure he can solve all arithmetic problems in geometry. The singular fusion between arithmetic and geometry in solving problems in Ozanam's manuscript is one of his original points. Ozanam himself explained it with these words:

> This is done to preserve the law of homogeneous, that is, in order to not deviate from the rules of geometry, which teaches us that there is no ratio between a line [dimension 1, for example $x$] and a plane [dimension 2, for example $x^2$], or between a plane and a solid [dimension 3, for example $x^3$], because these quantities are heterogeneous, that is, of different kinds; because in this way we can solve any arithmetic problem by geometry, as you will see in the first two books[39].

Ozanam also explained that he would use the word "equation" instead of equality because an equation was the comparison of two different quantities to make them equal, however an "equality" was a comparison between two quantities that are already identical[40].

> We have used the word equation, instead of the word equality, because an equation is the comparison that is made between two unequal quantities to make them equal, and an equality is the comparison of two really

[38] Our translation of [42], p. 5. "Nous avons pris les lettres $x$, $y$, $z$, $w$, pour les quantitez inconnues, & les autres lettres indifferemment pour les connues, & pour les indeterminées, excepté la lettre $l$, qui sera toujours prise pour l'unité, lorsqu'il s'agira de comparer ensemble par addition, ou par soutraction, deux grandeurs de divers genre, comme il arrive dans plusieurs questions de Diophante. Dans ce cas il est necessaire de multiplier la plus basse de ces deux quantitez par l'unité autant de fois qu'il en sera besoin pour la rendre aussy élevée que la plus haute, ou bien de diviser la plus haute par l'unité autant de fois qu'il sera necessaire pour la rendre homogene à la plus basse, ce qui se peut toujours faire sans changer la Question, parce que l'unité en multipliant ou en divisant n'aporte aucun changement".

[39] Our translation of [42], p. 5. "Cela se pratique pour conserver la loy des Homogenes, c'est à dire pour ne point s'eloigner des regles de la Geometrie, qui nous aprend qu'il n'y a aucune raison entre une ligne & un plan, ny entre un plan & un solide, & c. parce que ces grandeurs sont heterogenes, c'est à dire de different genre; car ainsy on peut resoudre tout probleme d'Arithmetique par Geometrie, comme vous verrez dans les deux premiers livres".

[40] For more information on the concept of equation see Cajori [8].

equal quantities. So when by reducing the equation we have made the two unequal quantities equal, this equation changes to equality[41].

Finally, he also offered some symbols that would be used throughout the text:

| | |
|---|---|
| $\sim$ | Denotes equality, the current sign $=$. |
| $\bigoplus$ | Denotes "greater than"; the actual $>$. |
| $\ominus$ | Denotes "less than"; the actual $<$. |
| $\cdots$ | Between two quantities, it is similar to the current absolute value of a difference. |

Moreover, he used without specifying it in the preface, the symbol :: to express the ratio between two quantities, as we have seen before, that is, $x, y :: a, b$ meant that $x$ and $y$ are in the same ratio as $a$ and $b$, that is $\frac{x}{y} = \frac{a}{b}$ in the current notation. He denoted the square powers as $xx$ in the manuscript, but also as $x^2$ in other works, like Descartes.

In Table 1, composed from the one made by Massa Esteve [33], a comparison can be observed between the notation used by Viète, Descartes and Ozanam.

| Signs | Viète (1590s) | Descartes (1637) | Ozanam (1700) |
|---|---|---|---|
| Equality | æqualis | $\propto$ | $\sim$, $\propto$ |
| Greater than | *Maior est* | *Plus grande* | $\bigoplus$ |
| Less than | *Minus est* | *Plus petite* | $\ominus$ |
| Product of $a$ and $b$ | $A$ in $B$ | $ab$ | $ab$ |
| Addition | *plus* | $+$ | $+$ |
| Subtraction | *minus* | $-$ | $-$ |
| Ratio | *ad* | à | :: |
| Square root | *VQ.* | $\sqrt{}$ | $\sqrt{}$ |
| Cubic root | *VC.* | $\sqrt{c}$ | $\sqrt[3]{}$ |
| Squares | A *quadratus*, A *quad* | $a^2$, $aa$ | $a^2$, $aa$ |
| Cubes | A *cubus*, A *cub* | $a^3$ | $a^3$ |
| Absolute value | | | $\cdots$ |

**Table 1** Comparative notation table among Viète, Descartes and Ozanam.

Furthermore Ozanam, also in the preface, explained that in various problems he had given at the end what he calls "the determination". It was about obtaining the possible values for the indeterminate variables (the parameters that are supposed to be known), so that the problem would not have irrational solutions, whenever it was possible, and never negative solutions, and he wrote:

[41] Our translation of [42], p. 5. "Nous nous sommes servy du mot Equation, plutôt que du mot Egalité, parce qu'une Equation c'est la comparaison que l'on fait entre deux grandeurs inégales pour les rendre égales, & qu'une Egalité est la comparaison de deux grandeurs veritablement égales. Ainsy lorsque par la reduction de l'Equation on a rendu égales les deux quantitez inégales, cette Equation se change en Egalité".

> [...]the numbers given in the problem, to make it possible, that is, not to solve it with irrational numbers, when this can happen, or with negative numbers, because in problems of numbers we do not admit negative solutions[...][42]

## 6 Ozanam's purposes and method

The Diophantus manuscript begins with a preface addressed to "Au lecteur" (To the reader) in which Ozanam introduced the work, explained his purpose, the content and, as shown, set out the notation to be used[43].

From the first lines of the preface it is clear that Ozanam was developing an ambitious project. To "reduce" Diophantus' classic treatise to specious, that is, to write it in the new algebraic language, was already, by itself, a very innovative task at that time. However, Ozanam proposed much more: to expand the Diophantine problems by giving them a general character, even with geometric applications and, what was undoubtedly more novel, to extract more questions from each of Diophantus' originals inspired by them, and solve them, of course, generally. For example, Ozanam added an amount of 51 questions to Question V of Book II, ranging from pages 194 to 259 of the manuscript.

Despite the fact that Ozanam's Diophantus was never published, it must have been a project that the author had conceived for a long time, so he stated it at the beginning of the preface, where he also explained the method he was going to apply, which is nothing else but the Viète's analytical method and wrote (see Figure 4):

> At last I offer you, dear reader, what I promised a long time ago, that is, the six books of Diophantus, not only reduced to specious, but even augmented, and solved not only with numbers indefinitely, but also with geometry, substituting continuous quantities instead of the given numbers, and of indeterminate letters, which remain in the indefinite solution of the problem. Various examples are seen in the first two books concerning determinate and indeterminate problems, which will serve as models for solving by imitation all other questions of the same nature. [Our translation; see Figure 4].

Ozanam continued explaining and insisting on his claim to solve all problems in a general way and clarifying that he had almost always used letters instead of numbers, stating that he had only resorted to numbers when they facilitated the calculation or provided a simpler solution. Again he follows the ideas on the new algebra introduced by Viète [55] in 1591.

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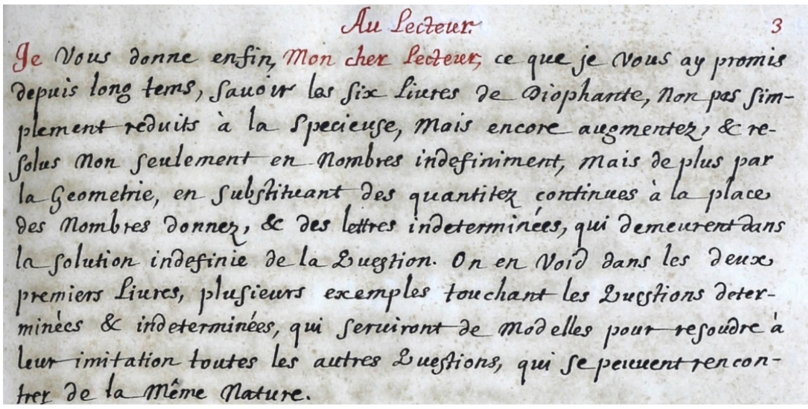
[42] Our translation of [42], p. 4. "[...]aux nombres donnez dans la question, pour la rendre possible, c'est à dire pour ne la pas resoudre en nombres irrationnels, quand cela est possible, n'y en nombres niez, parce que dans les questions de nombres on n'admet point de solutions negatives[...]".

[43] The preface has, like much of the manuscript, exquisite calligraphy and goes between pages 3 and 5.


Au Lecteur. 3

Je Vous donne enfin, Mon cher Lecteur, ce que je Vous ay promis depuis long tems, sçauoir les six Liures de Diophante, non pas simplement reduits à la Specieuse, mais encore augmentez, & resolus non seulement en Nombres indefiniment, mais de plus par la Geometrie, en substituant des quantitez continues à la place des Nombres donnez, & des lettres indeterminées, qui demeurent dans la solution indefinie de la Question. On en Void dans les deux premiers Liures, plusieurs exemples touchant les Questions determinées & indeterminées, qui seruiront de modelles pour resoudre à leur imitation toutes les autres Questions, qui se peuuent rencontrer de la même Nature.

**Fig. 4** Image of the beginning of the preface, p. 3 of [42].

> [...]but I have almost always done the initial steps according to the nature of the problem, to have an easier analysis and a more general solution.
> I have almost always used letters instead of numbers to offer the solution as general as possible, and to ignore any particular case: and if I have ever used numbers, it has been to have an easier calculation and a simpler solution [44].

This, he said, had forced himself to introduce a "Treatise on Simple, Double, and Triple Equalities", the results of which he would use to solve Diophantine problems. In the article [10] pp. 72-91, which we have already commented on, Cassinet made a complete description of the Treatise on Simple, Double and Triple Equalities contained in the manuscript. On the other hand, this Treatise was published by Ozanam in the second volume of the *New elements of algebra or general principles for solving all kinds of mathematical problems*[45] and covers Chapter II, whose title was *Simple, double and triple equalities: for the solution of indeterminate problems in rational numbers*[46].

In the preface, Ozanam stated how he had structured the resolution of each of the questions in the manuscript. In the first place, obviously, he formulated the question to be solved, written in red, and then gave names to the unknowns

[44] Our translation of [42], p. 3. "[...]mais j'ay presque toujours fait au commencement des positions conformes à la nature du Probleme, pour avoir une analyse plus aisée, & une solution plus generale.
J'ay mis presque par tout des lettres à la place des nombres, pour rendre la solution autant generale qu'il a été possible, & pour ne point faire de cas particulier: & si je me suis servy quelquefois des nombres, ça été pour avoir un calcul plus aisé, & une solution plus simple".

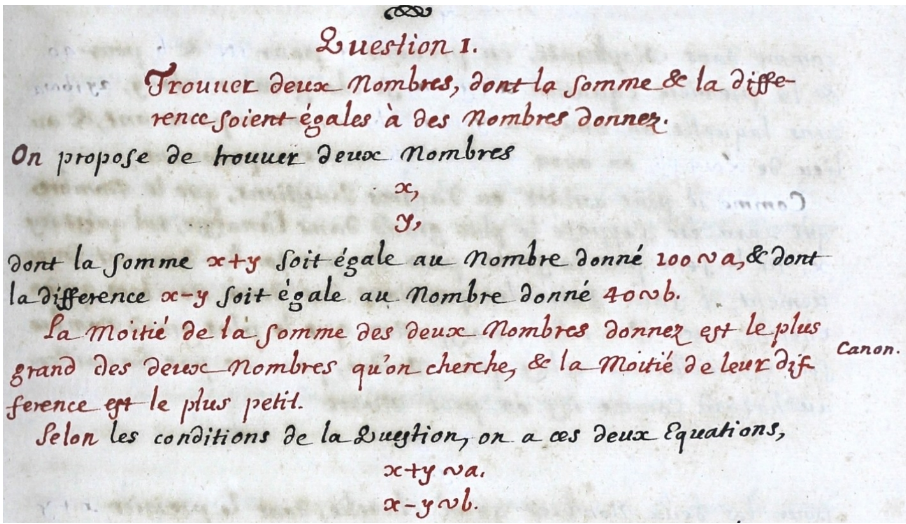
[45] *Nouveaux elements d'Algebre, ou Principes Generaux, Pour resoudre toutes sortes de Problems de Mathematique* [43]. This title shows, like Viète, the importance that Ozanam placed on algebra in solving "all problems". Importance that he also makes explicit in the preface of this work.

[46] *Des Simples, des Doubles, & des Triples Egalitez, Pour the solution des Problêmes indetérminez en nombres rationels.*

and formulated the question with algebraic language; this part is written in black except for algebra which is in red. Before solving it, also in red, he wrote what he called a "canon"; actually, this word appears in the margin, as it can be sean in Figure 5 and it is a simple rethorical formulation for the solution to the problem. In the preface he said regarding the canon:

> At the beginning of each Question I have added a general canon to solve it, and I have deduced it from the simplest solution, among several that I give in almost everything, to have a canon although simpler, but less general. I have thought that I should used it in this way, because a more general canon being longer loses its beauty and its usefulness[...]
> I have put the canon at the beginning and not at the end of the problem, to provoke in the reader the desire to know its origin and force him to study the different solutions, which follow the canon[...][47]

In Figure 5, we reproduce the beginning of Question I of Book I. Inside we can see how Ozanam began by stating the problem and making its algebraic formulation; then he wrote the canon that solved it and finally gave to the equations to be solved. From here the problem is solved. At the beginning he says [our translation of the content of Figure 5]:



Question I.

Trouver deux Nombres, dont la somme & la diffe-
rence soient égales à des Nombres donnez.

On propose de trouver deux Nombres

$x$,

$y$,

dont la somme $x+y$ soit égale au Nombre donné $100 \backsim a$, & dont
la difference $x-y$ soit égale au Nombre donné $40 \backsim b$.

La Moitié de la somme des deux Nombres donnez est le plus
grand des deux Nombres qu'on cherche, & la Moitié de leur dif-
ference est le plus petit. Canon.

Selon les conditions de la Question, on a ces deux Equations,

$x+y \backsim a$,

$x-y \backsim b$.

**Fig. 5** Question 1 of Book I, p. 7 of the manuscript [42].

[47] Our translation of [42], p. 4. "J'ay ajouté au commencement de chaque Question un canon general pour la resoudre, & je l'ay tirée de la solution la plus simple, entre plusieurs que je donne presque par tout, pour avoir un canon aussy plus simple, mais moins general. J'ay crû que j'en devois user de la sorte, parce qu'un canon plus general étant plus long perd sa beauté & son utilité[...]
J'ay mis ce canon plutôt au commencement qu'à la fin de la Question, pour donner l'envie au Lecteur d'en savoir l'origine, & l'obliger à étudier les solutions differentes, qui suivent le canon,[...]".

We put in italics the parts that in the manuscript are written in red which are, as we have mentioned previously in Section 4, the equations and other algebraic or numerical expressions, the formulations of the problems, each canon and the first word with the one each paragraph starts.

*Question I*

*To find two numbers, whose sum and whose difference are equal to two given numbers.*

It is proposed to find two numbers

$$x,$$
$$y,$$

whose sum $x + y$ is equal to the given number $100 \sim a$ [$100 = a$], and whose difference $x - y$ is equal to the given number $40 \sim b$ [$40 = b$].

*Canon* [Ozanam writes the word Canon in the margin of the text, see Figure 5]: *Half of the sum of the two given numbers is the greater of the numbers sought, and half of their difference is the lesser.*

*According* to the conditions of the question, we have the two equations,

$$x + y \sim a,$$
$$x - y \sim b.$$

It can be observed in the text that Ozanam wrote $100 \sim a$ and $40 \sim b$ where $a$ and $b$ are assumed to be given. These are the values originally used by Diophantus in his treatise. But in the manuscript this question (and all the others) is generally resolved for $a$ and $b$, although he checked for the numeric values above.

Finally, having solved the problem, with the title in the margin that says "Diophanthus' Method"[48] (see Figure 6), he added the solution of the problem following the style of the original reasoning of Diophantus.

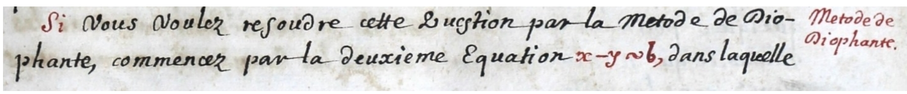

Si Vous Voulez resoudre cette Question par la Metode de Dio-
phante, commencez par la deuxieme Equation $x-y \sim b$, dans laquelle
Metode de Diophante.

**Fig. 6** Side note "Diophantus Method", p. 7, volume I of the manuscript [42].

Nevertheless, it is interesting what Ozanam said in his preface about Diophantus, his way of solving problems and his mathematical knowledge:

> [...] If I have not explained Diophantus' method in some places, it is because I have thought that it is easy to understand or too long to do for the specious; and in the most difficult places, I have shown the

[48] "Metode de Diophante".

origin of the steps that he has taken at the beginning, to satisfy at the same time one or more conditions of the Question. It will be seen that his steps were made more by chance, and by a knowledge that he had acquired more by a prolonged use of the property of numbers, than by a determined science, and by a truly specious one, since the theorems on which is based on for taking his steps, they are stated in a much more general way by the specious than those he proposes[49].

Ozanam's ambitious project was successfully completed with the handwriting of Diophantus' treatise, which, as we have seen, was recognized by distinguished mathematicians of the time and must have had some diffusion, although such success was not complete as it was not published.

## 7 Some examples in the "Diophantus" of Ozanam

As an example, we are going to look at Problem VIII of Book II of Diophantus[50], to see how Diophantus[51] solved it and be able to compare it with the solutions of Viète, Bachet and, later, Ozanam. The comparisons will allow us to verify how original, novel and rigorous Ozanam's work is. With regard to Viète, for the use of the new algebra already more developed, which Bachet did not use, and which allowed him to obtain more general solutions in a more comfortable way, as well as geometric approaches. As for Bachet, in addition to the fact that Ozanam used the new algebraic language and Bachet did not and translated it into Latin following the original text faithfully, becoming the Diophantus of reference at that time.

Diophantus formulated it in a general way:

"To find two square numbers whose sum is a given square".

For its resolution Diophantus took a specific number, 16, as the square that he wanted to decompose into the sum of two squares. However, Diophantus' resolution process and his way of reasoning presented certain general aspects that probably inspired the generalization of one of Viète's solutions and Bachet's solution. Both solved the question in a general way.

[49] Our translation of [42], p. 3. "[...] Que si je n'ay pas expliqué en quelques endroits la metode de Diophante, c'est parce que je l'ay crûe facile à concevoir, ou trop longue à pratiquer par la specieuse; & dans les endroits les plus difficiles, j'ay fait voir l'origine des positions qu'il a faites au commencement, puor satisfaire tout d'un coup à une, ou à plusieurs conditions de la Question. On y verra que ses positions ont été faites plutôt par hazard, & par une connoissance qu'il s'étoit acquise par un long usage de la proprieté des nombres, que par une science certaine, & par une veritable specieuse, puisque les Theoremes sur lesquels il se fonde pour faire ses positions, se trouvent enoncez par la specieuse beaucoup plus generalement qu'il ne les a proposez".

[50] This problem is well known, since it is the question on which Fermat wrote his famous note in the margin of Bachet's Diophantus that resulted in the known as Fermat's last theorem.

[51] See Tannery [53] or Heath [24].

Diophantus noticed that if 16 is the number that you want to decompose into the sum of two squares and the first of them was a square of "$\alpha\rho\iota\theta\mu o$ (arithm)"[52], the second of the squares searched would be the difference between 16 and the square of "arithm". In current algebraic language: if $x^2$ is the first square, the second will be $16 - x^2$.

Then Diophantus stated: "I form the square of any number of Arithm minus as many units as there are on the side of 16"[53], which means that the other square will be a multiple of the arithm minus "the side" of the square 16, that is, the square root of 16. In short, with our algebraic language, if the arithm is $x$, then

$$16 - x^2 = (mx - 4)^2$$

Diophantus took $m = 2$ and when developing remained $5x^2 = 16x$. Diophantus deduced, obviously not from an equation like the previous one, that the arithm is $x = \frac{16}{5}$ and the squares sought were $\frac{256}{25}$ and $\frac{144}{25}$.

## 7.1 The question VIII, Book II by Viète and by Bachet

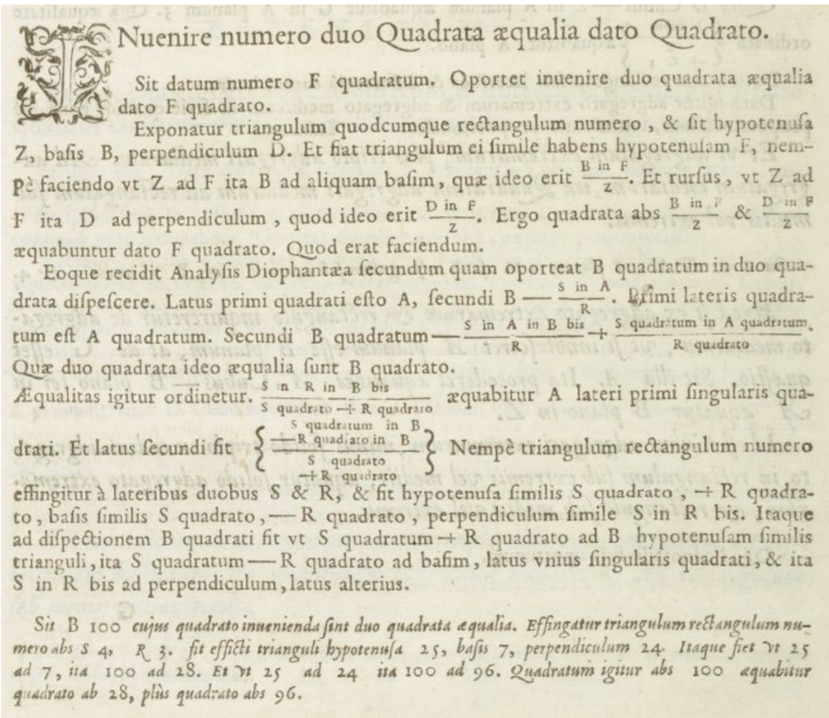
Viète innovated in the treatment of these problems; in particular, Viète solved problem VIII of Book II in a general way by following two different forms (strategies). In the first one, he carried out a geometric reasoning by using Thales and Pythagoras theorems ([56], p. 13) as follows: He named $F$ the number whose square he wanted to decompose into the sum of two other squares. Then he assumed known "a right triangle with hypotenuse $Z$, base $B$ and perpendicular $D$"[54]. He considered that $F$ was the hypotenuse of a right triangle similar to the previous one in such a way that if $Z$ is to $F$ like $B$ is to the base of the new triangle, then the base will be $\frac{B \text{ in } F}{Z}$ (Viète used "in" to represent multiplication). Reasoning in the same way with the perpendicular, it will be $\frac{D \text{ in } F}{Z}$. The squares of $\frac{B \text{ in } F}{Z}$ and $\frac{D \text{ in } F}{Z}$ are the squares searched for. We can see part of Viète's original text in Figure 7.

Viète then solved the problem again using, in his own words, the "Diophantus analysis", that is, the kind of reasoning that Diophantus did with the number 16, but in an abstract way. He changed the notation and called $B$ the number whose square, $B^2$, he wanted to decompose into the sum of two squares and said that if the first of them is $A$, the second will be of the form $B - \frac{S \text{ in } A}{R}$ for certain numbers $S$ and $R$, which in our notation would be $B - \frac{S}{R}A$. In current algebraic language, Viète's resolution was:

[52] "Arithm" can be translated as unknown or searched number.

[53] Heath [24] p. 145 "I form the square from any number of $\alpha\rho\iota\theta\mu o$ minus as many units as there are in the side of 16".

[54] Let us observe that Viète also used symbols for the known parameters, which allowed him to obtain general solutions.



INuenire numero duo Quadrata æqualia dato Quadrato.

Sit datum numero F quadratum. Oportet inuenire duo quadrata æqualia dato F quadrato.

Exponatur triangulum quodcumque rectangulum numero, & sit hypotenusa Z, basis B, perpendiculum D. Et fiat triangulum ei simile habens hypotenusam F, nempè faciendo vt Z ad F ita B ad aliquam basim, quæ ideo erit $\frac{\text{B in F}}{\text{Z}}$. Et rursus, vt Z ad F ita D ad perpendiculum, quod ideo erit $\frac{\text{D in F}}{\text{Z}}$. Ergo quadrata abs $\frac{\text{B in F}}{\text{Z}}$ & $\frac{\text{D in F}}{\text{Z}}$ æquabuntur dato F quadrato. Quod erat faciendum.

Eoque recidit Analysis Diophantæa secundum quam oporteat B quadratum in duo quadrata dispescere. Latus primi quadrati esto A, secundi B $-$ $\frac{\text{S in A}}{\text{R}}$. Primi lateris quadratum est A quadratum. Secundi B quadratum $-$ $\frac{\text{S in A in B bis}}{\text{R}}$ $+$ $\frac{\text{S quadratum in A quadratum}}{\text{R quadrato}}$. Quæ duo quadrata ideo æqualia sunt B quadrato.

Æqualitas igitur ordinetur. $\frac{\text{S in R in B bis}}{\text{S quadrato} + \text{R quadrato}}$ æquabitur A lateri primi singularis quadrati. Et latus secundi fit $\left\{\frac{\text{S quadratum in B} - \text{R quadrato in B}}{\text{S quadrato} + \text{R quadrato}}\right\}$ Nempè triangulum rectangulum numero effingitur à lateribus duobus S & R, & fit hypotenusa similis S quadrato, + R quadrato, basis similis S quadrato, $-$ R quadrato, perpendiculum simile S in R bis. Itaque ad dispectionem B quadrati fit vt S quadratum + R quadrato ad B hypotenusam similis trianguli, ita S quadratum $-$ R quadrato ad basim, latus vnius singularis quadrati, & ita S in R bis ad perpendiculum, latus alterius.

*Sit B 100 cujus quadrato inuenienda sint duo quadrata æqualia. Effingatur triangulum rectangulum numero abs S 4, R 3. fit effecti trianguli hypotenusa 25, basis 7, perpendiculum 24. Itaque fiet vt 25 ad 7, ita 100 ad 28. Et vt 25 ad 24 ita 100 ad 96. Quadratum igitur abs 100 æquabitur quadrato ab 28, plùs quadrato abs 96.*

**Fig. 7** Question VIII treated by Viète, p. 13, Liber Quartus Zeteticum I [56].

$$B^2 = A^2 + \left(B - \frac{S}{R}A\right)^2 = A^2 + B^2 - 2\frac{S}{R}AB + \frac{S^2}{R^2}A^2,$$

from where he deduced that $A = \dfrac{2SRB}{S^2 + R^2}$ and, therefore, that the second side sought is

$$\frac{R^2B - S^2B}{S^2 + R^2}. \tag{1}$$

It is interesting to know what Viète wrote and how he wrote it. Viète's innovation had two dimensions, the first one is to arrive to the solution by using geometric resources and the second the use of the new algebraic language that he himself introduced and which allowed him to obtain a general solution. Note that in Viète's symbolism "$R$ quadrato" is $R^2$ and "bis" means multiply by 2. Next, Viète solved the problem by following the analysis of Diophantus. We record the Viète's text[55]:

[55] Our translation of [56], p. 13 "Latus primi quadrati esto $A$, secundi $B - \frac{S \text{ in } A}{R}$. Primi lateris quadratum est $A$ quadratum. Secundi $B$ *quadratum* $- \frac{S \text{ in } A \text{ in } B \text{ bis}}{R} + \frac{S\ quadratum \text{ in } A\ quadratum}{R\ quadrato}$. Quae duo quadrata ideo aequalia sunt $B$ quadrato.

The first side of the square will be $A$ and the second will be $B - \dfrac{S \text{ in } A}{R}$[that is to say $B - \dfrac{SA}{R}$]. The first side squared will be $A$ *quadratum*. The second

$$B\ quadratum - \frac{S \text{ in } A \text{ in } B \text{ bis}}{R} + \frac{S\ quadratum \text{ in } A\ quadratum}{R\ quadrato}$$

[which in current algebraic language will be:

$$B^2 - 2\frac{S}{R}A + \frac{S^2}{R^2}A^2;$$

we continue to use current algebraic language to facilitate reading].
The sum of these two squares will be $B$ squared. Let the equality therefore be ordered. Then it will be equal to $A$, the first side

$$\frac{2SRB}{S^2 + R^2}$$

and the second side will be $\dfrac{S^2B - R^2B}{S^2 + R^2}$[56].

Although Viète only wrote what we have just seen, he actually followed the same reasoning of Diophantus that we have seen previously. Finally, he chose as an example $B = 100$ to decompose $B^2$ into the sum of two squares, taking $S = 4$ and $R = 3$, obtaining 28 and 96.

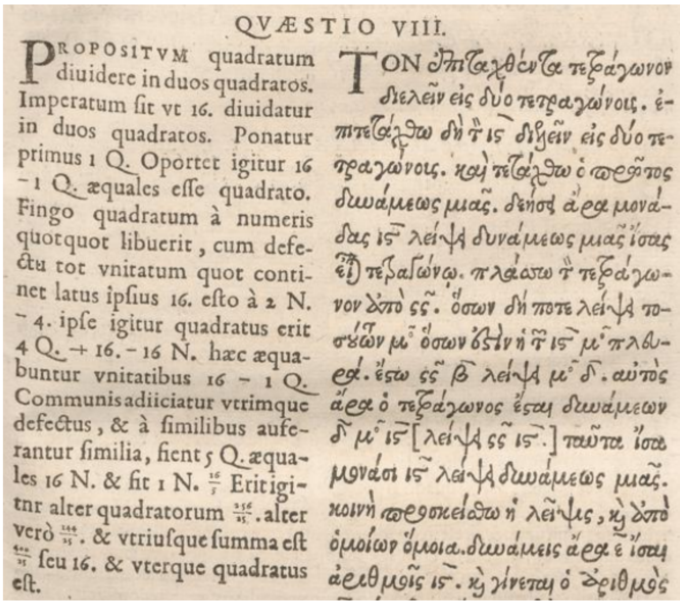
Later, in 1621, Bachet approached Problems VIII and IX in the same way due to their similarity[57] and did not do anything different from what Diophantus did, even by using the same number 16, as can be seen by comparing our above explanation over Diophantus' development and Figure 8, also using the number 16. In a long comment after both questions, occupying pages 86 to 89, Bachet explained in detail, rhetorically, the conditions that had to be met in order to obtain this relationship in general.

Bachet concluded with a proposal to solve the problem in general, so that if the side $D$ of the given square has to be decomposed into the sum of two squares $A + B = D$, with $A$ and $B$ in square proportion, that is $A/B$ a rational square, then the values $|A - B|$ and $2\sqrt{AB}$ will be the sought sides and he illustrated it with numerical examples. For instance, Bachet gives the example $D = 65$, and descomposes $D = A + B$ with $A = 1$, $B = 64$, or $A = 13$, $B = 52$, or $A = 16$, $B = 49$. Notice that in all cases $\dfrac{A}{B}$ is a rational square

Aequalitas igitur ordinetur. $\frac{S \text{ in } R \text{ in } B \text{ bis}}{S \text{ quadrato} + R \text{ quadrato}}$ aequabitur $A$ lateri primi singularis quadrati. Et latus secundi sit $\frac{-R \text{ quadrato in } B}{S \text{ quadrato } + R \text{ quadrato}}$".

[56] Note that the last expression on the second side of Viète does not match the expression (1); actually the signs change; Viète took the "absolute value"; in fact, if we do the calculations in the example with the expression (1), we have $-28$ instead of 28.

[57] Problem IX consists of "Given a number that is the sum of two squares, decompose it as the sum of two other square numbers".


QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quot continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$. & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

ΤΟΝ ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο τετραγώνους. ἐπιτετάχθω δὴ τὸν ις διελεῖν εἰς δύο τετραγώνους. καὶ τετάχθω ὁ πρῶτος δυνάμεως μιᾶς. δεήσει ἄρα μονάδας ις λεῖψει δυνάμεως μιᾶς ἴσας εἶναι τετραγώνῳ. πλάσσω τὸν τετράγωνον ἀπὸ ςς. ὅσων δή ποτε λεῖψει τοσούτων μ° ὅσων ἐστὶν ἡ τῶν ις μ° πλευρά. ἔστω ςς β λεῖψει μ° δ. αὐτὸς ἄρα ὁ τετράγωνος ἔσται δυνάμεων δ μ° ις [λεῖψει ςς ις.] ταῦτα ἴσα μονάσι ις λεῖψει δυνάμεως μιᾶς. κοινὴ προσκείσθω ἡ λεῖψις, καὶ ἀπὸ ὁμοίων ὅμοια. δυνάμεις ἄρα ε ἴσαι ἀριθμοῖς ις. καὶ γίνεται ὁ ἀριθμὸς

**Fig. 8** Question VIII treated by Bachet p. 85 [3].

($1/8^2$, $1/2^2, 4^2/7^2$, respectively); thus, taking the sides $|A-B|$ and $2\sqrt{AB}$ we obtain $|A-B| = 63$, $2\sqrt{AB} = 16$, or $|A-B| = 39$, $2\sqrt{AB} = 52$, or $|A-B| = 33$, $2\sqrt{AB} = 56$, respectively.

## 7.2 The question VIII, Book II by Ozanam

Like Viète and Bachet, Ozanam solved problems in general, but unlike Bachet he did use, with ease and efficiency, the power of the new algebraic language. Ozanam used the algebraic language with a notation closer to the current one and handled it with ease and with the mathematical power that this language allowed: he obtained clear results as a consequence of equations established from the conditions of each question. He even went further; he analyzed the solutions and the possible restrictions that they could have; for example, denominators that do not cancel out or negative values that he discarded, like other authors of the time, for not considering them as valid true solutions.

He also followed, as we have seen in Section 6, a defined approach to solving Diophantine problems; namely, establishment of the constitutive equation, which is determined from the conditions of the problem, the statement of a canon that explained how the solutions were obtained; the demonstration of the canon, which is nothing more than the solution of the equations and, in this case, as in so many other problems, a geometric vision of the question, together with the values of the original solutions of Diophantus. The transcription of how Ozanam approached problem VIII of Book II (later we will see three more problems) illustrates the great clarity and order with which he proceeded in his work and allows us to compare it with what Viète and Bachet did.

Figure 9 again shows the exquisite and clear style in which Ozanam wrote, in this case, at the beginning of the treatment of question VIII of Book II. Ozanam began by formulating the question in order to immediately establish the unknowns and to reformulate algebraically the problem[58].

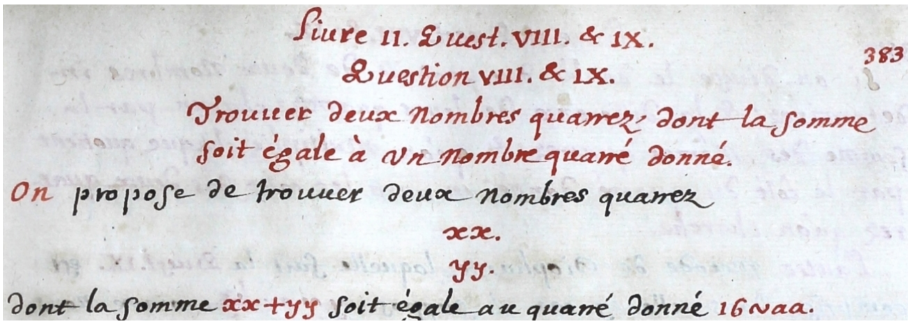

Liure II. Quest. VIII. & IX.
383
Question VIII. & IX.
Trouuer deux nombres quarrez dont la somme
soit égale à vn nombre quarré donné.
On propose de trouuer deux nombres quarrez
xx.
yy.
dont la somme xx+yy soit égale au quarré donné 16∾aa.

**Fig. 9** Beginning of question VIII, Book II. p. 383 of volume I of the manuscript [42].

We have already seen how the notation used by Ozanam was like, already quite close to the current one and we have also seen some examples of how he wrote with this notation. From now on, in order to make reading easier, we will use current algebraic notation.

*Question VIII* [Book II][59]

*To find two square numbers whose sum is equal to a given square number.*

*We* propose to find two square numbers

$$x^2.$$
$$y^2.$$

whose sum $x^2 + y^2$ equals the given square $16 = a^2$.

In all of Diophantus' original problems, Ozanam wrote the concrete numerical value that Diophantus used, 16 in this case. On the other hand, before continuing with the resolution of the problem, he wrote the canon, with the intention, declared in the introduction as we have seen, to arouse the interest of the reader, that is, to show the way to obtain the necessary numbers as they would appear in the solution he was about to find:

[58] Note that, as Ozanam himself explained, he approached questions VIII and IX at the same time due to the similarity of their approach as well as Bachet had done.

[59] Our translation of p. 383 of volume I of the manuscript [42]. "Trouver deux nombres quarrez, dont la somme soit égale à un nombre quarré donné.
On propose de trouver deux nombres quarrez $xx$. $yy$. dont la somme $xx + yy$ soit égale au quarré donné $16 \sim aa$".

*Canon. If each of the sides of a right triangle is multiplied by the side of the given square, and we divide each product by the hypotenuse, we will have the sides of the squares we are looking for*[60].

Once the canon was established, Ozanam proceeded to set analytically the equation, or equations, that were derived from the statement, with the aim of solving them:

*According* to the condition of the question, we will have the equation

$$x^2 + y^2 = a^2$$

from which we will get $a = \sqrt{x^2 + y^2}$. Thus, we will have to equal this power to the square $x^2 + y^2$, whose side we take $\left| x - \dfrac{by}{c} \right|$ or $x + \dfrac{by}{c}$, so that we will have $a = \left| x - \dfrac{by}{c} \right|$, or $a = x + \dfrac{by}{c}$; it will be found that $x = \left| \dfrac{b^2y - c^2y}{2bc} \right|$ and in consequence $a = \dfrac{b^2y + c^2y}{2bc}$[61].

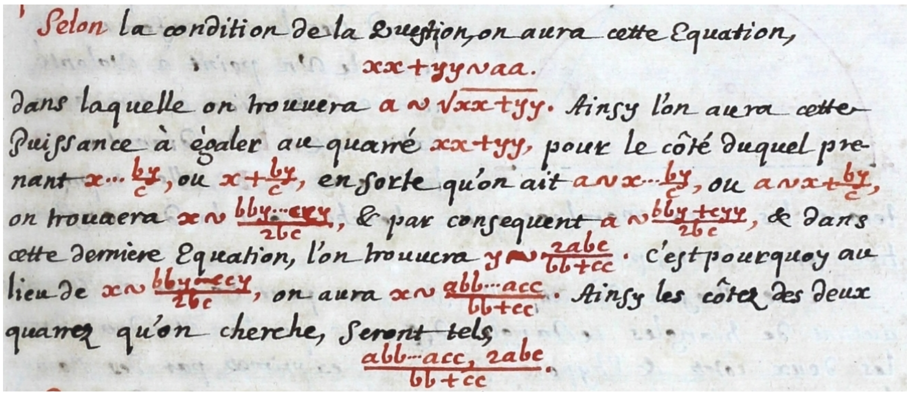

Selon la condition de la Question, on aura cette Equation,
$xx+yy \sim aa$.
dans laquelle on trouuera $a \sim \sqrt{xx+yy}$. Ainsy l'on aura cette Puissance à égaler au quarré $xx+yy$, pour le côté duquel prenant $x - \frac{by}{c}$, ou $x+\frac{by}{c}$, en sorte qu'on ait $a \sim x - \frac{by}{c}$, ou $a \sim x+\frac{by}{c}$, on trouuera $x \sim \frac{bby - ccy}{2bc}$, & par consequent $a \sim \frac{bby+ccy}{2bc}$, & dans cette derniere Equation, l'on trouuera $y \sim \frac{2abc}{bb+cc}$. c'est pourquoy au lieu de $x \sim \frac{bby-ccy}{2bc}$, on aura $x \sim \frac{abb-acc}{bb+cc}$. Ainsy les côtez des deux quarrez qu'on cherche, seront ~~tels~~
$\frac{abb-acc}{bb+cc}, \frac{2abc}{bb+cc}$.

**Fig. 10** Question VIII, Book II. p. 383 of volume I of the manuscript [42].

In current algebraic lenguage, notice that if $a = |x - \frac{by}{c}|$, then $x^2 + y^2 = a^2$ leads us to $y^2 = -\frac{2byx}{c} + \frac{b^2y^2}{c^2}$ or $x = \frac{b^2y - c^2y}{2bc}$. Then, returning to the expression of $a$,

$$a = \left| \frac{b^2y - c^2y}{2bc} - \frac{by}{c} \right| = \left| \frac{-b^2y - c^2y}{2bc} \right|$$

[60] Own translation p. 383 from volume I of the manuscript [42] "Si on multiplie chacun des deux côtez d'un triangle rectangle, par le côté du quarré donné, & qu'on divise chaque produit par l'hypotenuse; on aura les côtez des deux quarrez qu'on cherche".
[61] Our translation of p. 383 of volume I of the manuscript [42], see Figure 10.

$$\text{or } a = \frac{b^2y + c^2y}{2bc} \tag{2}$$

if we considerer that $b$, $c$ are positive numbers and that $y$ must be a "true" solution. On the other hand, if $a = x + \frac{by}{c}$, again from $x^2 + y^2 = a^2$ by developing the square and solving $x$, we obtain $x = \frac{c^2y - b^2y}{2bc}$, and consequently

$$a = x + \frac{by}{c} = \frac{c^2y - b^2y}{2bc} + \frac{by}{c}, \quad \text{or } a = \frac{b^2y + c^2y}{2bc}. \tag{3}$$

In both cases (2) and (3), we have

$$x = \frac{|b^2y - c^2y|}{2bc} \text{ and } a = \frac{b^2y + c^2y}{2bc}. \tag{4}$$

The way to express, from (4), the two numbers $x$, $y$ in the solution was

$$\frac{|ab^2 - ac^2|}{b^2 + c^2}, \quad \frac{2abc}{b^2 + c^2}.$$

This solution from Ozanam was purely algebraic, taking $b$ and $c$ as parameters. Next, Ozanam verified, by means of an example, that doing $a = 4$, he obtained the original solutions of Diophantus. For that he said that taking $b = 1$ and $c = 2$ he would obtain precisely those solutions $\frac{16\,,\,12}{5}$. Indeed, in this case,

$$x = \frac{|ab^2 - ac^2|}{b^2 + c^2} = \frac{|4 - 16|}{1 + 4} = \frac{12}{5}$$

and

$$y = \frac{2abc}{b^2 + c^2} = \frac{2 \cdot 4 \cdot 1 \cdot 2}{5} = \frac{16}{5}.$$

However, in addition to these two parameter values, it offered two other values, this time the irrationals numbers like $b = \sqrt{1/2}$, $c = \sqrt{9/2}$, also with $a = 4$, arriving again to $\frac{16, 12}{5}$.

Following the scheme described above, he continued by exposing what he called the Diophantus method, namely the type of reasoning that Diophantus himself used. But this time by getting the solutions in a general way in the algebraic language and by attaining the same solutions that he just found. Then he wrote the canon again. It was a new wording of the one he had already stated at the beginning:

> *The* preceding canon can be stated in another way as follows:
> *Canon. If you divide the double product of two indeterminate numbers and the difference of their squares, each one by the sum of the same squares, and multiply each quotient by the side of the given square, you will have the sides of the two squares you are looking for*[62].

[62] Our translation of p. 384 of volume I of the manuscript [42] "Si on divise le double du produit de deux nombres indeterminez, & la difference de leurs quarrez, chacun par la somme des mêmes quarrez, & qu'on multiplie chaque quotient par le côté du quarré donné; on aura les côtez des deux quarrez qu'on cherche".

He continued, with the title in the margin of "Diophantus' second method", briefly explaining that this second method is the one applied to the resolution of question IX that coincides with VIII. The only difference appears in the resolution.

It is evident that, working in this way, coinciding with the current one, the obtention of general solutions was assured; Ozanam sought to solve the problem, but even to obtain those general solutions and to reflect on such solutions and even in their geometric consequences. The differences with Viète and Bachet are thus revealed even more clearly.

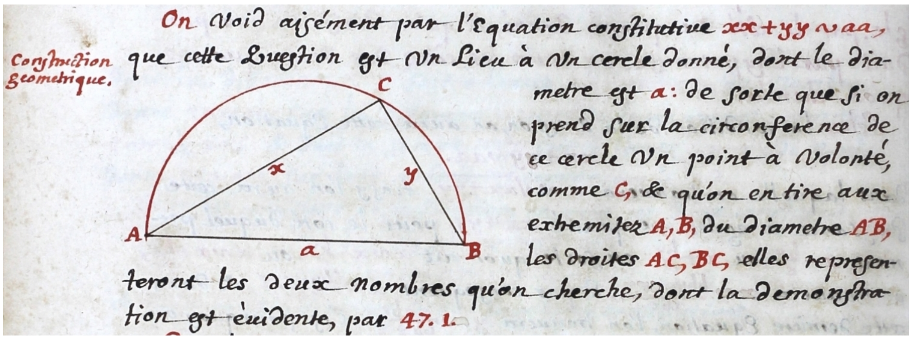
To end question VIII and with the title of "Geometric construction", Ozanam continued with his scheme, declared in the preface, of the development of Diophantus' problems, introducing one of the interesting novelties: the treatment or the geometric version of the problem. In this development he included an illustrative drawing made with extraordinary neatness and precision. This was also done in many other problems he dealt with as we will have the opportunity to see later.

Let us observe that Ozanam's geometric contributions to Diophantus' problems are not a geometric way of obtaining the solution, or solutions, as in the case of Viète, but a true connection between algebra and geometry; such a connection is derived directly from the algebraic solution and that he sees as loci in the same sense that they are considered at present, obtaining analytical equations of such loci[63]. This is the paramount innovation that does not appear in any of the mathematicians who dealt with Diophantus' Arithmetic before, and it offers surprising and elegant results that relate algebra and geometry.

> *Geometric construction.* It is easy to see from the constitutive equation $x^2 + y^2 = a^2$, that this question is a locus of a given circle, whose diameter is $a$: so that if it is taken on the circumference of this circle a point at will as $C$, and we draw from the extremes $A$, $B$ of the diameter $AB$, the segments $AC$, $BC$, they represent the numbers sought[...][64] [see Figure 11].

[63] We have already cited Ozanam's work from 1687 *Treatise of Loci* [40]. Ozanam developed loci in this treatise and already in the introduction he defines what he understands by loci: Our translation of p. 5 of [40] "When after fulfilling the conditions of a Problem, there is more than one unknown letter in the last Equation, the problem is a *Locus*; & when there are only two unknown letters left, this Locus is a *Line*, that will be straight line, that will be straight when the constructive Equation, because in the constitutive Equation each unknown letter will be one dimension and they never multiply each other;[...] But, if any of the two unknown letters has two dimensions, or else, when its Rectangle [product of two letters] is in the constitutive equation, this local line is a curve[...]", ("Lors qu'aprés avoir accompli les conditions d'un Probleme, il demeure dans la derniere Equation plus d'une lettre inconnuë, le Probleme est un Lieu; & quand il ne reste que deux lettres inconnuës, ce Lieu est une *Ligne*, qui sera droite, lorsque dans l'Equation constitutive chaque lettre inconnuë n'aura qu'une dimension, & qu'elles ne se multiplieront point ensemble;[...] Mais, si l'une de ces deux lettres inconnuës a deux dimensions, ou bien, quand leur Rectangle se trouve dans l'Equation constitutive, cette ligne locale est courbe[...]); he continued identifying that curve with a conic according to the type of constitutive equation.

[64] Our translation of p. 384 of volume I of the manuscript [42] in Figure 11.


Construction geometrique.

On void aisément par l'Equation constitutive $xx+yy \backsim aa$, que cette Question est vn Lieu à vn cercle donné, dont le diametre est $a$: de sorte que si on prend sur la circonference de ce cercle vn point à volonté, comme C, & qu'on en tire aux extremitez A, B, du diametre AB, les droites AC, BC, elles representeront les deux nombres qu'on cherche, dont la demonstration est évidente, par 47. 1.

**Fig. 11** Geometric construction for the question VII of the Book II p.384 of volume I of the manuscript [42]

He continued by describing by numbers the obtained right triangles which provided the solution of the problem to add that rational numbers could be obtained (logically, taking rational $b$ and $c$) and even integers that were solutions:

> *Through* this question we can find as many different right triangles as we want with numbers, whose two sides and the hypotenuse are expressed by two rational numbers, using this undefined right triangle,
>
> $$\frac{|ab^2-ac^2|}{b^2+c^2},\quad \frac{2abc}{b^2+c^2},\quad a$$
>
> which has just been found, and you can have integers in smaller terms, multiplying by $b^2+c^2$ and dividing by $a$, because then we will have this other right triangle
>
> $$|b^2-c^2|\,,\quad 2bc\,,\quad b^2+c^2$$
>
> whose indeterminate quantities $b$, $c$ are called generating numbers because they serve to form in numbers indefinitely a right triangle, according to the general canon[65].

Ozanam concluded the question by stating a new general canon based on what was done with the triangles, and illustrated it with an example, namely with Pytagorean triples:

[65] Our translation of p. 384 of volume I of the manuscript [42] "Par le moyen de cette Question, on peut trouver en nombres autant de triangles rectangles differens que l'on voudra, dont les deux côtez & l'hypotenuse soient exprimez par des nombres rationnels, en se servant de ce triangle rectangle indefiny, $\frac{abb\cdots acc\,,\,2abc}{bb+cc}$, $a$. qui vient d'être trouvé, & que l'on peut avoir en entiers & en moindres termes, en le multipliant par $bb+cc$, & en le divisant par $a$, car alors on aura cet autre triangle rectangle, $bb\cdots cc$, $2bc$, $bb+cc$, dont les deux quatintez indeterminées $b$, $c$, sont apelées nombres generateurs, parce qu'ils servent a former en nombres indefiniment un triangle rectangle, par ce canon general".

*Canon. The double of the product of the two generating numbers is one of the sides of the right triangle, the difference of their squares is the other side, and the sum of the same squares is the hypotenuse.*
If we assume $b = 2$, $c = 3$, the right triangle sought will be of this magnitude 5, 12, 13[66].

The study of this problem has allowed us to get to know Ozanam's rigorous style, as well as the richness of his algebraic and geometric solutions; and at the same time we have been able to compare their novelties with what Viète and Bachet had done.

To have a more complete vision of Ozanam's Diophantus and to be able to assess what his project could mean in algebraic innovation, rigor, and both geometric and algebraic consequences, we are going to see three more questions. We have chosen them because they can be representative of the style of the manuscript and thus they give a wide picture of his procedures. These are question I of Book II, question XIV of Book I and question XVIII of Book VI. The order in which they are presented also answers to a better understanding of the content of the treatise.

We intend to highlight the care, meticulousness and rigor with which Ozanam drew up his treatise, but first of all, the great wealth of mathematical contributions: the properly algebraic part, where he always respected Diophantus' original ideas, which he updated with the new algebraic language to obtain general solutions thanks to the use of this language, which he specified with the numerical examples of Diophantus; the study of the solutions obtained; and the geometric part that was novel and rich for its implications and for the relationship between algebra and geometry, obtaining loci, making use of cartesian geometry, using axes and drawing curves.

### 7.3 Question I. Book II

The first problem we have chosen is question I of Book II. Once again, the resolution scheme for each question adopted by Ozanam is clear: statement of the question, approach, canon, construction of the equation and resolution, numerical examples of Diophantus, other possible solutions and another canon and geometric construction. In this question, Bachet limits himself to solving it with the numerical case that Diophantus uses without adding any comment. We present the translation of Ozanam's question hereunder.

*Question 1.*

*To find two numbers such that the ratio of their sum to the sum of their squares is given.*
*We want* to find two numbers

[66] Our translation of p. 383 of volume I of the manuscript [42] "Le double du produit des deux nombres generateurs est l'un des deux côtez du triangle rectangle: la difference de leurs quarrez est l'autre côté: & la somme des mêmes quarrez est l'hypotenuse. Si l'on suppose $b \sim 2$, $c \sim 3$, le triangle rectangle qu'on cherche, sera cette grandeur 5, 12, 13.'"

$x$.
$y$.

whose sum $x+y$ is to the sum $x^2+y^2$ of its squares, such as $1=r$, to $10=s$[67].

As usual, Ozanam solved the problem in general for any given ratio $r/s$, but, we insist, he also wrote the original numerical values of Diophantus, in this case $r=1$, $s=10$. Then he wrote the canon to continue with the approach and resolution of the equation. Let us observe how Ozanam keeps multiplying by the unit, which he designates with $l$, to preserve the law of homogeneous:

*Canon: If the plane under [generated by] the sum of any two numbers and the second term of the ratio given is multiplied by each of those same numbers, and each solid is divided by the solid under [generated by] the first term and the sum of the squares of those same numbers, we will have the two numbers sought*[68].

*According* to the condition of the question, we will have the analogy ["analogy" is the word used bay Ozanam to express "proportion"]:

$$\frac{lx+ly}{x^2+y^2}=\frac{r}{s},$$

from which the equation follows $lsx+lsy=rx^2+ry^2$[69]. (5)

Later Ozanam offered a solution for $y$ which, although he did not mention it explicitly, is the positive solution which results from considering the previous equation, as a quadratic equation in $y$, $ry^2-sy+rx^2-sx=0$:

$$y=\frac{ls}{2r}+\sqrt{\frac{l^2s^2}{4r^2}+\frac{lsx}{r}-x^2}\ .$$

Ozanam continued by saying that for the solution obtained to be rational, the expression $\dfrac{s^2}{4r^2}+\dfrac{sx}{r}-x^2$ that appears in the previous radicand must be a perfect square and for that he said:

[67] Our translation of p. 169 of volume I of the manuscript [42] "Trouver deux nombres, tels que la raison de leur somme à la somme de leurs quarrez, soit donnée. On propose de trouver deux nombres $x$, $y$ dont la somme $x+y$, soit à la somme $xx+yy$ de leurs quarrez, comme $1\sim r$, à $10\sim s$".

[68] Thus, if $a$ and $b$ are arbitrary numbers, and $r/s$ is the given ratio, Ozanam tells us that it is necessary to construct the plane (dimension two) $(a+b)s$, and next to consider the solids (dimension three) $(a+b)sa, (a+b)sb$; finally, each of these numbers will be divided by the solid $(a^2+b^2)r$ in order to obtain the solution, namely $\frac{(a+b)sa}{(a^2+b^2)r}$ and $\frac{(a+b)sb}{(a^2+b^2)r}$.

[69] Our translation of p. 169 of volume I of the manuscript [42] "Canon: Si on multiplie le Plan sous la somme de deux nombres quelconques & le second terme de la raison donnée par chacun de ces mêmes nombres, & qu'on divise chaque solide par le solide sous le premier terme & la somme des quarrez des deux mêmes nombres; on aura les deux nombres qu'on cherche.
Selon la condition de la question, on aura cette analogie, $lx+ly, xx+yy :: r, s$ de laquelle on tire cette equation $lsx+lsy\sim rxx+ryy$".

> [...] and in order to have a rational solution, it will be necessary to equate this power $\dfrac{l^2s^2}{4r^2} + \dfrac{lsx}{r} - x^2$ to a square, for whose side [of the square] we take $\dfrac{ls}{2r} - \dfrac{ax}{b}$, or better $\dfrac{ax}{b} - \dfrac{ls}{2r}$ to have $y = \dfrac{ax}{b}$, will be find $x = \dfrac{abs + b^2s}{a^2r + b^2r}$ and instead of $y = \dfrac{ax}{b}$, we will have $y = \dfrac{abs + a^2s}{a^2r + b^2s}$. So the numbers searched will be[70]:
>
> $$\frac{abs + b^2s}{a^2r + b^2r}, \frac{abs + a^2s}{a^2r + b^2r}. \tag{6}$$

Though Ozanam did not make the calculations explicit so that the expression obtained inside the radical, he used the second of the possibilities to take $\dfrac{ax}{b} - \dfrac{ls}{2r}$ as the side of the square and established the equality

$$\left(\frac{ax}{b} - \frac{s}{2r}\right)^2 = \frac{s^2}{4r^2} + \frac{sx}{r} - x^2,$$

where the content of the first parentheses must have been taken by observing the expression on the right, being $a$ and $b$ what we could call parameters which he take integers. Expanding the square and calculating, he obtained the two numbers sought, which in current algebraic language are (bear in mind that $y = \dfrac{ax}{b}$):

$$x = \frac{abs + b^2s}{a^2r + b^2r}; \quad y = \frac{abs + a^2s}{a^2r + b^2r}.$$

As usual, once he solved the problem, he took Diophantus' original numerical values as an example to show how his algebraic solutions offered those of Diophantus. In this case, he found three different solutions depending on the values he took for the parameters $a$ and $b$. Specifically he wrote:

> *Since* we have assumed $r = 1$, $s = 10$, if we take $a = 1$, $b = 2$, the two numbers we are looking for will be 12, 6; and if we assume $a = 2$, $b = 3$, the two numbers that are searched will be $11\frac{7}{13}$, $7\frac{9}{13}$; but if $a = 3$, $b = 1$ is assumed, the two numbers searched are 4, 12[71].

[70] Our translation of p. 169 of volume I of the manuscript [42] "[...]& pour avoir une solution rationnelle, il faudra égaler au quarré cette Puissance $\frac{llss}{4rr} + \frac{lsx}{r} - xx$, pour le côté duquel prenant $\frac{ls}{2r} - \frac{ax}{b}$, ou mieux $\frac{ax}{b} - \frac{ls}{2r}$ pour avoir, $y \sim \frac{ax}{b}$, on trouvera $x \sim \frac{abs+bbs}{aar+bbr}$, & au lieu de $y \sim \frac{ax}{b}$, on aura $y \sim \frac{abs+aas}{aar+bbr}$. Ainsy les deux nombres qu'on cherche, seront tels, $\frac{abs+bbs\,,\,abs+aas}{aar+bbr}$".

[71] Our translation of pp. 169-170 of volume I of the manuscript [42] "Parce que nous avons supposé $r \sim 1$, $s \sim 10$ si l'on suppose $a \sim 1$, $b \sim 2$, les deux nombres qu'on cherche, seront de cette grandeur, 12, 6 & si l'on suppose $a \sim 2$, $b \sim 3$, les deux nombres qu'on cherche, seront de cette grandeur, $11\frac{7}{13}$, $7\frac{9}{13}$, mais si l'on suppose $a \sim 3$, $b \sim 1$, les deux nombres qu'on cherche, seront de cette grandeur, 4, 12".

Ozanam continued the algebraic part of the solution of the problem explaining that putting $y = \frac{ls}{2r} - \frac{ax}{b}$ instead to obtain the perfect square of the radicand, $\frac{ax}{b} - \frac{ls}{2r}$, doing the calculations in a similar way to the other case, the values obtained for the solution were (notice that now $y = \frac{s}{2r} + \left(\frac{s}{2r} - \frac{ax}{b}\right) = \frac{s}{r} - \frac{ax}{b}$):

$$\frac{b^2s + abs}{a^2r + b^2r}, \frac{b^2s - abs}{a^2r + b^2r}.$$

Then he wrote a new canon for this solution and made the corresponding numerical examples. But it should be noted that, with his usual thoroughness, he made a comment regarding this last solution, since the expression $b^2s - abs$ entails the possibility that the corresponding solution was negative:

> *This* second solution is not as general as the first because it has a restriction with respect to the first in the indeterminate variables $a$, $b$, which consists in that the first $a$ must be less than the second $b$ as it can easily be seen in the numerator $b^2s - abs$ of the second number found where we have $abs < b^2s$ and therefore $a < b$ when dividing by $bs$[...][72].

What we can call the algebraic part of the solution ends up here. From this point, Ozanam tackled the geometric part of the problem. On the question at hand, he wrote some comments regarding the similarity of its solution with question XXIV of the first Book and an introduction to the section that he called, in a title in the margin, "Geometric construction".

In the geometric construction we can see how Ozanam clearly identified the equations with loci. In addition, we will be able to observe how he does what nowadays we call a change of variable in order to handle simpler equations since the properties of the curves do not depend on where they are located. We will see a change of variable that translates the initial circumference to another with its center at the origin of coordinates, although Ozanam does not say so explicitly. This allows him to more comfortably handle both the equations and the curves that he associates or identifies with each equation. We can also see this domain of curves and equations in the examples which will be shown later.

Ozanam started from what he called a "constitutive equation", to wit, the equation obtained from the conditions of the question (see the expression (5)), in current algebraic language it would be $x^2 + y^2 - \frac{s}{r}x - \frac{s}{r}y = 0$, that he identified with a circumference. To obtain the radius, Ozanam did what

[72] Our translation of p. 171 of volume I of the manuscript [42] "Cette seconde solution n'est pas si generale que la premiere, puisqu'elle soufre une determination à l'égard des deux quantitez indetermineés $a$, $b$, qui est que la premiere $a$ doit être moindre que la seconde $b$, comme il est aisé de voir dans le numerateur $bbs - abs$, du second nombre trouvé où l'on a $abs \ominus bbs$, & par consequent $a \ominus b$ en divisant par $bs$[...].

today we would call two successive changes of variable; in the first place took $x = w + \dfrac{s}{2r}$ obtaining the expression $w^2 + y^2 - \dfrac{s^2}{4r^2} - \dfrac{s}{r}y = 0$; and then took $y = z + \dfrac{s}{2r}$ to get $w^2 + z^2 = \dfrac{s^2}{2r^2}$; therefore the radius was $\sqrt{\dfrac{s^2}{2r^2}} = \dfrac{1}{\sqrt{2}}\dfrac{s}{r}$. Currently we would say that he moved the center of the circumference from the point $\left(\dfrac{s}{2r}, \dfrac{s}{2r}\right)$ to the origin of coordinates. We remark that Ozanam will use this change of variables a little later. Ozanam's text was:

> *Since* this question is indeterminate, because we can give an infinity of different solutions, it must be *a Locus*, as can be seen in the constitutive equation $lsx + lsy = rx^2 + ry^2$, or $x^2 - \dfrac{lsx}{r} = \dfrac{lsy}{r} - y^2$, which is a locus in *the circle*, and to know the radius, let us take $x = w + \dfrac{ls}{2r}$, or $x - \dfrac{ls}{2r} = w$, and the equation $w^2 - \dfrac{l^2s^2}{4r^2} = \dfrac{lsy}{r} - y^2$ is obtained, or $y^2 - \dfrac{lsy}{r} = \dfrac{l^2s^2}{4r^2} - w^2$. Then we take $y = z + \dfrac{ls}{2r}$, or $y - \dfrac{ls}{2r} = z$ to have the last equation $z^2 - \dfrac{l^2s^2}{4r^2} = \dfrac{l^2s^2}{4r^2} - w^2$, or $z^2 = \dfrac{l^2s^2}{2r^2} - w^2$ which belongs to a circle whose radius is $\sqrt{\dfrac{l^2s^2}{2r^2}}$[73].

From here on, he continued with his "Geometric Construction", with this title in the margin of the text, accompanied by an illustration that we show in Figure 12, with an impeccable layout, which helps the reader to follow his argument; he described this construction in detail as can be seen in Figure 12. Realize that in the figure Ozanam identified the values $l$, $r$ and $s$ with segments $AO$, $AP$ and $AQ$, respectively, drawn to the left of the circle as well as the list of line identifications that he also writes on the left side, which are very useful to follow the reasoning.

> *To describe* the circle, we draw the right triangle $ABC$ whose sides $AB$, $BC$ measure $\dfrac{ls}{2r}$, which is the fourth proportional of the three lines $2r$, $l$ and $s$, that is, the three lines $2AP$, $AO$, $AQ$. Describe from the center A and passing by point C, a circumference of this circle which will be the Locus of the two numbers we are looking for. To determine the two numbers sought in segments, we draw the diameter $FG$ parallel to the

[73] Our translation of p. 171 of volume I of the manuscript [42] "Puisque donc cette Question est indeterminée, parce qu'on en peut donner une infinité de solutions differentes, elle doit être un Lieu, comme vous dans l'Equation constitutive $lsx + lsy \sim rxx + ryy$, ou $xx - \frac{lsx}{r} \sim \frac{lsy}{r} - yy$, qui est un Lieu au cercle; & pour en connoitre le rayon, supposez $x \sim w + \frac{ls}{2r}$, ou $x - \frac{ls}{2r} \sim w$, pour avoir cette autre Equation $ww - \frac{llss}{4rr} \sim \frac{lsy}{r} - yy$, ou $yy - \frac{lsy}{r} \sim \frac{llss}{4rr} - ww$. Supposez encore $y \sim z + \frac{ls}{2r}$, ou $y - \frac{ls}{2r} \sim z$, pour avoir cette derniere Equation $zz - \frac{llss}{4rr} \sim \frac{llss}{4rr} - ww$, ou $zz \sim \frac{llss}{2rr} - ww$, qui apartient à un cercle, dont le rayon est $\sqrt{\dfrac{llss}{2rr}}$.

segment $BC$ and take any point $D$ on the diameter $FG$ through which we draw the perpendicular $EH$ to the diameter $FG$ whose endpoint $E$ is on the circumference and the other endpoint $H$ is on line $BC$. The segments $EH$ and $CH$ will represent the two numbers we are looking for; so $CH$ will be $x$ and $EH$ will be $y$, as $x = \dfrac{ls}{2r} + w$ and $y = \dfrac{ls}{2r} + z$. Because $BH$, or $AD$, represents $w$ and $DE$, or $DI$, represents $z$, so that its square $z^2$ equals the Rectangle $FDG$ [74], whose value is $\dfrac{l^2 s^2}{2r^2} - w^2$ as a consequence of $FD = \sqrt{\dfrac{l^2 s^2}{2r^2}} - w$ and of $DG = \sqrt{\dfrac{l^2 s^2}{2r^2}} + w$, as it is easy to prove[75].

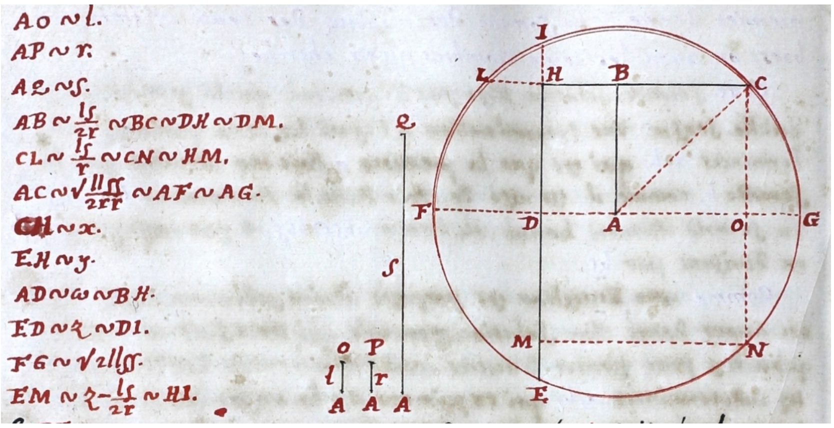



**Fig. 12** Geometric construction of question I Book II, p. 172 of volume II of the manuscript.

[74] Ozanam was referring to the area of the rectangle having sides $FD$ and $DG$. This statement is nothing other than the height theorem applied to the triangle $FIG$. Let us also keep in mind that $\sqrt{\frac{l^2 s^2}{2r^2}}$ is the radius of the circumference and has been taken $w = AD$.

[75] Our translation of pp. 171-172 of volume I of the manuscript [42] “Mais pour decrire ce cercle, faites le triangle rectangle $ABC$, dont chaque côté $AB$, $BC$, soit $\frac{ls}{2r}$, ou quatrieme proportionnel aux trois lignes $2r$, $l$, $s$, c’est à dire aux trois lignes $2AP$, $AO$, $AQ$, & decrivez du centre $A$, par le point $C$, une circonference de cercle, qui sera le Lieu qu’on cherche: & pour y determiner en lignes les deux nombres qu’on cherche, tirez le diametre $FG$, parallele à la ligne $BC$, & prenez sur ce diametre $FG$, un point quelconque $D$, par où vous tirerez au diametre $FG$, la perpendiculaire $EH$, qui se trouvera terminée en $E$, par la circonference du cercle, & en $H$, par la ligne $BC$: & les lignes $EH$, $CH$, representeront les deux nombres qu’on cherche; de sorte que $CH$ representera $x$, & $EH$ representera $y$, à cause de $x \sim \frac{ls}{2r} + w$, & de $y \sim \frac{ls}{2r} + z$. Car $BH$, ou $AD$, represente $w$, & $DE$, ou $DI$, represente $z$, afinque son quarré $zz$ soit égal au Rectangle $FDG$, qui vaut autant que $\frac{llss}{2rr} - ww$, à cause de $FD \sim \sqrt{\dfrac{llss}{2rr}} - w$ & de $DG \sim \sqrt{\dfrac{llss}{2rr}} + w$, comme il est aisé à demontrer”.

From here and writing in the left margin "Demonstration", he began the process to prove that, in fact, the indicated segments represent the numbers that are sought and wrote the following text, where his fidelity to the law of homogeneous is evidenced. Let us also bear in mind that, although in the case of the unknown $x$, or the given parameters, his notation for the square is $xx$, nevertheless when expressing the length of a segment, Ozanam uses a cosist notation; for intance, for the square of $EH$, Ozanam writes $EHq$ however, for easier reading we will use the current notation $EH^2$.

Let us take a closer look at Ozanam's demonstration to show his style of work. We show the generalization of the result and the locus of points that solve the problem by displaying its graph that we have already reproduced in Figure 12.

> *To show* that the two numbers represented by the lines $EH$, $CH$ satisfy the question, that is, the sum $EH + CH$, multiplied by the unit $AO$ [notice his concern about the law of homogeneous], namely $AO \cdot EH + AO \cdot CH$, is to the sum $EH^2 + CH^2$ of its squares, as $AP$ is to $AQ$, the lines $EH$, $CH$ are extended to the circumference in $I$ and in $L$ and we draw through $C$ the line $CN$ parallel to $EH$ that meets the circumference at $N$, through which we draw the line $MN$ parallel to the diameter $FG$[76].

Ozanam then wrote that "having this preparation done" he would go on to perform the demonstration he wanted. We schematically reproduce the steps of his demonstration and continue using current notation for ease of understanding.

First, he claims it is true that $HI = EH - CL$.

Indeed, if we remove the lines, also equal, $OC = ON$, or $DH = DM$ from the two equal lines $DE = DI$, then $HI = EM$ remains.

As $EM = EH - HM$, we will have $HI = EH - HM$ and as $HM = 2DH$, we have $HI = EH - 2DH$; furthermore, since $DH = AB$ and $AB = BC$, then $HI = EH - 2BC$ and finally, as $2BC = CL$, we conclude that $HI = EH - CL$, as claimed.

Secondly, he notices that twice the rectangle of sides $AB$, $AP$ is equal to the rectangle of sides $AO$, $AQ$, taking into account that Ozanam was referring to the area of both rectangles.

Indeed, as the four lines $2AP$, $AO$, $AQ$, $AB$ are proportional by the construction itself (remember that $AB = \frac{ls}{2r}$ is the fourth proportional of $2AP = 2r$, $AO = l$ and $AQ = s$), we have $2AB \cdot AP = AO \cdot AQ$, as he wanted to prove.

[76] Our translation of p. 172 of volume I of the manuscript [42] "Pour demontrer que les deux nombres representez par les lignes $EH$, $CH$, satisfont à la question, c'est à dire que leur somme $EH + CH$, multipliée par l'unité $AO$, savoir $AOEH + AOCH$, est à la somme $EHq + CHq$ de leurs quarrez, comme $AP$, est à $AQ$, prolongez les lignes $EH$, $CH$, jusqu'à la circonference du cercle en $I$, & en $L$, & tirez par le point $C$, à la ligne $EH$, la parallele $CN$, qui sera terminée par la circonference du cercle en $N$, par où vous tirerez la droite $MN$, parallele au diametre $FG$".

Having seen the above, since $HI = EH - CL$, the following analogy (proportion) can be established:

$$\frac{LH}{HI} = \frac{LH}{EH - CL}.$$

Ozanam continued:

> [...] if instead of the first two terms $LH, HI$, we put $EH, CH$, which are in the same ratio due to the nature of the circle[77], we will have another analogy $EH, CH :: LH, EH - CL$ and consequently the equality $EH^2 - CL \cdot EH = CH \cdot LH$ or what is the same $EH^2 - CL \cdot EH = CL \cdot CH - CH^2$ since $LH = CL - CH$. By antithesis [transposing terms] we obtain $CL \cdot EH + CL \cdot CH = EH^2 + CH^2$ or $2AB \cdot EH + 2AB \cdot CH = EH^2 + CH^2$, since $CL = 2AB$. If we give $AP$ [multiply by the magnitude $AP$] as the common height to each plane, we will have
>
> $$2AB \cdot AP \cdot EH + 2AB \cdot AP \cdot CH = AP \cdot EH^2 + AP \cdot CH^2$$
>
> and if instead of the plane $2AB \cdot AP$, we put the plane $AO \cdot AQ$, which we have proven to be the same, we will have this last equation
>
> $$AO \cdot AQ \cdot EH + AO \cdot AQ \cdot CH = AP \cdot EH^2 + AP \cdot CH^2$$
>
> and consequently the analogy [proportion]
>
> $$\frac{AO \cdot EH + AO \cdot CH}{EH^2 + CH^2} = \frac{AP}{AQ}$$
>
> as we wanted to prove[78].

In this form, Ozanam finished the matter of finding geometrically the numbers sought verifying the statement of the initial question and, in addition, ended the treatment of this question by expounding the following reflection about the law of homogeneity and the "nature" of the solution, one more sample of Ozanam's detailed and meticulous spirit, and his expertise in algebra and geometry; let us observe how he distinguishes and, at the same time, relates algebra and geometry with the obtaining of general solutions:

---

[77] Ozanam said that the equality of the proportion is in the nature of the circle since the two chords $LC$ and $EI$ intersect at the point $H$. The proportion is given by the well-known intersecting chords theorem.

[78] Our translation of pp. 172 and 175 of volume I of the manuscript [42] "[...]si à la place des deux premiers termes $LH$, $HI$, on met les deux $EH$, $CH$, qui sont en même raison, par la nature du cercle, on aura cette autre analogie, $EH, CH :: LH, EH - CL$, & par consequent cette égalité, $EHq - CL\,EH \sim CH\,LH$ ou $EHq - CL\,EH \sim CL\,CH - CHq$, à cause de $LH \sim CL - CH$, & par l'antitheze on aura celle-cy, $CL\,EH + CL\,CH \sim EHq + CHq$, ou $2AB\,EH + 2AB\,CH \sim EHq + CHq$, à cause de $CL \sim 2AB$, & si on donne à chaque Plan la hauteur commune $AP$, on aura $2AB\,AP\,EH + 2AB\,AP\,CH \sim AP\,EHq + AP\,CHq$, & si à la place du Plan $2AB\,AP$, on met le Plan $AO\,AQ$, qui luy a été demontre égal, on aura cette derniere equation $AO\,AQ\,EH + AO\,AQ\,CH \sim AP\,EHq + AP\,CHq$, & par consequent cette analogie, $AO\,EH + AO\,CH$, $EHq + CHq :: AP$, $AQ$. Ce qu'il faloit demontrer".

> [...]similarly to many other questions in the previous book, we have borrowed the unit $l$ to preserve the law of homogeneous, but we cannot take the unit $l$, and solve the question more elegantly, without having the need of squaring any power, in order to have a rational solution: that is, taking for the two numbers that are sought a point of view more in line with the nature of the question. Since when we put the two letters $x$, $y$ for the two numbers we are looking for, as these letters $x$, $y$ can represent lines and numbers and cannot be applied to lines without borrowing the unit $l$, then the position will be more natural and completely in accordance with the nature of the problem by taking two fractions, such as $\dfrac{a, b}{z}$ for the two numbers searched, for thus, they cannot represent lines, and then we will have according to the conditions of the question, the analogy
>
> $$\frac{az + bz}{a^2 + b^2} = \frac{r}{s}$$
>
> and consequently the following constitutive equation,
>
> $$asz + bsz = a^2r + b^2r,$$
>
> where we will get $z = \dfrac{a^2r + b^2r}{as + bs}$ and the two numbers being searched will be found in the same way as before.
> It is easy to see that in this way a condition is added to the question, which is to give a ratio at will, expressed here by the letters $a$, $b$ which we can assign whatever value we want[79].

So, Ozanam concluded the treatment of problem number I of Book II. With this problem, we have been able to follow his style of reasoning and the richness of his approaches, both geometric and algebraic. He also added two pages with two drawings of a circle in which he marked numerous examples of how to take the points and how to calculate the proportional quarter to which he had alluded. In Figure 13 we show the drawing corresponding to page 173 of volume I of the manuscript.

[79] Our translation of p. 175 of volume I of the manuscript [42] "[...]comme dans beaucoup d'autres Questions du livre precedent, emprunté l'unité $l$, pour observer la loy des Homogenes; mais on se peut passer d'emprunter l'unité $l$, & resoudre la Question plus élegantement, sans qu'il soit besoin d'égaler au quarré aucune Puissance, pour avoir une solution rationelle: savoir en faisant pour les deux nombres qu'on cherche, une position qui soit plus conforme à la nature de la Question. Car quand on a mis les deux lettres $x$, $y$, pour les deux nombres qu'on demande, comme ces deux lettres $x$, $y$, peuvent representer aussybien des lignes que des nombres, & que cette Question ne se peut point apliquer aux lignes sans emprunter l'unité, la position sera plus naturelle, & toutafait conforme à la nature du Probleme, en mettant deux fractions, comme par exemple $\frac{a,b}{z}$, pour les deux nombres qu'on cherche, car ainsy ils ne peuvent pas representer des lignes, & alors on aura selon la condition de la Question, cette analogie, $az + bz, aa + bb :: r, s$, & par consequent cette Equation constitutive, $asz + bsz \sim aar + bbr$, dans laquelle on trouvera $z \sim \frac{aar+bbr}{as+bs}$, & les deux nombres qu'on cherche, se trouveront les mêmes qu'auparavant. On voit aisément

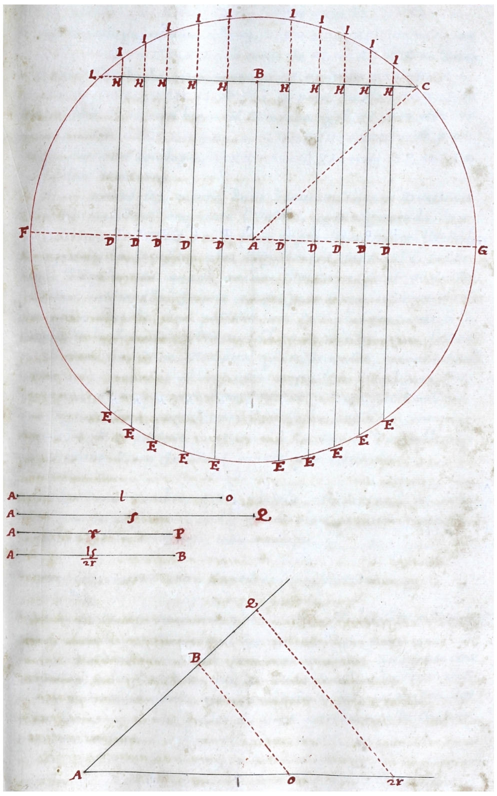


**Fig. 13** Geometric construction with examples of question I, Book II, p. 173 of the manuscript.

## 7.4 Book I. Question XIV

The following example corresponds to question XIV of Book I. We will pay special attention to the geometrical consequences that, as we will see, are very rich and show us, once again, the power of his algebraic treatment and the relationship he establishes between algebra and geometry.

*Question XIV*
*To find two numbers whose sum is in a given ratio to their product*[80].

Before addressing the resolution of this question, Ozanam reflected on the nature of the problem in which his respect for the law of homogeneous is once again evidenced. Ozanam points out once more the relation with the geometric construction from the arithmetic solution.

que par cette maniere on ajoute une condition à la Question, qui est qu'on leur donne une raison à volonté, laquelle est icy exprimée para la raison des deux lettres $a$, $b$, ausquelles on peut attribuer telle valeur que l'on voudra".

[80] Our translation of p.32 of volume I of the manuscript [42] "Trouver deux nombres, dont la somme soit à leur produit en raison donnée".

> *Since* it is a matter of comparing a simple number with a plane number, which is contrary to the law of homogeneous, we will consider the simple number as plane by multiplying it by the unit $l$, which does not change anything, to respect the law of the homogeneous. We will do this whenever we need to compare two heterogeneous quantities to carry out a geometric construction from the indefinite solution of the question [...].[81]

Ozanam explained that he had to find two numbers $x$ and $y$ such that $lx + ly$ is to their product $xy$ as $1 = a$ is to $3 = b$ (again the values $a = 1$ and $b = 3$ are Diophantus' originals)

$$\frac{lx + ly}{xy} = \frac{a}{b}.$$

From here, and after writing the corresponding canon, the equation $lbx + lby = axy$ was obtained and, although we will not go into the details, Ozanam solved the problem algebraically and found a general solution, so that if $x$ is one of the numbers, then $y = \dfrac{lbx}{ax - lb}$. Before continuing, Ozanam made the following observation about the result, because he did not admit the possibility of non-positive results:

> [...] this first number $x$ must be greater than $\dfrac{lb}{a}$ or $\dfrac{b}{a}$. Due to $ax - lb$ is the denominator of the second number found $\dfrac{lbx}{ax - lb}$, we have $ax > lb$: thus, dividing by $a$ we get $x > \dfrac{lb}{a}$[...][82].

From here, Ozanam dedicated himself to obtaining a variety of geometric consequences. First, he said that if we give to the letter $l$ any value other than 1, the question could be solved more generally with the equations:

$$lbx + lby = axy, \text{ or } \frac{lbx}{a} = xy - \frac{lby}{a} \tag{7}$$

which Ozanam identified with a "locus", specifically a hyperbola:

> *If* instead of assigning the unit to the letter $l$, another number is given, the Question will be solved in a more general way [...] and as there is an indeterminate letter $x$ here, this shows us that this Question is a

[81] Our translation of p.32 of volume I of the manuscript [42] "Parcequ'il s'agit icy de comparer un nombre simple avec un nombre plan, ce qui est contre la loy des Homogenes, nous concevrons ce nombre simple comme plan, en le multipliant par l'unité $l$, qui ne le changera point, pour observer la loy des Homogenes, ce que nous ferons toujours, quand il faudra comparer ensemble deux grandeurs heterogenes, pour tirer de la solution indefinie de la Question une construction geometrique[...]".

[82] Our translation of p.33 of volume I of the manuscript [42] "[...]ce premier nombre $x$ doit être plus grand que $\frac{lb}{a}$, ou $\frac{b}{a}$. Car dans le denominateur $ax - lb$ du second nombre trouvé $\frac{lbx}{ax-lb}$, on a $ax \oplus lb$: c'est pourquoy en divisant par $a$, on aura $x \oplus \frac{lb}{a}$[...]".

> *Locus*, namely, a *Locus in the Hyperbola between its asymptotes*, as seen by the preceding equation [equation (7)][...].[83]

To get the "reduced locus", he first made the change $x = z + \frac{lb}{a}$ and secondly $y = w + \frac{lb}{a}$ in order to obtain, by substituting in the previous equation (7), the reduced expression of the hyperbola $\frac{l^2b^2}{a^2} = zw$.

To describe the hyperbola, he made the following geometric construction that can be seen in Figure 14. He took $a = 1 = AO$, $b = 3 = AQ$, $l = 1 = AP$. Having fixed this, he took the angle "at will" (he takes it straight) $BAC$ determined by the asymptotes of the hyperbola $AC$ and $AB$. Then he took a point $D$ on the line $AC$, in such a way that $AD = \frac{lb}{a}$ or the fourth proportional of the lines $AO = a$, $AQ = b$ and $AP = l$, as seen in Figure 14. Through $D$ the line $DE$ of the same length as $AD$ and parallel to the asymptote $AB$ was drawn (the point $E$ is the vertex of the hyperbola). Then he drew the hyperbola of center $A$ that passes through $E$ and has $AB$ and $AC$ as asymptotes, which, he claimed, was the one that corresponded to the previous reduced equation $\frac{l^2b^2}{a^2} = zw$, and "accordingly" to the first equation $lbx + lby = axy$.

Thus if we take a point $N$ on $AC$ "in the direction" $C$, the line $MN$ parallel to the other asymptote $AB$ with $M$ on the hyperbola, we have $AN = z$ and $MN = w$, because the two lines $AD$, $DE$ are equal to $\frac{lb}{a}$ due to the way they have been constructed. But Ozanam had to find the required numbers $x$ and $y$ as lines, and for this purpose he wrote:

> *But* in order to find as lines the two numbers $x$, $y$, and firstly $y$, it is necessary to add to the line $MN = w$, the line $AD = \frac{lb}{a}$, since $y - \frac{lb}{a} = w$, or $y = \frac{lb}{a} + w$. For this purpose we extend the asymptote $AB$ to $H$, taking $AH = \frac{lb}{a}$, and at this point we draw the line $HI$, parallel to the asymptote $AC$, that meets the extension of $MN$ at the point $L$ and that provides $ML = y$.
> *To* find the other number $x$, the line $AD = \frac{lb}{a}$ is added to the line $AN$, or $HL = z$, since $x = z + \frac{lb}{a}$, which will be done by extending the line $HL$ up to $K$, taking $HK = \frac{lb}{a}$, to have $KL = x$. Thus the two numbers searched will be represented by the lines $KL$, $LM$ that can be found

[83] Our translation of p. 33 of volume I of the manuscript [42] "Si au lieu d'atribuer l'unité à la lettre $l$, on luy attribue tel autre nombre que l'on voudra, la Question sera resolue plus generalement[...] & comme il reste icy une lettre indeterminée $x$, cela fait connoitre que cette Question est un Lieu, geometrique, savoir un Lieu à l'Hyperbole entre ses asymptotes, comme l'on connoitra par l'Equation precedente [equation (7)][...]".

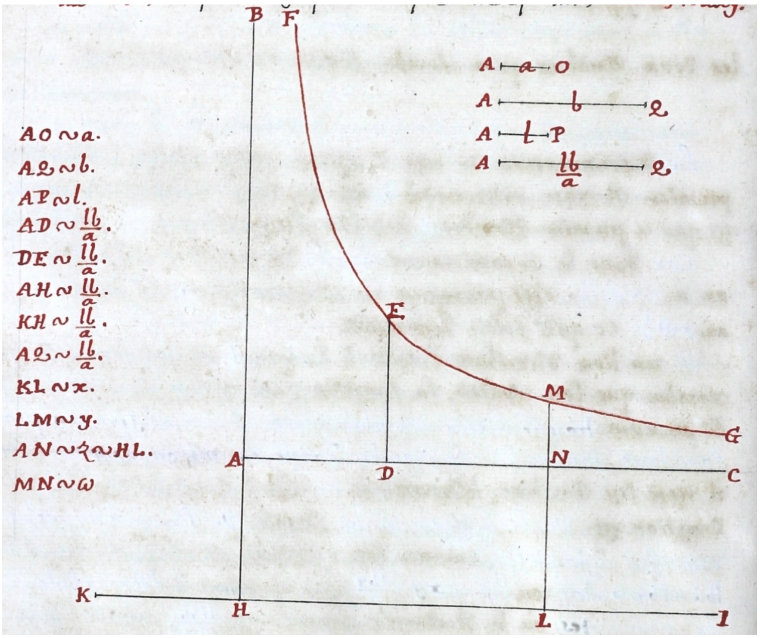



**Fig. 14** Hyperbola corresponding to question XIV of Book I, p.34 of volume I of the manuscript.

> in an infinity of different ways, taking the point $L$, indefinitely from $H$, towards $I$[...][84].

Once this "preparation" was done, he continued to demonstrate with the detailed and meticulous style that we have shown that, indeed, the numbers $x$ and $y$ that coincided with the segments $KL$ and $ML$, respectively, satisfy question XIV raised at the beginning, that is

$$\frac{KL \cdot AP + LM \cdot AP}{KLM} = \frac{AO}{AQ},$$

where $KLM$ is the rectangle formed by $KL$ y $LM$.

From a geometric point of view, Ozanam's task on this problem did not end here. He kept looking for more general solutions and relating his algebraic

[84] Our translation of p. 35 of volume I of the manuscript [42] "Mais pour trouver en lignes les deux nombres $x$, $y$, & premierement le nombre $y$, il faut ajouter à la ligne $MN \sim w$, la ligne $AD \sim \frac{lb}{a}$, à cause de $y - \frac{lb}{a} \sim w$, ou de $y \sim \frac{lb}{a} + w$. Pour cette fin, on prolongera l'asymptote $AB$, en $H$, en faisant $AH \sim \frac{lb}{a}$, & par le point on tirera à l'autre asymptote $AC$, la parallele indefinie $HI$, qui rencontrera la ligne $MN$, prolongée au point $L$, & donnera $ML \sim y$.
Pour trouver l'autre nombre $x$, on doit ajouter à la ligne $AN$, ou $HL \sim z$, la ligne $AD \sim \frac{lb}{a}$, à cause de $x \sim z + \frac{lb}{a}$, ce qui se fera en prolongeant la ligne $HL$, en $K$, & en faisant $HK \sim \frac{lb}{a}$, pour avoir $KL \sim x$. Ainsy les deux nombres qu'on cherche, seront representez par les deux lignes $KL$, $LM$, que l'on peut trouver en une infinité de manieres differentes, en prenant le point $L$, indefiniment depuis $H$, vers $I$[...]".

results to geometry. In this case, he kept aiming to eliminate the need to resort to the use of the $l$ unit and said:

> *In* order not to be obliged to resort to unity, and to have a totally undefined solution, that is, without any determination, we take
>
> $$\frac{x}{z}, \frac{y}{z}$$
>
> for the two numbers we are looking for and according to the condition of the question, we will have the proportion[85]
>
> $$\frac{\frac{x+y}{z}}{\frac{xy}{z^2}} = \frac{a}{b}.$$

This allowed him to compare two numbers of the same dimension and not a simple number with a plane number as in the previous case, and he obtained the equation

$$\frac{bx+by}{z} = \frac{axy}{z^2} \tag{8}$$

from where he obtained, after solving $z$ in the previous equation, the searched numbers

$$\frac{bx^2+bxy}{axy}, \frac{by^2+bxy}{axy}.$$

After writing the corresponding canon for this solution, where he described the previous numbers rhetorically, Ozanam observed that, of Question XIV, Book I, as this new solution depended on two indeterminate quantities $x$, $y$, such a solution was more general than the first in which knowing a number, say $x$, the second was obtained as $y = \frac{lbx}{ax-lb}$. He also transferred and related this dependence on two variables to its, let us say, geometric parallelism: it was a locus on a surface.

> [...] the question thus solved is a locus on the plane surface, namely, a part of a given hyperbola [a plane figure determined by the hyperbola] whose axis [distance between the vertices] is equal to the first given number $a$, and with parameter [distance between the focal points] to the second number given $b$[...][86]

Again he illustrated his geometric construction and the subsequent demonstration, with a graph of the hyperbola made with precision and an impeccable stroke, as shown in Figure 15, very useful to visualize Ozanam's description

[85] Our translation of p. 36 of volume I of the manuscript [42] "Pour n'être pas obligé d'emprunter l'unité, & pour avoir une solution toutafait indefinie, c'est à dire sans aucune determination, mettez $\frac{x,y}{z}$, pour les deux nombres qu'on cherche, & selon la condition de la Question, on aura cette analogie, $\frac{x+y}{z}, \frac{xy}{zz} :: a, b$".

[86] Our translation of p.36 of volume I of the manuscript [42] "[...] la Question ainsy resolue est un Lieu à la surface plane, savoir une partie d'une Hyperbole donnée, dont l'axe est égal au premier nombre donné $a$, & son parametre au second nombre donné $b$[...]".

in which we can observe, once again, the precision of his language and the elegance of his constructions. Let us also observe that in the right margin of the figure, as in the constructions that we have already seen, he describes the values representing the taken segments.

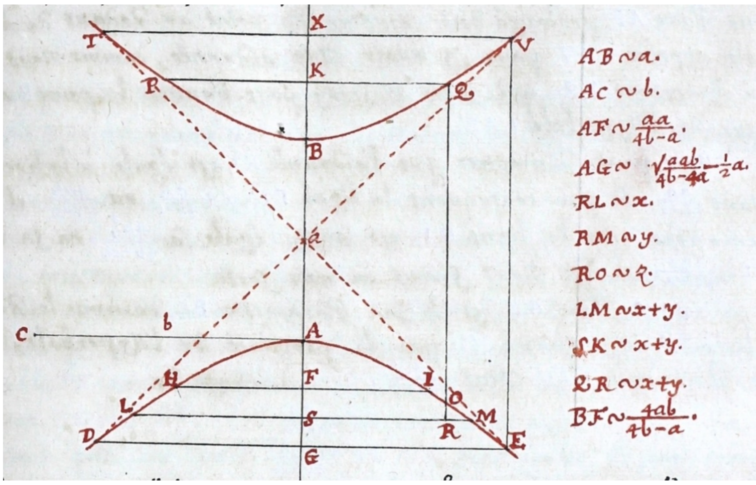



**Fig. 15** Second hyperbola in question XIV of Book I, p. 37.

*Geometric construction.* [Ozanam writes the words Geometric construction in the margin of the text] *We draw* the hyperbola $DAE$ whose axis $AB$ is equal to the first given number $a$, and its parameter $AC$ to the second given number $b$. On the $AB$ axis extended towards $A$, we take the line $AF = \dfrac{a^2}{4b-a}$ and the line $AG = \sqrt{\dfrac{a^2b}{4b-4a}} - \dfrac{1}{2}a$. We draw by the points $F$, $G$, on the $AB$ axis, the ordinates, $HFI$, $DGE$ [segments perpendicular to the $AB$ axis with extremes on the hyperbola], which delimit the local surface $DEIH$ in which we will determine an infinity of different ways of the three numbers $x$, $y$, $z$.
Taking, at our discretion, a point $S$ between the two $F$, $G$, we draw the segment $LSM$ parallel to the two $DE$, $HI$, and taking on the axis $AB$, extended towards $B$, the line $SK$, equal to the ordinate $LM$, with the point $K$ on the axis $AB$, and through this point $K$ the ordinate $PKQ$ is drawn with extremes at the points $P$, $Q$, from the opposite hyperbola $TBV$. Finally, we draw the line $QOR$ through $Q$ parallel to the $AB$ axis, which meets the hyperbola $DAE$ at point $O$, and the ordinate $LM$, below the hyperbola, at point $R$. Then the three numbers $x$, $y$, $z$ will be represented by the three lines $RL$, $RM$, $RO$[87].

[87] Our translation of pp. 36 and 37 of volume I of the manuscript [42] “Ayant decrit l’Hyperbole $DAE$, dont l’axe $AB$, soit égal au premier nombre donné $a$, & le Parametre

Next Ozanam made a complete, detailed and also long demonstration of what he had affirmed in the construction that we have just seen, namely the points $x$, $y$, $z$ determined by the lines $RL$, $RM$, $RO$, repectively, verify $\dfrac{bx+by}{z} = \dfrac{axy}{z^2}$, the equation (8) constructed at the beginning of this new solution to the problem, as well as the values he gives to the lines $BF$ or $AG$. For this, he relied on Proposition XIII on the hyperbola that Ozanam included in his work [38] on Conic Sections, and that, in current notation, states $\dfrac{QR \cdot RO}{LR \cdot RM} = \dfrac{a}{b}$. Furthermore, he proved that $HI = AF$ and that $DE = GX$.

Ozanam had not yet finished his task on question XIV of Book I, he continued saying that instead of a two-dimensional part of a hyperbola, as in this case, an ellipse could be obtained, changing some values from those he had taken in the case that we have just seen. The description is similar to that of the hyperbola and the graphic, once again impeccable, precise and beautiful, see Figure 16.

## 7.5 The Book VI. Question XVIII

When describing in Section 4 the content of the manuscript and the way in which each of the parts and books that comprise it are written and numbered, we have already said that, from Book III on, the way of writing changes, although as can be seen in Figure 17 the hand that writes all the texts is the same. This Figure 17, corresponds to Book VI, question XVIII. We have already said in the aforementioned section, that in this book, except for one, all the questions are dedicated to finding right triangles, which is none other than finding numbers with Pythagorean relationships. It is mentioning how Ozanam addressed and solved this question:

> Question XVIII
>
> To find a right triangle where the segment, which divides one of the two acute angles into two equal parts, is expressed by a rational number [88].

---

$BC$, au second nombre donné $b$, prenez sur l'axe $AB$, prolongé vers $A$, la ligne $AF \sim \frac{aa}{4b-a}$, & la ligne $AG \sim \sqrt{\frac{aab}{4b-4a}} - \frac{1}{2}a$, & tirez par les points $F$, $G$, à l'axe $AB$, les ordonnées $HFI$, $DGE$, qui termineront la surface locale $DEIH$, dans laquelle on determinera en une infinité de manieres differentes, les trois nombres indeterminez $x$, $y$, $z$, en cette sorte.
Ayant tiré par le point $S$, pris à discretion entre les deux $F$, $G$ la droite $LSM$, parallele aux deux $DE$, $HI$, & ayant pris sur l'axe $AB$ prolongé vers $B$, la ligne $SK$, égal à l'ordonnée $LM$, tirez par le point $K$, à l'axe $AB$, l'ordonnée $PKQ$, qui sera terminée aux poins $P$, $Q$, par l'Hyperbole opposée $TBV$, enfin tirez par le point $Q$, à l'axe $AB$, la parallele $QOR$, qui rencontrera l'Hyperbole $DAE$, au point $O$, & l'ordonnée $LM$, au dedans de l'Hyperbole au point $R$, & alors les trois nombres $x$, $y$, $z$, seront representez par les trois lignes $RL$, $RM$, $RO$".

[88] Our translation of p. 409 of volume II of the manuscript whose original text is shown in Figure 17.

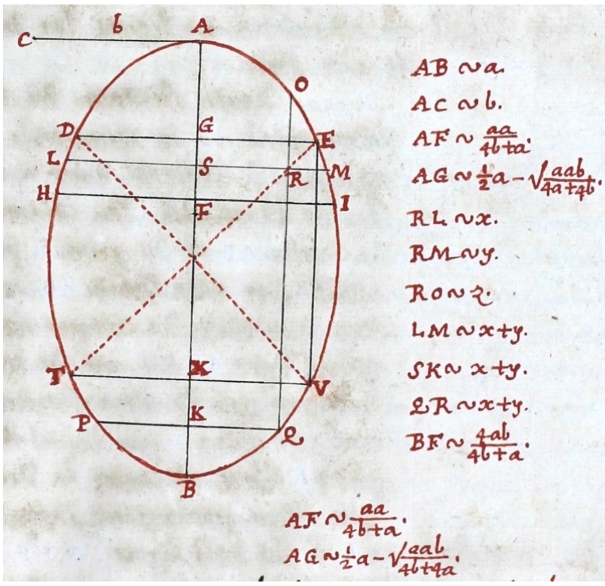
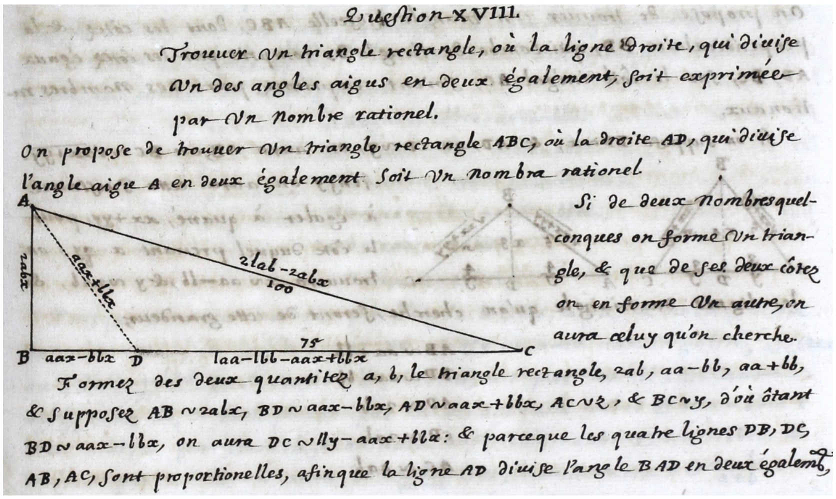



**Fig. 16** Ellipse corresponding to question XIV of Book I, p. 39.



Question XVIII.

Trouuer vn triangle rectangle, où la ligne droite, qui diuise vn des angles aigus en deux également, soit exprimée par vn nombre rationel.

On propose de trouuer vn triangle rectangle ABC, où la droite AD, qui diuise l'angle aigu A en deux également soit vn nombre rationel.

Si de deux nombres quelconques on forme vn triangle, & que de ses deux côtez on en forme vn autre, on aura celuy qu'on cherche.

Formez des deux quantitez a, b, le triangle rectangle, 2ab, aa−bb, aa+bb, & supposez AB ∼ 2abx, BD ∼ aax−bbx, AD ∼ aax+bbx, AC ∼ z, & BC ∼ y, d'où ôtant BD ∼ aax−bbx, on aura DC ∼ y−aax+bbx: & parceque les quatre lignes DB, DC, AB, AC, sont proportionelles, afinque la ligne AD diuise l'angle BAD en deux égalemẽt,

**Fig. 17** Beginning of Question XVIII of Book VI, page 409.

The aim is to find a right triangle $ABC$ where the line $AD$, which divides the angle $A$ into two equal parts, is a rational number (see again Figure 17). To solve the question, Ozanam said that if a triangle is formed from two numbers, considering its sides, it could form another triangle that would be the one we are looking for. For that he took two arbitrary numbers $a$ and $b$ (although Ozanam does not mention it explicitly, he considered such numbers as being integers) and affirmed that the three numbers $2ab$, $a^2-b^2$ and $a^2+b^2$ form a right triangle, since, indeed, they form a Pythagorean triple where the last of them is the hypotenuse.

Then he constructed the triangle $ABD$ in Figure 17, to obtain the new triangle $ABC$, being $AD$ the bisector angle of $A$. For this he took $AB = 2abx$, $BD = a^2x - b^2x$, $AC = z$, $BC = y$ and, since, $DC = l^2y - a^2x + b^2x$, where $x$ was a value he was looking for. He then argued that the four segments $DB$, $DC$, $AB$ and $AC$ are proportional since $AD$ is the bisector angle of $A$ (this is a known property of triangles so-called angle bisector theorem starting that $\dfrac{DB}{DC} = \dfrac{AB}{AC}$). In such a way that, in current language, he obtained the proportion $\dfrac{a^2x - b^2x}{l^2y - a^2x + b^2x} = \dfrac{2abx}{l^2z}$, from where he deduced that $z = \dfrac{2aby}{a^2 - b^2} - \dfrac{2abx}{l^2}$. From here, assuming that $y = la^2 - lb^2$ to give $z = 2lab - 2abx$ in integers, performing some algebraic operations and applying the Pythagorean relation to the sides $AC = z$, $AB = 2abx$ and $BC = y$, he arrived at the quadratic equation

$$4l^2a^2b^2 - 8la^2b^2x + 4a^2b^2x^2 = l^2a^4 - 2l^2a^2b^2 + l^2b^4 + 4a^2b^2x^2,$$

which yields $x = \dfrac{6a^2b^2 - a^4 - b^4}{8a^2b^2}$. With this value he completed the searched values. Specifically, the searched value $AD = a^2x + b^2x$ was

$$AD = \frac{5a^4b^2 + 5a^2b^4 - a^6 - b^6}{8a^2b^2}$$

or, as Ozanam said, in integers you have $AD = 5a^4b^2+5a^2b^4-a^6-b^6$ (assuming that it must be considered that the denominator is $8a^2b^2$).

Once finished, Ozanam added:

> On the occasion of this Question, we add here the following ones, where the first six are from Bachet, which he has only partially solved, except for the first three, because in the other three, he has not made the rational perpendicular. [89]

## 7.6 Questions added by Ozanam

To finish the description of the content of Diophantus, let us remark that Ozanam added numerous problems to many of Diophantus' original questions. Bachet had already made comments, generalizations, and added new questions to his Diophantus. But Ozanam went much further, adding considerably more new questions and, as Leibniz pointed out (see Section 3.1), solved them with a symbology and algebraic methods close to the current ones. In addition to obtaining general solutions, he provided a reflection on such solutions. He related

[89] Our translation of p. 410 of volume II of the manuscript [42] "A l'ocassion de cette Question, nous ajouterons icy les suivantes, dont les six premieres sont de Bachet, qu'il n'a reslues qu'en partie, excepté les trois premieres, parce que dans les trois autres, il n'a point fait la perpendiculaire rationelle".

the procedure and algebraic solutions to geometry, obtaining his "geometric determinations".

For example, question XXXIII of Book I has 24 questions added and numbered (pages 113 to 150 of the first volume of the manuscript); and even these questions incorporate others, such as number 24 which, in turn, has seven others added. Thus, Question XXXIII says:

> *Question XXXIII*
>
> *To find two numbers whose difference and whose product are equal to two given numbers: $x - y = a$, $xy = b$*[90].

Ozanam added a first problem (p. 114) which consisted in finding two numbers such that their product is the product of two given numbers and the sum of their squares is the product of two other given numbers: $xy = ab$; $x^2 + y^2 = cd$; and another problem (the second, p. 116) in which the difference is a given number and the sum of its squares is the product of two given numbers: $x - y = a$, $x^2 + y^2 = bc$; and so on until the 24 questions added in this case.

## 8 The manuscript after Ozanam's death

### 8.1 The "Eloge" of Fontenelle

After Ozanam's death, Bernard le Bovier de Fontenelle (1657-1757), permanent secretary of the Académie Royale des Sciences from 1697, published a panegyric in 1719 under the title *Eloge de M. Ozanam* [18], in which he collected biographical notes on Jacques Ozanam, as well as his main works and the most relevant aspects of his life. As a curiosity, it should be noted that, although the "Eulogy" placed Ozanam's death in 1717 and appeared published among the relevant events of that year, Ozanam actually died in 1718[91].

When Fontenelle wrote about Ozanam's main works, he referred only five of them among the more than 25 works and added a sixth work, precisely his Diophantus, which was never published and said that it was a manuscript located at the library by "M. le Chancelier", in clear reference to the former Chancellor of France, Henry François D'Aguesseau (1668-1751) [92].

> His main works are [...], a handwritten Diophantus that is in the hands of M. le Chancelier[...][93]

The relevance given at that time to the Diophantus of Ozanam is evident when an unpublished work was cited by Fontenelle among the most important

[90] Our translation of p. 113 of volume I of the manuscript [42] "Trouver deux nombres, dont la difference & le produit, soient égaux à des nombres donnez".

[91] C. Càndito states in [9] that it is a mistake.

[92] C. Càndito offers information on this in [9].

[93] Our translation [18], p. 90. "Ses principaux Ouvrages sont[...], un Diophante manuscrit qui est entre les mains de M. le Chancelier[...]".

of such a prolific author as Ozanam was. It was not the only time this happened with his Diophantus, already Leibniz and Father De Billy, as well as other authors as we will see, had appreciated its relevant and innovative nature.

## 8.2 *Histoire des mathématiques* of Montucla

The French J.E. Montucla (1725-1799) wrote a history of mathematics [36] that collected all mathematical knowledge up to his time. The work was divided into two volumes and the first edition was published in 1758. Volume number one dealt with the classic works, including the Arithmetic of Diophantus of Alexandria. Montucla wrote about what some French mathematicians had done or collected from Diophantus' work and highlighted Bachet's translation or that of Father De Billy, referring to the latter as a knowledgeable mathematician of Diophantus' Arithmetic. He also referred to Ozanam's work that De Billy highlighted with praise:

> M. Ozanam was also devoting himself to this work [the Arithmetic of Diophantus]; and in Father De Billy's judgment, he had made an extraordinary effort. He had written a treatise on the analysis of Diophantus, which exists only in manuscript, and which M. Daguesseau possessed in 1717, according to what appears in the History of the Academy of Sciences in the eulogy [the *Eloge* of Fontenelle] of this author. This work would have greatly contributed to his reputation, not as an ordinary mathematician, but as skillful and above those around him[94].

Montucla was referring to the *Eloge* of Fontenelle that we have seen before and, again, he emphasized the importance that the Diophantus of Ozanam must have at that time. In this case he relied on the opinion of another important mathematician, Father De Billy. Furthermore, he emphasized that it was this unpublished work that made Ozanam a skilled and unusual mathematician.

Montucla reissued in 1758 Ozanam's work *Récréations Mathématiques et Physiques* [45] published in 1694. In the reissue [46], Montucla wrote on the cover "New edition, totally revised and considerably increased" [95] and in page 59, wrote in terms similar to those we have just seen from his history of mathematics, that Ozanam had done a work on the Arithmetic of Diophantus that had never been published.

[94] Our translation [36] p. 321 "M. Ozanam se jettoit vers le même temps dans cette carriere; et au jugement du P. De Billi, il y prenoit un effor extraordinaire. Il avoit écrit un Traité de l'analyse de Diophante, qui n'existe qu'en manuscrit, et que possédoit M. Daguesseau en 1717, suivant ce que nous apprend l'Historien de l'Académie des Sciences dans l'éloge de cet Auteur. Cet ouvrage eût contribué davantage à sa réputation, non auprès du vulgaire des Mathématiciens, mais auprès des habiles gens, que la plûpart de ceux qu'on a de lui".

[95] Our translation of the cover of [46] "Nouvelle Edition, totalement refondue & considérablement augmentée".

### 8.3 The library of the Chancelier D'Aguesseau

Indeed, the Diophantus handwritten by Ozanam was in the library of M. le Chancelier D'Aguesseau[96], as Fontenelle had written in his Eloge to Ozanam and as later Montucla pointed out. This library was put up for sale in 1785 and was so voluminous and important that for this purpose a catalog was published [2] where the works it contained were collected in detail. Among them, the Diophantus appeared listed on page 165 with the number 2530:

> 2530 The six books of the Arithmetic of Diophantus of Alexandria, increased and reduced to specious by Ozanam.
> -Treatise of singles, doubles and triple equalities.
> -Treatise of loci for the solution of plane problems.
> -Treatise of minimums and maximums, for the same; two vol. Folio. Mss. that seem copied from the author's hand. This work seems important to us, and has not seen the light; but we do not know whether the three Treatises of Ozanam, indicated above, are not intended to fill a gap that is between pages 45 to 149 of the second volume, this Manuscript is imperfect: the leaves have been cut and the third issue of the third Book of Diophantus, or what serves to explain them. It is barely likely that other than Ozanam himself has cut these sheets, and he has undoubtedly deviated from completing this Manuscript only for some new work, or for a reason unknown to us[97].

Again, when referring to the manuscript in the library, the author of the catalog wrote that it gave him the impression that it was an important work but, perhaps due to his lack of mathematical knowledge and the absence of some pages, he did not say nothing else.

[96] Fontenelle explained in his *Eloge* [18] how the relationship between Ozanam and D'Aguesseau took place: Ozanam taught mathematics in Lyon to two foreigners who, at some point, told him that they could not return to Paris because they had not received the letters with the necessary credit. Ozanam selflessly lent them money without even demanding a receipt. These two persons must have been related to the father of the Chancelier of Paris, the noble D'Aguesseau, who, seeing Ozanam's generosity, ordered him to be brought to Paris, promising to help him make his mathematical skills known.

[97] Our translation [2] "2530 Les six livres de l'Arithmétique de Diophante d'Alexandrie, augmentés et réduits à la Spécieuse par Ozanam.
-Traité des simples, des doubles et des triples égalités.
-Traité des lieux géometriques pour la solution del Problêmes Plans.
-Traité de minimis et maximis, par le même; deux vol. in-fol. Mss. qui paroissent copiés au net de la main de l'Auteur. Cet Ouvrage nous semble important, et il n'a point vu le jour; mais nous ne saurions déguiser que si les trois Traités d'Ozanam, indiqués ci-dessu, ne sont pas destinés à remplir une lacune qui se trouve entre les pages 45 à 149 du second volume, ce Manuscrit seroit imparfait: les feuilles en ont été coupés et la trosiéme question du troisieme Livre de Diophante, ou ce qui sert à les expliquer. Il n'est guere probable qu'un autre qu'Ozanam lui-même ait arraché ces feuilles, et il n'a sans doute été détourné de completer ce Manuscrit que par quelque nouveau travail, ou par un motif qui nous est inconnu".

### 8.4 The missing 'Diophantus'

From the reference to the library sales catalog, the manuscript was lost. Cassinet [10] has written that the Belgian Paul ver Eecke (1867-1959), historian of mathematics, published in 1926 the first edition of his annotated French translation of Diophantus' Arithmetic and in 1959 a second edition was published [54]. In the introduction to this publication, ver Eecke reviewed the different editions and translations that he knew of Diophantus' work and referred to the manuscript, saying that it had been lost since its existence was referred in the sales catalog of the Chancelier library under the following terms:

> The Ozanam manuscript does not seem to have fallen into pious hands, since the investigation that has been carried out several times to find it in some private or public collection has never been successful[98].

Ver Eecke also related that the French mathematician E. Prouhet published a note in the journal *Intermédiaire des Chercheurs et des Curieux* [47] where he explained how he was able to buy a manuscript with an extraordinary calligraphy, which he supposed was written by Ozanam himself and that it only contained three propositions from the third Book of Diophantus and also that he had received a letter from the Italian nobleman Prince Boncompagni announcing to him the unsuccessfully search for the manuscript.

The manuscript remained unaccounted for until, as we have already said, Cassinet found it in Turin. His discovery has enabled us to complete the work he started at [10].

## 9 Conclusions

We can conclude, first of all, that Ozanam is remarkable for being an outstanding mathematician who was familiar with the mathematical works published at his time and used them in practice. He received justified praises from his contemporaries, Leibniz and De Billy, and also from Fontenelle and Montucla later on.

Ozanam used a very advanced algebraic language and he mastered it in such a way that he obtained the solutions of the problems posed by Diophantus, without apparent difficulty, and he added numerous additional questions. The effectiveness of his approach to equations and the elegance with which he obtained the solutions are evident throughout the many pages that form the manuscript.

On the other hand, the writing and drawings that illustrate the manuscript are also very noteworthy, as they were drawn in an impeccable way and without omitting any details. The order adopted in the writing and the clarity of

[98] Our translation of [54] p. LXXXV "Le manuscrit d'Ozanam ne paraît pas être tombé entre des mains pieuses, car les recherches entamées à plusieurs reprises pour la retrouver dans quelque collection privée ou publique n'ont jamais abouti".

its very prolific explanations, which we have already commented extensively, make the reading of Diophantus of Ozanam fluid and agile, as well as easily understandable, bridging the gap between the present time and the 17th century.

In addition, Ozanam's mathematical depth, probably based on the Jesuit tradition, and his mastery of algebra have also been revealed in the analysis he made of the solutions, not only because of the way he calculates them, but also because of how he studied the possible restrictions as well as possible alternatives or generalizations. Added to all of the above is his geometric ability in handling proportions, properties of conics, and in obtaining loci that he connected naturally with algebraic solutions, as we have seen in the examples of Section 7.

The identification that Ozanam used to made of equations with curves or with loci, like Descates, is very remarkable. This identification, which we have seen in the previously shown questions, includes to made changes of variables to place curves centered on the origin of coordinates or obtain what nowadays we call reduced or canonical equations of conic curves. It combines the effectiveness of algebra in obtaining solutions to geometric consequences, which emanate from a deep knowledge of both algebra and the geometry associated with equations and the plasticity and beauty of geometric constructions. In fact, Ozanam present us a very good example of mathematicians that contribute to the consolidation of algebrization of mathematics. Ozanam follows algebraic works by Viète and Descartes and use these in practice in original way.

Ozanam's text greatly improved the texts known at the time on the book of Diophantus, particularly, what was done by Viète, a mathematician who introduced the new analytical method and addressed some of Diophantus' problems. The algebraic language used by Ozanam is an adaptation of Viète's and Descartes', and allowed him to more comfortably approach the general resolution of Diophantine questions and work on his own solutions, as well as contribute with new questions to Diophantus. The notation is close to that of Descartes, and his writting preserves some vestiges of Viète, as the law of homogeneous or the cosist notation $EHq$ to represent the square of a segment.

The aforementioned algebraic language also improved the excellent Diophantus of Bachet, a reference at the time. Again the use of the new algebraic language, which Bachet did not use, allowed him to improve the results, comments, and added questions made by Bachet. Ozanam solved the problems algebraically from the beginning and therefore made a generalization of the results and added many more problems. But also, with the geometric determinations in a good part of the questions in Books I and II, he introduced a fundamental novelty connecting the algebraic problem and its solutions with the obtaining of loci. The ease with which he was obtaining the loci that responded to the algebraic solutions were surprising because of their elegance and the naturalness with which Ozanam connected algebra and geometry.

In summary, Ozanam's work scheme was novel and had great mathematical richness:

- Ozanam started with an arithmetic problem. Although Diophantus' statements are, in their origin, general, they are really solutions to specific numerical cases.
- This arithmetic problem was transformed into an algebraic problem by using equations with the Viète's method.
- Solving the equations led him to obtain general solutions.
- In many cases, he reflected and worked on such general solutions and added many additional questions that undoubtedly enriched Diophantus' legacy.
- In a good number of questions, especially in Books I and II, he obtained in a natural way what he called geometric determinations, which are nothing more than a connection between geometry and the algebraic solution of problems with loci specified with equations of curves.
- He identified in an efficient and natural way the equations obtained in the algebraic solution process with loci or curves that were associated with such equations. All this happened to the point of making changes of variable to obtain more comfortable curve equations.

Ozanam's Diophantus was not published in 1674, as Leibniz had previously announced. Not had it been published in 1687, thirteen years later, when Ozanam himself repeatedly mentioned it in his works [38], [39], [40], implying that this one would eventually be published. We have not found the reasons why it did not get published. Perhaps it was Ozanam himself who had to pay for the printing and that, given his battered economy, he was putting it off to such an extent that it never saw the light. However, after analysing the two different parts of the manuscript, which we have already described in Section 4, we could also infer that Ozanam decided to improve his work. Probably, due to some kind of unknown reasons, he could have not been able to finish that process, being this the reasosn explaining the unediting of the Diophantus.

It is thus clear that the objectives stated by Ozanam in the preface to Diophantus, which we have described and discussed in Section 6, were amply fulfilled in the manuscript text and in the mathematical content. Let us remember the beginning of the preface:

> *At last I* offer you, *dear reader*, what I promised a long time ago, that is, the six books of Diophantus, not only reduced to specious, but augmented, and solved not only always with numbers indefinitely, but also with geometry, substituting continuous quantities instead of the given numbers, and instead of indeterminate letters, which remain in the indefinite solution of the problem. Various examples are seen in the first two books concerning determinate and indeterminate problems, which will serve as models for solving all other questions of the same nature by imitation.

The analysis of this work shows that when ancient problems were solved with the new algebra at the end of the 17th century, it contributed to the development of what we now call modern mathematics. Ozanam knew how to understand Viète's algebra and Descartes's geometry and knew how to merge them and translate them into this ancient work, elevating algebra to a discipline, on the same level as arithmetic and geometry.

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