Terrain prickliness: theoretical grounds for low complexity viewsheds

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Abstract. An important task when working with terrain models is computing viewsheds: the parts of the terrain visible from a given viewpoint. When the terrain is modeled as a polyhedral terrain, the viewshed is composed of the union of all the triangle parts that are visible from the viewpoint. The complexity of a viewshed can vary significantly, from constant to quadratic in the number of terrain vertices, depending on the terrain topography and the viewpoint position.
In this work we study a new topographic attribute, the *prickliness*, that measures the number of local maxima in a terrain from all possible perspectives. We show that the prickliness effectively captures the potential of 2.5D terrains to have high complexity viewsheds, and we present near-optimal algorithms to compute the prickliness of 1.5D and 2.5D terrains. We also report on some experiments relating the prickliness of real word 2.5D terrains to the size of the terrains and to their viewshed complexity.

Keywords: Polyhedral Terrain · Visibility · Viewshed complexity.

1 Introduction

Digital terrain models are representations of (parts of) the earth's surface and are used to solve a variety of problems in geographic information science (GIS). An important task for which terrain models are used is visibility analysis: determining which parts of a terrain are visible from some other point(s). A point q on the terrain is said to be *visible* from p if the line segment $\overline{pq}$ does not intersect the interior of the terrain. The *viewshed* of p is the set of all points that are visible from a given *viewpoint* p. Viewsheds are useful, for example, in evaluating the visual impact of potential constructions [3], analyzing the coverage of an area by fire watchtowers [9], or measuring the scenic beauty of a landscape [20].

There are two major ways to represent terrains in GIS: as a *raster*, i.e., a rectangular grid where each cell stores an elevation, or as a *polyhedral terrain*, a triangulation of a set of points in the plane, where each point has an elevation. In this paper we focus on the latter, which is the standard model in computational geometry. A polyhedral terrain can be seen as an xy-monotone polyhedral surface in $\mathbb{R}^3$. This is sometimes called a *2.5D* terrain. From a theoretical standpoint, it will be also interesting to consider the simpler setting of 1.5D terrains, defined as x-monotone polygonal lines in $\mathbb{R}^2$. In a 2.5D polyhedral terrain, a viewshed is formed by the union of the maximal parts of each terrain triangle that are visible from at least one viewpoint. In 1.5D, a viewshed is composed of parts of terrain edges.

The computational study of viewsheds focuses on two main aspects: the complexity of the viewsheds, and their efficient computation. In a 1.5D terrain, the viewshed of one viewpoint can have a complexity that is linear on the number of vertices, and can be computed in linear time [8]. In contrast, in 2.5D terrains the complexity of the viewshed of one viewpoint can be quadratic in the number of vertices [14], which makes its computation too slow for most practical uses when terrains are large.

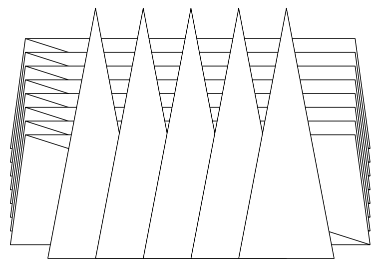

Fig. 1: A 2.5D terrain where the visible triangles form a viewshed of quadratic complexity.

A typical quadratic construction is shown schematically in Fig. 1, where the viewpoint would be placed at the center of projection, and both the number of vertical and horizontal triangles is $\Theta(n)$, for n terrain vertices. Notice that the vertical peaks form a grid-like pattern with the horizontal triangles, leading to $\Theta(n^2)$ visible triangle pieces. There exist several output-sensitive algorithms to compute the viewshed for a 2.5D terrain [10,18].

While a viewshed in a 2.5D terrain can have quadratic complexity, this seems to be uncommon in real terrains [1]. Indeed, there have been attempts to define theoretical conditions for a terrain to be "realistic" that guarantee, among others, that viewsheds cannot be that large. In particular, Moet et al. [17] showed that if terrain triangles satisfy certain "realistic" shape conditions, viewsheds have $O(n\sqrt{n})$ complexity. De Berg et al. [1] showed that similar conditions are enough to guarantee worst-case expected complexity of $\Theta(n)$ when the vertex heights are subject to uniform noise.

1.1 Viewsheds and peaks

The topography of the terrain has a strong influence on the potential complexity of the viewshed (something well-studied for sitting observers on terrains to maximize coverage, see e.g., [11]). To give an extreme example, in a totally concave[6] terrain, the viewshed of any viewpoint will be the whole terrain, so it will have a trivial description. Intuitively, to obtain a high complexity viewshed as in the figure above, one needs a large number of obstacles obstructing the visibility from the viewpoint, which in turn requires a somewhat rough topography.

In fact, it is well-established that viewsheds tend to be more complex in terrains that are more "rugged" [11]. This leads to the natural question of describing the terrain characteristics that can create high complexity (i.e., quadratic) viewsheds. Many topographic attributes have been proposed to capture different aspects of the shape or roughness of a terrain, such as the *terrain ruggedness index* [19], the *terrain shape index* [15], or the *fractal dimension* [13]. These measures focus on aspects like the amount of elevation change between adjacent parts of a terrain, its overall shape, or the terrain complexity. However, none of them is particularly aimed either to capture the possibility to produce high complexity viewsheds or to show any theoretical evidence for such a correlation.

One very simple and natural measure of the ruggedness of a terrain that is relevant for viewshed complexity is to simply count the number of local maxima, or "peaks", in a terrain. There has been evidence that areas with higher elevation difference, and hence, more peaks, cause irregularities in viewsheds [5,9], and this idea aligns with our theoretical understanding: the quadratic example from Fig. 1 is designed by creating an artificial row of peaks, and placing a viewpoint behind it. However, there is no theoretical correlation between the number of peaks and the viewshed complexity, as is easily seen by performing a simple trick: any terrain can be made arbitrarily flat

[6] In this work, as is common in terrain analysis but unlike functional analysis, a vertex v of a 2.5D terrain is *convex* (resp., *concave*) if there exists a non-vertical plane through v leaving all neighboring vertices below it (resp., above it). Convex/concave vertices of 1.5D terrains are defined analogously using lines rather than planes.

by scaling it in the z-dimension by a very small factor, and then tilted slightly—this results in a valid terrain without any peaks, but retains the same viewshed complexity. In fact, viewshed complexity is invariant under affine transformations of the terrain, and hence, any measure that has provable correlation with it must be affine-invariant as well. This is a common problem to establish theoretical guarantees on viewshed complexity, or to design features of "realistic" terrains in general [1,17].

1.2 Prickliness

In this work we explore a new topographic attribute: the *prickliness*. The definition follows directly from the above observations: it counts the number of peaks in a terrain, but does so *for every possible affine transformation* of the terrain.

Formally, let T be a polyhedral surface. We say that T is a *terrain* if the surface is xy-monotone; that is, if any vertical line intersects T at most once. Let A be an affine transformation. We define the number of *peaks* or *local maxima* of $A(T)$, $m(A(T))$, to be the number of internal and convex vertices of T[7] which are extremal in the z-direction in $A(T)$; that is, all adjacent vertices have a lower or equal z-coordinate. Let $\mathcal{A}(T)$ be the set of all affine transformations of T. Then we define the *prickliness* of T, $\pi(T)$, to be the maximum number of local maxima over all transformation of T;[8] that is, $\pi(T) = \max_{A \in \mathcal{A}(T)} m(A(T))$.

We start by observing that, essentially, the prickliness considers all possible *directions* in which the number of peaks are counted, and we can, in fact, provide an alternative definition that will be helpful in our analysis of the concept. Let $\boldsymbol{v}$ be a vector in $\mathbb{R}^3$. Let $\pi_{\boldsymbol{v}}(T)$ be the number of internal and convex vertices of T that are local maxima of T in direction $\boldsymbol{v}$; that is, the number of internal and convex vertices of T for which the local neighborhood does not extend further than that vertex in direction $\boldsymbol{v}$.

Observation 1 $\pi(T) = \max_{\boldsymbol{v}} \pi_{\boldsymbol{v}}(T)$

Proof. Clearly, for every vector $\boldsymbol{v}$ there exists an affine transformation A such that $m(A(T)) = \pi_{\boldsymbol{v}}(T)$: choose for A the rotation that makes $\boldsymbol{v}$ vertical. We will show that also for every affine transformation A there exists a vector $\boldsymbol{v}$ for which $m(A(T)) = \pi_{\boldsymbol{v}}(T)$. In particular this then implies that the maximum value of $m(A(T))$ over all A is equal to the maximum value of $\pi_{\boldsymbol{v}}(T)$ over all $\boldsymbol{v}$.

Let A be an affine transformation, and let H be the horizontal plane $z = 0$. Consider the transformed plane $H' = A^{-1}(H)$. Then any vertex of T which has the property that all neighbours are on the same side of H' in T, will be a local maximum or local minimum in $A(T)$. Now, choose for $\boldsymbol{v}$ the vector perpendicular to H' and pointing in the direction which will correspond to local maxima. □

Using this observation, we reduce the space of all possible affine transformations to essentially the 2-dimensional space of all possible directions. Further note that, since T is a terrain, for any $\boldsymbol{v}$ with a negative z-coordinate we have $\pi_{\boldsymbol{v}}(T) = 0$ by definition. This provides a natural way to visualize the prickliness of a terrain. Fig. 2 shows a small terrain and the resulting prickliness, shown as a function of the direction of $\boldsymbol{v}$.

[7] We explicitly only count vertices which are already convex in the *original* terrain, since some affine transformations will transform local minima / concave vertices of the original terrain into local maxima.

[8] It might happen that, in the affine transformation achieving prickliness, some of the vertices considered local maxima have neighbors at the same height, which might be considered non-desirable. However, under certain reasonable non-degeneracy assumptions for the terrain, there exists a small perturbation of that transformation giving one fewer local maxima and such that in that transformation all vertices considered local maxima have all neighbors at strictly lower height. An assumption guaranteeing this property is that no two edges of T have the same slope, and no two faces are parallel.

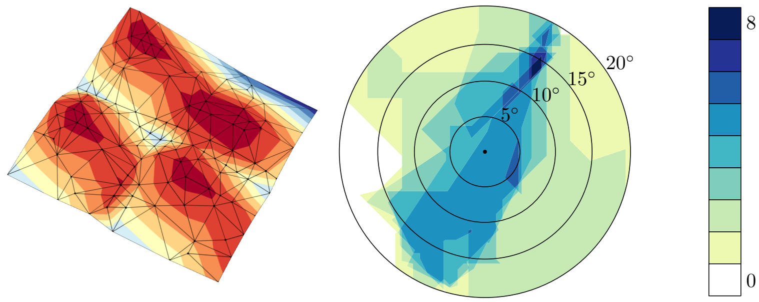

Fig. 2: (left) A polyhedral terrain T. Triangulation edges are shown in black, height (z-coordinates) are indicated using colors. (right) A visualization of the prickliness $\pi_{\boldsymbol{v}}(T)$ as a function of $\boldsymbol{v}$, in degrees from vertical $(0, 0, 1)$. The maximum prickliness is 8, attained at a direction about 13° north-east from the origin.

Results and organization The paper is organized as follows.

- In Section 2, we show that the prickliness of a 1.5D terrain and the viewshed complexity of a single viewpoint are not related: we given examples where one is constant and the other is linear.
- In contrast, we show in Section 3 that such a correlation does exist for 2.5D terrains: we show that the viewshed complexity of a single viewpoint cannot be higher than $O(n \cdot \pi(T))$, and this is tight.
- In Section 4, we consider the question of how to compute the prickliness of a 1.5D terrain. We provide an algorithm that runs in $O(n \log n)$ time, as well as a matching lower bound.
- In Section 5, we consider the question of how to compute the prickliness of a 2.5D terrain. We provide an $O(n^2)$ time algorithm, and show that this is near-optimal by proving that the problem is 3SUM-hard.
- Finally, in Section 6, we report on experiments that measure the prickliness of real terrains, and confirm the correlation between prickliness and viewshed complexity in practice.

2 Prickliness and viewshed complexity in 1.5D terrains

The prickliness of a 1.5D terrain and the viewshed complexity of a single viewpoint do not seem to be related. In order to show it, we need to introduce some notation.

For every internal and convex vertex v in T, we are interested in the vectors $\boldsymbol{w}$ such that v is a local maximum of T in direction $\boldsymbol{w}$. Note that these feasible vectors $\boldsymbol{w}$ can be represented as unit vectors, and then the feasible set becomes a region of the unit circle $\mathbb{S}^1$, which we denote by $se(v) \subset \mathbb{S}^1$. To find $se(v)$, for each edge e of T incident to v we consider the line ℓ through v which is perpendicular to e. Then we take the half-plane bounded by ℓ and opposite to e, and we translate it so that its boundary contains the origin. Finally, we intersect this half-plane with $\mathbb{S}^1$, which yields a half-circle. We intersect the two half-circles associated to the two edges of T incident to v, and obtain a sector of $\mathbb{S}^1$. For each direction $\boldsymbol{w}$ contained in the sector, the two corresponding edges do not extend further than v in direction $\boldsymbol{w}$. Thus, v is a local maxima in direction $\boldsymbol{w}$ and this sector indeed represents $se(v)$. See Fig. 3 for an example.

Theorem 2. *There exists a 1.5D terrain T with n vertices and constant prickliness, and a viewpoint on T with viewshed complexity $\Theta(n)$.*

Proof. The construction is illustrated in Fig. 4. From a point p, we shoot $n/2$ rays in the fourth quadrant of p such that the angle between any pair of consecutive rays is $2/n$. On the ith ray, there are two consecutive vertices of the terrain, namely, v_i and w_i. The vertices are placed so that $\angle w_{i-1} v_i w_i = 180 - 3/n$.

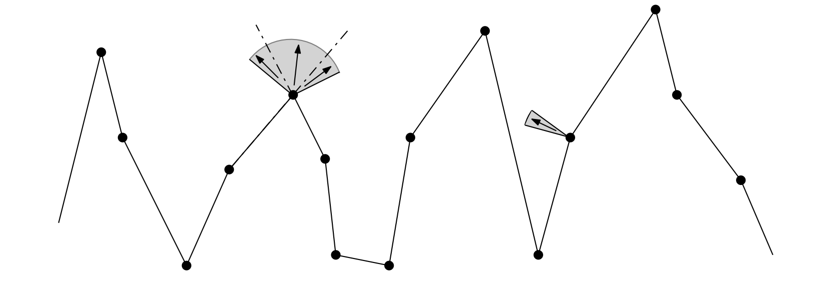

Fig. 3: In shaded, $se(v)$ for the corresponding vertices.

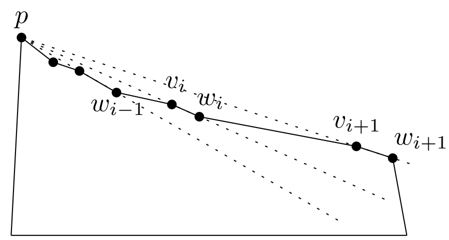

Fig. 4: Terrains with low prickliness can have high viewshed complexity.

For every i, we have that $se(v_i)$ has angle $3/n$, while $se(w_i)$ is empty because w_i is not convex. Since the angle between $w_{i-1}v_i$ and w_iv_{i+1} is $2/n$ and the angle between v_iw_i and $v_{i+1}w_{i+1}$ is also $2/n$, we have that $se(v_{i+1})$ can be obtained by rotating counterclockwise $se(v_i)$ by an angle of $2/n$. Thus, $se(v_i) \cap se(v_{i+1})$ has angle $1/n$, and $se(v_i) \cap se(v_{i+j})$ is empty for $j \geq 2$. We conclude that the prickliness of the terrain is constant.

If a viewpoint is placed very near p along the edge emanating to the right of p, then for every i the section $v_iw_iv_{i+1}$ contains a non-visible portion followed by a visible one. Hence, the complexity of the viewshed of the viewpoint is $\Theta(n)$. □

3 Prickliness and viewshed complexity in 2.5D terrains

Surprisingly, and in contrast to Theorem 2, we will show in Theorem 3 that in 2.5D there is a provable relation between prickliness and viewshed complexity.

We recall some terminology introduced in [7]. Let v be a vertex of T, and let p be a viewpoint. We denote by $\uparrow_p^v$ the half-line with origin at v in the direction of vector $\overrightarrow{pv}$. Now, let $e = uv$ be an edge of T. The *vase* of p and e, denoted $\uparrow_p^e$, is the region bounded by e, $\uparrow_p^u$, and $\uparrow_p^v$ (see Fig. 5).

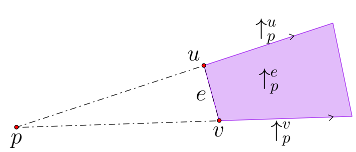

Fig. 5: A vase.

Vertices of the viewshed of p can have three types [7]. A vertex of type 1 is a vertex of T, of which there are clearly only n. A vertex of type 2 is the intersection of an edge of T and a vase. A vertex of type 3 is the intersection of a triangle of T and two vases. With the following two lemmas we will be able to prove Theorem 3.

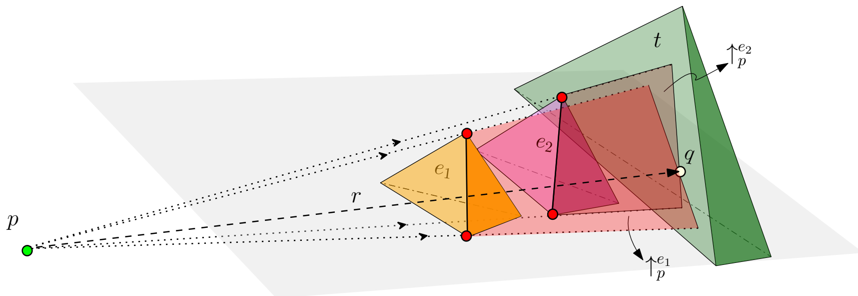

Fig. 6: The situation in the proof of Lemma 2.

Lemma 1. *There are at most* $O(n \cdot \pi(T))$ *vertices of type* 2.

Proof. Consider an edge e of T and let H be the plane spanned by e and p. Consider the viewshed of p on e. Let qr be a maximal invisible portion of e surrounded by two visible ones. Since q and r are vertices of type 2, the open segments pq and pr pass through a point of T. On the other hand, for any point x in the open segment qr, there exist points of T above the segment px. This implies that there is a continuous portion of T above H such that the vertical projection onto H of this portion lies on the triangle pqr. Such portion has a local maximum in the direction perpendicular to H which is a convex and internal vertex of T. In consequence, each invisible portion of e surrounded by two visible ones can be assigned to a distinct point of T that is a local maximum in the direction perpendicular to H. Hence, in the viewshed of p, e is partitioned into at most $2\pi(T) + 3$ parts.[9] □

Lemma 2. *There are at most* $O(n \cdot \pi(T))$ *vertices of type* 3.

Proof. Let q be a vertex of type 3 in the viewshed of p. Point q is the intersection between a triangle t of T and two vases, say, $\uparrow_p^{e_1}$ and $\uparrow_p^{e_2}$; see Fig. 6. Let r be the ray with origin at p and passing through q. Ray r intersects edges e_1 and e_2. First, we suppose that e_1 and e_2 do not share any vertex and, without loss of generality, we assume that $r \cap e_1$ is closer to p than $r \cap e_2$. Notice that $r \cap e_2$ is a vertex of type 2 because it is the intersection of e_2 and $\uparrow_p^{e_1}$, and $\uparrow_p^{e_1}$ partitions e_2 into a visible and an invisible portion. Thus, we charge q to $r \cap e_2$. If another vertex of type 3 was charged to $r \cap e_2$, then such a vertex would also lie on r. However, no point on r after q is visible from p because the visibility is blocked by t. Hence, no other vertex of type 3 is charged to $r \cap e_2$.

If e_1 and e_2 are both incident to a vertex v, since $t \cap \uparrow_p^{e_1} \cap \uparrow_p^{e_2}$ is a type 3 vertex, we have that r passes through v. Therefore, q is the first intersection point between r (which can be seen as the ray with origin at p and passing through v) and the interior of some triangle in T. Therefore, any vertex v of T creates at most a unique vertex of type 3 in this way. □

Theorem 3. *The complexity of a viewshed in a 2.5D terrain is* $O(n \cdot \pi(T))$.

Next we describe a construction showing that the theorem is best possible.

Theorem 4. *There exists a 2.5D terrain* T *with* n *vertices and prickliness* $\pi(T)$*, and a viewpoint on* T *with viewshed complexity* $\Theta(n \cdot \pi(T))$.

Proof. Consider the standard quadratic viewshed construction, composed of a set of front mountains and back triangles (Fig. 7 (left)). Notice that there can be at most $\pi(T)$ mountains "at the front". We add a surrounding box around the construction, see Fig. 7 (right), such that each vertex of the back triangles is connected to at least one vertex on this box. We set the elevation of the box so that it is higher than all the vertices of the back triangles, but lower than those of the front mountains. In this way, no vertex of the back triangles will be a local maximum in any direction, and all local maxima will come from the front. □

[9] We obtain $2\pi(T) + 3$ parts when the first and last portion of e are invisible; otherwise, we obtain fewer parts.

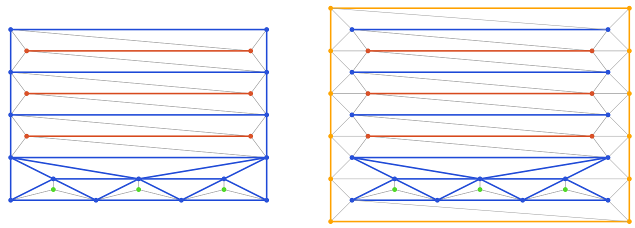

Fig. 7: Schematic top-down view of the classic quadratic construction (left), and the same adapted to have small prickliness (right). The camera with quadratic behaviour is at $(0, -\infty)$ (see Fig. 1 for the resulting view). Blue vertices/edges are low, red are medium hight, and green are high. The right construction introduces a new height (yellow) between medium and high, and changes the triangulation slightly, to ensure that all convex vertices in the construction are green.

4 Prickliness computation in 1.5D terrains

4.1 Algorithm

For every internal and convex vertex v in T, we compute $se(v)$ in constant time using the characterization given in Section 2. The prickliness of T is the maximum number of sectors of type $se(v)$ whose intersection is non-empty. We sort the bounding angles of the sectors in $O(n \log n)$ time, and obtain the maximum in a single pass. Thus, we obtain:

Theorem 5. *The prickliness of a 1.5D terrain can be computed in $O(n \log n)$ time.*

4.2 Lower bound

Now we show that $\Omega(n \log n)$ is also a lower bound for finding prickliness in a 1.5D terrain. The reduction is from the problem of checking distinctness of n integer elements, which has an $\Omega(n \log n)$ lower bound in the bounded-degree algebraic decision tree model [12,22].

Suppose we are given a set $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ of n integer elements, assumed w.l.o.g. to be positive. We multiply all elements of $\mathcal{S}$ by $180/(\max \mathcal{S} + 1)$ and obtain a new set $\mathcal{S}' = \{x'_1, x'_2, \ldots, x'_n\}$ such that $0 < x'_i < 180$, for each x'_i. We construct a terrain T that will be an instance of the prickliness problem. For each x'_i, we create in T a convex vertex v_i such that $se(v_i) = [x'_i - \varepsilon, x'_i + \varepsilon]$, where $\varepsilon = 18/(\max \mathcal{S} + 1)$, and such that its two neighbors are at distance 1 from v_i.[10] See Fig. 8 for an example. We denote the incident vertices to v_i to its left and right by w_i^l and w_i^r, respectively.

We arrange these convex vertices in the order of the elements in $\mathcal{S}'$ from left to right, and we place all of them at the same height. Then we place a dummy vertex u_i between every pair of consecutive vertices v_i and v_{i+1}, and connect u_i to w_i^r and w_{i+1}^l; see Fig. 8c. The height of u_i is chosen so that its two neighbors become concave vertices, and also so that $[\min(\mathcal{S}') - \varepsilon, \max(\mathcal{S}') + \varepsilon] \subseteq se(u_i)$. This is possible because moving u_i upwards increases its feasible region $se(u_i)$, the limit being $[0, 180]$. The following lemma allows us to prove Theorem 6 below.

Lemma 3. *The prickliness of T is n if and only if all elements in $\mathcal{S}$ are distinct.*

Proof. Every vertex of type u_i satisfies that $[\min(\mathcal{S}') - \varepsilon, \max(\mathcal{S}') + \varepsilon] \subseteq se(u_i)$. Thus the prickliness of T is at least $n - 1$. For every vertex of type v_i, $se(v_i)$ has an angle of 2ε. Finally, the vertices of type w_i^l and w_i^r are concave, so $se(w_i^l)$ and $se(w_i^r)$ are empty.

Consequently, the prickliness of T is n if and only the sectors of type $se(v_i)$ are pairwise disjoint, which happens if and only if the elements in $\mathcal{S}$ are all distinct. □

Theorem 6. *The problem of computing the prickliness of a 1.5D terrain has an $\Omega(n \log n)$ lower bound in the bounded-degree algebraic decision tree model.*

[10] We sometimes write $se(v) = [\alpha, \beta]$, where α and β are the angles bounding the sector.

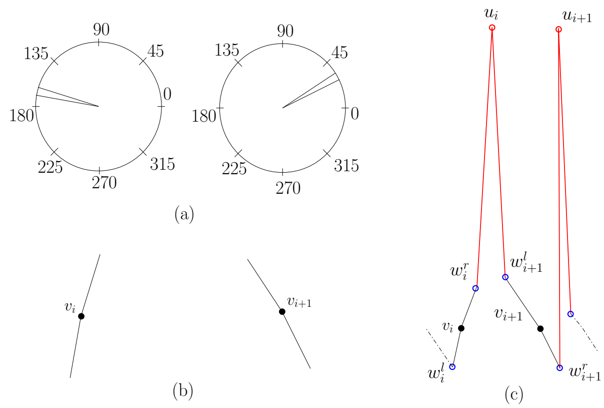

Fig. 8: (a) Sectors associated to the element set $\{160, 25\}$. (b) The corresponding convex vertices. (c) Construction of the terrain.

5 Prickliness computation in 2.5D terrains

5.1 Algorithm

A basic observation is that the sphere of potential directions is subdivided into n^2 cells by the n planes through the origin parallel to the triangles of T. For any pair of directions $\boldsymbol{v}$ and $\boldsymbol{w}$ in the same cell, the values of $\pi_{\boldsymbol{v}}$ and $\pi_{\boldsymbol{w}}$ are equal. This gives a trivial $O(n^3)$ time algorithm to calculate the prickliness: compute this subdivision, for each cell take a single vector, and for this vector count the number of local maxima (testing whether a vertex is a local maximum is a local operation that takes time proportional to the degree of the vertex; the sum of degrees of all vertices is linear).

We propose a faster algorithm that extends the idea from Section 4.1 to 2.5D terrains as follows: For every convex terrain vertex v, we compute the region of the unit sphere $\mathbb{S}^2$ containing all vectors $\boldsymbol{w}$ such that v is a local maximum of T in direction $\boldsymbol{w}$. As we will see, such a region is a cone and we denote it by $co(v)$. Furthermore, we denote the portion of $co(v)$ on the surface of the sphere by $co_{\mathbb{S}^2}(v)$.

In order to compute $co(v)$, we consider all edges of T incident to v. Let $e = vu$ be such an edge, and consider the plane orthogonal to e through v. Let H be the half-space which is bounded by this plane and does not contain u. We translate H so that the plane bounding it contains the origin; let H_e be the intersection of the obtained half-space with the unit sphere $\mathbb{S}^2$. The following property is satisfied: For any unit vector $\boldsymbol{w}$ in H_e, the edge e does not extend further than v in direction $\boldsymbol{w}$. We repeat this procedure for all edges incident to v, and consider the intersection $co(v)$ of all the obtained half-spheres H_e. For any unit vector $\boldsymbol{w}$ in $co(v)$, none of the edges incident to v extends further than v in direction $\boldsymbol{w}$. Since v is convex, this implies that v is a local maxima in direction $\boldsymbol{w}$.

Once we know all regions of type $co(v)$, computing the prickliness of T reduces to finding a unit vector that lies in the maximum number of such regions. To simplify, rather than considering these cones on the sphere, we extend them until they intersect the boundary of a unit cube $\mathbb{Q}$ centered at the origin. The conic regions of type $co(v)$ intersect the faces of $\mathbb{Q}$ forming (overlapping) convex regions. Notice that the problem of finding a unit vector that lies in the maximum number of regions of type $co(v)$ on $\mathbb{S}$ is equivalent to the problem of finding a point on the surface of $\mathbb{Q}$ that lies in the maximum number of "extended" regions of type $co(v)$. The second problem can be solved by computing the maximum overlap of convex regions using a topological sweep [4], for each face of the cube.

Computing the intersection between the extended regions of type $co(v)$ (for all convex vertices v) and the boundary of $\mathbb{Q}$ takes $O(n \log n)$ time, and topological sweep to find the maximum overlap takes $O(n^2)$ time. We obtain the following:

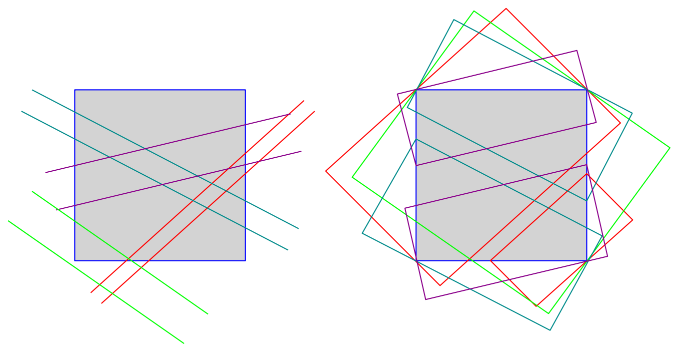

Fig. 9: Instance of the problem of coverage with strips (left), and the equivalent problem of coverage with rectangles (right).

Theorem 7. *The prickliness of a 2.5D terrain can be computed in $O(n^2)$ time.*

5.2 Lower bound

In this section we show that the problem of computing the prickliness of a 2.5D terrain is 3SUM-hard. This implies our result in Theorem 7 is likely to be close to optimal: The best-known algorithm for 3SUM runs in $O(n^2(\log\log n)^{O(1)}/(\log n)^2)$ time, and it is believed there are no significantly faster solutions [2].

The reduction is from the following problem, which is known to be 3SUM-hard [6]: Given a square Q and m strips of infinite length, does there exist a point in Q not covered by any of these m strips?

Let us take an instance of the problem above. The complement of each strip is given by two half-planes. Let us consider the $2m$ half-planes obtained in this way, together with the square Q. Answering the above question is equivalent to answering whether there exists a point in Q covered by m of the half-planes.

For every half-plane, if it does not intersect Q, we discard it. Otherwise, we replace the half-plane by the smallest-area rectangle that contains the intersection of Q with the half-plane; see Fig. 9. Then the problem becomes determining whether there exists a point in Q covered by m of these rectangles.

Let $\mathcal{A}$ be the arrangement containing Q and the rectangles. By construction, we have:

Observation 8 *Any point is covered by at most $m+1$ objects of $\mathcal{A}$. If a point is covered by $m+1$ objects of $\mathcal{A}$, it lies inside Q.*

Next we consider the plane containing $\mathcal{A}$ as a horizontal plane in $\mathbb{R}^3$ such that $(0,0,2)$ lies on Q. For every object in $\mathcal{A}$, we connect all its vertices to $(0,0,0)$, which gives a cone, and we intersect this cone with the unit sphere centered at $(0,0,0)$. The intersection of the cone with the surface of the sphere is referred to as the "projection" of the original object in $\mathcal{A}$. We denote the projection of Q by $\tilde{Q}$, and the projection of rectangle R_i by $\tilde{R}_i$. With some abuse of notation, we still refer to the objects $\tilde{R}_i$ as "rectangles". Let $\tilde{\mathcal{A}}$ be the arrangement containing the projection of the objects in $\mathcal{A}$. Observation 8 implies:

Observation 9 *Any point is covered by at most $m+1$ objects of $\tilde{\mathcal{A}}$. If a point is covered by $m+1$ objects of $\tilde{\mathcal{A}}$, it lies inside $\tilde{Q}$.*

We next construct a terrain T. The terrain contains *red*, *green* and *blue* vertices. We associate one red vertex to $\tilde{Q}$ and to each of the rectangles intersecting it. The red vertices are placed at height two, and the distance between any pair of them is at least three.

Each red vertex has four green vertices as neighbors, placed as follows. Let $\tilde{R}_i$ be one of the rectangles on the sphere (either $\tilde{Q}$ or one of the rectangles intersecting it), and let v_i be the red vertex of T associated to $\tilde{R}_i$. For each of the sides of $\tilde{R}_i$, consider the plane containing the side of $\tilde{R}_i$ and passing through the origin. Consider the half-space H that is bounded by this plane and contains $\tilde{R}_i$. Take a normal vector w of the plane that does not point towards H. Then place a green vertex x of T such that the vector $\boldsymbol{v_i x}$ is congruent to w and has length one.[1] By following this procedure for all sides of $\tilde{R}_i$, we obtain four green vertices of T which are adjacent to v_i. We denote this set of vertices by $N(v_i)$. There are no more vertices of T adjacent to v_i.

Lemma 4. *Vertex v_i satisfies:*

(a) It is a convex vertex of T.
(b) If we project v_i and $N(v_i)$ onto the XY-plane, then the convex hull of the projection of the vertices in $N(v_i)$ contains the projection of v_i.
(c) $co_{\mathbb{S}^2}(v_i) = \tilde{R}_i$.

Proof. The intersection of the four half-spaces associated to the four sides of $\tilde{R}_i$ forms a cone that can be described as the set of points $\bar{x} \in \mathbb{R}^3$ satisfying the equation $A\bar{x} \leq 0$, where A is a 4×3 matrix. It can also be described as $cone(W)$ –the conic hull of vectors in W–, where W is the set of four vectors pointing from the origin to the four endpoints of R_i.

Let $\tilde{W}$ be the 4×3 matrix where each row corresponds to the components of one of the vectors in W. Let $\tilde{A}$ be the set of four vectors corresponding to the four rows of A. By polar duality, the cones described by $\tilde{W}\bar{x} \leq 0$ and $cone(\tilde{A})$ are also the same.

Notice that the vectors of $\tilde{A}$ are the four vectors used to place the four neighbors of v_i in T. Therefore, (b) is equivalent to the following: If we project the endpoints of the vectors of $\tilde{A}$ onto the XY-plane, then the convex hull of the projected points contains the origin. We will prove that $(0, 0, -1) \in cone(\tilde{A})$, which implies this claim.

Since the four endpoints of $\tilde{R}_i$ have positive z-coordinate, $(0, 0, -1)$ satisfies the equation $\tilde{W}\bar{x} \leq 0$. Since the cones $\tilde{W}\bar{x} \leq 0$ and $cone(\tilde{A})$ are the same, we obtain $(0, 0, -1) \in cone(\tilde{A})$.

Regarding (a), let $\alpha \neq (0, 0, 0)$ be such that $A\alpha \leq 0$. Then $cone(\tilde{A})$ is contained in the half-space $\alpha^T x \leq 0$, which proves that v_i is a convex vertex of T.

By construction of $N(v_i)$ and the fact that v_i is a convex vertex of T, (c) follows. □

By Lemma 4b (and the facts that red vertices are at pairwise distance at least three and each red vertex has its green neighbors at distance one), the projection of the red and green vertices, and the edges among them, onto the XY-plane is a set of pairwise disjoint wheel graphs. Next, we triangulate this graph. Then, for every triangle defined by three green vertices, we place a blue vertex inside the triangle and we connect this vertex to the three vertices of the triangle (see Fig. 10). It only remains to specify the height of the blue vertices. Each blue vertex u is placed at a height high enough so that: (i) the three neighboring green vertices become non-convex vertices of T, and (ii) $co_{\mathbb{S}^2}(u)$ contains $\tilde{Q}$. Regarding (ii), if u is placed higher than all of its neighbors (which are all green, so they already have specified heights), then $co_{\mathbb{S}^2}(u)$ contains the north pole. By moving u upwards, $co(u)$ becomes bigger and bigger, the limit being the upper hemisphere of $\mathbb{S}$. Therefore, from some height onwards $co_{\mathbb{S}^2}(u)$ contains $\tilde{Q}$. Let $\tilde{\mathcal{A}}^+$ be the arrangement $\tilde{\mathcal{A}}$ augmented with $co_{\mathbb{S}^2}(u)$ for all blue vertices u. Let the number of blue vertices be k. By Observation 9,

Observation 10 *Any point is covered by at most $m + 1 + k$ objects of $\tilde{\mathcal{A}}^+$. If a point is covered by $m + 1 + k$ objects of $\tilde{\mathcal{A}}^+$, it lies inside $\tilde{Q}$.*

Lemma 5. *The prickliness of T is $m + k + 1$ if and only if there is a point in Q covered by m rectangles.*

Proof. Suppose that the prickliness of T is $m + k + 1$. This means that, when we compute $co(v)$ for all vertices v of T, a unit vector that is covered by the maximum number of such cones is covered by $m + k + 1$ cones. If v is a blue vertex, $co_{\mathbb{S}^2}(v)$ contains $\tilde{Q}$. If v is a green vertex, $co(v)$ is empty because v is a concave vertex of T. If v is a red vertex, either v is associated to $\tilde{Q}$ or $v = v_i$

[1] Since red vertices have height two, the obtained green vertex has positive height.

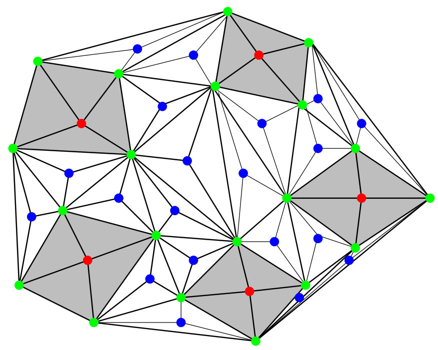

Fig. 10: Projection of T onto the XY-plane.

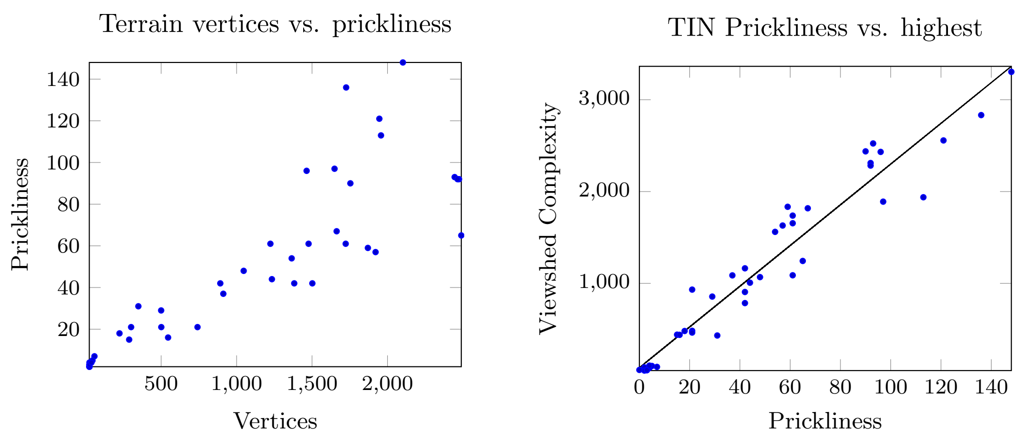

Fig. 11: (left) The prickliness values for the terrains we considered. (right) The relation between the (median) viewshed complextity of the high viewpoints in a terrain and its prickliness.

for some i. By Lemma 4c, in the first case $co_{\mathbb{S}^2}(v) = \tilde{Q}$, while in the second case $co_{\mathbb{S}^2}(v) = \tilde{R}_i$. Therefore, the problem of computing the prickliness of T is equivalent to the problem of finding a point in $\tilde{\mathcal{A}}^+$ covered by the maximum number of objects. By Observation 9, if the prickliness of T is $m + k + 1$, there is a point in $\tilde{Q}$ covered by $m + 1 + k$ objects of $\tilde{\mathcal{A}}^+$. This implies that there is a point in Q covered by m rectangles.

If there is a point in Q covered by m rectangles, there is a point in $\tilde{Q}$ covered by $m + 1 + k$ objects of $\tilde{\mathcal{A}}^+$. By Observation 9, there is no point in $\tilde{\mathcal{A}}^+$ covered by more than $m+1+k$ objects, so the prickliness of T is $m + k + 1$. $\square$

Thus we have the following:

Theorem 11. *The problem of computing the prickliness of a 2.5D terrain is 3SUM-hard.*

6 Experiments

We briefly report on some experiments relating the prickliness to the viewshed complexity in real world 2.5D terrains (see [16] for details). We considered a collection of 52 real-world terrains around the world. They varied in ruggedness, including mountainous regions (Rocky mountains, Himalaya), flat areas (farmlands in the Netherlands), and rolling hills (Sahara), and in number of vertices. For each of these terrains we computed the prickliness, and the viewsheds of nine "sufficiently separated" high points on the terrain. We refer to [16] on how we pick these points

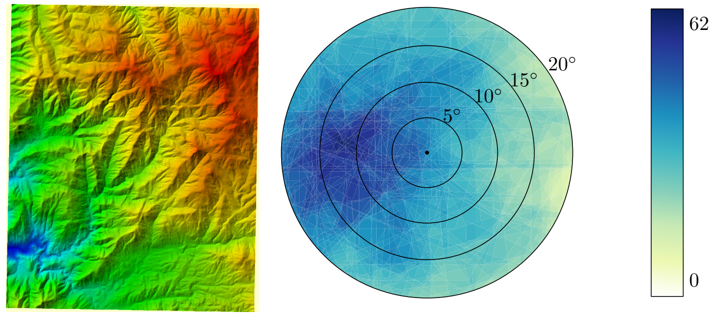

Fig. 12: A real-world terrain with 583 vertices from the neighborhood of California Hot Springs whose prickliness is only 62. On the right the value $\pi_{\boldsymbol{v}}$ for vectors near $(0, 0, 1)$.

exactly. Fig. 11 (left) shows the number of vertices in each of the terrains and their prickliness. We see that the prickliness is generally much smaller than the number of vertices. See also Fig. 12. Fig. 11 (right) then shows the relation between the prickliness and the viewshed complexities. The viewshed complexities seem to scale nicely with the prickliness. These results suggest that the prickliness may indeed be a valuable terrain descriptor.

7 Conclusions

We introduced the *prickliness* as a new measure of terrain ruggedness, and showed that for 2.5D terrains it has a direct correlation with viewshed complexity. As far as we know, this constitutes the first topographic attribute that has a provable connection with viewshed complexity, and is independent of the viewpoint location. Furthermore, we presented near-optimal algorithms to compute the prickliness of 1.5D and 2.5D terrains. We also performed some experiments indicating that, in real-world terrains, prickliness is significantly smaller than the number of vertices, and also that the correlation between prickliness and viewshed complexity translates into practice.

An intriguing question is whether the computation of the prickliness itself can be done more efficiently on real terrains, as our lower bound construction results in a very artificial terrain. It is conceivable that one could achieve this assuming, for instance, *realistic terrain* conditions [17], or a *slope* condition [21]. Finally, extending our study to viewsheds of multiple viewpoints, whose complexity was studied recently for the first time [7], is another interesting direction for further research.

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