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ROBUST STABILITY IN THE PROBLEM OF MOTION OF AN INVERTED PENDULUM WITH A VIBRATING SUSPENSION POINT

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The work is devoted to the study of the robust stability of the motion of an inverted pendulum with a suspension point oscillating according to a sinusoidal law. The case of perturbations of the amplitude of oscillations is considered, an estimate is given for the magnitude of the perturbation at which the stability of motion is preserved.

Introduction. The motion of an inverted pendulum, the suspension point of which oscillates according to a sinusoidal law along a straight line making a small angle α with the vertical, is described by the following equation:

$$\varphi'' + \varepsilon\varphi' + \frac{g - a\omega^2 \sin(\omega t)\cos\alpha}{l}\sin\varphi - \frac{a\omega^2 \sin(\omega t)\sin\alpha}{l}\cos\varphi = 0, \tag{1}$$

where $\varphi = \varphi(t)$ is the angle of deviation of the pendulum from the lower vertical position of equilibrium, ε is the friction coefficient, l is the length of the pendulum, g is the acceleration of gravity, ω is the oscillation frequency of the suspension point, and a is the amplitude of the suspension point oscillations.

It is well known that, in case of zero angle $\alpha = 0$, at a sufficiently large oscillation frequency $\omega \gg 1$ and sufficiently small amplitude of oscillations of the suspension point $\frac{a}{l} \ll 1$, the upper equilibrium position of the pendulum becomes stable. This result was predicted back in 1908 by the English mathematician Stephenson (see [1]) and was first rigorously proved by Bogolyubov in 1942 (see [2]). Namely, he demonstrated that, if the above conditions on the amplitude and frequency are satisfied and $a\omega > \sqrt{2gl}$, the upper vertical position $\varphi(t) = \pi$ is asymptotically stable. The last condition means that the maximum oscillation velocity of the suspension point must exceed the velocity of a free fall of a body from the height equal to the pendulum length. A clear demonstration of this phenomenon is provided by the Kapitsa's installation [3].

At present, there are various approaches to the proof of the theorem on the stability of the upper equilibrium position of the pendulum at zero angle $\alpha = 0$ (see [4]), the classical proof is carried out using the averaging method [5, 6]. In papers [7, 8, 9] the stability problem for an inverted pendulum in the case $\alpha \geq 0$ is solved using a special boundary value problem for the Lyapunov differential equation and the principle of contracting mappings (see, for example, [10]). This approach is remarkable in that it allows not only to establish the very fact of motion stability, but also to obtain the region of attraction of the solution and an estimate of the stabilization rate at $t \to \infty$. Developing the [7, 9] approach and using the technique described in [11], it is also possible to obtain some results on robust stability (that is, stability with respect to perturbations of the coefficients of the equation) in the problem of motion of an inverted pendulum with a vibrating suspension point, which is the subject of this work.

Task formulation. Consider the movement of the pendulum in the case of a zero angle $\alpha = 0$. Then the equation (1) takes the following form

$$\varphi'' + \varepsilon\varphi' + \frac{g - a\omega^2 \sin(\omega t)}{l}\sin\varphi = 0.$$

Shifting the angle by π ($\varphi := \varphi + \pi$) and writing the result equation in the form of a system, we get

$$\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & -\varepsilon\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \begin{pmatrix}0\\ \frac{g - a\omega^2 \sin(\omega t)}{l}\sin\varphi_1\end{pmatrix}. \tag{2}$$

Here $\varphi_1 = \varphi$ and $\varphi_2 = \dot{\varphi}$. Due to the angle shift, the study of the upper vertical equilibrium position of the pendulum is reduced to the study of the zero solution of system (2).

Our goal is to study the robust stability of the system (2) with respect to the perturbation Δa of the amplitude of the suspension point oscillations, i.e. we consider the following perturbed system

$$\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & -\varepsilon\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \begin{pmatrix}0\\ \frac{g - (a + \Delta a)\omega^2 \sin(\omega t)}{l}\sin\varphi_1\end{pmatrix}. \tag{3}$$

Preliminary information. Consider a linear homogeneous system of differential equations with periodic coefficients

$$\frac{dy}{dt} = A(t)y, \quad t \geq 0, \tag{4}$$

where $A(t)$ is a continuous T-periodic matrix of size $N \times N$.

In this paper, we rely on the criterion of the asymptotic stability of the zero solution for the system (4), formulated in terms of the solvability of the following special boundary value problem for the Lyapunov differential equation

$$\frac{dH}{dt} + HA(t) + A(t)^*H = -C(t), \quad 0 \leq t \leq T, \tag{5}$$

$$H(0) = H(T). \tag{6}$$

According to this criterion (see [12]), if the zero solution of the system (4) is asymptotically stable, then for any matrix $C(t)$ continuous on $[0, T]$ and such that

$$C(t) = C(t)^* > 0, \quad t \in [0, T],$$

there exists a unique Hermitian positive definite solution $H(t)$ of the boundary problem (5)-(6).

We will further assume that the zero solution of the system (4) is asymptotically stable. Let us denote

$$\Delta = \left(1 - \exp\left(-\int_0^T \frac{1}{2}\|H(\eta)\|^{-1} d\eta\right)\right)^{-1}, \tag{7}$$

$$\mu(H) = \max_{\tau\in[0,T]} \|H(\tau)\| \max_{\xi\in[0,T]} \|H^{-1}(\xi)\|. \tag{8}$$

In our work, in the study of robust stability in the problem of motion of an inverted pendulum, we will use the following result from [11] reformulated in terms of the asymptotic stability of the zero solution:

Theorem 1. *Let the zero solution of the system (4) with T-periodic coefficients be asymptotically stable. If a T-periodic matrix $A_1(t)$ satisfies the following condition*

$$q = 2T\Delta\sqrt{\mu(H)} \max_{t\in[0,T]} \|A_1(t)\| < 1, \tag{9}$$

then the zero solution of the system with perturbed coefficients

$$\frac{dy}{dt} = \big(A(t) + A_1(t)\big)y, \quad t \geq 0$$

is also asymptotically stable.

Results. We begin the study of the robust stability of the system (2) with the study of the robust stability of its linear approximation, i.e. we consider $\sin\varphi_1 \approx \varphi_1$ and the corresponding system is the following

$$\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g - a\omega^2\sin(\omega t)}{l} & -\varepsilon \end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = A(t)\vec{\varphi}. \tag{10}$$

The corresponding perturbed system has the form

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{g - (a+\Delta a)\,\omega^2\sin(\omega t)}{l} & -\varepsilon \end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ \frac{g - a\omega^2\sin(\omega t)}{l} & -\varepsilon \end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{\Delta a\omega^2\sin(\omega t)}{l} & 0 \end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \big(A(t) + A_1(t)\big)\vec{\varphi}. \end{aligned} \tag{11}$$

The following result holds.

Theorem 2. *Let $\frac{a}{l} \ll 1$, $\omega \gg 1$ and $a\omega > \sqrt{2gl}$, i.e. the zero solution of the system (10) is asymptotically stable. If*

$$|\Delta a| < \frac{l}{4\pi\omega\Delta\sqrt{\mu(H)}}, \tag{12}$$

where $H(t)$ is Hermitian positive definite solution of the problem (5)-(6) with the matrix $A(t)$ corresponding to the system (10) and the values Δ and $\mu(H)$ are defined by (7) and (8), respectively, then the zero solution of the system (11) with the perturbed amplitude is also asymptotically stable.

This theorem follows directly from the theorem 1 applied to the systems (10), (11) and namely the inequality (9). For initial nonlinear system (2) similar result is true.

Theorem 3. *Let $\frac{a}{l} \ll 1$, $\omega \gg 1$ and $a\omega > \sqrt{2gl}$, i.e. the zero solution of the system (10) is asymptotically stable. If*

$$|\Delta a| < \frac{l}{4\pi\omega\Delta\sqrt{\mu(H)}},$$

where $H(t)$ is Hermitian positive definite solution of the problem (5)-(6) with the matrix $A(t)$ corresponding to the system (10) and the values Δ and $\mu(H)$ are defined by (7) and (8), respectively, then the zero solution of the nonlinear system (3) with the perturbed amplitude is also asymptotically stable.

In the proof of this theorem we first show with the use of the results from [7] about asymptotic stability of zero solutions for quasilinear systems with periodic coefficients, that from asymptotic stability of the zero solution of the linear system (10) it follows that the zero solution of the nonlinear system (2) is also asymptotically stable. Thus, the statement of the theorem 3 is indeed a result of robust stability (from the asymptotic stability of the zero solution of the nonlinear system (2) follows the asymptotic stability of the zero solution of the perturbed nonlinear system (3) when the estimate (12) on the value of Δa holds). And then, using again the results from [7], we prove the statement of the theorem 3.

Conclusion. The paper studies the question of the robust stability of the motion of an inverted pendulum, the suspension point of which oscillates according to a sinusoidal law along a vertical line. Namely, the case of a perturbed vibration amplitude is considered. An estimate is given for the magnitude of the perturbation of the amplitude at which the stability of motion is preserved. The research uses the approaches from [7] and [11].

In the future, the question of robust stability with respect to other parameters of the pendulum motion (the oscillation frequency of the suspension point ω and the length of the pendulum l) is raised, and the case of oscillations of the suspension point along an inclined straight line ($\alpha > 0$) should be studied. In addition, there stays the problem of finding a solution to a special boundary value problem arising in the study of the Lyapunov differential equation (at least approximately).

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