Linear l^2 decoupling through multilinear estimates

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Tiivistelmä — Referat — Abstract

In this thesis we study the article by J. Bourgain and C. Demeter called *A study guide for the l^2 decoupling theorem*. In particular, we hope to give an in detail exposition to certain results from the aforementioned research article so that this text combined with the master's thesis *On the l^2 decoupling theorem* by Jaakko Sinko covers the l^2 decoupling theorem comprehensively in the range $2 \leq p \leq \frac{2n}{n-1}$. The results in this text also self-sufficiently motivate the use of the extension operator and explain why it is possible to prove linear decouplings with multilinear estimates.

We begin the thesis by giving the basic notation and highlighting some useful results from analysis and linear algebra that are later used in the thesis.

In the second chapter we introduce and prove a certain multilinear Kakeya inequality, which asserts an upper bound for the overlap of neighbourhoods of nearly axis parallel lines in $\mathbb{R}^n$ that point in different directions. In the next chapter this is applied to prove a multilinear cube inflation inequality, which is one of the main mechanisms in the proof of the l^2 decoupling theorem.

In the fourth chapter we study two forms of linear decoupling. One that is defined by an extension operator and one that defined via Fourier restriction. The main result of this chapter is that the former is strong enough to produce decoupling inequalities that are of the latter form.

The fifth chapter is reserved for comparing linear and multilinear decouplings. Here we use the main result of the previous chapter to prove that multilinear estimates can produce linear decouplings, if the lower dimensional decoupling constant is somehow contained. This paves the way for the induction proof of the l^2 decoupling theorem.

Avainsanat — Nyckelord — Keywords

Decoupling, Fourier analysis, Harmonic analysis, Multilinear Kakeya inequality

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Chapter 1

Introduction

1.1 Preface

Decoupling is a powerful Fourier analytic tool with many applications in number theory, partial differential equations and geometric measure theory. In a general setting we have possibly infinite elements $(x_i)_{i\in I}$ in a normed space $(X, \|\cdot\|_X)$ and we study the smallest constant C_{dec} that satisfies

$$\|\sum_{i\in I} x_i\|_X \leq C_{dec}(\sum_{i\in I}\|x_i\|_X^q)^{\frac{1}{q}}.$$

A common example of this is when X is the space of square integrable functions, $q = 2$ and x_j have disjointly supported Fourier transforms. In this case using orthogonality one can deduce an equality $\|\sum_{i\in I} x_i\|_X = (\sum_{i\in I}\|x_i\|_X^2)^{\frac{1}{2}}$. A natural question is to ask what happens in the absence of Hilbert space orthogonality, i.e., in the case $X = L^p$ and $q = 2$. This particular case is called $l^2(L^p)$ decoupling or for the sake of brevity l^2 decoupling. The l^2 decoupling theorem is a theorem that yields an upper bound for a certain l^2 decoupling constant and this bound is sharp up to $\delta^{-\varepsilon}$ losses.

The first instance of decoupling was by Thomas Wolff. In the year 2000 Wolff published an article [17], where he proved a sharp l^pL^p decoupling inequality for the cone and used it to get local smoothing estimates on L^p. In a more resent context decoupling was strongly developed by Jean Bourgain in 2013 in his paper [4], where he combined an induction on scales argument with multilinear restriction to prove the l^2 decoupling theorem in the range $2 \leq p \leq \frac{2n}{n-1}$. Two years later the ground breaking l^2 decoupling theorem for the full range $2 \leq p \leq \frac{2(n+1)}{n-1}$ was proven by Jean Bourgain and Ciprian Demeter in [6]. Possibly the most remarkable application is from [8], where l^2 decoupling is used to prove the eighty-year-old main conjecture in Vinogradov's mean value theorem. This result is an upper bound for the number of solutions to a certain system of Diophantine equations. Other resent works include connections to the Riemann zeta function [3] and connections to the Schrödinger equation [7] and [10]. In the year 2020 Demeter published a book [9] that covers decoupling in a broad range of generality.

The main objective of this thesis is to provide a highly detailed exposition to chapters 8 and 9 from the article [5] *A study guide to the l^2 decoupling theorem* published in 2017 by Jean Bourgain and Ciprian Demeter. Along the way we will also shed some light to chapter 5 of the same article. The intention is that the details presented in this thesis makes the results more accessible for beginners. The only thing that we will leave uninformed is how can one reduce the equation of the line L in chapter 8 of [5]. This text combined with the Master thesis of Jaakko Sinko [14] covers the l^2 decoupling theorem exhaustively in the range $2 \leq p \leq \frac{2n}{n-1}$ excluding only the aforementioned reduction. In other words, only the final chapter of [5] is left uninvestigated.

In addition to the in-depth details, noteworthy contributions of this thesis include: construction of tempered distributions and Schwartz functions used in the proofs, steps needed to be taken from the multilinear Kakeya inequality of [11] so that one gets the multilinear Kakeya inequality from [5] and careful treatment of Fourier transforms of non-integrable functions so that we will not need heuristics like $\widehat{E_Q g}(\lambda)$ (see chapter 9 of [5]).

We begin the thesis by equipping ourselves with results from functional analysis, linear algebra and real analysis. This includes the construction of Schwartz functions and tempered distributions used in the later proofs of the thesis.

In the second chapter we will present and prove the multilinear Kakeya inequality, which is an upper bound on the overlap properties of certain neighbourhoods of lines in $\mathbb{R}^n$.

We will start the third chapter by presenting the extension operator E_Q and the weight functions $w_{B,E}$. Then we will use the multilinear Kakeya inequality from the previous chapter to prove a multilinear cube inflation inequality.

In the fourth chapter we define the linear decoupling constant and consider the relations between two different formulations of linear l^2 decoupling. This chapter includes a decoupling inequality that makes use of Fourier restriction.

The fifth and final chapter is about comparing linear and multilinear decouplings. We will start this by introducing the multilinear decoupling constant. The thesis is finalized with a reverse decoupling constant inequality that makes use of the Fourier restriction decoupling from the previous section.

To summarize the argument of the article [5] *A study guide to the l^2 decoupling theorem*, two types of mechanisms are used to decouple. The first is l^2L^2 decoupling, which is proved using the Hilbert space properties of L^2 and the second is cube inflation, which relies on the multilinear Kakeya inequality. These two mechanisms are combined to make a multiscale inequality, which is used to estimate the multilinear decoupling constant. These multilinear estimates can be turned into linear decouplings via the reverse decoupling constant inequality, which is proved using an alternative formulation of linear decoupling. In this thesis we cover the Multilinear Kakeya inequality and the cube inflation result. We also study the alternative form of linear decoupling and use it to prove the reverse decoupling constant inequality. The masters thesis of Jaakko Sinko [14] covers the l^2L^2 decoupling and the application of all these results to get the l^2 decoupling theorem in the slightly truncated range $2 \leq p \leq \frac{2n}{n-1}$.

I am very grateful to my thesis supervisor Professor Tuomas Hytönen for introducing me to the topic and for patiently answering the questions that I had.

1.2 Notation

For the natural numbers we include zero and denote $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}_+ = \{1, 2, \dots\}$. Throughout this text $n \in \mathbb{N}_+$. Often we restrict n to be even bigger, if the notation does not otherwise make sense, for example, whenever we denote $\mathbb{R}^{n-1}$, we assume that $n \geq 2$.

Throughout this text we write $A \lesssim B$, if $A \leq CB$ for some fixed constant $0 < C < \infty$. We also write $A \sim B$, if $B \lesssim A \lesssim B$. When the constant C is associated with this notation, it is called implicit. Furthermore, we denote $A \lesssim_{p_1,\cdots,p_n} B$, when we allow the implicit constant to depend on the parameters $\{p_1, \cdots, p_n\}$. However, we let the implicit constant depend on dimension n and the Lebesque index p. These can be considered as fixed parameters and thus we will in general not write $\lesssim_{n,p}$.

Cubes have a big role in the later chapters of this thesis. In order to make unique partitions of cubes into smaller cubes, we make a restriction that throughout the cubes that appear in the statement of the theorems, lemmas, propositions and definitions have side length in $2^{\mathbb{Z}}$. These cubes are usually noted by B, Q, Δ, q. However, in proofs we might construct cubes that have arbitrary side lengths. The notation $B(c_B, R)$ means that the center of the cube is c_B and the side length is R. By $\text{Part}_\alpha(Q)$ we denote the unique partition of a cube Q into cubes of side length α. All cubes will have sides that are parallel to the coordinate axes. This means that we can uniquely define a cube using only the center and side length. The weight function associated to a cube $B = B(c_B, R)$ is defined by

$$w_{B,E}(x) := \left(1 + \frac{|x - c_B|}{R}\right)^{-E}.$$

Since B is reserved for cubes, we will denote the euclidean ball in $\mathbb{R}^n$ centered at x with radius r by $B_\circ^n(x, r)$. The Euclidean norm is denoted simply by $|\cdot|$, i.e., for $x \in \mathbb{R}^n$ we write $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

For a set S, we denote cS by the dilation of S by a factor of c around its center, where $c \in \mathbb{R}_+$.

We define $\pi_j \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ to be the projection map that forgets the j-th coordinate, i.e.,

$$\pi_j(x_1, \dots, x_n) \coloneqq (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

For a discrete set A, we denote the cardinality of A with $\#A$. When we are mentioning measurability whilst not explicitly introducing a measure, we are considering the Lebesque measure in $\mathbb{R}^n$ which we denote by m_n. If there is no risk of confusion, we might also use $|\cdot|$ instead of m_n.

For a function $g \colon Q \to \mathbb{C}$ the extension operator E_Q is defined by the formula

$$E_Q g(x) \coloneqq \int_Q g(\xi) e(x_1\xi_1 + \dots + x_{n-1}\xi_{n-1} + |\xi|^2 x_n) \, \mathrm{d}\xi.$$

For averaged integrals we use notations

$$\fint_\Omega f \, \mathrm{d}m_n \coloneqq \frac{1}{m_n(\Omega)} \int_\Omega f \, \mathrm{d}m_n = \frac{1}{|\Omega|} \int_\Omega f \, \mathrm{d}m_n.$$

For a measurable set $\Omega \subset \mathbb{R}^n$ and a measurable function f, we write

$$\|f\|_{L^p(\Omega)} \coloneqq \left(\int_\Omega |f|^p \, \mathrm{d}m_n \right)^{\frac{1}{p}}, \qquad 0 < p < \infty$$

and

$$\|f\|_{L^\infty(\Omega)} \coloneqq \inf\{c > 0 : m_n(\{x \in \Omega : |f(x)| > c\}) = 0\}.$$

If $\Omega = \mathbb{R}^n$, then we also use $\|f\|_p = \|f\|_{L^p(\Omega)}$. We characterize the L^p-spaces by

$$L^p(\Omega) \coloneqq \{f \colon \Omega \to \mathbb{C} \text{ measurable} : \|f\|_{L^p(\Omega)} < \infty\}, \qquad 0 < p \le \infty.$$

The local L^p-spaces are denoted by

$$L^p_{loc}(\Omega) \coloneqq \{f \colon \Omega \to \mathbb{C} \text{ measurable} : \|f\|_{L^p(K)} < \infty, \forall K \subset \Omega \text{ compact}\}, \qquad 0 < p \le \infty.$$

Other variants of L^p norms we use include

$$\|f\|_{L^p_\sharp(B)} \coloneqq \left(\frac{1}{m_n(B)} \int_B |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

$$\|f\|_{L^p(w_{B,E})} \coloneqq \left(\int_{\mathbb{R}^n} |f(x)|^p w_{B,E}(x) \, \mathrm{d}x \right)^{\frac{1}{p}}$$

and

$$\|f\|_{L^p_\sharp(w_{B,E})} := \left(\frac{1}{m_n(B)}\int_{\mathbb{R}^n} |f(x)|^p w_{B,E}(x)\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

For a sequence $(a_i)_{i\in I}$, we write

$$\|(a_i)_{i\in I}\|_{l^p} := \|a_i\|_{l^p} := \left(\sum_{i\in I}|a_i|^p\right)^{\frac{1}{p}}, \qquad 0<p<\infty.$$

If there is no risk of confusion, we will also write $\|a_i\|_p = \|a_i\|_{l^p}$.

For the operator norm of a linear operator $L\colon V\to W$ between normed spaces $(V,\|\cdot\|_V)$ and $(W,\|\cdot\|_W)$, we use the notation

$$\|L\|_{op} := \inf\{c>0 : \|Lv\|_W \le c\|v\|_V \text{ for all } v\in V\}.$$

We also reserve the notation $\langle\cdot,\cdot\rangle$ for functionals. For a functional T and a function ϕ, the expression $\langle T,\phi\rangle$ stands for the value that T sends ϕ to.

A multi-index is a sequence of natural numbers $(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$. The norm of a multi-index is defined by $|\alpha| := \sum_{i=1}^n \alpha_i$ and we have the following notations $x^\beta := x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}$, $\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $\binom{\alpha}{\gamma} := \prod_{i=1}^n\binom{\alpha_i}{\gamma_i}$ and $\gamma\le\alpha \Leftrightarrow \gamma_i\le\alpha_i$, $\forall i$.

The class of Schwartz test functions is

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi\in C^\infty(\mathbb{R}^n) : \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^N|\partial^\alpha\varphi(x)|<\infty \quad \forall N\in\mathbb{N}\},$$

where

$$C^\infty(\mathbb{R}^n) := \{\varphi\colon\mathbb{R}^n\to\mathbb{C} : \partial^\alpha\varphi \text{ exists and is continuous for all } \alpha\in\mathbb{N}^n\}.$$

Furthermore, the topology of the space $\mathcal{S}(\mathbb{R}^n)$ is induced by the collection of seminorms

$$p_N(\varphi) := \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}(1+|x|^2)^N|\partial^\alpha\varphi(x)|.$$

The space of continuous linear mappings from $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{C}$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$. We also call the elements of $\mathcal{S}'(\mathbb{R}^n)$ tempered distributions.

Throughout the thesis we will write $e(t) := e^{2\pi i t}$, where $t\in\mathbb{R}$.

For an integrable function f the convention of Fourier transform we use is

$$\widehat{f}(\xi) := \mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)e(-\xi\cdot x)\,\mathrm{d}x.$$

We use the same notations $\widehat{f}$ and $\mathcal{F}f$ in the general case where f is a tempered distribution. The analogous inverse Fourier transform is denoted by $\check{f}$ and $\mathcal{F}^{-1}f$.

1.3 Preliminary results

In this section, we present and recall some general results that highlight the techniques we use to prove the results in this text.

Proposition 1.1. *Let L be a linear operator $L\colon V \to W$ between normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, where $V \neq \{0\}$. For the operator norms*

$$\|L\|_S := \sup\{\|Lv\|_W : v \in V,\ \ \|v\|_V = 1\},$$
$$\|L\|_B := \sup\{\|Lv\|_W : v \in V,\ \ \|v\|_V \leq 1\},$$
$$\|L\|_F := \sup\{\frac{\|Lv\|_W}{\|v\|_V} : v \in V,\ \ \|v\|_V \neq 0\},$$

we have

$$\|L\|_{op} = \|L\|_S = \|L\|_B = \|L\|_F.$$

Proof. The properties of supremum directly gives us that $\|L\|_S \leq \|L\|_B$. If $\|v\|_V \leq 1$ and $\|v\|_V \neq 0$, then $\|Lv\|_W \leq \frac{\|Lv\|_W}{\|v\|_V} \leq \|L\|_F$. Furthermore, $\|v\|_V = 0$ implies $\|Lv\|_W = 0$. Thus we have $\|L\|_B \leq \|L\|_F$. Also if $\|v\|_V \neq 0$, then linearity implies that $\frac{\|Lv\|_W}{\|v\|_V} = \|L(\frac{v}{\|v\|_V})\|_W \leq \|L\|_S$ and hence $\|L\|_F \leq \|L\|_S$. Combining these arguments we get that $\|L\|_S = \|L\|_B = \|L\|_F$.

Now it suffices to show that $\|L\|_{op} = \|L\|_F$. We know that $\|L\|_F \geq \frac{\|Lv\|_W}{\|v\|_V} \Leftrightarrow \|Lv\|_W \leq \|L\|_F\|v\|_V$ and hence $\|L\|_F \geq \|L\|_{op}$. For the other direction notice that for all $\varepsilon > 0$ there exists $v' \in V$, such that $\|L\|_{op} \geq \frac{\|Lv'\|_W}{\|v'\|_V} \geq \|L\|_F - \varepsilon$. Letting $\varepsilon \to 0$ gives us $\|L\|_{op} \geq \|L\|_F$ and we are done. □

The following lemma gives us a quick way to check that a bounded operator is bilipschitz.

Lemma 1.2. *Let L be a bounded linear operator $L\colon V \to W$ between normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. If the inverse operator L^{-1} exists and is bounded, then both L and L^{-1} are K-bilipschitz with $K = \max\{\|L\|_{op}, \|L^{-1}\|_{op}\}$.*

Proof. Since L and L^{-1} are bounded, we have $\|L\|_{\mathrm{op}} < \infty$ and $\|L^{-1}\|_{\mathrm{op}} < \infty$. Thus

$$\begin{aligned} &\|u\|_V = \|L^{-1}Lu\|_V \leq \|L^{-1}\|_{op}\|Lu\|_W \leq \|L^{-1}\|_{op}\|L\|_{op}\|u\|_V \\ \Leftrightarrow\quad & \frac{\|u\|_V}{\|L^{-1}\|_{op}} \leq \|Lu\|_W \leq \|L\|_{op}\|u\|_V \\ \Rightarrow\quad & \frac{1}{\max\{\|L\|_{op}, \|L^{-1}\|_{op}\}}\|u\|_V \leq \|Lu\|_W \leq \max\{\|L\|_{op}, \|L^{-1}\|_{op}\}\|u\|_V \end{aligned}$$

and similarly

$$\frac{1}{\max\{\|L\|_{op}, \|L^{-1}\|_{op}\}}\|u'\|_W \leq \|L^{-1}u'\|_V \leq \max\{\|L\|_{op}, \|L^{-1}\|_{op}\}\|u'\|_W.$$

□

In chapter 2 we will use singular value decomposition to estimate the operator norms of a certain linear transformation. We refer to [15] chapter 5 section 6 for the proof of the following well known result.

Theorem 1.3 (Singular value decomposition)**.** *An $n \times m$ matrix M can be factored as*

$$M = U\Sigma V^\top,$$

where U is an $n \times n$ orthogonal matrix and V is an $m \times m$ orthogonal matrix and Σ is a $n \times m$ diagonal matrix with non-negative entries. The number of positive entries in Σ is the same as the rank of M.

The next theorem generalizes the regular Hölder inequality for multiple products.

Theorem 1.4 (Generalized Hölder inequality)**.** *Assume that $p_1, \ldots, p_{n+1} \in \,]0, \infty]$ are such that*

$$\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{p_{n+1}}.$$

Then, for all measurable functions $f_1, \ldots, f_n$, we have

$$\Big\|\prod_{j=1}^{n} f_j\Big\|_{p_{n+1}} \leq \prod_{j=1}^{n} \|f_j\|_{p_j}.$$

Proof. Let $p_i \in \,]0, \infty[$, for $i = 1, 2, \ldots, n$. We proceed by induction on the number of products n. Assume that $n = 1$, we now have

$$\frac{1}{p_1} = \frac{1}{p_2} \Leftrightarrow p_2 = p_1.$$

Thus we immediately get

$$\|f_1\|_{p_2} = \|f_1\|_{p_1}.$$

For the general case $n = k$ assume that the inequality holds for $n = k - 1$ i.e., for

$$\sum_{j=1}^{k-1} \frac{1}{p_j} = \frac{1}{q_k}$$

we have

$$\Big\|\prod_{j=1}^{k-1} f_j\Big\|_{q_k} \le \prod_{j=1}^{k-1} \|f_j\|_{p_j}.$$

Since

$$\frac{1}{\frac{p_k}{p_{k+1}}} + \frac{1}{\frac{q_k}{p_{k+1}}} = \frac{p_{k+1}}{p_k} + \frac{p_{k+1}}{q_k} = p_{k+1}\left(\frac{1}{p_k} + \frac{1}{q_k}\right) = p_{k+1}\left(\sum_{j=1}^{k} \frac{1}{p_j}\right) = 1,$$

we can use $\frac{p_k}{p_{k+1}}, \frac{q_k}{p_{k+1}}$ as Hölder conjugates. Thus

$$\begin{aligned}\Big\|\prod_{j=1}^{k} f_j\Big\|_{p_{k+1}} = \left(\Big\|\prod_{j=1}^{k} f_j^{p_{k+1}}\Big\|_1\right)^{\frac{1}{p_{k+1}}} &\le \left(\Big\|\prod_{j=1}^{k-1} f_j^{p_{k+1}}\Big\|_{\frac{q_k}{p_{k+1}}} \|f_k^{p_{k+1}}\|_{\frac{p_k}{p_{k+1}}}\right)^{\frac{1}{p_{k+1}}} \\ &= \Big\|\prod_{j=1}^{k-1} f_j\Big\|_{q_k} \|f_k\|_{p_k} \le \prod_{j=1}^{k} \|f_j\|_{p_j}.\end{aligned}$$

Lastly, assume that $p_i = \infty$ for $i \in I$, where I is an arbitrary subset of $\{1, \dots, n\}$. Now we have

$$\sum_{i \notin I} \frac{1}{p_i} = \frac{1}{p_{n+1}}$$

and hence the problem reduces to the previous case

$$\Big\|\prod_{j=1}^{n} f_j\Big\|_{p_{n+1}} \le \prod_{i \in I} \|f_i\|_\infty \Big\|\prod_{i \notin I} f_i\Big\|_{p_{n+1}} \le \prod_{j=1}^{n} \|f_j\|_{p_j}$$

and we are done. □

Note that the same proof can be done for sequences $a_{k,1}, \dots, a_{k,n} \in \mathbb{C}^{\mathbb{N}}$ when $p_i \in]0, \infty[$, for $i = 1, \dots, n+1$.

The following is a simple yet effective upper bound for Schwartz test functions.

Lemma 1.5. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $s > 0$. Then there exists an index $N = N(s) \in \mathbb{N}$ such that*

$$|\varphi(x)| \lesssim_{s, p_N(\varphi)} \frac{1}{(1+|x|)^s},$$

for all $x \in \mathbb{R}^n$.

Proof. If $|x| \le 1$, then

$$(1+|x|)^s |\varphi(x)| \le 2^s |\varphi(x)| \lesssim_s p_0(\varphi) \le p_N(\varphi) \sim_{p_N(\varphi)} 1.$$

If $|x| > 1$, then choose $N = \lceil s \rceil$ and calculate

$$\begin{aligned}(1 + |x|)^s |\varphi(x)| &\le (1 + |x|^2)^s |\varphi(x)| \\ &\le (1 + |x|^2)^N |\varphi(x)| \\ &\le p_N(\varphi) \lesssim_{p_N(\varphi)} 1.\end{aligned}$$

□

Remark 1.6. By the definition of $\mathcal{S}(\mathbb{R}^n)$, the dependence on $p_N(\varphi)$ in lemma 1.5 is harmless as long as φ does not depend on anything critical. For contrast, the Schwartz function in lemma 1.8 is dependent on a critical variable and hence lemma 1.5 is not enough to give the estimate in lemma 1.8 part *b*.

Next we will prove the existence of an important Schwartz function. The following bump function will be used in most of the Schwartz function constructions in this thesis. In some cases, we can also find a uniform estimate for the Fourier transform of the product of this function and a polynomial.

Lemma 1.7. *Let $K \subset \mathbb{R}^n$ be compact, $V \subset \mathbb{R}^n$ open and $K \subset V$. If V is bounded, then there exists a function $f \in \mathcal{S}(\mathbb{R}^n)$, such that*

$$|f| \le 1, \qquad \mathrm{Supp}(f) \subset V \quad \textit{and} \quad f = 1 \ \textit{on}\ K.$$

Proof. Define $\varepsilon = \frac{d(K, \partial V)}{2}$ and $U_\varepsilon := \{x \in \mathbb{R}^n : d(K, x) < \varepsilon\}$. Consider the function $h \in C^\infty(\mathbb{R})$ that is defined by $h(x) = e^{-\frac{1}{x}} \mathbb{1}(x > 0)$. We define $\eta \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(x) = \frac{h(1 - |x|^2)}{\int_{B_\circ^n(0,1)} h(1 - |t|^2)\,\mathrm{d}t}$$

and $\eta_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta(\frac{x}{\varepsilon}).$$

Now $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\mathrm{Supp}(\eta_\varepsilon) \subset B_\circ^n(0, \varepsilon)$ and $\|\eta_\varepsilon\|_1 = 1$. All that is left is to check that

$$f(x) = \mathbb{1}_{U_\varepsilon} * \eta_\varepsilon(x) = \int_{U_\varepsilon} \eta_\varepsilon(x - y)\,\mathrm{d}y$$

is the desired function.

By the dominated convergence theorem we can differentiate under the integral sign to see that f is smooth. For every $x \in \mathbb{R}^n$, we have

$$|f(x)| \le \|\eta_\varepsilon\|_1 = 1.$$

If $x \notin V$ and $y \in U_\varepsilon$, then $|x-y| > \varepsilon$ and thus

$$f(x) = \int_{U_\varepsilon} \eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{U_\varepsilon} 0\,\mathrm{d}y = 0.$$

Furthermore, if $x \in K$, then we have that $B^n_\circ(0,\varepsilon) \subset \{x-y : y \in U_\varepsilon\} =: x - U_\varepsilon$ and thus

$$f(x) = \int_{U_\varepsilon} \eta_\varepsilon(x-y)\,\mathrm{d}y = \int_{x-U_\varepsilon} \eta_\varepsilon(z)\,\mathrm{d}z = \int_{B^n_\circ(0,\varepsilon)} \eta_\varepsilon(z)\,\mathrm{d}z = \|\eta_\varepsilon\|_1 = 1.$$

□

Lemma 1.8. *Assume that $a = e^{-c}$, for some $c \sim 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function that is supported in $]-a, a[$ (see lemma 1.7 for the existence of such f) and define $g(x) = f(x)x^k$, where $k \in \mathbb{N}$. Then we have*

(a) $\|\frac{\mathrm{d}^j}{\mathrm{d}x^j} g\|_\infty \lesssim_j 1$,

(b) $|\widehat{g}(\xi)| \lesssim_M (1+|\xi|)^{-M}$, *for all $M > 0$.*

Particularly, the above estimates are uniform with respect to k.

Proof. Note that for $k = 0$, the results follow from lemma 1.5. Thus we can assume that $k \in \mathbb{N}_+$ and hence $a < k$.

(a) By the Leibniz formula for differentiation of products, we have

$$\begin{aligned}
\frac{\mathrm{d}^j}{\mathrm{d}x^j} g(x) &= \sum_{l=0}^{j} \binom{j}{l} \frac{\mathrm{d}^l}{\mathrm{d}x^l} x^k \frac{\mathrm{d}^{j-l}}{\mathrm{d}x^{j-l}} f(x) \\
&= \sum_{l=0}^{\min(j,k)} \binom{j}{l} \frac{\mathrm{d}^l}{\mathrm{d}x^l} x^k \frac{\mathrm{d}^{j-l}}{\mathrm{d}x^{j-l}} f(x) \\
&= \sum_{l=0}^{\min(j,k)} \binom{j}{l} \frac{k!}{(k-l)!} x^{k-l} \frac{\mathrm{d}^{j-l}}{\mathrm{d}x^{j-l}} f(x).
\end{aligned}$$

Taking L^∞ norms from both sides leads to

$$\begin{aligned}
\left\|\frac{\mathrm{d}^j}{\mathrm{d}x^j}g\right\|_\infty &\le \sum_{l=0}^{\min(j,k)} \binom{j}{l}\frac{k!}{(k-l)!}a^{k-l}\left\|\frac{\mathrm{d}^{j-l}}{\mathrm{d}x^{j-l}}f\right\|_\infty \\
&\le \sum_{l=0}^{\min(j,k)} j!\,k^l a^{k-l} \sup_{0\le m\le j}\left\|\frac{\mathrm{d}^m}{\mathrm{d}x^m}f\right\|_\infty \\
&\sim_j \sum_{l=0}^{\min(j,k)} \frac{k^j a^k}{k^{j-l}a^l} \\
&\le \sum_{l=0}^{\min(j,k)} \frac{k^j a^k}{a^{j-l}a^l} \\
&\le (j+1)\,\frac{k^j a^k}{a^j} \sim_j k^j a^{k-j}.
\end{aligned}$$

The L'Hospital rule applied j times yields

$$\lim_{k\to\infty} k^j a^{k-j} = \lim_{k\to\infty}\frac{k^j}{a^j a^{-k}} = \lim_{k\to\infty}\frac{j!}{(-a\ln a)^j a^{-k}} = \frac{j!}{(-a\ln a)^j}\lim_{k\to\infty} a^k = 0,$$

which implies that there exists a real number k_* that maximizes the function $k\mapsto k^j a^{k-j}$ on $[0,\infty[$. A routine analysis of zeros of the derivatives reveals that $k_* = -\frac{j}{\ln a} = \frac{j}{c}$. Hence

$$\left\|\frac{\mathrm{d}^j}{\mathrm{d}x^j}g\right\|_\infty \lesssim_j k_*^j\, a^{k_*-j} = \left(\frac{j}{c}\right)^j e^{cj-j} \sim_j 1.$$

We have now proven the first part.

(b) If $|y|\le 1$, then

$$(1+|y|)^M|\widehat{g}(y)| \le 2^M|\widehat{g}(y)| \sim_M |\widehat{g}(y)| \le \|g\|_1 \sim 1.$$

If $|y|>1$, then repeated integration by parts and part (a) yields

$$\begin{aligned}
(1+|y|)^M|\widehat{g}(y)| &\lesssim_M |y|^M|\widehat{g}(y)| \\
&= \left|\int_{\mathbb{R}} g(x)y^M e^{2\pi ixy}\,\mathrm{d}x\right| \\
&= \left|\int_{-1}^{1} g(x)y^M e^{2\pi ixy}\,\mathrm{d}x\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_{-1}^{1} \frac{\mathrm{d}g(x)}{\mathrm{d}x} y^M \frac{e^{-2\pi i x y}}{-2\pi i y} \,\mathrm{d}x \right| \\
&\quad \vdots \\
&= \left| \int_{-1}^{1} \frac{\mathrm{d}^{\lceil M \rceil} g(x)}{\mathrm{d}x^{\lceil M \rceil}} y^M \frac{e^{-2\pi i x y}}{(-2\pi i y)^{\lceil M \rceil}} \,\mathrm{d}x \right| \\
&\leq \left\| \frac{\mathrm{d}^{\lceil M \rceil} g}{\mathrm{d}x^{\lceil M \rceil}} \right\|_\infty \left| y^{M-\lceil M \rceil} \int_{-1}^{1} \frac{e^{-2\pi i x y}}{(-2\pi i)^{\lceil M \rceil}} \,\mathrm{d}x \right| \\
&\leq \left\| \frac{\mathrm{d}^{\lceil M \rceil} g}{\mathrm{d}x^{\lceil M \rceil}} \right\|_\infty \left| \int_{-1}^{1} \frac{e^{-2\pi i x y}}{(-2\pi i)^{\lceil M \rceil}} \,\mathrm{d}x \right| \\
&\lesssim_M \left| \int_{-1}^{1} \frac{e^{-2\pi i x y}}{(-2\pi i)^{\lceil M \rceil}} \,\mathrm{d}x \right| \leq \frac{2}{(2\pi)^{\lceil M \rceil}} \leq 1
\end{aligned}
$$

and we are done.

□

Note that essentially the same proof gives us lemma 1.8 part b for the inverse Fourier operator.

If $\Omega \subset \mathbb{R}^n$, $f \in L^1_{loc}(\Omega)$ and $\varphi \colon \Omega \to \mathbb{C}$ is a compactly supported smooth function, then T_f is a functional that is defined by the formula

$$
\langle T_f, \varphi \rangle := \int_\Omega f(x)\varphi(x) \,\mathrm{d}x. \tag{1.9}
$$

When we are discussing distributions the symbol f will stand for T_f.

Remark 1.10. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then in formula (1.9) it is not enough to say that f is locally integrable. However, if $f \in L^p(\mathbb{R}^n)$, then (1.9) defines a continuous linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{C}$, i.e., the linear mapping is a tempered distribution.

The following result states that formula (1.9) is an injection in $L^1_{loc}(\mathbb{R}^n)$. We will use this to convert distributional equalities into classical ones. The result is true also in $L^1_{loc}(\Omega)$, but we will not need this.

Lemma 1.11. *Let $f, g \in L^1_{loc}(\mathbb{R}^n)$. Then $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for every compactly supported smooth function φ implies that $f(x) = g(x)$ for almost every $x \in \mathbb{R}^n$.*

Proof. We choose $\varphi_x(y) = \eta_\varepsilon(x - y)$, where η_ε is the standard mollifier defined in the proof of lemma 1.7.

$$\begin{aligned}\langle f, \varphi_x \rangle &= \int_{\mathbb{R}^n} f(y)\varphi_x(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(y)\eta_\varepsilon(x-y)\,\mathrm{d}y \\ &= f * \eta_\varepsilon(x).\end{aligned}$$

We also have

$$\lim_{\varepsilon\to 0} \|f * \eta_\varepsilon - f\|_{L^1(K)} = 0,$$

for every compact $K \subset \mathbb{R}^n$. In particular, this is true for K_i, where $(K_i)_{i\in\mathbb{N}}$ is a compact exhaustion of $\mathbb{R}^n$. Then there exists a subsequence ε_1 such that $f * \eta_{\varepsilon_1} \to f$ almost everywhere in K_1. For this sequence we have

$$\lim_{\varepsilon_1\to 0} \|f * \eta_{\varepsilon_1} - f\|_{L^1(K_2)} = 0$$

and hence there exists a subsequence ε_2 of ε_1 of such that $f * \eta_{\varepsilon_2} \to f$ almost everywhere in $K_2 \supset K_1$. Continuing this inductively yields that there exists a subsequence ε_n such that $f * \eta_{\varepsilon_n} \to f$ almost everywhere in K_i for all $i \in \mathbb{N}$ and hence we also have $f * \eta_{\varepsilon_n} \to f$ almost everywhere in $\mathbb{R}^n$.

On the other hand we similarly have $g * \eta_\varepsilon \to g$ almost everywhere along a subsequence ε_{n_j} of the sequence ε_n. Since the limit of a subsequence equals the limit of the original sequence, we now get that for almost every $x \in \mathbb{R}^n$, we have

$$f(x) \leftarrow f * \eta_{\varepsilon_{n_j}}(x) = g * \eta_{\varepsilon_{n_j}}(x) \to g(x).$$

Thus we must have $f(x) = g(x)$ for almost every $x \in \mathbb{R}^n$. □

The next two lemmas are slightly generalized versions of well known results. The first is a generalized version of the Fourier inversion formula.

Lemma 1.12. *Let $f \in L^\infty(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n)$. Then we have*

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e(\xi \cdot x)\,\mathrm{d}\xi$$

for almost every $x \in \mathbb{R}^n$.

Proof. Since $\widehat{f} \in L^1(\mathbb{R}^n)$ we know that the expression

$$\int_{\mathbb{R}^n} \widehat{f}(\xi)e(x \cdot \xi)\,\mathrm{d}\xi$$

converges to a continuous function f_1. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and calculate by Fubini's theorem that

$$\begin{aligned}\langle \mathcal{F}^{-1}\widehat{f}, \varphi\rangle &= \langle \widehat{f}, \mathcal{F}^{-1}\varphi\rangle \\ &= \int_{\mathbb{R}^n} \widehat{f}(\xi) \int_{\mathbb{R}^n} \varphi(x) e(\xi \cdot x)\,\mathrm{d}x\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \varphi(x) \int_{\mathbb{R}^n} \widehat{f}(\xi) e(\xi \cdot x)\,\mathrm{d}\xi\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f_1(x)\varphi(x)\,\mathrm{d}x \\ &= \langle f_1, \varphi\rangle.\end{aligned}$$

Thus f_1 is the distributional inverse Fourier transform of $\widehat{f}$. This combined with the Fourier inverse formula of $\mathcal{S}'(\mathbb{R}^n)$ gives that

$$\langle f, \varphi\rangle = \langle f_1, \varphi\rangle.$$

Now lemma 1.11 yields that $f = f_1$ almost everywhere in the classical sense. Thus we have proven the inversion formula

$$f(x) = f_1(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e(x \cdot \xi)\,\mathrm{d}\xi,$$

for almost every $x \in \mathbb{R}^n$. □

In the second generalization we weaken the assumptions of the convolution theorem.

Lemma 1.13. *Let $\varphi, \phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in L^\infty(\mathbb{R}^n)$. Then*

$$\langle \mathcal{F}(f * \varphi), \phi\rangle = \langle \widehat{f}\,\widehat{\varphi}, \phi\rangle.$$

Proof. Denote $R\varphi(x) \coloneqq \varphi(-x)$. The Fourier inversion formula implies that

$$R\varphi(x) = \mathcal{F}(\widehat{\varphi})(x). \tag{1.14}$$

Furthermore, Fubini's theorem and the above equality gives

$$\mathcal{F}\left(\widehat{\eta} * \widehat{\psi}\right) = \mathcal{F}(\widehat{\eta})\mathcal{F}(\widehat{\psi}) = R(\eta)R(\psi) = R(\eta\psi) = \mathcal{F}(\mathcal{F}(\eta\psi)), \qquad \text{for all } \eta, \psi \in \mathcal{S}(\mathbb{R}^n).$$

Taking inverse Fourier transforms from both sides yields

$$\widehat{\eta} * \widehat{\psi} = \mathcal{F}(\eta\psi), \qquad \text{for all } \eta, \psi \in \mathcal{S}(\mathbb{R}^n).$$

Plugging $\eta = \widehat{\varphi}$ and $\psi = \phi$ to the above equation yields

$$\mathcal{F}(\widehat{\varphi}) * \widehat{\phi} = \mathcal{F}(\widehat{\varphi}\phi). \tag{1.15}$$

Now using Fubini's theorem, (1.14) and (1.15) we can calculate that

$$\begin{aligned}
\langle \mathcal{F}(f * \varphi), \phi \rangle &= \langle f * \varphi, \widehat{\phi} \rangle \\
&= \int_{\mathbb{R}^n} f * \varphi(x) \widehat{\phi}(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y) \varphi(x - y) \widehat{\phi}(x) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} f(y) \int_{\mathbb{R}^n} \varphi(x - y) \widehat{\phi}(x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} f(y) \, R\varphi * \widehat{\phi}(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} f(y) \, \mathcal{F}(\widehat{\varphi}) * \widehat{\phi}(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} f(y) \, \mathcal{F}(\widehat{\varphi}\phi)(y) \, \mathrm{d}y \\
&= \langle f, \mathcal{F}(\widehat{\varphi}\phi) \rangle = \langle \widehat{f}, \widehat{\varphi}\phi \rangle = \langle \widehat{f}\widehat{\varphi}, \phi \rangle.
\end{aligned}$$

□

Lastly, we will aim to prove the existence of two different kinds of tempered distributions and then calculate their Fourier transforms. However, before we can do this, we need a result which gives us a way to check for the continuity of a linear mapping from $\mathcal{S}(\mathbb{R}^n)$ to itself. We will give the result without proof.

Theorem 1.16. *A linear mapping $T \colon \mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{S}(\mathbb{R}^n)$ is continuous if and only if for every $N \in \mathbb{N}$ there exists a number $M \in \mathbb{N}$ and a constant $C_{N,M}$ such that*

$$p_N(Tf) \leq C_{N,M} p_M(f).$$

The proof of the above result can be found in [1] theorem 12.2.

Let $\tau_h f(x) = f(x + h)$, where f is a function. If the following integrals converge, then a change of variables gives

$$\int_{\mathbb{R}^n} f(x + h) \phi(x) \, \mathrm{d}x = \int_{\mathbb{R}^n} f(x) \phi(x - h) \, \mathrm{d}x.$$

If f, g and ϕ are functions, then we have

$$\int_{\mathbb{R}^n} g(x)f(x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)g(x)\phi(x)\,\mathrm{d}x.$$

This motivates the following lemma.

Lemma 1.17. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$. Assume also that for every multi-index α there exists constants M_α and C_α such that*

$$|\partial^\alpha g(x)| \leq C_\alpha (1+|x|^2)^{M_\alpha}. \tag{1.18}$$

Then the functionals gf and $\tau_h f$ defined by

$$\langle gf, \phi \rangle \coloneqq \langle f, g\phi \rangle$$

and

$$\langle \tau_h f, \phi \rangle \coloneqq \langle f, \tau_{-h}\phi \rangle$$

respectively, are continuous in $\mathcal{S}'(\mathbb{R}^n)$, i.e., they are a tempered distributions.

Proof. Since $f \in \mathcal{S}'(\mathbb{R}^n)$, it suffices to show that the mappings

$$\phi \mapsto g\phi$$

and

$$\phi \mapsto \tau_{-h}\phi$$

are continuous in the Schwartz space topology. We start with the first mapping. Applying the Leibniz rule for differentiation and (1.18) gives

$$\begin{aligned}
p_N(g\phi) &= \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |\partial^\alpha \left(g(x)\phi(x)\right)| \\
&= \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |\sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \partial^\gamma g(x) \partial^{\alpha-\gamma}\phi(x)| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |\partial^\gamma g(x)||\partial^{\alpha-\gamma}\phi(x)| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{N+M} C|\partial^{\alpha-\gamma}\phi(x)| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} C p_{N+M}(\phi) = C_N p_{N+M}(\phi),
\end{aligned}$$

where $M := \max_{|\alpha|\le N} M_\alpha$, $C := \max_{|\alpha|\le N} C_\alpha$ and $C_N = 2^N C$.

For the second mapping we calculate

$$\begin{aligned}
p_N(\tau_{-h}\phi) &= \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^N |\partial^\alpha \phi(x-h)| \\
&= \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|h+x-h|^2)^N |\partial^\alpha \phi(x-h)| \\
&\le \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+2|h|^2+2|x-h|^2)^N |\partial^\alpha \phi(x-h)| \\
&\le \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} \Big(2(1+2|h|^2)+2(1+2|h|^2)|x-h|^2\Big)^N |\partial^\alpha \phi(x-h)| \\
&= 2^N(1+2|h|^2)^N \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x-h|^2)^N |\partial^\alpha \phi(x-h)| \\
&= C_{N,h}\, p_N(\phi),
\end{aligned}$$

where $C_{N,h} := 2^N(1+2|h|^2)^N$. An application of theorem 1.16 concludes the proof of the lemma. □

Define $e_h(x) := e(x \cdot h)$ and notice that by the above lemma we have $f \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow e_h f \in \mathcal{S}'(\mathbb{R}^n)$. Next we calculate the Fourier transforms of $\tau_h f$ and $e_h f$.

Lemma 1.19. *Assume that f is a tempered distribution. Then*

$$\widehat{\tau_h f} = e_h \widehat{f} \qquad \textit{and} \qquad \widehat{e_h f} = \tau_{-h}\widehat{f}.$$

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. We calculate that

$$\langle \widehat{\tau_h f}, \phi\rangle = \langle \tau_h f, \widehat{\phi}\rangle = \langle f, \tau_{-h}\widehat{\phi}\rangle. \tag{1.20}$$

On the other hand we have

$$\begin{aligned}
\tau_{-h}\widehat{\phi}(x) &= \int_{\mathbb{R}^n} \phi(\xi) e(-(x-h)\cdot\xi)\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \phi(\xi) e_h(\xi) e(-x\cdot\xi)\,\mathrm{d}\xi = \widehat{e_h\phi}(x)
\end{aligned}$$

and plugging this into (1.20) gives

$$\langle \widehat{\tau_h f}, \phi\rangle = \langle f, \widehat{e_h\phi}\rangle = \langle \widehat{f}, e_h\phi\rangle = \langle e_h\widehat{f}, \phi\rangle$$

and since $\phi \in \mathcal{S}(\mathbb{R}^n)$ was arbitrary, we have proven the left side equation.

For the right side equation we calculate

$$\langle \widehat{e_h f}, \phi \rangle = \langle e_h f, \widehat{\phi} \rangle = \langle f, e_h \widehat{\phi} \rangle \tag{1.21}$$

and by a change of variables we have

$$\begin{aligned} e_h \widehat{\phi}(x) &= \int_{\mathbb{R}^n} \phi(\xi) e(-\xi \cdot x) e(x \cdot h) \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \phi(\xi) e(-x \cdot (\xi - h)) \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \phi(\xi + h) e(-x \cdot \xi) \,\mathrm{d}\xi = \widehat{\tau_h \phi}(x). \end{aligned} \tag{1.22}$$

Combining (1.21) and (1.22) gives

$$\langle \widehat{e_h f}, \phi \rangle = \langle f, \widehat{\tau_h \phi} \rangle = \langle \widehat{f}, \tau_h \phi \rangle = \langle \tau_{-h} \widehat{f}, \phi \rangle$$

and we are done since $\phi \in \mathcal{S}(\mathbb{R}^n)$ was arbitrary. $\square$

Chapter 2

Multilinear Kakeya inequality

In this chapter we give the statement of multilinear Kakeya inequality and prove it. Heuristically, the multilinear Kakeya inequality is a geometric estimate about the overlap of almost axis parallel cylindrical tubes in $\mathbb{R}^n$ pointing in various directions. Roughly, the inequality conveys that the tubes pointing in different directions cannot overlap excessively.

This estimate was originally proven, in a slightly more general setting, by Bennett, Carberry and Tao in [2]. The result was sharpened in [12] by Larry Guth to cover the whole range of the former conjecture. In [12] techniques from algebraic topology are applied to show that theorem 2.1 holds without the multiplier S^ε. However, in order to prove the l^2 decoupling theorem from [5] the version that we will prove is equally useful.

This chapter will mainly follow the more elementary approach of the paper [11] by Larry Guth called *A short proof of the multilinear Kakeya inequality* and every formulation for theorems, lemmas and corollaries (excluding corollary 2.15) will be taken from [11]. The formulation of corollary 2.15 is from [5].

2.1 Statement of the inequality

We will first construct the cylindrical tubes that we discussed earlier. In order to do this, we consider lines $l_{j,a}$ in $\mathbb{R}^n$. Here $j \in \{1, \dots, n\}$ tells us which axis the line is nearly parallel to and $a = 1, \dots, N_j$, where N_j is the number of lines that are nearly parallel to x_j-axis. Now the tubes $T_{j,a}$ are given by the indicator function of the 1-neighbourhood of the line $l_{j,a}$, i.e.,

$$T_{j,a}(x) \coloneqq \mathbb{1}\{x \in \mathbb{R}^n : d(x, l_{j,a}) \le 1\}.$$

Now we can establish the statement of multilinear Kakeya inequality.

Theorem 2.1 (Multilinear Kakeya inequality)**.** *Let $Q_S \subset \mathbb{R}^n$ be an arbitrary cube of side length S. Suppose that each line $l_{j,a}$ in $\mathbb{R}^n$ makes an angle of at most $\frac{1}{10n}$ with the x_j-axis. Then for any $\varepsilon > 0$ and any $S \geq 1$, the following inequality holds*

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \lesssim_\varepsilon S^\varepsilon \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}. \tag{2.2}$$

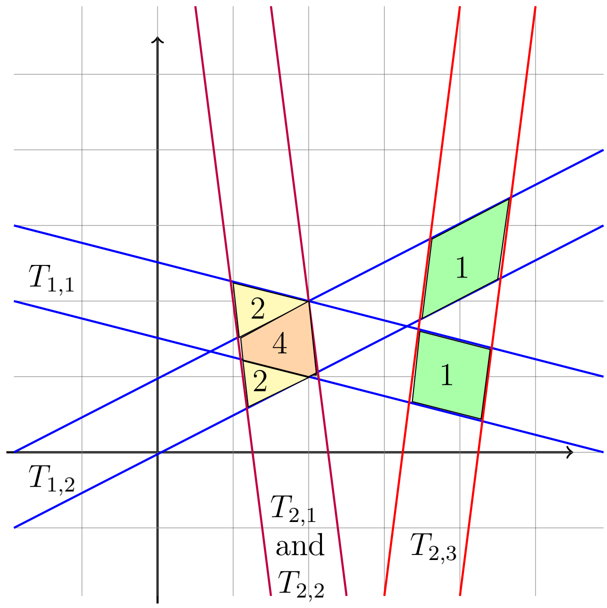

Figure 2.1: An example of the contribution of overlapping tubes on left-hand side of the multilinear Kakeya inequality in $\mathbb{R}^2$. The left-hand side equates to the sum of the weighted areas of the colorful regions.

2.2 Angle reduction

The strategy for proving theorem 2.1 is reducing the angle from $\frac{1}{10n}$ to an even smaller angle δ. This motivates the following theorem.

Theorem 2.3. *For every $\varepsilon > 0$, there exists some $\delta \sim_\varepsilon 1$ such that the following holds. Suppose that each line $l_{j,a}$ in $\mathbb{R}^n$ makes an angle of at most δ with the x_j-axis. Then for* *any $S \geq 1$ and any cube Q_S of side length S, we have*

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \lesssim_{\varepsilon} S^{\varepsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

First we show that theorem 2.3 implies theorem 2.1. Let e_j be unit vector in the x_j direction, and let $S_j \subset \mathbb{S}^{n-1}$ be the e_j centered spherical cap of radius $\frac{1}{10n}$, i.e., $S_j := \{x \in \mathbb{S}^{n-1} : |x - e_j| \leq \frac{1}{10n}\}$. If the direction vector of a line $l_{j,a}$ hits S_j, then we write $l_{j,a} \in S_j$. By the assumptions of theorem 2.1, we have that $l_{j,a} \in S_j$. For a given $\varepsilon > 0$, we choose such a $\delta > 0$ that satisfies theorem 2.3. We cover the cap S_j with smaller caps S_{j,β_j} of radius $\frac{\delta}{2}$. We also make the covering so that the number of caps S_{j,β_j} is at most a constant dependent only on ε. This is possible since our choice of δ will essentially depend only on ε. We want to break the left-hand side of the inequality (2.2) into contributions from different caps S_{j,β_j}. Denote

$$A_{j,\beta_j} := \{a \in \{1, \dots, N_j\} : l_{j,a} \in S_{j,\beta_j}\}.$$

By concavity, we have the following inequality

$$\begin{aligned} \int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x &= \int_{Q_S} \prod_{j=1}^{n} \left(\sum_{\beta_j} \sum_{a \in A_{j,\beta_j}} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \\ &\leq \int_{Q_S} \prod_{j=1}^{n} \sum_{\beta_j} \left(\sum_{a \in A_{j,\beta_j}} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x. \end{aligned} \tag{2.4}$$

We abbreviate

$$Y_{j,\beta_j}(x) := \sum_{a \in A_{j,\beta_j}} T_{j,a}(x).$$

Now by (2.4) and the distributive law we have

$$\begin{aligned} \int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x &\leq \int_{Q_S} \prod_{j=1}^{n} \sum_{\beta_j} Y_{j,\beta_j}(x)^{\frac{1}{n-1}} \mathrm{d}x \\ &= \int_{Q_S} \left(\sum_{\beta_1} Y_{1,\beta_1}(x)^{\frac{1}{n-1}} \right) \cdots \left(\sum_{\beta_n} Y_{n,\beta_n}(x)^{\frac{1}{n-1}} \right) \mathrm{d}x \end{aligned}$$

$$= \int_{Q_S} \sum_{\beta_1} \cdots \sum_{\beta_n} Y_{1,\beta_1}(x)^{\frac{1}{n-1}} \cdots Y_{n,\beta_n}(x)^{\frac{1}{n-1}} \, dx$$

$$= \int_{Q_S} \sum_{\beta_1,\ldots,\beta_n} \prod_{j=1}^{n} Y_{j,\beta_j}(x)^{\frac{1}{n-1}}$$

$$= \sum_{\beta_1,\ldots,\beta_n} \int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a\in A_{j,\beta_j}} T_{j,a}(x) \right)^{\frac{1}{n-1}} \tag{2.5}$$

Now we want to control (2.5) with theorem 2.3. If S_{j,β_j} contains e_j, then theorem 2.3 applies directly. If not, then we make a linear transformation L that maps the center of S_{j,β_j} to e_j for every $j \in \{1,\ldots,n\}$. This implies the assumptions of theorem 2.3. We claim that this transformation distorts the lengths of vectors at most by a factor of 2.

Denote the center of S_{j,β_j} with v_{j,β_j} and define $M := [v_{1,\beta_1} \ldots v_{n,\beta_n}] \in \mathbb{R}^{n\times n}$. Now M defines a linear mapping such that $Me_j = v_{j,\beta_j}$. For $x = (x_1,\ldots,x_n) = \sum_{i=1}^n x_i e_i$ we have

$$|Mx| \geq |x| - |Mx - x| = |x| - |\sum_{i=1}^{n} x_i M e_i - x_i e_i|$$

$$= |x| - |\sum_{i=1}^{n} x_i v_{i,\beta_i} - x_i e_i| \geq |x| - \sum_{i=1}^{n} |x_i||v_{i,\beta_i} - e_i|$$

$$\geq |x| - \frac{1}{10n}\left(\sum_{i=1}^{n} 1^2\right)^{\frac{1}{2}} |x| \geq |x|\left(1 - \frac{1}{10\sqrt{n}}\right) \geq \frac{1}{2}|x|. \tag{2.6}$$

In the third to last inequality we used the Cauchy-Schwarz inequality and the fact that $|v_{j,\beta_j} - e_j| \leq \frac{1}{10n}$.

From (2.6) we see that $\operatorname{Ker} M = \{0\}$ and hence the inverse map $L = M^{-1}$ exists and is also a bijection. By the bijectivity of L, we have

$$|Ly| = |x| \leq 2|Mx| = 2|y|, \qquad \text{for all } y \in \mathbb{R}^n.$$

We have now shown that the transformation L distorts the lengths of vectors at most by a factor of 2 and hence volumes are distorted at most by a factor of 2^n.

Thus we can estimate (2.5) further using theorem 2.3

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} dx \leq \sum_{\beta_1,\ldots,\beta_n} \int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a\in A_{j,\beta_j}} T_{j,a}(x) \right)^{\frac{1}{n-1}} dx$$

$$
\begin{aligned}
&\lesssim_\varepsilon \sum_{\beta_1,\ldots,\beta_n} S^\varepsilon \prod_{j=1}^n (\#A_{j,\beta_j})^{\frac{1}{n-1}} \\
&= S^\varepsilon \sum_{\beta_1,\ldots,\beta_n} (\#A_{1,\beta_1})^{\frac{1}{n-1}} \cdots (\#A_{n,\beta_n})^{\frac{1}{n-1}} \\
&= S^\varepsilon \prod_{j=1}^n \sum_{\beta_j} (\#A_{j,\beta_j})^{\frac{1}{n-1}}.
\end{aligned}
$$

Using Hölder inequality for sequences with conjugates $\frac{n-1}{n-2}$ and $n-1$ gives

$$
\begin{aligned}
S^\varepsilon \prod_{j=1}^n \sum_{\beta_j} (\#A_{j,\beta_j})^{\frac{1}{n-1}} &\le S^\varepsilon \prod_{j=1}^n \left(\sum_{\beta_j} \#A_{j,\beta_j} \right)^{\frac{1}{n-1}} \left(\sum_{\beta_j} 1^{\frac{n-1}{n-2}} \right)^{\frac{n-2}{n-1}} \\
&\lesssim_\varepsilon S^\varepsilon \prod_{j=1}^n \left(\sum_{\beta_j} \#A_{j,\beta_j} \right)^{\frac{1}{n-1}} = S^\varepsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}
\end{aligned}
$$

In the last inequality we used the fact that the number of caps S_{j,β_j} is at most a constant that depends only on ε.

2.3 Axis parallel case

Before proving the angle reducted case, we will study the case where every line is exactly parallel to one of the coordinate axes. In this case the multilinear Kakeya inequality follows from the so called Loomis-Whitney inequality. Our proof for the Loomis-Whitney inequality will follow the argument from section 6.1 of [9].

Theorem 2.7 (Loomis-Whitney inequality)**.** *Suppose that $f_j \colon \mathbb{R}^{n-1} \to \mathbb{R}$ are measurable functions. Then the following inequality holds*

$$
\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\pi_j(x))^{\frac{1}{n-1}} \,\mathrm{d}x \le \prod_{j=1}^n \|f_j\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}}.
$$

Proof. Let $h \colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a function. We know that $\int h \le \int |h|$ and $\|h\|_{L^1(\mathbb{R}^{n-1})} = \||h|\|_{L^1(\mathbb{R}^{n-1})}$. Thus we can assume that f_j are non-negative. Furthermore, it suffices to prove that for measurable non-negative $g_j \colon \mathbb{R}^{n-1} \to \mathbb{R}$, we have

$$
\int_{\mathbb{R}^n} \prod_{j=1}^n g_j(\pi_j(x)) \,\mathrm{d}x \le \prod_{j=1}^n \|g_j\|_{L^{n-1}(\mathbb{R}^{n-1})}, \tag{2.8}
$$

since plugging $g_j = f_j^{\frac{1}{n-1}}$ to (2.8) gives us the result. We start proving (2.8) from the base case $n = 2$. Applying Tonelli's theorem we have

$$\int_{\mathbb{R}^2} \prod_{j=1}^{2} g_j(\pi_j(x))\,\mathrm{d}x = \int_{\mathbb{R}^2} g_1(x_2)g_2(x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \prod_{j=1}^{2} \|g_j\|_{L^1(\mathbb{R})}.$$

Now assume that the theorem holds for $n = k-1$, that is, for measurable $g_j' \colon \mathbb{R}^{k-2} \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^{k-1}} \prod_{j=1}^{k-1} g_j'(\pi_j(x))\,\mathrm{d}x \leq \prod_{j=1}^{k-1} \|g_j'\|_{L^{k-2}(\mathbb{R}^{k-2})}.$$

Now for the general case $n = k$ we define

$$h(x_1,\ldots,x_{k-1}) = \int_{\mathbb{R}} \prod_{j=1}^{k-1} g_j(\pi_j(x))\,\mathrm{d}x_k.$$

Now applying the Hölder inequality with $p = k-1$ and $q = \frac{k-1}{k-2}$, we get

$$\begin{aligned}\int_{\mathbb{R}^k} \prod_{j=1}^{k} g_j(\pi_j(x))\,\mathrm{d}x &= \int_{\mathbb{R}^{k-1}} h(x_1,\ldots,x_{k-1})g_k(\pi_k(x))\,\mathrm{d}x_{k-1}\ldots\mathrm{d}x_1 \\ &\leq \|h\|_{L^{\frac{k-1}{k-2}}(\mathbb{R}^{k-1})}\|g_k\|_{L^{k-1}(\mathbb{R}^{k-1})}. \end{aligned} \tag{2.9}$$

We briefly denote $x_j^* \coloneqq \pi_j(x_1,\ldots,x_{k-1})$. By using the generalized Hölder inequality with conjugates $p_k = 1$ and $p_j = k-1$ for all $j \in \{1,\ldots,k-1\}$ and the induction hypothesis on the functions defined by

$$g_j'(x_j^*) \coloneqq \left(\int_{\mathbb{R}} g_j(\pi_j(x))^{k-1}\,\mathrm{d}x_k\right)^{\frac{1}{k-2}},$$

we get

$$\begin{aligned}\|h\|_{L^{\frac{k-1}{k-2}}(\mathbb{R}^{k-1})}^{\frac{k-1}{k-2}} &= \int_{\mathbb{R}^{k-1}} \left(\int_{\mathbb{R}} \prod_{j=1}^{k-1} g_j(\pi_j(x))\,\mathrm{d}x_k\right)^{\frac{k-1}{k-2}} \mathrm{d}x_{k-1}\ldots\mathrm{d}x_1 \\ &= \int_{\mathbb{R}^{k-1}} \left\|\prod_{j=1}^{k-1} g_j \circ \pi_j(x_1,\ldots,x_{k-1},\cdot)\right\|_{L^1(\mathbb{R})}^{\frac{k-1}{k-2}} \mathrm{d}x_{k-1}\ldots\mathrm{d}x_1 \\ &\leq \int_{\mathbb{R}^{k-1}} \prod_{j=1}^{k-1} \|g_j \circ \pi_j(x_1,\ldots,x_{k-1},\cdot)\|_{L^{k-1}(\mathbb{R})}^{\frac{k-1}{k-2}} \mathrm{d}x_{k-1}\ldots\mathrm{d}x_1\end{aligned}$$

$$= \int_{\mathbb{R}^{k-1}} \prod_{j=1}^{k-1} \left(\int_{\mathbb{R}} g_j(\pi_j(x))^{k-1} \, \mathrm{d}x_k \right)^{\frac{1}{k-2}} \mathrm{d}x_{k-1} \ldots \mathrm{d}x_1$$

$$= \int_{\mathbb{R}^{k-1}} \prod_{j=1}^{k-1} g'_j(x_j^*) \, \mathrm{d}x_{k-1} \ldots \mathrm{d}x_1 \le \prod_{j=1}^{k-1} \|g'_j\|_{L^{k-2}(\mathbb{R}^{k-2})}.$$

On the last line we used the induction hypothesis. Thus we have

$$\|h\|_{L^{\frac{k-1}{k-2}}(\mathbb{R}^{k-1})} \le \prod_{j=1}^{k-1} \|g'_j\|_{L^{k-2}(\mathbb{R}^{k-2})}^{\frac{k-2}{k-1}}$$

$$= \prod_{j=1}^{k-1} \left(\int_{\mathbb{R}^{k-2}} g'_j(x_j^*)^{k-2} \, \mathrm{d}x_j^* \right)^{\frac{1}{k-1}}$$

$$= \prod_{j=1}^{k-1} \left(\int_{\mathbb{R}^{k-1}} g_j(\pi_j(x))^{k-1} \, \mathrm{d}\pi_j(x) \right)^{\frac{1}{k-1}} = \prod_{j=1}^{k-1} \|g_j\|_{L^{k-1}(\mathbb{R}^{k-1})}. \tag{2.10}$$

Finally plugging (2.10) to (2.9), we get the desired result.

□

If the line $l_{j,a}$ is parallel to the x_j-axis, then the other coordinates in lines $l_{j,a}$ are constant and we can define $l_{j,a}$ with y_a for some $y_a \in \mathbb{R}^{n-1}$. Thus

$$T_{j,a}(x) = \mathbb{1}\{x \in \mathbb{R}^n : d(x, l_{j,a}) \le 1\}$$
$$= \mathbb{1}\{x \in \mathbb{R}^n : d\left(\pi_j(x), y_a\right) \le 1\} = \mathbb{1}_{B_\circ^{n-1}(y_a,1)}(\pi_j(x)).$$

Applying the Loomis-Whitney inequality with $f_j(\pi_j(x)) = \sum_{a=1}^{N_j} T_{j,a}(x)$ gives

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x = \int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j^{\frac{1}{n-1}} \, \mathrm{d}x \le \prod_{j=1}^{n} \|f_j\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}}$$
$$= |B_\circ^{n-1}(0,1)|^{\frac{n}{n-1}} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}} \sim \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

2.4 Proof of Multilinear Kakeya inequality

Now that we have proven the inequality for axis parallel tubes, all that is left, is to control the effect that happens when the tubes have a small tilt in them. We consider the sequence of scales $\{\delta^{-1}, \delta^{-2}, \dots\}$. The idea is to use these scales to work our way up to the arbitrary scale S. To set up this argument, we need to generalize our tubes a bit. We define

$$T_{j,a,W} := \mathbb{1}\{x \in \mathbb{R}^n : d(x, l_{j,a}) \leq W\} \text{ and } f_{j,W} := \sum_{a=1}^{N_j} T_{j,a,W}.$$

The next lemma will give us a tool to jump between scales.

Lemma 2.11. *Suppose that $l_{j,a}$ are lines with angle at most δ from the x_j-axis. If $S \geq \delta^{-1}W$ and Q_S is any cube of side length S, then*

$$\int_{Q_S} \prod_{j=1}^{n} f_{j,W}(x)^{\frac{1}{n-1}} \, \mathrm{d}x \leq C_n \delta^n \int_{Q_S} \prod_{j=1}^{n} f_{j,\delta^{-1}W}(x)^{\frac{1}{n-1}} \, \mathrm{d}x.$$

Proof. We divide Q_S into subcubes Q of side length between $\frac{1}{20n}\delta^{-1}W$ and $\frac{1}{10n}\delta^{-1}W$. It suffices to prove the lemma for each Q. Since the side length of Q is at most $\frac{1}{10n}\delta^{-1}W$ and the angle between one of the coordinate axes and $l_{j,a}$ is at most δ, we can cover any tube $T_{j,a,W}$ in Q with an axis parallel tube $\widetilde{T}_{j,a,2W}$ in Q (see figure 2.2), i.e., $T_{j,a,W}(x) \leq \widetilde{T}_{j,a,2W}(x)$ for all $x \in Q$. This implies

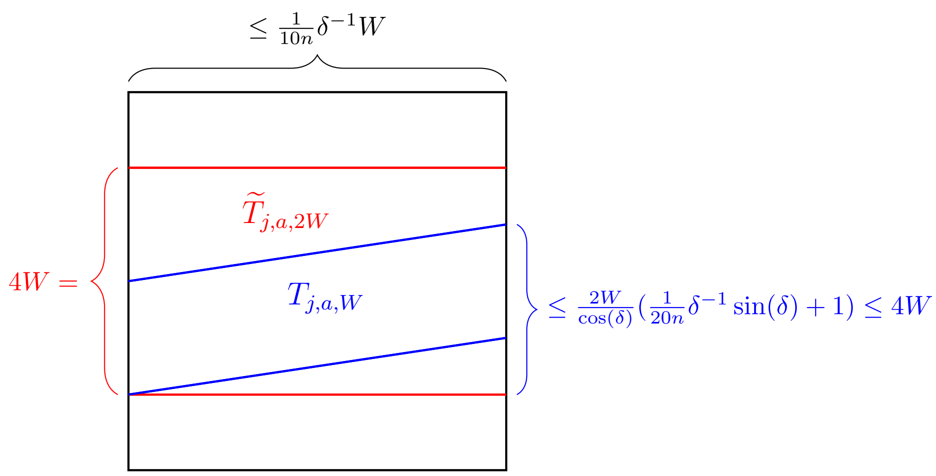

Figure 2.2: Covering an almost axis parallel tube with an axis parallel tube in $\mathbb{R}^2$. This can also be interpreted as a projection from higher dimensions.

$$\int_Q \prod_{j=1}^{n} f_{j,W}(x)^{\frac{1}{n-1}} \, \mathrm{d}x = \int_Q \prod_{j=1}^{n} \left(\sum_{a=1}^{N_{j,Q}} T_{j,a,W}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \leq \int_Q \prod_{j=1}^{n} \left(\sum_{a=1}^{N_{j,Q}} \widetilde{T}_{j,a,2W}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x,$$

where $N_{j,Q}$ is the number of tubes $T_{j,a,W}$ that intersect Q. Now since the tubes $\widetilde{T}_{j,a,2W}$ are axis parallel, we can use the Loomis-Whitney inequality as in section 2.3 to get

$$\begin{aligned}\int_Q \prod_{j=1}^n \sum_{a=1}^{N_{j,Q}} \left(\widetilde{T}_{j,a,2W}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x &\le \prod_{j=1}^n \left\| \sum_{a=1}^{N_{j,Q}} \widetilde{T}_{j,a,2W}(x) \right\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}} \\ &= |B_\circ^{n-1}(0,2W)|^{\frac{n}{n-1}} \prod_{j=1}^n N_{j,Q}^{\frac{1}{n-1}} \\ &= C_n \left(W^{n-1}\right)^{\frac{n}{n-1}} \prod_{j=1}^n N_{j,Q}^{\frac{1}{n-1}} = C_n W^n \prod_{j=1}^n N_{j,Q}^{\frac{1}{n-1}}.\end{aligned}$$

Since the side length of Q is at most $\frac{1}{10n}\delta^{-1}W$, it follows that the diameter of Q is at most $\frac{1}{10\sqrt{n}}\delta^{-1}W$. Thus if $T_{j,a,W}$ intersects Q, then $T_{j,a,\delta^{-1}W}$ is identically 1 in Q and therefore

$$N_{j,Q} \le \sum_{a=1}^{N_j} T_{j,a,\delta^{-1}W}(x), \qquad \text{for } x \in Q,$$

which implies that

$$\begin{aligned}C_n W^n \prod_{j=1}^n N_{j,Q}^{\frac{1}{n-1}} &\le C_n W^n ⨍_Q \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-1}W}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x \\ &\le \frac{C_n W^n}{(\frac{1}{20n}\delta^{-1}W)^n} \int_Q \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-1}W}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x \\ &= C_n' \delta^n \int_Q \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-1}W}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x \\ &= C_n' \delta^n \int_Q \prod_{j=1}^n f_{j,\delta^{-1}W}(x)^{\frac{1}{n-1}} \mathrm{d}x.\end{aligned}$$

□

Now we are ready for the proof of theorem 2.3.

Proof of theorem 2.3. Suppose first that $S = \delta^{-M}$, where $M \in \mathbb{N}_+$. Using lemma 2.11 repeatedly we get

$$\begin{aligned}\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} &= \int_{Q_S} \prod_{j=1}^{n} f_{j,1}^{\frac{1}{n-1}} \le C_n \delta^n \int_{Q_S} \prod_{j=1}^{n} f_{j,\delta^{-1}}^{\frac{1}{n-1}} \\ &\le C_n^2 \delta^{2n} \int_{Q_S} \prod_{j=1}^{n} f_{j,\delta^{-2}}^{\frac{1}{n-1}} \le \cdots \\ &\le C_n^M \delta^{Mn} \int_{Q_S} \prod_{j=1}^{n} f_{j,\delta^{-M}}^{\frac{1}{n-1}} = C_n^M \fint_{Q_S} \prod_{j=1}^{n} f_{j,\delta^{-M}}^{\frac{1}{n-1}}\end{aligned}$$

Since $f_{j,\delta^{-M}} \le N_j$, we know that

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \le C_n^M \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

To control the C_n^M term, we calculate

$$S = \delta^{-M} \Leftrightarrow \ln S = M \ln \delta^{-1} \Leftrightarrow M = \frac{\ln S}{\ln \delta^{-1}}$$

and therefore

$$C_n^M = e^{\ln(C_n^M)} = e^{M \ln C_n} = (e^{\ln S})^{\frac{\ln C_n}{\ln \delta^{-1}}} = S^{\frac{\ln C_n}{\ln \delta^{-1}}}.$$

We choose a small enough $\delta = \delta(\varepsilon)$ so that $\frac{\ln C_n}{\ln \delta^{-1}} \le \varepsilon$. For $S = \delta^{-M}$, we have now proven that

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \le S^{\varepsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

For an arbitrary $S \ge 1$, we chose an integer $M \ge 0$ such that $\delta^{-M} \le S \le \delta^{-M-1}$ and cover Q_S with cubes Q_δ of side length δ^{-M}. Denote set of cubes Q_δ by F_Q. Since $\# F_Q \le (\delta^{-1})^n \lesssim_\varepsilon 1$, we get

$$\begin{aligned}\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x &\le \sum_{Q_\delta \in F_Q} \int_{Q_\delta} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \\ &\le \sum_{Q_\delta \in F_Q} (\delta^{-M})^{\varepsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}} \lesssim_\varepsilon S^{\varepsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.\end{aligned}$$

□

2.5 Some generalizations

We make two minor generalizations of theorem 2.1. One argument adds weights on each tube and the other generalizes the angle restriction. We also present a result that we are going to use in chapter 3. This third result follows directly from the first two corollaries.

Corollary 2.12. *Let Q_S, $l_{j,a}$ and $T_{j,a}$ be as in theorem 2.1. Suppose also that $w_{j,a} \geq 0$ are numbers. Define*

$$f_j(x) := \sum_{a=1}^{N_j} w_{j,a} T_{j,a}(x).$$

Then for any $\varepsilon > 0$ and any $S \geq 1$, we have

$$\int_{Q_S} \prod_{j=1}^{n} f_j(x)^{\frac{1}{n-1}} \, \mathrm{d}x \lesssim_\varepsilon S^\varepsilon \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} w_{j,a} \right)^{\frac{1}{n-1}}.$$

Proof. Since the tubes in theorem 2.1 do not have to be distinct, we have the result for positive integer weights by including tubes multiple times. For rational weights, we scale the theorem in the following way. Consider weights $w_{j,a} = \frac{m_{j,a}}{k_{j,a}}$, where $k_{j,a}, m_{j,a} \in \mathbb{N}_+$ for each $a = \{1, \ldots, N_j\}$ and $j = \{1, \ldots, n\}$. Since the theorem holds for integer weights, we know that

$$\int_{Q_S} \prod_{j=1}^{n} \left(\Big(m_{j,1} \prod_{i \neq 1} k_{j,i} \Big) T_{j,1}(x) + \ldots \; + \Big(m_{j,N_j} \prod_{i \neq N_j} k_{j,i} \Big) T_{j,N_j}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x$$
$$\lesssim_\varepsilon S^\varepsilon \prod_{j=1}^{n} \left(m_{j,1} \prod_{i \neq 1} k_{j,i} + \cdots + m_{j,N_j} \prod_{i \neq N_j} k_{j,i} \right)^{\frac{1}{n-1}}.$$

Now dividing both sides by $\prod_{j=1}^{n} \prod_{i=1}^{N_j} k_{j,i}^{\frac{1}{n-1}}$, we get the result for rational weights. Since both sides are continuous with respect to the weights and the rationals are a dense subset of $\mathbb{R}$, the result follows for non-negative real weights.

□

This next corollary gives us a more general condition for the angle of the lines.

Corollary 2.13. *Let $l_{j,a}$, $T_{j,a}$ and Q_S be as in theorem 2.1. Let $S_j \subset \mathbb{S}^{n-1}$ and suppose that $l_{j,a} \in S_j$. Suppose that for any vectors $v_j \in S_j$, we have $|\det([v_1, \ldots, v_n])| \geq \nu$, where*

$0 < \nu < 1$. *Then for any* $\varepsilon > 0$ *and any* $S \geq 1$, *we have*

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \lesssim_{\varepsilon,\nu} S^{\varepsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

Proof. This result follows from theorem 2.3 with a similar argument to the one in section 2.2. We cover S_j with caps $S_{j,\beta}$ of small radius $\rho > 0$. We can choose that the number of caps is $\lesssim_{\varepsilon,\nu} 1$. We pick a sequence of caps $S_{1,\beta_1}, \ldots, S_{n,\beta_n}$ and change coordinates so that the center v_{j,β_j} of S_{j,β_j} is mapped to the standard basis vector e_j for each $j = \{1, \ldots, n\}$.

Define M and L similarly as in section 2.2. First we show that $\|M\|_{op} \leq \sqrt{n}$. For $x = (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i e_i$, applying the Cauchy-Schwarz inequality we have

$$|Mx| = |\sum_{i=1}^{n} x_i M e_i| \leq \sum_{i=1}^{n} |x_i||Me_i| = \sum_{i=1}^{n} |x_i| \cdot 1 \leq |x| \left(\sum_{i=1}^{n} 1^2 \right)^{\frac{1}{2}} = \sqrt{n}|x|.$$

Thus $\|M\|_{op} \leq \sqrt{n}$.

By the singular value decomposition we can write $M = U\Sigma V^{\top}$ and $L = V\Sigma^{-1}U^{\top}$, where U and V are orthogonal matrices and Σ is a diagonal matrix that has entries $\{\sigma_1, \ldots, \sigma_n\}$, where $\sigma_j > 0$ for each $j \in \{1, \ldots, n\}$. Since for an orthogonal matrix O we can compute

$$|Ov| = \sqrt{v^{\top} O^{\top} O v} = \sqrt{v^{\top} v} = |v|, \qquad \text{for all } v \in \mathbb{R}^n,$$

we get

$$\begin{aligned} \sup_{x \neq 0} \frac{|Mx|}{|x|} = \sup_{x \neq 0} \frac{|U\Sigma V^{\top} x|}{|x|} &= \sup_{x \neq 0} \frac{|\Sigma V^{\top} x|}{|x|} \\ &= \sup_{y \neq 0} \frac{|\Sigma y|}{|Vy|} = \sup_{y \neq 0} \frac{|\Sigma y|}{|y|} \\ &= \sup_{z=1} |\Sigma z| = \max\{\sigma_j : j \in \{1, \ldots, n\}\} \end{aligned}$$

and similarly

$$\|L\|_{op} = \max\{\frac{1}{\sigma_j} : j \in \{1, \ldots, n\}\}.$$

We can also compute

$$|\det(M)| = |\det(U)\det(\Sigma)\det(V^{\top})| = |\det(\Sigma)| = \sigma_1 \cdots \sigma_n.$$

Since $\|M\|_{op} \leq \sqrt{n}$, we have that $\sigma_j \leq \sqrt{n}$ for each $j \in \{1, \ldots, n\}$. Thus for every $j \in \{1, \ldots, n\}$, we have

$$\nu \leq |\det(M)| = \sigma_1 \cdots \sigma_n \leq n^{\frac{n-1}{2}} \sigma_j$$
$$\Rightarrow \frac{1}{\sigma_j} \leq \nu^{-1} n^{\frac{n-1}{2}}.$$

Hence,

$$\|L\|_{op} = \max\{\frac{1}{\sigma_j} : j \in \{1, \ldots, n\}\} \lesssim \nu^{-1}.$$

We have now shown that $\|L\|_{op} \lesssim \nu^{-1}$. If we choose $\rho \leq n^{-\frac{n-1}{2}} \nu\delta$, then the image of $S_{j,\beta}$ in the new coordinates is contained in a cap of radius δ as in theorem 2.3. Now summing over different choices of caps $S_{1,\beta_1}, \ldots, S_{n,\beta_n}$ and applying theorem 2.3, we get the desired result

$$\int_{Q_S} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \lesssim_{\varepsilon,\nu} S^{\varepsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

□

Remark 2.14. Notice that the same proof can be done with rectangular 1-tubes, i.e.,

$$T_{j,a}(x) = \mathbb{1}\{x \in \mathbb{R}^n : d_{max}(l_{j,a}, x) \leq 1\},$$

where $d_{max}(l_{j,a}, x) = \inf \left\{ \max_j |x_j - y_j| : y \in l_{j,a} \right\}$. In the proof of the next corollary we will use this definition of $T_{j,a}$.

Corollary 2.15. *Let $R \in 2^{\mathbb{N}_+}$, $c \sim 1$, $\varepsilon > 0$ and $0 < \nu < 1$. Consider n families $\mathcal{P}_j$ of rectangular tiles P in $\mathbb{R}^n$ having the following properties.*

1. *Each P has $n-1$ side lengths equal to $\sqrt{R}$ and one side length equal to R which points in the direction of the unit vector v_P.*
2. *For each $P_i \in \mathcal{P}_i$, we have $|\det([v_{P_1}, \ldots, v_{P_n}])| \geq \nu$.*
3. *All tiles are subsets of a fixed cube B_{cR} that has a side length of cR.*

Then we have

$$\fint_{B_{cR}} \left(\prod_{j=1}^{n} F_j(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \lesssim_{\nu,\varepsilon} R^{\varepsilon} \prod_{j=1}^{n} \left(\fint_{B_{cR}} F_j(x) \right)^{\frac{1}{n-1}} \mathrm{d}x$$

for all functions F_j of the form

$$F_j(x) = \sum_{P \in \mathcal{P}_j} c_P \mathbb{1}_P(x), \qquad c_P \geq 0.$$

Proof. We can cover each tile P with $\left(\sqrt{R}\right)^{n-1}$ essentially disjoint tubes $T_{j,a}$ that are parallel to the longest side of the tile. The weights of the tubes are chosen so that the tubes that are covering a tile have the same weight as the tile. This means that there are $R^{\frac{n-1}{2}}$ tubes $T_{j,a}$ with weight $w_{j,a} = c_P$.

Now applying the previous corollaries, we have

$$\begin{aligned}
\int_{B_{cR}} \left(\prod_{j=1}^{n} \sum_{P \in \mathcal{P}_j} c_P \mathbb{1}_P(x) \right)^{\frac{1}{n-1}} \mathrm{d}x &\leq \int_{B_{cR}} \left(\prod_{j=1}^{n} \sum_{a=1}^{N_j} w_{j,a} T_{j,a}(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \\
&\lesssim_{\nu,\varepsilon} R^{\varepsilon} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} w_{j,a} \right)^{\frac{1}{n-1}} \\
&= R^{\varepsilon} \prod_{j=1}^{n} \left(R^{\frac{n-1}{2}} \sum_{P \in \mathcal{P}_j} c_P \right)^{\frac{1}{n-1}} \\
&= R^{\varepsilon} \prod_{j=1}^{n} \left(R^{-1} \sum_{P \in \mathcal{P}_j} R^{\frac{n+1}{2}} c_P \right)^{\frac{1}{n-1}} \\
&= R^{\varepsilon} R^{-\frac{n}{n-1}} \prod_{j=1}^{n} \left(\sum_{P \in \mathcal{P}_j} m_n(P) c_P \right)^{\frac{1}{n-1}} \\
&= R^{\varepsilon} R^{-\frac{n}{n-1}} \prod_{j=1}^{n} \left(\int_{B_{cR}} \sum_{P \in \mathcal{P}_j} c_P \mathbb{1}_P(x) \, \mathrm{d}x \right)^{\frac{1}{n-1}}.
\end{aligned}$$

Multiplying both sides with $m_n(B_{cR})^{-1}$, we get

$$\begin{aligned}
\fint_{B_{cR}} \left(\prod_{j=1}^{n} F_j(x)\right)^{\frac{1}{n-1}} \mathrm{d}x &\lesssim_{\nu,\varepsilon} R^{\varepsilon} R^{-\frac{n}{n-1}} \frac{1}{m_n(B_{cR})} \prod_{j=1}^{n} \left(\int_{B_{cR}} \sum_{P\in\mathcal{P}_j} c_P \mathbb{1}_P(x)\,\mathrm{d}x\right)^{\frac{1}{n-1}} \\
&= R^{\varepsilon} R^{-\frac{n}{n-1}} m_n(B_{cR})^{\frac{1}{n-1}} \prod_{j=1}^{n} \left(\fint_{B_{cR}} \sum_{P\in\mathcal{P}_j} c_P \mathbb{1}_P(x)\,\mathrm{d}x\right)^{\frac{1}{n-1}} \\
&\sim R^{\varepsilon} R^{-\frac{n}{n-1}} R^{\frac{n}{n-1}} \prod_{j=1}^{n} \left(\fint_{B_{cR}} F_j(x)\,\mathrm{d}x\right)^{\frac{1}{n-1}} \\
&= R^{\varepsilon} \prod_{j=1}^{n} \left(\fint_{B_{cR}} F_j(x)\,\mathrm{d}x\right)^{\frac{1}{n-1}}.
\end{aligned}$$

In the first equality we used

$$a \prod_{j=1}^{n} b^{\frac{1}{n-1}} = \prod_{j=1}^{n} (a^{\frac{n-1}{n}} b)^{\frac{1}{n-1}} = \prod_{j=1}^{n} (a^{\frac{-1}{n}} ab)^{\frac{1}{n-1}} = a^{\frac{-1}{(n-1)}} \prod_{j=1}^{n} (ab)^{\frac{1}{n-1}}$$

with $a = \frac{1}{m_n(B_{cR})}$ and $b = \int_{B_{cR}} \sum_{P\in\mathcal{P}_j} c_P \mathbb{1}_P(x)\,\mathrm{d}x$. □

Chapter 3

Multilinear decoupling device

The main objective of this chapter is theorem 3.15, which we call cube inflation. This result is a multilinear estimate that can be used to create multilinear decouplings. In the first section we study the extension operator E_Q and particular weight functions $w_{B,E}$ that will be central in the rest of the thesis. The second section, where we prove theorem 3.15, follows the ninth chapter from [5]. The strategy for proving theorem 3.15 will rely on dyadic pigeonholing and the multilinear Kakeya inequality. As seen in the tenth chapter of [5] and in more detail in the seventh and eight chapter of [14] this multilinear estimate can be paired with l^2L^2 decoupling to create a partial decoupling that is used to estimate a certain decoupling constant.

3.1 Extension operator for the paraboloid and weight functions

Before we can dive into multilinear decouplings, we must first look at a few definitions that are essential for the l^2 decoupling theorem. Let $\mathbb{P}^{n-1}$ be a paraboloid in $\mathbb{R}^n$ that is defined

$$\mathbb{P}^{n-1} := \{(\xi, |\xi|^2) : \xi \in [0,1]^{n-1}\}.$$

Notice that now $\pi_n[\mathbb{P}^{n-1}] = [0,1]^{n-1}$.

Definition 3.1. Let $0 < \nu < 1$. Cubes $Q_1, \dots, Q_n \subset [0,1]^{n-1}$ are called ν-transverse, if the n-dimensional parallelepiped, that is spanned by unit normal vectors $n(P_i)$, has volume greater than ν, for each $P_i \in \mathbb{P}^{n-1}$ with $\pi_n(P_i) \in Q_i$.

In this chapter the above definition works as way to check the determinant condition of the multilinear Kakeya inequality from corollary 2.15. If cubes $Q_1, \dots, Q_n$ are ν-

transverse, then tiles that have long sides pointing in the direction of the unit normal vectors $n(P_i)$ satisfy the second condition from corollary 2.15.

Inspired by the ν-transverse definition, we will now calculate the unit normal vectors of $\mathbb{P}^{n-1}$. The surface $\mathbb{P}^{n-1}$ is parametrized by the zeros of the function $F(\xi, z) = z - |\xi|^2$, where $\xi \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$. Now the gradient of F gives us the upward unit normal vectors

$$n(P_i) = \frac{\nabla F}{|\nabla F|} = \frac{(-2\xi_1, \dots, -2\xi_{n-1}, 1)}{\sqrt{\sum_{j=1}^{n}(-2\xi_j)^2 + 1}} = \frac{(-2\xi_1, \dots, -2\xi_{n-1}, 1)}{\sqrt{4|\xi|^2 + 1}}.$$

Next we will present and study the extension operator and the weight functions. Recall that $e(t) = e^{2\pi i t}$, where $t \in \mathbb{R}$.

Definition 3.2. Let $Q \subset [0,1]^{n-1}$ be a cube and $g\colon Q \to \mathbb{C}$. We also stipulate that $g \in L^1(Q)$. Now we define

$$E_Q g(x) = \int_Q g(\xi) e(x_1\xi_1 + \cdots + x_{n-1}\xi_{n-1} + |\xi|^2 x_n)\, \mathrm{d}\xi.$$

From the integrability condition of g we immediately get that $E_Q g \in L^\infty(\mathbb{R}^n)$. Indeed, $|E_Q g(x)| \le \|g\|_{L^1(Q)} < \infty$.

Definition 3.3. Let B be a cube in $\mathbb{R}^n$ centered at c_B and with side length R. The associated weight function for B is defined

$$w_{B,E}(x) = \frac{1}{\left(1 + \frac{|x - c_B|}{R}\right)^E},$$

where $x \in \mathbb{R}^n$ and $E \in \mathbb{R}$.

The exponent E is chosen so that $w_{B,E}$ satisfies various integrability conditions and it turns out that those conditions are fulfilled with $E \ge 100n$ and we will assume this inequality for the rest of the thesis (we might also at times explicitly assume smaller lower bounds). The implicit constants will often harmlessly depend on E. As E is a fixed number, we will write $\lesssim_E$ as $\lesssim$ for the rest of the thesis. Also in this chapter we will write $w_{B,E} = w_B$.

Remark 3.4. If $x \in B$, then $w_B(x) \sim 1$. Indeed, if $x \in B$, then $|x - c_B| \le \sqrt{n}\frac{R}{2} \sim R$. Hence

$$w_B(x) = \left(1 + \frac{|x - c_B|}{R}\right)^{-E} \gtrsim 2^{-E} \sim 1$$

and it always holds that $w_B(x) \le 1$.

We recall the notations

$$\|f\|_{L^p_\sharp(B)} = \left(\frac{1}{m_n(B)} \int_B |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

$$\|f\|_{L^p(w_B)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w_B(x) \, \mathrm{d}x \right)^{\frac{1}{p}}$$

and

$$\|f\|_{L^p_\sharp(w_B)} = \left(\frac{1}{m_n(B)} \int_{\mathbb{R}^n} |f(x)|^p w_B(x) \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Since $E_Q g$ is essentially bounded, we have $\|E_Q g\|_{L^p_\sharp(B)} < \infty$. Using the following lemma one can easily prove that $\|E_Q g\|_{L^p(w_B)}$ and $\|E_Q g\|_{L^p_\sharp(w_B)}$ are also finite.

Lemma 3.5. *Let $E > n$. For all cubes $B = B(c_B, R)$ in $\mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} w_B(x) \, \mathrm{d}x \sim R^n.$$

Proof. We can write

$$\int_{\mathbb{R}^n} w_B(x) \, \mathrm{d}x = \int_{|x-c_B| \le R} w_B(x) \, \mathrm{d}x + \sum_{k=1}^{\infty} \int_{2^{k-1}R < |x-c_B| \le 2^k R} w_B(x) \, \mathrm{d}x.$$

We will treat the cases separately. Now

$$\int_{|x-c_B| \le R} w_B(x) \, \mathrm{d}x \le \int_{|x-c_B| \le R} 1 \, \mathrm{d}x \sim R^n$$

and for every $k \in \mathbb{N}_+$, we have

$$\begin{aligned} \int_{2^{k-1}R < |x-c_B| \le 2^k R} w_B(x) \, \mathrm{d}x &= \int_{2^{k-1} < |x-c_B| \le 2^k R} \frac{1}{\left(1 + \frac{|x-c_B|}{R}\right)^E} \, \mathrm{d}x \\ &\le \frac{1}{(1+2^{k-1})^E} \int_{|x-c_B| \le 2^k R} 1 \, \mathrm{d}x \\ &\le \frac{1}{(2^{k-1})^E} \int_{|x-c_B| \le 2^k R} 1 \, \mathrm{d}x \\ &\sim \frac{1}{2^{kE}} \int_{|x-c_B| \le 2^k R} 1 \, \mathrm{d}x \\ &\sim \frac{2^{kn}}{2^{kE}} R^n = 2^{k(n-E)} R^n. \end{aligned}$$

We have now shown that

$$\int_{\mathbb{R}^n} w_B(x)\,\mathrm{d}x \lesssim R^n \left(1 + \sum_{k=1}^{\infty} 2^{k(n-E)}\right) \sim R^n$$

The other direction is a straightforward calculation

$$R^n \sim 2^{-E} R^n \sim \int_{|x-c_B|\le R} 2^{-E}\,\mathrm{d}x \le \int_{|x-c_B|\le R} (1 + \frac{|x-c_B|}{R})^{-E}\,\mathrm{d}x \le \int_{\mathbb{R}^n} w_B(x)\,\mathrm{d}x.$$

□

Integration of a weight function over an integration domain that is restricted to a dilated cube can be estimated with another weight function.

Lemma 3.6. *Let $c_n \sim 1$ and $S \ge R > 0$. For all cubes $B' = B'(c_{B'}, S)$ in $\mathbb{R}^n$ we have*

$$R^{-n} \int_{c_n B'} w_{B(u,R)}(x)\,\mathrm{d}x \lesssim w_{B'}(u),$$

for all $u \in \mathbb{R}^n$.

Proof. If $|c_{B'} - u| \le \sqrt{n} c_n S$, then $w_{B'}(u) = (1 + \frac{|c_{B'}-u|}{S})^{-E} \ge (1 + \sqrt{n} c_n)^{-E} \sim 1$. Now lemma 3.5 gives

$$R^{-n} \int_{c_n B'} w_{B(u,R)}(x)\,\mathrm{d}x \le R^{-n} \int_{\mathbb{R}^n} w_{B(u,R)}(x)\,\mathrm{d}x \sim 1 \lesssim w_{B'}(u)$$

as wanted.

If $|c_{B'} - u| > \sqrt{n} c_n S$, then we write

$$\begin{aligned} R^{-n} \int_{c_n B'} w_{B(u,R)}(x)\,\mathrm{d}x &= R^{-n} \int_{B(c_{B'}, c_n S)} (1 + \frac{|x-u|}{R})^{-E}\,\mathrm{d}x \\ &= R^{-n} \int_{B(0, c_n S)} (1 + \frac{|y + c_{B'} - u|}{R})^{-E}\,\mathrm{d}y. \end{aligned} \tag{3.7}$$

Since $|y| \le \sqrt{n} \frac{c_n}{2} S \le \sqrt{n} c_n S < |c_{B'} - u|$, we have

$$\frac{|y + c_{B'} - u|}{R} \le \frac{(|c_{B'} - u| + |y|)}{R} \le 2\frac{|c_{B'} - u|}{R}$$

and since $\frac{1}{2}|c_{B'} - u| > \sqrt{n} \frac{c_n}{2} S \ge |y|$, we have

$$\frac{|y + c_{B'} - u|}{R} \ge \frac{(|c_{B'} - u| - |y|)}{R} \ge \frac{1}{2}\frac{|c_{B'} - u|}{R}.$$

We have shown

$$\frac{|x + c_{B'} - u|}{R} \sim \frac{|c_{B'} - u|}{R},$$

which implies

$$\frac{|x + c_{B'} - u|}{R} + 1 \sim \frac{|c_{B'} - u|}{R} + 1.$$

Furthermore, since $\frac{|c_{B'}-u|}{R} > \sqrt{n} c_n \frac{S}{R} \gtrsim 1$, we have

$$\frac{|c_{B'} - u|}{R} + 1 \sim \frac{|c_{B'} - u|}{R}.$$

Now we get

$$(1 + \frac{|x + c_{B'} - u|}{R})^{-E} \sim (1 + \frac{|c_{B'} - u|}{R})^{-E} \sim (\frac{|c_{B'} - u|}{R})^{-E}$$

and for the integral

$$\begin{aligned} R^{-n} \int_{B(0, c_n S)} (1 + \frac{|x + c_{B'} - u|}{R})^{-E} \,\mathrm{d}x &\sim R^{-n} (\frac{|c_{B'} - u|}{R})^{-E} \int_{B(0, c_n S)} \mathrm{d}x \\ &\sim R^{E-n} S^n |c_{B'} - u|^{-E} \\ &\leq S^E |c_{B'} - u|^{-E} \\ &= (\frac{|c_{B'} - u|}{S})^{-E} \sim w_{B'}(u). \end{aligned}$$

Combining the above estimate with (3.7) gives us the result. □

Lemma 3.8. *Let $B = B(c_B, R)$. If $y \in B(x, R)$, then $w_B(y) \sim w_B(x)$.*

Proof. By symmetry, it suffices to show that $w_B(y) \lesssim w_B(x)$.

Since $y \in B(x, R)$, we have

$$\begin{aligned} w_B(x) &= \left(1 + \frac{|x - c_B|}{R}\right)^{-E} \\ &\geq \left(1 + \frac{|x - y|}{R} + \frac{|y - c_B|}{R}\right)^{-E} \\ &\geq \left(2 + \frac{|y - c_B|}{R}\right)^{-E} \\ &\sim \left(1 + \frac{|y - c_B|}{R}\right)^{-E} = w_B(y). \end{aligned}$$

□

The proofs for both of the inequalities in the following lemma can be found in [14] lemma 4.2.1. We will give a slightly different approach to the first inequality.

Lemma 3.9. *Let $B = B(c_B, R)$ be a cube in $\mathbb{R}^n$. Let $\mathcal{B}$ be an essential partition of $\mathbb{R}^n$ with cubes $B' = B'(c_{B'}, R)$. Then*

$$w_B(x) \lesssim \sum_{B'\in\mathcal{B}} \mathbb{1}_{B'}(x) w_B(c_{B'}) \tag{3.10}$$

and

$$\sum_{B'\in\mathcal{B}} w_{B'}(x) w_B(c_{B'}) \lesssim w_B(x). \tag{3.11}$$

Proof. Let $x \in \mathbb{R}^n$. Since $\mathcal{B}$ is an essential partition of $\mathbb{R}^n$, there exists $B'_x \in \mathcal{B}$ such that $x \in B'_x$. By lemma 3.8 we have

$$w_B(x) \sim w_B(c_{B'_x}) \leq \sum_{B'\in\mathcal{B}} \mathbb{1}_{B'}(x) w_B(c_{B'}).$$

□

Lemma 3.12. *Let $E > n$. Then*

$$\mathbb{1}_B \lesssim \sum_{\Delta\in\mathcal{B}} w_\Delta \lesssim w_B,$$

for all cubes $B = B(c, R)$ and all essential partitions $\mathcal{B}$ of B with cubes Δ of side length R', where $0 < R' \leq R$.

Proof. Proof can be found in [14] lemmas 4.1.1. and 4.1.5. □

A covering $\mathcal{B}$ is finitely overlapping, if $\sum_{B\in\mathcal{B}} \mathbb{1}_B \lesssim 1$. Lemma 3.9 is also true for finitely overlapping covers $\mathcal{B}$ and lemma 3.12 holds true when the cover is finitely overlapping and the cubes in the covering intersect with the original cube B. Working with essential partitions will only harmlessly restrict the side lengths of the partition cubes in future arguments.

Lastly, we estimate a single variable convolution of a one dimensional weight function and an n-dimensional weight function.

Lemma 3.13. *Let $B := B(c_B, R) \subset \mathbb{R}^n$ and $B^1(0, R) \subset \mathbb{R}^1$, then for $x \in \mathbb{R}^n$ we have*

$$R^{-1} \int_{\mathbb{R}} w_{B^1(0,R)}(x_n - y) w_B(\pi_n(x), y)\, \mathrm{d}y \lesssim w(x).$$

Proof. Throughout the proof we will use

$$\int_{\mathbb{R}} (1+|z|)^{-E} \,\mathrm{d}z = \frac{2}{E-1} \sim 1.$$

We will first assume that c_B is the origin. Now it suffices to show that

$$R^{-1} \int_{\mathbb{R}} (1+\frac{|x_n - y|}{R})^{-E} (1+\frac{|(\pi_n(x), y)|}{R})^{-E} (1+\frac{|x|}{R})^{E} \,\mathrm{d}y \lesssim 1. \tag{3.14}$$

We denote $x_R := \frac{x}{R}$ and change variables twice $z = \frac{y}{R}$ and $z' = (x_R)_n - z$ to get

$$\begin{aligned} R^{-1} \int_{\mathbb{R}} (1+\frac{|x_n - y|}{R})^{-E} (1+\frac{|(\pi_n(x), y)|}{R})^{-E} (1+\frac{|x|}{R})^{E} \,\mathrm{d}y \\ &= \int_{\mathbb{R}} (1+|(x_R)_n - z|)^{-E} (1+|(\pi_n(x_R), z)|)^{-E} (1+|x_R|)^{E} \,\mathrm{d}z \\ &= \int_{\mathbb{R}} (1+|z'|)^{-E} (1+|x_R - z_n')|)^{-E} (1+|x_R|)^{E} \,\mathrm{d}z', \end{aligned}$$

where $z_n' := (0, \ldots, 0, z') \subset \mathbb{R}^n$. If $|x_R| \leq 1$, then

$$\int_{\mathbb{R}} (1+|z'|)^{-E} (1+|x_R - z_n')|)^{-E} (1+|x_R|)^{E} \,\mathrm{d}z' \leq 2^E \int_{\mathbb{R}} (1+|z'|)^{-E} \,\mathrm{d}z' \sim 1$$

and (3.14) follows. We can now assume that $|x_R| > 1$ and write

$$\int_{\mathbb{R}} = \int_{|x_R - z_n'| > \frac{|x_R|}{2}} + \int_{|x_R - z_n'| \leq \frac{|x_R|}{2}}.$$

We can straightforwardly estimate the first integral

$$\begin{aligned} I_1 &:= \int_{|x_R - z_n'| > \frac{|x_R|}{2}} (1+|z'|)^{-E} (1+|x_R - z_n')|)^{-E} (1+|x_R|)^{E} \,\mathrm{d}z' \\ &\leq \frac{(1+|x_R|)^E}{(1+\frac{|x_R|}{2})^E} \int_{|x_R - z_n'| > \frac{|x_R|}{2}} (1+|z'|)^{-E} \,\mathrm{d}z' \\ &\leq 2^{-E} \int_{\mathbb{R}} (1+|z'|)^{-E} \,\mathrm{d}z' \sim 1. \end{aligned}$$

For the second integral we have $|x_R - z_n'| \leq \frac{|x_R|}{2}$ and thus

$$|z'| \geq |x_R| - |x_R - z_n'| \geq \frac{|x_R|}{2}.$$

Now we get

$$\begin{aligned}
I_2 &:= \int_{|x_R - z_n'| \leq \frac{|x_R|}{2}} (1+|z'|)^{-E}(1+|x_R - z_n')|)^{-E}(1+|x_R|)^E \,\mathrm{d}z' \\
&\leq \frac{(1+|x_R|)^E}{(1+\frac{|x_R|}{2})^E} \int_{\mathbb{R}} (1+|x_R - z_n')|)^{-E} \,\mathrm{d}z' \\
&\leq 2^{-E} \int_{\mathbb{R}} (1+|(x_R)_n - z')|)^{-E} \,\mathrm{d}z' \\
&= 2^{-E} \int_{\mathbb{R}} (1+|z)|)^{-E} \,\mathrm{d}z \sim 1.
\end{aligned}$$

We have now proven that for $|x_R| > 1$ we have

$$\int_{\mathbb{R}} (1+|z'|)^{-E}(1+|x_R - z_n')|)^{-E}(1+|x_R|)^E \,\mathrm{d}z' = I_1 + I_2 \lesssim 2 \sim 1$$

and again (3.14) follows. This ends the proof for the case where c_B is the origin.

For the general case we change variables and apply the origin centered version

$$\begin{aligned}
R^{-1} &\int_{\mathbb{R}} w_{B^1(0,R)}(x_n - y) w_B(\pi_n(x), y) \,\mathrm{d}y \\
&= R^{-1} \int_{\mathbb{R}} w_{B^1(0,R)}(x_n - y) w_{B(\bar{0},R)}((\pi_n(x), y) - c_B) \,\mathrm{d}y \\
&= R^{-1} \int_{\mathbb{R}} w_{B^1(0,R)}(x_n - (c_B)_n - y') w_{B(\bar{0},R)}(\pi_n(x) - \pi_n(c_B), y') \,\mathrm{d}y' \\
&\lesssim w_{B(\bar{0},R)}(x - c_B) = w_B(x).
\end{aligned}$$

□

3.2 Cube inflation

The multilinear Kakeya inequality is central in the proof of the following key result which gives us a tool to make decouplings. The result is not strictly speaking a decoupling since the cubes $Q_{i,1}$ under the paraboloid $\mathbb{P}^{n-1}$ are unaffected. However, the scale of the cubes associated to the weights increases from δ^{-1} to δ^{-2}. In many texts, see for example theorem 6.6 in [8], similar inequalities are referred to as ball inflation. Since we are dealing with cubes, it is sensible to call the following result cube inflation.

Theorem 3.15. *Let $p \geq \frac{2n}{n-1}$, $\varepsilon \in (0,\infty)$ and $\delta \in 2^{-\mathbb{N}_+}$. Consider ν-transverse cubes $Q_1,\dots,Q_n \subset [0,1]^{n-1}$. Let B be an arbitrary cube with side length δ^{-2} and let $\mathcal{B}$ be the unique partition of B into cubes of side length δ^{-1}, i.e., $\mathcal{B} = \mathrm{Part}_{\delta^{-1}}(B)$. Then for each $g \in L^1([0,1]^{n-1})$ we have*

$$\frac{1}{\#\mathcal{B}} \sum_{\Delta\in\mathcal{B}} \left(\prod_{i=1}^{n} \left(\sum_{Q_{i,1}\in\mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}} g\|^2_{L_\sharp^{\frac{p(n-1)}{n}}(w_\Delta)} \right)^{\frac{1}{2}} \right)^{\frac{p}{n}}$$
$$\lesssim_{\varepsilon,\nu} \delta^{-\varepsilon} \left(\prod_{i=1}^{n} \left(\sum_{Q_{i,1}\in\mathrm{Part}_\delta(Q_i)} \|E_{Q_{i,1}} g\|^2_{L_\sharp^{\frac{p(n-1)}{n}}(w_B)} \right)^{\frac{1}{2}} \right)^{\frac{p}{n}}.$$

We will begin with a lemma

Lemma 3.16. *Consider the cubes $B = B(c_B,\delta^{-2})$ and $Q = Q(c_Q,\delta)$. Furthermore, let $\mathcal{T}_Q$ be a covering of B that consists of pairwise disjoint mutually parallel tiles $T_Q \subset 4B$. They are rectangles with $n-1$ short sides of length δ^{-1} and one longer side of length δ^{-2}, pointing in the direction of the normal $n(c_Q)$ to the paraboloid $\mathbb{P}^{n-1}$ at c_Q. Let $T_Q(x)$ be the tile containing x. Define also*

$$F_Q(x) = \sup_{y\in(1+\sqrt{n})T_Q(x)} \|E_Q g\|_{L^q_\sharp(w_B(y,\delta^{-1}))}, \qquad \text{for} \qquad x \in \bigcup_{T_Q\in\mathcal{T}_Q} T_Q.$$

Then we have

$$\|F_Q\|^q_{L^q_\sharp(4B)} \lesssim \|E_Q g\|^q_{L^q_\sharp(w_B)}. \tag{3.17}$$

Proof. First we show that we may assume that $Q = [-\frac{\delta}{2},\frac{\delta}{2}]^{n-1} =: Q_0$. We define

$$L = \begin{bmatrix} \mathbb{I}_{n-1} & 2\sigma \\ \mathbf{0}^\top & 1 \end{bmatrix},$$

where $\mathbf{0} = (0,\dots,0) \in \mathbb{R}^{n-1}$, $\mathbb{I}_{n-1}$ is the unit matrix in $\mathbb{R}^{n-1\times n-1}$ and $\sigma \in \mathbb{R}^{n-1}$ satisfies $Q = Q_0 + \sigma$. We also notice that

$$Lu = (u_1 + 2\sigma_1 u_n, \dots, u_{n-1} + 2\sigma_{n-1}u_n, u_n).$$

The inverse of L is

$$L^{-1} = \begin{bmatrix} \mathbb{I}_{n-1} & -2\sigma \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

and we have $|\det L| = 1 = |\det L^{-1}|$. Now we can compute

$$\begin{aligned}
|E_Q g(u)| &= \left| \int_Q g(\xi) e\big((\xi, |\xi|^2) \cdot u\big) \, \mathrm{d}\xi \right| \\
&= \left| \int_{Q_0} g(\xi+\sigma) e\big((\xi+\sigma, |\xi+\sigma|^2) \cdot u\big) \, \mathrm{d}\xi \right| \\
&= \left| \int_{Q_0} g(\xi+\sigma) e\big((\xi+\sigma) \cdot \pi_n(u) + \big(|\xi|^2 + 2\xi \cdot \sigma + |\sigma|^2\big)\, u_n\big) \, \mathrm{d}\xi \right| \\
&= \left| \int_{Q_0} g(\xi+\sigma) e\big((\sigma, |\sigma|^2) \cdot u + \xi \cdot \pi_n(u) + 2\xi \cdot \sigma u_n + |\xi|^2 u_n\big) \, \mathrm{d}\xi \right| \\
&= \left| \int_{Q_0} g(\xi+\sigma) e\big((\sigma, |\sigma|^2) \cdot u + \xi \cdot (\pi_n(u) + 2\sigma u_n) + |\xi|^2 u_n\big) \, \mathrm{d}\xi \right| \\
&= \left| e\big((\sigma, |\sigma|^2) \cdot u\big) \int_{Q_0} g(\xi+\sigma) e\big((\xi, |\xi|^2) \cdot Lu\big) \, \mathrm{d}\xi \right| \\
&= \left| \int_{Q_0} g(\xi+\sigma) e\big((\xi, |\xi|^2) \cdot Lu\big) \, \mathrm{d}\xi \right| = |E_{Q_0} G(Lu)|
\end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} |E_Q g(u)|^q w_{B(y,\delta^{-1})}(u) \, \mathrm{d}u = \int_{\mathbb{R}^n} |E_{Q_0} G(v)|^q w_{B(y,\delta^{-1})}(L^{-1}v) \, \mathrm{d}v \tag{3.18}$$

where $G(u) := g(u+\sigma)$.

For any $x \in \mathbb{R}^n$, we know that

$$|Lx| = |x + (2x_n\sigma, 0)| \le |x| + 2|x_n||\sigma| \le (1 + 2|\sigma|)|x| \le 3\sqrt{n-1}|x|$$

and similarly $\|L^{-1}\|_{op} \le 3\sqrt{n-1}$. Thus by lemma 1.2, the operator L is bilipschitz and we have $|y - L^{-1}v| \sim |L(y - L^{-1}v)| = |Ly - v|$. This implies that

$$\begin{aligned}
w_{B(y,\delta^{-1})}(L^{-1}v) &= \big(1 + \delta|y - L^{-1}v|\big)^{-E} \\
&\sim (1 + \delta|Ly - v|)^{-E} = w_{B(Ly,\delta^{-1})}(v).
\end{aligned} \tag{3.19}$$

Plugging (3.19) to (3.18), we get

$$\int_{\mathbb{R}^n} |E_Q g(u)|^q w_{B(y,\delta^{-1})}(u) \, \mathrm{d}u \sim \int_{\mathbb{R}^n} |E_{Q_0} G(v)|^q w_{B(Ly,\delta^{-1})}(v) \, \mathrm{d}v,$$

which is equivalent with

$$\|E_Q g\|_{L^q_\sharp(w_{B(y,\delta^{-1})})} \sim \|E_{Q_0} G\|_{L^q_\sharp(w_{B(Ly,\delta^{-1})})}.$$

Recalling our calculations of the unit normal vectors of $\mathbb{P}^{n-1}$, we have $n(c_Q) = \frac{(-2\sigma,1)}{\sqrt{1+4|\sigma|^2}}$ and $L(n(c_Q)) = \frac{1}{\sqrt{1+4|\sigma|^2}} e_n = \frac{1}{\sqrt{1+4|\sigma|^2}} n(c_{Q_0})$. Recall also that T_Q is a $\delta^{-2} \times (\delta^{-1})^{n-1}$ tile where the longest side is parallel to $n(c_Q)$. Thus $L(T_Q)$ is a parallelepiped with longest side parallel to $n(c_{Q_0})$. Since $\|L\|_{op} \leq 3\sqrt{n-1}$ and for any $a, b \in (1+\sqrt{n})T_Q(x)$ we have

$$|a-b| \leq \sqrt{(n-1)((1+\sqrt{n})\delta^{-1})^2 + ((1+\sqrt{n})\delta^{-2})^2} \leq (\sqrt{n}+n)\delta^{-2},$$

we get that

$$|L(a-b)| \leq \|L\|_{op}|a-b| \leq 3(n^{\frac{3}{2}}+n)\delta^{-2}.$$

Since $6n^{\frac{3}{2}} \geq 3n^{\frac{3}{2}}+3n+\frac{1}{2}$ holds for $n \geq 2$, the above deduction and $L(T_Q(x)) \cap T_{Q_0}(Lx) \neq \varnothing$ imply that $L((1+\sqrt{n})T_Q(x)) \subset 12n^{\frac{3}{2}} T_{Q_0}(Lx)$. See figure 3.1 below for the worst case scenario in $\mathbb{R}^2$.

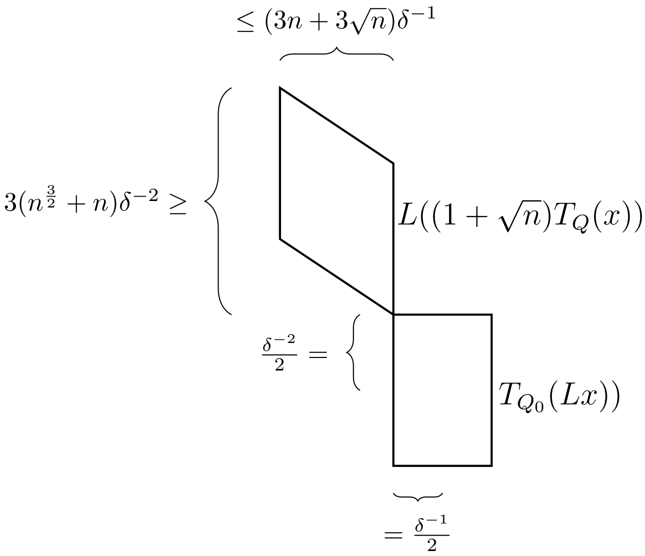

Figure 3.1: Containment of $L((1+\sqrt{n})T_Q(x))$ with a dilation of $T_{Q_0}(Lx))$ in $\mathbb{R}^2$. The lengths are from the n-dimensional case.

Now we can estimate

$$\begin{aligned} F_Q(x) &= \sup_{y\in(1+\sqrt{n})T_Q(x)} \|E_Q g\|_{L^q_\sharp(w_B(y,\delta^{-1}))} \\ &\sim \sup_{y\in(1+\sqrt{n})T_Q(x)} \|E_{Q_0} G\|_{L^q_\sharp(w_B(Ly,\delta^{-1}))} \\ &= \sup_{y\in L((1+\sqrt{n})T_Q(x))} \|E_{Q_0} G\|_{L^q_\sharp(w_B(y,\delta^{-1}))} \\ &\leq \sup_{y\in 12n^{\frac{3}{2}}T_{Q_0}(Lx)} \|E_{Q_0} G\|_{L^q_\sharp(w_B(y,\delta^{-1}))} =: F_0(Lx) \end{aligned}$$

and thus on the left-hand side of (3.17) we get

$$\int_{4B} |F_Q(x)|^q \,\mathrm{d}x \lesssim \int_{4B} |F_0(Lx)|^q \,\mathrm{d}x = \int_{L(4B)} |F_0(y)|^q \,\mathrm{d}y \leq \int_{12nB'} |F_0(y)|^q \,\mathrm{d}y,$$

where $B' := B\big(L(c_B), \delta^{-2}\big)$. Furthermore, we can change variables back on right-hand side of (3.17) to see that

$$\begin{aligned} \|E_{Q_0} G\|^q_{L^q_\sharp(w_{B'})} &= \frac{1}{m_n(B')} \int_{\mathbb{R}^n} |E_{Q_0} G(v)|^q w_{B'}(v) \,\mathrm{d}v \\ &= \frac{1}{m_n(B')} \int_{\mathbb{R}^n} |E_Q g(u)|^q w_{B'}(Lu) \,\mathrm{d}u \\ &\sim \frac{1}{m_n(B)} \int_{\mathbb{R}^n} |E_Q g(u)|^q w_B(u) \,\mathrm{d}u = \|E_Q g\|^q_{L^q_\sharp(w_B)}. \end{aligned}$$

Hence in order to prove the lemma, it suffices to prove

$$\|F_0\|^q_{L^q_\sharp(12nB')} \lesssim \|E_{Q_0} G\|^q_{L^q_\sharp(w_{B'})}. \tag{3.20}$$

We will abuse notation by writing $Q = Q_0$, $g = G$ and $F_0 = F_Q$. We fix $x \in \mathbb{R}^n$ and let $y \in 12n^{\frac{3}{2}}T_Q(x)$. The reduction of Q implies that $T_Q(x)$ has sides parallel to the coordinate axes and the longest side is parallel to the x_n-axis. This means that $y = x + y'$ with $|y'_j| \lesssim \delta^{-1}$ for $1 \leq j < n$ and $|y'_n| \lesssim \delta^{-2}$. For the rest of the proof we simplify notation by writing $\widetilde{a} := \pi_n(a)$, for $a \in \mathbb{R}^n$. Now we change variables twice to get

$$\begin{aligned} \|E_Q g\|^q_{L^q(w_{B(y,\delta^{-1})})} &= \int_{\mathbb{R}^n} |E_Q g(z)|^q w_{B(y,\delta^{-1})}(z) \,\mathrm{d}z \\ &= \int_{\mathbb{R}^n} |E_Q g(u+y)|^q w_{B(y,\delta^{-1})}(y+u) \,\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} |E_Q g(u+y)|^q w_{B(0,\delta^{-1})}(u)\,\mathrm{d}u \\
&= \int_{\mathbb{R}^n} |E_Q g(x_1+u_1+y_1', \dots, x_n+u_n+y_n')|^q w_{B(0,\delta^{-1})}(u)\,\mathrm{d}u \\
&= \int_{\mathbb{R}^n} |E_Q g(\widetilde{x}+\widetilde{u}, x_n+u_n+y_n')|^q w_{B(0,\delta^{-1})}(\widetilde{u}-\widetilde{y}', u_n)\,\mathrm{d}u.
\end{aligned}
$$

Furthermore, by triangle inequality

$$
\delta|(\widetilde{u}-\widetilde{y}'), u_n)| \le \delta|u| + \delta|\widetilde{y}'| \le \delta|u| + \delta\sum_{j=1}^{n-1}|y_j'| \lesssim \delta|u| + 1
$$

and

$$
\delta|u| = \delta|(\widetilde{u}-\widetilde{y}'+\widetilde{y}', u_n)| \le \delta|(\widetilde{u}-\widetilde{y}', u_n)| + \delta|\widetilde{y}'| \lesssim \delta|(\widetilde{u}-\widetilde{y}', u_n)| + 1.
$$

Hence

$$
\begin{aligned}
&1+\delta|u| \sim 1+\delta|(\widetilde{u}-\widetilde{y}', u_n)| \\
\Leftrightarrow\ & w_{B(0,\delta^{-1})}(u) \sim w_{B(0,\delta^{-1})}(\widetilde{u}-\widetilde{y}', u_n)
\end{aligned}
$$

and so we have

$$
\begin{aligned}
\|E_Q g\|^q_{L^q(w_{B(y,\delta^{-1})})} &\sim \int_{\mathbb{R}^n} |E_Q g(\widetilde{x}+\widetilde{u}, x_n+u_n+y_n')|^q w_{B(0,\delta^{-1})}(u)\,\mathrm{d}u \\
&= \int_{\mathbb{R}^n} |E_Q g(x_1+u_1, \dots, x_{n-1}+u_{n-1}, x_n+u_n+y_n')|^q w_{B(0,\delta^{-1})}(u)\,\mathrm{d}u. \qquad (3.21)
\end{aligned}
$$

Now we may write

$$
\begin{aligned}
|E_Q g(\widetilde{x}+\widetilde{u}, x_n+u_n+y_n')| &= \left|\int_Q g(\lambda) e\big((\widetilde{x}+\widetilde{u}, x_n+u_n+y_n')\cdot(\lambda, |\lambda|^2)\big)\,\mathrm{d}\lambda\right| \\
&= \left|\int_Q g(\lambda) e\big((x+u)\cdot(\lambda, |\lambda|^2)\big) e(y_n'|\lambda|^2)\,\mathrm{d}\lambda\right| \\
&= \left|\int_Q g(\lambda) e\big((x+u)\cdot(\lambda, |\lambda|^2)\big) \sum_{k=0}^{\infty} \frac{(2\pi i y_n')^k}{k!}|\lambda|^{2k}\,\mathrm{d}\lambda\right| \\
&= \sum_{k=0}^{\infty} \frac{(2\pi)^k |y_n'|^k}{k!} \left|\int_Q g(\lambda) e\big((x+u)\cdot(\lambda, |\lambda|^2)\big)|\lambda|^{2k}\,\mathrm{d}\lambda\right|
\end{aligned}
$$

$$\lesssim \sum_{k=0}^{\infty} \frac{(2\pi)^k(\delta^{-2})^k}{k!} \left| \int_Q g(\lambda) e\big((x+u)\cdot(\lambda,|\lambda|^2)\big)|\lambda|^{2k}\,\mathrm{d}\lambda \right|$$

$$= \sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} \left| \int_Q g(\lambda) e\big((x+u)\cdot(\lambda,|\lambda|^2)\big) \left(\frac{|\lambda|^2}{n\delta^2}\right)^k \mathrm{d}\lambda \right|$$

$$= \sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} \left| M_k(E_Q g)(x+u) \right|. \tag{3.22}$$

We were able to interchange the series and integral due to dominated convergence theorem. Above we denoted

$$M_k(E_Q g)(z) := \int_Q g(\lambda) e\big(z\cdot(\lambda,|\lambda|^2)\big) m_k\left(\frac{|\lambda|^2}{n\delta^2}\right) \mathrm{d}\lambda,$$

where $z \in \mathbb{R}^n$ and $m_k(t) = t^k \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(t)$. We were able to insert the cutoff since $\lambda \in Q = [-\frac{\delta}{2}, \frac{\delta}{2}]^{n-1}$ implies that

$$|\lambda|^2 \le \sum_{j=1}^{n-1} \frac{\delta^2}{4} = \frac{\delta^2(n-1)}{4} \le \frac{\delta^2 n}{2} \Leftrightarrow \frac{|\lambda|^2}{n\delta^2} \le \frac{1}{2}.$$

Plugging (3.22) to (3.21) we get

$$\begin{aligned} \|E_Q g\|^q_{L^q(w_{B(y,\delta^{-1})})} &\lesssim \sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} \int_{\mathbb{R}^n} |M_k(E_Q g)(x+u)|^q w_{B(0,\delta^{-1})}(u)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} \int_{\mathbb{R}^n} |M_k(E_Q g)(u)|^q w_{B(0,\delta^{-1})}(u-x)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} \int_{\mathbb{R}^n} |M_k(E_Q g)(u)|^q w_{B(x,\delta^{-1})}(u)\,\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} \|M_k(E_Q g)\|^q_{L^q(w_{B(x,\delta^{-1})})}. \end{aligned}$$

Multiplying by δ^n, taking supremums over $y \in 12n^{\frac{3}{2}} T_Q(x)$ and mean value integrating both sides we get

$$\begin{aligned}\|F_Q\|^q_{L^q_\sharp(12nB')} &\lesssim \sum_{k=0}^{\infty}\frac{(2n\pi)^k}{k!}\delta^n \fint_{12nB'} \|M_k(E_Qg)\|^q_{L^q(w_{B(x,\delta^{-1})})}\,\mathrm{d}x\\ &= \sum_{k=0}^{\infty}\frac{(2n\pi)^k}{k!}\frac{\delta^n}{m_n(12nB')}\int_{\mathbb{R}^n}|M_k(E_Qg)(u)|^q\int_{12nB'} w_{B(x,\delta^{-1})}(u)\,\mathrm{d}x\,\mathrm{d}u\\ &\sim \sum_{k=0}^{\infty}\frac{(2n\pi)^k}{k!}\frac{1}{m_n(B')}\int_{\mathbb{R}^n}|M_k(E_Qg)(u)|^q\delta^n\int_{12nB'} w_{B(x,\delta^{-1})}(u)\,\mathrm{d}x\,\mathrm{d}u.\end{aligned} \tag{3.23}$$

The equality above is due to Tonelli's theorem. Notice that $w_{B(x,\delta^{-1})}(u) = w_{B(u,\delta^{-1})}(x)$ and by lemma 3.6 we have

$$\delta^n\int_{12nB'} w_{B(u,\delta^{-1})}(x)\,\mathrm{d}x \lesssim w_{B'}(u),$$

for all $u\in\mathbb{R}^n$. Plugging this into (3.23) yields

$$\|F_Q\|^q_{L^q_\sharp(12nB')} \lesssim \sum_{k=0}^{\infty}\frac{(2n\pi)^k}{k!}\|M_k(E_Qg)\|^q_{L^q_\sharp(w_{B'})}. \tag{3.24}$$

We can replace the function m_k with a compactly supported smooth function m_k^* that is defined by

$$m_k^*(t) = t^k b(t),$$

where $b\in\mathcal{S}(\mathbb{R})$ satisfies $b=1$ on $K=[-\frac{1}{2},\frac{1}{2}]$ and $\mathrm{supp}(b)\subset V = \left]-\frac{3}{4},\frac{3}{4}\right[$. See lemma (1.7) for the existence of such b. By lemma 1.8, we have

$$|\widecheck{m_k^*}(x_n)| \lesssim_M (1+|x_n|)^{-M}, \qquad \text{for all } M>0, \tag{3.25}$$

where the implicit constant is independent of k.

Let M_k^* be the operator that is like M_k with the exception that m_k is replaced by m_k^*. We can now apply Fourier inversion formula, change of variables and Fubini to write

$$\begin{aligned}|M_k(E_Qg)(x)| &= |M_k^*(E_Qg)(x)|\\ &= \left|\int_Q g(\lambda)e\big(x\cdot(\lambda,|\lambda|^2)\big)m_k^*\left(\frac{|\lambda|^2}{n\delta^2}\right)\,\mathrm{d}\lambda\right|\\ &= \left|\int_Q g(\lambda)e\big(x\cdot(\lambda,|\lambda|^2)\big)\int_{\mathbb{R}} e\left(-\frac{|\lambda|^2}{n\delta^2}z\right)\widecheck{m_k^*}(z)\,\mathrm{d}z\,\mathrm{d}\lambda\right|\\ &= \left|\int_Q g(\lambda)e\big(x\cdot(\lambda,|\lambda|^2)\big)n\delta^2\int_{\mathbb{R}} e(-|\lambda|^2 y)\widecheck{m_k^*}\left(n\delta^2 y\right)\,\mathrm{d}y\,\mathrm{d}\lambda\right|\end{aligned}$$

$$
\begin{aligned}
&\sim \left| \int_{\mathbb{R}} \int_Q g(\lambda) e\big(x \cdot (\lambda, |\lambda|^2)\big) e(-|\lambda|^2 y) \,\mathrm{d}\lambda \;\; \delta^2 \widecheck{m_k^*}\,\big(n\delta^2 y\big) \,\mathrm{d}y \right| \\
&= \left| \int_{\mathbb{R}} E_Q g(\widetilde{x}, x_n - y) \delta^2 \widecheck{m_k^*}\,\big(n\delta^2 y\big) \,\mathrm{d}y \right| \\
&\leq \int_{\mathbb{R}} |E_Q g(\widetilde{x}, x_n - y)| h_{\delta^2}(y) \,\mathrm{d}y = |E_Q g| \odot h_{\delta^2}(x),
\end{aligned}
$$

where $\odot$ denotes the convolution with respect to the last variable x_n and $h_{\delta^2}(x_n) := \delta^2 |\widecheck{m_k^*}\,(n\delta^2 x_n)\,|$. We can normalize h_{δ^2} with

$$
c := \frac{1}{n} \|\widecheck{m_k^*}\|_1 \lesssim \frac{1}{n} \int_{\mathbb{R}} (1 + |x|)^{-2} \,\mathrm{d}x \leq \frac{1}{n} \int_{\mathbb{R}} (1 + x^2)^{-1} \,\mathrm{d}x = \frac{\pi}{n} \sim 1,
$$

so that $\int_{\mathbb{R}} h_{\delta^2}/c = 1$ and use the Jensen inequality to get

$$
\begin{aligned}
|M_k(E_Q g)(x)|^q &\lesssim \big(|E_Q g| \odot h_{\delta^2}(x)\big)^q \\
&= \left(c\,|E_Q g| \odot \frac{h_{\delta^2}(x)}{c}\right)^q \\
&\leq c^q |E_Q g|^q \odot \frac{h_{\delta^2}(x)}{c} \lesssim |E_Q g|^q \odot h_{\delta^2}(x).
\end{aligned}
$$

We recall that $B' = B(L(c_B), \delta^{-2})$. Then we multiply both sides with $w_{B'}$ and integrate to get

$$
\|M_k(E_Q g)\|_{L^q(w_{B'})}^q \lesssim \int_{\mathbb{R}^n} |E_Q g|^q \odot h_{\delta^2}(x) w_{B'}(x) \,\mathrm{d}x.
$$

Using Fubini's theorem we can write

$$
\begin{aligned}
\int_{\mathbb{R}^n} |E_Q g|^q \odot h_{\delta^2}(x) w_{B'}(x) \,\mathrm{d}x &= \int_{\mathbb{R}^n} \int_{\mathbb{R}} |E_Q g(\widetilde{x}, y)|^q h_{\delta^2}(x_n - y) w_{B'}(x) \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} |E_Q g(\widetilde{x}, y)|^q \int_{\mathbb{R}} h_{\delta^2}(x_n - y) w_{B'}(x) \,\mathrm{d}x_n \,\mathrm{d}(\widetilde{x}, y)
\end{aligned}
$$

and equation (3.25) with $M = E$ allows us to write

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |E_Q g(\widetilde{x}, y)|^q \int_{\mathbb{R}} h_{\delta^2}(x_n - y) w_{B'}(x) \,\mathrm{d}x_n \,\mathrm{d}(\widetilde{x}, y) \\
&\qquad \lesssim \int_{\mathbb{R}^n} |E_Q g(\widetilde{x}, y)|^q \delta^2 \int_{\mathbb{R}} (1 + n\delta^2 |x_n - y|)^{-E} w_{B'}(x) \,\mathrm{d}x_n \,\mathrm{d}(\widetilde{x}, y) \\
&\qquad \leq \int_{\mathbb{R}^n} |E_Q g(\widetilde{x}, y)|^q \delta^2 \int_{\mathbb{R}} (1 + \delta^2 |y - x_n|)^{-E} w_{B'}(x) \,\mathrm{d}x_n \,\mathrm{d}(\widetilde{x}, y) \\
&\qquad = \int_{\mathbb{R}^n} |E_Q g(\widetilde{x}, y)|^q \delta^2 \int_{\mathbb{R}} w_{B^1(0,\delta^{-2})}(y - x_n) w_{B'}(x) \,\mathrm{d}x_n \,\mathrm{d}(\widetilde{x}, y).
\end{aligned}
$$

By lemma 3.13 we have

$$\delta^2 \int_{\mathbb{R}} w_{B^1(0,\delta^{-2})}(y - x_n) w_{B'}(x)\, \mathrm{d}x_n \lesssim w_{B'}(\widetilde{x}, y)$$

and this means that we have

$$\int_{\mathbb{R}^n} |E_Q g(\widetilde{x}, y)|^q \int_{\mathbb{R}} h_{\delta^2}(x_n - y) w_{B'}(x)\, \mathrm{d}x_n\, \mathrm{d}(\widetilde{x}, y) \lesssim \int_{\mathbb{R}^n} |E_Q g(\widetilde{x}, y)|^q w_{B'}(\widetilde{x}, y)\, \mathrm{d}(\widetilde{x}, y).$$

Merging our recent estimates, we have

$$\|M_k(E_Q g)\|^q_{L^q(w_{B'})} \lesssim \|E_Q g\|^q_{L^q(w_{B'})}.$$

Finally, combining this with (3.24) we get

$$\|F_Q\|^q_{L^q_\sharp(12nB')} \lesssim \sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} \|E_Q g\|_{L^q_\sharp(w_{B'})} \sim \|E_Q g\|_{L^q_\sharp(w_{B'})},$$

which is exactly (3.20). The last relation is due to $\sum_{k=0}^{\infty} \frac{(2n\pi)^k}{k!} = e^{2\pi n}$. □

Now we are ready to prove theorem 3.15.

Proof of theorem 3.15. To make notions more bearable, we abbreviate

$$\sum_{Q_{i,1}} := \sum_{Q_{i,1} \in \mathrm{Part}_\delta(Q_i)},$$

$$q := \frac{p(n-1)}{n},$$

$$e(Q_{i,1}, B) := \|E_{Q_{i,1}} g\|_{L^q_\sharp(w_B)}$$

and

$$e_i^* := \max_{Q_{i,1} \in \mathrm{Part}_\delta(Q_i)} e(Q_{i,1}, B).$$

Let $C = C(n,p) \in \mathbb{N}_+$. The idea is to use dyadic pigeonholing to restrict the cubes $Q_{i,1}$. We notice that each cube $Q_{i,1}$ satisfies either

$$e(Q_{i,1}, B) \leq \delta^C e_i^* \tag{3.26}$$

or

$$2^k \delta^C e_i^* < e(Q_{i,1}, B) \leq 2^{k+1} \delta^C e_i^*, \qquad \text{for some } k \in \mathbb{N}. \tag{3.27}$$

Since $e(Q_{i,1}, B) \leq e_i^*$ always holds, we can forget each k that satisfies $2^k \delta^C \geq 1$. Since

$$2^k \delta^C < 1 \quad \Leftrightarrow \quad \log_2(2^k \delta^C) < 0 \quad \Leftrightarrow \quad k - C \log_2 \delta^{-1} < 0 \quad \Leftrightarrow \quad k < C \log_2 \delta^{-1} =: A,$$

we now know that (3.27) is equivalent with

$$e_i^* 2^{-k'-1} < e(Q_{i,1}, B) \leq 2^{-k'} e_i^*, \qquad k' \in \{0, 1, \dots, A-1\}.$$

We denote

$$b_{-1,i}(\Delta) := \sum_{\substack{Q_{i,1} \\ e(Q_{i,1},B) \leq \delta^C e_i^*}} e(Q_{i,1}, \Delta)^2$$

for each $i = 1, \dots, n$ and

$$b_{k,i}(\Delta) := \sum_{\substack{Q_{i,1} \\ e_i^* 2^{-k-1} < e(Q_{i,1},B) \leq 2^{-k} e_i^*}} e(Q_{i,1}, \Delta)^2$$

for each $i = 1, \dots, n$ and $k = 0, \dots, A-1$. Thus we have

$$\begin{aligned}
\sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^2 &= \sum_{\substack{Q_{i,1} \\ e(Q_{i,1},B) \leq \delta^C e_i^*}} e(Q_{i,1}, \Delta)^2 \\
&\qquad + \sum_{k=0}^{A-1} \sum_{\substack{Q_{i,1} \\ e_i^* 2^{-k-1} < e(Q_{i,1},B) \leq 2^{-k} e_i^*}} e(Q_{i,1}, \Delta)^2 \\
&= \sum_{k=-1}^{A-1} b_{k,i}(\Delta)
\end{aligned}$$

This implies that

$$\frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^n \left(\sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^2 \right)^{\frac{1}{2}} \right)^{\frac{p}{n}} = \frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^n \sum_{k=-1}^{A-1} b_{k,i}(\Delta) \right)^{\frac{p}{2n}}. \tag{3.28}$$

By the distributive law we calculate

$$\begin{aligned}
\prod_{i=1}^{n}\sum_{k=-1}^{A-1} b_{k,i}(\Delta) &= \left(\sum_{k_1=-1}^{A-1} b_{k_1,1}(\Delta)\right)\cdots\left(\sum_{k_n=-1}^{A-1} b_{k_n,n}(\Delta)\right)\\
&= \sum_{k_1=-1}^{A-1}\cdots\sum_{k_n=-1}^{A-1} b_{k_1,1}(\Delta)\cdots b_{k_n,n}(\Delta)\\
&=: \sum_{\vec{k}\in K}\prod_{i=1}^{n} b_{k_i,i}(\Delta),
\end{aligned}$$

where on the last line we denoted $\vec{k} \coloneqq (k_1,\ldots,k_n)$ and $K \coloneqq \{-1,0,1,\ldots,A-1\}^n$. If $\frac{p}{2n} \le 1$, then convexity yields

$$\begin{aligned}
\sum_{\Delta\in\mathcal{B}}\left(\sum_{\vec{k}\in K}\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}} &\le \sum_{\Delta\in\mathcal{B}}\sum_{\vec{k}\in K}\left(\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}\\
&= \sum_{\vec{k}\in K}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}\\
&\le (1+A)^n \max_{\vec{k}\in K}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}.
\end{aligned}$$

If $\frac{p}{2n} > 1$, then Hölder inequality with conjugates $\frac{p}{2n}$ and $\frac{p}{p-2n}$ gives

$$\begin{aligned}
\sum_{\Delta\in\mathcal{B}}\left(\sum_{\vec{k}\in K}\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}} &\le \sum_{\Delta\in\mathcal{B}}\left[\left(\sum_{\vec{k}\in K}\prod_{i=1}^{n} b_{k_i,i}(\Delta)^{\frac{p}{2n}}\right)^{\frac{2n}{p}}\left(\sum_{\vec{k}\in K}1\right)^{1-\frac{2n}{p}}\right]^{\frac{p}{2n}}\\
&= \sum_{\Delta\in\mathcal{B}}\sum_{\vec{k}\in K}\left(\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}\left(\sum_{\vec{k}\in K}1\right)^{\frac{p}{2n}-1}\\
&\le (1+A)^{\frac{p}{2}-n}\sum_{\Delta\in\mathcal{B}}\sum_{\vec{k}\in K}\left(\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}\\
&\le (1+A)^{\frac{p}{2}}\max_{\vec{k}\in K}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n} b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}.
\end{aligned}$$

Overall we now have

$$\sum_{\Delta\in\mathcal{B}}\left(\sum_{\vec{k}\in K}\prod_{i=1}^{n}b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}} \leq (1+A)^{\max\{\frac{p}{2},n\}}\max_{\vec{k}\in K}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}$$

and plugging the above to (3.28) gives that

$$\begin{aligned}\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}\left(\sum_{Q_{i,1}}e(Q_{i,1},\Delta)^2\right)^{\frac{1}{2}}\right)^{\frac{p}{n}} &\leq (1+A)^{\max\{\frac{p}{2},n\}}\frac{1}{\#\mathcal{B}}\max_{\vec{k}\in K}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}\\ &\lesssim_{\varepsilon}\delta^{-\varepsilon}\frac{1}{\#\mathcal{B}}\max_{\vec{k}\in K}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}}.\end{aligned}$$

Thus we see that in order to prove the theorem, it suffices to show that for each combination $\vec{k}=(k_1,\dots,k_n)\in K=\{-1,0,1,\dots,A-1\}^n$ we have

$$\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{n}b_{k_i,i}(\Delta)^{\frac{p}{2n}} \lesssim_{\varepsilon,\nu}\delta^{-\varepsilon}\left(\prod_{i=1}^{n}\left(\sum_{Q_{i,1}}e(Q_{i,1},B)^2\right)^{\frac{1}{2}}\right)^{\frac{p}{n}} \tag{3.29}$$

In order to show (3.29), we will consider two cases.

1. At least one $k_i=-1$.
2. Every index k_i is non-negative.

In the first case say $k_j=-1$. We notice that for each $\Delta\in\mathcal{B}$ and each $Q_{i,1}$, we have

$$\begin{aligned}\delta^{\frac{n}{q}}e(Q_{i,1},\Delta) &= \delta^{\frac{n}{q}}\left(\delta^n\int_{\mathbb{R}^n}|E_{Q_{i,1}}g(x)|^q w_\Delta(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &\lesssim \delta^{\frac{n}{q}}\left(\delta^n\int_{\mathbb{R}^n}|E_{Q_{i,1}}g(x)|^q w_B(x)\,\mathrm{d}x\right)^{\frac{1}{q}} = e(Q_{i,1},B),\end{aligned}$$

where we used $w_\Delta(x)\lesssim w_B(x)$, which is an immediate corollary of lemma 3.12. For the cubes $Q'_{i,1}$ in $b_{k_j,j}$, the above calculation combined with (3.26) gives

$$\max_{\Delta\in\mathcal{B}}e(Q'_{j,1},\Delta)\lesssim\delta^{C-\frac{n}{q}}e_j^*$$

and for each $Q_{i,1}$ we have

$$\max_{\Delta\in\mathcal{B}} e(Q_{i,1},\Delta) \lesssim \delta^{-\frac{n}{q}} e_i^*.$$

Hence

$$b_{k_j,j} = \sum_{\substack{Q_{j,1} \\ e(Q_{j,1},B)\le \delta^C e_j^*}} e(Q_{j,1},\Delta)^2 \le \delta^{2C-\frac{2n}{q}-n}(e_j^*)^2$$

and for $i \neq j$

$$b_{k_i,i} \le \sum_{Q_{i,1}} e(Q_{i,1},\Delta)^2 \le \delta^{-\frac{2n}{q}-n}(e_i^*)^2.$$

Above we used the fact that the number of cubes $Q_{i,1}$ is less or equal to δ^{-n}. Thus by choosing $C \ge \frac{n}{p}(\frac{np}{q}+\frac{np}{2}) = \frac{n^3}{p(n-1)}+\frac{n^2}{2}$ we get

$$\begin{aligned}
\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^n b_{k_i,i}(\Delta)\right)^{\frac{p}{2n}} &\le \frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\delta^{2C-\frac{2n}{q}-n}(e_j^*)^2\prod_{i\neq j}\delta^{-\frac{2n}{q}-n}(e_i^*)^2\right)^{\frac{p}{2n}} \\
&= \delta^{C\frac{p}{n}-\frac{np}{q}-n\frac{p}{2}}\left(\prod_{i=1}^n (e_i^*)^2\right)^{\frac{p}{2n}} \\
&\le \delta^{C\frac{p}{n}-\frac{np}{q}-\frac{np}{2}}\left(\prod_{i=1}^n\left(\sum_{Q_{i,1}} e(Q_{i,1},B)^2\right)^{\frac{1}{2}}\right)^{\frac{p}{n}} \\
&\le \left(\prod_{i=1}^n\left(\sum_{Q_{i,1}} e(Q_{i,1},B)^2\right)^{\frac{1}{2}}\right)^{\frac{p}{n}}.
\end{aligned}$$

If more indices k_i are equal to -1, then we are able to choose a smaller C. This ends the proof for the first case.

For the second case let $N_{k,i}$ be the number of cubes $Q_{i,1}$ in $b_{k,i}$. We denote the set of these cubes by $B_{k,i}$ and abbreviate $\sum_{Q_{i,1}} = \sum_{Q_{i,1}\in B_{k,i}}$. If $p = \frac{2n}{n-1}$, then $q = \frac{\frac{2n}{n-1}(n-1)}{n} = 2$ and $\frac{p}{2n} = \frac{1}{n-1}$, so we immediately have

$$\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^n b_{k_i,i}(\Delta)^{\frac{p}{2n}} = \frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^n\left(\sum_{Q_{i,1}} e(Q_{i,1},\Delta)^2\right)^{\frac{1}{2}}\right)^{\frac{p}{n}}$$

$$= \frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^{n} \sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^q \right)^{\frac{1}{n-1}}.$$

If $p > \frac{2n}{n-1}$, then we can use Hölder inequality on $e(Q_{i,1}, \Delta)^2$ with conjugates $p' := \frac{p(n-1)}{p(n-1)-2n}$ and $q' := \frac{p(n-1)}{2n}$. This gives us

$$\|e(Q_{i,1}, \Delta)^2\|_{l_1} \leq \|1\|_{l_{p'}} \|e(Q_{i,1}, \Delta)^2\|_{l_{q'}} = N_{k,i}^{\frac{1}{p'}} \|e(Q_{i,1}, \Delta)^2\|_{l_{q'}}$$

$$\Leftrightarrow$$

$$\frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^{n} \|e(Q_{i,1}, \Delta)^2\|_{l_1}^{\frac{1}{2}} \right)^{\frac{p}{n}} \leq \left(\prod_{i=1}^{n} N_{k,i}^{\frac{1}{2p'}} \right)^{\frac{p}{n}} \frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^{n} \|e(Q_{i,1}, \Delta)^2\|_{l_{q'}}^{\frac{1}{2}} \right)^{\frac{p}{n}}, \tag{3.30}$$

where we denoted $\|e(Q_{i,1}, \Delta)^2\|_{l_r} := \left(\sum_{Q_{i,1}} (e(Q_{i,1}, \Delta)^2)^r \right)^{\frac{1}{r}}$ for $1 \leq r < \infty$. We notice that (3.30) is equivalent with

$$\frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \prod_{i=1}^{n} b_{k_i,i}(\Delta)^{\frac{p}{2n}} = \frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^{n} \left(\sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^2 \right)^{\frac{1}{2}} \right)^{\frac{p}{n}}$$

$$\leq \left(\prod_{i=1}^{n} N_{k,i}^{\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{n}} \frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^{n} \sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^q \right)^{\frac{1}{n-1}}. \tag{3.31}$$

Next we want to control the expression

$$\frac{1}{\#\mathcal{B}} \sum_{\Delta \in \mathcal{B}} \left(\prod_{i=1}^{n} \sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^q \right)^{\frac{1}{n-1}}.$$

In order to estimate to above expression, we will recreate the setting of lemma 3.16. For each cube $Q_{i,1}$ centered at $c_{Q_{i,1}}$, we cover B with a family $\mathcal{T}_{Q_{i,1}}$ of pairwise disjoint, mutually parallel tiles $T_{Q_{i,1}}$. They are rectangles with $n-1$ short sides of length δ^{-1} and one longer side of length δ^{-2}, pointing in the direction of the normal $n(c_{Q_{i,1}})$ to the paraboloid $\mathbb{P}^{n-1}$ at $c_{Q_{i,1}}$. Moreover, we know that these tiles are inside the cube $4B$. We let $T_{Q_{i,1}}(x)$ be the tile containing x.

We define

$$F_Q(x) = \sup_{y \in (1+\sqrt{n})T_Q(x)} e(Q, B(y, \delta^{-1})), \qquad \text{for } x \in \bigcup_{T_Q \in \mathcal{T}_Q} T_Q.$$

Assume that $x \in \Delta$. Recall that Δ is a cube of side length δ^{-1} and thus the diameter of Δ is $\sqrt{n}\delta^{-1}$. Since $x \in \Delta \cap T_Q(x)$, we now have $c_\Delta \in (1+\sqrt{n})T_Q(x)$ (see figure 3.2).

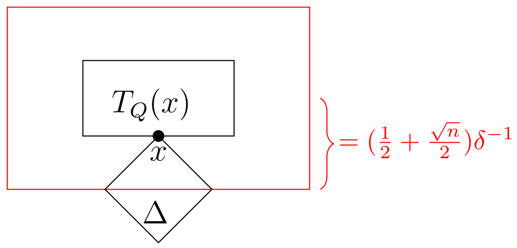

Figure 3.2: Inclusion $c_\Delta \in (1+\sqrt{n})T_Q(x)$ in $\mathbb{R}^2$ for the worst case scenario. Lengths are from the n-dimensional case.

This implies $e(Q,\Delta) \leq F_Q(x)$, which means that

$$\begin{aligned}
&\left(\prod_{i=1}^{n}\sum_{Q_{i,1}} e(Q_{i,1},\Delta)^q\right)^{\frac{1}{n-1}} \leq \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}} \\
\Rightarrow &\left(\prod_{i=1}^{n}\sum_{Q_{i,1}} e(Q_{i,1},\Delta)^q\right)^{\frac{1}{n-1}} \leq \fint_\Delta \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x \\
\Rightarrow &\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}\sum_{Q_{i,1}} e(Q_{i,1},\Delta)^q\right)^{\frac{1}{n-1}} \leq \sum_{\Delta\in\mathcal{B}}\fint_\Delta \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x \\
\Leftrightarrow &\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}\sum_{Q_{i,1}} e(Q_{i,1},\Delta)^q\right)^{\frac{1}{n-1}} \leq \frac{1}{\#\mathcal{B}}\frac{1}{m_n(\Delta)}\int_B \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x \\
\Leftrightarrow &\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}\sum_{Q_{i,1}} e(Q_{i,1},\Delta)^q\right)^{\frac{1}{n-1}} \leq \frac{1}{|B|}\int_B \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x \\
\Rightarrow &\frac{1}{\#\mathcal{B}}\sum_{\Delta\in\mathcal{B}}\left(\prod_{i=1}^{n}\sum_{Q_{i,1}} e(Q_{i,1},\Delta)^q\right)^{\frac{1}{n-1}} \lesssim \fint_{4B} \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} F^q_{Q_{i,1}}(x)\right)^{\frac{1}{n-1}} \mathrm{d}x.
\end{aligned}$$

Moreover, the function $F^q_{Q_{i,1}}$ is constant on each tile $T_{Q_{i,1}} \in \mathcal{F}_{Q_{i,1}}$. Indeed, if $x, z \in T_{Q_{i,1}}$, then $T_{Q_{i,1}}(x) = T_{Q_{i,1}}(z)$. Since the cubes $Q_1, \ldots, Q_n$ are ν-transverse, we can utilize the multilinear Kakeya inequality from corollary 2.15 applied to the functions $F_i(x) = \sum_{Q_{i,1}} F_{Q_{i,1}}^q(x)$ to get

$$\fint_{4B} \left(\prod_{i=1}^n \sum_{Q_{i,1}} F_{Q_{i,1}}^q(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \lesssim_{\varepsilon,\nu} \delta^{-\varepsilon} \prod_{i=1}^n \left(\fint_{4B} \sum_{Q_{i,1}} F_{Q_{i,1}}^q(x) \,\mathrm{d}x \right)^{\frac{1}{n-1}}$$

$$= \delta^{-\varepsilon} \left(\prod_{i=1}^n \sum_{Q_{i,1}} \fint_{4B} F_{Q_{i,1}}^q(x) \,\mathrm{d}x \right)^{\frac{1}{n-1}}.$$

An application of lemma 3.16 gives

$$\|F_{Q_{i,1}}\|_{L^q_\sharp(4B)}^q \lesssim e(Q_{i,1}, B)^q$$

and hence

$$\delta^{-\varepsilon} \left(\prod_{i=1}^n \sum_{Q_{i,1}} \fint_{4B} F_{Q_{i,1}}^q(x) \,\mathrm{d}x \right)^{\frac{1}{n-1}} \lesssim \delta^{-\varepsilon} \left(\prod_{i=1}^n \sum_{Q_{i,1}} e(Q_{i,1}, B)^q \right)^{\frac{1}{n-1}}.$$

Combining the estimates, starting from (3.31), leads to

$$\begin{aligned} \frac{1}{\#\mathcal{B}} \sum_{\Delta\in\mathcal{B}} \prod_{i=1}^n b_{k_i,i}(\Delta)^{\frac{p}{2n}} &= \frac{1}{\#\mathcal{B}} \sum_{\Delta\in\mathcal{B}} \left(\prod_{i=1}^n \left(\sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^2 \right)^{\frac{1}{2}} \right)^{\frac{p}{n}} \\ &\leq \left(\prod_{i=1}^n N_{k,i}^{\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{n}} \frac{1}{\#\mathcal{B}} \sum_{\Delta\in\mathcal{B}} \left(\prod_{i=1}^n \sum_{Q_{i,1}} e(Q_{i,1}, \Delta)^q \right)^{\frac{1}{n-1}} \\ &\lesssim \left(\prod_{i=1}^n N_{k,i}^{\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{n}} \fint_{4B} \left(\prod_{i=1}^n \sum_{Q_{i,1}} F_{Q_{i,1}}^q(x) \right)^{\frac{1}{n-1}} \mathrm{d}x \\ &\lesssim_{\varepsilon,\nu} \delta^{-\varepsilon} \left(\prod_{i=1}^n N_{k,i}^{\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{n}} \left(\prod_{i=1}^n \sum_{Q_{i,1}} \fint_{4B} F_{Q_{i,1}}^q(x) \,\mathrm{d}x \right)^{\frac{1}{n-1}} \\ &\lesssim \delta^{-\varepsilon} \left(\prod_{i=1}^n N_{k,i}^{\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{n}} \left(\prod_{i=1}^n \sum_{Q_{i,1}} e(Q_{i,1}, B)^q \right)^{\frac{1}{n-1}}. \end{aligned} \tag{3.32}$$

Recalling the restriction on the cubes $Q_{i,1}$, for a cube $Q \in B_{k,i}$, we have

$$2^{-k_i-1}e_i^* < e(Q,B) \leq 2^{-k_i}e_i^* \Rightarrow e(Q,B) \sim 2^{-k_i}e_i^*$$

and thus

$$\begin{aligned}
\left(\prod_{i=1}^{n} N_{k,i}^{\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{n}} \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} e(Q_{i,1},B)^q\right)^{\frac{1}{n-1}} &\sim \left(\prod_{i=1}^{n} N_{k,i}^{\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{n}} \left(\prod_{i=1}^{n}\sum_{Q_{i,1}} (2^{-k_i}e_i^*)^q\right)^{\frac{1}{n-1}} \\
&= \left(\prod_{i=1}^{n} N_{k,i}^{\frac{p}{2n}-\frac{1}{n-1}}\right) \left(\prod_{i=1}^{n} N_{k,i}\,(2^{-k_i}e_i^*)^q\right)^{\frac{1}{n-1}} \\
&= \left(\prod_{i=1}^{n} N_{k,i}^{\frac{p}{2n}}\right) \left(\prod_{i=1}^{n} (2^{-k_i}e_i^*)^{\frac{p}{n}}\right) \\
&= \prod_{i=1}^{n} \left(N_{k,i}^{\frac{1}{2}} 2^{-k_i}e_i^*\right)^{\frac{p}{n}} \\
&= \prod_{i=1}^{n} \left(\left(N_{k,i}\,(2^{-k_i}e_i^*)^2\right)^{\frac{1}{2}}\right)^{\frac{p}{n}} \\
&= \prod_{i=1}^{n} \left(\Big(\sum_{Q_{i,1}} (2^{-k_i}e_i^*)^2\Big)^{\frac{1}{2}}\right)^{\frac{p}{n}} \\
&\sim \left(\prod_{i=1}^{n} \left(\sum_{Q_{i,1}} e(Q_{i,1},B)^2\right)^{\frac{1}{2}}\right)^{\frac{p}{n}},
\end{aligned}$$

which we can plug into (3.32) and (3.29) follows for the second case. This ends the proof of the theorem. □

Chapter 4

Equivalent formulations of linear decoupling

We will begin this chapter by defining the linear decoupling constant $\mathrm{Dec}_n(\delta, p, E)$ that is associated to the extension operator E_Q. In literature regarding decoupling, an alternative formulation is often used, where the decoupling constant is defined without E_Q and the decoupling is done using Fourier restriction. For the sake of exposition, let the decoupling constant related to the alternative formulation be $\mathrm{Dec}_n(\mathcal{F})$. The main result of this chapter (theorem 4.27) shows that these two formulations are essentially equivalent in the sense that an upper bound for $\mathrm{Dec}_n(\delta, p, E)$ will imply an upper bound for $\mathrm{Dec}_n(\mathcal{F})$. This motivates the use of the extension operator E_Q and the associated decoupling constant. Another motivation for theorem 4.27 is that in the fifth chapter we will use it to prove an important reverse decoupling constant inequality.

4.1 The linear decoupling constant

In this section we will define the linear decoupling constant and introduce a lemma that we will use to simplify the proofs of decoupling inequalities. Throughout the rest of the thesis we will assume that $\delta \in 4^{-\mathbb{N}}$.

Definition 4.1. For $p \geq 2$, let $\mathrm{Dec}_n(\delta, p, E) = \mathrm{Dec}(\delta, p, E)$ be the smallest constant that satisfies

$$\|Eg\|_{L^p(w_{B,E})} \leq \mathrm{Dec}(\delta, p, E) \left(\sum_{Q \in \mathrm{Part}_{\sqrt{\delta}}([0,1]^{n-1})} \|E_Q g\|^2_{L^p(w_{B,E})} \right)^{\frac{1}{2}}, \tag{4.2}$$

for each cube B with side length δ^{-1}.

The exact statement of the l^2 decoupling theorem is that if $2 \leq p \leq \frac{2(n+1)}{n-1}$, then we have $\mathrm{Dec}(\delta, p, E) \lesssim_\varepsilon \delta^{-\varepsilon}$. In contrast, it is a straightforward application of Minkowski's inequality and Cauchy-Schwarz inequality to show that $\delta^{-\frac{n-1}{4}}$ is a constant that satisfies the inequality in the definition. With a little more work one can comfortably show that the decoupling constant $\mathrm{Dec}(\delta, p, E)$ is well defined. Together these results imply that $\mathrm{Dec}(\delta, p, E) \leq \delta^{-\frac{n-1}{4}}$. The details of these computations can be found in [14] section 2.6.

Next we will look at a lemma which can be used to simplify the inequalities that are of the form (4.2).

Lemma 4.3. *Fix $E > n$ and $R > 0$. Let $\mathcal{W}$ be the space of non-negative integrable functions on $\mathbb{R}^n$. Assume that operators $O_1, O_2 \colon \mathcal{W} \to [0, \infty]$ satisfy the following four properties:*

(W1) $O_1(\mathbb{1}_B) \lesssim O_2(w_{B,E})$ *for all cubes B of side length R.*

(W2) $O_1(\sum_{i=1}^{\infty} \alpha_i u_i) \leq \sum_{i=1}^{\infty} \alpha_i O_1(u_i)$ *for all $u_i \in \mathcal{W}$ and $\alpha_i > 0$ such that $\sum_{i=1}^{\infty} \alpha_i u_i \in \mathcal{W}$.*

(W3) $O_2(\sum_{i=1}^{\infty} \alpha_i u_i) \geq \sum_{i=1}^{\infty} \alpha_i O_2(u_i)$ *for all $u_i \in \mathcal{W}$ and $\alpha_i > 0$ such that $\sum_{i=1}^{\infty} \alpha_i u_i \in \mathcal{W}$.*

(W4) If $u \leq v$, then $O_i(u) \leq O_i(v)$, where $i = 1, 2$.

Then

$$O_1(w_{B,E}) \lesssim O_2(w_{B,E})$$

for all cubes B with side length R.

Proof. This is a relatively immediate consequence of lemma 3.9 after we check that the implicit constants do not affect the operators too much. Hence, we will begin with proving that the operators preserve scalar multiplication, i.e.,

$$O_i(\alpha u) = \alpha O_i(u) \tag{4.4}$$

where $\alpha > 0$, $u \in \mathcal{W}$ and $i = 1, 2$.

By $(W2)$ and $(W3)$, we have

$$O_1(0) = O_1(\sum_{i=2}^{\infty} (\frac{1}{2})^i \cdot 0) \leq \frac{1}{2} O_1(0)$$

and

$$O_2(0) = O_2(\sum_{i=0}^{\infty} (\frac{1}{2})^i \cdot 0) \geq 2 O_2(0).$$

Thus $O_i(0) = 0$ or $O_i(0) = \infty$. If $O_2(0) = \infty$, then by (*W4*) we have $O_2(w_{B,E}) = \infty$ and the result is trivial. If $O_1(0) = \infty$, then by (*W4*) and (*W1*) we again have $O_2(w_{B,E}) = \infty$.

We can now assume that $O_1(0) = 0 = O_2(0)$. This means that setting $u_i = 0$ for $i > 1$ in (*W2*) gives us $O_1(\alpha u) \leq \alpha O_1(u)$ for all $u \in \mathcal{W}$ and $\alpha > 0$. Applying this twice, we get

$$O_1(\alpha u) \leq \alpha O_1(u) = \alpha O_1(\alpha^{-1}\alpha u) \leq O_1(\alpha u) \Leftrightarrow O_1(\alpha u) = \alpha O_1(u).$$

Similar calculations for O_2 proves (4.4). Clearly (4.4) and (W4) shows that $u \lesssim v$ implies $O_i(u) \lesssim O_i(v)$. Now applying inequalities (3.10) and (3.11) of lemma 3.9, gives us

$$\begin{aligned}
O_1(w_{B,E}) &\lesssim O_1\left(\sum_{B'\in\mathcal{B}} w_{B,E}(c_{B'})\mathbb{1}_{B'}(x)\right) && (3.10)\\
&\leq \sum_{B'\in\mathcal{B}} w_{B,E}(c_{B'})O_1(\mathbb{1}_{B'}(x)) && (\text{W2})\\
&\lesssim \sum_{B'\in\mathcal{B}} w_{B,E}(c_{B'})O_2(w_{B',E}(x)) && (\text{W1})\\
&\leq O_2\left(\sum_{B'\in\mathcal{B}} w_{B,E}(c_{B'})w_{B',E}(x)\right) && (\text{W3})\\
&\lesssim O_2(w_{B,E}). && (3.11)
\end{aligned}$$

□

The main interest of lemma 4.3 is that if $f, f_1, f_2, \ldots$ are measurable and $v \in \mathcal{W}$, then

$$O_1(v) := \|f\|^p_{L^p(v)} \qquad \text{with } p \geq 1$$

and

$$O_2(v) := \left(\sum_{j\in\mathbb{N}_+} \|f_j\|^2_{L^p(v)}\right)^{\frac{p}{2}} \qquad \text{with } p \geq 2$$

satisfy conditions (*W2*), (*W3*) and (*W4*). Indeed, both satisfy (*W4*) by the increasing nature of the integral. Condition (*W2*) follows from the monotone convergence theorem

$$\begin{aligned}
O_1(\sum_{i=1}^{\infty} \alpha_i u_i) &= \int_{\mathbb{R}^n} |f|^p \lim_{k\to\infty} \sum_{i=1}^{k} \alpha_i u_i \, \mathrm{d}m_n \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \alpha_i \int_{\mathbb{R}^n} |f|^p u_i \, \mathrm{d}m_n = \sum_{i=1}^{\infty} \alpha_i O_1(u_i).
\end{aligned}$$

We briefly denote $s := \sum_{i=1}^{\infty} \alpha_i u_i$. For condition (*W3*) note that similarly as above we have

$$\|(f_j)^p\|_{L^1(s)} = \sum_{i=1}^{\infty} \|(f_j)^p\|_{L^1(\alpha_i u_i)}.$$

Using this we write

$$O_2(s) = \left(\sum_{j \in \mathbb{N}_+} \|f_j\|^2_{L^p(s)} \right)^{\frac{p}{2}} = \left(\sum_{j \in \mathbb{N}_+} \|(f_j)^p\|^{\frac{2}{p}}_{L^1(s)} \right)^{\frac{p}{2}} = \left\| \sum_{i=1}^{\infty} \|(f_j)^p\|_{L^1(\alpha_i u_i)} \right\|_{l^{\frac{2}{p}}}. \tag{4.5}$$

The monotone convergence theorem for series and the continuity of the functions $x \to x^a$, where $a > 0$, imply that

$$\left\| \sum_{i=1}^{\infty} \|(f_j)^p\|_{L^1(\alpha_i u_i)} \right\|_{l^{\frac{2}{p}}} = \lim_{k \to \infty} \left\| \sum_{i=1}^{k} \|(f_j)^p\|_{L^1(\alpha_i u_i)} \right\|_{l^{\frac{2}{p}}}.$$

Note that $\frac{2}{p} \leq 1$. Combining the above with (4.5) and applying Minkovski's inequality in $l^{\frac{2}{p}}$ we get

$$\begin{aligned}
O_2(s) &= \lim_{k \to \infty} \left\| \sum_{i=1}^{k} \|(f_j)^p\|_{L^1(\alpha_i u_i)} \right\|_{l^{\frac{2}{p}}} \\
&\geq \lim_{k \to \infty} \sum_{i=1}^{k} \left\| \|(f_j)^p\|_{L^1(\alpha_i u_i)} \right\|_{l^{\frac{2}{p}}} \\
&= \sum_{i=1}^{\infty} \left\| \|(f_j)^p\|_{L^1(\alpha_i u_i)} \right\|_{l^{\frac{2}{p}}} \\
&= \sum_{i=1}^{\infty} \alpha_i \left\| \|(f_j)^p\|_{L^1(u_i)} \right\|_{l^{\frac{2}{p}}} \\
&= \sum_{i=1}^{\infty} \alpha_i \left(\sum_{j \in \mathbb{N}_+} \|(f_j)^p\|^{\frac{2}{p}}_{L^1(u_i)} \right)^{\frac{p}{2}} \\
&= \sum_{i=1}^{\infty} \alpha_i \left(\sum_{j \in \mathbb{N}_+} \|f_j\|^2_{L^p(u_i)} \right)^{\frac{p}{2}} = \sum_{i=1}^{\infty} \alpha_i O_2(u_i),
\end{aligned}$$

which is ($W3$). We have now shown that in order to prove

$$\|f\|_{L^p(w_{B,E})}^p \lesssim \left(\sum_{j\in\mathbb{N}_+} \|f_j\|_{L^p(w_{B,E})}^2\right)^{\frac{p}{2}},$$

by lemma 4.3 it suffices to prove

$$\|f\|_{L^p(B)}^p \lesssim \left(\sum_{j\in\mathbb{N}_+} \|f_j\|_{L^p(w_{B,E})}^2\right)^{\frac{p}{2}}.$$

Lastly we note the immediate fact that if some operators satisfy ($W2$) and ($W4$) or ($W3$) and ($W4$), then positive linear combinations of these operators also satisfy ($W2$) and ($W4$) or ($W3$) and ($W4$) respectively.

4.2 Decoupling via Fourier restriction

In this section we will show how to decouple a function using its Fourier restriction and the linear decoupling constant associated to E_Q. The main idea is to divide the Fourier support of a function f into disjoint sets A and use the Fourier restrictions of f into A to form a decoupling inequality.

Definition 4.6. Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be a function that has an integrable Fourier transform. Then the Fourier restriction of f to R is defined by

$$f_R(x) := \int_R \widehat{f}(\xi) e(x\cdot\xi)\,\mathrm{d}\xi.$$

In the above definition we allow that $\widehat{f} \in \mathcal{S}'(\mathbb{R}^n)$ is a distribution that is defined by an integrable function f^* with the formula $\phi \mapsto \int \phi f^*$. In this case we abuse notation by $\widehat{f}(\xi) := f^*(\xi)$.

The Fourier restriction decoupling will work for functions that are Fourier supported in a specific neighbourhood of the paraboloid $\mathbb{P}^{n-1}$.

Definition 4.7. For a number $r \in \,]0,1[$ and a cube $Q \subset [0,1]^{n-1}$ the r-neighbourhood of $\mathbb{P}^{n-1}$ above Q is

$$N_r(Q) := \{(\xi_1,\xi_2,\dots\xi_{n-1},\xi_1^2+\cdots+\xi_{n-1}^2+t) : (\xi_1,\dots,\xi_{n-1}) \in Q \text{ and } 0 \le t \le r\}.$$

Before we tackle the decoupling, we will introduce four lemmas. The first lemma is about breaking the n-dimensional weight function into products of one dimensional weight functions.

Lemma 4.8. *Let $B_R \coloneqq B(c, R) \subset \mathbb{R}^n$ and $W \geq 0$. Then*

$$w_{B_R,W}(x) \leq \prod_{i=1}^{n} \left(1 + \frac{|x_i - c_i|}{R}\right)^{-W_i},$$

where $W_i \geq 0$ and $\sum_{i=1}^{n} W_i \leq W$.

Proof. The proof is a straightforward calculation

$$\begin{aligned} w_{B_R,W}(x) = \left(1 + \frac{|x - c|}{R}\right)^{-W} &\leq \left(1 + \frac{\sqrt{\sum_{j=1}^{n}(x_j - c_j)^2}}{R}\right)^{-\sum_{i=1}^{n} W_i} \\ &= \prod_{i=1}^{n} \left(1 + \frac{\sqrt{\sum_{j=1}^{n}(x_j - c_j)^2}}{R}\right)^{-W_i} \\ &\leq \prod_{i=1}^{n} \left(1 + \frac{|x_i - c_i|}{R}\right)^{-W_i}. \end{aligned}$$

□

Then we introduce a lemma that allows us to write the weight function as a convolution of two weight functions when the cubes are centered at the origin.

Lemma 4.9. *Let $B_R \coloneqq B(0, R) \subset \mathbb{R}^n$ and $E > n$. For $0 < R' \leq R$ we have*

$$w_{B_R,E} * \left(\frac{1}{(R')^n} w_{B_{R'},E}\right)(x) \lesssim w_{B_R,E}(x) \tag{4.10}$$

and

$$R^n w_{B_R,E}(x) \lesssim \mathbb{1}_{B_R} * w_{B_R,E}(x). \tag{4.11}$$

Proof. We will first prove (4.10). Throughout the proof we will use

$$\int_{\mathbb{R}^n} (1 + |z|)^{-E} \, \mathrm{d}z \sim 1,$$

which is a consequence of lemma 3.5. The above calculation uses the assumption $E > n$. The idea of the proof is exactly same as in lemma 3.13. Let $x_R = \frac{x}{R}$. By a change of variables $z = \frac{y}{R'}$ we can write

$$\frac{w_{B_R,E} * \left(\frac{1}{(R')^n} w_{B_{R'},E}\right)(x)}{w_{B_R,E}(x)} = \frac{1}{(R')^n} \int_{\mathbb{R}^n} (1 + \frac{|x-y|}{R})^{-E} (1 + \frac{|y|}{R'})^{-E} (1 + \frac{|x|}{R})^E \, \mathrm{d}y$$
$$= \int_{\mathbb{R}^n} (1 + |x_R - \frac{zR'}{R}|)^{-E} (1 + |z|)^{-E} (1 + |x_R|)^E \, \mathrm{d}z.$$

Thus it suffices to prove

$$\int_{\mathbb{R}^n} (1 + |x_R - \frac{zR'}{R}|)^{-E} (1 + |z|)^{-E} (1 + |x_R|)^E \, \mathrm{d}z \lesssim 1.$$

If $|x_R| \le 1$, then

$$\int_{\mathbb{R}^n} (1 + |x_R - \frac{zR'}{R}|)^{-E} (1 + |z|)^{-E} (1 + |x_R|)^E \, \mathrm{d}z \le 2^E \int_{\mathbb{R}^n} (1 + |z|)^{-E} \, \mathrm{d}z \sim 1.$$

We will now assume that $|x_R| > 1$ and bisect the integral

$$\int_{\mathbb{R}^n} = \int_{|x_R - \frac{zR'}{R}| > \frac{|x_R|}{2}} + \int_{|x_R - \frac{zR'}{R}| \le \frac{|x_R|}{2}}.$$

For the first integral we have

$$\begin{aligned} I_1 := \int_{|x_R - \frac{zR'}{R}| > \frac{|x_R|}{2}} & (1 + |x_R - \frac{zR'}{R}|)^{-E} (1 + |z|)^{-E} (1 + |x_R|)^E \, \mathrm{d}z \\ & \le \frac{(1 + |x_R|)^E}{(1 + \frac{|x_R|}{2})^E} \int_{|x_R - \frac{zR'}{R}| > \frac{|x_R|}{2}} (1 + |z|)^{-E} \, \mathrm{d}z \\ & \le 2^E \int_{|x_R - \frac{zR'}{R}| > \frac{|x_R|}{2}} (1 + |z|)^{-E} \, \mathrm{d}z \\ & \lesssim \int_{\mathbb{R}^n} (1 + |z|)^{-E} \, \mathrm{d}z \sim 1. \end{aligned}$$

In the second region of integration we have $|x_R - \frac{zR'}{R}| \le \frac{|x_R|}{2}$, which implies

$$|z\frac{R'}{R}| \ge |x_R| - |x_R - \frac{zR'}{R}| \ge \frac{|x_R|}{2} \Leftrightarrow |z| \ge \frac{R}{2R'} |x_R|.$$

Thus

$$\begin{aligned}
I_2 &:= \int_{|x_R - \frac{zR'}{R}| \le \frac{|x_R|}{2}} (1+|x_R - \frac{zR'}{R}|)^{-E}(1+|z|)^{-E}(1+|x_R|)^E \,\mathrm{d}z \\
&\le \frac{(1+|x_R|)^E}{\left(1+\frac{R}{2R'}|x_R|\right)^E} \int_{|x_R - \frac{zR'}{R}| \le \frac{|x_R|}{2}} (1+|x_R - \frac{zR'}{R}|)^{-E} \,\mathrm{d}z \\
&\le \frac{(1+|x_R|)^E}{\left(\frac{R}{2R'}|x_R|\right)^E} \int_{\mathbb{R}^n} (1+|x_R - \frac{zR'}{R}|)^{-E} \,\mathrm{d}z \\
&= \left(\frac{2R'}{R}\right)^E \frac{(1+|x_R|)^E}{(|x_R|)^E} \int_{\mathbb{R}^n} (1+|x_R - \frac{zR'}{R}|)^{-E} \,\mathrm{d}z \\
&\sim \left(\frac{R'}{R}\right)^E \int_{\mathbb{R}^n} (1+|x_R - \frac{zR'}{R}|)^{-E} \,\mathrm{d}z \\
&= \left(\frac{R'}{R}\right)^E \left(\frac{R}{R'}\right)^n \int_{\mathbb{R}^n} (1+|x_R - u|)^{-E} \,\mathrm{d}u \\
&= \left(\frac{R'}{R}\right)^{E-n} \int_{\mathbb{R}^n} (1+|v|)^{-E} \,\mathrm{d}v \lesssim 1.
\end{aligned}$$

We have now shown that for $|x_R| > 1$ we have

$$\int_{\mathbb{R}^n} (1+|x_R - \frac{zR'}{R}|)^{-E}(1+|z|)^{-E}(1+|x_R|)^E \,\mathrm{d}z = I_1 + I_2 \lesssim 2 \sim 1.$$

This ends the proof of (4.10).

In order to show (4.11) we again let $x_R := \frac{x}{R}$. A change of variables $z = \frac{y}{R}$ allows us to write

$$\begin{aligned}
\frac{\mathbb{1}_{B_R} * w_{B_R,E}(x)}{R^n w_{B_R,E}(x)} &= \frac{1}{R^n} \int_{B_R} \left(1 + \frac{|x-y|}{R}\right)^{-E} \left(1+\frac{|x|}{R}\right)^E \,\mathrm{d}y \\
&= \int_{B_1} (1+|x_R - z|)^{-E} (1+|x_R|)^E \,\mathrm{d}z \\
&\ge \int_{B_1} (1+|x_R|+|z|)^{-E} (1+|x_R|)^E \,\mathrm{d}z \\
&\ge \left(\frac{1+|x_R|}{1+\frac{\sqrt{n}}{2}+|x_R|}\right)^E \\
&= (1+\frac{\sqrt{n}}{2})^{-E} \left(\frac{1+|x_R|}{1+\frac{|x_R|}{1+\frac{\sqrt{n}}{2}}}\right)^E
\end{aligned}$$

$$\geq (1 + \frac{\sqrt{n}}{2})^{-E} \sim 1.$$

Multiplying both sides by $R^n w_{B_R,E}(x)$ proves (4.11). □

Lemma 4.9 immediately yields the following corollary.

Corollary 4.12. *Let f be measurable and $B_R := B(0,R)$. For $1 \leq p < \infty$ and $E > n$ we have*

$$\|f\|^p_{L^p(w_{B_R,E})} \sim \int_{\mathbb{R}^n} \|f\|^p_{L^p_\sharp(B(y,R))} w_{B_R,E}(y)\,\mathrm{d}y.$$

Proof. By applying Tonelli's theorem, remark 3.4 and (4.10) with $R' = R$ we can write

$$\begin{aligned}
\int_{\mathbb{R}^n} \|f\|^p_{L^p_\sharp(B(y,R))} w_{B_R,E}(y)\,\mathrm{d}y &= \int_{\mathbb{R}^n} \frac{1}{R^n} \int_{\mathbb{R}^n} |f(x)|^p \mathbb{1}_{B_R}(x-y) w_{B_R,E}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} |f(x)|^p \frac{1}{R^n} \mathbb{1}_{B_R} * w_{B_R,E}(x)\,\mathrm{d}x \\
&\lesssim \int_{\mathbb{R}^n} |f(x)|^p \frac{1}{R^n} w_{B_R,E} * w_{B_R,E}(x)\,\mathrm{d}x \\
&\lesssim \|f\|^p_{L^p(w_{B_R,E})}.
\end{aligned}$$

For the other direction we apply (4.11) to get

$$\begin{aligned}
\int_{\mathbb{R}^n} \|f\|^p_{L^p_\sharp(B(y,R))} w_{B_R,E}(y)\,\mathrm{d}y &= \int_{\mathbb{R}^n} |f(x)|^p \frac{1}{R^n} \mathbb{1}_{B_R} * w_{B_R,E}(x)\,\mathrm{d}x \\
&\gtrsim \|f\|^p_{L^p(w_{B_R,E})}.
\end{aligned}$$

□

For the following two lemmas we will restrict the dimension to $n = 2$. This reduction is justified by the fact that the results will be used only in this case. Furthermore, the reduction should only simplify the notation.

Lemma 4.13. *Let $E > n$, $W \geq 2E + p + 1$, $l \in \mathbb{N}$ and $R \geq 1$. Assume that B_R is cube of side length R that is centered at the origin and let f be a Schwartz function in $\mathbb{R}^2$ that satisfies the derivative bound*

$$\|\partial^a_{\xi_1} \partial^b_{\xi_2} f\|_\infty \lesssim_{a,b} \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}} |y_2|\right)^a R^b, \tag{4.14}$$

for some integers $a, b > E$.

Assume also that f is supported in a set S that satisfies $m_2(S) \lesssim R^{-\frac{3}{2}}$ and $|f(\xi)| \leq 1$. Then

$$\|\mathcal{F}^{-1}f\|_{L^1(\mathbb{R}^2)} \lesssim 1 + R^{-1}|y_2| \tag{4.15}$$

and

$$\int_{\mathbb{R}^2} |\mathcal{F}^{-1}f| * \left(R^{-2}\mathbb{1}_{B_R}\right)(y_1 - x_1, -x_2)(1 + R^{-1}|y_2|)^{p-1} w_{B_R,W}(y)\,\mathrm{d}y \lesssim w_{B_R,E}(x), \tag{4.16}$$

for all $x \in \mathbb{R}^2$.

Proof. The first step is to show

$$|\mathcal{F}^{-1}f(x)| \lesssim R^{-\frac{1}{2}}\left(1 + \frac{|x_1|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-E} R^{-1}\left(1 + \frac{|x_2|}{R}\right)^{-E}. \tag{4.17}$$

An application of this and lemma 3.5 with $n = 1$ gives us

$$\|\mathcal{F}^{-1}f\|_{L^1(\mathbb{R}^2)} \lesssim \int_{\mathbb{R}^2} R^{-\frac{1}{2}}\left(1 + \frac{|x_1|}{R^{\frac{1}{2}}(1 + R^{-1}|y_2|)}\right)^{-E} R^{-1}\left(1 + \frac{|x_2|}{R}\right)^{-E} \mathrm{d}x \sim 1 + R^{-1}|y_2|.$$

In order to show (4.17) we will consider four cases. If $|x_1| > R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|$ and $|x_2| > R$, then utilizing (4.14) and the fact that f is supported in a set of area $\lesssim R^{-\frac{3}{2}}$, multiple integration by parts gives us

$$\begin{aligned}
|\mathcal{F}^{-1}f(x)| &= \left|\int_{\mathbb{R}^2} f(\xi) e^{2\pi i(x_1\xi_1 + x_2\xi_2)}\,\mathrm{d}\xi\right| \\
&= \left|\int_{\mathbb{R}^2} \partial_{\xi_1}^a \partial_{\xi_2}^b (f(\xi)) \frac{e^{2\pi i(x_1\xi_1 + x_2\xi_2)}}{(2\pi i x_1)^a (2\pi i x_2)^b}\,\mathrm{d}\xi\right| \\
&\lesssim_{a,b} \frac{1}{|x_1|^a |x_2|^b}\left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|\right)^a R^b R^{-\frac{3}{2}} \\
&= R^{-\frac{1}{2}} \frac{\left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|\right)^a}{|x_1|^a} R^{-1} \frac{R^b}{|x_2|^b} \\
&= R^{-\frac{1}{2}}\left(\frac{|x_1|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-a} R^{-1}\left(\frac{|x_2|}{R}\right)^{-b} \\
&\sim_{a,b} R^{-\frac{1}{2}}\left(1 + \frac{|x_1|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-a} R^{-1}\left(1 + \frac{|x_2|}{R}\right)^{-b}.
\end{aligned}$$

If $|x_1| \leq R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|$ and $|x_2| \leq R$, then we can directly estimate

$$\begin{aligned}
|\mathcal{F}^{-1}f(x)| &= \left| \int_{\mathbb{R}^2} f(\xi) e^{2\pi i(x_1\xi_1 + x_2\xi_2)} \,\mathrm{d}\xi \right| \\
&\leq \int_{\mathbb{R}^2} |f(\xi)| \,\mathrm{d}\xi \\
&\lesssim R^{-\frac{3}{2}} \\
&\sim_{a,b} R^{-\frac{1}{2}} \left(1 + \frac{|x_1|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-a} R^{-1} \left(1 + \frac{|x_2|}{R}\right)^{-b}.
\end{aligned}$$

If $|x_1| > R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|$ and $|x_2| \leq R$, then we integrate by parts with respect to ξ_1 and estimate ξ_2 directly to get

$$\begin{aligned}
|\mathcal{F}^{-1}f(x)| &= \left| \int_{\mathbb{R}^2} f(\xi) e^{2\pi i(x_1\xi_1 + x_2\xi_2)} \,\mathrm{d}\xi \right| \\
&= \left| \int_{\mathbb{R}^2} \partial_{\xi_1}^a (f(\xi)) \frac{e^{2\pi i(x_1 s + x_2 t)}}{(2\pi i x_1)^a} \,\mathrm{d}\xi \right| \\
&\lesssim_a \frac{\left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|\right)^a}{|x_1|^a} R^{-\frac{3}{2}} \\
&\lesssim_{a,b} R^{-\frac{1}{2}} \left(1 + \frac{|x_1|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-a} R^{-1} \left(1 + \frac{|x_2|}{R}\right)^{-b}.
\end{aligned}$$

Similarly if $|x_1| \leq R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|$ and $|x_2| > R$, then

$$\begin{aligned}
|\mathcal{F}^{-1}f(x)| &= \left| \int_{\mathbb{R}^2} f(\xi) e^{2\pi i(x_1\xi_1 + x_2\xi_2)} \,\mathrm{d}\xi \right| \\
&= \left| \int_{\mathbb{R}^2} \partial_{\xi_2}^b (f(\xi)) \frac{e^{2\pi i(x_1\xi_1 + x_2\xi_2)}}{(2\pi i x_2)^b} \,\mathrm{d}\xi \right| \\
&\lesssim_b \frac{R^b}{|x_2|^b} R^{-\frac{3}{2}} \\
&\lesssim_{a,b} R^{-\frac{1}{2}} \left(1 + \frac{|x_1|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-a} R^{-1} \left(1 + \frac{|x_2|}{R}\right)^{-b}.
\end{aligned}$$

We have proven that

$$|\mathcal{F}^{-1}f(x)| \lesssim_{a,b} R^{-\frac{1}{2}} \left(1 + \frac{|x_1|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-a} R^{-1} \left(1 + \frac{|x_2|}{R}\right)^{-b}.$$

Since $a, b > E$, the above gives (4.17).

Then we prove (4.16). Let $I_r = [-\frac{r}{2}, \frac{r}{2}]$ for $r > 0$ and recall that $B_R = I_R \times I_R$. Define functions $\phi_1, \phi_2 \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ by $\phi_1(z) := R^{-\frac{1}{2}} \left(1 + \frac{|z|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-E}$ and $\phi_2(z) := R^{-1} w_{I_R,E}(z)$. Now (4.17) can be written as

$$|\mathcal{F}^{-1} f(x)| \lesssim \phi_1(x_1)\phi_2(x_2). \tag{4.18}$$

Note also that $(1 + R^{-1}|y_2|)^{p-1} = w_{I_R,1-p}(y_2)$. Using (4.18) and lemma 4.9 we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mathcal{F}^{-1} f| * \left(R^{-2} \mathbb{1}_{B_R}\right)(y_1 - x_1, -x_2) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\, \mathrm{d}y \\
&\quad \lesssim \phi_2 * \left(\frac{1}{R} \mathbb{1}_{I_R}\right)(-x_2) \int_{\mathbb{R}^2} \phi_1 * \left(\frac{1}{R} \mathbb{1}_{I_R}\right)(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\, \mathrm{d}y \\
&\quad \lesssim \phi_2 * \left(\frac{1}{R} w_{I_R,E}\right)(-x_2) \int_{\mathbb{R}^2} \phi_1 * \left(\frac{1}{R} \mathbb{1}_{I_R}\right)(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\, \mathrm{d}y \\
&\quad \lesssim \frac{1}{R} w_{I_R,E}(x_2) \int_{\mathbb{R}^2} \phi_1 * \left(\frac{1}{R} \mathbb{1}_{I_R}\right)(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\, \mathrm{d}y. \qquad (4.19)
\end{aligned}$$

Next we will show that

$$\int_{\mathbb{R}^2} \phi_1 * \left(\frac{1}{R} \mathbb{1}_{I_R}\right)(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\, \mathrm{d}y \lesssim R\, w_{I_R,E}(x_1). \tag{4.20}$$

We will consider three regions

$$\begin{aligned}
U_1 &:= \{y : |y_2| \leq R\}, \\
U_2 &:= \bigcup_{k=1}^{R^{\frac{1}{2}}-1} \{y : kR < |y_2| \leq (k+1)R\} \text{ and} \\
U_3 &:= \bigcup_{k \geq 0} \{y : 2^k R^{\frac{3}{2}} < |y_2| \leq 2^{k+1} R^{\frac{3}{2}}\}.
\end{aligned}$$

Note that for $a \geq 1$ and $b \geq 0$, we have

$$(1 + \frac{b}{a})^{-E} \leq a^E (1 + b)^{-E}. \tag{4.21}$$

In the first region we have

$$\phi_1(z) = R^{-\frac{1}{2}} \left(1 + \frac{|z|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-E} \leq 2^E R^{-\frac{1}{2}} \left(1 + \frac{|z|}{R^{\frac{1}{2}}}\right)^{-E} \sim \frac{1}{R^{\frac{1}{2}}} w_{I_{\sqrt{R}},E}(z),$$

for all $z \in \mathbb{R}$. Using this and lemma 4.9 with $R' = R^{\frac{1}{2}}$ gives

$$\left(\frac{1}{R}\mathbb{1}_{I_R}\right) * \phi_1(y_1 - x_1) \lesssim \frac{1}{R} w_{I_R,E} * \left(\frac{1}{R^{\frac{1}{2}}} w_{I_{\sqrt{R}},E}\right)(y_1 - x_1) \lesssim \frac{1}{R} w_{I_R,E}(y_1 - x_1)$$

Recall that $W \geq 2E + p + 1 > E + p + 2$. Now an application of lemma 4.8 with $W_1 = E$ and $W_2 = p + 2$ and lemma 4.9 with $R' = R$ yields

$$\begin{aligned}
&\int_{U_1} \phi_1 * \left(\frac{1}{R}\mathbb{1}_{I_R}\right)(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\qquad = \int_{U_1} \left(\frac{1}{R}\mathbb{1}_{I_R}\right) * \phi_1(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\qquad \lesssim \int_{U_1} \frac{1}{R} w_{I_R,E}(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\qquad \leq \int_{U_1} \frac{1}{R} w_{I_R,E}(x_1 - y_1) w_{I_R,E}(y_1) w_{I_R,1-p}(y_2) w_{I_R,p+2}(y_2)\,\mathrm{d}y \\
&\qquad \leq \int_{\mathbb{R}^2} \frac{1}{R} w_{I_R,E}(x_1 - y_1) w_{I_R,E}(y_1) w_{I_R,1-p}(y_2) w_{I_R,p+2}(y_2)\,\mathrm{d}y \\
&\qquad = w_{I_R,E} * \left(\frac{1}{R} w_{I_R,E}\right)(x_1) \int_{\mathbb{R}} w_{I_R,3}(y_2)\,\mathrm{d}y_2 \\
&\qquad \lesssim w_{I_R,E}(x_1) \int_{\mathbb{R}} w_{I_R,3}(y_2)\,\mathrm{d}y_2 \sim R\, w_{I_R,E}(x_1).
\end{aligned}$$

The last relation is an application of lemma 3.5.

For the second region we note that for each $k \in [1, R^{\frac{1}{2}}[$ and y such that $kR < |y_2| \leq (k+1)R$, we have

$$\phi_1(z) = R^{-\frac{1}{2}} \left(1 + \frac{|z|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-E} \leq R^{-\frac{1}{2}} \left(1 + \frac{|z|}{R^{\frac{1}{2}}(2+k)}\right)^{-E} \lesssim R^{-\frac{1}{2}} w_{I_{k\sqrt{R}},E}(z),$$

for all $z \in \mathbb{R}$. Therefore lemma 4.9 with $R' = kR^{\frac{1}{2}}$ gives

$$\left(\frac{1}{R}\mathbb{1}_{I_R}\right) * \phi_1(y_1 - x_1) \lesssim \frac{k}{R} w_{I_R,E} * \left(\frac{1}{kR^{\frac{1}{2}}} w_{I_{k\sqrt{R}},E}\right)(y_1 - x_1) \lesssim \frac{k}{R} w_{I_R,E}(y_1 - x_1).$$

Similarly as in the first region we now have

$$\begin{aligned}
\int_{U_2} \left(\frac{1}{R}\mathbb{1}_{I_R}\right) * \phi_1(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&= \sum_{k=1}^{R^{\frac{1}{2}}-1} \int_{kR<|y_2|\le(k+1)R} \left(\frac{1}{R}\mathbb{1}_{I_R}\right) * \phi_1(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\lesssim \sum_{k=1}^{R^{\frac{1}{2}}-1} \frac{k}{R} \int_{kR<|y_2|\le(k+1)R} w_{I_R,E}(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\le \sum_{k=1}^{R^{\frac{1}{2}}-1} \frac{k}{R} w_{I_R,E} * w_{I_R,E}(x_1) \int_{kR<|y_2|\le(k+1)R} w_{I_R,3}(y_2)\,\mathrm{d}y_2 \\
&\lesssim \sum_{k=1}^{R^{\frac{1}{2}}-1} k\, w_{I_R,E}(x_1) \int_{kR<|y_2|\le(k+1)R} (1 + R^{-1}|y_2|)^{-3}\,\mathrm{d}y_2 \\
&\le \sum_{k=1}^{R^{\frac{1}{2}}-1} k\, w_{I_R,E}(x_1) \int_{kR<|y_2|\le(k+1)R} k^{-3}\,\mathrm{d}y_2 \\
&\sim \sum_{k=1}^{R^{\frac{1}{2}}-1} \frac{R}{k^2}\, w_{I_R,E}(x_1) \\
&= R\, w_{I_R,E}(x_1) \sum_{k=1}^{R^{\frac{1}{2}}-1} \frac{1}{k^2} \\
&\le R\, w_{I_R,E}(x_1) \frac{6}{\pi^2} \sim R\, w_{I_R,E}(x_1).
\end{aligned}$$

Finally, for the last region we note that for each $k \ge 0$ and y such that $2^k R^{\frac{3}{2}} < |y_2| \le 2^{k+1} R^{\frac{3}{2}}$, we have

$$\phi_1(z) = R^{-\frac{1}{2}} \left(1 + \frac{|z|}{R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y_2|}\right)^{-E} \le R^{-\frac{1}{2}} \left(1 + \frac{|z|}{R(1 + 2^k)}\right)^{-E} \lesssim R^{-\frac{1}{2}} w_{I_{2^k R},E}(z),$$

for all $z \in \mathbb{R}$. Lemma 4.9 with $2^k R$ as the bigger side length and R as the smaller side length gives that

$$\mathbb{1}_{I_R} * \left(\frac{1}{R}\phi_1\right)(y_1 - x_1) \lesssim R^{-\frac{1}{2}} \left(\frac{1}{R} w_{I_R,E}\right) * w_{I_{2^k R},E}(y_1 - x_1) \lesssim R^{-\frac{1}{2}} w_{I_{2^k R},E}(y_1 - x_1).$$

Since $W \geq 2E+p+1$, we can apply lemma 4.8 with $W_1 = E$ and $W_2 = E+p+1$. This and a similar use of lemma 4.9 as above yields

$$\begin{aligned}
&\int_{U_3} \left(\frac{1}{R}\mathbb{1}_{I_R}\right) * \phi_1(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&= \sum_{k=0}^{\infty} \int_{2^k R^{\frac{3}{2}} < |y_2| \leq 2^{k+1} R^{\frac{3}{2}}} \left(\frac{1}{R}\mathbb{1}_{I_R}\right) * \phi_1(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\lesssim \sum_{k=0}^{\infty} \int_{2^k R^{\frac{3}{2}} < |y_2| \leq 2^{k+1} R^{\frac{3}{2}}} R^{-\frac{1}{2}} w_{I_{2^k R},E}(y_1 - x_1) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\leq \sum_{k=0}^{\infty} R^{-\frac{1}{2}} w_{I_{2^k R},E} * w_{I_R,E}(x_1) \int_{2^k R^{\frac{3}{2}} < |y_2| \leq 2^{k+1} R^{\frac{3}{2}}} w_{I_R,E+2}(y_2)\,\mathrm{d}y_2 \\
&\lesssim \sum_{k=0}^{\infty} R^{\frac{1}{2}} w_{I_{2^k R},E}(x_1) \int_{2^k R^{\frac{3}{2}} < |y_2| \leq 2^{k+1} R^{\frac{3}{2}}} (1 + \frac{|y_2|}{R})^{-E-2}\,\mathrm{d}y_2 \\
&\leq \sum_{k=0}^{\infty} R^{\frac{1}{2}} w_{I_{2^k R},E}(x_1) 2^k R^{\frac{3}{2}} (2^k R^{\frac{1}{2}})^{-E-2} \\
&= \sum_{k=0}^{\infty} w_{I_{2^k R},E}(x_1) (2^k)^{-E-1} R^{-\frac{E-2}{2}} \\
&\leq \sum_{k=0}^{\infty} w_{I_R,E}(x_1) 2^{Ek} 2^{-Ek-k} R^{-\frac{E-2}{2}} \\
&= R^{-\frac{E-2}{2}} w_{I_R,E}(x_1) \sum_{k=0}^{\infty} 2^{-k} \lesssim R\, w_{I_R,E}(x_1).
\end{aligned}$$

In the second to last inequality we used (4.21). This ends the proof of (4.20).

Plugging (4.20) to (4.19) yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mathcal{F}^{-1} f| * \left(R^{-2}\mathbb{1}_{B_R}\right)(y_1 - x_1, -x_2) w_{I_R,1-p}(y_2) w_{B_R,W}(y)\,\mathrm{d}y \\
&\lesssim (1 + \frac{|x_1|}{R})^{-E} (1 + \frac{|x_2|}{R})^{-E} \\
&= (1 + \frac{|x_1|}{R} + \frac{|x_2|}{R} + \frac{|x_1 x_2|}{R})^{-E} \\
&\leq (1 + \frac{|x_1|}{R} + \frac{|x_2|}{R})^{-E} \\
&\sim (1 + \frac{|x|}{R})^{-E} = w_{B_R,E}(x),
\end{aligned}$$

which conclude the proof of the lemma. ∎

Lastly, we will give an example of a function that satisfies the conditions of the previous lemma. See lemmas (1.7) and (1.8) for the existence of the functions η and M_l in the following lemma.

Lemma 4.22. *Let $b, l \in \mathbb{N}$ and let a be an even integer. Let $C > 0$ be a constant, $R \geq 1$, $y \in \mathbb{R}$ and let $m_{l,R}$ be a function in $\mathbb{R}^2$ that is defined by*

$$m_{l,R}(\xi) = e(\xi_1^2 y) M_l(\frac{R(\xi_2 - \xi_1^2)}{2C}) \eta(R\xi_2) \eta(R^{\frac{1}{2}}\xi_1),$$

where η is a compactly supported smooth function that satisfies $|\eta| \leq 1$ and is equal to 1 on $[0, 1+C]$ and M_l is a compactly supported smooth function that agrees with the function $x \mapsto x^l$ on $[0, \frac{1}{2}]$. Furthermore, we stipulate that $|M_l| \leq 1$ and the derivative bound

$$\left\| \frac{\mathrm{d}^k}{\mathrm{d}x^k} M_l \right\|_\infty \lesssim_k 1$$

holds for all $k \in \mathbb{N}$.

Then $m_2(\mathrm{supp}(m_{l,R})) \lesssim R^{-\frac{3}{2}}$ and

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b m_{l,R}\|_\infty \lesssim_{a,b} \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y| \right)^a R^b,$$

uniformly over l.

Proof. The measure of the support arises naturally from the fact that $m_1(\mathrm{supp}(\eta)) \sim 1$. We will first look at the more general case where $a \in \mathbb{N}$. We will begin with showing by induction that

(4.23)

$$\partial_{\xi_1}^a M_l \left(\frac{R(\xi_2 - \xi_1^2)}{2C} \right) = \begin{cases} R^{\frac{a+1}{2}} \xi_1 \sum_{k=0}^{\frac{a-1}{2}} c_{a,k} M_l^{\left(k+\frac{a+1}{2}\right)} \left(\frac{R(\xi_2-\xi_1^2)}{2C} \right) (R\xi_1^2)^k, & \text{if } a \text{ is odd,} \\ R^{\frac{a}{2}} \sum_{k=0}^{\frac{a}{2}} c_{a,k} M_l^{\left(k+\frac{a}{2}\right)} \left(\frac{R(\xi_2-\xi_1^2)}{2C} \right) (R\xi_1^2)^k, & \text{if } a \text{ is even,} \end{cases}$$

where $c_{a,k}$ are constants.

Denote $A(\xi) := \frac{R(\xi_2 - \xi_1^2)}{2C}$ and notice that $\partial_{\xi_1} A(\xi) = -\frac{R\xi_1}{C}$. For the base of our induction we calculate

$$\partial_{\xi_1} M_l(A(\xi)) = M_l'(A(\xi)) \left(-\frac{R\xi_1}{C} \right) = R\xi_1 \left(-\frac{1}{C} M_l'(A(\xi)) \right).$$

Now assume (4.23) when $a = 2r$, where $r \in \mathbb{N}$. Let $\widetilde{a} := a + 1$, then

$$
\begin{aligned}
\partial_{\xi_1}^{\widetilde{a}} M_l(A(\xi)) = \partial_{\xi_1}^{2r+1} M_l(A(\xi)) &= \partial_{\xi_1}\left(\partial_{\xi_1}^{2r} M_l(A(\xi))\right) \\
&= \partial_{\xi_1}\left(R^{\frac{a}{2}} \sum_{k=0}^{\frac{a}{2}} c_{a,k} M_l^{\left(k+\frac{a}{2}\right)}(A(\xi))(R\xi_1^2)^k\right) \\
&= R^r \sum_{k=0}^{r} c_{a,k} \partial_{\xi_1}\left(M_l^{(k+r)}(A(\xi))(R\xi_1^2)^k\right). \qquad (4.24)
\end{aligned}
$$

We calculate the derivative inside the sum

$$
\begin{aligned}
&\partial_{\xi_1}\left(M_l^{(k+r)}(A(\xi))(R\xi_1^2)^k\right) \\
&\quad = M_l^{(k+r+1)}(A(\xi))(R\xi_1^2)^k\left(-\frac{R\xi_1}{C}\right) + M_l^{(k+r)}(A(\xi))2kR^k\xi_1^{2k-1} \\
&\quad = M_l^{(k+\frac{\widetilde{a}-1}{2}+1)}(A(\xi))(R\xi_1^2)^k\left(-\frac{R\xi_1}{C}\right) + 2kM_l^{(k+\frac{\widetilde{a}-1}{2})}(A(\xi))(R\xi_1^2)^{k-1}R\xi_1 \\
&\quad = R\xi_1\left(\left(-\frac{1}{C}\right) M_l^{(k+\frac{\widetilde{a}+1}{2})}(A(\xi))(R\xi_1^2)^k + 2kM_l^{(k-1+\frac{\widetilde{a}+1}{2})}(A(\xi))(R\xi_1^2)^{k-1}\right).
\end{aligned}
$$

Plugging the above to (4.24) gives us

$$
\begin{aligned}
\partial_{\xi_1}^{\widetilde{a}} M_l(A(\xi)) &= R^{r+1}\xi_1 \sum_{k=0}^{r} c'_{a,k} M_l^{\left(k+\frac{\widetilde{a}+1}{2}\right)}(A(\xi))(R\xi_1^2)^k \\
&= R^{\frac{\widetilde{a}+1}{2}}\xi_1 \sum_{k=0}^{\frac{\widetilde{a}-1}{2}} c'_{a,k} M_l^{\left(k+\frac{\widetilde{a}+1}{2}\right)}(A(\xi))(R\xi_1^2)^k,
\end{aligned}
$$

as wanted. If we assume (4.23) for $a = 2r + 1$, then

$$
\begin{aligned}
\partial_{\xi_1}^{\widetilde{a}} M_l(A(\xi)) &= \partial_{\xi_1}\left(R^{\frac{a+1}{2}}\xi_1 \sum_{k=0}^{\frac{a-1}{2}} c_{a,k} M_l^{(k+\frac{a+1}{2})}(A(\xi))(R\xi_1^2)^k\right) \\
&= R^{\frac{\widetilde{a}}{2}} \sum_{k=0}^{\frac{\widetilde{a}}{2}-1} c_{a,k} \partial_{\xi_1}\left(M_l^{(k+\frac{\widetilde{a}}{2})}(A(\xi))R^k\xi_1^{2k+1}\right). \qquad (4.25)
\end{aligned}
$$

Again we calculate the derivative inside the sum

$$\begin{aligned}
&\partial_{\xi_1}\left(M_l^{(k+\frac{\tilde{a}}{2})}(A(\xi))R^k\xi_1^{2k+1}\right)\\
&\qquad = M_l^{(k+\frac{\tilde{a}}{2}+1)}(A(\xi))\left(-\frac{R\xi_1}{C}\right)R^k\xi_1^{2k+1}+(2k+1)M_l^{(k+\frac{\tilde{a}}{2})}(A(\xi))R^k\xi_1^{2k}\\
&\qquad = \left(-\frac{1}{C}\right)M_l^{(k+1+\frac{\tilde{a}}{2})}(A(\xi))(R\xi_1^2)^{k+1}+(2k+1)M_l^{(k+\frac{\tilde{a}}{2})}(A(\xi))(R\xi_1^2)^k\\
&\qquad =: -\frac{1}{C}G(k+1)+(2k+1)G(k),
\end{aligned}$$

where

$$G(k) := M_l^{(k+\frac{\tilde{a}}{2})}(A(\xi))(R\xi_1^2)^k.$$

Plugging this to (4.25) we get

$$\partial_{\xi_1}^{\tilde{a}}M_l(A(\xi)) = R^{\frac{\tilde{a}}{2}}\sum_{k=0}^{\frac{\tilde{a}}{2}-1}c_{a,k}\left(-\frac{1}{C}G(k+1)+(2k+1)G(k)\right)$$

Notice that in the first term G we have $1\le k+1\le\frac{\tilde{a}}{2}$ and in the second term G we have $0\le k\le\frac{\tilde{a}}{2}-1$ and hence we can write

$$\partial_{\xi_1}^{\tilde{a}}M_l(A(\xi)) = R^{\frac{\tilde{a}}{2}}\sum_{k=0}^{\frac{\tilde{a}}{2}}c'_{a,k}G(k) = R^{\frac{\tilde{a}}{2}}\sum_{k=0}^{\frac{\tilde{a}}{2}}c'_{a,k}M_l^{\left(k+\frac{\tilde{a}}{2}\right)}(A(\xi))(R\xi_1^2)^k$$

which ends the proof of (4.23). Now we go back to the case where a is even. From (4.23) we get

$$\partial_{\xi_1}^a\partial_{\xi_2}^b M_l\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right) = R^{\frac{a}{2}+b}\sum_{k=0}^{\frac{a}{2}}c_{a,b,k}M_l^{\left(k+\frac{a}{2}+b\right)}\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)(R\xi_1^2)^k \tag{4.26}$$

where $c_{a,b,k}$ are constants. Note that $\eta(R\xi_1^2)\neq 0$ implies $R\xi_1^2\lesssim 1$ and in this case the derivative bound of M_l and (4.26) assert that

$$\sup_{\xi\in\mathbb{R}^2}\left|\partial_{\xi_1}^a\partial_{\xi_2}^b M_l\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)\right| \lesssim_{a,b} R^{\frac{a}{2}+b}\le\left(R^{\frac{1}{2}}+R^{-\frac{1}{2}}|y|\right)^a R^b.$$

A similar calculation shows that

$$\partial_{\xi_1}^a e(\xi_1^2 y) = \begin{cases} e(\xi_1^2 y)y^{\frac{a+1}{2}}\xi_1\sum_{k=0}^{\frac{a-1}{2}}c_{a,k}(\xi_1^2 y)^k, & \text{if } a \text{ is odd},\\ e(\xi_1^2 y)y^{\frac{a}{2}}\sum_{k=0}^{\frac{a}{2}}c_{a,k}(\xi_1^2 y)^k, & \text{if } a \text{ is even}.\end{cases}$$

and hence

$$\partial_{\xi_1}^a \partial_{\xi_2}^b e(\xi_1^2 y) = 0.$$

If $|y| \le R$, then for even a we have

$$\sup_{\xi\in\mathbb{R}^2} \left|\partial_{\xi_1}^a \partial_{\xi_2}^b e(\xi_1^2 y)\right| \lesssim_a |y|^{\frac{a}{2}} \le R^{\frac{a}{2}} \le \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y|\right)^a R^b.$$

If $|y| > R$, then in the support of η for even a we have

$$\sup_{\xi\in\mathbb{R}^2} \left|\partial_{\xi_1}^a \partial_{\xi_2}^b e(\xi_1^2 y)\right| \lesssim_a |y\xi_1|^a \lesssim (R^{-\frac{1}{2}}|y|)^a \le \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y|\right)^a R^b.$$

The other two terms are easy

$$\partial_{\xi_1}^a \eta(R^{\frac{1}{2}}\xi_1) = \eta^{(a)}(R^{\frac{1}{2}}\xi_1)R^{\frac{a}{2}}, \quad \partial_{\xi_2}^b \eta(R\xi_2) = \eta^{(b)}(R\xi_2)R^b$$

and

$$\partial_{\xi_1}^a \partial_{\xi_2}^b \eta(R^{\frac{1}{2}}\xi_1) = \partial_{\xi_2}^a \partial_{\xi_2}^b \eta(R\xi_1) = 0.$$

Thus

$$\sup_{\xi\in\mathbb{R}^2} \left|\partial_{\xi_1}^a \partial_{\xi_2}^b \eta(R^{\frac{1}{2}}\xi_1)\right| \lesssim_b R^{\frac{a}{2}} \le \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y|\right)^a R^b$$

and

$$\sup_{\xi\in\mathbb{R}^2} \left|\partial_{\xi_1}^a \partial_{\xi_2}^b \eta(R\xi_2)\right| \lesssim_a R^b \le \left(R^{\frac{1}{2}} + R^{-\frac{1}{2}}|y|\right)^a R^b.$$

Now the Leibniz rule for differentiation of products gives us the result. □

Now we are ready to get to the meat of this chapter.

Theorem 4.27. *Let $C > 0$ be a constant, $R \in 4^{\mathbb{N}}$ and $p \ge 2$. Let $\Gamma_n(E)$ be a large enough constant depending on E and n. Then the following statement is true for every $W \ge \Gamma_n(E)$. For each $f \in L^\infty(\mathbb{R}^n)$ with an integrable Fourier transform supported in $N_{\frac{C}{R}}([0,1]^{n-1})$ and for each cube $B_R \subset \mathbb{R}^n$ we have*

$$\|f\|_{L^p(w_{B_R,E})} \lesssim \mathrm{Dec}_n(R^{-1}, p, W) \left(\sum_{Q\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1]^{n-1})} \|f_{N_{\frac{C}{R}}(Q)}\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}}. \tag{4.28}$$

Again, we will reduce the dimension to $n = 2$. However, this should only simplify the notation. In this case $\Gamma_2(E) = 2E + p + 1$ will suffice.

Proof. By lemma 4.3 it suffices to prove the inequality (4.28) with $\|Eg\|_{L^p(B_R)}$ on the left-hand side. We may also assume that B_R is centered at the origin. Indeed, we translate the center c_{B_R} to the origin

$$\begin{aligned}\|f\|_{L^p(B_R)} &= \left(\int_{B_R} |f(x)|^p \, \mathrm{d}x\right)^p \\ &= \left(\int_{B(0,R)} |f(x+c_{B_R})|^p \, \mathrm{d}x\right)^p = \|\tau_{c_{B_R}} f\|_{L^p(w_{B(0,R),E})},\end{aligned}$$

where $\tau_h f(x) := f(x+h)$, for $h \in \mathbb{R}^2$. On the right-hand side of (4.28) we would then have

$$\operatorname{Dec}_n(R^{-1}, p, W) \left(\sum_{Q \in \operatorname{Part}_{R^{-\frac{1}{2}}}([0,1]^{n-1})} \|(\tau_{c_{B_R}} f)_{N_{\frac{C}{R}}(Q)}\|^2_{L^p(w_{B(0,R),E})} \right)^{\frac{1}{2}}. \tag{4.29}$$

Since by lemma 1.21 we have $\widehat{\tau_{c_{B_R}} f} = e_{c_{B_R}} \widehat{f} \in L^1$, the Fourier restriction becomes

$$(\tau_{c_{B_R}} f)_{N_{\frac{C}{R}}(Q)}(x) = \int_{N_{\frac{C}{R}}(Q)} \widehat{f}(\xi) e((x+c_{B_R}) \cdot \xi) \, \mathrm{d}\xi = f_{N_{\frac{C}{R}}(Q)}(x+c_{B_R})$$

and translating $B(0,R)$ back to B_R gives

$$\begin{aligned}\|(\tau_{c_{B_R}} f)_{N_{\frac{C}{R}}(Q)}\|_{L^p(w_{B(0,R),E})} &= \int_{\mathbb{R}^2} |f_{N_{\frac{C}{R}}(Q)}(x+c_{B_R})|^p w_{B(0,R),E}(x) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} |f_{N_{\frac{C}{R}}(Q)}(x)|^p w_{B_R,E}(x) \, \mathrm{d}x = \|f_{N_{\frac{C}{R}}(Q)}\|_{L^p(w_{B_R,E})}\end{aligned}$$

and plugging this to (4.29) gives us the wanted result.

We now assume that B_R is centered at the origin. In order to bring out the decoupling constant, we must rewrite the function f by using Eg with some function g. By lemma 1.12 we have

$$f(x) = \int_{\mathbb{R}^2} \widehat{f}(\xi) e(x \cdot \xi) \, \mathrm{d}\xi = \int_{N_{\frac{C}{R}}([0,1])} \widehat{f}(\xi) e(x \cdot \xi) \, \mathrm{d}\xi,$$

for almost every $x \in \mathbb{R}^2$.

A change of variables $(\xi_1, \xi_2) = (s, s^2 + t)$ allows us to write

$$\int_{N_{\frac{C}{R}}([0,1])} \widehat{f}(\xi) e(x \cdot \xi) \, \mathrm{d}\xi = \int_0^1 \int_0^{\frac{C}{R}} \widehat{f}(s, s^2+t) e(sx_1 + s^2 x_2) e(t x_2) \, \mathrm{d}t \, \mathrm{d}s.$$

Using Taylor expansion we write

$$e(x_2t)=\sum_{j=0}^{\infty}\frac{(i2\pi x_2t)^j}{j!}=\sum_{j=0}^{\infty}\frac{(i2\pi C)^j}{j!}\left(\frac{2x_2}{R}\right)^j\left(\frac{Rt}{2C}\right)^j$$

and hence for almost every $x\in\mathbb{R}^2$, we have

$$\begin{aligned}f(x)&=\sum_{j=0}^{\infty}\frac{(i4\pi C)^j}{j!}\left(\frac{x_2}{R}\right)^j\int_0^1\int_0^{\frac{C}{R}}\widehat{f}(s,s^2+t)e(sx_1+s^2x_2)\left(\frac{Rt}{2C}\right)^j\,\mathrm{d}t\,\mathrm{d}s\\&=\sum_{j=0}^{\infty}\frac{(i4\pi C)^j}{j!}\left(\frac{x_2}{R}\right)^j\int_0^1\int_0^{\frac{C}{R}}\widehat{f}(s,s^2+t)\left(\frac{Rt}{2C}\right)^j\,\mathrm{d}t\,e(sx_1+s^2x_2)\,\mathrm{d}s.\end{aligned}$$

The above implies that for almost every $x\in B_R$ we can write

$$|f(x)|\leq\sum_{j=0}^{\infty}\frac{(4\pi C)^j}{j!}|Eg_j(x)|,$$

where

$$g_j(s)=\int_0^{\frac{C}{R}}\widehat{f}(s,s^2+t)\left(\frac{Rt}{2C}\right)^j\,\mathrm{d}t.$$

Then we use the definition of the decoupling constant to write

$$\begin{aligned}\|f\|_{L^p(B_R)}&\leq\sum_{j=0}^{\infty}\frac{(4\pi C)^j}{j!}\|Eg_j\|_{L^p(B_R)}\lesssim\sum_{j=0}^{\infty}\frac{(4\pi C)^j}{j!}\|Eg_j\|_{L^p(w_{B_R},W)}\\&\leq\mathrm{Dec}_2(R^{-1},p,W)\sum_{j=0}^{\infty}\frac{(4\pi C)^j}{j!}\left(\sum_{Q\in\mathrm{Part}_{R^{-\frac{1}{2}}}([0,1])}\|E_Qg_j\|^2_{L^p(w_{B_R},W)}\right)^{\frac{1}{2}}.\end{aligned}$$

If we can prove

$$\|E_Qg_j\|_{L^p(w_{B_R},W)}\lesssim\|f_{N_{\frac{C}{R}}(Q)}\|_{L^p(w_{B_R},E)},\tag{4.30}$$

for all $j\geq 0$, then we are done. Indeed, then we would have

$$
\begin{aligned}
\|f\|_{L^p(B_R)} &\lesssim \mathrm{Dec}_2(R^{-1},p,W)\sum_{j=0}^{\infty}\frac{(4\pi C)^j}{j!}\left(\sum_{Q\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1])}\|E_Q g_j\|^2_{L^p(w_{B_R,W})}\right)^{\frac{1}{2}}\\
&\lesssim \mathrm{Dec}_2(R^{-1},p,W)\sum_{j=0}^{\infty}\frac{(4\pi C)^j}{j!}\left(\sum_{Q\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1])}\|f_{N_{\frac{C}{R}}(Q)}\|^2_{L^p(w_{B_R,E})}\right)^{\frac{1}{2}}\\
&= e^{4\pi C}\,\mathrm{Dec}_2(R^{-1},p,W)\left(\sum_{Q\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1])}\|f_{N_{\frac{C}{R}}(Q)}\|^2_{L^p(w_{B_R,E})}\right)^{\frac{1}{2}}\\
&\sim \mathrm{Dec}_2(R^{-1},p,W)\left(\sum_{Q\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1])}\|f_{N_{\frac{C}{R}}(Q)}\|^2_{L^p(w_{B_R,E})}\right)^{\frac{1}{2}}.
\end{aligned}
$$

We will denote $Q_0 := [0, R^{-\frac{1}{2}}]$. The original Q is of the form $Q = [u, u + R^{-\frac{1}{2}}]$, for $u \in [0, 1 - R^{-\frac{1}{2}}]$. We will now try to write the left-hand side of (4.30) using Q_0. We calculate

$$
\begin{aligned}
|E_Q g_j(x)| &= \left|\int_u^{u+R^{-\frac{1}{2}}} g_j(s)e(sx_1+s^2x_2)\,\mathrm{d}s\right|\\
&= \left|\int_0^{R^{-\frac{1}{2}}} g_j(s+u)e((s+u)x_1+(s+u)^2x_2)\,\mathrm{d}s\right|\\
&= \left|e(ux_1+u^2x_2)\int_0^{R^{-\frac{1}{2}}} g_j(s+u)e(sx_1+(s^2+2su)x_2)\,\mathrm{d}s\right|\\
&= \left|\int_0^{R^{-\frac{1}{2}}} g_j(s+u)e(s(x_1+2ux_2)+s^2x_2)\,\mathrm{d}s\right|\\
&= \left|\int_{Q_0} g_{j,u}(s)e(s(x_1+2ux_2)+s^2x_2)\,\mathrm{d}s\right| = |E_{Q_0}g_{j,u}(Lx)|,
\end{aligned}
$$

where $g_{j,u}(s) := g_j(s+u)$ and $L := \begin{bmatrix} 1 & 2u \\ 0 & 1 \end{bmatrix}$. Notice that $\det L = 1$ and $L^{-1} := \begin{bmatrix} 1 & -2u \\ 0 & 1 \end{bmatrix}$. For $x \in \mathbb{R}^n$, we have $|Lx| \le 3|x|$ and $|L^{-1}x| \le 3|x|$. Thus lemma 1.2 gives us that L^{-1} is

3-bilipschitz. Now since B_R is centered at the origin, a change of variables gives that

$$\begin{aligned}\|E_Q g_j\|^p_{L^p(w_{B_R,W})} &= \|(E_{Q_0} g_{j,u}) \circ L\|^p_{L^p(w_{B_R,W})} \\ &= \int_{\mathbb{R}^2} |E_{Q_0} g_{j,u}(Lx)|^p w_{B_R,W}(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} |E_{Q_0} g_{j,u}(x)|^p w_{B_R,W}(L^{-1}x)\,\mathrm{d}x \\ &\sim \int_{\mathbb{R}^2} |E_{Q_0} g_{j,u}(x)|^p w_{B_R,W}(x)\,\mathrm{d}x = \|E_{Q_0} g_{j,u}\|^p_{L^p(w_{B_R,W})} \qquad (4.31)\end{aligned}$$

and by corollary 4.12 we have that

$$\|E_{Q_0} g_{j,u}\|^p_{L^p(w_{B_R,W})} \sim \int_{\mathbb{R}^2} \|E_{Q_0} g_{j,u}\|^p_{L^p_\sharp(B(y,R))} w_{B_R,W}(y)\,\mathrm{d}y. \qquad (4.32)$$

Notice that

$$\begin{aligned}E_{Q_0} g_{j,u}(x) &= \int_{Q_0}\int_0^{\frac{C}{R}} \widehat{f}(s+u,(s+u)^2+t)\left(\frac{Rt}{2C}\right)^j e(sx_1+s^2x_2)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{Q_0}\int_0^{\frac{C}{R}} \widehat{f}(s+u,s^2+t+2su+u^2)\left(\frac{Rt}{2C}\right)^j e(sx_1+s^2x_2)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{N_{\frac{C}{R}}(Q_0)} \widehat{f}(\xi_1+u,\xi_2+2\xi_1 u+u^2)\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)^j e((\xi_1^2-\xi_2)x_2)e(x\cdot\xi)\,\mathrm{d}\xi. \qquad (4.33)\end{aligned}$$

We denote $r := (u,u^2)$ and $L_r(\xi) := L^\top\xi + r$ and note that

$$(\xi_1+u,\xi_2+2\xi_1 u+u^2) = L_r(\xi).$$

We also denote $\widehat{h}(\xi) := \widehat{f}\circ L_r(\xi)$, which means that

$$\widehat{f}(\xi_1+u,\xi_2+2\xi_1 u+u^2) = \widehat{h}(\xi).$$

Combining this with (4.33) we get

$$E_{Q_0} g_{j,u}(x) = \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi)\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)^j e((\xi_1^2-\xi_2)x_2)e(x\cdot\xi)\,\mathrm{d}\xi \qquad (4.34)$$

For $x \in B(y,R)$ we write

$$e((\xi_1^2-\xi_2)x_2) = e((\xi_1^2-\xi_2)y_2)e((\xi_1^2-\xi_2)(x_2-y_2))$$

and another Taylor expansion for $e((\xi_1^2-\xi_2)(x_2-y_2))$ gives

$$\begin{aligned} e((\xi_1^2-\xi_2)(x_2-y_2)) &= \sum_{k=0}^{\infty} \frac{(-i2\pi(x_2-y_2)(\xi_2-\xi_1^2))^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-i2\pi C)^k}{k!} \left(\frac{2(x_2-y_2)}{R}\right)^k \left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)^k \end{aligned}$$

and plugging this into (4.34) yields that for $x \in B(y,R)$ we have

$$|E_{Q_0} g_{j,u}(x)| \leq \sum_{k=0}^{\infty} \frac{(4\pi C)^k}{k!} \left| \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) \left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)^{j+k} e((\xi_1^2-\xi_2)y_2) e(x\cdot\xi)\, \mathrm{d}\xi \right|.$$

Recall the notation $e(x\cdot\xi) = e_\xi(x)$. Taking $L^p_\sharp(B(y,R))$ norms from both sides gives

$$\begin{aligned} &\|E_{Q_0} g_{j,u}\|^p_{L^p_\sharp(B(y,R))} \\ &\leq \sum_{k=0}^{\infty} \frac{(4\pi C)^k}{k!} \left\| \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) \left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)^{j+k} e((\xi_1^2-\xi_2)y_2) e_\xi \, \mathrm{d}\xi \right\|^p_{L^p_\sharp(B(y,R))}. \end{aligned} \tag{4.35}$$

We denote $l \coloneqq j+k$. Next we will show that

$$\begin{aligned} &\int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) \left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)^{l} e((\xi_1^2-\xi_2)y_2) e(x\cdot\xi)\, \mathrm{d}\xi \\ &= \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) e(\xi_1^2 y_2) M_l \left(\frac{R(\xi_2-\xi_1^2)}{2C}\right) \eta(R\xi_2)\eta(R^{\frac{1}{2}}\xi_1) e(\xi_1 x_1 + \xi_2(x_2-y_2))\, \mathrm{d}\xi, \end{aligned} \tag{4.36}$$

where η is a compactly supported smooth function that is equal to 1 on $[0, 1+C]$ and M_l is a compactly supported smooth function that agrees with the function $x \mapsto x^l$ on $[0, \frac{1}{2}]$ and satisfies the derivative bound

$$\left\| \frac{\mathrm{d}^a}{\mathrm{d}x^a} M_l \right\|_\infty \lesssim_a 1.$$

Recall lemma 1.8 for the existence of such functions M_l. Note that if $\xi \in N_{\frac{C}{R}}(Q_0) = \{\xi \colon \xi_1 \in [0, R^{-\frac{1}{2}}],\, \xi_1^2 \leq \xi_2 \leq \xi_1^2 + \frac{C}{R}\}$, then we have

$$0 \leq \frac{R(\xi_2-\xi_1^2)}{2C} \leq \frac{1}{2},$$

$$0 \le R^{\frac{1}{2}}\xi_1 \le 1$$

and

$$0 \le R\xi_2 \le R\xi_1^2 + C \le 1 + C.$$

This means that we are able to replace $\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)^l$ with $M_l\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)$ and insert the function $\eta(R\xi_2)\eta(R^{\frac{1}{2}}\xi_1)$ into the integrand on the left-hand side of (4.36). Then writing

$$e((\xi_1^2-\xi_2)y_2)e(x\cdot\xi) = e(\xi_1^2 y_2)e(\xi_1 x_1 + \xi_2(x_2-y_2))$$

gives us (4.36).

Denote $m_l(\xi) := e(\xi_1^2 y_2)M_l\left(\frac{R(\xi_2-\xi_1^2)}{2C}\right)\eta(R\xi_2)\eta(R^{\frac{1}{2}}\xi_1)$. Plugging (4.36) to (4.35) yields

$$\|E_{Q_0}g_{j,u}\|^p_{L^p_\sharp(B(y,R))} \le \sum_{k=0}^\infty \frac{(4\pi C)^k}{k!}\left\|\int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi)m_l(\xi)e(\xi_1x_1+\xi_2(x_2-y_2))\,\mathrm{d}\xi\right\|^p_{L^p_{\sharp,x}(B(y,R))},$$

where the extra x on the subscript of the norm on the right-hand side indicates that the integration is done over the variable x. Plugging this to (4.32) and applying monotone convergence theorem gives

$$\begin{aligned}
&\|E_{Q_0}g_{j,u}\|^p_{L^p(w_{B_R,W})}\\
&\sim \int_{\mathbb{R}^n}\|E_{Q_0}g_{j,u}\|^p_{L^p_\sharp(B(y,R))}w_{B_R,W}(y)\,\mathrm{d}y\\
&\le \int_{\mathbb{R}^2}\sum_{k=0}^\infty\frac{(4\pi C)^k}{k!}\left\|\int_{N_{\frac{C}{R}}(Q_0)}\widehat{h}(\xi)m_l(\xi)e(\xi_1x_1+\xi_2(x_2-y_2))\,\mathrm{d}\xi\right\|^p_{L^p_{\sharp,x}(B(y,R))}w_{B_R,W}(y)\,\mathrm{d}y
\end{aligned}$$

(4.37)

$$= \sum_{k=0}^\infty\frac{(4\pi C)^k}{k!}\int_{\mathbb{R}^2}\left\|\int_{N_{\frac{C}{R}}(Q_0)}\widehat{h}(\xi)m_l(\xi)e(\xi_1x_1+\xi_2(x_2-y_2))\,\mathrm{d}\xi\right\|^p_{L^p_{\sharp,x}(B(y,R))}w_{B_R,W}(y)\,\mathrm{d}y.$$

Denote $H := h_{N_{\frac{C}{R}}(Q_0)}$. We aim to prove

(4.38)

$$\int_{\mathbb{R}^2}\left\|\int_{N_{\frac{C}{R}}(Q_0)}\widehat{h}(\xi)m_l(\xi)e(\xi_1x_1+\xi_2(x_2-y_2))\,\mathrm{d}\xi\right\|^p_{L^p_{\sharp,x}(B(y,R))}w_{B_R,W}(y)\,\mathrm{d}y \lesssim \|H\|^p_{L^p(w_{B_R,E})},$$

uniformly over $l \geq 0$. We start by applying Fourier inversion theorem, Fubini's theorem and a change of variables to write

$$\begin{aligned}
&\int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) m_l(\xi) e(\xi_1 x_1 + \xi_2(x_2 - y_2))\, \mathrm{d}\xi \\
&\qquad = \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) e(\xi_1 x_1 + \xi_2(x_2 - y_2)) \int_{\mathbb{R}^2} \widehat{m_l}(\gamma) e(\gamma \cdot \xi)\, \mathrm{d}\gamma\, \mathrm{d}\xi \\
&\qquad = \int_{\mathbb{R}^2} \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) e\big(\xi \cdot ((x_1, x_2 - y_2) + \gamma)\big)\, \mathrm{d}\xi\, \widehat{m_l}(\gamma)\, \mathrm{d}\gamma \\
&\qquad = \int_{\mathbb{R}^2} H((x_1, x_2 - y_2) + \gamma) \widehat{m_l}(\gamma)\, \mathrm{d}\gamma \\
&\qquad = \int_{\mathbb{R}^2} H((x_1, x_2 - y_2) - \gamma) \widehat{m_l}(-\gamma)\, \mathrm{d}\gamma \\
&\qquad = \int_{\mathbb{R}^2} H((x_1, x_2 - y_2) - \gamma) \mathcal{F}^{-1} m_l(\gamma)\, \mathrm{d}\gamma \\
&\qquad = H * \mathcal{F}^{-1} m_l(x_1, x_2 - y_2).
\end{aligned}$$

Now we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left\| \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) m_l(\xi) e(\xi_1 x_1 + \xi_2(x_2 - y_2))\, \mathrm{d}\xi \right\|^p_{L^p_{\sharp,x}(B(y,R))} w_{B_R,W}(y)\, \mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^2} \left\| H * \mathcal{F}^{-1} m_l(x_1, x_2 - y_2)) \right\|^p_{L^p_{\sharp,x}(B(y,R))} w_{B_R,W}(y)\, \mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |H * \mathcal{F}^{-1} m_l(x_1, x_2 - y_2)|^p \mathbb{1}_{B(y,R)}(x)\, \mathrm{d}x\, R^{-2} w_{B_R,W}(y)\, \mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |H * \mathcal{F}^{-1} m_l(x_1, x_2 - y_2)|^p \mathbb{1}_{B_R}(x - y)\, \mathrm{d}x\, R^{-2} w_{B_R,W}(y)\, \mathrm{d}y \\
&\qquad = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |H * \mathcal{F}^{-1} m_l(x)|^p \mathbb{1}_{B_R}(x_1 - y_1, x_2)\, \mathrm{d}x\, R^{-2} w_{B_R,W}(y)\, \mathrm{d}y. \qquad (4.39)
\end{aligned}$$

The last equality above is due to change of variables. Hölder inequality with p and $p' := \frac{p}{p-1}$ as conjugates gives

$$\begin{aligned}
|H * \mathcal{F}^{-1} m_l(x)| &\leq \int_{\mathbb{R}^2} |H(x - \eta)| |\mathcal{F}^{-1} m_l(\eta)|\, \mathrm{d}\eta \\
&= \int_{\mathbb{R}^2} |H(x - \eta)| |\mathcal{F}^{-1} m_l(\eta)|^{\frac{1}{p}} |\mathcal{F}^{-1} m_l(\eta)|^{\frac{1}{p'}}\, \mathrm{d}\eta
\end{aligned}$$

$$\leq \left(\int_{\mathbb{R}^2} |H(x-\eta)|^p |\mathcal{F}^{-1}m_l(\eta)| \,\mathrm{d}\eta\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^2} |\mathcal{F}^{-1}m_l(\eta)|\right)^{\frac{1}{p'}}$$
$$= \left(\int_{\mathbb{R}^2} |H(x-\eta)|^p |\mathcal{F}^{-1}m_l(\eta)| \,\mathrm{d}\eta\right)^{\frac{1}{p}} \|\mathcal{F}^{-1}m_l\|_{L^1(\mathbb{R}^2)}^{\frac{p-1}{p}}.$$

Now (4.15) from lemma 4.13 gives

$$\begin{aligned}|H * \mathcal{F}^{-1}m_l(x)|^p &\leq |H|^p * |\mathcal{F}^{-1}m_l|(x) \,\|\mathcal{F}^{-1}m_l\|_{L^1(\mathbb{R}^2)}^{p-1} \\ &\lesssim |H|^p * |\mathcal{F}^{-1}m_l|(x)\,(1+R^{-1}|y_2|)^{p-1}.\end{aligned}$$

Plugging this to (4.39) and applying Tonelli's theorem results in

$$\begin{aligned}&\int_{\mathbb{R}^2} \left\|\int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) m_l(\xi) e(\xi_1 x_1 + \xi_2(x_2-y_2)) \,\mathrm{d}\xi\right\|_{L^p_{\sharp,x}(B(y,R))}^p w_{B_R,W}(y)\,\mathrm{d}y \\ &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |H|^p * |\mathcal{F}^{-1}m_l|(x) \mathbb{1}_{B_R}(x_1-y_1,x_2)\,\mathrm{d}x R^{-2}(1+R^{-1}|y_2|)^{p-1} w_{B_R,W}(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} |H(\eta)|^p \\ &\qquad \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{F}^{-1}m_l(x-\eta)| R^{-2}\mathbb{1}_{B_R}(x_1-y_1,x_2)(1+R^{-1}|y_2|)^{p-1} w_{B_R,W}(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\eta.\end{aligned} \tag{4.40}$$

By change of variables $\eta' = x - \eta$, symmetry of $\mathbb{1}_{B_R}$ and Tonelli's theorem, we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{F}^{-1}m_l(x-\eta)| R^{-2}\mathbb{1}_{B_R}(x_1-y_1,x_2)(1+R^{-1}|y_2|)^{p-1} w_{B_R,W}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{F}^{-1}m_l(\eta')| R^{-2}\mathbb{1}_{B_R}(\eta-(y_1,0)+\eta')(1+R^{-1}|y_2|)^{p-1} w_{B_R,W}(y)\,\mathrm{d}y\,\mathrm{d}\eta' \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathcal{F}^{-1}m_l(\eta')| R^{-2}\mathbb{1}_{B_R}(-\eta+(y_1,0)-\eta')(1+R^{-1}|y_2|)^{p-1} w_{B_R,W}(y)\,\mathrm{d}y\,\mathrm{d}\eta' \\ &= \int_{\mathbb{R}^2} |\mathcal{F}^{-1}m_l| * \left(R^{-2}\mathbb{1}_{B_R}\right)(y_1-\eta_1,-\eta_2)(1+R^{-1}|y_2|)^{p-1} w_{B_R,W}(y)\,\mathrm{d}y \\ &\lesssim w_{B_R,E}(\eta),\end{aligned}$$

where the last line is justified by lemma 4.13. Note that here we also needed lemma 4.22 to know that the assumptions of lemma 4.13 are satisfied. Plugging the above to (4.40) gives us

$$\int_{\mathbb{R}^2} \left\| \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) m_l(\xi) e(\xi_1 x_1 + \xi_2(x_2 - y_2)) \,\mathrm{d}\xi \right\|^p_{L^p_{\sharp,x}(B(y,R))} w_{B_R,W}(y) \,\mathrm{d}y$$
$$\lesssim \int_{\mathbb{R}^2} |H(\eta)|^p w_{B_R,E}(\eta) \,\mathrm{d}\eta,$$

which is exactly (4.38).

Now using (4.31), (4.37) and (4.38) we can write

$$\begin{aligned}
&\|E_Q g_j\|^p_{L^p(w_{B_R,W})} \\
&\lesssim \sum_{k=0}^{\infty} \frac{(4\pi C)^k}{k!} \int_{\mathbb{R}^2} \left\| \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) m_l(\xi) e(\xi_1 x_1 + \xi_2(x_2 - y_2)) \,\mathrm{d}\xi \right\|^p_{L^p_{\sharp,x}(B(y,R))} w_{B_R,W}(y) \,\mathrm{d}y \\
&\lesssim \sum_{k=0}^{\infty} \frac{(4\pi C)^k}{k!} \|H\|^p_{L^p(w_{B_R,E})} \\
&= e^{4\pi C} \|h_{N_{\frac{C}{R}}(Q_0)}\|^p_{L^p(w_{B_R,E})} \\
&\sim \|h_{N_{\frac{C}{R}}(Q_0)}\|^p_{L^p(w_{B_R,E})}.
\end{aligned}$$

All that is left from the proof of (4.30) and simultaneously the theorem is

$$\|h_{N_{\frac{C}{R}}(Q_0)}\|_{L^p(w_{B_R,E})} \sim \|f_{N_{\frac{C}{R}}(Q)}\|_{L^p(w_{B_R,E})}.$$

We translate Q_0 back to Q

$$\begin{aligned}
h_{N_{\frac{C}{R}}(Q_0)}(x) &= \int_{N_{\frac{C}{R}}(Q_0)} \widehat{h}(\xi) e(x \cdot \xi) \,\mathrm{d}\xi \\
&= \int_0^{R^{-\frac{1}{2}}} \int_0^{\frac{C}{R}} \widehat{h}(s, s^2 + t) e(x \cdot (s, s^2 + t)) \,\mathrm{d}t \,\mathrm{d}s \\
&= \int_u^{u+R^{-\frac{1}{2}}} \int_0^{\frac{C}{R}} \widehat{h}(s - u, (s-u)^2 + t) e(x \cdot (s - u, (s-u)^2 + t)) \,\mathrm{d}t \,\mathrm{d}s \\
&= \int_{N_{\frac{C}{R}}(Q)} \widehat{h}(\xi_1 - u, \xi_2 - 2u\xi_1 + u^2) e(x \cdot (\xi_1 - u, \xi_2 - 2u\xi_1 + u^2)) \,\mathrm{d}\xi \\
&= e(x \cdot (-u, u^2)) \int_{N_{\frac{C}{R}}(Q)} \widehat{h}(\xi_1 - u, \xi_2 - 2u\xi_1 + u^2) e(x \cdot (\xi_1, \xi_2 - 2u\xi_1)) \,\mathrm{d}\xi
\end{aligned}$$

$$= e(x\cdot(-u,u^2))\int_{N_{\frac{C}{R}(Q)}} \widehat{h}(\xi_1-u,\xi_2-2u\xi_1+u^2)e((x_1-2ux_2,x_2)\cdot\xi)\,\mathrm{d}\xi$$

$$= e(x\cdot(-u,u^2))\int_{N_{\frac{C}{R}(Q)}} \widehat{h}(\xi_1-u,\xi_2-2u\xi_1+u^2)e(L^{-1}x\cdot\xi)\,\mathrm{d}\xi. \tag{4.41}$$

Recall that $L = \begin{bmatrix} 1 & 2u \\ 0 & 1 \end{bmatrix}$, $r = (u, u^2)$ and $L_r(\xi) = L^\top\xi + r$. Thus

$$(\xi_1-u,\xi_2-2u\xi_1+u^2) = (L^{-1})^\top\xi - (L^{-1})^\top r = L_r^{-1}(\xi).$$

Since we also defined $\widehat{h} = \widehat{f}\circ L_r$, we now have

$$\widehat{h}(\xi_1-u,\xi_2-2u\xi_1+u^2) = \widehat{f}\circ L_r\circ L_r^{-1}(\xi) = \widehat{f}(\xi).$$

Combining the above with (4.41) we get

$$|h_{N_{\frac{C}{R}(Q_0)}}(x)| = \left|\int_{N_{\frac{C}{R}(Q)}} \widehat{f}(\xi)e(L^{-1}x\cdot\xi)\,\mathrm{d}\xi\right| = \left|f_{N_{\frac{C}{R}(Q)}}(L^{-1}x)\right|$$

and a familiar calculation using the fact that L is 3-bilipschitz and B_R is centered at the origin gives

$$\|h_{N_{\frac{C}{R}(Q_0)}}\|_{L^p(w_{B_R},E)} = \|f_{N_{\frac{C}{R}(Q)}}\circ L^{-1}\|_{L^p(w_{B_R},E)} \sim \|f_{N_{\frac{C}{R}(Q)}}\|_{L^p(w_{B_R},E)}$$

as needed. This concludes the proof of theorem 4.27.

□

Chapter 5

Linear versus multilinear decouplings

We will compare linear and multilinear decouplings. In order to do this, we need to present the multilinear decoupling constant. We will see that it is straightforward to dominate this new decoupling constant with the linear decoupling constant. Naturally, one may ponder whether some kind of reverse inequality exists and the main aspiration of this chapter is to prove that such an inequality does indeed exist. The reverse inequality turns out to be possible, if the lower dimensional linear decoupling constant is under control. The second section is dedicated to estimating $\|Eg\|_{L^p(w_{B,E})}$ when this condition holds. The greatest motivator for the reverse inequality is that it provides an important step in the proof of the l^2 decoupling theorem.

5.1 Multilinear decoupling constant

In this section we will introduce the multilinear decoupling constant and show the easier decoupling constant inequality which indicates that the linear decoupling constant dominates the multilinear one.

Definition 5.1. For $p \geq 2$, $\delta \in 4^{-\mathbb{N}}$, $m \in \mathbb{N}_+$ and $0 < \nu < 1$, let $\mathrm{Dec}_n(\delta, p, \nu, m, E) = \mathrm{Dec}(\delta, p, \nu, m, E)$ be the smallest constant that satisfies

$$\left(\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B)} \left(\prod_{i=1}^{n} \|E_{Q_i} g\|^p_{L^p(w_{\Delta,10E})} \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$

$$\leq \mathrm{Dec}(\delta, p, \nu, m, E) \left(\prod_{i=1}^{n} \sum_{q_i \in \mathrm{Part}_{\sqrt{\delta}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{B,E})} \right)^{\frac{1}{2n}}, \tag{5.2}$$

for each cube B with side length δ^{-1} and all ν-transverse cubes $Q_i \subset [0,1]^{n-1}$ with equal side lengths μ satisfying $\mu \geq \delta^{2^{-m}}$.

Notice that in addition to the decoupling, the scale of the cube related to the weight function is inflated similarly as in the cube inflation inequality of theorem 3.15. Notice also that the lower bound on the size of μ is more significant than the minimal lower bound $\mu \geq \sqrt{\delta}$ that is required for the partition $\text{Part}_{\sqrt{\delta}}(Q_i)$ to make sense. The lower bound $\mu \geq \delta^{2^{-m}}$ is used later in the proof of the l^2 decoupling theorem, where a similar estimation is done for $\text{Part}_{\delta^{2^{-m}}}(Q_i)$. This restriction is not needed in the results of this thesis.

We will show that when the parameters δ, p, ν, m, E and μ satisfy the assumptions from definition 5.1, we have that

$$\text{Dec}(\delta, p, \nu, m, E) \lesssim \text{Dec}(\frac{\delta}{\mu^2}, p, 3E). \tag{5.3}$$

The sequence version of the generalized Hölder inequality from theorem 1.4 with

$$p_i = \begin{cases} p_i = 1, & 1 \leq i \leq n \\ p_i = \frac{1}{n}, & i = n+1 \end{cases}$$

applied to the sequences $b_j := a_{k,j}$ gives us

$$\sum_{k=1}^{K} \prod_{j=1}^{n} a_{k,j}^{\frac{1}{n}} = \| \prod_{j=1}^{n} b_j \|_{l^{\frac{1}{n}}}^{\frac{1}{n}} \leq \prod_{j=1}^{n} \|b_j\|_{l^1}^{\frac{1}{n}} = \prod_{j=1}^{n} \left(\sum_{k=1}^{K} a_{k,j} \right)^{\frac{1}{n}}, \tag{5.4}$$

for $0 \leq a_{k,j} < \infty$. With the assumptions of definition 5.1 we have

$$\|E_{Q_i} g\|_{L^p(w_{B,E})} \leq \text{Dec}(\frac{\delta}{\mu^2}, p, 3E) \left(\sum_{q_i \in \text{Part}_{\sqrt{\delta}}(Q_i)} \|E_{q_i} g\|_{L^p(w_{B,E})}^2 \right)^{\frac{1}{2}}. \tag{5.5}$$

A detailed proof for the above linear decoupling inequality can be found in [14] chapter 5 proposition 5.2.3. Now applying (5.4), lemma 3.12 and (5.5) we get

$$\left(\sum_{\Delta \in \text{Part}_{\mu^{-1}}(B)} \left(\prod_{i=1}^{n} \|E_{Q_i} g\|_{L^p(w_{\Delta,10E})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$

$$\leq \left(\prod_{i=1}^{n} \left(\sum_{\Delta \in \text{Part}_{\mu^{-1}}(B)} \|E_{Q_i} g\|_{L^p(w_{\Delta,10E})}^p \right)^{\frac{1}{n}} \right)^{\frac{1}{p}}$$

$$= \left(\prod_{i=1}^{n} \left(\|E_{Q_i} g\|^p_{L^p(\sum_{\Delta \in \mathrm{Part}_{\mu^{-1}}(B)} w_{\Delta,10E})} \right)^{\frac{1}{p}} \right)^{\frac{1}{n}}$$

$$\lesssim \left(\prod_{i=1}^{n} \|E_{Q_i} g\|_{L^p(w_{B,10E})} \right)^{\frac{1}{n}}$$

$$\leq \left(\prod_{i=1}^{n} \mathrm{Dec}(\frac{\delta}{\mu^2}, p, 3E) \left(\sum_{q_i \in \mathrm{Part}_{\sqrt{\delta}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{B,10E})} \right)^{\frac{1}{2}} \right)^{\frac{1}{n}}$$

$$= \mathrm{Dec}(\frac{\delta}{\mu^2}, p, 3E) \left(\prod_{i=1}^{n} \sum_{q_i \in \mathrm{Part}_{\sqrt{\delta}}(Q_i)} \|E_{q_i} g\|^2_{L^p(w_{B,10E})} \right)^{\frac{1}{2n}},$$

which gives us (5.3). This fact implies that the multilinear decoupling constant is well defined.

5.2 Estimating $\|Eg\|_{L^p(w_{B,E})}$ when the lower dimensional decoupling constant is under control

This section is devoted to a collection of inequalities that we will use in the final section to prove the reverse decoupling constant inequality. Before we present these inequalities, we will introduce two lemmas. The first lemma will aid in discovering ν-transverse cubes.

Lemma 5.6. *If $P_j = (\xi^{(j)}, \|\xi^{(j)}\|^2) \in \mathbb{P}^{n-1}$ for $j = 1, \dots, n$, then the volume V_P of a parallelepiped spanned by the unit normal vectors $n(P_j)$ is comparable to the volume V_s of the $(n-1)$-simplex with vertices $\xi^{(j)} \in [0,1]^{n-1}$. In particular, we have*

$$\frac{1}{(4n+1)^{\frac{n}{2}}} V_s \leq \frac{1}{2^{n-1}(n-1)!} V_P \leq V_s.$$

Proof. We refer to [16] for the proof of the fact that the volume of an $(n-1)$-simplex is given by

$$V_s = \frac{1}{(n-1)!} \left| \det \left(\begin{bmatrix} \xi^{(1)} & \cdots & \xi^{(n)} \\ 1 & \cdots & 1 \end{bmatrix} \right) \right|.$$

The volume of the parallelepiped is

$$V_P = \left| \det \left(\left[n(P_1) \cdots n(P_n) \right] \right) \right| = \left| \det \left(\begin{bmatrix} -\frac{2\xi^{(1)}}{c_1} & \cdots & -\frac{2\xi^{(n)}}{c_n} \\ \frac{1}{c_1} & \cdots & \frac{1}{c_n} \end{bmatrix} \right) \right|,$$

where $c_j = \sqrt{4\|\xi^{(j)}\|^2 + 1}$ for every $j = 1, \dots, n$. First we can extract the multiplier -2 out of the first $n-1$ rows and get

$$V_P = \left| \det\left(\begin{bmatrix} -\frac{2\xi^{(1)}}{c_1} & \cdots & -\frac{2\xi^{(n)}}{c_n} \\ \frac{1}{c_1} & \cdots & \frac{1}{c_n} \end{bmatrix} \right) \right| = 2^{n-1} \left| \det\left(\begin{bmatrix} \frac{\xi^{(1)}}{c_1} & \cdots & \frac{\xi^{(n)}}{c_n} \\ \frac{1}{c_1} & \cdots & \frac{1}{c_n} \end{bmatrix} \right) \right|.$$

Then we extract the constants c_j out of the columns and derive that

$$V_P = 2^{n-1} \left| \det\left(\begin{bmatrix} \xi^{(1)} & \cdots & \xi^{(n)} \\ 1 & \cdots & 1 \end{bmatrix} \right) \right| \prod_{j=1}^{n} \frac{1}{c_j} = 2^{n-1}(n-1)!\, V_s \prod_{j=1}^{n} \frac{1}{c_j}.$$

Since $\xi^{(j)} \in [0,1]^{n-1}$, we have $1 \le c_j \le \sqrt{4n+1}$ for every $j = 1, \dots, n$. Thus

$$\frac{2^{n-1}(n-1)!}{(4n+1)^{\frac{n}{2}}} V_s \le V_P \le 2^{n-1}(n-1)!\, V_s$$

and dividing the inequalities with $2^{n-1}(n-1)!$ gives us the wanted result. □

The following lemma can be seen as manifestation of the uncertainty principle, which asserts that the extension operator is constant at scale K when the frequency is supported on a scale K^{-1}.

Lemma 5.7. *Let B_K be a cube of side length K and $\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^{n-1})$. Then*

$$\sup_{x \in B_K} |E_\alpha g(x)| \lesssim \left(\frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(x)|^p w_{B_K, 10E}(x) \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Proof. By lemma 1.7 there exists a Schwartz function f that is 1 on the cube $B(0, 3n)$. Let φ be the inverse Fourier transform of f. Now we have that $\widehat{\varphi} = 1$ on $B(0, 3n)$. We define

$$\varphi_K(x) = \frac{1}{K^n} \varphi(\frac{x}{K}) e(x \cdot z_\alpha) = \frac{1}{|B_K|} \varphi(\frac{x}{K}) e(x \cdot z_\alpha),$$

where $z_\alpha = (c_\alpha, |c_\alpha|^2)$ and c_α is the center of the cube α. Now by change of variables

$$\begin{aligned} \widehat{\varphi_K}(\xi) &= \int_{\mathbb{R}^n} \varphi_K(x) e(-x \cdot \xi) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \frac{1}{K^n} \varphi(\frac{x}{K}) e(x \cdot z_\alpha) e(-x \cdot \xi) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \varphi(y) e(Ky \cdot z_\alpha) e(-Ky \cdot \xi) \, \mathrm{d}y \end{aligned}$$

$$= \int_{\mathbb{R}^n} \varphi(y) e(-Ky \cdot (\xi - z_\alpha))\, \mathrm{d}y$$
$$= \widehat{\varphi}(K(\xi - z_\alpha)) = \widehat{\varphi}(\frac{\xi - z_\alpha}{K^{-1}}).$$

Hence $\widehat{\varphi} = 1$ on $B(z_\alpha, 3nK^{-1})$.

We arque that the support of the Fourier transfrom of $E_\alpha g$ is in $B(z_\alpha, 3nK^{-1})$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. We apply Fubini's theorem and Fourier inversion theorem to calculate that

$$\begin{aligned}\langle \widehat{E_\alpha g}, \phi \rangle = \langle E_\alpha g, \widehat{\phi} \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \mathbb{1}_\alpha(\xi) g(\xi) e(x \cdot (\xi, |\xi|^2)) \widehat{\phi}(x)\, \mathrm{d}\xi\, \mathrm{d}x \\ &= \int_{\mathbb{R}^{n-1}} \mathbb{1}_\alpha(\xi) g(\xi) \int_{\mathbb{R}^n} \widehat{\phi}(x) e(x \cdot (\xi, |\xi|^2))\, \mathrm{d}x\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^{n-1}} \mathbb{1}_\alpha(\xi) g(\xi) \phi(\xi, |\xi|^2)\, \mathrm{d}\xi.\end{aligned}$$

Now we see that

$$\operatorname{supp}(\widehat{E_\alpha g}) \subset \{(\xi, |\xi|^2) \in \mathbb{R}^n : \xi \in \alpha\}.$$

For $\xi \in \alpha$ we have

$$|(\xi, |\xi|^2) - z_\alpha| \leq |\xi - c_\alpha| + \left||\xi|^2 - |c_\alpha|^2\right| \leq \sqrt{n-1} K^{-1} + \left||\xi|^2 - |c_\alpha|^2\right|$$

and

$$\begin{aligned}\left||\xi|^2 - |c_\alpha|^2\right| &\leq \sum_{i=1}^{n-1} |\xi_i^2 - (c_\alpha)_i^2| \\ &= \sum_{i=1}^{n-1} |(\xi_i - (c_\alpha)_i)(\xi_i + (c_\alpha)_i)| \\ &\leq \sum_{i=1}^{n-1} K^{-1} |\xi_i + (c_\alpha)_i| \\ &\leq K^{-1} \sum_{i=1}^{n-1} (|\xi_i| + |(c_\alpha)_i|) \\ &\leq 2(n-1) K^{-1}.\end{aligned}$$

Thus we have

$$\operatorname{supp}(\widehat{E_\alpha g}) \subset B(z_\alpha, 3nK^{-1}).$$

This means that by lemma 1.13 we have $\widehat{E_\alpha g * \varphi_K} = \widehat{E_\alpha g}\, \widehat{\varphi_K} = \widehat{E_\alpha g}$ which implies $E_\alpha g * \varphi_K = E_\alpha g$ in the distributional sense. Since the functions $E_\alpha g * \varphi_K$ and $E_\alpha g$ are continuous, lemma 1.11 implies that we also have $E_\alpha g * \varphi_K = E_\alpha g$ in the classical sense. Then applying lemma 1.5 with $s = 10E$, we get

$$\begin{aligned} |E_\alpha g(x)| &= \left| \int_{\mathbb{R}^n} E_\alpha g(y) \varphi_K(x-y)\, \mathrm{d}y \right| \\ &= \frac{1}{|B_K|} \left| \int_{\mathbb{R}^n} E_\alpha g(y) \varphi(\frac{x-y}{K}) e((x-y)\cdot z_\alpha)\, \mathrm{d}y \right| \\ &\leq \frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(y)| \left| \varphi(\frac{x-y}{K}) \right| \mathrm{d}y \\ &\lesssim \frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(y)| \left(1 + \frac{|x-y|}{K}\right)^{-10E} \mathrm{d}y. \end{aligned} \tag{5.8}$$

For $x \in B_K$, we can apply lemma 3.8 to get

$$(1 + \frac{|x-y|}{K})^{-10E} = w_{B_K,10E}(y + c_{B_K} - x) \sim w_{B_K,10E}(y)$$

and plugging this to (5.8) gives that for all $x \in B_K$, we have

$$|E_\alpha g(x)| \lesssim \frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(y)|\, w_{B_K,10E}(y)\, \mathrm{d}y. \tag{5.9}$$

An application of Hölder inequality with coefficients p and p' yields

$$\begin{aligned} &\frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(y)|\, w_{B_K,10E}(y)\, \mathrm{d}y \\ &\quad = \int_{\mathbb{R}^n} \frac{|E_\alpha g(y)|\, w_{B_K,10E}(y)^{\frac{1}{p}}}{|B_K|^{\frac{1}{p}}} \frac{w_{B_K,10E}(y)^{\frac{1}{p'}}}{|B_K|^{\frac{1}{p'}}}\, \mathrm{d}y \\ &\quad \leq \left(\frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(y)|^p\, w_{B_K,10E}(y)\, \mathrm{d}y \right)^{\frac{1}{p}} \left(\frac{1}{|B_K|} \int_{\mathbb{R}^n} w_{B_K,10E}(y)\, \mathrm{d}y \right)^{\frac{1}{p'}} \\ &\quad \sim \left(\frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(y)|^p\, w_{B_K,10E}(y)\, \mathrm{d}y \right)^{\frac{1}{p}}. \end{aligned}$$

The last relation is due to lemma 3.5. Combining this with (5.9) gives us

$$\sup_{x \in B_K} |E_\alpha g| \lesssim \left(\frac{1}{|B_K|} \int_{\mathbb{R}^n} |E_\alpha g(y)|^p\, w_{B_K,10E}(y)\, \mathrm{d}y \right)^{\frac{1}{p}}$$

as wanted. □

For the rest of this section we will restrict the dimension to $n = 3$. At the end of this chapter we will discuss the distinctions in the proofs for other dimensions. A key step in the proof of the reverse decoupling constant inequality is the following proposition which gives us an estimate on the left-hand side of (4.2) when the lower dimensional linear decoupling constant is bounded by any negative power of the scale δ.

Proposition 5.10. *Let $p \geq 2$, $g \in L^1([0,1]^2)$ and assume $\mathrm{Dec}_2(\delta, p, \Gamma_2(10E)) \lesssim_\varepsilon \delta^{-\varepsilon}$. Then there exists a constant C and C_ε such that for each $m \geq 1$ and each $R \geq K^{2^m}$ such that $\frac{R}{K} \in \mathbb{N}$, we have*

$$\|Eg\|_{L^p(w_{B_R,E})} \leq C_\varepsilon K^\varepsilon \left[\left(\sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} + \left(\sum_{\beta \in \mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} \right] + K^C \, \mathrm{Dec}_3(R^{-1}, p, K^{-2}, m, E) \left(\sum_{\Delta \in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1]^2)} \|E_\Delta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}}.$$

Proof. By lemma 4.3 it suffices to prove the inequality with $\|Eg\|_{L^p(B_R)}$ on the left-hand side. Cover B_R with a family $\mathrm{Part}_K(B_R)$ of cubes in $\mathbb{R}^3$ with side length K. For each $\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)$ and each $B_K \in \mathrm{Part}_K(B_R)$ we define

$$c_\alpha(B_K) = \left(\frac{1}{|B_K|} \int_{\mathbb{R}^3} |E_\alpha g(x)|^p w_{B_K,10E}(x) \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Note that lemma 5.7 gives us that

$$\sup_{x \in \mathbb{R}^3} |E_\alpha g(x)| \lesssim c_\alpha(B_K). \tag{5.11}$$

Let $\alpha^* := \alpha^*(B_K) \in \mathrm{Part}_{K^{-1}}([0,1]^2)$ be the square that maximizes $c_\alpha(B_K)$. Let L be a line in the (ξ_1, ξ_2) plane and define

$$S_L = \{(\xi_1, \xi_2) \in [0,1]^2 : \mathrm{dist}(\mathrm{L}, (\xi_1, \xi_2)) \leq \frac{\mathrm{C_L}}{\mathrm{K}}\}.$$

Define also

$$S_{\mathrm{big}} = \{\alpha : c_\alpha(B_K) \geq K^{-2} c_{\alpha^*}(B_K)\}.$$

We will show that for each $B_K \in \mathrm{Part}_K(B_R)$ there exists a line $L = L(B_K) \subset \mathbb{R}^2$ such that for all $x \in B_K$

$$|Eg(x)| \leq$$

$$C_1\, c_{\alpha^*}(B_K) \tag{5.12}$$

$$+ K^{C_2} \max_{\substack{\alpha_1,\alpha_2,\alpha_3 \\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} c_{\alpha_i}(B_K) \right)^{\frac{1}{3}} \tag{5.13}$$

$$+ \left| \sum_{\alpha \subset S_L} E_\alpha g(x) \right|, \tag{5.14}$$

where C_1 and C_2 are constants. Define $U_{\alpha^*} := \{\alpha \in S_{\text{big}} : \operatorname{dist}(\alpha, \alpha^*) \geq \frac{10}{K}\}$. We distinguish three scenarios

1. U_{α^*} is empty.
2. U_{α^*} is not empty and there exists an $\alpha \in S_{\text{big}}$ that intersects the complement of S_L.
3. U_{α^*} is not empty and all $\alpha \in S_{\text{big}}$ are in S_L.

Clearly these scenarios cover all possible situations.

In the first scenario we do not need S_L. We can directly bound $|Eg(x)|$ with (5.12) as follows

$$\begin{aligned}
|Eg(x)| &\leq \sum_{\alpha} |E_\alpha g(x)| \\
&= \sum_{\alpha \in S_{\text{big}}} |E_\alpha g(x)| + \sum_{\alpha \notin S_{\text{big}}} |E_\alpha g(x)| \\
&\leq C' \left(\sum_{\alpha \in S_{\text{big}}} c_\alpha(B_K) + \sum_{\alpha \notin S_{\text{big}}} c_\alpha(B_K) \right) \\
&\leq C' \left(\sum_{\alpha \in S_{\text{big}}} c_{\alpha^*}(B_K) + \sum_{\alpha \notin S_{\text{big}}} K^{-2} c_{\alpha^*}(B_K) \right) \\
&\leq C' \left(23^2 c_{\alpha^*}(B_K) + K^2 K^{-2} c_{\alpha^*}(B_K) \right) \\
&= 530 C'\, c_{\alpha^*}(B_K),
\end{aligned}$$

where C' is the maximum of the implicit constants from the estimates of the form (5.11).

Now we assume that U_{α^*} is not empty. Then the line L is determined by centers of the squares α_1 and α_2 that are furthest apart among all possible pairs in S_{big}. Since U_{α^*} is not empty, the distance between these squares is at least $\frac{10}{K}$.

We claim that in the second scenario (5.13) suffices. To show this we let α_3 be a square that intersects the complement of S_L. Let T be a triangle that is determined by three points that are in different squares α_i. The area of the triangle has the lower bound

$$m_2(T) \geq \frac{1}{2}\frac{10}{K}(\frac{C_L}{K} - \frac{\sqrt{2}}{K}) = \frac{5(C_L - \sqrt{2})}{K^2}.$$

An application of lemma 5.6 shows that choosing $C_L > \sqrt{2} + \frac{9}{10}$ gives us that α_1, α_2 and α_3 are K^{-2}-transverse. Now

$$|Eg(x)| \leq \sum_{\alpha} |E_\alpha g(x)| \leq C' \left(\sum_{\alpha} c_{\alpha^*}(B_K) \right) = C' K^2 c_{\alpha^*}(B_K) \leq C' K^4 c_{\alpha_i}(B_K),$$

for $i = 1, 2, 3$. Multiplying both sides with respect to i and taking a cube roots and maximums gives us

$$|Eg(x)| \leq C' K^4 \max_{\substack{\alpha_1, \alpha_2, \alpha_3 \\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} c_{\alpha_i}(B_K) \right)^{\frac{1}{3}}.$$

Since $\log_K(C') \leq \log_2(C')$ for all $C' \geq 1$, we get

$$\begin{aligned} |Eg(x)| &\leq C' K^4 \max_{\substack{\alpha_1, \alpha_2, \alpha_3 \\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} c_{\alpha_i}(B_K) \right)^{\frac{1}{3}} \\ &= K^{\log_K(C')+4} \max_{\substack{\alpha_1, \alpha_2, \alpha_3 \\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} c_{\alpha_i}(B_K) \right)^{\frac{1}{3}} \\ &\leq K^{\log_2(C')+4} \max_{\substack{\alpha_1, \alpha_2, \alpha_3 \\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} c_{\alpha_i}(B_K) \right)^{\frac{1}{3}} \end{aligned}$$

as wanted.

In the third scenario (5.12) and (5.14) will suffice. Indeed,

$$\begin{aligned}
|Eg(x)| &= \left|\sum_{\alpha} E_\alpha g(x)\right| \\
&\leq \left|\sum_{\alpha\subset S_L} E_\alpha g(x)\right| + \sum_{\alpha\notin S_{\text{big}}} |E_\alpha g(x)| \\
&\leq \left|\sum_{\alpha\subset S_L} E_\alpha g(x)\right| + C'\left(\sum_{\alpha\notin S_{\text{big}}} c_\alpha(B_K)\right) \\
&\leq \left|\sum_{\alpha\subset S_L} E_\alpha g(x)\right| + C'\left(\sum_{\alpha\notin S_{\text{big}}} K^{-2} c_{\alpha^*}(B_K)\right) \\
&\leq \left|\sum_{\alpha\subset S_L} E_\alpha g(x)\right| + C' c_{\alpha^*}(B_K).
\end{aligned}$$

Next we are going to prove that (5.12)-(5.14) implies

$$\|Eg\|_{L^p(B_K)} \lesssim_\varepsilon$$

$$K^\varepsilon\left[\left(\sum_{\alpha\in\text{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} + \left(\sum_{\beta\in\text{Part}_{K^{-\frac{1}{2}}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}\right]$$

$$+ K^{C_2} \max_{\substack{\alpha_1,\alpha_2,\alpha_3 \\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} \|E_{\alpha_i} g\|_{L^p(w_{B_K,10E})}\right)^{\frac{1}{3}}. \tag{5.15}$$

In the first case scenario we have

$$|Eg(x)| \leq C_1 c_{\alpha^*}(B_K)$$

$$\Rightarrow \quad \|Eg\|_{L^p(B_K)} = \left(\int_{B_K} |Eg(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq C_1 |B_K|^{\frac{1}{p}} c_{\alpha^*}(B_K) = C_1 \|E_{\alpha^*} g\|_{L^p(w_{B_K,10E})}$$

$$\Leftrightarrow \quad \|Eg\|_{L^p(B_K)} \leq C_1 \left(\|E_{\alpha^*} g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \|Eg\|_{L^p(B_K)} \leq C_1 \left(\sum_{\alpha\in\text{Part}_{K^{-1}}([0,1])^2} \|E_\alpha g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}.$$

The second scenario yields

$$
\begin{aligned}
|Eg(x)| &\leq K^{C_2} \max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} c_{\alpha_i}(B_K) \right)^{\frac{1}{3}} \\
\Leftrightarrow |Eg(x)| &\leq K^{C_2} |B_K|^{-\frac{1}{p}} \max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} \|E_{\alpha_i}\|_{L^p(w_{B_K,10E})} \right)^{\frac{1}{3}} \\
\Rightarrow \|Eg\|_{L^p(B_K)} &\leq K^{C_2} \max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} \|E_{\alpha_i}\|_{L^p(w_{B_K,10E})} \right)^{\frac{1}{3}}.
\end{aligned}
$$

For the third scenario we translate B_K so that it is centered in the origin. Then on the left side of (5.15) we have

$$
\int_{B_K} |Eg(t)|^p \, \mathrm{d}t = \int_{[-\frac{K}{2},\frac{K}{2}]^3} |Eg(u + c_{B_K})|^p \, \mathrm{d}u
$$

and changing the variables back on the right-hand side of (5.15) yields

$$
\begin{aligned}
\int_{\mathbb{R}^2} |E_\alpha g(u + c_{B_K})|^p w_{[-\frac{K}{2},\frac{K}{2}]^3,10E}(u) \, \mathrm{d}u &= \int_{\mathbb{R}^2} |E_\alpha g(t)|^p w_{[-\frac{K}{2},\frac{K}{2}]^3,10E}(t - c_{B_K}) \, \mathrm{d}u \\
&= \int_{\mathbb{R}^2} |E_\alpha g(t)|^p w_{B_K,10E}(t) \, \mathrm{d}t.
\end{aligned}
$$

Thus we may assume that $B_K = [-\frac{K}{2}, \frac{K}{2}]^3$. We then calculate that

$$
\begin{aligned}
|Eg(x)| &\leq \left| \sum_{\alpha \subset S_L} E_\alpha g(x) \right| + C' c_{\alpha^*}(B_K) \\
\Rightarrow \qquad \|Eg\|_{L^p(B_K)} &\leq \left\| \sum_{\alpha \subset S_L} E_\alpha g \right\|_{L^p(B_K)} + C' |B_K|^{\frac{1}{p}} c_{\alpha^*}(B_K).
\end{aligned}
$$

Identically as in the first scenario we have

$$
C'|B_K|^{\frac{1}{p}} c_{\alpha^*}(B_K) \leq C' \left(\sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1])^2} \|E_\alpha g\|^2_{L^p(w_{B_K,10E})} \right)^{\frac{1}{2}}.
$$

All that is left is to control the term $\left\| \sum_{\alpha \subset S_L} E_\alpha g \right\|_{L^p(B_K)}$ with the right-hand side of (5.15).

We will assume that $L = \{\xi : \xi_2 = 0\}$. This is essentially the same assumption as in [5]. Unfortunately, we will take this simplification of notation as granted and leave out the details of the general case.

We cover S_L by pairwise disjoint rectangles U that have side lengths $C_L K^{-1}$ and $K^{-\frac{1}{2}}$, with the longer side parallel to L. Denote $\widetilde{S}_L := \bigcup_{\alpha \subset S_L} \alpha$ and $\widetilde{g} = g \mathbb{1}_{\widetilde{S}_L}$. These simplify the notation as we can write $\sum_{\alpha \subset S_L} E_\alpha g = E_{S_L} \widetilde{g}$. Next we will prove the following claim.

$$\text{(5.16)} \qquad \textbf{Claim:} \qquad \|E_{S_L}\widetilde{g}\|_{L^p(B_K)} \lesssim K^\varepsilon \left(\sum_U \|E_U \widetilde{g}\|^2_{L^p(w_{B_K},10E)} \right)^{\frac{1}{2}}.$$

In order to prove this claim, we will apply theorem 4.27 in the lower dimension $n = 2$. The lower dimensional decoupling constant that arises from theorem 4.27 will be contained by assumption. Thus the idea is to find a lower dimensional function that satisfies the assumptions of theorem 4.27 and is somehow similar to $E_{S_L}\widetilde{g}$.

We study the following change of variables

$$(s,t) = (\xi_1, \xi_1^2 + \xi_2^2) \Leftrightarrow (\xi_1, \xi_2) = (s, \sqrt{t - s^2}).$$

The Jacobian determinant of this transformation is

$$\det \left(\begin{bmatrix} 1 & 0 \\ \frac{\partial(\sqrt{t-s^2})}{\partial s} & \frac{1}{2\sqrt{t-s^2}} \end{bmatrix} \right) = \frac{1}{2\sqrt{t-s^2}}$$

and hence we can write

$$\begin{aligned} E_{S_L}\widetilde{g}(x) &= \int_{\mathbb{R}^2} \widetilde{g}(\xi) e(x \cdot (\xi, |\xi|^2)) \, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^2} \widetilde{g}(s, \sqrt{t-s^2}) e(x \cdot (s, \sqrt{t-s^2}, t)) \frac{1}{2\sqrt{t-s^2}} \, \mathrm{d}(s,t) \\ &= \int_{\mathbb{R}^2} \widetilde{g}(s, \sqrt{t-s^2}) e(\sqrt{t-s^2}\, x_2) \frac{1}{2\sqrt{t-s^2}} e(sx_1 + tx_3) \, \mathrm{d}(s,t). \end{aligned}$$

Note that in the above equalities the information about the area of integration is included in the function $\widetilde{g}$. Denote $E_A \widetilde{g}_{x_2}(x_1, x_3) := E_A \widetilde{g}(x_1, x_2, x_3)$, where A is a set. From the above calculation we see that for a fixed x_2 the inverse Fourier transform of the function $g^*_{x_2}$, that is defined by

$$(s,t) \mapsto \widetilde{g}(s, \sqrt{t-s^2}) e(\sqrt{t-s^2}\, x_2) \frac{1}{2\sqrt{t-s^2}},$$

is $E_{S_L}\widetilde{g}_{x_2}$. Thus the Fourier inverse formula of $\mathcal{S}'(\mathbb{R}^2)$ gives that $\mathcal{F}\left(E_{S_L}\widetilde{g}_{x_2}\right) = g^*_{x_2}$, which is supported in the set

$$\{(s,t) : (s, \sqrt{t-s^2}) \in \widetilde{S}_L\}.$$

We have

$$S_L = \{(s,r) : |r| \leq \frac{C_L}{K}\}$$

and hence we get

$$\sqrt{t-s^2} \leq \frac{C_L}{K} \Leftrightarrow s^2 \leq t \leq s^2 + \left(\frac{C_L}{K}\right)^2,$$

which implies that $(s,t) \in N_{\left(\frac{C_L}{K}\right)^2}([0,1])$, i.e., the Fourier transform of $E_{S_L}\widetilde{g}_{x_2}$ is supported in $N_{\left(\frac{C_L}{K}\right)^2}([0,1])$. This means that theorem 4.27 yields

$$\|E_{S_L}\widetilde{g}_{x_2}\|_{L^p(\widetilde{B}_K)} \lesssim$$
$$\mathrm{Dec}_2(K^{-1}, p, \Gamma_2(10E)) \left(\sum_{Q \in \mathrm{Part}_{K^{-\frac{1}{2}}}([0,1])} \| \left(E_{S_L}\widetilde{g}_{x_2}\right)_{N_{\left(\frac{C_L}{K}\right)^2(Q)}} \|^2_{L^p(w_{\widetilde{B}_K,10E})} \right)^{\frac{1}{2}},$$

where we denoted $\widetilde{B}_K := [-\frac{K}{2}, \frac{K}{2}]^2$ in the (x_1, x_3) plane. Next we calculate the Fourier restriction

$$\begin{aligned}
&\left(E_{S_L}\widetilde{g}_{x_2}\right)_{N_{\left(\frac{C_L}{K}\right)^2(Q)}}(x_1,x_3) \\
&\quad = \int_{N_{\left(\frac{C_L}{K}\right)^2(Q)}} \widetilde{g}(s, \sqrt{t-s^2}) e(\sqrt{t-s^2}x_2) \frac{1}{2\sqrt{t-s^2}} e(sx_1 + tx_3) \,\mathrm{d}(s,t) \\
&\quad = \int_Q \int_0^{\frac{C_L}{K}} \widetilde{g}(\xi_1,\xi_2) e(\xi_2 x_2) e(\xi_1 x_1 + |\xi|^2 x_3) \,\mathrm{d}\xi_2 \,\mathrm{d}\xi_1 \\
&\quad = \int_U \widetilde{g}(\xi_1,\xi_2) e(x \cdot (\xi, |\xi|^2)) \,\mathrm{d}\xi = E_U \widetilde{g}_{x_2}(x_1,x_3),
\end{aligned}$$

where $U = \{(\xi_1,\xi_2) : \xi_1 \in Q,\, \xi_2 \in [0, \frac{C_L}{K}]\}$. We now have proven

$$\|E_{S_L}\widetilde{g}x_2\|_{L^p(\widetilde{B}_K)} \lesssim \mathrm{Dec}_2(K^{-1}, p, \Gamma_2(10E)) \left(\sum_U \|E_U \widetilde{g}_{x_2}\|^2_{L^p(w_{\widetilde{B}_K,10E})} \right)^{\frac{1}{2}},$$

which by assumption implies

$$\|E_{S_L}\widetilde{g}_{x_2}\|_{L^p(\widetilde{B}_K)} \lesssim_\varepsilon K^\varepsilon \left(\sum_U \|E_U\widetilde{g}_{x_2}\|^2_{L^p(w_{\widetilde{B}_K,10E})}\right)^{\frac{1}{2}}. \tag{5.17}$$

Raising the left-hand side of (5.17) to the power p and integrating over $x_2 \in [-\frac{K}{2}, \frac{K}{2}]$ gives

$$\int_{-\frac{K}{2}}^{\frac{K}{2}} \|E_{S_L}\widetilde{g}_{x_2}\|^p_{L^p(\widetilde{B}_K)}\,\mathrm{d}x_2 = \|E_{S_L}\widetilde{g}\|^p_{L^p(B_K)}$$

and applying Minkovski's inequality in $L^{\frac{p}{2}}$ we see that the same procedure on the right-hand side gives

$$\begin{aligned}
K^{\varepsilon p}\int_{-\frac{K}{2}}^{\frac{K}{2}} \left(\sum_U \|E_U\widetilde{g}_{x_2}\|^2_{L^p(w_{\widetilde{B}_K,10E})}\right)^{\frac{p}{2}}\mathrm{d}x_2 &= K^{\varepsilon p}\left\|\sum_U \|E_U\widetilde{g}_{x_2}\|^2_{L^p(w_{\widetilde{B}_K,10E})}\right\|^{\frac{p}{2}}_{L^{\frac{p}{2}}([-\frac{K}{2},\frac{K}{2}])} \\
&\leq K^{\varepsilon p}\left(\sum_U \left\|\|E_U\widetilde{g}_{x_2}\|^2_{L^p(w_{\widetilde{B}_K,10E})}\right\|_{L^{\frac{p}{2}}([-\frac{K}{2},\frac{K}{2}])}\right)^{\frac{p}{2}} \\
&= K^{\varepsilon p}\left(\sum_U \left\|\|E_U\widetilde{g}_{x_2}\|^p_{L^p(w_{\widetilde{B}_K,10E})}\right\|^{\frac{2}{p}}_{L^1([-\frac{K}{2},\frac{K}{2}])}\right)^{\frac{p}{2}}.
\end{aligned}$$

By lemma 3.8, we have $w_{\widetilde{B}_K,10E}(x_1,x_3) = w_{B_K,10E}(x_1,0,x_3) \sim w_{B_K,10E}(x_1,x_2,x_3)$ for $x_2 \in [-\frac{K}{2}, \frac{K}{2}]$ and thus

$$\begin{aligned}
&\left\|\,\|E_U\widetilde{g}_{x_2}\|^p_{L^p(w_{\widetilde{B}_K,10E})}\right\|_{L^1([-\frac{K}{2},\frac{K}{2}])} \\
&\quad= \int_{-\frac{K}{2}}^{\frac{K}{2}}\int_{\mathbb{R}^2} |E_U\widetilde{g}(x_1,x_2,x_3)|^p w_{\widetilde{B}_K,10E}(x_1,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_3\,\mathrm{d}x_2 \\
&\quad= \int_{\mathbb{R}^3} |E_U\widetilde{g}(x_1,x_2,x_3)|^p \mathbb{1}_{[-\frac{K}{2},\frac{K}{2}]}(x_2) w_{\widetilde{B}_K,10E}(x_1,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&\quad\lesssim \int_{\mathbb{R}^3} |E_U\widetilde{g}(x)|^p w_{B_K,10E}(x)\,\mathrm{d}x = \|E_U\widetilde{g}\|^p_{L^p(w_{B_K,10E})}.
\end{aligned}$$

Thus the procedure modifies the right-hand side of (5.17) to

$$K^{\varepsilon p}\int_{-\frac{K}{2}}^{\frac{K}{2}}\left(\sum_U \|E_U\widetilde{g}_{x_2}\|^2_{L^p(w_{\widetilde{B}_K,10E})}\right)^{\frac{p}{2}}\mathrm{d}x_2 \le K^{\varepsilon p}\left(\sum_U \Big\| \|E_U\widetilde{g}_{x_2}\|^p_{L^p(w_{\widetilde{B}_K,10E})}\Big\|^{\frac{2}{p}}_{L^1([-\frac{K}{2},\frac{K}{2}])}\right)^{\frac{p}{2}}$$
$$\lesssim K^{\varepsilon p}\left(\sum_U \|E_U\widetilde{g}\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{p}{2}}.$$

This proves our **claim**

$$\|E_{S_L}\widetilde{g}\|_{L^p(B_K)} \lesssim_\varepsilon K^\varepsilon\left(\sum_U \|E_U\widetilde{g}\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}.$$

In order to get (5.15) in the third scenario, we still need to estimate the expression $\left(\sum_U \|E_U\widetilde{g}\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}$ further. Let V be the unique square in $\mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)$ that covers U. By the restriction on the line L, we have $U\cap\widetilde{S}_L = U$ and thus we can write

$$E_U\widetilde{g}(x) = E_U g(x) = E_V g(x) - E_{V\setminus U}.$$

By Minkovski's inequalities in L^p and l^2 we get

$$\left(\sum_U \|E_U\widetilde{g}\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} \le \left(\sum_V \|E_V g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} + \left(\sum_{V\setminus U} \|E_{V\setminus U} g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}.$$

Since every V is in $\mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)$, we can trivially estimate the first term

$$\left(\sum_V \|E_V g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} \le \left(\sum_{\beta\in\mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}.$$

For the second term we divide the rectangles $V\setminus U$ into squares $V'\in\mathrm{Part}_{K^{-1}}([0,1]^2)$ by using Minkovski's inequality

$$\|E_{V\setminus U}\|_{L^p(w_{B_K,10E})} \le \sum_{V'} \|E_{V'}\|_{L^p(w_{B_K,10E})}.$$

For each $V \setminus U$ we need $K^{\frac{1}{2}}(K^{\frac{1}{2}} - C_L)$ squares V'. Since $(V \setminus U) \cap S_L = \varnothing$, we have $V' \notin S_{\text{big}}$ and hence we can estimate further

$$\begin{aligned}
\|E_{V\setminus U}\|_{L^p(w_{B_K,10E})} &\leq \sum_{V'} \|E_{V'}\|_{L^p(w_{B_K,10E})} \\
&\leq \sum_{V'} K^{-2}\|E_{\alpha^*}\|_{L^p(w_{B_K,10E})} \\
&\leq K^{-1}\|E_{\alpha^*}\|_{L^p(w_{B_K,10E})}.
\end{aligned}$$

Since the number of rectangles $V \setminus U$ is $K^{\frac{1}{2}}$, we get

$$\begin{aligned}
\left(\sum_{V\setminus U} \|E_{V\setminus U}g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} &\leq \sum_{V\setminus U} \|E_{V\setminus U}g\|_{L^p(w_{B_K,10E})} \\
&\leq \sum_{V\setminus U} K^{-1}\|E_{\alpha^*}g\|_{L^p(w_{B_K,10E})} \\
&= K^{-\frac{1}{2}}\|E_{\alpha^*}g\|_{L^p(w_{B_K,10E})} \\
&\leq \|E_{\alpha^*}g\|_{L^p(w_{B_K,10E})}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left(\sum_{U} \|E_U\widetilde{g}\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} &\leq \left(\sum_{V} \|E_V g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} + \left(\sum_{V\setminus U} \|E_{V\setminus U}g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\beta\in \text{Part}_{K^{-\frac{1}{2}}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} + \|E_{\alpha^*}g\|_{L^p(w_{B_K,10E})}.
\end{aligned}$$

We conclude that (5.15) holds in the third scenario as

$$\begin{aligned}
&\|Eg\|_{L^p(B_K)} \leq \|\sum_{\alpha\subset S_L} E_\alpha g\|_{L^p(B_K)} + C'\|E_{\alpha^*}g\|_{L^p(w_{B_K,10E})} \\
&\lesssim_\varepsilon K^\varepsilon \left[\left(\sum_{\beta\in\text{Part}_{K^{-\frac{1}{2}}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}} + \left(\sum_{\alpha\in\text{Part}_{K^{-1}}([0,1])^2} \|E_\alpha g\|^2_{L^p(w_{B_K,10E})}\right)^{\frac{1}{2}}\right]
\end{aligned}$$

and we are done proving (5.15).

Using $w_{B_K,10E} \le w_{B_K,E}$, inequality (5.15) implies

$$\|Eg\|_{L^p(B_K)} \lesssim_\varepsilon$$

$$K^\varepsilon \left(\sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_K,E})} \right)^{\frac{1}{2}} + K^\varepsilon \left(\sum_{\beta \in \mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_K,E})} \right)^{\frac{1}{2}}$$

$$+ K^{C_2} \max_{\substack{\alpha_1,\alpha_2,\alpha_3 \\ K^{-2}\text{-transverse}}} \left(\prod_{i=1}^{3} \|E_{\alpha_i} g\|_{L^p(w_{B_K,10E})} \right)^{\frac{1}{3}}. \tag{5.18}$$

Raising the left-hand side of the above inequality to the power p, summing over $B_K \in \mathrm{Part}_K(B_R)$ and then raising it to the power $\frac{1}{p}$ gives

$$\left(\sum_{B_K \in \mathrm{Part}_K(B_R)} \|Eg\|^p_{L^p(B_K)} \right)^{\frac{1}{p}} = \|Eg\|_{L^p(B_R)}. \tag{5.19}$$

For the first term on the right-hand side the same process gives

$$K^\varepsilon \left(\sum_{B_K \in \mathrm{Part}_K(B_R)} \left(\sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_K,E})} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}$$

$$= K^\varepsilon \left(\sum_{B_K \in \mathrm{Part}_K(B_R)} \left(\sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)} \left(\int_{\mathbb{R}^3} |E_\alpha g(x)|^p w_{B_K,E} \,\mathrm{d}x \right)^{\frac{2}{p}} \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}$$

$$= K^\varepsilon \left(\sum_{B_K \in \mathrm{Part}_K(B_R)} \left\| \int_{\mathbb{R}^3} |E_\alpha g(x)|^p w_{B_K,E}(x) \,\mathrm{d}x \right\|_{l^{\frac{2}{p}}} \right)^{\frac{1}{p}}$$

$$\le K^\varepsilon \left(\left\| \int_{\mathbb{R}^3} |E_\alpha g(x)|^p \sum_{B_K \in \mathrm{Part}_K(B_R)} w_{B_K,E}(x) \,\mathrm{d}x \right\|_{l^{\frac{2}{p}}} \right)^{\frac{1}{p}}$$

$$\lesssim K^\varepsilon \left(\left\| \int_{\mathbb{R}^3} |E_\alpha g(x)|^p w_{B_R,E}(x) \,\mathrm{d}x \right\|_{l^{\frac{2}{p}}} \right)^{\frac{1}{p}}$$

$$= K^\varepsilon \left(\sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}}.$$

The first inequality is due to Minkovski's inequality in $l^{\frac{2}{p}}$ and the second inequality is justified by lemma 3.12. The second term is dealt identically. Thus

$$K^{\varepsilon}\left(\sum_{B_K\in \mathrm{Part}_K(B_R)}\left(\sum_{\alpha\in \mathrm{Part}_{K^{-1}}([0,1]^2)}\|E_\alpha g\|^2_{L^p(w_{B_K},E)}\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\lesssim K^{\varepsilon}\left(\sum_{\alpha\in \mathrm{Part}_{K^{-1}}([0,1]^2)}\|E_\alpha g\|^2_{L^p(w_{B_R},E)}\right)^{\frac{1}{2}} \tag{5.20}$$

and

$$K^{\varepsilon}\left(\sum_{B_K\in \mathrm{Part}_K(B_R)}\left(\sum_{\beta\in \mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)}\|E_\beta g\|^2_{L^p(w_{B_K},E)}\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\lesssim K^{\varepsilon}\left(\sum_{\beta\in \mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)}\|E_\beta g\|^2_{L^p(w_{B_R},E)}\right)^{\frac{1}{2}} \tag{5.21}$$

For the third term we calculate that

$$K^{C_2}\left(\sum_{B_K\in \mathrm{Part}_K(B_R)}\max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-2}\text{-transverse}}}\left(\prod_{i=1}^{3}\|E_{\alpha_i}g\|^p_{L^p(w_{B_K},10E)}\right)^{\frac{1}{3}}\right)^{\frac{1}{p}}$$

$$\le K^{C_2}\left(\sum_{B_K\in \mathrm{Part}_K(B_R)}\sum_{\alpha_i}\left(\prod_{i=1}^{3}\|E_{\alpha_i}g\|^p_{L^p(w_{B_K},10E)}\right)^{\frac{1}{3}}\right)^{\frac{1}{p}}$$

$$= K^{C_2}\left(\sum_{\alpha_i}\sum_{B_K\in \mathrm{Part}_K(B_R)}\left(\prod_{i=1}^{3}\|E_{\alpha_i}g\|^p_{L^p(w_{B_K},10E)}\right)^{\frac{1}{3}}\right)^{\frac{1}{p}}$$

$$\le \sum_{\alpha_i}K^{C_2}\left(\sum_{B_K\in \mathrm{Part}_K(B_R)}\left(\prod_{i=1}^{3}\|E_{\alpha_i}g\|^p_{L^p(w_{B_K},10E)}\right)^{\frac{1}{3}}\right)^{\frac{1}{p}},$$

where the sum is taken over all K^{-2}-transverse triples that are in $\mathrm{Part}_{K^{-1}}([0,1]^2)$. Then by (5.2) we have

$$\left(\sum_{B_K\in \mathrm{Part}_K(B_R)}\left(\prod_{i=1}^{3}\|E_{\alpha_i}g\|^p_{L^p(w_{B_K,10E})}\right)^{\frac{1}{3}}\right)^{\frac{1}{p}}$$

$$\leq \mathrm{Dec}_3(R^{-1},p,K^{-2},m,E)\left(\prod_{i=1}^{3}\sum_{q_i\in \mathrm{Part}_{R^{-\frac{1}{2}}}(\alpha_i)}\|E_{q_i}g\|^2_{L^p(w_{B_R,E})}\right)^{\frac{1}{6}}.$$

Since $\alpha_i\subset[0,1]^2$, we have

$$\sum_{q_i\in \mathrm{Part}_{R^{-\frac{1}{2}}}\alpha_i}\|E_{q_i}g\|^2_{L^p(w_{B_R,E})}\leq \sum_{\Delta\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1]^2)}\|E_{\Delta}g\|^2_{L^p(w_{B_R,E})}$$

and thus we further have

$$K^{C_2}\left(\sum_{B_K\in \mathrm{Part}_K(B_R)}\max_{\substack{\alpha_1,\alpha_2,\alpha_3\\ K^{-2}\text{-transverse}}}\left(\prod_{i=1}^{3}\|E_{\alpha_i}g\|^p_{L^p(w_{B_K,10E})}\right)^{\frac{1}{3}}\right)^{\frac{1}{p}}$$

$$\leq \sum_{\alpha_i}K^{C_2}\,\mathrm{Dec}_3(R^{-1},p,K^{-2},m,E)\left(\sum_{\Delta\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1]^2)}\|E_{\Delta}g\|^2_{L^p(w_{B_R,E})}\right)^{\frac{1}{2}}$$

$$\leq K^{C_2+6}\,\mathrm{Dec}_3(R^{-1},p,K^{-2},m,E)\left(\sum_{\Delta\in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1]^2)}\|E_{\Delta}g\|^2_{L^p(w_{B_R,E})}\right)^{\frac{1}{2}}. \tag{5.22}$$

The last inequality is due to the fact that $\#\,\mathrm{Part}_{K^{-1}}([0,1]^2)=K^2$.

Combining (5.19), (5.20), (5.21) and (5.22) means that (5.18) becomes the inequality in the proposition once we stipulate that C_ε is the maximum of all of the implicit constants in the argument and $C=C_2+6$. □

The following is analogous to proposition 5.10 for smaller partitioning cubes. This will allow us to iterate the scale of the partitioning cubes from big to small, which will give us the main argument for the proof of theorem 5.27.

Proposition 5.23. *Let $\tau \in \mathrm{Part}_\delta([0,1]^2)$, where $\delta \geq R^{-\frac{1}{2}}K^{2^{m-1}}$. Assume that for all $\delta' < 1$, we have $\mathrm{Dec}_2(\delta', p, \Gamma_2(10E)) \lesssim_\varepsilon \delta'^{-\varepsilon}$. Then if $R \geq K^{2^m}$ and $\frac{R}{K} \in \mathbb{N}$, we have*

$$\|E_\tau g\|_{L^p(w_{B_R,E})} \leq C_\varepsilon K^\varepsilon \left(\left(\sum_{\alpha \in \mathrm{Part}_{\delta K^{-1}}(\tau)} \|E_\alpha g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} + \left(\sum_{\beta \in \mathrm{Part}_{\delta K^{-\frac{1}{2}}}(\tau)} \|E_\beta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} \right) + K^C \,\mathrm{Dec}_3((R\delta^2)^{-1}, p, K^{-2}, m, 3E) \left(\sum_{\Delta \in \mathrm{Part}_{R^{-\frac{1}{2}}}(\tau)} \|E_\Delta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}},$$

where the constants C and C_ε are independent of δ, R, τ and K.

The idea is to extend the parabola over τ to the whole parabola $\mathbb{P}^2$ via an affine transformation. In [5], this technique is referred to as parabolic rescaling.

Proof. We write $\tau = a + [0,\delta]^2$, for some $a \in \mathbb{R}^2$. We consider a change of variables $\xi' = \frac{\xi - a}{\delta}$. Notice that the image of τ in this mapping is $[0,1]^2$. Hence, using the fact that $|c+b|^2 = |c|^2 + |b|^2 + 2c \cdot b$, for $c, b \in \mathbb{R}^n$, gives

$$\begin{aligned}
|E_\tau g(x)| &= \left| \int_\tau g(\xi) e\big(x \cdot (\xi, |\xi|^2)\big)\, \mathrm{d}\xi \right| \\
&= \left| \delta^2 \int_{[0,1]^2} g(\xi'\delta + a) e\big(x \cdot (\xi'\delta + a, |\xi'\delta + a|^2)\big)\, \mathrm{d}\xi' \right| \\
&= \left| e(x \cdot (a, |a|^2)) \delta^2 \int_{[0,1]^2} g(\xi'\delta + a) e\big(x \cdot (\xi_1'\delta, \xi_2'\delta, \delta^2|\xi'|^2 + 2\delta a \cdot \xi)\big)\, \mathrm{d}\xi' \right| \\
&= \left| \delta^2 \int_{[0,1]^2} g(\xi'\delta + a) e\Big(x_1\xi_1'\delta + x_2\xi_2'\delta + x_3(\delta^2|\xi'|^2 + \sum_{i=1}^2 2\delta a_i \xi_i')\Big)\, \mathrm{d}\xi' \right| \\
&= \left| \delta^2 \int_{[0,1]^2} g(\xi'\delta + a) e\big(\sum_{i=1}^2 \xi_i'\delta(x_i + 2a_i x_3) + x_3\delta^2|\xi'|^2\big)\, \mathrm{d}\xi' \right| \\
&= \left| \delta^2 \int_{[0,1]^2} g(\xi'\delta + a) e\Big(\big((x_1 + 2a_1x_3)\delta, (x_2 + 2a_2x_3)\delta, \delta^2 x_3\big) \cdot (\xi', |\xi'|^2)\Big)\, \mathrm{d}\xi' \right| \\
&=: \delta^2 |Eg'(Tx)|. \qquad (5.24)
\end{aligned}$$

In the last line we denoted $g'(\xi) := g(\xi\delta + a)$ and $Tx := \big((x_1+2a_1x_3)\delta, (x_2+2a_2x_3)\delta, \delta^2 x_3\big)$. Note that the Jacobian determinant of T is δ^4.

Then we cover the image of B_R under T with a pairwise disjoint family of cubes B' with side length $\delta^2 R$ that satisfy

$$\mathbb{1}_{T(B_R)}(x) \lesssim \sum_{B'} w_{B',3E}(x) \lesssim w_{B_R,E}(T^{-1}x).$$

The proof of the existence of the above covering can be found in [14] section 5.2 lemma 5.2.2. Now we change variables to get

$$\begin{aligned}
\|E_\tau g\|_{L^p(B_R)} &= \delta^2 \left(\int_{B_R} |Eg'(Tx)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\
&= \delta^2 \left(\delta^{-4} \int_{T(B_R)} |Eg'(y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}} \\
&\lesssim \delta^{2-\frac{4}{p}} \left(\int_{\mathbb{R}^3} |Eg'(y)|^p \sum_{B'} w_{B',3E}(y) \,\mathrm{d}y \right)^{\frac{1}{p}} \\
&= \delta^{2-\frac{4}{p}} \left(\sum_{B'} \|Eg'\|^p_{L^p(w_{B',3E})} \right)^{\frac{1}{p}}. \qquad (5.25)
\end{aligned}$$

Since $\delta \geq R^{-\frac{1}{2}} K^{2^{m-1}} \Leftrightarrow \delta^2 R \geq K^{2^m}$, proposition 5.10 yields

$$\begin{aligned}
\|Eg'\|_{L^p(w_{B',3E})} \leq C_\varepsilon K^\varepsilon \Bigg(&\Bigg(\sum_{\alpha \in \mathrm{Part}_{K^{-1}}([0,1]^2)} \|E_\alpha g'\|^2_{L^p(w_{B',3E})} \Bigg)^{\frac{1}{2}} \\
&+ \Bigg(\sum_{\beta \in \mathrm{Part}_{K^{-\frac{1}{2}}}([0,1]^2)} \|E_\beta g'\|^2_{L^p(w_{B',3E})} \Bigg)^{\frac{1}{2}} \Bigg) \\
&+ K^C \operatorname{Dec}_3((\delta^2 R)^{-1}, p, K^{-2}, m, 3E) \Bigg(\sum_{\Delta \in \mathrm{Part}_{(\delta^2 R)^{-\frac{1}{2}}}([0,1]^2)} \|E_\Delta g'\|^2_{L^p(w_{B',3E})} \Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Remark 5.26. In the above inequality we must choose $\Gamma_2(10E)$ to be at least $60E+, p+1$. This comes from the fact that we must apply theorem 4.27 with the weight exponent being $30E$ and thus in lemma 4.13 we need $W \geq 60E + p + 1$.

Plugging the first term into (5.25) gives

$$\delta^{2-\frac{4}{p}}\left(\sum_{B'}C_\varepsilon^p K^{\varepsilon p}\left(\sum_{\alpha\in\mathrm{Part}_{K^{-1}}([0,1]^2)}\|E_\alpha g'\|^2_{L^p(w_{B',3E})}\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\begin{aligned}
&= C_\varepsilon K^\varepsilon \delta^{2-\frac{4}{p}}\left(\sum_{B'}\left\|\|E_\alpha g'\|^p_{L^p(w_{B',3E})}\right\|_{l^{\frac{2}{p}}}\right)^{\frac{1}{p}}\\
&\le C_\varepsilon K^\varepsilon \delta^{2-\frac{4}{p}}\left(\left\|\sum_{B'}\|E_\alpha g'\|^p_{L^p(w_{B',3E})}\right\|_{l^{\frac{2}{p}}}\right)^{\frac{1}{p}}\\
&= C_\varepsilon K^\varepsilon \delta^{2-\frac{4}{p}}\left(\left\|\|E_\alpha g'\|^p_{L^p(\sum_{B'}w_{B',3E})}\right\|_{l^{\frac{2}{p}}}\right)^{\frac{1}{p}}\\
&\lesssim C_\varepsilon K^\varepsilon \delta^{2-\frac{4}{p}}\left(\left\|\|E_\alpha g'\|^p_{L^p(w_{B_R,E}\circ T^{-1})}\right\|_{l^{\frac{2}{p}}}\right)^{\frac{1}{p}}\\
&= C_\varepsilon K^\varepsilon \delta^{2-\frac{4}{p}}\left(\sum_{\alpha\in\mathrm{Part}_{K^{-1}}([0,1]^2)}\|E_\alpha g'\|^2_{L^p(w_{B_R,E}\circ T^{-1})}\right)^{\frac{1}{2}}.
\end{aligned}$$

Then changing the variables back gives

$$\begin{aligned}
\delta^{2-\frac{4}{p}}\|E_\alpha g'\|_{L^p(w_{B_R,E}\circ T^{-1})} &= \delta^2\left(\int_{\mathbb{R}^3}|E_\alpha g'(y)|^p w_{B_R,E}(T^{-1}y)\delta^{-4}\,\mathrm{d}y\right)^{\frac{1}{p}}\\
&= \delta^2\left(\int_{\mathbb{R}^3}|E_\alpha g'(Tx)|^p w_{B_R,E}(x)\,\mathrm{d}x\right)^{\frac{1}{p}}
\end{aligned}$$

and as $\xi\mapsto\xi\delta+a$ maps cubes in $\mathrm{Part}_{K^{-1}}([0,1]^2)$ into cubes in $\mathrm{Part}_{\delta K^{-1}}(\tau)$, we get similarly as in (5.24) that

$$\begin{aligned}
\delta^2|E_\alpha g'(Tx)| &= \left|\delta^2\int_\alpha g(\xi'\delta+a)e\big(Tx\cdot(\xi',|\xi'|^2)\big)\,\mathrm{d}\xi'\right|\\
&= \left|\int_{\alpha'} g(\xi)e\big(x\cdot(\xi,|\xi|^2)\big)\,\mathrm{d}\xi\right| = |E_{\alpha'}g(x)|,
\end{aligned}$$

where $\alpha'\in\mathrm{Part}_{\delta K^{-1}}(\tau)$.

The same deduction applies to the other two terms and thus we have shown that

$$\|E_\tau g\|_{L^p(B_R)} \le C_\varepsilon K^\varepsilon \left(\left(\sum_{\alpha \in \mathrm{Part}_{\delta K^{-1}}(\tau)} \|E_\alpha g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} + \left(\sum_{\beta \in \mathrm{Part}_{\delta K^{-\frac{1}{2}}}(\tau)} \|E_\beta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} \right) + K^C \mathrm{Dec}_3((R\delta^2)^{-1}, p, K^{-2}, m, 3E) \left(\sum_{\Delta \in \mathrm{Part}_{R^{-\frac{1}{2}}}(\tau)} \|E_\Delta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}}$$

and now lemma 4.3 gives us the result. □

5.3 Reverse decoupling constant inequality

In section 5.1 we showed that one can dominate the multilinear decoupling constant with the linear decoupling constant. In this section we will show the following reverse inequality which states that we can somehow dominate the linear decoupling constant with the multilinear decoupling constant. The proof is essentially multiple iterations of proposition 5.23 and careful accounting of the additional coefficients that arise from the iteration. The same proof applies in all dimensions as long as propositions 5.10 and 5.23 have analogous counterparts in $\mathbb{R}^n$ for $n \neq 3$.

Theorem 5.27. *Let $E \ge 100n$ and $\varepsilon > 0$. Let also $\Gamma_n(E)$ be a large enough constant dependent on E and n. Assume that one of the following holds*

- $n = 2$.
- $n \ge 3$ *and* $\mathrm{Dec}_{n-1}(\delta, p, \Gamma_{n-1}(10E)) \lesssim_{\varepsilon,E} \delta^{-\varepsilon}$.

Then for each $0 < \nu < 1$, such that $\frac{R}{\nu^{-\frac{1}{2}}} \in \mathbb{N}$, there exists a function $\epsilon(\nu) = \epsilon(\nu, p, E)$ with $\lim_{\nu \to 0} \epsilon(\nu) = 0$ and a constant $C_{\nu,m}$ such that for each $m \ge 1$, we have

$$\mathrm{Dec}_n(R^{-1}, p, E) \le C_{\nu,m} R^{\epsilon(\nu)} \left(1 + \sup_{\substack{1 \le R' \le R \\ E^* \in \{E, 3E\}}} \mathrm{Dec}_n(R'^{-1}, p, \nu, m, E^*) \right)$$

for each $R \ge \nu^{-2^{m-1}}$.

The harmless restriction of $\frac{R}{\nu^{-\frac{1}{2}}} \in \mathbb{N}$ comes from working with essential partitions in lemma 3.12 (see the end of the proof of proposition 5.10 on the bottom of page 105 and the top of page 106). With finitely overlapping covers this restriction is not needed. Note also that theorem 5.27 is custom made for the induction proof of the l^2 decoupling theorem. For $n \geq 3$ we assume an inequality which is similar to the l^2 decoupling theorem in the lower dimensional space $\mathbb{R}^{n-1}$. We will prove the theorem for $n = 3$.

Proof. Let $K = \nu^{-\frac{1}{2}}$ and $R \geq K^{2^m} = \nu^{-2^{m-1}}$. Our plan is to iterate proposition 5.23 starting from scale $\delta = 1$ until we reach scales $\delta = R^{-\frac{1}{2}}K^{2^{m-1}}$ and $\delta = R^{-\frac{1}{2}}K^{2^{m-1}-\frac{1}{2}}$. Each iteration lowers the scale from δ to at least $\delta K^{-\frac{1}{2}}$. This gives us intuition to the fact that we must iterate at most $M := \log_K R - 2^m$ times since

$$K^{-\frac{M}{2}} = (K^{-\frac{1}{2}})^{\log_K R - 2^m} = \frac{1}{(K^{\frac{1}{2}})^{\log_K R}} K^{2^{m-1}} = \frac{1}{(K^{\log_K R})^{\frac{1}{2}}} K^{2^{m-1}} = R^{-\frac{1}{2}} K^{2^{m-1}}.$$

We note that $M \geq 0$, since by assumption we have $R^{-\frac{1}{2}}K^{2^{m-1}} = (R^{-1}K^{2^m})^{\frac{1}{2}} \leq 1$ and $K > 1$. We denote $\mathrm{MDec}_3(\delta) := \sup_{E^* \in \{E, 3E\}} \mathrm{Dec}_3(\delta, p, K^{-2}, m, E^*)$ and

$$I_\delta(Q) := \left(\sum_{\alpha \in \mathrm{Part}_\delta Q} \|E_\alpha g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}},$$

where Q is a cube. Furthermore denote $I_\delta([0,1]^2) := I_\delta$. Now we can use proposition 5.10 to write

$$\begin{aligned} \|Eg\|_{L^p(w_{B_R,E})} &\leq C_\varepsilon K^\varepsilon (I_{K^{-1}} + I_{K^{-\frac{1}{2}}}) + K^C \mathrm{Dec}_3(R^{-1}, p, K^{-2}, m, E) I_{R^{-\frac{1}{2}}} \\ &\leq C_\varepsilon K^\varepsilon (I_{K^{-1}} + I_{K^{-\frac{1}{2}}}) + K^C \mathrm{MDec}_3(R^{-1}) I_{R^{-\frac{1}{2}}}. \end{aligned} \tag{5.28}$$

We want to retain the $I_{R^{-\frac{1}{2}}}$ terms so we will use proposition 5.23 to $\|E_\alpha g\|_{L^p(w_{B_R,E})}$ in the terms $I_{K^{-1}}$ and $I_{K^{-\frac{1}{2}}}$.

For an arbitrary scale $\delta \geq R^{-\frac{1}{2}}K^{2^{m-1}}$ proposition 5.23 and the Minkovski's inequality of l^2 gives that

$$I_\delta = \left(\sum_{\alpha \in \mathrm{Part}_\delta([0,1]^2)} \|E_\alpha g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{\alpha \in \mathrm{Part}_\delta([0,1]^2)} \left[C_\varepsilon K^\varepsilon \big(I_{\delta K^{-1}}(\alpha) + I_{\delta K^{-\frac{1}{2}}}(\alpha) \big) + K^C \mathrm{MDec}_3(R^{-1}\delta^{-2}) I_{R^{-\frac{1}{2}}}(\alpha) \right]^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{\alpha \in \mathrm{Part}_\delta([0,1]^2)} C_\varepsilon^2 K^{2\varepsilon} I_{\delta K^{-1}}(\alpha)^2 \right)^{\frac{1}{2}} + \left(\sum_{\alpha \in \mathrm{Part}_\delta([0,1]^2)} C_\varepsilon^2 K^{2\varepsilon} I_{\delta K^{-\frac{1}{2}}}(\alpha)^2 \right)^{\frac{1}{2}}$$

$$+ \left(\sum_{\alpha \in \mathrm{Part}_\delta([0,1]^2)} K^{2C} \, \mathrm{MDec}_3(R^{-1}\delta^{-2})^2 I_{R^{-\frac{1}{2}}}(\alpha)^2 \right)^{\frac{1}{2}}$$

and for every scale $\delta' \leq \delta$, we have the following property

$$\left(\sum_{\alpha \in \mathrm{Part}_\delta([0,1]^2)} I_{\delta'}(\alpha)^2 \right)^{\frac{1}{2}} = \left(\sum_{\alpha \in \mathrm{Part}_\delta([0,1]^2)} \sum_{\alpha' \in \mathrm{Part}_{\delta'} \alpha} \|E_{\alpha'} g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}}$$

$$= \left(\sum_{\alpha' \in \mathrm{Part}_{\delta'}([0,1]^2)} \|E_{\alpha'} g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} = I_{\delta'}.$$

Since $C_\varepsilon K^\varepsilon \geq 1$, we now get

$$I_\delta \leq C_\varepsilon K^\varepsilon \left(I_{\delta K^{-1}} + I_{\delta K^{-\frac{1}{2}}} + K^C \, \mathrm{MDec}_3(R^{-1}\delta^{-2}) I_{R^{-\frac{1}{2}}} \right), \tag{5.29}$$

for every scale $\delta \geq R^{-\frac{1}{2}} K^{2^{m-1}} = K^{-\frac{M}{2}}$. We note that since $1 \geq \delta \geq R^{-\frac{1}{2}} K^{2^{m-1}}$ we have $R^{-1} \leq R^{-1}\delta^{-2} \leq K^{-2^m} \leq 1$ and thus we may estimate

$$\mathrm{MDec}_3(R^{-1}\delta^{-2}) \leq \sup_{1 \leq R' \leq R} \mathrm{MDec}_3((R')^{-1}) =: \mathrm{MDec}\,.$$

Now (5.29) simplifies to

$$I_\delta \leq C_\varepsilon K^\varepsilon \left(I_{\delta K^{-1}} + I_{\delta K^{-\frac{1}{2}}} + K^C \, \mathrm{MDec} \; I_{R^{-\frac{1}{2}}} \right), \tag{5.30}$$

for each $\delta \geq R^{-\frac{1}{2}} K^{2^{m-1}} = K^{-\frac{M}{2}}$.

The problem with the above estimate is that the lower bound for the scale δ brings technical difficulties to the argument. To get rid of these difficulties we will look for an inequality that holds also with smaller scales than $K^{-\frac{M}{2}}$. The following is a simplification of the authors original argument that was suggested by T. Hytönen. To facilitate this argument, we denote

$$I_{R^*} := \max\{I_{K^{-\frac{M}{2}}}, I_{K^{-\frac{M+1}{2}}}\} = \max\{I_{R^{-\frac{1}{2}} K^{2^{m-1}}}, I_{R^{-\frac{1}{2}} K^{2^{m-1}-\frac{1}{2}}}\}$$

and

$$J_\delta := \begin{cases} I_\delta, & \text{if } \delta > K^{-\frac{M}{2}}, \\ I_{R^*}, & \text{if } \delta \le K^{-\frac{M}{2}}, \end{cases}$$

for each $\delta = K^{-\frac{N}{2}}$, where $N \in \mathbb{N}$. We now claim that (5.30) implies

$$J_\delta \le C_\varepsilon K^\varepsilon \left(J_{\delta K^{-1}} + J_{\delta K^{-\frac{1}{2}}} + K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right) \tag{5.31}$$

To show this, we consider four cases. If $K^{-1}\delta > K^{-\frac{M}{2}}$, then all J terms in (5.31) coincide with the corresponding I terms and (5.30) is exactly (5.31). If $\delta \le K^{-\frac{M}{2}}$, then all J terms in (5.31) are equal to I_{R^*} and (5.31) is clear, since $C_\varepsilon K^\varepsilon \ge 1$. Now it only remains to consider $\delta \in \{K^{-\frac{M}{2}+\frac{1}{2}}, K^{-\frac{M}{2}+1}\}$. If $\delta = K^{-\frac{M}{2}+1}$, then

$$J_\delta = I_\delta, \qquad J_{\delta K^{-\frac{1}{2}}} = I_{\delta K^{-\frac{1}{2}}}, \qquad J_{\delta K^{-1}} = I_{R^*} \ge I_{K^{-\frac{M}{2}}} = I_{\delta K^{-1}},$$

so that from (5.30) we get (5.31). Lastly, if $\delta = K^{-\frac{M}{2}+\frac{1}{2}}$, then

$$J_\delta = I_\delta, \qquad J_{\delta K^{-\frac{1}{2}}} = I_{R^*} \ge I_{K^{-\frac{M}{2}}} = I_{\delta K^{-\frac{1}{2}}}, \qquad J_{\delta K^{-1}} = I_{R^*} \ge I_{K^{-\frac{M+1}{2}}} = I_{\delta K^{-1}}$$

and again from (5.30) we get (5.31). This concludes the proof of (5.31).

Let us stress the fact that in (5.31) there is no lower bound for δ, which means that it is best to iterate (5.31). Let us note that $\mathrm{Part}_1([0,1]^2) = \{[0,1]^2\}$ and hence

$$I_1 = \|Eg\|_{L^p(w_{B_R,E})}.$$

If $M > 0$, then $J_1 = I_1$. If $M = 0$, then $J_1 = I_{R^*} = \max\{I_1, I_{K^{-\frac{1}{2}}}\} \ge I_1$. Thus in any case, we have

$$I_1 \le J_1.$$

At each iteration the scale decreases by a small step of $K^{-\frac{1}{2}}$, a big step of K^{-1} or it jumps to the scale $R^{-\frac{1}{2}}$. The terms of this last type is no linger iterated, but new such terms are generated at each iteration. After $N \le M$ iterations we may end up in one of the following scenarios. We may have taken $s \le N$ iterations with small step $K^{-\frac{1}{2}}$ and $N-s$ iterations with big step K^{-1}, resulting in a final scale $K^{-\frac{s}{2}}K^{-N+s} = K^{-(N-\frac{s}{2})}$. The possible order of such iterations may have occurred in $\binom{N}{s}$ different ways. On the other hand, after any number of small step and large step iterations before the scales $K^{-\frac{M}{2}}$ and $K^{-\frac{M+1}{2}}$, there is a possibility of taking a jump to the scale $R^{-\frac{1}{2}}$ as the last iteration. Note also that both small step and large step iterations produce a factor $C_\varepsilon K^\varepsilon$, whereas a jump to the scale $R^{-\frac{1}{2}}$ (which can only happen once, as the last iteration) produces the factor K^C MDec.

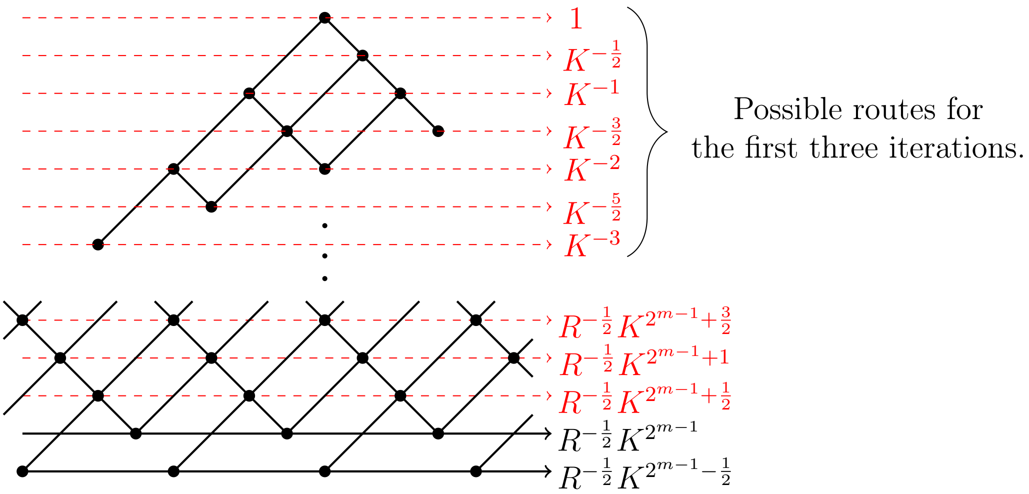

Figure 5.1: The iteration tree that describes how the scale decreases with every iteration.

Estimating the possible routes of iteration, we are led to the following bound

$$J_1 \leq (C_\varepsilon K^\varepsilon)^N \left(\sum_{s=0}^{N} \binom{N}{s} J_{K^{-(N-\frac{s}{2})}} + 2^N K^C \operatorname{MDec} I_{R^{-\frac{1}{2}}} \right), \tag{5.32}$$

for each $N \in \mathbb{N}$. While we have already sketched the combinatorial reasons behind the above estimate, let us still provide a rigorous proof by induction on N.

Base case $\mathbf{N = 0}$: In this case (5.32) reduces to

$$J_1 \leq J_1 + K^C \operatorname{MDec} I_{R^{-\frac{1}{2}}},$$

which is trivially true.

The induction step: We assume that (5.32) holds for some $N \geq 0$. Since $\sum_{s=0}^{N} \binom{N}{s} = 2^N$, an application of (5.31) gives that

$$\begin{aligned}
\sum_{s=0}^{N} \binom{N}{s} J_{K^{-(N-\frac{s}{2})}} &\leq C_\varepsilon K^\varepsilon \sum_{s=0}^{N} \binom{N}{s} \left(J_{K^{-(N+1-\frac{s}{2})}} + J_{K^{-(N+\frac{1}{2}-\frac{s}{2})}} + K^C \operatorname{MDec} I_{R^{-\frac{1}{2}}} \right) \\
&= C_\varepsilon K^\varepsilon \Bigg[\sum_{s=0}^{N} \binom{N}{s} \left(J_{K^{-(N+1-\frac{s}{2})}} + J_{K^{-(N+\frac{1}{2}-\frac{s}{2})}} \right) \\
&\qquad\qquad + \sum_{s=0}^{N} \binom{N}{s} K^C \operatorname{MDec} I_{R^{-\frac{1}{2}}} \Bigg]
\end{aligned}$$

$$= C_\varepsilon K^\varepsilon \left[\sum_{s=0}^{N} \binom{N}{s} \left(J_{K^{-(N+1-\frac{s}{2})}} + J_{K^{-(N+\frac{1}{2}-\frac{s}{2})}} \right) + 2^N K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right]. \tag{5.33}$$

Furthermore, we have

$$\sum_{s=0}^{N} \binom{N}{s} J_{K^{-(N+\frac{1}{2}-\frac{s}{2})}} = \sum_{s=0}^{N} \binom{N}{s} J_{K^{-(N+1-\frac{s+1}{2})}} = \sum_{t=1}^{N+1} \binom{N}{t-1} J_{K^{-(N+1-\frac{t}{2})}}.$$

Thus by denoting $\binom{N}{-1} = \binom{N}{N+1} = 0$ and applying Pascal's formula $\binom{N}{s} + \binom{N}{s-1} = \binom{N+1}{s}$, we get

$$\begin{aligned} \sum_{s=0}^{N} \binom{N}{s} \left(J_{K^{-(N+1-\frac{s}{2})}} + J_{K^{-(N+\frac{1}{2}-\frac{s}{2})}} \right) &= \sum_{s=0}^{N+1} \left[\binom{N}{s} + \binom{N}{s-1} \right] J_{K^{-(N+1-\frac{s}{2})}} \\ &= \sum_{s=0}^{N+1} \binom{N+1}{s} J_{K^{-(N+1-\frac{s}{2})}}. \end{aligned}$$

Substituting this into (5.33) yields

$$\sum_{s=0}^{N} \binom{N}{s} J_{K^{-(N-\frac{s}{2})}} \le C_\varepsilon K^\varepsilon \left[\sum_{s=0}^{N+1} \binom{N+1}{s} J_{K^{-(N+1-\frac{s}{2})}} + 2^N K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right].$$

This combined with the induction assumption and $C_\varepsilon K^\varepsilon \ge 1$ results in

$$\begin{aligned} J_1 \le& (C_\varepsilon K^\varepsilon)^N \left(\sum_{s=0}^{N} \binom{N}{s} J_{K^{-(N-\frac{s}{2})}} + 2^N K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right) \\ &\le (C_\varepsilon K^\varepsilon)^N \Bigg(C_\varepsilon K^\varepsilon \Bigg[\sum_{s=0}^{N+1} \binom{N+1}{s} J_{K^{-(N+1-\frac{s}{2})}} \\ &\qquad\qquad + 2^N K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \Bigg] + 2^N K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \Bigg) \\ &\le (C_\varepsilon K^\varepsilon)^{N+1} \left(\sum_{s=0}^{N+1} \binom{N+1}{s} J_{K^{-(N+1-\frac{s}{2})}} + 2^{N+1} K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right). \end{aligned}$$

This concludes the proof of (5.32) by induction.

By choosing $N = M$ in (5.32), we have $N - \frac{s}{2} = M - \frac{s}{2} \geq M - \frac{M}{2} = \frac{M}{2}$ and hence all J terms on the right-hand side of (5.32) are equal to I_{R^*}. This means that we get

$$\begin{aligned}
I_1 \leq J_1 &\leq (C_\varepsilon K^\varepsilon)^M \left(\sum_{s=0}^{M} \binom{M}{s} I_{R^*} + 2^M K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right) \\
&= (C_\varepsilon K^\varepsilon)^M \left(2^M I_{R^*} + 2^M K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right) \\
&= (2C_\varepsilon K^\varepsilon)^M \left(I_{R^*} + K^C \,\mathrm{MDec}\; I_{R^{-\frac{1}{2}}} \right) \qquad (5.34)
\end{aligned}$$

Recall that $I_{R^*} = \max\{I_{R^{-\frac{1}{2}}K^{2^{m-1}}}, I_{R^{-\frac{1}{2}}K^{2^{m-1}-\frac{1}{2}}}\}$. Next we will show that

$$I_{R^*} \leq K^{2^{m-1}} I_{R^{-\frac{1}{2}}}. \qquad (5.35)$$

An application of Minkovski's inequality and Cauchy-Schwartz inequality gives

$$\begin{aligned}
\|E_\beta g\|_{L^p(w_{B_R,E})} &\leq \sum_{\beta' \in \mathrm{Part}_{R^{-\frac{1}{2}}} \beta} \|E_{\beta'} g\|_{L^p(w_{B_R,E})} \\
&\leq \left(\sum_{\beta' \in \mathrm{Part}_{R^{-\frac{1}{2}}} \beta} 1 \right)^{\frac{1}{2}} \left(\sum_{\beta' \in \mathrm{Part}_{R^{-\frac{1}{2}}} \beta} \|E_{\beta'} g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} \\
&= K^{2^{m-1}} \left(\sum_{\beta' \in \mathrm{Part}_{R^{-\frac{1}{2}}} \beta} \|E_{\beta'} g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}},
\end{aligned}$$

where $\beta \subset [0,1]^2$ is a cube of side length $R^{-\frac{1}{2}} K^{2^{m-1}}$. Thus

$$\begin{aligned}
I_{R^{-\frac{1}{2}}K^{2^{m-1}}} &= \left(\sum_{\beta \in \mathrm{Part}_{R^{-\frac{1}{2}}K^{2^{m-1}}}([0,1]^2)} \|E_\beta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\beta \in \mathrm{Part}_{R^{-\frac{1}{2}}K^{2^{m-1}}}([0,1]^2)} K^{2^m} \sum_{\beta' \in \mathrm{Part}_{R^{-\frac{1}{2}}} \beta} \|E_{\beta'} g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} \\
&= K^{2^{m-1}} \left(\sum_{\Delta \in \mathrm{Part}_{R^{-\frac{1}{2}}}([0,1]^2)} \|E_\Delta g\|^2_{L^p(w_{B_R,E})} \right)^{\frac{1}{2}} = K^{2^{m-1}} I_{R^{-\frac{1}{2}}}.
\end{aligned}$$

For the smaller scale $\delta = R^{-\frac{1}{2}}K^{2^{m-1}-\frac{1}{2}}$ we similarly have

$$I_{R^{-\frac{1}{2}}K^{2^{m-1}-\frac{1}{2}}} \leq K^{2^{m-1}-\frac{1}{2}} I_{R^{-\frac{1}{2}}} \leq K^{2^{m-1}} I_{R^{-\frac{1}{2}}}.$$

We have now shown (5.35) and plugging it to (5.34) gives

$$\begin{aligned} \|Eg\|_{L^p(w_{B_R,E})} = I_1 &\leq (2C_\varepsilon K^\varepsilon)^M \left(K^{2^{m-1}} I_{R^{-\frac{1}{2}}} + K^C \,\mathrm{MDec}\, I_{R^{-\frac{1}{2}}}\right) \\ &\leq (2C_\varepsilon K^\varepsilon)^M K^{2^{m-1}+C}(1+\mathrm{MDec}) I_{R^{-\frac{1}{2}}} \end{aligned} \tag{5.36}$$

Recall that $K = \nu^{-\frac{1}{2}}$ and $M = \log_K R - 2^m$. Using this we calculate that

$$\begin{aligned} (2C_\varepsilon K^\varepsilon)^M K^{2^{m-1}+C} &\leq (2C_\varepsilon K^\varepsilon)^{\log_K R} \nu^{-2^{m-2}-\frac{C}{2}} \\ &= R^{\log_K(2C_\varepsilon)+\varepsilon} \nu^{-2^{m-2}-\frac{C}{2}} \\ &= R^{-2\log_\nu(2C_\varepsilon)+\varepsilon} \nu^{-2^{m-2}-\frac{C}{2}}. \end{aligned} \tag{5.37}$$

The first equality is justified by

$$C^{\log_K R} = (K^{\log_K C})^{\log_K R} = (K^{\log_K R})^{\log_K C} = R^{\log_K C}, \quad C > 0$$

and the last equality is due to the change of base formula of the logarithm

$$\log_K C = \log_{\nu^{-\frac{1}{2}}} C = \frac{\log_\nu C}{\log_\nu \nu^{-\frac{1}{2}}} = -2\log_\nu C, \quad C > 0.$$

Recall that we denoted $\mathrm{MDec} = \sup_{1 \leq R' \leq R} \mathrm{MDec}_3\left((R')^{-1}\right)$. Plugging (5.37) to (5.36) gives us

$$\|Eg\|_{L^p(w_{B_R,E})} \leq C_{\nu,m} R^{\epsilon(\nu)} \left(1 + \sup_{1 \leq R' \leq R} \mathrm{MDec}_3\left((R')^{-1}\right)\right) I_{R^{-\frac{1}{2}}},$$

where we choose $C_{\nu,m} = \nu^{-2^{m-2}-\frac{C}{2}}$ and $\epsilon(\nu) = -2\log_\nu(2C_\varepsilon) + \varepsilon$. It remains to check that $\lim_{\nu \to 0} \epsilon(\nu) = 0$. Since ε is arbitrary, it suffices to check that $\log_\nu(2C_\varepsilon) \to 0$ as $\nu \to 0$. This follows directly from the change of base formula

$$\log_\nu(2C_\varepsilon) = \frac{\ln 2C_\varepsilon}{\ln \nu} \underset{\nu \to 0}{\longrightarrow} 0.$$

□

As a recap, the proof of theorem 5.27 for $n = 3$ utilized the fact that the contribution coming from the squares β that exist near a line can be controlled by a lower dimensional decoupling constant $\mathrm{Dec}_2(\delta, p) \lesssim_\varepsilon \delta^{-\varepsilon}$ via the Fourier support decoupling theorem 4.27. When $n \geq 4$, the contribution coming from the cubes near a hyperplane H in $[0,1]^{n-1}$ will be similarly controlled by $\mathrm{Dec}_{n-1}(\delta, p)$. This is due to the fact that the maximum and minimum curvatures of the paraboloid over H are comparable to 1 uniformly over the hyperplanes H. When $n = 2$ there is no such lower dimensional contribution.

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