$\mathcal{P}$-Fredholmness of Band-dominated Operators, and its Equivalence to Invertibility of Limit Operators and the Uniform Boundedness of Their Inverses

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Tiivistelmä – Referat – Abstract

In the thesis ”P-Fredholmness of Band-dominated Operators, and its Equivalence to Invertibility of Limit Operators and the Uniform Boundedness of Their Inverses”, we present the generalization of the classical Fredholm-Riesz theory with respect to a sequence of approximating projections on direct sums of spaces. The thesis is a progessive introduction to understanding and proving the core result in the generalized Fredholm-Riesz theory, which is stated in the title. The stated equivalence has been further improved and it can be generalized further by omitting either the initial condition of richness of the operator or the uniform boundedness criterion. Our focal point is on the elementary form of this result.

We lay the groundwork for the classical Fredholm-Riesz theory by introducing compact operators and defining Fredholmness as invertibility on modulo compact operators. Thereafter we introduce the concept of approximating projections in infinite direct sums of Banach spaces, that is we operate continuous operators with a sequence of projections which approach the identity operator in the limit and examine whether we have convergence in the norm sense. This method yields us a way to define P-compactness, P-strong converngence and finally P-Fredholmness.

We introduce the notion of limit operators operators by first shifting, then operating and then shifting back an operator with respect to an element in a sequence and afterwards investigating what happens in the P-strong limit of this sequence. Furthermore we define band-dominated operators as uniform limits of linear combinations of simple multiplication and shift operators. In this subspace of operators we prove that indeed for rich operators the core result holds true.

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Chapter 1

Introduction

We assume the reader has a basic understanding of the concepts of functional analysis, such as Hahn-Banach theorems, spaces of operators and norms, Banach-Steinhaus theorem and duality. We start building on this with the concept of compact operators and their properties. This is a necessary backround for introducing the Fredholm-Riesz theory which is the motivator for our main goal since we will work in the realm of generalizing compactness and Fredholm-Riesz theory and the new concepts the process gives rise to.

Let $\mathcal{L}(X, Y)$ be the space of bounded linear operators from X to Y and $\mathcal{K}(X, Y)$ the space of compact operators from X to Y. Whenever $X = Y$, we denote $\mathcal{L}(X, X) = \mathcal{L}(X)$ and $\mathcal{K}(X, X) = \mathcal{K}(X)$. In classical Fredholm-Riesz theory we deal with bounded linear operators with finite-dimensional kernels and cokernels. These type of operators are called Fredholm-operators and they have powerful properties. For example, the definition of Fredholm-operators be equivalent with invertibility modulo compact-operators. This equivalence is called the Atkinson theorem, which states that an operator $A \in \mathcal{L}(X)$ is Fredholm if and only if it is invertible in the quotient space $\mathcal{L}(X)/\mathcal{K}(X)$.

The space in which we work, is the l^p-space $E = l^p(Z^N, X)$, where X is a fixed Banach space. We generalize the concept of compactness by operating with a fixed sequence $\mathcal{P}$ of projections $(P_1, P_2, ...)$, from E to E, where $P_n \to I$ as $n \to \infty$. For $\mathcal{P}$-compactness of an operator $K \in \mathcal{L}(E)$ we require for both $P_n K$ and $K P_n$ to approach K in the norm-sense as $n \to \infty$. These so called $\mathcal{P}$-compact operators create their own space $\mathcal{K}(E, \mathcal{P})$ of operators on X which is the basis for the concepts of $\mathcal{P}$-strong convergence, $\mathcal{P}$-Fredholmness and the generalization of the space $\mathcal{L}(E)$ into $\mathcal{L}(E, \mathcal{P})$.

Our main results involve the new concepts of band-dominated operators and limit operators. Band operators are finite linear combinations of shift and multiplication operators. Band-dominated operators in turn are limits of band operators. From band operators we continue to limit operators. The limit operator A_h of $A \in \mathcal{L}(E, \mathcal{P})$ is the $\mathcal{P}$-strong limit of the sequence of operators $(V_{-h(n)}AV_{h(n)})_{n\in\mathbb{N}}$ with respect to a sequence $h(0), h(1), h(2), ... \in \mathbb{Z}^N$ where $V_h(n)$ is the shift in the direction of $h(n)$. The existence of limit operators depends on the chosen sequence $(h(n))_{n\in\mathbb{N}}$ and the notion of richness tells about how commonly the limit operator exists. An operator $A \in \mathcal{L}(E)$ is said to be rich if for every sequence $h \subset \mathbb{Z}^N$ there is a subsequence g for which the limit operator A_g exists.

The goal of this thesis is to describe the groundwork in more detail and prove the claim that for a rich band-dominated operator A there is an equivalence between $\mathcal{P}$-Fredholmness and the property that its limit operators A_h are invertible modulo $\mathcal{K}(E, \mathcal{P})$ and also that the inverses are uniformly bounded. This generalization of the classical Fredholm-Riesz theory was introduced and discussed in detail in the book [1] by Vladimir Rabinovich, Steffen Roch and Bernd Silbermann. Further advances on this topic can be found on the publications [5, 6] of Markus Seidel and Marko Lindner, where they show that it is possible to generalize this core result even further. Specifically the theorem holds true also when omitting either the richness criterion or the uniform boundedness criterion. This raises the question whether it would be possible to omit both richness and uniform boundedness criteria simultaneously, however these recent results and ideas are beyond the scope of this thesis.

Chapter 2

Fredholm-Riesz theory

In this chapter we review the basics of the classical Fredholm-Riesz theory, which will serve as a model case for our later generalization. We will start with the compact operators. First in a specific case of the space $C(X)$ and then go through the their properties and implications in general Banach spaces. Thereafter we continue with the classical notion of Fredholm operators and their main properties.

2.1 Definitions and notations

Definition 2.1.1 (Banach spaces)**.** A complete normed vector space X is called a Banach space. With completeness we mean that all Cauchy sequences in X converge in X with respect to the norm. In other words, if the sequence $(x_i) \subset X$ is Cauchy, then there exists $x \in X$ such that

$$\|x_i - x\|_X \to 0, \text{ as } n \to \infty.$$

2.2 Compact operators on $C(X)$

Before considering compact operators between Banach spaces we first look at a special case $C(X)$, where X is a compact topological space.

Definition 2.2.1 (Pre-compactness)**.** Let (X, d) be a metric space with the distance metric d. The space (X, d) is called pre-compact if for all $\varepsilon > 0$ there exists a finite cover $\{A_1, ..., A_n\}$ of X such that for all $k = 1, ..., n$ the diameters of the sets,

$$\operatorname{diam}(A_k) := \sup_{x,y \in A_k} d(x, y) < \varepsilon$$

and also

$$X = \bigcup_{k=1}^{n} A_k.$$

Equivalently, (X, d) is pre-compact if for all $\varepsilon > 0$ there exists a finite set of points $x_1, ..., x_n \in X$ such that the open balls $\{B(x_1, \varepsilon), ..., B(x_n, \varepsilon)\}$ with radii ε cover X. Here we denote open balls as $B(x_0, r) := \{x \in X : d(x, x_0) < \varepsilon\}$. In other words

$$X \subset \bigcup_{k=1}^{n} B(x_k, \varepsilon).$$

Theorem 2.2.2. Let (X, d) be a metric space. Then the following conditions are equivalent:

(a) X is compact, that is every open cover $\mathcal{D}$ of X has a finite subcover of X.

(b) Any sequence $(x_i) \subset X$ has a converging subsequence (x_{i_k}) in X, that is X is sequentially compact.

(c) X is pre-compact and complete.

Proof. The proof of (a) $\Leftrightarrow$ (b) is found in Väisälä: Topologia I [7]

(b) $\Rightarrow$ (c)

Let X be sequentially compact. From the material of Funktionaalianalyysin peruskurssi [2] we know that if a Cauchy sequence has a converging subsequence, then the whole sequence converges to the same point as the subsequence. Also from the assumption of sequential compactness we know that every Cauchy sequence in X has a converging subsequence. Hence we deduce that X is complete. For the rest of the proof we follow [4].

Furthermore, to prove pre-compactness we make a counterassumption. Let us assume X is not pre-compact. Thus there exists an $\varepsilon_0 > 0$ for which there are no finite set of open balls $\{B_1, ..., B_k\}$ with radii ε_0 such that $X \subset \bigcup_{j=1}^{k} B_j$. We construct a sequence (x_n) as follows. Pick $x_1 \in X$. Since $X \not\subset B(x_1, \varepsilon_0)$, we can pick $x_2 \in X$ such that $d(x_1, x_2) \geq \varepsilon_0$. Given $\{x_1, ..., x_n\} \subset X$, we can pick $x_{n+1} \in X$ such that $d(x_j, x_{n+1}) \geq \varepsilon_0$, for all $1 \leq j \leq n$ since $X \not\subset \bigcup_{j=1}^{n} B(x_j, \varepsilon_0)$. Now by the way the sequence $(x_n) \subset X$ was constructed, we have $d(x_j, x_k) \geq \varepsilon_0$ for all $j \neq k$. This means that the sequence (x_n) has no converging subsequences which again implies that X is not compact. This is a contradiction. Hence X is pre-compact.

(c) $\Rightarrow$ (b) Suppose X is pre-compact and complete. Let (x_n) be a sequence in X. Fix $\varepsilon = 1$. Since X is pre-compact we have a finite cover $\{A_1, ..., A_n\}$

of X for which $\text{diam}(A_k) < 1$ for $1 \leq k \leq n$. Since the cover is finite we know that at least one A_k contains infinitely many terms of (x_n) so we choose $x_{n_1} \in A_k$ and denote the corresponding A_k as X_1.

Similarly, since a subset of a pre-compact space is pre-compact, we know that X_1 is pre-compact and thus fixing $\varepsilon = \frac{1}{2}$ gives us a new cover $\{A_1, ..., A_n\}$ of X_1 such that $\text{diam}(A_k) < \frac{1}{2}$ for $1 \leq k \leq n$. Again there is an A_k which contains infinitely many terms of (x_n). We denote this set A_k by X_2 and choose such a $x_{n_2} \in X_2$ that $n_2 > n_1$.

Continuing in this manner, given $x_{n_1}, ..., x_{n_m}$ and $X_m \subset ... \subset X_1 \subset X$, and such that the set X_m contains infinitely many terms of (x_n). Let $\varepsilon = \frac{1}{m+1}$ and cover X_m with a finite cover $\{A_1, ..., A_n\}$, such that $\text{diam}(A_k) < \frac{1}{m+1}$ for $1 \leq k \leq n$. Again one of the sets A_k contains infinitely many terms of (x_n) and we denote it X_{m+1} and pick $x_{n_{m+1}}$ in X_{m+1} such that $n_{m+1} > n_m$. Observe that for any $j \geq m$ we have $x_{n_j} \in X_m$ and thus $d(x_{n_j}, x_{n_i}) < \frac{1}{m}$ for all $i, j \geq m$. Hence the sequence (x_n) has a Cauchy subsequence (x_{n_j}) and from the completeness of X follows that the subsequence (x_{n_j}) converges. Since (x_n) was arbitrary we conclude that X is sequentially compact and hence compact. □

Definition 2.2.3 (Equicontinuity)**.** Let (X, d) and (Y, d') be metric spaces and H be a collection of mappings from X to Y. We say that H is equicontinuous at $x \in X$ if for all $\varepsilon > 0$ there exists an open neighborhood $V \subset X$ of x, such that $d'(f(y), f(x)) < \varepsilon$, for any $f \in H$ and $y \in V$.

Definition 2.2.4. Let H be a family of mappings $X \longrightarrow \mathbb{K}$. We say that H is pointwise bounded if for each $x \in X$ there is such an $M(x) < \infty$, that

$$|f(x)| \leq M(x), \text{ for all } f \in H.$$

Definition 2.2.5. Let H be a family of mappings $X \longrightarrow \mathbb{K}$ and let $A \subset X$. For a subset $A \subset X$, H is called uniformly bounded in A if there exists such an $M < \infty$ that

$$|f(x)| \leq M, \text{ for all } f \in H \text{ and } x \in A.$$

Let X be a Banach space. A subset $A \subset X$ is called relatively compact if its closure $\overline{A}$ is compact. With these concepts we formulate the classical result of Cesare Arzelà and Giulio Ascoli.

Note that we define the sup-norm $\|\cdot\|_\infty$ for a function f in $C(X, \mathbb{K})$ as $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

Theorem 2.2.6 (Arzela-Ascoli)**.** Let X be a compact topological space and $H \subset C(X, \mathbb{K}) := C(X)$. Then H is relatively compact in $(C(X), \|\cdot\|_\infty)$ if and only if the family H is equicontinuous and pointwise bounded in X.

Proof. We refer the proof to the course material of Funktionaalianalyysin peruskurssi at page 177. [2] □

This theorem is a powerful tool and in our case, we now have a way to identify compact sets in $C(X)$.

The following classical example from [2] completes the introduction of compactness in $C(X)$ and presents a very important operator which is used in the setting of Fredholm equations. Fredholm equations are a class of integral equations and the attempts to solve these gave rise to the Fredholm-Riesz theory.

Example 2.2.7. Let $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be a continuous kernel function. Then the operator $T : C(0,1) \longrightarrow C(0,1)$ defined as follows

$$Tf(x) = \int_0^1 K(x,y)f(y)dy, \text{ for } x \in [0,1] \text{ and } f \in C(0,1)$$

is compact, that is, the image of the closed unit ball

$$T(B_{C(0,1)}) = \{Tf : f \in C(0,1), \|f\|_\infty \leq 1\}$$

is relatively compact in $C(0,1)$. Here we denote $C([0,1])$ as $C(0,1)$.

Proof. The kernel function K is continuous in the compact set $[0,1] \times [0,1]$, thus K is bounded and uniformly continuous. Since, for all $x \in [0,1]$

$$|Tf(x)| \leq \int_0^1 |K(x,y)f(y)|dy \leq \int_0^1 \|K\|_\infty \|f\|_\infty dy \leq C \|f\|_\infty ,$$

where $C = \|K\|_\infty = \max\{|K(x,y)| : x,y \in [0,1]\} < \infty$, it follows that $T(B_{C(0,1)})$ is uniformly bounded, as

$$\|Tf\|_\infty \leq C \|f\|_\infty \leq C,$$

for $\|f\|_\infty \leq 1$.

Let $x_0 \in [0,1]$ and $\varepsilon > 0$. Since K is uniformly continuous there exists such a $\delta > 0$ that

$$|K(x,y) - K(x_0,y)| < \varepsilon, \text{ whenever } |x - x_0| < \delta \text{ and } y \in [0,1].$$

Thus for all $f \in B_{C(0,1)}$ it holds that

$$\begin{aligned} |Tf(x) - Tf(x_0)| &\leq \int_0^1 |K(x,y) - K(x_0,y)||f(y)|dy \\ &\leq \varepsilon \int_0^1 \|f\|_\infty dy \leq \varepsilon, \end{aligned}$$

whenever $|x - x_0| < \delta$.

These results imply that $T(B_{C(0,1)})$ is uniformly bounded and equicontinuous, thus by Arzela-Ascolis theorem, $T(B_{C(0,1)})$ is relatively compact and T is a compact operator. □

2.3 Compact operators on general Banach spaces

Here we will generalize the concept of compact operators from $C(X)$ to general Banach spaces. We denote the closed unit ball in X as $B_X := \{x \in X : \|x\| \leq 1\}$.

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the class of bounded linear operators $S : X \to Y$ equipped with the operator norm

$$\|S\|_{\mathcal{L}(X,Y)} = \sup\{\|Sx\|_Y : x \in X, \|x\|_X \leq 1\}.$$

Definition 2.3.1 (Compact operator)**.** Let X and Y be Banach spaces. A linear operator $T \in \mathcal{L}(X,Y)$ is compact if the closure of the image of the closed unit ball B_X;

$$\overline{TB_X} = \overline{\{Tx \ : \ x \in B_X\}}$$

is compact in Y. We also denote by $\mathcal{K}(X,Y)$ the set of all compact operators $T \in \mathcal{L}(X,Y)$.

We can thus by Theorem 2.2.2 check if an operator $T \in \mathcal{L}(X,Y)$ is compact by checking whether the image TB_X is pre-compact in the Banach space Y.

The following two properties from [2] show that $\mathcal{K}(X,Y)$ is an operator ideal.

Theorem 2.3.2. Let X and Y be Banach spaces. Then $\mathcal{K}(X,Y)$ is a closed linear subspace of $\mathcal{L}(X,Y)$.

Proof. Let $S, T \in \mathcal{K}(X,Y)$ and let $\varepsilon > 0$ be arbitrary. Thus there exists vectors $x_1..., x_m \in Y$ and $y_1, ..., y_n \in Y$ such that we have $SB_X \subset \bigcup_{i=1}^m B(x_i, \varepsilon)$ and $TB_X \subset \bigcup_{j=1}^n B(y_j, \varepsilon)$, where B_X is the closed unit ball in X. Hence if we pick any vector $z \in B_X$, then $\|Sz - x_i\|_Y < \varepsilon$ and $\|Tz - y_j\|_Y < \varepsilon$ for some $1 \leq i \leq m,\ 1 \leq j \leq n$. By the triangle inequality we have

$$\|(S+T)z - (x_i + y_j)\|_Y \leq \|Sz - x_i\|_Y + \|Tz - y_j\|_Y < 2\varepsilon.$$

Thus

$$(S+T)B_X \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} B(x_i + y_j, 2\varepsilon).$$

We conclude that $S + T \in \mathcal{K}(X,Y)$.

Now suppose $S \in \mathcal{K}(X,Y)$ and $\lambda \in \mathbb{K}$. Given $\varepsilon > 0$ there exists vectors $x_1, ..., x_m \in Y$ such that $SB_X \subset \bigcup_{i=1}^m B(x_i, \varepsilon)$. For any $z \in B_X$ we have $\|Sz - x_i\|_Y < \epsilon$ for some $1 \leq i \leq m$. Thus we have

$$\|\lambda Sz - \lambda x_i\|_Y = |\lambda| \, \|Sz - x_i\|_Y < |\lambda|\varepsilon.$$

Hence $\lambda SB_X \subset \bigcup_{i=1}^m B(\lambda x_i, |\lambda|\varepsilon)$ and $\lambda S \in \mathcal{K}(X,Y)$.

Finally assume $S \in \overline{\mathcal{K}(X,Y)}$. Hence there exists a sequence of operators (S_n) in $\mathcal{K}(X,Y)$ such that $\|S_n - S\| \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$ and such an $n \in \mathbb{N}$ that $\|S_n - S\| < \varepsilon$. Since S_n is compact, there exists vectors $x_1, ..., x_m$ such that $S_n B_X \subset \bigcup_{i=1}^m B(x_i, \varepsilon)$. By the triangle inequality, for any $z \in B_X$ there is such an index $1 \leq i \leq m$ that we have

$$\|Sz - x_i\|_Y \leq \|Sz - S_n z\|_Y + \|S_n z - x_i\| \leq \|S - S_n\| + \varepsilon \leq 2\varepsilon.$$

Thus $SB_X \subset \bigcup_{i=1}^m B(x_i, 2\varepsilon)$ and $S \in \mathcal{K}(X,Y)$.

These three results combined prove that $\mathcal{K}(X,Y)$ is indeed a closed linear subspace of $\mathcal{L}(X,Y)$. □

Theorem 2.3.3. Let X, Y, X_1 and Y_1 be Banach spaces and $T \in \mathcal{K}(X,Y)$ a compact operator. If $S \in \mathcal{L}(Y, Y_1)$ is a continuous operator, then the composed operator $ST \in \mathcal{K}(X, Y_1)$ is compact. Moreover if $R \in \mathcal{L}(X_1, X)$ is a continuous operator, then the composed operator $TR \in \mathcal{K}(X_1, Y)$ is compact.

Proof. Fix $\varepsilon > 0$. Since $T \in \mathcal{K}(X,Y)$ is compact, there exists vectors $x_1, ..., x_m \in Y$ such that $TB_X \subset \bigcup_{i=1}^m B(x_i, \varepsilon)$.

Since $R \in \mathcal{L}(X_1, X)$ is continuous, for any $z \in B_{X_1}$ we have $\|Rz\|_X \leq \|R\| \, \|z\|_{X_1}$. Thus

$$RB_{X_1} \subset \|R\| \, B_X.$$

Hence we have

$$TRB_{X_1} \subset \|R\| \, TB_X \subset \bigcup_{i=1}^m B_Y(\|R\| \, x_i, \|R\| \, \varepsilon),$$

which implies that $TR \in \mathcal{K}(X_1, Y)$ is compact.

Furthermore if $S \in \mathcal{L}(Y, Y_1)$ is continuous, then $SB_Y \subset \|S\| \, B_{Y_1}$ and additionally

$$SB_Y(x_i, \varepsilon) = S(x_i + \varepsilon B_Y) = Sx_i + \varepsilon SB_Y \subset Sx_i + \varepsilon \|S\| \, B_{Y_1}$$

for all $1 \leq i \leq m$. Hence we have

$$STB_X \subset S(\bigcup_{i=1}^m B_Y(x_i, \varepsilon)) \subset \bigcup_{i=1}^m SB_Y(x_i, \varepsilon) \subset \bigcup_{i=1}^m B_{Y_1}(Sx_i, \|S\| \, \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary it follows that $ST \in \mathcal{K}(X, Y_1)$ is compact. □

The following lemma from [2] is needed to show that compact sets are quite uncommon in infinite dimensional Banach spaces. In fact such basic sets as closed balls B_X are not compact when the dimension of X is infinite.

Lemma 2.3.4 (Riesz's lemma)**.** Let X be a normed space and $M \subsetneq X$ a closed linear subspace. Then for all $\varepsilon > 0$ there exists such a vector $x \in X$ that $\|x\|_X = 1$ and

$$dist(x, M) = \inf_{m \in M} \|x - m\|_X > 1 - \varepsilon.$$

Proof. In the proof, to avoid making the formulae hard to read, we simplify the norm $\|\cdot\|_X$ as $\|\cdot\|$. Fix $\varepsilon > 0$ and $z \in X \setminus M$. Thus clearly $dist(z, M) > 0$, since M is closed. Now choose such an $m \in M$ that

$$dist(z, M) \leq \|z - m\| < \frac{dist(z, M)}{1 - \varepsilon},$$

which is possible from the definition of the infimum.

Choose $x = \frac{z-m}{\|z-m\|}$. Now $\|x\| = \left\|\frac{z-m}{\|z-m\|}\right\| = 1$ and $x = \frac{z}{\|z-m\|} - \frac{m}{\|z-m\|} \notin M$, since $z \in X \setminus M$.

Thus for any $n \in M$ we have

$$\|x - n\| = \left\|\frac{z - m}{\|z - m\|} - n\right\| = \frac{1}{\|z - m\|} \|z - (m + n\|z - m\|)\|$$

and since $m + n\|z - m\|$ is an element in M, we can estimate as follows

$$\|x - n\| \geq \frac{1}{\|z - m\|} dist(z, M) > \frac{1 - \varepsilon}{dist(z, M)} dist(z, M) = 1 - \varepsilon.$$

□

We can now state and prove the fact, that interestingly, closed balls are never compact in infinite dimensional spaces.

Corollary 2.3.5. If X is an infinite dimensional Banach space, then the closed unit ball $B_X = \{x \in X : \|x\|_X \leq 1\} \subset X$ is not compact. Moreover, any closed ball $\overline{B}(x_0, r) \subset X$ is not compact.

Proof. Let $\dim(X) = \infty$. Fix such a vector $x_1 \in B_X$ that $\|x_1\|_X = 1$. Now according to Riesz's lemma, there exists a vector $x_2 \in B_X$ with $\|x_2\|_X = 1$ and $dist(x_2, span\{x_1\}) > \frac{1}{2}$.

Assume we have chosen vectors $x_1, ..., x_n \in B_X$ where $\|x_j\|_X = 1$ and

$$dist(x_j, span\{x_1, ..., x_{j-1}\}) > \frac{1}{2},$$

for all $2 \leq j \leq n$. Since $\dim(X) = \infty$ and $span\{x_1, ..., x_n\}$ is a finite dimensional closed linear subspace of X, according to Riesz's lemma there exists such a vector $x_{n+1} \in B_X$ that $\|x_{n+1}\|_X = 1$ and

$$dist(x_{n+1}, span\{x_1, ..., x_n\}) > \frac{1}{2}.$$

By repeating this argument in this manner we construct a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with the property that

$$\|x_j - x_k\|_X > \frac{1}{2}, \text{ whenever } j \neq k.$$

Hence the sequence $(x_n)_{n\in\mathbb{N}}$ does not have any converging subsequences. Thus we conclude that the cosed unit ball B_X is not compact.

Furthermore,

$$\overline{B}(x_0, r) = x_0 + rB_X$$

and the mapping $x \mapsto x_0 + rx$ is a homeomorphism. Since homeomorphisms preserve compactness, no closed balls $\overline{B}(x_0, r)$ can be compact. □

Definition 2.3.6 (Strong convergence)**.** Let X be a Banach space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ be a sequence of continuous operators on X such that there exists an operator $A \in \mathcal{L}(X)$ for which

$$\|A_n x - Ax\|_X \to 0, \text{ for all } x \in X \text{ as } n \to \infty.$$

Then A is called the strong limit of $(A_n)_{n\in\mathbb{N}}$ and we can say that $A_n \to A$ strongly.

The following theorem from [1] gives us a connection between compactness and strong convergence.

Theorem 2.3.7. Let $A_n, A \in \mathcal{L}(X)$ be continuous operators in X, for $n \in \mathbb{N}$. Then $A_n \to A$ strongly if and only if

$$\|A_n T - AT\|_{\mathcal{L}(X)} \to 0, \text{ for all } T \in \mathcal{K}(X) \text{ as } n \to \infty.$$

Proof. "$\Rightarrow$" Let $A_n \to A$ strongly and let T be a compact operator. Thus the set $M := \{Tx : \|x\|_X \leq 1\}$ is relatively compact in X.

We argue by a counterargument. Suppose that

$$\|A_n T - AT\|_{\mathcal{L}(X)} = \sup_{\|x\|_X \leq 1} \|A_n Tx - ATx\|_X = \sup_{y \in M} \|A_n y - Ay\|_X$$

does not converge to zero as $n \to \infty$. Hence there exists also such an $\varepsilon > 0$ and an infinite sequence $(y_n)_{n\in\mathbb{N}} \subset M$ that

$$\|A_n y_n - Ay_n\| > \varepsilon, \text{ for } n \in \mathbb{N}.$$

Since M is relatively compact there exists a converging subsequence $(z_n)_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ which converges to some $z \in X$ as $n \to \infty$. Thus for every $n \in N$ the following holds.

$$\begin{aligned}\varepsilon &< \|A_n z_n - Az_n\|_X = \|(A_n - A)z_n\|_X \\ &\leq \|(A_n - A)(z_n - z)\|_X + \|(A_n - A)z\|_X \\ &\leq \sup_k \|A_k - A\|_{\mathcal{L}(X)} \|z_n - z\|_X + \|(A_n - A)z\|_X .\end{aligned}$$

From the Banach-Steinhaus theorem we obtain that $\sup_k \|A_k - A\|_{\mathcal{L}(X)}$ is finite. Thus, since $A_n \to A$ strongly and since $\|z_n - z\|_X \to 0$ as $n \to \infty$, the right-hand side converges to zero as $n \to \infty$. This is a contradiction. Hence, since $T \in \mathcal{K}(X)$ was arbitrary,

$$\|A_n T - AT\|_{\mathcal{L}(X)} \to 0, \text{ for all } T \in \mathcal{K}(X).$$

"$\Leftarrow$" Assume that $\|A_n T - AT\|_{\mathcal{L}(X)} \to 0$ as $n \to \infty$ and let $\overline{0} \neq x \in X$. The Hahn-Banach theorem, see [2], allows us to choose such a linear functional $f \in X^*$ that $\|f\|_{X^*} = 1$ and $f(x) = \|x\|_X$.

Let us consider the operator

$$K_x y := f(y)x, \text{ where } y \in X.$$

Since x is fixed and the range of f is $\mathbb{R}$, the operator K_x has rank one, i.e. $\dim(K_x X) = 1$, hence the operator K_x is compact. Furthermore we have

$$\|K_x\|_{\mathcal{L}(X)} = \sup_{y\in X} \|f(y)x\|_X = \|f\|_{X^*} \|x\|_X = \|x\|_X .$$

Since

$$\begin{aligned}K_{A_n x}(y) - K_{Ax}(y) &= f(y)A_n x - f(y)Ax = (A_n - A)(f(y)x) \\ &= (A_n - A)K_x(y),\end{aligned}$$

we get that

$$\begin{aligned}\|A_n x - Ax\|_X &= \|K_{A_n x - Ax}\|_{\mathcal{L}(X)} = \|K_{A_n x} - K_{Ax}\|_{\mathcal{L}(X)} \\ &= \|A_n K_x - AK_x\|_{\mathcal{L}(X)} ,\end{aligned}$$

which converges to zero as $n \to \infty$ by our assumption. Hereby $A_n \to A$ strongly. □

2.4 Fredholm-Riesz theory

In this section we introduce Fredholm operators and some of their basic properties. This section is based on the book by [3] Caradus, Pfaffenberger and Yood with the exception being the proof of Theorem 2.4.3., which follows [2].

Definition 2.4.1 (Fredholm operator)**.** A bounded linear operator $A \in \mathcal{L}(X)$ is called Fredholm if its kernel $\ker A$ and cokernel $\operatorname{coker} A := X/\operatorname{Im} A$ are finite-dimensional.

Theorem 2.4.2. If $A \in \mathcal{L}(X)$ is a Fredholm operator, then the image $\operatorname{Im} A$ of A is closed.

Proof. Assume that $A \in \mathcal{L}(X)$ is Fredholm. Thus the $\operatorname{coker} A$ is finite-dimensional and X is the direct sum of the image of A and a closed finite-dimensional linear subspace $M \subset X$, and we write $X = \operatorname{Im} A \oplus M$. Define an operator $T_0 : X/\ker A \times M \to X$ as

$$T_0(x + \ker A, m) = Ax + m,$$

for $x \in X$ and $m \in M$. The mapping T_0 has full range, $\operatorname{Im} T_0 = X$, and is one-to-one since if $T_0(x + \ker A, m) = Ax + m = 0$ then $m = 0$ and $Ax = 0$, that is $x \in \ker A$. Thus the operator T_0 is a continuous bijection. This again implies that T_0 has a continuous inverse T_0^{-1}, hence there exists such a constant $C > 0$ that

$$\|Ax + n\|_X \geq C \,\|(x + \ker A, n)\|_{X/\ker A \times M} = C(\|x\|_{X/\ker A} + \|n\|_X).$$

Choosing here $n = 0$ yields us

$$\|Ax\|_X \geq C \,\|x\|_{X/\ker A}\,.$$

Hence, also the mapping $T : X/\ker A \to X$,

$$T(x + \ker A) := Ax,$$

has a continuous inverse. Since $X/\ker A$ is a Banach space, thus the image of T, $\operatorname{Im} T = \operatorname{Im} A$ is closed. $\square$

The following classical result from [2] is needed for the proof of the characterization of a Fredholm operator as being invertible modulo the compact operators.

Theorem 2.4.3. The operator $I+K \in \mathcal{L}(X)$ is a Fredholm operator, whenever $K \in \mathcal{K}(X)$ is compact.

Proof. Suppose $K \in \mathcal{K}(X)$ is compact.

First we prove that the kernel $\ker(I+K)$ has finite dimension. We argue by a counterargument. Suppose that the operator $I+K$ has infinite-dimensional kernel, that is $\dim(\ker(I+K)) = \infty$. By Riesz's lemma there exists a sequence of normalized vectors $(e_n)_{n\in\mathbb{N}} \subset \ker(I+K)$ with $\|e_n\|_X = 1$ for all $n \in \mathbb{N}$ and

$$\|e_n - e_m\|_X \geq \frac{1}{2},$$

whenever $n \neq m$.

For the vectors $e_n \in \ker(I+K)$, we have

$$(I+K)e_n = 0 \Leftrightarrow e_n = -Ke_n \text{ for all } n \in \mathbb{N}.$$

The operator $K \in \mathcal{K}(X)$ is compact, thus the sequence $(-Ke_n)_{n\in\mathbb{N}}$ has a converging subsequence and hence also the sequence $(e_n)_{n\in\mathbb{N}}$ has a converging subsequence. This is a contradiction. Thus the dimension of $\ker(I+K)$ is finite.

We still need to show that the image $\mathrm{Im}(I+K)$ is closed in order to prove the second claim. Assume $y \in \overline{\mathrm{Im}(I+K)}$ is arbitrary. We want to prove that $y \in \mathrm{Im}(I+K)$. By our assumption there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$y_n = (I+K)x_n \to y, \text{ as } n \to \infty.$$

From the previous part of the proof we know that $\ker(I+K)$ is finite dimensional and hence closed. Thus we can find a $z_n \in \ker(I+K)$ for any $n \in \mathbb{N}$ such that

$$dist\Big(x_n, \ker(I+K)\Big) = \|x_n - z_n\|_X .$$

Additionally, since $z_n \in \ker(I+K)$ we have

$$(I+K)(x_n - z_n) = (I+K)x_n - \underbrace{(I+K)z_n}_{=0} = y_n \to y,$$

as $n \to \infty$.

Next we want to show that the set $\{x_n - z_n : \ n \in \mathbb{N}\}$ is bounded. For this we make a counterassumption that there exists such a subsequence that

$$\left\|x_{n_j} - z_{n_j}\right\|_X \to \infty, \text{ as } n \to \infty.$$

We normalize these vector by setting $v_n = \frac{x_n - z_n}{\|x_n - z_n\|_X} \in B_X$ for $n \in \mathbb{N}$. Then

$$v_{n_j} + Kv_{n_j} = \frac{1}{\left\|x_{n_j} - z_{n_j}\right\|_X}(I+K)(x_{n_j} - z_{n_j}) \to 0,$$

as $j \to \infty$ since the sequence $(I+K)(x_{n_j} - z_{n_j})$ is bounded. Since K is compact, also $-K$ is compact and there exists a subsequence $(v_{n_{j_k}})$ such that $-Kv_{n_{j_k}} \to z$ as $k \to \infty$. Since as we showed before $v_{n_j} + Kv_{n_j} \to 0$ we have

$$v_{n_{j_k}} = v_{n_{j_k}} + Kv_{n_{j_k}} - Kv_{n_{j_k}} \to z, \text{ as } k \to \infty.$$

From continuity it follows that $Kv_{n_{j_k}} \to Kz$, thus $Kz = -z$. Hence

$$(I+K)z = z - z = 0$$

and $z \in \ker(I+K)$. On the other hand $z_n \in \ker(I+K)$ and

$$\|x_n - z_n\|_X = dist\Big(x_n, \ker(I+K)\Big),$$

for $n \in \mathbb{N}$. Thus

$$\begin{aligned} dist\Big(v_n, \ker(I+K)\Big) &= dist\Big(\frac{x_n - z_n}{\|x_n - z_n\|_X}, \ker(I+K)\Big) \\ &= \frac{1}{\|x_n - z_n\|_X} dist\Big(x_n, \ker(I+K)\Big) = 1. \end{aligned}$$

This means that $\|v_n - z\| \geq 1$ for all $n \in \mathbb{N}$, which is a contradiction since we previously showed that $v_{n_j} \to z$ as $j \to \infty$. Hence we conclude that $\{x_n - z_n : \ n \in \mathbb{N}\}$ is bounded that is

$$\|x_n - z_n\|_X \leq C < \infty,$$

for all $n \in \mathbb{N}$.

Since K is compact, the closure $\overline{\{Kv : \ \|v\|_X \leq C\}}$ is a compact subset of X. Thus there exists a converging subsequence

$$K(x_{n_j} - z_{n_j}) \to u \in X, \text{ as } j \to \infty.$$

Hence

$$\begin{aligned} x_{n_j} - z_{n_j} &= x_{n_j} - z_{n_j} + K(x_{n_j} - z_{n_j}) - K(x_{n_j} - z_{n_j}) \\ &= (I+K)(x_{n_j} - z_{n_j}) - K(x_{n_j} - z_{n_j}) \to y - u, \end{aligned}$$

as $j \to \infty$. Thus from continuity we obtain

$$(I+K)(y-u) = \lim_{j\to\infty} (I+K)(x_{n_j} - z_{n_j}) = \lim_{j\to\infty} y_{n_j} = y.$$

Hence $y = (I+K)(y-u) \in \operatorname{Im}(I+K)$ and $\operatorname{Im}(I+K)$ is closed.

To prove that $I+K$ has finite-dimensional cokernel, consider the mapping $Q : X \to X/\operatorname{Im}(I+K)$, where $Qx = x + \operatorname{Im}(I+K)$, for $x \in X$. Notice, that since $\operatorname{Im}(I+K)$ is closed, the quotient space $X/\operatorname{Im}(I+K)$ is a Banach space. By the way Q is defined we have $0 = Q(I+K) = Q + QK$, hence by Theorem 2.3.3. the operator $Q = -QK$ is compact as a composite operator of a bounded linear operator and a compact operator. Pick an $x+\operatorname{Im}(I+K) \in B_{x/\operatorname{Im}(I+K)}$ in the closed unit ball of $X/\operatorname{Im}(I+K)$, meaning that

$$\|x + \operatorname{Im}(I+K)\|_{X/\operatorname{Im}(I+K)} = dist(x, \operatorname{Im}(I+K)) \leq 1.$$

Hence there exists such an $m \in \operatorname{Im}(I+K)$ that $\|x-m\|_X < 2$ and additionally $Q(x-m) = x + \operatorname{Im}(I+K)$. This implies that

$$\frac{1}{2} B_{X/\operatorname{Im}(I+K)} \subset \overline{QB_X}.$$

Since Q is compact, by definition $\overline{QB_X}$ is a compact set, and as a closed subset of a compact set also $\frac{1}{2}B_{X/\operatorname{Im}(I+K)}$ is compact. Additionally, since the map $x \mapsto \frac{1}{2}x$ is a homeomorphism, the closed unit ball $B_{X/\operatorname{Im}(I+K)}$ is compact. By Corollary 2.3.5, closed unit balls of infinite-dimensional Banach spaces are not compact, hence the cokernel $\operatorname{coker} X/\operatorname{Im}(I+K)$ has finite dimension.

We have proved that $I+K$ has finite-dimensional kernel and cokernel, hence by definition $I+K$ is Fredholm. □

The following classical result about Fredholm operators is often called Atkinson's theorem.

Theorem 2.4.4. The operator $A \in \mathcal{L}(X)$ is Fredholm if and only if there exists an operator $B \in \mathcal{L}(X)$ such that

$$\begin{cases} BA = I + K_1 \\ AB = I + K_2, \end{cases}$$

where $K_1, K_2 \in \mathcal{K}(X)$ are compact operators.

Proof. "$\Rightarrow$" Let $A \in \mathcal{L}(X)$ be a Fredholm operator. Thus there exists subspaces X_1 and X_2 such that we can write X as

$$X = \ker A \oplus X_1 = \operatorname{Im} A \oplus X_2.$$

The subspaces X_1 and X_2 are also closed since we can write X_1 as $X_1 = T(X/\ker A)$ where $T : X/\ker A \to X_1$ is the induced map from the Banach space $X/\ker A$ to X_1 which is defined by $T(x + \ker A) := x$. Additionally X_2 is a finite dimensional subspace of a Banach space and hence closed.

Denote the operator $A' := A|_{X_1}$, that is, A restricted to X_1. The operator A' is one-to-one and thus invertible onto the range $\operatorname{Im} A$. Define the operator $B : X \to X$ as follows: $B = (A')^{-1}$ in $\operatorname{Im} A$ and $B = 0$ in X_2. Now B is continuous by the open mapping theorem and hence in $\mathcal{L}(X)$.

To choose the compact operators K_1 and K_2, consider the maps BA and AB.

$$BA : X \xrightarrow{A} \operatorname{Im} A \xrightarrow{B} X_1 \text{ and}$$
$$AB : X \xrightarrow{B} X_1 \xrightarrow{B} \operatorname{Im} A.$$

Hence by choosing $K_1 = \begin{cases} -I, & \text{in} \ker A \\ 0, & \text{in } X_1 \end{cases}$ and $K_2 = \begin{cases} 0, & \text{in} \operatorname{Im} A \\ -I, & \text{in } X_2 \end{cases}$ gives us the desired result:

$$\begin{cases} BA = I + K_1 \\ AB = I + K_2. \end{cases}$$

The operators K_1 and K_2 have their values in finite dimensional subspaces of X, hence they have finite rank and thus are compact.

"$\Leftarrow$" Assume there exists such a $B \in \mathcal{L}(X)$ and $K_1, K_2 \in \mathcal{K}(X)$ that

$$\begin{cases} BA = I + K_1 \\ AB = I + K_2. \end{cases}$$

Thus we have

$$\begin{aligned} \ker(I + K_1) = \ker AB = \{x \in X : BAx = 0\} \\ \supset \{x \in X : Ax = 0\} = \ker A. \end{aligned}$$

By lemma 2.4.3 the operator $I + K_1$ is Fredholm and hence also $\ker A$ has finite dimension.

Additionally we have

$$\begin{aligned} \operatorname{Im}(I + K_2) = \operatorname{Im} AB = \{ABx : x \in X\} \\ \subset \{Ax : x \in X\} = \operatorname{Im} A. \end{aligned}$$

Again by Lemma 2.4.3 the operator $I + K_2$ is Fredholm and hence has finite-dimensional cokernel $\operatorname{coker}(I + K_2) = X/\operatorname{Im}(I + K_2)$. Hence also $\operatorname{coker} A = X/\operatorname{Im} A$ has finite dimension.

The operator A has finite-dimensional kernel and cokernel, so by definition it is a Fredholm operator. □

The above theorem can also be formulated differently, that is, $A \in \mathcal{L}(X)$ is a Fredholm operator if and only if A is invertible in $\mathcal{L}(X)/\mathcal{K}(X)$. Here the quotient space $\mathcal{L}(X)/\mathcal{K}(X)$ is called the Calkin algebra of X. In practice this means that Fredholm operators are precisely the operators which are invertible modulo the compact operators. We note here that Theorem 2.4.4. holds also for operators $A : X \to Y$ with a similar proof.

Chapter 3

Approximate projections and limit operators

3.1 Band dominated operators

We will concentrate on the setting of the sequence space $l^p(\mathbb{Z}^N, X)$, where $N \in \mathbb{N}$, X is a Banach space and $1 < p < \infty$. A vector-valued sequence $x = (x_i) : \mathbb{Z}^N \longrightarrow X$ belongs to $l^p(\mathbb{Z}^N, X)$ if the norm

$$\|x\|_p := (\sum_{i \in \mathbb{Z}^N} \|x_i\|_X^p)^{\frac{1}{p}} < \infty.$$

An easy modification of the scalar-valued argument shows that the space $l^p(\mathbb{Z}^N, X)$ equipped with the norm $\|x\|_p$ becomes a Banach space.

We will denote $E = l^p(\mathbb{Z}^N, X)$, where $1 < p < \infty$ and X is a Banach space to simplify the definitions. Some of the results and properties in this chapter hold also for $p = 1$ or $p = \infty$, but it is not relevant for this thesis.

Example 3.1.1. Let $X = C([0, 1])$ be the space of continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$ equipped with the regular sup-norm $\|\cdot\|_\infty$. Define $f_n \in X$ as

$$f_n(x) = \frac{\sin(|n|\pi x)}{|n|}, \text{ for } n \in \mathbb{Z}^2 \setminus \{0\},$$

where $|n| = \sum_{k=1}^2 n_k$ is the sum of the components of $n = (n_1, n_2) \in Z^2$ and

$$f_n \equiv 0, \text{ for } n = 0.$$

Now the sequence $f = (f_n)_{n \in \mathbb{Z}^2}$ belongs to the space $E = l^p(\mathbb{Z}^2, X)$, for $p > 2$.

Proof. Let f be as above. Consider the sup-norm $\|f_n\|_\infty$ for all $n \in \mathbb{Z}^2 \setminus \{0\}$. We have

$$\|f_n\|_\infty = \sup_{x\in[0,1]} |f_n(x)| = \sup_{x\in[0,1]} |\frac{\sin(|n|\pi x)}{|n|}|$$

and since $|n| \geq 1$ for all $n \in \mathbb{Z}^2 \setminus \{0\}$, we know that

$$\sup_{x\in[0,1]} |\frac{\sin(|n|\pi x)}{|n|}| = \frac{1}{|n|}.$$

Hence,

$$\|f\|_p^p = \sum_{n\in Z^2} \|f_n\|_\infty^p = \sum_{n\in Z^2\setminus\{0\}} \|f_n\|_\infty^p = \sum_{n\in Z^2\setminus\{0\}} \frac{1}{|n|^p}.$$

Now the sum is a bit more tricky to calculate than a normal one-dimensional p-series, so we will estimate it from above and below by breaking the summation into rings

$$A_k = S_k \setminus S_{k-1}, \text{ for } k \geq 1,$$

where $S_k = \{(n_1, n_2) \in \mathbb{Z}^2 \ : |n_1| \leq k, |n_2| \leq k\}$ for all $k \geq 0$. We have the properties

$$\bigcup_{k=1}^{\infty} A_k = \mathbb{Z}^2 \setminus \{0\}$$

and

$$A_k \cap A_j = \emptyset, \text{ for all } k \neq j.$$

Thus we obtain

$$\|f\|_p^p = \sum_{k=1}^{\infty} \sum_{n\in A_k} \frac{1}{|n|^p}.$$

Next fix $k \in \mathbb{N}$. In the ring A_k there are $2(2k+1)+2(2k-1) = 8k$ points, and if $n = (n_1, n_2) \in A_k$ then $k^p \leq |n|^p = (|n_1| + |n_2|)^p \leq 2^p k^p$. Hence we can estimate as an upper bound

$$\sum_{n\in A_k} \frac{1}{|n|^p} \leq \frac{8k}{k^p} = 8k^{1-p}$$

and as a lower bound

$$\sum_{n\in A_k} \frac{1}{|n|^p} \geq \frac{8k}{(2k)^p} = 2^{3-p}k^{1-p}.$$

With these estimates we are almost finished. By using the upper bound and the lower bound we obtain

$$2^{3-p}\sum_{k=1}^{\infty} k^{1-p} \leq \|f\|_p^p = \sum_{k=1}^{\infty}\sum_{n\in A_k}\frac{1}{|n|^p} \leq 8\sum_{k=1}^{\infty} k^{1-p},$$

where we know that $\sum_{k=1}^{\infty} k^{1-p} = M < \infty$ if and only if $p > 2$.

Hence $f \in E = l^p(\mathbb{Z}^2, X)$ if and only if $p > 2$. □

We shall define some basic operators in $E = l^p(\mathbb{Z}^N, X)$ which are shift operators and multiplication operators. These operators form together a larger class of operators, the band operators. See also [1].

Definition 3.1.2 (Shift operators)**.** In this thesis we will exclusively use the notation V_k, $k \in \mathbb{Z}^N$ for the shift operators, defined as follows:

If $x = (x_i) \in E$, then $V_k : E \longrightarrow E$ where

$$V_k x = V_k(x_i) = (x_{i-k}).$$

Note that the indices are in $\mathbb{Z}^N$, that is

$$i - k = (i_1 - k_1, ..., i_N - k_N) \text{ for } i, k \in \mathbb{Z}^N.$$

Definition 3.1.3 (Multiplication operators)**.** Let $a = (a_i) \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X))$ be a bounded sequence of operators $a_i \in \mathcal{L}(X)$ indexed by $i \in \mathbb{Z}^N$. Define the multiplication operator $aI : E \longrightarrow E$ by

$$aIx = (a_i x_i) \text{ for } x = (x_i) \in E.$$

Now we can define band operators and band-dominated operators with the help of the shift and multiplication operators.

Definition 3.1.4. Any finite sum of form

$$\sum_k a_k V_k : E \to E, \text{ where } a_k \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X)) \text{ and } k \in \mathbb{Z}^N$$

is called a band operator on E.

That is, for $(x_i) \in E = l^p(\mathbb{Z}^N, X)$, we have

$$\sum_k a_k V_k(x_i) = \sum_k a_k(x_{i-k}) = \sum_k (a_k^{(i)} x_{i-k}),$$

where $a_k^{(i)} : X \to X$ is the i:th element of the sequence $a_k = (a_k^{(i)})_{i\in\mathbb{Z}^N}$.

Moreover, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of band operators on E. The uniform limit $A : E \to E$, for which $\|A - A_n\|_{\mathcal{L}(E)} \to 0$ as $n \to \infty$, is called a band-dominated operator.

We denote the class of band-dominated operators with respect to the space E as $\mathcal{A}_E$.

3.2 Approximate projections

Definition 3.2.1. Let $U \subset \mathbb{Z}^N$. We define a natural projection operator P_U: $l^p(\mathbb{Z}^N, X) \longrightarrow l^p(\mathbb{Z}^N, X)$ by

$$(x_i) \mapsto (P_U x_i) := \begin{cases} x_i & \text{if } i \in U \\ 0 & \text{if } i \notin U. \end{cases}$$

We also have the complementary projection $Q_U := I - P_U$.

The most relevant projections for us are the canonical projections where $U_n = \{-n, ..., n\}^N$ for $n \in \mathbb{N}$. For simplicity we denote $P_n = P_{U_n}$ and $Q_n = I - P_n$. From these canonical projections we construct the sequence

$$\mathcal{P} = (P_1, P_2, P_3, ...)$$

which is the necessary tool for the theory of approximate projections, see [1].

Proposition 3.2.2. $\mathcal{P}$ is a perfect approximate identity, that is, we have

$$P_n \longrightarrow I \text{ strongly on } E \text{ and } P_n^* \longrightarrow I^* \text{ strongly on } E^*.$$

Proof. Let $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$ be as defined above. Now fix $x \in E = l^p(\mathbb{Z}^N, X)$. The property that the tail-sum of all $x = (x_m)_{m \in \mathbb{Z}^N} \in E$ approaches 0, $\sum_{|m| \geq n} \|x_m\|_X^p \to 0$ as $n \to \infty$, where $|m| = \max\{|m_k| : k \in \mathbb{Z}^N\}$, directly implies

$$\|P_n x - Ix\|_E = (\sum_{m \notin U_n} \|x_m\|_X^p)^{\frac{1}{p}} \to 0, \text{ as } n \to \infty,$$

for all $x \in E$. Thus $P_n \longrightarrow I$ strongly.

Now fix such an $x \in E$ that $\|x\|_E \leq 1$. We have

$$|\langle x, P_n^* x^* - I^* x^* \rangle| = |\langle (P_n - I)x, x^* \rangle|$$

Since $x = (x_k)_{k \in \mathbb{Z}^N} \in E = l^p(\mathbb{Z}^N, X)$, we have $x^* = (x_k^*)_{k \in \mathbb{Z}^N} \in E^* = l^q(\mathbb{Z}^N, X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus we can apply Hölder's inequality as follows

$$|\langle (P_n - I)x, x^* \rangle| \leq \sum_{|j| \geq n+1} |\langle x_j, x_j^* \rangle|$$

$$\leq \left(\sum_{|j| \geq n+1} \|x_j\|_X^p \right)^{\frac{1}{p}} \left(\sum_{|j| \geq n+1} \left\| x_j^* \right\|_{X^*}^q \right)^{\frac{1}{q}} \to 0,$$

as $n \to \infty$. Since $x \in B_E$ was arbitrary we conclude that

$$\|(P_n^* - I^*)x^*\| = \sup_{\|x\|_E \le 1} |\langle x, P_n^* x^* - I^* x^* \rangle| \to 0,$$

as $n \to \infty$.

Thus we have proved both $P_n \longrightarrow I$ strongly in E and $P_n^* \longrightarrow I^*$ strongly in E^*. □

The following three definitions from [1] are analogies of compactness, Fredholmness and strong convergence in the setting of the more general $\mathcal{P}$-theory. Recall that $E = l^p(\mathbb{Z}^N, X)$.

Definition 3.2.3 ($\mathcal{P}$- compactness)**.** Let $K \in \mathcal{L}(E)$ be a bounded linear operator on E. K is called $\mathcal{P}$-compact if both $\|KP_n - K\| \longrightarrow 0$ and $\|P_n K - K\| \longrightarrow 0$ as $n \longrightarrow \infty$.

We will denote the space of $\mathcal{P}$-compact operators by $\mathcal{K}(E, \mathcal{P})$ and also define the space $\mathcal{L}(E, \mathcal{P})$ to consist of all operators $A \in \mathcal{L}(E)$ for which AK and KA are $\mathcal{P}$-compact when $K \in \mathcal{K}(E, \mathcal{P})$.

By definition, this means that $P_n KA \to KA$, $KAP_n \to KA$, $P_n AK \to AK$ and $AKP_n \to AK$ as $n \to \infty$ for every $K \in \mathcal{K}(E, \mathcal{P})$.

Definition 3.2.4 ($\mathcal{P}$- strong convergence)**.** Let $(A_n) \subset \mathcal{L}(E)$ be a sequence of bounded operators on E. The sequence (A_n) converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(E)$ if for any $K \in \mathcal{K}(E, \mathcal{P})$ we have both $\|K(A_n - A)\|_{\mathcal{L}(E)} \to 0$ and $\|(A_n - A)K\|_{\mathcal{L}(E)} \to 0$ as $n \to \infty$. In this case we will denote $A = \mathcal{P}\text{-}\lim_{n \longrightarrow \infty} A_n$.

The following lemma from [1] is useful for checking whether an operator belongs to $\mathcal{L}(E, \mathcal{P})$ or not.

Lemma 3.2.5. Let $A \in \mathcal{L}(E)$. Then A belongs to $\mathcal{L}(E, \mathcal{P})$ if and only if,

$$\|P_m A Q_n\|_{\mathcal{L}(E)} \to 0 \text{ and } \|Q_n A P_m\|_{\mathcal{L}(E)} \to 0 \text{ as } n \to \infty,$$

for every $m \in \mathbb{N}$.

Proof. "⇒" Let $A \in \mathcal{L}(E, \mathcal{P})$. Since $P_m \in \mathcal{K}(E, \mathcal{P})$ for all $m \in \mathbb{N}$ we know that $P_m A, AP_m \in \mathcal{K}(E, \mathcal{P})$, for all $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$. By the definition of $\mathcal{P}$-compactness both

$$\|P_m A Q_n\|_{\mathcal{L}(E)} = \|P_m A - P_m A P_n\|_{\mathcal{L}(E)} \to 0,$$
$$\|Q_n A P_m\|_{\mathcal{L}(E)} = \|P_n A P_m - A P_m\|_{\mathcal{L}(E)} \to 0,$$

as $n \to \infty$.

"$\Leftarrow$" Suppose $\|P_m A Q_n\|_{\mathcal{L}(E)} \to 0$ and $\|Q_n A P_m\|_{\mathcal{L}(E)} \to 0$ as $n \to \infty$, for every $m \in \mathbb{N}$. Let $K \in \mathcal{K}(E, \mathcal{P})$ be a $\mathcal{P}$-compact operator. Given $\varepsilon > 0$ first choose such a $k \in \mathbb{N}$ that

$$\|K - P_k K\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2\,\|A\|_{\mathcal{L}(E)}},$$

and then choose such an $M \in \mathbb{N}$ that

$$\|Q_n A P_k\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2\,\|K\|_{\mathcal{L}(E)}}, \text{ for all } n \geq M.$$

It follows that

$$\begin{aligned}
\|AK - P_n AK\|_{\mathcal{L}(E)} &= \|Q_n AK\|_{\mathcal{L}(E)} = \|Q_n AK - Q_n A P_k K + Q_n A P_k K\|_{\mathcal{L}(E)} \\
&\leq \|Q_n A\|_{\mathcal{L}(E)} \|K - P_k K\|_{\mathcal{L}(E)} + \|Q_n A P_k\|_{\mathcal{L}(E)} \|K\|_{\mathcal{L}(E)} \\
&< \frac{\varepsilon \|Q_n A\|_{\mathcal{L}(E)}}{2\,\|A\|_{\mathcal{L}(E)}} + \frac{\varepsilon \|K\|_{\mathcal{L}(E)}}{2\,\|K\|_{\mathcal{L}(E)}} \\
&\leq \frac{\varepsilon \|A\|_{\mathcal{L}(E)}}{2\,\|A\|_{\mathcal{L}(E)}} + \frac{\varepsilon \|K\|_{\mathcal{L}(E)}}{2\,\|K\|_{\mathcal{L}(E)}} = \varepsilon.
\end{aligned}$$

For KA, let us again fix $\varepsilon > 0$. Now choose as above such $k, M \in \mathbb{N}$ that

$$\|K - K P_k\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2\,\|A\|_{\mathcal{L}(E)}}$$

and

$$\|P_k A Q_n\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2\,\|K\|_{\mathcal{L}(E)}}, \text{ for all } n \geq M.$$

Then similarly as before

$$\begin{aligned}
\|KA - P_n KA\|_{\mathcal{L}(E)} &= \|KAQ_n\|_{\mathcal{L}(E)} = \|KAQ_n - KP_k A Q_n + K P_k A Q_n\|_{\mathcal{L}(E)} \\
&\leq \|AQ_n\|_{\mathcal{L}(E)} \|K - KP_k\|_{\mathcal{L}(E)} + \|P_k A Q_n\|_{\mathcal{L}(E)} \|K\|_{\mathcal{L}(E)} \\
&< \frac{\varepsilon \|AQ_n\|_{\mathcal{L}(E)}}{2\,\|A\|_{\mathcal{L}(E)}} + \frac{\varepsilon \|K\|_{\mathcal{L}(E)}}{2\,\|K\|_{\mathcal{L}(E)}} \\
&\leq \frac{\varepsilon \|A\|_{\mathcal{L}(E)}}{2\,\|A\|_{\mathcal{L}(E)}} + \frac{\varepsilon \|K\|_{\mathcal{L}(E)}}{2\,\|K\|_{\mathcal{L}(E)}} = \varepsilon.
\end{aligned}$$

The cases AKP_n and KAP_n are analogous.

We conclude that $AK, KA \in \mathcal{K}(E, \mathcal{P})$ and hence $A \in \mathcal{L}(E, \mathcal{P})$. □

The following theorem, following along [1], tells us about the structure and relations of the spaces $\mathcal{L}(E,\mathcal{P})$ and $\mathcal{K}(E,\mathcal{P})$. In particular, we verify that $\mathcal{L}(E,\mathcal{P})$ is a Banach algebra and that $\mathcal{K}(E,\mathcal{P})$ is a closed ideal of $\mathcal{L}(E,\mathcal{P})$, analogously to the classical setting of continuous and compact operators.

Theorem 3.2.6. Let $(A_n)_{n\in\mathbb{N}}$ be a bounded sequence of operators in $\mathcal{L}(E,\mathcal{P})$ that converges $\mathcal{P}$-strongly to $A \in \mathcal{L}(E)$. Then

(a) $A \in \mathcal{L}(E,\mathcal{P})$.

(b) if $A \in \mathcal{L}(E,\mathcal{P})$ is invertible, then $A^{-1} \in \mathcal{L}(E,\mathcal{P})$.

(c) $\mathcal{L}(E,\mathcal{P})$ is an inverse-closed closed subalgebra of $\mathcal{L}(E)$, and in particular $\mathcal{L}(E,\mathcal{P})$ is a unital Banach algebra. Moreover, $\mathcal{K}(E,\mathcal{P})$ is a closed ideal of $\mathcal{L}(E,\mathcal{P})$.

Proof. (a) Let $(A_n)_{n\in\mathbb{N}}$ be as above. Since $A_n \to A$ $\mathcal{P}$-strongly, also KA_n and A_nK converge $\mathcal{P}$-strongly to KA and AK respectively for all $K \in \mathcal{K}(E,\mathcal{P})$:

Fix $K_0, K \in \mathcal{K}(E,\mathcal{P})$. Then we have

$$\|K_0(KA_n - KA)\|_{\mathcal{L}(E)} \leq \|K_0\|_{\mathcal{L}(E)} \|KA_n - KA\|_{\mathcal{L}(E)} \to 0, \text{ as } n \to \infty \text{ and}$$
$$\|(KA_n - KA)K_0\|_{\mathcal{L}(E)} \leq \|KA_n - KA\|_{\mathcal{L}(E)} \|K_0\|_{\mathcal{L}(E)}, \text{ as } n \to \infty.$$

Similarly the case where $A_nK \to AK$ converges $\mathcal{P}$-strongly.

(b) Let $A \in \mathcal{L}(E,\mathcal{P})$ and let A be invertible in $\mathcal{L}(E)$. Thus $A^{-1} \in \mathcal{L}(E)$. Fix $K \in \mathcal{K}(E,\mathcal{P})$. Now

$$\begin{aligned}\left\|A^{-1}KP_n - A^{-1}K\right\|_{\mathcal{L}(E)} &= \left\|(A^{-1})^2AKP_n - (A^{-1})^2AK\right\|_{\mathcal{L}(E)}\\ &\leq \left\|(A^{-1})^2\right\|_{\mathcal{L}(E)} \|AKP_n - AK\|_{\mathcal{L}(E)} \to 0.\end{aligned}$$

Thus $A^{-1}K$ is $\mathcal{P}$-compact for all $K \in \mathcal{K}(E,\mathcal{P})$. Similarly

$$\begin{aligned}\left\|KA^{-1}P_n - KA^{-1}\right\|_{\mathcal{L}(E)} &= \left\|KA(A^{-1})^2P_n - KA(A^{-1})^2\right\|_{\mathcal{L}(E)}\\ &\leq \left\|(A^{-1})^2\right\|_{\mathcal{L}(E)} \|AKP_n - AK\|_{\mathcal{L}(E)} \to 0.\end{aligned}$$

Hence KA^{-1} is $\mathcal{P}$-compact for all $K \in \mathcal{K}(E,\mathcal{P})$. The cases with P_n and K on left-hand side are analogous. This shows that $A^{-1} \in \mathcal{L}(E,\mathcal{P})$. Hence $\mathcal{L}(E,\mathcal{P})$ is inverse-closed in $\mathcal{L}(E)$.

(c) To prove that $\mathcal{L}(E,\mathcal{P})$ is closed in $\mathcal{L}(E)$ it suffices to prove that $\mathcal{K}(E,\mathcal{P})$ is closed. To prove this let $(K_m)_{m\in\mathbb{N}}$ be a sequence of $\mathcal{P}$-compact operators which converge to K in operator norm and fix $\varepsilon > 0$. Remember that we have $Q_n = I - P_n$. To show $K \in \mathcal{K}(E, \mathcal{P})$, choose such $r, N \in \mathbb{N}$ that $\|K - K_r\|_{\mathcal{L}(E)} \sup_n \|Q_n\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2}$ and $\|K_r Q_n\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2}$ for $n \geq N$. Now we have

$$\begin{aligned}\|K - KP_n\|_{\mathcal{L}(E)} = \|KQ_n\|_{\mathcal{L}(E)} &= \|KQ_n - K_rQ_n + K_rQ_n\|_{\mathcal{L}(E)} \\ &\leq \|KQ_n - K_rQ_n\|_{\mathcal{L}(E)} + \|K_rQ_n\|_{\mathcal{L}(E)} \\ &\leq \|K - K_r\|_{\mathcal{L}(E)} \sup_n \|Q_n\|_{\mathcal{L}(E)} + \|K_rQ_n\|_{\mathcal{L}(E)} < \varepsilon.\end{aligned}$$

Similarly $\|K - P_nK\|_{\mathcal{L}(E)}$ as $n \to \infty$. Thus we conclude that $\mathcal{K}(E, \mathcal{P})$ is closed. For a sequence $(A_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{L}(E, \mathcal{P})$, which converges to A, the sequences $(KA_n)_{n\in\mathbb{N}}$ and $(A_nK)_{n\in\mathbb{N}}$ are sequences of $\mathcal{P}$-compact operators which converge to KA and AK respectively. Thus by what has been shown above we conclude that also $\mathcal{L}(E, \mathcal{P})$ is closed.

To show that $\mathcal{L}(E, \mathcal{P})$ is a subalgebra of $\mathcal{L}(E)$ we need only simple calculations. Let $A, B \in \mathcal{L}(E, \mathcal{P})$. This means that $AK, KA, BK, KB \in \mathcal{K}(E, \mathcal{P})$ and thus

$$\begin{aligned}\|(A + B)KP_n - (A + B)K\|_{\mathcal{L}(E)} &= \|AKP_n + BKP_n - AK + BK\|_{\mathcal{L}(E)} \\ &\leq \|AKP_n - AK\|_{\mathcal{L}(E)} + \|BKP_n - BK\|_{\mathcal{L}(E)} \to 0.\end{aligned}$$

Next fix $m \in \mathbb{N}$. Now for any $r \in \mathbb{N}$ holds

$$\begin{aligned}\|P_mABQ_n\|_{\mathcal{L}(E)} &\leq \|P_mAP_rBQ_n\|_{\mathcal{L}(E)} + \|P_mAQ_rBQ_n\|_{\mathcal{L}(E)} \\ &\leq C\,\|P_rBQ_n\|_{\mathcal{L}(E)} + D\,\|P_mAQ_r\|_{\mathcal{L}(E)},\end{aligned}$$

with C and D being constants. Now we can choose such an r to make the second term as small as desired and choose n_0 large enough that the first term is as small as desired for all $n \geq n_0$. The case of Q_nABP_m is analogous. Thus by Lemma 3.2.5 we conclude that $AB \in \mathcal{L}(E, \mathcal{P})$. Lastly for $\lambda \in \mathbb{R}$

$$\|\lambda AKP_n - \lambda AK\|_{\mathcal{L}(E)} = |\lambda|\,\|AKP_n - AK\|_{\mathcal{L}(E)} \to 0.$$

The cases with P_n and K on left-hand side are analogous. Hence $\mathcal{L}(E, \mathcal{P})$ is a subalgebra of $\mathcal{L}(E)$.

All this combined with the fact that $\mathcal{L}(E)$ is a Banach space imply that $\mathcal{L}(E, \mathcal{P})$ is a Banach algebra. Additionally, by the way $\mathcal{L}(E, \mathcal{P})$ is defined, for any $S, T \in \mathcal{L}(E, \mathcal{P})$ and any $K \in \mathcal{K}(E, \mathcal{P})$ we have $SK \in \mathcal{K}(E, \mathcal{P})$ and $KT \in \mathcal{K}(E, \mathcal{P})$, thus making $\mathcal{K}(E, \mathcal{P})$ a closed ideal of $\mathcal{L}(E, \mathcal{P})$. □

An example of a non-trivial operator in $\mathcal{L}(E, \mathcal{P})$ is the shift operator V_k, $k \in \mathbb{Z}^N$. This will be crucial for the properties of limit operators which will be defined in the next section.

Example 3.2.7. The shift operators, $V_k \in \mathcal{L}(E)$ belongs to $\mathcal{L}(E, \mathcal{P})$, for all $k \in \mathbb{Z}^N$.

Proof. Let us fix $x = (x_i)_{i\in\mathbb{Z}^N} \in E$ and $m \in \mathbb{N}$. Recall $U_n = \{-n, ..., n\}^N$ from before. We use the notation $(x_i)_{i\in U_n}$ for

$$(x_i)_{i\in U_n} = \begin{cases} x_i, \text{ for } i \in U_n \\ 0, \text{ otherwise.} \end{cases}$$

Now

$$\begin{aligned} P_m V_k Q_n x = (P_m V_k - P_m V_k P_n)x = P_m V_k (x_i)_{i\in\mathbb{Z}^N} - P_m V_k (x_i)_{i\in U_n} \\ = P_m (x_{i-k})_{i\in Z^N} - P_m (x_{i-k})_{i\in U_n} = (x_{i-k})_{i\in U_m} - (x_{i-k})_{i\in U_n\cap U_m} \\ = (x_{i-k})_{i\in U_m\setminus U_n} = 0, \text{ for } n \geq m. \end{aligned}$$

Similarly for $Q_n V_k P_m$. Thus

$$\|P_m V_k Q_n\|_{\mathcal{L}(E)} \to 0 \text{ and } \|Q_n V_k P_m\|_{\mathcal{L}(E)} \to 0 \text{ as } n \to \infty.$$

Hence according to Lemma 3.2.5. we get that $V_k \in \mathcal{L}(E, \mathcal{P})$ for all $k \in \mathbb{Z}^N$. □

For the next theorem we will introduce a new class of functions on R^N called

$$BUC(R^N) := \{\varphi : R^N \to \mathbb{C} \mid \varphi \text{ is bounded and uniformly continuous}\}$$

and a new "hat"-notation, where $\hat{\varphi} : \mathbb{Z}^N \to \mathbb{C}$ is the restriction to $\mathbb{Z}^N$ of the function φ from $\mathbb{R}^N$ to $\mathbb{C}$ and $\varphi_{t,r}(x) = \varphi(t(x - r))$ for $x, r \in \mathbb{R}^N$ and $t > 0$. We omit the subscript r for $r = 0$ so that $\varphi_{t,0}(x) = \varphi_t(x)$. Hence $\hat{\varphi}_{t,r}$ and $\hat{\varphi}_t$ represent the restrictions to $\mathbb{Z}^N$ of the functions $\varphi_{t,r}$ and φ_t respectively. Note, that here $\hat{\varphi}_{t,r}I$ is the multiplication operator $(x_i) \mapsto (\hat{\varphi}_{t,r}(i)x_i)$ on E and similarly for $\hat{\varphi}_t I$.

Theorem 3.2.8. Let $A \in \mathcal{L}(E)$, where $E = l^p(\mathbb{Z}^N, X)$. The following conditions are equivalent:

(a) A is band-dominated

(b) For every $\varphi \in BUC(\mathbb{R}^N)$,

$$\lim_{t\to 0} \sup_{r\in\mathbb{R}^N} \|\hat{\varphi}_{t,r} A - A\hat{\varphi}_{t,r} I\|_{\mathcal{L}(E)} = 0$$

(c) For every $\varphi \in BUC(\mathbb{R}^N)$,

$$\lim_{t \to 0} \|\hat{\varphi}_t A - A\hat{\varphi}_t I\|_{\mathcal{L}(E)} = 0$$

Proof. The proof is skipped here, but can be found in the book [1] (Theorem 2.1.6. page 36) by Rabinovich, Roch and Silbermann. □

Recall the notation for the space of band-dominated operators $\mathcal{A}_E$, Definition 2.1.4. The following theorem from [1] gives us that the inverses of band-dominated operators are also band-dominated operators.

Theorem 3.2.9. The algebra $\mathcal{A}_E$ is inverse closed in $\mathcal{L}(E)$.

Proof. Let $A \in \mathcal{A}_E$ be invertible in $\mathcal{L}(E)$. Define $\varphi_t = \varphi_{t,0}$ for $\varphi \in BUC(\mathbb{R}^N)$. Thus it holds that

$$\begin{aligned}\left\|\hat{\varphi}_t A^{-1} - A^{-1}\hat{\varphi}_t I\right\|_{\mathcal{L}(E)} &= \left\|A^{-1}A\hat{\varphi}_t A^{-1} - A^{-1}\hat{\varphi}_t A A^{-1}\right\|_{\mathcal{L}(E)} \\ &\leq \left\|A^{-1}\right\|_{\mathcal{L}(E)} \|A\hat{\varphi}_t I - \hat{\varphi}_t A\|_{\mathcal{L}(E)} \left\|A^{-1}\right\|_{\mathcal{L}(E)} \\ &= \left\|A^{-1}\right\|_{\mathcal{L}(E)}^2 \|\hat{\varphi}_t A - A\hat{\varphi}_t I\|_{\mathcal{L}(E)} \to 0, \text{ as } t \to 0.\end{aligned}$$

Hence by theorem 3.2.8 (c) also $A^{-1} \in \mathcal{A}_E$ is band-dominated. [1] □

Recall that $\mathcal{K}(E, \mathcal{P}) \subset \mathcal{L}(E, \mathcal{P})$ is a closed ideal which makes the quotient space $\mathcal{L}(E, \mathcal{P})/\mathcal{K}(E, \mathcal{P})$ into a Banach quotient algebra.

Definition 3.2.10 ($\mathcal{P}$- Fredholmness)**.** An operator $A \in \mathcal{L}(E, \mathcal{P})$ is called a $\mathcal{P}$- Fredholm operator, if the coset $A + \mathcal{K}(E, \mathcal{P})$ is invertible in the quotient algebra $\mathcal{L}(E, \mathcal{P})/\mathcal{K}(E, \mathcal{P})$.

The above definition means that A is $\mathcal{P}$-Fredholm if and only if there exist operators $C, D \in \mathcal{L}(E, \mathcal{P})$ and $K_1, K_2 \in \mathcal{K}(E, \mathcal{P})$ such that

$$AC = I + K_1 \text{ and } DA = I + K_2.$$

3.3 Limit operators

For it to make sense to define the concept of a limit operator, we need to define a certain class of sequences from [1]. Let $h = (h(n))_{n\in\mathbb{N}} \subset \mathbb{Z}^N$ be such a sequence that $|h(n)|$ tends to infinity as n tends to infinity. Here we consider the norm in $\mathbb{Z}^N$ as the maximum norm, $|m| = \max_{1\leq n\leq N} |m_n|$, for $m = (m_1, ..., m_N) \in \mathbb{Z}^N$. For such sequences $h := (h(n))_{n\in\mathbb{N}}$ we define the class

$$\mathcal{H} := \{h = (h(n))_{n\in\mathbb{N}} \subset \mathbb{Z}^N \ : \ |h(n)| \to \infty, \text{ as } n \to \infty\}.$$

For $\mathcal{P}$-strong convergence see Definition 3.2.4.

Definition 3.3.1 (Limit operators)**.** Let $h \in \mathcal{H}$ and $A \in \mathcal{L}(E, \mathcal{P})$. If the $\mathcal{P}$-strong limit

$$A_h = \mathcal{P}\text{-}\lim_{n \longrightarrow \infty} V_{-h(n)} A V_{h(n)}$$

exists, we define $A_h \in \mathcal{L}(E)$ to be the limit operator of A with respect to the sequence h. Here $V_{h(n)}$ is the shift operator for $h(n) \in \mathbb{Z}^N$ from Definition 3.1.2. that is, $(x_i) \underset{V_{h(n)}}{\mapsto} (x_{i-h(n)})$.

We also define the operator spectrum $\sigma_{op}(A)$ of A as

$$\begin{gathered} \sigma_{op}(A) = \{A_h : h \in \mathcal{H} \text{ and the } \mathcal{P}\text{-strong limit} \\ A_h =: \mathcal{P}\text{-}\lim_{n \longrightarrow \infty} V_{-h(n)} A V_{h(n)} \text{ exists}\}. \end{gathered}$$

The following theorem from [1] introduces some elementary properties of limit operators.

Theorem 3.3.2. Let $h \in \mathcal{H}$, and let $A, B \in \mathcal{L}(E, \mathcal{P})$ be operators for which the limit operators A_h and B_h exist. Then:

(a) $\|A_h\|_{\mathcal{L}(E)} \leq C \|A\|_{\mathcal{L}(E)}$, where C is independent of h and A.

(b) the limit operators $(A+B)_h$ and $(AB)_h$ exist and we get the algebraic identities $(A+B)_h = A_h + B_h$ and $(AB)_h = A_h B_h$.

(c) if A is invertible, then A_h is invertible, the limit operator $(A^{-1})_h$ exists and $(A^{-1})_h = (A_h)^{-1}$.

(d) the limit operator $(A^*)_h$ with respect to $\mathcal{P}^* = (P_n^*)_{n \in \mathbb{N}}$ on E^* exists and $(A^*)_h = (A_h)^*$.

Proof. (a) Let $A_l, A_r : \mathcal{K}(E, \mathcal{P}) \to \mathcal{K}(E, \mathcal{P})$ represent left (l) and right (r) multiplication with the operator A, that is $A_l(S) = AS$ and $A_r(S) = SA$, for $S \in \mathcal{K}(E, \mathcal{P})$. We want to prove first that

$$\begin{gathered} \|A_r\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} \leq \|A\|_{\mathcal{L}(E)} \leq C \|A_r\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))}, \\ \|A_l\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} \leq \|A\|_{\mathcal{L}(E)} \leq C^3 \|A_l\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} \end{gathered}$$

where $C = \sup_n \|P_n\|_{\mathcal{L}(E)}$. The first part of these inequalities are trivial since

$$\|A_r\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} = \sup_{\|K\|_{\mathcal{L}(E)} \leq 1} \|KA\|_{\mathcal{L}(E)} \leq \|A\|_{\mathcal{L}(E)}$$

and identically for A_l. For the second inequality for A_r let $\varepsilon > 0$. Choose such an $x_0 \in E$ with $\|x_0\| = 1$ that

$$\|Ax_0\|_E \geq \|A\|_{\mathcal{L}(E)} - \varepsilon.$$

This is possible, since by the definition of supremum for any $\varepsilon > 0$ we can always find such a $y_0 \in B_E$ that $\|Ay_0\|_E \geq \sup_{y\in B_E} \|Ay\|_E - \varepsilon$ and by normalizing with $y_1 = \frac{y_0}{\|y_0\|_E}$ we attain $\|y_1\|_E = 1$ and

$$\|Ay_1\|_E = \frac{1}{\|y_0\|_E} \|Ay_0\|_E \geq \|Ay_0\|_E \geq \sup_{y\in B_E} \|Ay\|_E - \varepsilon.$$

Next, by a similar argument let $P_n \in \mathcal{P}$ be such that

$$\|P_n A x_0\|_E \geq \|Ax_0\|_E - \varepsilon.$$

Since $\mathcal{P} \subset \mathcal{K}(E, \mathcal{P})$ we have that

$$\|A_r\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} = \sup_{K\in\mathcal{K}(E,\mathcal{P})/\{0\}} \frac{\|KA\|_{\mathcal{L}(E)}}{\|K\|_{\mathcal{L}(E)}} \geq \frac{\|P_nA\|_{\mathcal{L}(E)}}{\|P_n\|_{\mathcal{L}(E)}} \geq \frac{\|P_nA\|_{\mathcal{L}(E)}}{\sup_n \|P_n\|_{\mathcal{L}(E)}}$$
$$\geq \frac{\|P_nAx_0\|_E}{\sup_n \|P_n\|_{\mathcal{L}(E)}} \geq \frac{1}{\sup_n \|P_n\|_{\mathcal{L}(E)}} (\|A\|_{\mathcal{L}(E)} - 2\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary obtain the desired result

$$\|A\|_{\mathcal{L}(E)} \leq C \|A_r\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))},$$

where $C = \sup_n \|P_n\|_{\mathcal{L}(E)}$

For the second inequality in the A_l case, let $\varepsilon > 0$. As shown before there exists such an m that

$$\|P_mA\|_{\mathcal{L}(E)} = \frac{\|P_mA\|_{\mathcal{L}(E)}}{\|P_m\|_{\mathcal{L}(E)}} \geq \frac{1}{C} \|A\|_{\mathcal{L}(E)} - \varepsilon.$$

Lemma 3.2.5. allows us to choose such an n that

$$\|P_mAQ_n\|_{\mathcal{L}(E)} = \|P_mA - P_mAP_n\|_{\mathcal{L}(E)} < \varepsilon.$$

Hence now we have that

$$C^2 \|A_l\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} \geq C^2 \frac{\|AP_n\|_{\mathcal{L}(E)}}{\|P_n\|_{\mathcal{L}(E)}} \geq C^2 \frac{\|AP_n\|_{\mathcal{L}(E)}}{C} \geq C \|AP_n\|_{\mathcal{L}(E)}$$
$$\geq \|P_mAP_n\|_{\mathcal{L}(E)} \geq \|P_mA\|_{\mathcal{L}(E)} - \|P_mAQ_n\|_{\mathcal{L}(E)} \geq \frac{1}{C} \|A\|_{\mathcal{L}(E)} - 2\varepsilon,$$

where $C = \sup_n \|P_n\|_{\mathcal{L}(E)}$. Thus, since $\varepsilon > 0$ was arbitrary, we have that

$$\|A\|_{\mathcal{L}(E)} \leq C^3 \|A_l\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))}.$$

Hence we have proved the necessary inequalities and as a result we see that $\mathcal{P}$-strong convergence is essentially equivalent to regular strong convergence of the left- and right multiplication operators on $\mathcal{K}(E,\mathcal{P})$. Thus by applying the Banach -Steinhaus theorem to a sequence $(A_n^{(r)}) \subset \mathcal{L}(\mathcal{K}(E,\mathcal{P}))$, for which $A_n^{(r)} \to A_r$ strongly in $\mathcal{K}(E,\mathcal{P})$, we obtain

$$\|A_r\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} \leq \liminf_{n\to\infty} \left\|A_n^{(r)}\right\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))}$$

and additionally for the sequence $(A_n) \subset \mathcal{L}(E,\mathcal{P})$ for which $A_n \to A$ $\mathcal{P}$-strongly in E as $n \to \infty$ we have a similar result

$$\|A\|_{\mathcal{L}(E)} \leq C\,\|A_r\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} \leq \liminf_{n\to\infty} C \left\|A_n^{(r)}\right\|_{\mathcal{L}(\mathcal{K}(E,\mathcal{P}))} \leq \liminf_{n\to\infty} C\,\|A_n\|_{\mathcal{L}(E)}\,.$$

It follows that, since $V_{-h(n)}AV_{h(n)} \to A_h$ $\mathcal{P}$-strongly in E we have

$$\begin{aligned}\|A_h\|_{\mathcal{L}(E)} &\leq \liminf_{n\to\infty} C \left\|V_{-h(n)}AV_{h(n)}\right\|_{\mathcal{L}(E)} \\ &\leq \liminf_{n\to\infty} C \underbrace{\left\|V_{h(n)}\right\|^2_{\mathcal{L}(E)}}_{=1} \|A\|_{\mathcal{L}(E)} = C\,\|A\|_{\mathcal{L}(E)}\,,\end{aligned}$$

where $C = \sup_n \|P_n\|_{\mathcal{L}(E)}$. Thus we have proved that $\|A_h\|_{\mathcal{L}(E)} \leq C\,\|A\|_{\mathcal{L}(E)}$ and that the constant C is independent from h and A. In our case actually $C = 1$, but if one uses a more general approximate projection class $\mathcal{P}$, this might not be the case. However this is not within the scope of this thesis.

(b) Fix $P_m \in \mathcal{P}$. Now

$$\begin{aligned}&\left\|\Big(V_{-h(n)}(A+B)V_{h(n)} - (A_h + B_h)\Big)P_m\right\|_{\mathcal{L}(E)} \\ &= \left\|\Big(V_{-h(n)}AV_{h(n)} + V_{-h(n)}BV_{h(n)} - (A_h + B_h)\Big)P_m\right\|_{\mathcal{L}(E)} \\ &\leq \left\|(V_{-h(n)}AV_{h(n)} - A_h)P_m\right\|_{\mathcal{L}(E)} + \left\|(V_{-h(n)}BV_{h(n)} - B_h)P_m\right\|_{\mathcal{L}(E)} \to 0,\end{aligned}$$

as $n \to \infty$ for all $m \in \mathbb{N}$. The proof where P_m is on the left hand side is similar. Thus the limit operator $(A+B)_h = A_h + B_h$ exists.

Also, since $I = V_0 = V_{h(n)}V_{-h(n)}$ we have

$$\begin{aligned}&\left\|(V_{-h(n)}(AB)V_{h(n)} - (A_hB_h))P_m\right\|_{\mathcal{L}(E)} \\ &= \left\|(V_{-h(n)}AV_{h(n)}V_{-h(n)}BV_{h(n)} - (A_hB_h))P_m\right\|_{\mathcal{L}(E)} \to 0\end{aligned}$$

as $n \to \infty$ for all $m \in \mathbb{N}$. The proof where P_m is on the left hand side is similar. Thus the limit operator $(AB)_h = A_hB_h$ exists.

(c) Let $A \in \mathcal{L}(E, \mathcal{P})$ be invertible. Because $\mathcal{L}(E, \mathcal{P})$ is inverse closed, also $A^{-1} \in \mathcal{L}(E, \mathcal{P})$. We know from Example 3.2.7 that $V_{h(n)} \in \mathcal{L}(E, \mathcal{P})$ for all $n \in \mathbb{N}$ and by Theorem 3.2.6 (c) the product of two operators in $\mathcal{L}(E, \mathcal{P})$ is also in $\mathcal{L}(E, \mathcal{P})$ since $\mathcal{L}(E, \mathcal{P})$ is a subalgebra. Thus $V_{-h(n)}A^{-1}V_{h(n)} \in \mathcal{L}(E, \mathcal{P})$, for all $n \in \mathbb{N}$. Additionally

$$(V_{-h(n)}AV_{h(n)})^{-1} = V_{h(n)}^{-1}A^{-1}V_{-h(n)}^{-1} = V_{-h(n)}A^{-1}V_{h(n)},$$

for all $n \in \mathbb{N}$. Theorem 3.2.6.(a) states that $\mathcal{P}$-strong limits of sequences of operators in $\mathcal{L}(E, \mathcal{P})$ are in $\mathcal{L}(E, \mathcal{P})$, so that $A_h^{-1} \in \mathcal{L}(E, \mathcal{P})$ exists and

$$A_h^{-1} = (A^{-1})_h.$$

(d) Suppose $A_n \to A$ $\mathcal{P}$-strongly. Let $K^* \in \mathcal{K}(E^*, \mathcal{P}^*)$, $x \in E$ and $x^* \in E^*$. Now,

$$\begin{aligned}\left\| K^*(V_{-h(n)}AV_{h(n)} - A_h)^* \right\|_{E^*} &= \sup_{\|x\|_E \le 1} |\langle x, K^*(V_{h(n)}^*A^*V_{-h(n)}^* - (A_h)^*)x^* \rangle| \\ &= \sup_{\|x\|_E \le 1} |\langle Kx, V_{h(n)}^*A^*V_{-h(n)}^*x^* \rangle - \langle Kx, (A_h)^*x^* \rangle| \\ &= \sup_{\|x\|_E \le 1} |\langle V_{-h(n)}AV_{h(n)}Kx, x^* \rangle - \langle A_hKx, x^* \rangle| \\ &= \sup_{\|x\|_E \le 1} |\langle (V_{-h(n)}AV_{h(n)} - A_h)Kx, x^* \rangle \to 0,\end{aligned}$$

since $V_{-h(n)}AV_{h(n)} \to A_h$ $\mathcal{P}$-strongly as $n \to \infty$.

Furthermore, for any $n \in \mathbb{N}$, we have

$$(V_{-h(n)}AV_{h(n)})^* = V_{h(n)}^*A^*V_{-h(n)}^* = V_{-h(n)}A^*V_{h(n)}.$$

This is because the dual space of $E = l^p(\mathbb{Z}^N, X)$ is $E^* = l^q(\mathbb{Z}^N, X^*)$, with q satisfying $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < \infty$. Hence we can find the adjoint of the shift operator $V_{h(n)} = (x_{i-h(n)})_{i \in \mathbb{Z}^N}$ by operating on a linear functional, that is

$$\langle V_{h(n)}x, x^* \rangle = \sum_{i \in \mathbb{Z}^N} x_i^*(x_{i-h(n)}) = \sum_{i \in \mathbb{Z}^N} x_{i+h(n)}^*(x_i) = \langle x, V_{-h(n)}^*x^* \rangle.$$

From this we can solve that indeed $V_{h(n)}^* = V_{-h(n)}$ in the dual space E^*.

Hence combining these result we conclude that the limit operator $(A^*)_h$ exists and furthermore that $(A^*)_h = (A_h)^*$. $\square$

Next we will introduce a concept of richness of an operator, regarding the operator spectrum and how plentiful it is. We equate the notion of a rich operator with the notion an operator with rich operator spectrum.

Definition 3.3.3 (Richness)**.** Let $A \in \mathcal{L}(E, \mathcal{P})$ be an operator. The operator A is said to have a rich operator spectrum, if for every sequence $h \in \mathcal{H}$ there exists a subsequence $g \subset h$ for which the limit operator $A_g = \mathcal{P}\text{-}\lim_{n\longrightarrow\infty} V_{-g(n)}AV_{g(n)}$ with respect to the subsequence g exists.

We denote the class of rich operators with a dollar sign \$ superscript, for example $\mathcal{L}^{\$}(E, \mathcal{P}) := \{A \in \mathcal{L}(E, \mathcal{P}) :\ A \text{ is rich}\}$ and $\mathcal{A}_E^{\$} := \{A \in \mathcal{A}_E : A \text{ is rich}\}$.

The most trivial examples of rich operators are the identity operator I and the shift operator V_k, since for any $h \in \mathcal{H}$ we have $V_{-h(m)}IV_{h(m)} = I$ and $V_{-h(m)}V_kV_{h(m)} = V_{-h(m)+k+h(m)} = V_k$.

The following lemma from [1] will be useful for proving our main result and it shows that the $\mathcal{P}$-compact operators form a closed ideal in $\mathcal{L}^{\$}(E, \mathcal{P})$.

Lemma 3.3.4. Let $K \in \mathcal{K}(E, \mathcal{P})$ be a $\mathcal{P}$-compact operator and $h \in \mathcal{H}$. Then K_h is the zero-operator.

Proof. Fix $P_m \in \mathcal{P}$. Consider

$$\begin{aligned} \left\|V_{-h(k)}KV_{h(k)}P_m\right\|_{\mathcal{L}(E)} &= \left\|V_{-h(k)}K(P_n + I - P_n)V_{h(k)}P_m\right\|_{\mathcal{L}(E)} \\ &\le \left\|V_{-h(k)}KP_nV_{h(k)}P_m\right\|_{\mathcal{L}(E)} + \left\|V_{-h(k)}KQ_nV_{h(k)}P_m\right\|_{\mathcal{L}(E)} \\ &\le C\left\|P_nV_{h(k)}P_m\right\|_{\mathcal{L}(E)} + C\left\|KQ_n\right\|_{\mathcal{L}(E)}, \end{aligned}$$

for some constant C. Given any $\varepsilon > 0$, choose such $n, k_0 \in \mathbb{N}$ that $\|KQ_n\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2C}$ and $\left\|P_nV_{h(k)}P_m\right\|_{\mathcal{L}(E)} < \frac{\varepsilon}{2C}$ for $k \ge k_0$.

Thus,

$$\left\|V_{-h(k)}KV_{h(k)}P_m\right\|_{\mathcal{L}(E)} < 2C\frac{\varepsilon}{2C} = \varepsilon$$

and K_h is the zero-operator. □

Since $K_h = 0$ for any $h \in \mathcal{H}$, clearly $K_g = 0$ for all $g \subset h$, thus $\mathcal{K}(E, \mathcal{P}) \subset \mathcal{L}^{\$}(E, \mathcal{P})$. Additionally, the fact that $\mathcal{K}(E, \mathcal{P})$ is a closed ideal of $\mathcal{L}(E, \mathcal{P})$ implies that $\mathcal{K}(E, \mathcal{P})$ is indeed a closed ideal of $\mathcal{L}^{\$}(E, \mathcal{P})$.

The goal of this thesis is to prove the following fundamental characterization of $\mathcal{P}$-Fredholmness for rich band-dominated operators A on $E = l^p(\mathbb{Z}^N, X)$. According to [8] this result was first proven by B.V. Lange and V.S. Rabinovich for $1 < p < \infty$. The condition that the inverses

$$\{A_h^{-1} : h \in \mathcal{H} \text{ and } A_h \text{ exists}\}$$

are uniformly bounded was removed quite recently by Lindner and Seidel, see [6], however this stronger result is outside of the scope of this thesis. The rest of this thesis is dedicated to our main theorem and to the actual proof of it. From now on we follow along the lines of the proof presented in [1].

Theorem 3.3.5. Let A be a rich band-dominated operator. Then A is $\mathcal{P}$-Fredholm if and only if all limit operators A_h of A are invertible and the inverses $\{A_h^{-1} : h \in \mathcal{H}\}$ are uniformly bounded, meaning that there exists $C < \infty$ such that

$$\left\|A_h^{-1}\right\|_{\mathcal{L}(E)} < C,$$

for all $h \in \mathcal{H}$ such that A_h exists.

Proof. We first prove the easier part of the theorem. "$\Rightarrow$"

Let $A \in \mathcal{L}(E, \mathcal{P})$ be a $\mathcal{P}$-Fredholm operator, that is there exists operators $D \in \mathcal{L}(E, \mathcal{P})$ and $T_1, T_2 \in \mathcal{K}(E, \mathcal{P})$ such that $DA = I + T_1$ and $AD = I + T_2$. If $h \in \mathcal{H}$ is a sequence such that the limit operator A_h exists, then for all $K \in \mathcal{K}(E, \mathcal{P})$ holds

$$\begin{aligned} K &= V_{-h(n)} I V_{h(n)} K = V_{-h(n)}(DA - T_1)V_{h(n)}K \\ &= V_{-h(n)} D V_{h(n)} V_{-h(n)} A V_{h(n)} K - V_{-h(n)} T_1 V_{h(n)} K \end{aligned}$$

where we have the estimate $\left\|V_{-h(n)} D V_{h(n)}\right\|_{\mathcal{L}(E)} \leq \|D\|_{\mathcal{L}(E)} =: C$ independently of the sequence $h \in \mathcal{H}$, so that

$$\|K\|_{\mathcal{L}(E)} \leq C \left\|V_{-h(n)} A V_{h(n)} K\right\|_{\mathcal{L}(E)} + \left\|V_{-h(n)} T_1 V_{h(n)} K\right\|_{\mathcal{L}(E)}.$$

As n tends to infinity, $V_{-h(n)} A V_{h(n)} \to A_h$ in $\mathcal{P}$-limit and by lemma 3.3.4, since T_1 is a $\mathcal{P}$-compact operator, we know that $V_{-h(n)} T_1 V_{h(n)}$ tends to the zero-operator. Hence

$$\|K\|_{\mathcal{L}(E)} \leq C \|A_h K\|_{\mathcal{L}(E)}.$$

Similarly for $A^* \in \mathcal{L}(E^*, \mathcal{P}^*)$ we have

$$\begin{aligned} K^* &= V_{-h(n)} I V_{h(n)} K^* = V_{-h(n)}(D^* A^* - T_2^*)V_{h(n)}K^* \\ &= V_{-h(n)} D^* V_{h(n)} V_{-h(n)} A^* V_{h(n)} K^* - V_{-h(n)} T_2^* V_{h(n)} K^*, \end{aligned}$$

where $K^* \in \mathcal{K}(E^*, \mathcal{P}^*)$. As before we have

$$\|K^*\|_{\mathcal{L}(E^*)} \leq C \left\|V_{-h(n)}A^*V_{h(n)}K^*\right\|_{\mathcal{L}(E^*)} + \left\|V_{-h(n)}T_2^*V_{h(n)}K^*\right\|_{\mathcal{L}(E^*)}$$

and by letting $n \to \infty$ we obtain

$$\|K^*\|_{\mathcal{L}(E^*)} \leq C \|A_h^*K^*\|_{\mathcal{L}(E)}.$$

Proposition 3.2.2. states that $P_n \to I$ strongly in E, that is $\|P_nx - x\|_E \to 0$ as $n \to \infty$ for all $x \in E$. Additionally Theorem 2.3.7. states that this is equivalent to $\|P_nK - K\|_{\mathcal{L}(E)} \to 0$ as $n \to \infty$ for all $K \in \mathcal{K}(E)$. Thus compact operators are also $\mathcal{P}-$compact. The same argument applies in the dual case $\mathcal{K}(E^*)$. Hereby, we obtain the fact that both $\mathcal{K}(E) \subseteq \mathcal{K}(E, \mathcal{P})$ and $\mathcal{K}(E^*) \subseteq \mathcal{K}(E^*, \mathcal{P}^*)$ hold. Thus we can replace K with a rank one operator similarly as in the proof of Theorem 2.3.7 to get the following estimates

$$\|x\|_E \leq C \|A_hx\|_E \text{ and } \|f\|_{E^*} \leq C' \|A_h^*f\|_{E^*},$$

for all $x \in E$ and $f \in E^*$. This means that

$$\ker A_h = \{0\} = \ker A_h^*.$$

Firstly, $\ker A_h = \{0\}$ implies that the limit operator A_h is an injection. Furthermore $\ker A_h^* = \{0\}$ means that the set

$$\begin{aligned}\ker A_h^* &= \{x^* \in E^* : A_h^*x^* = 0\} \\ &= \{x^* \in E^* : \langle x, A_h^*x^*\rangle = 0, \text{ for all } x \in E\} \\ &= \{x^* \in E^* : \langle A_hx, x^*\rangle = 0, \text{ for all } x \in E\} = \{0\},\end{aligned}$$

is the singleton containing zero. We claim that A_h is a surjection, that is $\operatorname{Im} A_h = E$. We argue this by counterargument.

Assume that $\operatorname{Im} A_h \neq E$. Since A_h is bounded from below, we have that $\operatorname{Im} A_h$ is a closed linear subspace of E. Thus there exists an $x_0 \in E$ such that $x_0 \notin \operatorname{Im} A_h$, $x_0 \neq 0$ and $dist(x_0, \operatorname{Im} A_h) = 1$. Hahn-Banach theorem [2] states that there exists a linear functional $y^* \in E^*$ such that

$$\begin{aligned}&y^*(\operatorname{Im} A_h) = \{0\} \text{ and} \\ &y^*(x_0) = \langle x_0, y^*\rangle = 1.\end{aligned}$$

Now $\langle x_0, y^*\rangle \neq 0$, but $\langle x, A_h^*y^*\rangle = \langle A_hx, y^*\rangle = 0$ for all $x \in E$. Hence $\overline{0} \neq y^* \in \ker A_h^*$ and the kernel of A_h^* would not be a singleton, that is $\ker A_h^* \neq \{0\}$. This is a contradiction. Hence, $\operatorname{Im} A_h = E$ and A_h is a surjection. Thus A_h is a bijection and hence invertible. Finally, we choose an arbitrary $x \in E \setminus \{0\}$. There exists a $y \in E$, such that $x = A_h^{-1}(y)$ and applied with $\|x\|_E \leq C \|A_h x\|_E$ yields us the following estimate,

$$\left\|A_h^{-1}(y)\right\|_E \leq C \left\|A_h(A_h^{-1}y)\right\|_E = C \|y\|_E .$$

Taking the supremum over $\|y\|_E \leq 1$ we obtain the result

$$\left\|A_h^{-1}\right\|_{\mathcal{L}(E)} \leq C.$$

□

To prove the other direction of the equivalency we need to introduce some more building blocks and lemmas.

We start by fixing two continuous functions $\varphi, \psi : \mathbb{R} \to [0,1]$ such that:

$$\varphi(x) = \begin{cases} 1, \text{ for } |x| \leq \frac{1}{3} \\ \text{positive, for } \frac{1}{3} < |x| < \frac{2}{3} \\ 0, \text{ for } \frac{2}{3} \leq |x| \end{cases}$$

and

$$\psi(x) = \begin{cases} 1, \text{ for } |x| \leq \frac{3}{4} \\ \text{positive, for } \frac{3}{4} < |x| < \frac{4}{5} \\ 0, \text{ for } \frac{4}{5} \leq |x|. \end{cases}$$

Suppose that the families $\{\varphi_\alpha^2\}$ and $\{\psi_\alpha^2\}$, where $\varphi_\alpha^2(x) = \varphi_\alpha(x)\varphi_\alpha(x)$ and $\varphi_\alpha(x) = \varphi(x-\alpha)$ for $\alpha \in \mathbb{Z}$, form a partition of unity on $\mathbb{R}$, that is

$$\sum_{\alpha\in\mathbb{Z}} \varphi_\alpha^2(x) = 1, \text{ for all } x \in \mathbb{R},$$

and respectively for ψ_α. This can be forced by choosing continuous functions $f : \mathbb{R} \to [0,1]$ similarly as φ above and $g : \mathbb{R} \to [0,1]$ similarly as ψ above and then defining

$$\varphi(x) := \sqrt{\frac{f(x)}{\sum_{\alpha\in\mathbb{Z}} f(x-\alpha)}} \text{ and } \psi(x) := \sqrt{\frac{g(x)}{\sum_{\alpha\in\mathbb{Z}} g(x-\alpha)}}$$

hence the families $\{\varphi_\alpha^2\}$ and $\{\psi_\alpha^2\}$ form a partition of unity on $\mathbb{R}$, such that $\varphi(x) \geq 0$ and $\psi(x) \geq 0$ while still preserving the properties as originally defined. Hence let φ and ψ be as originally defined in this proof.

For an arbitrary dimension $N \in \mathbb{N}$, $x = (x_1, ..., x_N) \in \mathbb{R}^N$ we define $\varphi^{(N)}(x) := (\varphi(x_1), ..., \varphi(x_N))$ and $\psi^{(N)}(x) := (\psi(x_1), ..., \psi(x_N))$ and analogously $\varphi_\alpha^{(N)}(x) := \varphi^{(N)}(x - \alpha)$ and $\psi_\alpha^{(N)}(x) := \psi^{(N)}(x - \alpha)$. Finally, for $R > 0$, define

$$\varphi_{\alpha,R}^{(N)}(x) = \varphi_\alpha^{(N)}\left(\frac{x}{R}\right) = \varphi^{(N)}\left(\frac{x}{R} - \alpha\right)$$

and respectively

$$\psi_{\alpha,R}^{(N)}(x) = \psi_\alpha^{(N)}\left(\frac{x}{R}\right) = \psi^{(N)}\left(\frac{x}{R} - \alpha\right).$$

For these functions we have the property

$$\psi_{\alpha,R}^{(N)}\varphi_{\alpha,R}^{(N)} = \varphi^{(N)}\left(\frac{x}{R} - \alpha\right)\psi^{(N)}\left(\frac{x}{R} - \alpha\right) = \varphi_{\alpha,R}^{(N)}$$

for all $\alpha \in \mathbb{Z}^N$ and $R > 0$, since by substituting $y_k = \frac{x_k}{R} - \alpha$ to the previous equation, we see clearly that $\varphi(y_k) \geq 0$ only when $\psi(y_k) = 1$ and $\varphi(y_k) = 0$ otherwise. In addition the families $\{\varphi_{\alpha,R}^{(N)}\}_{\alpha \in \mathbb{Z}^N}$ and $\{\psi_{\alpha,R}^{(N)}\}_{\alpha \in \mathbb{Z}^N}$ form a partition of unity on $\mathbb{R}^N$ for every fixed R. We also remind of the "hat"-notation, where $\hat{\varphi}_{\alpha,R}^{(N)} : \mathbb{Z}^N \to [0,1]^N$ is the restriction to $\mathbb{Z}^N$ of the function $\varphi_{\alpha,R}^{(N)}$ from $\mathbb{R}^N$ to $[0,1]^N$.

The following three lemmas are needed towards the yet unproven direction of the equivalence. Note that we use the notation $\hat{\varphi}_{\alpha,R}^{(N)}I$ for the multiplication operator $\hat{\varphi}_{\alpha,R}^{(N)}I : E \to E$ where $\hat{\varphi}_{\alpha,R}^{(N)}Iu = (\hat{\varphi}_{\alpha,R}^{(N)}(k)u_k)_{k \in \mathbb{Z}^N}$, for all $u \in E$ and respectively for $\hat{\psi}_{\alpha,R}^{(N)}I$.

Lemma 3.3.6. Let $\varphi_{\alpha,R}^{(N)}$ and $\psi_{\alpha,R}^{(N)}$ be as constructed above. If $\{A_\alpha\}$ is a bounded family of operators in $\mathcal{L}(E, \mathcal{P})$, then the series

$$\sum_{\alpha \in \mathbb{Z}^N} \hat{\varphi}_{\alpha,R}^{(N)} A_\alpha \hat{\psi}_{\alpha,R}^{(N)} I$$

converges $\mathcal{P}$-strongly in $\mathcal{L}(E)$ for every fixed $R > 0$, and

$$\left\| \sum_{\alpha \in \mathbb{Z}^N} \hat{\varphi}_{\alpha,R}^{(N)} A_\alpha \hat{\psi}_{\alpha,R}^{(N)} I \right\|_{\mathcal{L}(E)} \leq 2^N \sup_{\alpha \in \mathbb{Z}^N} \|A_\alpha\|_{\mathcal{L}(E)}.$$

Proof. We start with the dimension $N = 1$. Fix $R > 0$ and let $\alpha, \beta \in 2\mathbb{Z}$ be even whole numbers such that $\alpha \neq \beta$.

Fix $a \in \mathbb{R}$ such that $\varphi_{\alpha,R}(a) = \varphi(\frac{a}{R} - \alpha) > 0$. Then we have

$$\varphi_{\beta,R}(a) = \varphi(\frac{a}{R} - \beta) = \varphi(\frac{a}{R} - \alpha + 2N),$$

for $N \in \mathbb{Z} \setminus \{0\}$. Substituting $y = \frac{a}{R} - \alpha$ and from the definition of φ we see that $\varphi_{\beta,R}(a) = \varphi(y + 2N)$ must be zero, since the $2N$ component translates a outside of the interval $-\frac{2}{3} \leq y \leq \frac{2}{3}$. The proof is similar for ψ in place of φ. Thus we have

$$supp(\varphi_{\alpha,R}) \cap supp(\varphi_{\beta,R}) = supp(\psi_{\alpha,R}) \cap supp(\psi_{\beta,R}) = \emptyset.$$

The same fact is true also whenever $\alpha, \beta \in 2\mathbb{Z} + 1$ are odd.

Thus for any $u \in E = l^p(\mathbb{Z}, \mathcal{L}(X))$ we have

$$\left\| \sum_{\alpha \in 2\mathbb{Z}} \hat{\varphi}_{\alpha,R} A_\alpha \hat{\psi}_{\alpha,R} u \right\|_E^p = \sum_{\alpha \in 2\mathbb{Z}} \left\| \hat{\varphi}_{\alpha,R} A_\alpha \hat{\psi}_{\alpha,R} u \right\|_E^p$$

$$\leq \sum_{\alpha \in 2\mathbb{Z}} \left\| A_\alpha \hat{\psi}_{\alpha,R} u \right\|_E^p \leq \sup_{\alpha \in \mathbb{Z}} \|A_\alpha\|_{\mathcal{L}(E)}^p \sum_{\alpha \in 2\mathbb{Z}} \left\| \hat{\psi}_{\alpha,R} u \right\|_E^p \leq \sup_{\alpha \in \mathbb{Z}} \|A_\alpha\|_{\mathcal{L}(E)}^p \|u\|_E^p .$$

Thus the series converges strongly in $\mathcal{L}(E)$. Taking the supremum over $u \in E$ with $\|u\|_E \leq 1$ yields us

$$\left\| \sum_{\alpha \in 2\mathbb{Z}} \hat{\varphi}_{\alpha,R} A_\alpha \hat{\psi}_{\alpha,R} I \right\|_{\mathcal{L}(E)} \leq \sup_{\alpha \in \mathbb{Z}} \|A_\alpha\|_{\mathcal{L}(E)} .$$

Similar estimates are also valid whenever $\alpha \in 2\mathbb{Z}+1$ is odd, thus we have the estimate

$$\left\| \sum_{\alpha \in \mathbb{Z}} \hat{\varphi}_{\alpha,R} A_\alpha \hat{\psi}_{\alpha,R} I \right\|_{\mathcal{L}(E)}$$

$$\leq \left\| \sum_{\alpha \in 2\mathbb{Z}} \hat{\varphi}_{\alpha,R} A_\alpha \hat{\psi}_{\alpha,R} I \right\|_{\mathcal{L}(E)} + \left\| \sum_{\alpha \in 2\mathbb{Z}+1} \hat{\varphi}_{\alpha,R} A_\alpha \hat{\psi}_{\alpha,R} I \right\|_{\mathcal{L}(E)} \leq 2 \sup_{\alpha \in \mathbb{Z}} \|A_\alpha\|_{\mathcal{L}(E)} .$$

In the case of $N > 1$ we use induction. By writing the points $x \in \mathbb{R}^N$ as $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ we have the properties

$$\varphi_{\alpha,R}^{(N)}(x) I = \varphi_{\alpha,R}^{(N-1)}(x') I \varphi_{\alpha,R}^{(1)}(x_N) I$$

and

$$\psi_{\alpha,R}^{(N)}(x) I = \psi_{\alpha,R}^{(N-1)}(x') I \psi_{\alpha,R}^{(1)}(x_N) I.$$

Thus by defining

$$B_{\alpha_N} := \sum_{\alpha' \in \mathbb{Z}^{N-1}} \varphi_{\alpha,R}^{(N-1)} A_{(\alpha',\alpha_N)} \psi_{\alpha,R}^{(N-1)} I$$

we can write

$$\left\| \sum_{\alpha \in \mathbb{Z}^N} \varphi_{\alpha,R}^{(N)} A_\alpha \psi_{\alpha,R}^{(N)} I \right\|_{\mathcal{L}(E)} = \left\| \sum_{\alpha_N \in \mathbb{Z}} \varphi_{\alpha,R}^{(1)} B_{\alpha_N} \psi_{\alpha,R}^{(1)} I \right\|_{\mathcal{L}(E)}$$

Since B_{α_N} is also in $\mathcal{L}(E,\mathcal{P})$ and by using the induction assumption

$$\|B_{\alpha_N}\|_{\mathcal{L}(E)} \leq 2^{N-1} \sup_{\alpha' \in \mathbb{Z}^{N-1}} \left\| A_{(\alpha',\alpha_N)} \right\|_{\mathcal{L}(E)} \leq 2^{N-1} \sup_{\alpha \in \mathbb{Z}^N} \|A_\alpha\|_{\mathcal{L}(E)},$$

it follows that B_{α_N} is bounded. Applying the result of the one-dimensional case we get

$$\left\| \sum_{\alpha_N \in \mathbb{Z}} \varphi_{\alpha,R}^{(1)} B_{\alpha_N} \psi_{\alpha,R}^{(1)} I \right\|_{\mathcal{L}(E)} \leq 2 \sup_{\alpha_N \in \mathbb{Z}} \|B_{\alpha_N}\|_{\mathcal{L}(E)} \leq 2^N \sup_{\alpha \in \mathbb{Z}^N} \|A_\alpha\|_{\mathcal{L}(E)}.$$

Hereby the series $\sum_{\alpha \in \mathbb{Z}^N} \varphi_{\alpha,R}^{(N)} A_\alpha \psi_{\alpha,R}^{(N)} I$ converges strongly in $\mathcal{L}(E)$ and defines a bounded linear operator on E.

Finally fixing $m \in \mathbb{N}$, for any $\alpha \in \mathbb{Z}^N$ large enough, we have $P_m \hat{\varphi}_{\alpha,R}^{(N)} I = 0 = \hat{\psi}_{\alpha,R}^{(N)} P_m$, thus $P_m \hat{\varphi}_{\alpha,R}^{(N)} A_\alpha \hat{\psi}_{\alpha,R}^{(N)} I$ and $\hat{\varphi}_{\alpha,R}^{(N)} A_\alpha \hat{\psi}_{\alpha,R}^{(N)} P_m$ are zero-operators for all $\alpha \in \mathbb{Z}^N$ with $|\alpha| \geq M$ for a large enough M. Hence the series $\sum_{\alpha \in \mathbb{Z}^N} \varphi_{\alpha,R}^{(N)} A_\alpha \psi_{\alpha,R}^{(N)} I$ converges also in the sense of the $\mathcal{P}$-strong convergence (You can replace K with P_m). □

The above result is also true when one switches the places of $\varphi_{\alpha,R}^{(N)}$ and $\psi_{\alpha,R}^{(N)}$.

Corollary 3.3.7. Let $\varphi_{\alpha,R}^{(N)}$ and $\psi_{\alpha,R}^{(N)}$ be as constructed above. If $\{A_\alpha\}$ is a bounded family of operators in $\mathcal{L}(E,\mathcal{P})$, then the series

$$\sum_{\alpha \in \mathbb{Z}^N} \hat{\psi}_{\alpha,R}^{(N)} A_\alpha \hat{\varphi}_{\alpha,R}^{(N)} I$$

converges in the $\mathcal{P}$-strong topology of E for every fixed $R > 0$, and

$$\left\| \sum_{\alpha \in \mathbb{Z}^N} \hat{\psi}_{\alpha,R}^{(N)} A_\alpha \hat{\varphi}_{\alpha,R}^{(N)} I \right\|_{\mathcal{L}(E)} \leq 2^N \sup_{\alpha \in \mathbb{Z}^N} \|A_\alpha\|_{\mathcal{L}(E)}.$$

Proof. The proof is identical with the above proof. The only difference is switching the places of $\varphi^{(N)}_{\alpha,R}$ and $\psi^{(N)}_{\alpha,R}$. $\square$

The purpose of the following lemma is to construct regularizers, that is Fredholm-inverses, of $\mathcal{P}$-Fredholm band-dominated operators.

Lemma 3.3.8. Let $A \in \mathcal{A}_E$ be a band-dominated operator in E and let $\psi^{(N)}_{\alpha,R}$ be as above. Suppose there exists an $M > 0$ such that, for all natural numbers $R \in \mathbb{N}$ there exists a $\rho(R) > 0$ such that, for all $\alpha \in \mathbb{Z}^N$ with $|\alpha| \geq \rho(R)$ there are operators $B_{\alpha,R}$ and $C_{\alpha,R}$ in $\mathcal{L}(E, \mathcal{P})$ with $\|B_{\alpha,R}\|_{\mathcal{L}(E)} \leq M$, $\|C_{\alpha,R}\|_{\mathcal{L}(E)} \leq M$ and

$$B_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} I = \hat{\psi}^{(N)}_{\alpha,R} A C_{\alpha,R} = \hat{\psi}^{(N)}_{\alpha,R} I.$$

Then the operator A is $\mathcal{P}$-Fredholm, and the $\mathcal{P}$-essential norm $\|\cdot\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})}$ of the regularizers of A is not greater than $2^{N+1}M$

Proof. Assume that the family $\{B_{\alpha,R} : |\alpha| \geq \rho(R)\}$ is uniformly bounded with the constant M. By the lemma 3.3.6 and the fact that $\psi^{(N)}_{\alpha,R}\varphi^{(N)}_{\alpha,R} = \varphi^{(N)}_{\alpha,R}$ the series

$$\sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I = \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} \hat{\psi}^{(N)}_{\alpha,R} I$$

converges $\mathcal{P}$-strongly to a certain operator B_R with $\|B_R\|_{\mathcal{L}(E)} \leq 2^N M$. By theorem 3.2.6 we know that B_R belongs to $\mathcal{L}(E, \mathcal{P})$.

Furthermore, since $\operatorname{dist}(supp(\varphi^{(N)}_{\alpha,R}), supp(1 - \psi^{(N)}_{\alpha,R})) = |\frac{3}{4} - \frac{2}{3}|R = \frac{R}{12}$ we know that

$$\lim_{R\to\infty} \operatorname{dist}(supp(\varphi^{(N)}_{\alpha,R}), supp(1 - \psi^{(N)}_{\alpha,R})) = \infty$$

and hence $\hat{\varphi}^{(N)}_{\alpha,R} A = \hat{\varphi}^{(N)}_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} I$ for large enough $R > 0$ and all $\alpha \in \mathbb{Z}^N$. Thus for large enough R we have

$$\begin{aligned}
B_R A &= \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} A = \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} I \\
&= \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} I - \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I \\
&\quad + \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I \\
&= \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I + T_R,
\end{aligned}$$

where T_R is defined as

$$T_R = \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R}[\hat{\varphi}^{(N)}_{\alpha,R}I, A]\hat{\psi}^{(N)}_{\alpha,R}I,$$

and $[\hat{\varphi}^{(N)}_{\alpha,R}I, A] = \hat{\varphi}^{(N)}_{\alpha,R}IA - A\hat{\varphi}^{(N)}_{\alpha,R}I$ is the commutator of $\hat{\varphi}^{(N)}_{\alpha,R}I$ and A. From the assumption $B_{\alpha,R}A\hat{\psi}^{(N)}_{\alpha,R}I = \hat{\psi}^{(N)}_{\alpha,R}I$ we get

$$\begin{aligned}\sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} B_{\alpha,R}A\hat{\psi}^{(N)}_{\alpha,R}\hat{\varphi}^{(N)}_{\alpha,R}I + T_R &= \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R}\hat{\psi}^{(N)}_{\alpha,R}\hat{\varphi}^{(N)}_{\alpha,R}I + T_R \\ &= \sum_{|\alpha|\geq\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I + T_R = I - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I + T_R.\end{aligned}$$

First suppose $A = \sum_{k\in\Omega} \alpha_k V_k$, for a suitable finite subset $\Omega \subset \mathbb{Z}^N$, is a band operator. Hence we know that $B_{\alpha,R}[\hat{\varphi}^{(N)}_{\alpha,R}I, A]$ is in $\mathcal{L}(E, \mathcal{P})$ also. Thus by lemma 3.3.6 T_R converges $\mathcal{P}$-strongly in $\mathcal{L}(E)$ and by Theorem 3.2.6 we get that T_R is also in $\mathcal{L}(E, \mathcal{P})$. From theorem 3.2.8 we find that since $\varphi^{(N)}_{\alpha,R} \in BUC(\mathbb{R}^N)$,

$$\lim_{R\to\infty} \left\|\hat{\varphi}^{(N)}_{\alpha,R}A - A\hat{\varphi}^{(N)}_{\alpha,R}I\right\|_{\mathcal{L}(E)} = \lim_{R\to\infty} \left\|[\hat{\varphi}^{(N)}_{\alpha,R}I, A]\right\|_{\mathcal{L}(E)} = 0$$

uniformly with respect to $\alpha \in \mathbb{Z}^N$. Hence $\|T_R\|_{\mathcal{L}(E)}$ tends to zero as R tends to infinity.

Now fix a large enough R such that $\|T_R\|_{\mathcal{L}(E)} < \frac{1}{2}$. Since $T_R \in \mathcal{L}(E, \mathcal{P})$, clearly also $I + T_R \in \mathcal{L}(E, \mathcal{P})$ and we know from the Neumann series that $I + T_R$ is invertible on E and

$$\left\|(I + T_R)^{-1}\right\|_{\mathcal{L}(E)} \leq \sum_{k=0}^{\infty} \left\|T^k_R\right\|_{\mathcal{L}(E)} = \frac{1}{1 - \|T_R\|_{\mathcal{L}(E)}} < 2.$$

Also since $\mathcal{L}(E, \mathcal{P})$ is inverse closed in $\mathcal{L}(E)$ we know that $(I + T_R)^{-1} \in \mathcal{L}(E, \mathcal{P})$. Multiplying the above identity

$$B_R A = I - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I + T_R = I + T_R - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I$$

with $(I + T_R)^{-1}$ from the left-hand side we attain

$$\begin{aligned}(I + T_R)^{-1} B_R A &= (I + T_R)^{-1}(I + T_R - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I) \\ &= I - (I + T_R)^{-1} \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I.\end{aligned}$$

Since the $(\hat{\varphi}^{(N)}_{\alpha,R})^2 I$ is a finite-rank operator for any $\alpha \in \mathbb{Z}^N$ and $R > 0$, it is also $\mathcal{P}$-compact. Hence $\sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I \in \mathcal{K}(E, \mathcal{P})$. Because $(I + T_R)^{-1}$ belongs to $\mathcal{L}(E, \mathcal{P})$ we finally attain

$$K_1 := -(I + T_R)^{-1} \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I \in \mathcal{K}(E, \mathcal{P}).$$

The $\mathcal{P}$-compactness of K_1 implies that

$$(I + T_R)^{-1} B_R A = I + K_1, \quad K_1 \in \mathcal{K}(E, \mathcal{P})$$

and thus $(I + T_R)^{-1} B_R$ is a left-sided regularizer for A with

$$\left\| (I + T_R)^{-1} B_R \right\|_{\mathcal{L}(E)} \leq 2 \left\| B_R \right\|_{\mathcal{L}(E)} \leq 2^{N+1} M.$$

For the right-sided regularizer the proof has a similar structure. The series

$$C_R := \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} C_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I = \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} C_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} \hat{\psi}^{(N)}_{\alpha,R} I$$

converges $\mathcal{P}$-strongly in E with $\|C_R\|_{\mathcal{L}(E)} \leq 2^N M$ and for large enough $R > 0$ we have $A \hat{\varphi}^{(N)}_{\alpha,R} I = \hat{\psi}^{(N)}_{\alpha,R} A \hat{\varphi}^{(N)}_{\alpha,R} I$.

Thus

$$\begin{aligned} AC_R &= \sum_{|\alpha|\geq\rho(R)} A\hat{\varphi}^{(N)}_{\alpha,R} C_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I = \sum_{|\alpha|\geq\rho(R)} \hat{\psi}^{(N)}_{\alpha,R} A \hat{\varphi}^{(N)}_{\alpha,R} C_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I \\ &= \sum_{|\alpha|\geq\rho(R)} \hat{\psi}^{(N)}_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} A C_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I + S_R = \sum_{|\alpha|\geq\rho(R)} \hat{\varphi}^{(N)}_{\alpha,R} \hat{\psi}^{(N)}_{\alpha,R} A C_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I + S_R \\ &= \sum_{|\alpha|\geq\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I + S_R = I - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I + S_R, \end{aligned}$$

where

$$S_R := \sum_{|\alpha|\geq\rho(R)} \hat{\psi}^{(N)}_{\alpha,R} [A, \hat{\varphi}^{(N)}_{\alpha,R} I] C_{\alpha,R} \hat{\varphi}^{(N)}_{\alpha,R} I$$

and $[A, \hat{\varphi}^{(N)}_{\alpha,R} I] = A\hat{\varphi}^{(N)}_{\alpha,R} I - \hat{\varphi}^{(N)}_{\alpha,R} A$ being the commutator. By the same arguments as T_R above, S_R converges $\mathcal{P}$-strongly in E, S_R belongs in $\mathcal{L}(E, \mathcal{P})$ and $\|S_R\|_{\mathcal{L}(E)} \to 0$ as R tends to infinity.

Fix R such that $\|S_R\|_{\mathcal{L}(E)} < \frac{1}{2}$. By the Neumann series $I + S_R$ is invertible and $\|(I + S_R)^{-1}\|_{\mathcal{L}(E)} \leq 2$. Multiplying

$$AC_R = I + S_R - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I$$

from right-hand side with $(I+S_R)^{-1}$ yields us

$$\begin{aligned} AC_R(I+S_R)^{-1} &= (I+S_R - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 I)(I+S_R)^{-1} \\ &= I - \sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 (I+S_R)^{-1}. \end{aligned}$$

Thus with

$$K_2 := -\sum_{|\alpha|<\rho(R)} (\hat{\varphi}^{(N)}_{\alpha,R})^2 (I+S_R)^{-1}$$

being $\mathcal{P}$-compact we have

$$AC_R(I+S_R)^{-1} = I + K_2, \quad K_2 \in \mathcal{K}(E,\mathcal{P}).$$

Hence $C_R(I+S_R)^{-1}$ is a right sided regularizer of A with

$$\left\| C_R(I+S_R)^{-1} \right\|_{\mathcal{L}(E)} \le 2^N M \left\| (I+S_R)^{-1} \right\|_{\mathcal{L}(E)} \le 2^{N+1} M.$$

For the case of band-dominated operators, assume that a band-dominated operator $A \in \mathcal{L}(E)$ satisfies our assumptions and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of band operators with $\|A - A_n\|_{\mathcal{L}(E)} < \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} B_{\alpha,R} A_n \hat{\psi}^{(N)}_{\alpha,R} I &= B_{\alpha,R} A \hat{\psi}^{(N)}_{\alpha,R} I - B_{\alpha,R}(A - A_n)\hat{\psi}^{(N)}_{\alpha,R} I \\ &= \hat{\psi}^{(N)}_{\alpha,R} I - B_{\alpha,R}(A-A_n)\hat{\psi}^{(N)}_{\alpha,R} I = (I - B_{\alpha,R}(A-A_n))\hat{\psi}^{(N)}_{\alpha,R} I \end{aligned}$$

and $\|B_{\alpha,R}(A-A_n)\|_{\mathcal{L}(E)} \le M \|A - A_n\|_{\mathcal{L}(E)} < \frac{M}{n}$. For large enough n we have $\|B_{\alpha,R}(A-A_n)\|_{\mathcal{L}(E)} < 1$, so that by the Neumann series the operator $I - B_{\alpha,R}(A-A_n)$ is invertible with

$$\left\| (I - B_{\alpha,R}(A-A_n))^{-1} \right\|_{\mathcal{L}(E)} \le \sum_{k=0}^{\infty} \left\| (B_{\alpha,R}(A-A_n))^k \right\|_{\mathcal{L}(E)} \le \frac{1}{1-\frac{M}{n}}.$$

Define $B^{(n)}_{\alpha,R} := (I - B_{\alpha,R}(A-A_n))^{-1} B_{\alpha,R}$, so that one has

$$\left\| B^{(n)}_{\alpha,R} \right\|_{\mathcal{L}(E)} = \left\| (I - B_{\alpha,R}(A-A_n))^{-1} B_{\alpha,R} \right\|_{\mathcal{L}(E)} \le \frac{M}{1-\frac{M}{n}}$$

and

$$\begin{aligned} B^{(n)}_{\alpha,R} A_n \hat{\psi}^{(N)}_{\alpha,R} I &= (I - B_{\alpha,R}(A-A_n))^{-1} B_{\alpha,R} A_n \hat{\psi}^{(N)}_{\alpha,R} I \\ &= (I - B_{\alpha,R}(A-A_n))^{-1}(I - B_{\alpha,R}(A-A_n))\hat{\psi}^{(N)}_{\alpha,R} I = \hat{\psi}^{(N)}_{\alpha,R} I. \end{aligned}$$

Thus A_n satisfies the assumptions with respect to a constant $M_n := \frac{M}{1-\frac{M}{n}}$. By what has been proven above, such band operators A_n are $\mathcal{P}$-Fredholm and the $\mathcal{P}$-essential norms of their regularizers are bounded by $2^{N+1}M_n$.

Since $\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})$ is a unital Banach algebra, by defining

$$a_n := A_n + \mathcal{K}(E,\mathcal{P}) \text{ and } a := A + \mathcal{K}(E,\mathcal{P})$$

we have $\|a - a_n\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} \to 0$ as $n \to \infty$. Because the algebra $\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})$ is inverse closed in the Calkin algebra $\mathcal{L}(E,\mathcal{P})/\mathcal{K}(E,\mathcal{P})$ we know that a^{-1} is in $\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})$.

The sequence $(M_n)_{n\in\mathbb{N}}$ tends to M as n tends to infinity and thus for large enough n we have

$$\sup_{n\in\mathbb{N}} \left\|a_n^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} = \sup_{n\in\mathbb{N}} \left\|A_n^{-1}\right\|_{\mathcal{L}(E)} \leq \sup_{n\in\mathbb{N}} M_n \to M,$$

as $n \to \infty$. Hence, since $\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})$ is a unital Banach algebra, a^{-1} exists. Furthermore, note that

$$a^{-1}(a_n - a)a_n^{-1} = a^{-1}a_n a_n^{-1} - a^{-1}a a_n^{-1} = a^{-1} - a_n^{-1}$$

for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} \left\|a^{-1} - a_n^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} &= \left\|a^{-1}(a_n - a)a_n^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} \\ \leq \left\|a^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} &\|a_n - a\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} \left\|a_n^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} \to 0, \end{aligned}$$

as $n \to \infty$, since

$$\left\|a_n^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} \leq \frac{M}{1 - \frac{M}{n}} \leq 2M$$

for all $n \geq 2M$. Thies means that $a_n^{-1} \to a^{-1}$ as $n \to \infty$, so that by continuity

$$\left\|a^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} = \lim_{n\to\infty} \left\|a_n^{-1}\right\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} \leq 2^{N+1}M.$$

Recall that $\|a_n^{-1}\|_{\mathcal{A}_E/\mathcal{K}(E,\mathcal{P})} \leq 2^{N+1}M$ by the first part of the argument, where a_n^{-1} denotes the $\mathcal{P}$-Fredholm regularizer of the band operator A_n.

□

Lemma 3.3.9. Let $A \in \mathcal{A}_E$ be a band-dominated operator and suppose the limit operator A_h with respect to the sequence $h \in \mathcal{H}$ exists and is invertible.

Then, for each function $\varphi \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X))$ with finite support, there is a number m_0 such that, for all $m \geq m_0$ there are operators B_m and C_m in $\mathcal{A}_E$ with properties $\|B_m\|_{\mathcal{L}(E)} \leq 2\,\|(A_h)^{-1}\|_{\mathcal{L}(E)}$, $\|C_m\|_{\mathcal{L}(E)} \leq 2\,\|(A_h)^{-1}\|_{\mathcal{L}(E)}$ and

$$B_m A V_{h(m)}\varphi V_{-h(m)} = V_{h(m)}\varphi V_{-h(m)} A C_m = V_{h(m)}\varphi V_{-h(m)}.$$

Proof. Given φ in $l^\infty(\mathbb{Z}^N, \mathcal{L}(X))$ with finite support, we choose a sequence $\chi \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X))$ with finite support such that $\chi\varphi = \varphi$, that is $\chi_m\varphi_m = \varphi_m$ for all $m \in \mathbb{Z}^N$. For $\chi_m : X \to X$, with $\chi_m \neq 0$ one may choose $\chi_m = I_X$ to be the identity operator on X.

From the definition of a limit operator we have

$$V_{-h(m)}AV_{h(m)}\chi I = A_h\chi I + T_m,$$

where $T_m := (V_{-h(m)}AV_{h(m)} - A_h)\chi I \in \mathcal{A}_E$ are band-dominated operators on E with $\|T_m\|_{\mathcal{L}(E)}$ tending to zero as m tends to infinity.

Now by multiplying with A_h^{-1} on the left-hand side we obtain

$$A_h^{-1}V_{-h(m)}AV_{h(m)}\chi I = A_h^{-1}A_h\chi I + A_h^{-1}T_m.$$

Additionally, by multiplying with $\varphi V_{-h(m)}$ on the right-hand side and using the fact $\chi\varphi = \varphi$, we get

$$A_h^{-1}V_{-h(m)}AV_{h(m)}\varphi V_{-h(m)} = (I + A_h^{-1}T_m)\varphi V_{-h(m)}.$$

By assumption A_h^{-1} exists and is bounded, thus by theorem 3.2.9 we can pick m_0 such that $\left\|A_h^{-1}T_m\right\|_{\mathcal{L}(E)} < \frac{1}{2}$ for $m \geq m_0$. Hence by the Neumann series $I + A_h^{-1}T_m$ is invertible and

$$\left\|(I + A_h^{-1}T_m)^{-1}\right\|_{\mathcal{L}(E)} \leq \sum_{k=0}^{\infty} \left\|(A_h^{-1}T_m)^k\right\|_{\mathcal{L}(E)} = \frac{1}{1 - \left\|A_h^{-1}T_m\right\|_{\mathcal{L}(E)}} < 2.$$

Now by multiplying the equation

$$A_h^{-1}V_{-h(m)}AV_{h(m)}\varphi V_{-h(m)} = (I + A_h^{-1}T_m)\varphi V_{-h(m)}$$

with $V_{h(m)}(I + A_h^{-1}T_m)^{-1}$ from the left-hand side yields us

$$V_{h(m)}(I + A_h^{-1}T_m)^{-1}A_h^{-1}V_{-h(m)}AV_{h(m)}\varphi V_{-h(m)} = V_{h(m)}\varphi V_{-h(m)}.$$

Define $B_m \in \mathcal{A}_E$ as $B_m := V_{h(m)}(I + A_h^{-1}T_m)^{-1}A_h^{-1}V_{-h(m)}$. Thus we now have

$$B_mAV_{h(m)}\varphi V_{-h(m)} = V_{h(m)}\varphi V_{-h(m)}$$

and because $\left\|V_{h(m)}\right\|_{\mathcal{L}(E)} = \left\|V_{-h(m)}\right\|_{\mathcal{L}(E)} = 1$ we have $\|B_m\|_{\mathcal{L}(E)} \leq 2\left\|A_h^{-1}\right\|_{\mathcal{L}(E)}$.

For the right-hand side case we choose such a $\chi \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X))$ with finite support that $\varphi\chi = \varphi$. Thus from the definition of a limit operator we have

$$\chi IV_{-h(m)}AV_{h(m)} = \chi IA_h + S_m,$$

where $S_m := \chi I(V_{-h(m)}AV_{h(m)} - A_h)$ tends to zero when $m \to \infty$. Multiplying from both sides analogously as in the left-hand side case we obtain

$$\begin{aligned}\chi I V_{-h(m)}AV_{h(m)} &= \chi I A_h + S_m \\ \Leftrightarrow \chi I V_{-h(m)}AV_{h(m)}A_h^{-1} &= \chi I(I + S_m A_h^{-1}) \\ \Leftrightarrow V_{h(m)}\varphi V_{-h(m)}AV_{h(m)}A_h^{-1} &= V_{h(m)}\varphi(I + S_m A_h^{-1}).\end{aligned}$$

Choose again a large enough m_0 so that we have $\left\|S_m A_h^{-1}\right\|_{\mathcal{L}(E)} < \frac{1}{2}$ for $m \geq m_0$ and thus from the Neumann series we know that $I + S_m A_h^{-1}$ is invertible and $\left\|(I + S_m A_h^{-1})^{-1}\right\|_{\mathcal{L}(E)} \leq 2$. Finally by multiplying both sides with $(I + S_m A_h^{-1})^{-1}V_{-h(m)}$ from the right-hand side we attain

$$V_{h(m)}\varphi V_{-h(m)}AV_{h(m)}A_h^{-1}(I + S_m A_h^{-1})^{-1}V_{-h(m)} = V_{h(m)}\varphi V_{-h(m)}.$$

Hence, defining C_m as $C_m := V_{h(m)}A_h^{-1}(I + S_m A_h^{-1})^{-1}V_{-h(m)} \in \mathcal{A}_E$ we get the desired result

$$V_{h(m)}\varphi V_{-h(m)}AC_m = V_{h(m)}\varphi V_{-h(m)}$$

with $\|C_m\|_{\mathcal{L}(E)} \leq 2\left\|A_h^{-1}\right\|_{\mathcal{L}(E)}$. [1] □

With the help of these lemmas we are ready to prove the other direction of theorem 3.3.5.

Proof. Let $A \in \mathcal{A}_E^{\$}$ be a rich band-dominated operator. Suppose that all limit operators of A are invertible and let the inverses of the limit operators be uniformly bounded, meaning that

$$M_A := \sup_{n\in\mathbb{N}}\left\{\left\|A_h^{-1}\right\|_{\mathcal{L}(E)} : \; A_h \in \sigma_{op}(A)\right\} < \infty.$$

We will prove the claim by a counter assumption. Thus we assume that A is missing either the right or left regularizer. We prove the left-regularizer case here and the proof for missing right-regularizer is handled similarly. Lemma 3.3.8 implies that for $M := 2M_A$, there is a natural number $R \in \mathbb{N}$ such that for all $\rho(R) > 0$ there is an $\alpha_1 \in \mathbb{Z}^N$ with $|\alpha_1| > \rho(R)$ and

$$BA\hat{\psi}_{\alpha_1,R}I \neq \hat{\psi}_{\alpha_1,R}I$$

for all $B \in \mathcal{L}(E, \mathcal{P})$ with $\|B\|_{\mathcal{L}(E)} \leq M$.

We construct a sequence as follows. Suppose $\alpha_1, ..., \alpha_N \in \mathbb{Z}^N$ have been constructed. By choosing $\rho(R) = |\alpha_{k-1}|$, there exists an $\alpha_k \in \mathbb{Z}^N$ such that $|\alpha_k| > \rho(R)$ with

$$BA\hat{\psi}_{\alpha_k,R}I \neq \hat{\psi}_{\alpha_k,R}I,$$

for all $B \in \mathcal{L}(E, \mathcal{P})$ with $\|B\|_{\mathcal{L}(E)} \leq M$. Continuing with this construction we obtain a sequence $(\alpha_k)_{k\in\mathbb{N}}$ with $|\alpha_k| \to \infty$ as k tends to infinity and

$$BA\hat{\psi}_{\alpha_k,R}I \neq \hat{\psi}_{\alpha_k,R}I,$$

for all $k \in \mathbb{N}$ and for all $B \in \mathcal{L}(E, \mathcal{P})$ with $\|B\|_{\mathcal{L}(E)} \leq M$.

Since A is rich, for the sequence $h := (\alpha_k R)_{k\in\mathbb{N}} \in \mathcal{H}$ there exists a subsequence $g = (\alpha_{k_m} R)_{m\in\mathbb{N}}$ which tends to infinity and such that the limit operator A_g exists. By our assumption A_g is invertible and $\left\|A_g^{-1}\right\|_{\mathcal{L}(E)} \leq M_A$. Thus by lemma 3.3.9 for every function $\xi \in l^\infty(\mathbb{Z}^N, \mathcal{L}(X)$ with finite support there is a number $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ there is an operator B_m in $\mathcal{A}_E \subset \mathcal{L}(E, \mathcal{P})$ with $\|B_m\|_{\mathcal{L}(E)} \leq 2\left\|A_g^{-1}\right\|_{\mathcal{L}(E)} \leq 2M_A$ and

$$B_m A V_{g(m)} \xi V_{-g(m)} = V_{g(m)} \xi V_{-g(m)}.$$

Choosing $\xi := \hat{\psi}_{0,R}$ and any $f \in E$ we get

$$\begin{aligned} & V_{g(m)} \xi V_{-g(m)} f(x) = V_{\alpha_{k_m}R} \xi V_{-\alpha_{k_m}R} f(x) \\ &= V_{\alpha_{k_m}R} \hat{\psi}_{0,R}(x) V_{-\alpha_{k_m}R} f(x) = V_{\alpha_{k_m}R} \hat{\psi}_{0,R}(x) f(x + \alpha_{k_m}R) \\ &= \hat{\psi}_{0,R}(x - \alpha_{k_m}R) f(x + \alpha_{k_m}R - \alpha_{k_m}R) \\ &= \hat{\psi}(\frac{x}{R} - \alpha_{k_m}) f(x) = \hat{\psi}_{\alpha_{k_m},R} f(x). \end{aligned}$$

Thus we have

$$V_{g(m)} \xi V_{-g(m)} = \hat{\psi}_{\alpha_{k_m},R} I$$

and as a consequence also

$$B_m A \hat{\psi}_{\alpha_{k_m},R} I = \hat{\psi}_{\alpha_{k_m},R} I$$

for a certain $B_m \in \mathcal{L}(E, \mathcal{P})$ such that $\|B_m\|_{\mathcal{L}(E)} \leq 2M_A = M$. This is a contradiction with our construction of the sequence $(\alpha_k)_{k\in\mathbb{N}}$ and thus we conclude that A is $\mathcal{P}$-Fredholm. $\square$

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