Game-Theoretic Semantics for ATL+ with Applications to Model Checking

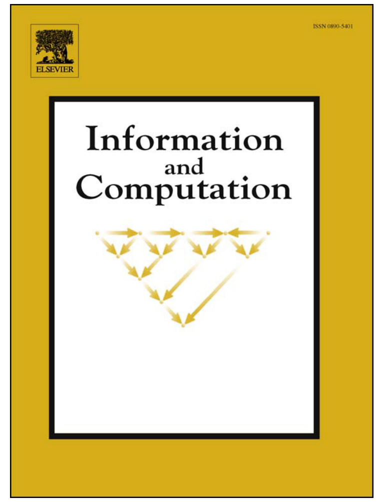
Goranko, Valentin

2021-02

Goranko , V , Kuusisto , A & Rönnholm , R 2021 , ' Game-Theoretic Semantics for ATL+ with Applications to Model Checking ' , Information and Computation , vol. 276 , 104554 . https://doi.org/10.1016/j.ic.2020

http://hdl.handle.net/10138/342064
https://doi.org/10.1016/j.ic.2020.104554

Journal Pre-proof



Game-Theoretic Semantics for ATL$^+$ with Applications to Model Checking

Valentin Goranko, Antti Kuusisto, Raine Rönnholm

PII: S0890-5401(20)30042-0
DOI: https://doi.org/10.1016/j.ic.2020.104554
Reference: YINCO 104554

To appear in: *Information and Computation*

Received date: 2 April 2018
Revised date: 16 February 2019
Accepted date: 8 June 2019

Game-Theoretic Semantics for ATL$^+$ with Applications to Model Checking

Valentin Goranko[a], Antti Kuusisto[b], Raine Rönnholm[c]

[a] Stockholm University, Sweden; and University of Johannesburg, South Africa (visiting professorship)
[b] Tampere University, Finland; and University of Helsinki, Finland
[c] Tampere University, Finland

Abstract

We develop a game-theoretic semantics (GTS) for the fragment ATL$^+$ of the alternating-time temporal logic ATL*, thereby extending the recently introduced GTS for ATL. We show that the game-theoretic semantics is equivalent to the standard compositional semantics of ATL$^+$ with perfect-recall strategies. Based on the new semantics, we provide an analysis of the memory and time resources needed for model checking ATL$^+$ and show that strategies of the verifier that use only a very limited amount of memory suffice. Furthermore, using the GTS, we provide a new algorithm for model checking ATL$^+$ and identify a natural hierarchy of tractable fragments of ATL$^+$ that substantially extend ATL.

Keywords: game-theoretic semantics, alternating-time temporal logic, algorithmic model checking, tractable fragments, finite memory strategies

1. Introduction

The full Alternating-time Temporal Logic ATL* [1] is one of the main logical systems used for formalising and verifying strategic reasoning about agents in multi-agent systems. It is very expressive, and that expressiveness comes at a high (2-EXPTIME) price of computational complexity of model checking. Its basic fragment ATL—which can be regarded as the multi-agent extension of

Email addresses: valentin.goranko@philosophy.su.se (Valentin Goranko), antti.kuusisto@tuni.fi (Antti Kuusisto), raine.ronnholm@tuni.fi (Raine Rönnholm)

CTL—has, on the other hand, tractable model checking, but its expressiveness is rather limited. In particular, ATL only allows expressing strategic objectives of the type $\langle\langle A \rangle\rangle\, \Phi$ where Φ is a simple temporal goal involving a single temporal operator.

The intermediate fragment ATL^+ naturally emerges as a good alternative, essentially extending ATL to allow expressing strategic objectives that are Boolean combinations of simple temporal goals. The price for this is the reasonably higher computational complexity of model checking, namely, PSPACE-completeness [2]. Still, the PSPACE-completeness result alone gives a rather crude estimate of the amount of computational resources, such as memory, needed for model checking ATL^+.

Main ideas and contributions. In this paper we take an alternative approach to the semantic analysis and model checking of fragments of ATL^*, concentrating in particular on fragments of ATL^+. Our analysis is not based on the standard compositional semantics, but instead, we present and study a new, game-theoretic semantics (GTS). The main aims and contributions of the paper are three-fold:

1. We introduce a game-theoretic semantics for ATL^+ and prove it equivalent to the standard (perfect-recall) compositional semantics.

2. We propose new model checking algorithms for ATL^+ and some of its fragments. To this end, we use the new GTS developed here rather than the standard semantics. Based on this novel perspective on model checking, we identify a hierarchy of new fragments of ATL^+ with tractable model checking.

3. We analyse, with the help of GTS, the use of memory resources in ATL^+.

The main part of the paper consists of a detailed presentation and analysis of the new GTS for ATL^+. In particular, we obtain results similar to those in our earlier work [3, 4], where we defined a GTS for ATL. We establish, inter alia, the surprising result that it is always sufficient to consider *finite paths only* when formulae are evaluated via GTS, *even when investigating infinite models*. Since we are dealing with ATL^+ as opposed to ATL, a range of new technical ideas and mechanisms are needed due to shifting focus from single temporal goals to multiple, simultaneous goals.

Furthermore, the approach via GTS enables us, among other things, to perform a more detailed analysis of the resources needed for evaluating ATL^+-formulae than the algorithm from [2]. The algorithm presented in [2] employs a combination of a path construction procedure for checking strategic formulae $\langle\langle A\rangle\rangle\,\Phi$ on one hand, and the standard labelling algorithm on the other hand. Our model checking algorithm for ATL^+ follows uniformly a procedure directly based on GTS only. The GTS-based produre enables us to identify and correct a flaw in the model checking procedure of [2]. Yet, the PSPACE upper bound result of [2] is easily confirmed by our algorithm, and we provide a new, simple proof of that result. In addition to new methods, we employ some nice ideas from [2].

As a new complexity result obtained via GTS, we identify a natural hierarchy of fragments of ATL^+ that extend ATL and have *tractable* (PTIME-complete) model checking. The hierarchy is based on bounding the *Boolean strategic width* of formulae. We denote the new fragments in the hierarchy by ATL^k for different positive integers k. The fragment ATL^k contains those formulae of ATL^+ where subformulae $\langle\langle A\rangle\rangle\,\Phi$ are restricted such that Φ is a Boolean combination of at most k formulae. Note that thus ATL^1 corresponds to plain ATL.

The current paper is the journal version of [5], and here we extend [5] by, inter alia, including a range of new results on systems of *bounded semantics* based on *finite transducers*. We analyse the amount of memory resources needed for winning strategies and establish tight lower and upper bounds for it. We notice that in transducer based semantics, an exponential amount of memory with respect to formula size is required. However, only a linear amount of this is actually used in any concrete single evaluation process of a formula. Based on this we argue that the transducer-based approach does not give a complete analysis of the requirement of memory resources. Indeed, a full look-up table of potentially needed memory resources has to be encoded in a transducer. An alternative approach would be to replace the look-up table by an equivalent computing device and then relate the needed memory resources to the memory used by the device.

The current paper directly extends the results in [3, 4], where a GTS for ATL was considered, and relates quite closely also to the GTS-based work presented in [6, 7, 8]. In particular, the current article extends the work in [3, 4] in the way described next. Firstly, several new ideas and technical notions, such as the *role of a seeker* and the use of a *truth function*, are introduced in this article in order to enable the transition from ATL to ATL^+ in a GTS setting. Secondly, a useful and generally elucidating link between our GTS and *Büchi games* is identified. Thirdly, we show how to use the new upgraded semantics in a model checking procedure for ATL^+ and the fragments ATL^k. This would not have been possible with the semantics of [3, 4].

There are several related works concerning the extensions of ATL as well as game-theoretic semantics; we mention here some of such papers. Game-theoretic semantics for first order logic has been proposed by Hintikka [9] and Lorenzen [10]. GTS-like approaches have been used to solve decision problems of, e.g., fragments of Strategy Logic (especially with respect to the so-called "behavioral semantics") in [11, 12]. The idea about imposing time bounds for temporal operators has been studied in e.g. [13, 14, 15, 16].

Structure of the paper. After the preliminaries in Section 2, we define a bounded, finitely bounded, and unbounded game-theoretic semantics for ATL^+ in Section 3. In Section 4 we analyse the novel systems of GTS. In Section 5 we prove equivalence of the bounded and unbounded versions with the standard compositional semantics of ATL^+ with perfect-recall strategies. In Section 6 we apply the GTS to the model checking problem for ATL^+ and identify a hierarchy of tractable fragments of the logic. In Section 7 we study the transducer-based bounded memory semantics for these fragments. Section 8 concludes the paper.

2. Preliminaries

In this section we define concurrent game models and the syntax and the (perfect-recall) semantics for ATL^+. We also introduce some new terminology and notations that will be used later in this paper.

Definition 2.1. A **concurrent game model** (CGM) is a tuple

$$\mathcal{M} := (\mathbb{A}\text{gt}, \text{St}, \Pi, \text{Act}, d, o, v)$$

which consists of:

– The following non-empty sets: **agents** $\mathbb{A}\text{gt} = \{a_1, \ldots, a_k\}$, **states** St, **proposition symbols** Π, and **actions** Act;

– The following functions: an **action function** $d : \mathbb{A}\text{gt} \times \text{St} \to \mathcal{P}(\text{Act}) \setminus \{\emptyset\}$ which assigns a non-empty set of actions available to each agent at each state; a **transition function** o which assigns an **outcome state** $o(q, \vec{\alpha})$ to each state $q \in \text{St}$ and admissible action profile $\vec{\alpha}$ (a tuple of actions $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ such that $\alpha_i \in d(a_i, q)$ for each $a_i \in \mathbb{A}\text{gt}$); and finally, a **valuation function** $v : \Pi \to \mathcal{P}(\text{St})$.

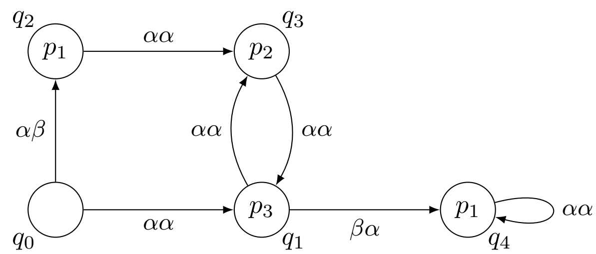
We use symbols $p, p_0, p_1, \ldots$ to denote proposition symbols and $q, q_0, q_1, \ldots$ to denote states. Sets of agents are called **coalitions**. The complement $\overline{A} = \mathbb{A}\text{gt} \setminus A$ of a coalition A is the **opposing coalition** of A. The set $\mathsf{action}(A, q)$ of action tuples available to coalition A at state $q \in \text{St}$ is defined as

$$\mathsf{action}(A, q) := \{(\alpha_i)_{a_i \in A} \mid \alpha_i \in d(a_i, q) \text{ for each } a_i \in A\}.$$

Example 2.2. Let $\mathcal{M}^* = (\mathbb{A}\text{gt}, \text{St}, \Pi, \text{Act}, d, o, v)$, where:

$$\mathbb{A}\text{gt} = \{a_1, a_2\},\ \text{St} = \{q_0, q_1, q_2, q_3, q_4\};$$
$$\Pi = \{p_1, p_2, p_3\},\ \text{Act} = \{\alpha, \beta\};$$
$$d(a_2, q_0) = d(a_1, q_1) = \{\alpha, \beta\} \text{ and else } d(a_i, q_i) = \{\alpha\};$$
$$o(q_0, \alpha\alpha) = q_1,\ o(q_0, \alpha\beta) = q_2,\ o(q_1, \alpha\alpha) = q_3,\ o(q_1, \beta\alpha) = q_4,$$
$$o(q_2, \alpha\alpha) = q_3,\ o(q_3, \alpha\alpha) = q_1 \text{ and } o(q_4, \alpha\alpha) = q_4;$$
$$v(p_1) = \{q_2, q_4\},\ v(p_2) = \{q_3\} \text{ and } v(p_3) = \{q_1\}.$$

$\mathcal{M}^*$:



Definition 2.3. Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a CGM. A **path** in $\mathcal{M}$ is a sequence $\Lambda : \mathbb{N} \to \mathrm{St}$ of states such that for each $n \in \mathbb{N}$, we have $\Lambda[n+1] = o(\Lambda[n], \vec{\alpha})$ for some admissible action profile $\vec{\alpha}$ in $\Lambda[n]$. A **finite path** (aka **history**) is a finite prefix sequence of a path in $\mathcal{M}$. We let $\mathsf{paths}(\mathcal{M})$ denote the set of all paths in $\mathcal{M}$ and $\mathsf{paths}_{\mathrm{fin}}(\mathcal{M})$ the set of all finite paths in $\mathcal{M}$.[1]

A **positional strategy** of an agent $a \in \mathbb{A}\mathrm{gt}$ is a function $s_a : \mathrm{St} \to \mathrm{Act}$ such that $s_a(q) \in d(a, q)$ for each $q \in \mathrm{St}$. A **perfect-recall strategy**, or hereafter just **strategy**, of an agent $a \in \mathbb{A}\mathrm{gt}$ is a function $s_a : \mathsf{paths}_{\mathrm{fin}}(\mathcal{M}) \to \mathrm{Act}$ such that $s_a(\lambda) \in d(a, \lambda[k])$ for each $\lambda \in \mathsf{paths}_{\mathrm{fin}}(\mathcal{M})$, where $\lambda[k]$ is the last state in λ. A **collective strategy** S_A for a coalition $A \subseteq \mathbb{A}\mathrm{gt}$ is a tuple of individual strategies, one for each agent in A. With $\mathsf{paths}(q, S_A)$ we denote the set of all paths emerging in plays beginning from q where the agents in A follow the strategy S_A.

The formulae of ATL^+ are defined with the help of the following grammar:

State formulae: $\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\!\langle A \rangle\!\rangle\, \Phi \quad (p \in \Pi)$

Path formulae: $\Phi ::= \varphi \mid \neg\Phi \mid \Phi \vee \Phi \mid \mathsf{X}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi$

Only state formulae are considered to be actual ATL^+ formulae. Path formulae are only auxiliary for defining the semantics.

Other Boolean connectives are defined as usual, and furthermore, $\mathsf{F}\,\varphi$, $\mathsf{G}\,\varphi$ and $\varphi\,\mathsf{R}\,\psi$ are abbreviations for $\top\,\mathsf{U}\,\varphi$, $\neg(\top\,\mathsf{U}\,\neg\varphi)$, and $\neg(\neg\varphi\,\mathsf{U}\,\neg\psi)$ respectively. With Φ and Ψ we refer to path formulae only; φ, ψ, and χ refer to any formulae.

[1]Note that, accordingly this terminology, a "path" always refers to an infinite path. We use this terminology since we mostly consider infinite paths.

Definition 2.4. Let $\mathcal{M}$ be a CGM. **Truth** of state and path formulae of ATL^+ is defined, respectively, with respect to states $q \in St$ and paths $\Lambda \in \mathsf{paths}(\mathcal{M})$, inductively as follows, where φ, ψ are state formulae:

- $\mathcal{M}, q \models p$ iff $q \in v(p)$ (for $p \in \Pi$).
- $\mathcal{M}, q \models \neg\varphi$ iff $\mathcal{M}, q \not\models \varphi$.
- $\mathcal{M}, q \models \varphi \vee \psi$ iff $\mathcal{M}, q \models \varphi$ or $\mathcal{M}, q \models \psi$.
- $\mathcal{M}, q \models \langle\langle A \rangle\rangle\, \Phi$ iff there exists a collective strategy S_A for the coalition A such that $\mathcal{M}, \Lambda \models \Phi$ for each $\Lambda \in \mathsf{paths}(q, S_A)$.
- $\mathcal{M}, \Lambda \models \varphi$ iff $\mathcal{M}, \Lambda[0] \models \varphi$.
- $\mathcal{M}, \Lambda \models \neg\Phi$ iff $\mathcal{M}, \Lambda \not\models \Phi$.
- $\mathcal{M}, \Lambda \models \Phi \vee \Psi$ iff $\mathcal{M}, \Lambda \models \Phi$ or $\mathcal{M}, \Lambda \models \Psi$.
- $\mathcal{M}, \Lambda \models \mathsf{X}\, \varphi$ iff $\mathcal{M}, \Lambda[1] \models \varphi$.
- $\mathcal{M}, \Lambda \models \varphi\, \mathsf{U}\, \psi$ iff there exists $i \in \mathbb{N}$ such that $\mathcal{M}, \Lambda[i] \models \psi$ and $\mathcal{M}, \Lambda[j] \models \varphi$ for all $j < i$.

The **set of subformulae**, $\mathrm{SUB}(\varphi)$, of a formula φ is defined as usual. Subformulae with a temporal operator as the main connective will be called **temporal subformulae**, while subformulae with $\langle\langle\rangle\rangle$ as the main connective are **strategic subformulae**. The subformula Φ of a formula $\varphi = \langle\langle A \rangle\rangle\, \Phi$ is called the **temporal objective of** φ. We also define the set $At(\Phi)$ of **relative atoms** of Φ as follows:

- $At(\chi \vee \chi') = At(\chi) \cup At(\chi')$ and $At(\neg\chi) = At(\chi)$.
- $At(\langle\langle A \rangle\rangle\, \chi) = \{\langle\langle A \rangle\rangle\, \chi\}$ and $At(p) = \{p\}$ for $p \in \Pi$.
- $At(\chi\, \mathsf{U}\, \chi') = \{\chi\, \mathsf{U}\, \chi'\}$ and $At(\mathsf{X}\, \chi) = \{\mathsf{X}\, \chi\}$.

We say that $\chi \in At(\Phi)$ occurs **positively** (resp. **negatively**) **in** Φ if χ has an occurrence in the scope of an even (resp. odd) number of negations in Φ. We denote by $\mathrm{SUB}_{At}(\Phi)$ the subset of $\mathrm{SUB}(\Phi)$ containing all relative atoms of Φ and also all Boolean combinations χ of these relative atoms such that $\chi \in \mathrm{SUB}(\Phi)$.

Example 2.5. Let $\varphi^* := \langle\!\langle a_1 \rangle\!\rangle\, \Psi$, where

$$\Psi := (\neg \mathsf{X}\, p_3 \wedge \langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1) \;\vee\; (\mathsf{F}\, p_1 \wedge (\neg p_1)\, \mathsf{U}\, p_2).$$

Written without using abbreviations, Ψ becomes

$$\neg(\neg\neg \mathsf{X}\, p_3 \vee \neg \langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1) \vee \neg(\neg(\top\, \mathsf{U}\, p_1) \vee \neg((\neg p_1)\, \mathsf{U}\, p_2)).$$

Here $At(\Psi) = \{\mathsf{X}\, p_3, \langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1, \top\, \mathsf{U}\, p_1, (\neg p_1)\, \mathsf{U}\, p_2\}$, where $\langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1$ is a state formula and the rest are path formulae. The formula $\mathsf{X}\, p_3$ occurs negatively in Ψ and the rest of the formulae in $At(\Psi)$ occur positively in Ψ.

3. Game-theoretic semantics

In this section we define *bounded*, *finitely bounded* and *unbounded evaluation games* for ATL^+. These games give rise to three different systems of semantics, namely, the *bounded*, *finitely bounded* and *unbounded* GTS for ATL^+.

These systems of semantics were defined for plain ATL already in [4]. The principal difference between the bounded and unbounded GTS is that the bounded variant *forces games to end after a finite number of steps.* This is a significant difference achieved, as we shall see, via requiring the players to choose ordinal numbers that can intuitively be considered to determine upper bounds for game durations (see Example 4.8). In the unbounded semantics, no such ordinals are used, and the games can continue for infinitely many rounds.

As discussed in [4], the difference between bounded and unbounded semantics is directly analogous to the difference between for-loops and while-loops. Indeed, for-loops require an extra parameter that determines the number of loop iterations, and while-loops can possibly loop infinitely long.

Having both the bounded and unbounded semantics at our disposal will prove beneficial in Section 6 where we discuss model checking. Indeed, we shall need the unbounded semantics for connecting fragments of ATL^+ to Büchi games and thereby obtaining novel tractability results. On the other hand, we shall need the bounded semantics for our proof strategy of Theorem 6.1 which confirms the PSPACE-completeness of ATL^+ model checking.

The unbounded and bounded semantics will be proved equivalent below. (The equivalence holds on the condition that the players are allowed to use sufficiently large ordinals in bounded games.) The *finitely* bounded semantics is not equivalent to these two systems of semantics. The difference between the finitely bounded and bounded semantics is that the parameters with which the players force the games to be finite are possibly infinite ordinals in bounded semantics and finite ordinals in finitely bounded semantics. The finitely bounded and bounded semantics are equivalent over finite models but not over infinite ones. The reason for introducing finitely bounded semantics is that it provides a novel, interesting perspective on ATL and ATL^+ while still being equivalent over finite (but not infinite) models with the standard semantics.

Below we shall use some terminology and notational conventions introduced in [4].

3.1. Evaluation games: informal description

Given a CGM $\mathcal{M}$, a state q_{in} and a state formula φ, the **evaluation game** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ is, intuitively, a formal debate between two opponents, **Eloise** ($\mathbf{E}$) and **Abelard** ($\mathbf{A}$), about whether the formula φ is true at the state q_{in} in the model $\mathcal{M}$. Eloise claims that φ is true, so she (initially) adopts the role of a **verifier** in the game, and Abelard tries to prove the formula false, so he is (initially) the **falsifier**. These roles (verifier, falsifier) can swap in the course of the game when negations are encountered in the formula. If $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$, then $\overline{\mathbf{P}}$ denotes the **opponent** of $\mathbf{P}$, i.e., $\overline{\mathbf{P}} \in \{\mathbf{E}, \mathbf{A}\} \setminus \{\mathbf{P}\}$.

We now provide an intuitive account of the *bounded* evaluation game and the *bounded* GTS for ATL^+. The intuitions underlying the finitely bounded and unbounded GTS are similar. A reader unfamiliar with the concept of GTS may find it useful to consult, for example, [17] for GTS in general and [3] or [4] for ATL-specific GTS. The GTS for ATL^+ presented here follows the general principles of GTS, with the main original feature being the treatment of strategic formulae $\langle\langle A\rangle\rangle\,\Phi$. We first give an *informal* account of the way such formulae are treated in our evaluation games. Formal definitions and some concrete examples will be given further, beginning from Section 3.2.

The evaluation of ATL^+ formulae of the type $\langle\langle A\rangle\rangle\,\Phi$ in a given model is based on constructing finite paths in that model. The following two ideas are central.

Firstly, the path formula Φ in $\langle\langle A\rangle\rangle\,\Phi$ can be *divided* into *goals for the verifier* ($\mathbf{V}$), these being the relative atoms $\psi \in At(\Phi)$ that occur *positively* in Φ, and *goals for the falsifier* ($\overline{\mathbf{V}}$), these being the relative atoms $\psi \in At(\Phi)$ that occur *negatively* in Φ. (Some relative atoms may occur both positively and negatively and thus be goals for both players.) For simplicity, let us assume for now that Φ is in negation normal form and all the atoms in $At(\Phi)$ are temporal formulae of the type $\mathsf{F}\,p$. Then the verifier's goals are eventuality statements $\mathsf{F}\,p$, while the falsifier's goals are statements $\mathsf{F}\,p'$ that occur negated; note that the negation of $\mathsf{F}\,p'$ is equivalent to the safety statement $\mathsf{G}\,\neg p'$. The verifier wishes to *verify* her/his goals. The falsifier, likewise, wants to *verify* her/his goals, i.e., the falsifier wishes to *falsify* the related safety statements.

Secondly, on any given path, every temporal goal associated with $\langle\langle A\rangle\rangle\,\Phi$ has a unique "finite determination point" where that goal can be verified by the player to whom the goal belongs. This means the following: If a goal $\mathsf{F}\,p$ of the verifier is true on an infinite path Λ, then there necessarily exists an earliest point q on that path where the fact that "$\mathsf{F}\,p$ holds on Λ" becomes *verified* simply because p is true at q. Indeed, the first point of Λ where p is true is the finite determination point q of $\mathsf{F}\,p$. Once $\mathsf{F}\,p$ has been verified, it will remain *true on* Λ, no matter what happens on the path after q. Similarly, concerning falsifier's goals, if $\mathsf{G}\,\neg p'$ is false (and thus $\mathsf{F}\,p'$ true) on an infinite path Λ', there is a unique point where $\mathsf{G}\,\neg p'$ first becomes *falsified*, that point being the first state q' of Λ' where p' is true. That point q' is the finite determination point of the goal

$\mathsf{F}\,p'$ of the falsifier. Furthermore, $\mathsf{G}\,\neg p'$ will remain false on the path no matter what happens further. (Note that there is no analogous finite determination point for ATL^*-formulae such as $\langle\!\langle A\rangle\!\rangle \mathsf{GF}p$ on a given infinite path. Note also that we discussed only the simple temporal goals $\mathsf{F}\,p$ and $\mathsf{F}\,p'$ for simplicity, but every temporal goal—as long as it can be verified by the player to whom the goal belongs—does indeed have a finite determination point. This will become clear below.)

Now, the game-theoretic evaluation procedure of an ATL^+-formula $\langle\!\langle A\rangle\!\rangle\,\Phi$ proceeds roughly as follows. The verifier is controlling the agents in the coalition A and the falsifier controls the agents in the **opposing coalition** $\overline{A} = \mathbb{A}\mathrm{gt} \setminus A$. The players start constructing a path. (Each transition from one state to another is carried out according to the process "$\mathsf{Step\,phase}$" defined formally in Section 3.2.2.) The verifier is first given a chance to verify some of her/his goals in Φ. The falsifier tries to prevent this and to possibly verify some of her/his own goals instead. During this path construction/verification process, the verifier is said to have the role of the **seeker**. A player is allowed to stay as the seeker for only a finite number of rounds. This is ensured by requiring the seeker to announce an ordinal[2], called **timer**[3], before the path construction process begins, and then lower the ordinal each time a new state is reached. The process ends when the ordinal becomes zero or when the seeker is satisfied, having verified some of her/his goals. Since ordinals are well-founded, the process must terminate.

After the verifier has ended her/his seeker turn, the falsifier may either end the game or take the role of the seeker. If (s)he decides to become the seeker, then (s)he sets a new timer and the path construction process continues for some finite number of rounds. When the falsifier is satisfied, having verified some of her/his goals, the verifier may again take the seeker's role, and so on. Thus, the verifier and falsifier take turns being the seeker, trying to reach (verify) their

[2]To see why finite ordinals do not suffice in general relates to infinite branching, see Example 4.8.

[3]Note that the term "timer" is used here differently from [3, 4].

goals. The number of these alternations is bounded by a **seeker turn counter** which is a finite number that equals the total number of goals in Φ. (The formal description of seeker turn alternation is given in the clause "Deciding whether to continue and adjusting the timer" in Section 3.2.2.)

Each time a goal in Φ becomes verified, this is recorded in a **truth function** T. (The recording of verified goals is described formally in the process "Adjusting the truth function" defined in Section 3.2.2.) The truth function carries the following information at any stage of the game:

- The verifier's goals that have been verified.
- The falsifier's goals that have been verified.
- All other goals remain **open**.

When neither of the players wants to become the seeker, or when the seeker turn counter becomes zero, the path construction process *ends* and the players play a standard Boolean evaluation game[4] on Φ by using the truth values given by T; the open goals are given truth values as follows:

- The verifier's open goals are *(so far) not verified* and thus considered *false*.
- Likewise, the falsifier's open goals are *(so far) not verified* and thus considered *false*. Recall here that the falsifier's goals occur in the scope of a negation.

Next we consider the conditions when a player is "satisfied" with the current status of the truth function T—and thus wants to end the game—and when (s)he is "unsatisfied" and wants to continue the game as the seeker. Note that when

[4]The Boolean game determines the truth value of any given formula of propositional logic for any given set of truth values for its atoms. Obviously the verifier wins the game if and only if the formula is true (with the given truth values), and otherwise the falsifier wins. The game is described in any elementary reference on game-theoretic semantics, e.g., in [17]. Note here that while Φ is not strictly speaking a formula of propositional logic, the function T gives interpretations to a set of subformulae of Φ so that it can be treated as propositional.

the path construction ends, then every goal is given a Boolean truth value based on the truth function T, as described above. With these values, the formula Φ is either true or false. If Φ is true with the current values based on T, then the verifier can win the Boolean game for Φ; dually, if Φ is not true with the values based on T, then the falsifier can win the Boolean game for Φ. Hence the players want to take the role of the seeker in order to modify the truth function T in such a way that the truth of Φ with respect to T changes from false to true (making $\mathbf{V}$ satisfied) or from true to false (making $\overline{\mathbf{V}}$ satisfied).

The truth value of Φ with respect to T can keep changing when T is modified, but only a finite number of changes is possible. Indeed, the maximum number of such truth alternations is the total number of goals in Φ.

In the general case, formulae of the type $\varphi \,\mathsf{U}\, \psi$, $\mathsf{X}\,\varphi$ and (state formulae) φ may also occur in $At(\Phi)$ as goals, and Φ does not have to be in negation normal form. Formulae of the type $\varphi \,\mathsf{U}\, \psi$ can be either verified, by showing that ψ is true, or falsified, by showing that φ is not true. State formulae φ can only be verified at the initial state and the next-state-formulae $\mathsf{X}\,\varphi$ can only be verified at the second state on the path traveled.

3.2. Evaluation games: formal description

Now we will present the **bounded evaluation game** which uses the **bounded transition game** as a subgame for evaluating strategic subformulae. Interleaved with the definition we will provide, in *italics*, a running example that uses $\mathcal{M}^*$ and φ^* from Examples 2.2 and 2.5 respectively.

3.2.1. Rules of the bounded evaluation game

Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a CGM, $q_{in} \in \mathrm{St}$ a state, φ a state formula and $\Gamma > 0$ an ordinal called a **timer bound**. The Γ**-bounded evaluation game** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ between the players $\mathbf{A}$ and $\mathbf{E}$ is defined as follows.

A **location** of the game is a tuple $(\mathbf{P}, q, \psi, T)$ where $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$, $q \in \mathrm{St}$ is a state, ψ is a subformula of φ and T is a **truth function**, mapping some subset of SUB(φ) into $\{\top, \bot, \mathsf{open}\}$.[5]

The **initial location** of the game is $(\mathbf{E}, q_{in}, \varphi, T_{in})$, where T_{in} is the empty function. In every location $(\mathbf{P}, q, \psi, T)$, the player $\mathbf{P}$ is called the **verifier** and $\overline{\mathbf{P}}$ the **falsifier** for that location. Intuitively, q is the current state of the game and T encodes truth values of formulae on a path that has been constructed earlier in the game.

Each location is associated with exactly one of the rules **1**–**6** given below. First we provide the rules for locations $(\mathbf{P}, q, \psi, T)$ where ψ is either a proposition symbol or has a Boolean connective as its main operator:

1. A location $(\mathbf{P}, q, p, T)$, where $p \in \Pi$, is an **ending location** of the evaluation game. If $T \neq \emptyset$, then $\mathbf{P}$ wins the game if $T(p) = \top$ and else $\overline{\mathbf{P}}$ wins. Respectively, if $T = \emptyset$, then $\mathbf{P}$ wins if $q \in v(p)$ and else $\overline{\mathbf{P}}$ wins.

2. From a location $(\mathbf{P}, q, \neg\psi, T)$ the game moves to the location $(\overline{\mathbf{P}}, q, \psi, T)$.

3. In a location $(\mathbf{P}, q, \psi \vee \theta, T)$ the player $\mathbf{P}$ chooses one of the locations $(\mathbf{P}, q, \psi, T)$ and $(\mathbf{P}, q, \theta, T)$, which becomes the next location of the game.

We then define the rules of the evaluation game for locations with strategic formulae as follows.

4. Suppose a location $(\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \Phi, T)$ is reached.

- If $T \neq \emptyset$, then this location is an ending location where $\mathbf{P}$ wins if $T(\langle\!\langle A \rangle\!\rangle\, \Phi) = \top$ and else $\overline{\mathbf{P}}$ wins.
- If $T = \emptyset$, then the evaluation game enters a **transition game** $\mathbf{g}(\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$. The transition game is a subgame to be defined later on. The transition game eventually reaches an **exit location** $(\mathbf{P}', q', \psi, T')$, and the evaluation game continues from that location. Note that an *exit location*

[5] We note here that the values of T are only modified during transition games and that T is always a function mapping the subformulae of φ (into $\{\top, \bot, \mathsf{open}\}$) that are relevant for the transition game that is played. We also note that $T \neq \emptyset$ will hold precisely during transition games and when playing a Boolean game resulting from a transition game.

only ends the transition game, so exit locations of transition games and *ending locations* of the evaluation game are different concepts.

The rules corresponding to the temporal connectives are defined using the truth function T (updated in an earlier transition game) as follows.

5. A location $(\mathbf{P}, q, \varphi \,\mathsf{U}\, \psi, T)$ is an ending location of the evaluation game. $\mathbf{P}$ wins if $T(\varphi \,\mathsf{U}\, \psi) = \top$ and else $\overline{\mathbf{P}}$ wins.

6. Likewise, a location $(\mathbf{P}, q, \mathsf{X}\, \varphi, T)$ is an ending location. $\mathbf{P}$ wins if $T(\mathsf{X}\, \varphi) = \top$ and otherwise $\overline{\mathbf{P}}$ wins.

These are the rules of the evaluation game. We note that the timer bound Γ will be used only in transition games. If $\Gamma = \omega$, we say that the evaluation game is **finitely bounded**.

The initial location of the finitely bounded evaluation game $\mathcal{G}(\mathcal{M}^*, q_0, \varphi^*, \omega)$ (see Examples 2.2 and 2.5) is $(\mathbf{E}, q_0, \langle\!\langle a_1 \rangle\!\rangle\, \Psi, \emptyset)$, from where the transition game $\boldsymbol{g}(\mathbf{E}, q_0, \langle\!\langle a_1 \rangle\!\rangle\, \Psi, \omega)$ begins.

3.2.2. Rules of the bounded transition game

Recall that transition games are subgames of evaluation games. Their purpose is to evaluate the truth of strategic subformulae, in a game-like fashion.

Now we give a detailed description of transition games.[6] A **transition game** $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$, where $\mathbf{V} \in \{\mathbf{A}, \mathbf{E}\}$, $q_0 \in \mathrm{St}$, $\langle\!\langle A \rangle\!\rangle\, \Phi \in \mathsf{ATL}^+$ and $\Gamma > 0$ is an ordinal, is defined as follows. $\mathbf{V}$ is called **the verifier in the transition game**. The game $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$ is based on **configurations**, i.e., tuples $(\mathbf{S}, q, T, n, \gamma, x)$, where the player $\mathbf{S} \in \{\mathbf{E}, \mathbf{A}\}$ is called the **seeker**; q is the **current state**; $T : At(\Phi) \to \{\top, \bot, \mathsf{open}\}$ is a **truth function**; $n \in \mathbb{N}$ is a **seeker turn counter** ($n \leq |At(\Phi)|$); γ is an ordinal called **timer**; and $x \in \{\, \mathbf{i}, \mathbf{ii}, \mathbf{iii} \,\}$

[6] A transition game for ATL^+ is similar to the *embedded game* introduced in [4] for the GTS of ATL. The role of the seeker $\mathbf{S}$ here is similar to the role of the controller in the embedded game.

is an index showing the current **phase** of the transition game. The game $\mathbf{g}(\mathbf{V}, q_0, \langle\langle A \rangle\rangle\, \Phi, \Gamma)$ begins at the **initial configuration** $(\mathbf{V}, q_0, T_0, |At(\Phi)|, \Gamma, \mathbf{i})$, with $T_0(\chi) = \mathsf{open}$ for all $\chi \in At(\Phi)$.

The transition game $\boldsymbol{g}(\mathbf{E}, q_0, \langle\langle a_1 \rangle\rangle\, \Psi, \omega)$ begins from the initial configuration $(\mathbf{E}, q_0, T_0, 4, \omega, \mathbf{i})$, since $|At(\Psi)| = 4$. (Note that the timer is initially ω in transition games occurring within finitely bounded evaluation games, but the timer will always have a finite value thereafter.)

The transition game then proceeds by iterating the phases **i**, **ii** and **iii**, which we first describe informally; detailed formal definitions are given afterwards.

i. Adjusting the truth function: In this phase the players make claims on the truth of state formulae at the current state q. If $\mathbf{P}$ makes some claim, then the opponent $\overline{\mathbf{P}}$ may either: (1) accept the claim, whence truth function is updated accordingly, or (2) challenge the claim. In the latter case the *transition game ends* and truth of the claim is verified/falsified in a continued evaluation game.

ii. Deciding whether to continue and adjusting the timer: Here the current seeker $\mathbf{S}$ may either continue her/his seeker turn and lower the value of the timer, or end her/his seeker turn. If $\mathbf{S}$ chooses the latter option, then the opponent $\overline{\mathbf{S}}$ of the seeker may either (1) take the role of the seeker and announce a new initial value for the timer or (2) end the transition game, whence the formula Φ is evaluated based on current values of the truth function.

iii. Step phase: Here the verifier $\mathbf{V}$ chooses actions for the agents in the coalition in A at the current state q. Then $\overline{\mathbf{V}}$ chooses actions for the agents in the opposing coalition $\overline{A}$. After the resulting transition to a new state q' has been made, the game continues again with phase **i**.

We now describe the phases **i**, **ii** and **iii** in technical detail:

i. Adjusting the truth function.

Suppose the current configuration is $(\mathbf{S}, q, T, n, \gamma, \mathbf{i})$. Then the truth function T is updated by considering, one by one, each formula $\chi \in At(\Phi)$ in some fixed order[7]. If $T(\chi) \neq \mathsf{open}$, then the value χ cannot be updated. Else the value of χ may be modified according to the rules **A** – **C** below.

A. *Updating T on temporal formulae with* U: Suppose that $\varphi \,\mathsf{U}\, \psi \in At(\Phi)$. Now first the verifier **V** may *claim that ψ is true* at the current state q. If **V** makes that claim, then $\overline{\mathbf{V}}$ chooses either of the following:

- $\overline{\mathbf{V}}$ *accepts* the claim of **V**, whence the truth function is updated so that $\varphi \,\mathsf{U}\, \psi$ is assigned value $\top$ ($\varphi \,\mathsf{U}\, \psi$ becomes **verified**), hereafter indicated by $\varphi \,\mathsf{U}\, \psi \mapsto \top$.

- $\overline{\mathbf{V}}$ *challenges* the claim of **V**, whence the transition game ends at the **exit location** $(\mathbf{V}, q, \psi, \emptyset)$. (We note that, here and further, when a transition game ends, the evaluation game continues from the related exit location and the evaluation game will *never* return to the same exited transition game again.)

If **V** does not claim that ψ is true at q, then $\overline{\mathbf{V}}$ may make that same claim (that ψ is true at q). If $\overline{\mathbf{V}}$ makes that claim, then the same two steps above concerning *accepting* and *challenging* are followed, but with **V** and $\overline{\mathbf{V}}$ swapped everywhere.

Suppose then that neither of the players claims that ψ is true at q. Then first **V** can *claim that φ is false* at q. If **V** makes that claim, then $\overline{\mathbf{V}}$ chooses either of the following:

- $\overline{\mathbf{V}}$ accepts the claim, whence the truth function is updated so that $\varphi \,\mathsf{U}\, \psi \mapsto \bot$ ($\varphi \,\mathsf{U}\, \psi$ becomes **falsified**).

- $\overline{\mathbf{V}}$ challenges the claim, whence the transition game ends at the exit location $(\overline{\mathbf{V}}, q, \varphi, \emptyset)$.

[7] We will see that the order here is irrelevant for the existence of winning strategies in the evaluation game. This is simply because the player with a winning strategy can make all the claims that are true and challenge all the other claims—regardless of the order in which the formulae are considered.

If $\mathbf{V}$ does not claim that φ is false at q, then $\overline{\mathbf{V}}$ may make that claim. If (s)he does, then the same steps as those above are followed, but with $\mathbf{V}$ and $\overline{\mathbf{V}}$ swapped.

B. *Updating T on proposition symbols and strategic formulae*: The truth function can be updated on proposition symbols $p \in At(\Phi)$ and formulae $\langle\langle A' \rangle\rangle \, \Psi \in At(\Phi)$ only when the phase **i** is executed for the first time (so, $q = q_0$). In this case, given such a formula χ, first $\mathbf{V}$ can claim that χ is true at q. Now, if $\overline{\mathbf{V}}$ accepts this claim, then the truth function is updated s.t. $\chi \mapsto \top$. If $\overline{\mathbf{V}}$ challenges the claim, then the transition game ends at the exit location $(\mathbf{V}, q, \chi, \emptyset)$. If $\mathbf{V}$ does not claim that χ is true at q, then $\overline{\mathbf{V}}$ may make that claim. If (s)he does, then the same steps are followed, but with $\mathbf{V}$ and $\overline{\mathbf{V}}$ swapped.

C. *Updating T on formulae with* X: The truth function can be updated on formulae of type $\mathsf{X}\,\psi \in At(\Phi)$ only when phase **i** is executed for the second time in the transition game (so, q is some successor of q_0). First $\mathbf{V}$ can claim that ψ is true at q. If $\overline{\mathbf{V}}$ accepts that claim, then the truth function is updated s.t. $\mathsf{X}\,\psi \mapsto \top$. If $\overline{\mathbf{V}}$ challenges the claim, then the transition game ends at the exit location $(\mathbf{V}, q, \psi, \emptyset)$. If $\mathbf{V}$ does not claim that ψ is true at q, then $\overline{\mathbf{V}}$ can make that claim. If (s)he does, the same steps are followed, but with $\mathbf{V}$ and $\overline{\mathbf{V}}$ swapped.

Note that in points **B** and **C**, the formulae cannot be mapped to $\bot$ by the truth function T. But if these formulae are left with the value open, then they will be considered false by default if the transition game ends in stage **ii** (and the Boolean game is played). Intuitively this is because if no player has claimed these formulae to be true, then the players have agreed that they are indeed false.

If neither player makes any claim which would update the value of a formula $\chi \in At(\Phi)$, then the value of χ is indeed left open. Once the values of the truth function T have been updated (or left as they are) for all formulae in $At(\Phi)$, a new truth function T' is obtained. The transition game then moves to the new configuration $(\mathbf{S}, q, T', n, \gamma, \mathbf{ii})$.

In the configuration $(\mathbf{E}, q_0, T_0, 4, \omega, \mathbf{i})$ *the players begin adjusting* T_0 *for which initially* $T_0(\chi) = \mathsf{open}$ *for every* $\chi \in At(\Psi)$*. Since it is the first round of the transition game, the value of* $\mathsf{X}\, p_3$ *cannot be modified, but the value of* $\langle\langle a_2 \rangle\rangle \mathsf{X}\, p_1$ *can be modified. Suppose that Eloise claims that* $\langle\langle a_2 \rangle\rangle \mathsf{X}\, p_1$ *is true at* q_0*. Now Abelard could challenge the claim, whence the transition game would end and the evaluation game would continue from location* $(\mathbf{E}, q_0, \langle\langle a_2 \rangle\rangle \mathsf{X}\, p_1, \emptyset)$ *(which leads to a new transition game* $\boldsymbol{g}(\mathbf{E}, q_0, \langle\langle a_2 \rangle\rangle \mathsf{X}\, p_1, \omega)$*). Suppose Abelard does not challenge the claim. Then* $\langle\langle a_2 \rangle\rangle \mathsf{X}\, p_1$ *is mapped to* $\top$*.*

Since $\mathsf{F}\, p_1$ *and* $(\neg p_1) \mathsf{U}\, p_2$ *occur positively in* Φ*, Eloise has interest only to verify them and Abelard has interest only to falsify them. Eloise could verify* $\mathsf{F}\, p_1$ *by claiming that* p_1 *is true, or verify* $(\neg p_1) \mathsf{U}\, p_2$ *by claiming that* p_2 *is true. But, if Eloise makes either of these claims, then Abelard wins the whole evaluation game by challenging, since* $q_0 \notin v(p_1) \cup v(p_2)$*. Suppose that Eloise does not make any claims. Now, Abelard could claim that* $\neg p_1$ *is not true, in order to falsify* $(\neg p_1) \mathsf{U}\, p_2$*. But if he does that, he loses the evaluation game if Eloise challenges, since* $q_0 \notin v(p_1)$*. Suppose that Abelard does not make any claims either. Then the transition game proceeds to configuration* $(\mathbf{E}, q_0, T, 4, \omega, \mathbf{ii})$*, where* $T(\langle\langle a_2 \rangle\rangle \mathsf{X}\, p_1) = \top$ *and* $T(\chi) = \mathsf{open}$ *for the other* $\chi \in At(\Psi)$*.*

ii. Deciding whether to continue and adjusting the timer.

Suppose that a configuration $(\mathbf{S}, q, T, n, \gamma, \mathbf{ii})$ has been reached. Assume first that $\gamma \neq 0$. Then the seeker $\mathbf{S}$ can choose whether to continue the transition game as the seeker. If yes, then $\mathbf{S}$ chooses some ordinal $\gamma' < \gamma$ and the transition game continues from $(\mathbf{S}, q, T, n, \gamma', \mathbf{iii})$. If $\mathbf{S}$ does not want to continue, or if $\gamma = 0$, then one of the following applies.

(a) Suppose that $n \neq 0$. Then the player $\overline{\mathbf{S}}$ chooses whether she wishes to continue the transition game. If yes, then $\overline{\mathbf{S}}$ chooses some ordinal $\gamma' < \Gamma$ (so, $\overline{\mathbf{S}}$ in fact *resets* the timer value) and the transition game continues from $(\overline{\mathbf{S}}, q, T, n-1, \gamma', \mathbf{iii})$. Otherwise the transition game ends at the exit location $(\mathbf{V}, q, \Phi, T)$.

(b) Suppose that $n = 0$. Then the transition game ends at the exit location

$(\mathbf{V}, q, \Phi, T)$.

In $(\mathbf{E}, q_0, T, 4, \omega, \mathbf{ii})$ *Eloise may decide whether to continue the transition game as the seeker. Suppose that Eloise does not continue, whence Abelard may now either become the seeker and continue the transition game, or end it. If Abelard ends the transition game, then the evaluation game is continued from* $(\mathbf{E}, q_0, \Psi, T)$*. But because* $T(\mathsf{X}\, p_3) = \mathsf{open}$ *and* $T(\langle\!\langle a_2 \rangle\!\rangle \mathsf{X}\, p_1) = \top$*, Eloise can then win the evaluation game by choosing the left disjunct of* Ψ *(note that with these values of* T *Eloise is then guaranteed to win the evaluation game). Suppose thus that Abelard decides to become the seeker, whence he chooses some* $m < \omega$ *and the next configuration is* $(\mathbf{A}, q_0, T, 3, m, \mathbf{iii})$.

iii. Step phase[8]

Suppose that the configuration is $(\mathbf{S}, q, T, n, \gamma, \mathbf{iii})$.

(a) First, $\mathbf{V}$ chooses an action $\alpha_i \in d(a_i, q)$ for each $a_i \in A$.

(b) Then, $\overline{\mathbf{V}}$ chooses an action $\alpha_i \in d(a_i, q)$ for each $a_i \in \overline{A}$.

The resulting action profile produces a **successor state** $q' := o(q, \alpha_1, \ldots, \alpha_k)$. The transition game then moves to the configuration $(\mathbf{S}, q', T, n, \gamma, \mathbf{i})$.

In the configuration $(\mathbf{A}, q_0, T, 3, m, \mathbf{iii})$ *Eloise (who is the verifier* $\mathbf{V}$*) first chooses action for agent* a_1*, then Abelard chooses action for agent* a_2*, which produces either successor state* q_1 *or* q_2*. Then the transition game continues from the configuration* $(\mathbf{A}, q_j, T, 3, m, \mathbf{i})$*, where* $j \in \{1, 2\}$.

This concludes the definition of the rules for the phases **i, ii** and **iii** in the transition game $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$.

Suppose that the transition game continues from the configuration $(\mathbf{A}, q_2, T', 3, m, \boldsymbol{i})$*. Since it is the second round of the transition game, Abelard could now try to verify* $\mathsf{X}\, p_3$ *by claiming that* p_3 *is true at* q_2*. However, then Eloise could win the evaluation game by challenging this claim. But if Abelard does not try to verify* $\mathsf{X}\, p_3$ *at that configuration,*

[8]The procedure in this phase is analogous to the *step game*, $\mathsf{step}(\mathbf{V}, A, q)$, which was introduced for the GTS for ATL ([3, 4]).

then the value of $\mathsf{X}\, p_3$ *will stay* open*. Hence, when Abelard decides to end his turn as the seeker or when the timer* m *is lowered to* 0*, then Eloise can end the transition game and win the evaluation game from a location of the form* $(\mathbf{E}, \Psi, q', T'')$*.*

Suppose now that the transition game continues from the configuration $(\mathbf{A}, q_1, T', 3, m, \boldsymbol{i})$*. Suppose that Abelard verifies* $\mathsf{X}\, p_3$ *by claiming that* p_3 *is true at* q_1 *and that Eloise does not challenge that claim. If the transition game now ended at location* $(\mathbf{E}, q_1, \Psi, T'')$*, where* $T''(\mathsf{X}\, p_3) = \top$*, Abelard could win the resumed evaluation game. Thus, if Abelard decides to quit the transition game, then Eloise wants to continue as the seeker from configuration* $(\mathbf{E}, q_1, T'', 2, m', \boldsymbol{iii})$ *for some* $m' < \omega$*. Then Eloise can choose action* α *for agent* a_1 *and lower the timer to* 2*, whence the next configuration is* $(\mathbf{E}, q_3, T'', 2, 2, \boldsymbol{i})$*. Eloise can then verify* $(\neg p_1)\, \mathsf{U}\, p_2$ *at it by claiming that* p_2 *is true at* q_3*. Furthermore, Eloise can move via* q_1 *to* q_4 *and verify* $\mathsf{F}\, p_1$ *there, before the timer reaches* 0*. Then Eloise will win when the evaluation game is continued from a location of the form* $(\mathbf{E}, q_4, \Psi, T''')$*.*

3.2.3. The unbounded evaluation game

Let $\mathcal{G}(\mathcal{M}, q, \varphi, \Gamma)$ be a Γ-bounded evaluation game. We can define a corresponding **unbounded evaluation game**, $\mathcal{G}(\mathcal{M}, q, \varphi)$, by replacing transition games $\mathbf{g}(\mathbf{V}, q, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$ with **unbounded transition games**, $\mathbf{g}(\mathbf{V}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$; these are played with the same rules as $\mathbf{g}(\mathbf{P}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$ except that timers γ are not used in them. Instead, the players can keep the role of the seeker for arbitrarily long and thus the game may last for an infinite number of rounds. In the case of an infinite play, *the player who took the last seeker turn loses the entire evaluation game.* (Recall that the number of seeker turn alternations is bounded by the number $|At(\Phi)|$.)

3.3. Defining the game-theoretic semantics

In this section we define game-theoretic semantics for ATL^+ by equating truth of formulae with the existence of a winning strategy for Eloise in the corresponding evaluation game. We begin with the following remark which will be relevant for the notion of positional strategies in evaluation games.

Remark 3.1. The description of transition games above is based on a simplified notion of configurations. The phases **i–iii** consist of several "subphases" and more information should be encoded into configurations. The full notion of configuration should also include:

– In phase **i**, a counter indicating the relative atom currently under consideration by the players; flags for each player indicating whether and what claim (s)he has made on the truth of the current relative atom; a 3-bit flag indicating if it is the first, second, or some later round in the transition game.

– For phase **ii**, a flag whether the current seeker wants to continue, and for phase **iii**, a record of the current choice of actions for the agents in A by $\mathbf{V}$.

For technical simplicity, we omit these formal details. Note that these additional details are similar for bounded and unbounded games.

Hereafter a **position** in an evaluation game will mean either a location of the form $(\mathbf{P}, q, \varphi, T)$ or a configuration in the fully extended form described in the remark above. By this definition, at every position, only one of the players (Abelard or Eloise) has a move to choose. Thus, the entire evaluation game—including transition games as subgames—is a turn-based game of perfect information.

By a **game tree** $T_{\mathcal{G}}$ of an evaluation game $\mathcal{G}$, we mean the tree whose nodes correspond to all positions arising in $\mathcal{G}$, and every branch of which corresponds to a possible play of $\mathcal{G}$ (including transition games as subgames). Note that, in the case of *unbounded* evaluation games, some of these plays may be infinite, but only because an embedded transition game does not terminate, in which case a winner in the entire evaluation game is uniquely assigned according to the rules in Section 3.2.3. However, in the case of *bounded* evaluation games, all the paths in $T_{\mathcal{G}}$ are guaranteed to be finite since ordinals (used as timers) are well-founded.

The formal definitions of players' memory-based strategies in the evaluation games games are defined as expected, based on histories of positions. As usual, a strategy for a player $\mathbf{P}$ is called **winning** if, following that strategy, $\mathbf{P}$ is guaranteed to win regardless of how $\overline{\mathbf{P}}$ plays. A strategy is **positional** if it depends only on the current position. We can also define strategies for transition games that arise within evaluation games; note that these are substrategies for the strategies in evaluation games. A strategy τ for a transition game is called winning for $\mathbf{P}$ if

- every exit location that can be reached with τ is a winning location for $\mathbf{P}$ in the evaluation game that continues from the exit location,

and additionally

- in the alternative scenario where the transition game continues infinitely long while τ is followed (which is possible only in unbounded games), the player $\mathbf{P}$ is ***not*** the player who holds the (necessarily last) seeker's turn that lasts infinitely long.

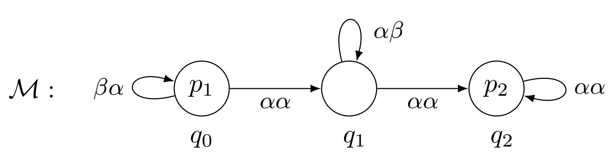
Definition 3.2. Let $\mathcal{M}$ be a CGM, $q \in \mathrm{St}$, $\varphi \in \mathsf{ATL}^+$ and Γ an ordinal. **Truth of φ in the Γ-bounded** ($\Vdash_\Gamma$), resp. **unbounded** ($\Vdash$) GTS is defined as follows:

$$\mathcal{M}, q \Vdash_\Gamma \varphi \text{ (resp. } \mathcal{M}, q \Vdash \varphi) \text{ iff Eloise has a positional winning strategy in } \mathcal{G}(\mathcal{M}, q, \varphi, \Gamma) \text{ (resp. } \mathcal{G}(\mathcal{M}, q, \varphi)).$$

We will show later, in Section 4.1, that evaluation games are determined with positional strategies. Therefore, even if we allowed perfect-recall strategies in the truth definition above, we would obtain equivalent semantics.

Example 3.3. Consider the CGM $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$, where:

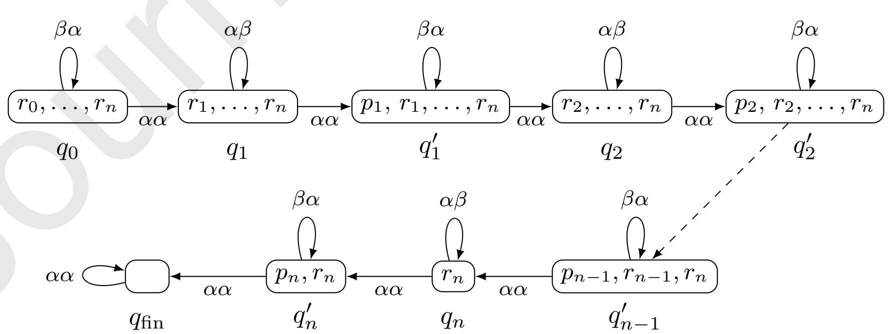
$$\mathbb{A}\mathrm{gt} = \{1, 2\},\ \mathrm{St} = \{q_0, q_1, q_2\},\ \Pi = \{p_1, p_2\},\ \mathrm{Act} = \{\alpha, \beta\}$$
$$d(1, q_0) = d(2, q_1) = \{\alpha, \beta\};\ d(a, q_i) = \{\alpha\} \text{ in all other cases;}$$
$$o(q_0, \beta\alpha) = q_0,\ o(q_0, \alpha\alpha) = o(q_1, \alpha\beta) = q_1,\ o(q_1, \alpha\alpha) = o(q_2, \alpha\alpha) = q_2$$
$$v(p_1) = \{q_0\} \text{ and } v(p_2) = \{q_2\}.$$



Let $\varphi := \langle\langle a_2 \rangle\rangle (\mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$ (here $\mathsf{G}\, p_1 = \neg \mathsf{F} \neg p_1$). We describe a winning strategy for Eloise in the unbounded evaluation game $\mathcal{G}(\mathcal{M}, q_0, \varphi)$. Eloise immediately ends her turn as the seeker and does not make claims while being at q_0. If Abelard makes any claims at q_0, she challenges those claims. If Abelard ends the transition game at q_0, Eloise wins the evaluation game by choosing the disjunct $\neg \mathsf{F} \neg p_1$, as now the value of $\mathsf{F} \neg p_1$ is open. Suppose that Abelard forces a transition to q_1 by choosing α for the agent a_1. If he claims $\neg p_1$ is true at q_1, Eloise does not challenge. If Abelard ends his seeker turn at q_1, Eloise becomes the seeker. At q_1 she forces a transition to q_2, by choosing α for a_2. Then she verifies $\mathsf{F}\, p_2$ by claiming that p_2 is true at q_2 and ends her turn as the seeker after that. If the transition game ends at q_2, she wins by choosing $\mathsf{F}\, p_2$, whose value is $\top$. Note that by following this strategy, Eloise cannot stay as the seeker for infinitely long.

We will see later that there is never a need for a larger number than $|At(\Phi)|$ of seeker turn alternations in a transition game for a formula $\langle\langle A \rangle\rangle\, \Phi$. In Example 3.3 we saw that there are cases where exactly $|At(\Phi)|$ seeker alternations are needed in the corresponding transition game. The following example generalizes the setting of Example 3.3 by showing that no fixed upper bound for the number of seeker alternations suffices for all transitions games.

Example 3.4. Let $\varphi_k = \langle\langle a_2 \rangle\rangle\, \Psi_k$, where $\Psi_k := \mathsf{G}\, r_0 \vee \bigvee_{1 \leq i \leq k} (\mathsf{F}\, p_i \wedge \mathsf{G}\, r_i)$. Consider the following CGM $\mathcal{M}$ (c.f. the model in Example 3.3).



At q_0 Eloise wants to end her seeker turn immediately as $\mathsf{G}\, r_0$ is “still” true. When Abelard becomes the seeker, he wants to make a transition to q_1 and falsify $\mathsf{G}\, r_0$ there. Since Abelard has then no reason to continue as the seeker, he gives the seeker turn to Eloise. Now Eloise wants to make a transition to q'_1 in order to verify $\mathsf{F}\, p_1$; since $\mathsf{G}\, r_1$ is still true, Eloise has then no reason to continue as the seeker. We may suppose that the transition game continues like this, so that the seeker role is swapped *after every transition* and $\mathsf{F}\, p_i$ are verified and $\mathsf{G}\, r_i$ are falsified. When Abelard finally becomes the seeker at q'_n, the maximum number of $|At(\Psi_k)| = 2k + 1$ seeker turn alternations have been used. Then Abelard makes a transition to q'_n, falsifies $\mathsf{G}\, r_n$ and wins the Boolean game for Ψ_k with the values of the (fully updated) truth function.

4. Analysing evaluation games

In this section we will analyse the properties of the evaluation games of ATL^+. We first prove positional determinacy of both bounded and unbounded evaluation games. Then we find so-called stable timer bounds for bounded evaluation games and show that with them, the bounded GTS becomes equivalent to the unbounded GTS. Finally we present the notion of a regular strategy which will be needed for proving the equivalence of GTS and the standard compositional semantics of ATL^+ in the next section.

4.1. Positional determinacy

Here we prove positional determinacy of both bounded and unbounded evaluation games. Recall here that positions are either locations in evaluation games or configurations in transition games—in the extended sense which was discussed in Remark 3.1. The positional determinacy of bounded games is easy to prove since their game tree is well-founded.

Proposition 4.1. Bounded evaluation games are determined and the winner has a positional winning strategy.

Proof. (Sketch) Since ordinals are well-founded and they must decrease during transition games, it is easy to see that the game tree is well-founded. Thus positional determinacy follows easily, essentially by backward induction. □

We will prove positional determinacy of unbounded evaluation games by showing that they can be translated into corresponding Büchi-games (which are known to be positionally determined). This correspondence between unbounded evaluation games and Büchi-games is also interesting in its own right and we will be use it later in Section 6.2 for proving tractability of certain natural fragments of ATL^+.

Proposition 4.2. Unbounded evaluation games are determined and the winner has a positional winning strategy.

Proof. We will show that unbounded evaluation games are essentially Büchi-games (see, e.g., [18]). We first discuss the case where the underlying CGM $\mathcal{M}$ is finite. We follow the technicalities for Büchi-games from [19], which gives an excellently detailed and to-the-point presentation of the related basic notions.

Take a triple $(\mathcal{M}, q, \varphi)$, where $\mathcal{M}$ is a finite CGM, q a state of $\mathcal{M}$, and φ a formula of ATL^+. We will convert this triple into a Büchi-game BG such that $\mathcal{M}, q \Vdash \varphi$ iff player 2 has a winning strategy in BG from a position of BG determined by the initial location of $\mathcal{G}(\mathcal{M}, q, \varphi)$. The required Büchi-game BG corresponds almost exactly to the unbounded evaluation game $\mathcal{G}(\mathcal{M}, q, \varphi)$. The set of states of BG is the finite set of positions in $\mathcal{G}(\mathcal{M}, q, \varphi)$. The states of BG assigned to player 1 (resp., player 2) of BG are the positions where Abelard (resp., Eloise) is to move. The edges of the binary transition relation E of BG correspond to the changes of positions in $\mathcal{G}(\mathcal{M}, q, \varphi)$. Also, E is defined such that ending locations in the evaluation game connect (only) to themselves via E. This ensures that every state of BG has a successor state.

We set a *co-Büchi-objective* such that an infinite play of BG is winning for player 2 iff the set of states visited infinitely often is a subset of the union of the following sets of states of BG:

1. States of BG corresponding to configurations of the transition games where Abelard is the seeker.

2. States of BG corresponding to such ending locations in the game $\mathcal{G}(\mathcal{M}, q, \varphi)$ where Eloise has already won.

Clearly, Eloise (resp., Abelard) has a positional winning strategy in the evaluation game $\mathcal{G}(\mathcal{M}, q, \varphi)$ iff player 2 (resp., player 1) in BG has a positional winning strategy from the state of BG corresponding to initial location of $\mathcal{G}(\mathcal{M}, q, \varphi)$. Finite Büchi games enjoy positional determinacy (see e.g. [19]), which completes the case of finite $\mathsf{CGM}s$. For infinite $\mathsf{CGM}s$, the argument is the same but requires positional determinacy of Büchi games on infinite game graphs. That fact is well-known and follows easily from Theorem 4.3 of [20]. □

By the positional determinacy, we have the following consequence: if Eloise (Abelard) has a perfect-recall strategy in a bounded or unbounded evaluation game (or transition game), then she (he) has a positional winning strategy in that game.

4.2. Finding stable timer bounds

In this section study which timer bounds are "stable" for a given model. Intuitively this means that a timer bound Γ is stable for a model $\mathcal{M}$ if neither of the players can benefit from announcing timers that are higher than (or equal to) Γ. We will see that, by finding stable timer bounds, we can make the bounded GTS equivalent to the unbounded GTS. Moreover, the identification of stable timer bounds for finite models will be necessary for our model checking proofs in Section 6.

Every unbounded transition game (and even evaluation game) gives rise to a **semi-bounded** variant of that game, defined so that Eloise is forced to use ordinals when being the seeker. The ordinals are lowered in the way identical to bounded games. Abelard is not forced to use ordinals, but he naturally loses the game if he stays as the seeker for infinitely long. Obviously, an analogous semi-bounded game, where only Abelard has to use ordinals, can easily be defined. It will be clear from the context which variant of the semi-bounded game is meant.

A timer bound Γ is **stable** for an unbounded transition game $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A\rangle\!\rangle\,\Phi)$ if the player with a winning strategy in $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A\rangle\!\rangle\,\Phi)$ can, in fact, win the corresponding semi-bounded game using timers below Γ.

We first identify stable timer bounds for *finite* models.

Proposition 4.3. Let $\mathcal{M}$ be a finite CGM, $q_0 \in \mathrm{St}$ a state and $\Phi \in \mathsf{ATL}^+$ a path formula. Then $k := |\,\mathrm{St}\,| \cdot |At(\Phi)|$ is a stable timer bound for $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi)$.

Proof. We give a detailed sketch of proof. Let $c = (\mathbf{E}, q, T, n, x)$ be a configuration for an *unbounded* game (so no timer is listed). Suppose that the exit location $(\mathbf{V}, q, \Phi, T)$ is not a winning location for Eloise. Then she wants to stay as the seeker until the truth function is modified to a new truth function T' that makes Φ true. Since T is updated state-wise, it is not beneficial for Eloise to go in loops such that T is not updated. Hence, if Eloise has a winning strategy from c, then she has a winning strategy in which T is updated at least once every $|\,\mathrm{St}\,|$ rounds. Since T can be updated at most $|At(\Phi)|$ times, we see that a timer greater than $k = |\,\mathrm{St}\,| \cdot |At(\Phi)|$ is not needed. Symmetrically, if Abelard has a winning strategy from c, he can win by using timers below Γ. $\square$

Corollary 4.4. *If $\mathcal{M}$ is a finite* CGM*, the unbounded* GTS *is equivalent on $\mathcal{M}$ to the Γ-bounded* GTS *when $\Gamma \geq |\,\mathrm{St}\,| \cdot |\varphi|$.*

Proof. Let $\Gamma \geq |\,\mathrm{St}\,| \cdot |\varphi|$. Suppose first that $\mathcal{M}, q \Vdash \varphi$. By Proposition 4.3 Eloise can win the evaluation game using timers smaller than Γ when being the seeker in the transition games within $\mathcal{G}(\mathcal{M}, q, \varphi)$. Hence clearly $\mathcal{M}, q \Vdash_\Gamma \varphi$.

Suppose then that $\mathcal{M}, q \nVdash \varphi$. By Proposition 4.2, Abelard has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi)$. Thus, by Proposition 4.3, Abelard can win $\mathcal{G}(\mathcal{M}, q, \varphi)$ using timers smaller than Γ when being the seeker in transition games. Hence, Abelard clearly has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, \Gamma)$ and thus $\mathcal{M}, q \nVdash_\Gamma \varphi$. $\square$

In order to find stable timer bounds for infinite models, we give the following definition (cf. Definition 4.12 in [3]).

Definition 4.5. Let $\mathcal{M}$ be a CGM and let $q \in \mathrm{St}$. The **branching degree of** q, $\mathsf{BD}(q)$, is the cardinality of the set of outcome states from q:

$$\mathsf{BD}(q) := \mathsf{card}(\{o(q, \vec{\alpha}) \mid \vec{\alpha} \in \mathsf{action}(\mathbb{A}\mathrm{gt}, q)\}).$$

The **regular branching bound of** $\mathcal{M}$, or $\mathsf{RBB}(\mathcal{M})$, is *the smallest infinite regular cardinal* κ *such that* $\kappa > \mathsf{BD}(q)$ *for every* $q \in \mathrm{St}$. Note that $\mathsf{RBB}(\mathcal{M}) = \omega$ if and only if $\mathcal{M}$ is **image-finite**.

If $c = (\mathbf{S}, q, T, n, x)$ is a configuration in an unbounded transition game and γ is an ordinal, we use the notation $c[\gamma] := (\mathbf{S}, q, T, n, \gamma, x)$.

Proposition 4.6. Let $\mathcal{M}$ be a CGM, $q_0 \in \mathrm{St}$ and $\Phi \in \mathsf{ATL}^+$ a path formula. Then $\mathsf{RBB}(\mathcal{M})$ is a stable timer bound for $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle \, \Phi)$.

Proof. Suppose that Eloise has a winning strategy τ in the unbounded transition game $\mathbf{g}(\mathcal{M}, q_0, \langle\!\langle A \rangle\!\rangle \, \Phi)$ (the reasoning is symmetrical if Abelard has a winning strategy). We need to supplement τ with announcements of ordinals below $\mathsf{RBB}(\mathcal{M})$, when Eloise takes the role of the seeker, and with instructions on lowering the ordinal after every transition while she is the seeker.

Let c be any configuration that can be reached with τ and where Eloise has just begun a turn as the seeker. Let $T_{\mathbf{g},c}$ be the tree that is formed by all of those paths of configurations, starting from c in which Eloise *stays as the seeker* and plays according to τ (note that in the leafs, Eloise ends her seeker turn or the transition game ends). Since τ is a winning strategy, every path in $T_{\mathbf{g},c}$ must be finite, and thus $T_{\mathbf{g},c}$ is well-founded. We prove the following claim by well-founded induction on $T_{\mathbf{g},c}$:

$$\text{For every } c' \in T_{\mathbf{g},c}, \text{ there is an ordinal } \gamma < \mathsf{RBB}(\mathcal{M})$$
$$\text{s.t. } c'[\gamma] \text{ is a winning configuration for Eloise.}$$

Furthermore, Eloise can use these ordinals in the so obtained winning strategy (extending τ) from c.

We choose $\gamma = 0$ for every leaf on $T_{\mathbf{g},c}$. Suppose then that c' is not a leaf. By the inductive hypothesis, the claim holds for every configuration that can be reached with τ by a transition from c'. We now define γ to be the *successor of the supremum* of these ordinals. Since $\mathsf{RBB}(\mathcal{M})$ is regular, we have $\gamma < \mathsf{RBB}(\mathcal{M})$. Thus $c'[\gamma]$ is a winning configuration for Eloise. □

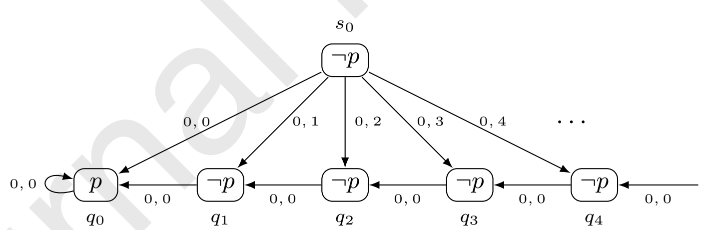
In [4] we show for ATL that $\mathsf{RBB}(\mathcal{M})$ is (in a sense) the least ordinal which is guaranteed to be stable for all transition games in the model $\mathcal{M}$. The same result holds (with the same proof) also for the transition games of ATL^+.

From Proposition 4.6, it follows that when the regular branching bound of the given model is used as a timer bound Γ, then the Γ-bounded GTS becomes equivalent to the unbounded GTS. The proof for this claim is analogous to the proof of Corollary 4.4.

Corollary 4.7. *The unbounded* GTS *is equivalent on* $\mathcal{M}$ *to the* Γ*-bounded* GTS *when* $\Gamma \geq \mathsf{RBB}(\mathcal{M})$.

Consequently, finite timers suffice in image-finite models. However, the finitely bounded GTS (with $\Gamma = \omega$) is not generally equivalent to the unbounded GTS. See the following example.

Example 4.8 (Cf. Example 3.7 in [3])**.** Consider the image infinite concurrent game model $\mathcal{M}$ which is displayed in the figure below. (The labels α_1, α_2 on the edges correspond to the actions of the agents a_1 and a_2, respectively, resulting in the corresponding transition.)



Here we clearly have $\mathcal{M}, s_0 \Vdash \langle\!\langle a_1 \rangle\!\rangle \mathsf{F}\, p$ since every path from s_0 will eventually reach the state q_0 where p is true. However, $\mathcal{M}, s_0 \nVdash_\omega \langle\!\langle a_1 \rangle\!\rangle \mathsf{F}\, p$ since for any value $n < \omega$ for the timer, chosen by Eloise, Abelard can choose n for the first action of agent a_2 and then it will take $n+1$ rounds to reach q_0.

Because $\mathsf{RBB}(\mathcal{M}) = \aleph_1$ (equal to $2^{\aleph_0}$ if we assume the continuum hypothesis), by Corollary 4.7 we have $\mathcal{M}, s_0 \Vdash_{\aleph_1} \langle\!\langle a_1 \rangle\!\rangle \mathsf{F}\, p$. However, in this particular model, we also have $\mathcal{M}, s_0 \Vdash_{\omega+1} \langle\!\langle a_1 \rangle\!\rangle \mathsf{F}\, p$ since Eloise can win the game by first choosing

ω for the value of the timer and then lowering its value to $n < \omega$ which corresponds the the action which Abelard first chooses for the agent a_2.

4.3. Regular strategies

Here we define a notion of a regular strategy which will be important for the proofs in the next sections. We only define this concept for Eloise for the transition games in which Eloise is the verifier. This suffices for our needs, but the definition—and the related Lemma 4.10—could easily be generalized for both players and all kinds of transition games.

Definition 4.9. A strategy τ for Eloise in a transition game $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$ is **regular**, if the following properties hold:

(i) τ instructs Eloise to make all the claims which are valid (by the respective GTS). Moreover, τ instructs Eloise to challenge all the claims which Abelard makes. (Note that this latter condition is safe for Eloise since she is given the chance to make every claim first and thus, by the first condition, *Abelard can only make claims which are false.*)

(ii) τ instructs Eloise to try to end the transition game (by ending her seeker turn or by not taking a new seeker turn) always when the truth function T has winning values for Eloise.

(iii) Actions chosen by τ for the agents in A are independent of the current seeker $\mathbf{S}$ and seeker turn counter $n \in \mathbb{N}$ in configurations. That is, τ assigns the same choice for configurations $(\mathbf{P}, q, T, n, \mathbf{iii})$ and $(\mathbf{P}', q, T, n', \mathbf{iii})$.

Note that the conditions (i)-(iii) together imply that *all* the actions chosen by a regular strategy are independent of the current seeker $\mathbf{S}$ and seeker turn counter $n \in \mathbb{N}$ in configurations. Thus these additional parameters cannot be used for "signalling" any information.

Lemma 4.10. *If Eloise has a winning strategy in a transition game $\boldsymbol{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$, then she has a regular winning strategy in that game.*

Proof. Suppose that Eloise has a winning strategy τ in $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$. We first modify τ so that it is a winning strategy from every configuration which is a winning configuration for Eloise in the transition game (note that τ could make "bad choices" for configurations which are not reachable by τ). We then make τ a regular winning strategy by doing the following modifications in the given order.

(1) If τ does not satisfy the regularity property (i), then we simply first modify it so that Eloise makes all the claims which are true by GTS; it is clear that we end up in Eloise's winning exit location if Abelard challenges these new claims. Moreover, we then redefine τ to challenge all the claims made by Abelard; since all of these claims must now be false by GTS, it follows from the determinacy of evaluation games that every challenge by Eloise leads into an exit location which is winning for her. After these modifications, τ is still a winning strategy and it now satisfies the regularity property (i).

(2) Let $c = (\mathbf{P}, q, T, n, \mathbf{ii})$ be a configuration such that $(\mathbf{E}, q, \Phi, T)$ is a winning location for Eloise, but τ does not instruct Eloise to try to end the transition game at c (by ending her seeker turn or by not taking a new seeker turn). We then redefine τ to instruct Eloise to try to end the game at c. If Abelard also wants to end the game, then we reach a winning exit location for Eloise. If Abelard does not want to end the game, then the game continues from a configuration c' that must be winning for Eloise. After doing this change for all configurations conflicting the regularity property (ii), τ now satisfies the properties (i) and (ii).

(3) In order to satisfy the regularity condition (iii), will first modify τ in various ways and then show that the modified strategy satisfies the condition (iii).

We first redefine τ at configurations of the form $(\mathbf{E}, q, T, n, \mathbf{iii})$ so that it selects the same actions as at the configuration $(\mathbf{E}, q, T, |At(\Phi)|, \mathbf{iii})$. Similarly we define τ at $(\mathbf{A}, q, T, n, \mathbf{iii})$ to select the same actions as at $(\mathbf{A}, q, T, |At(\Phi)| - 1, \mathbf{iii})$. By doing this procedure for all configurations, τ becomes independent of the seeker turn counter n. Note that the truth function can be updated at most $|At(\Phi)|$ many times and, by condition (ii), T gets updated after every seeker turn alternation. Therefore it is impossible that Eloise would now lose the game due to the seeker turn counter becoming zero. Hence τ is still a winning strategy after these modifications.

Let then $c = (\mathbf{A}, q, T, n, \mathbf{iii})$ be a configuration such that T is not winning for Eloise (in the corresponding Boolean game). Now also the configuration $c' = (\mathbf{E}, q, T, n-1, \mathbf{iii})$ is a winning configuration for Eloise since Abelard could end his seeker turn at $(\mathbf{A}, q, T, n, \mathbf{ii})$. We then modify τ so that it makes the same choice at c as at c'. We do this modification for all configurations c of this type.

As arbitrary choices may be assigned to the configurations that are not reachable by τ, it suffices that we check the regularity condition (iii) only for the configurations reachable by τ. Suppose for the sake of contradiction that τ assigns different actions for A in configurations $c = (\mathbf{P}, q, T, n, \mathbf{iii})$ and $c' = (\mathbf{P}', q, T, n', \mathbf{iii})$—reachable by τ—such that $c \neq c'$. Since τ is independent of the seeker turn counter (by the modifications above), we must have $\mathbf{P} \neq \mathbf{P}'$. By symmetry we may assume that $\mathbf{P} = \mathbf{E}$ and $\mathbf{P}' = \mathbf{A}$.

Suppose first that T is winning for Eloise. Now, by the condition (ii), τ instructs Eloise to end her seeker turn at $(\mathbf{E}, q, T, n, \mathbf{ii})$, and hence the configuration c cannot be reached with τ. Suppose then that T is not winning for Eloise. Recall that we have defined τ to make the same choice at c' as at the configuration $c'' = (\mathbf{E}, q, T, n'-1, \mathbf{iii})$. But this is impossible since τ is independent of the seeker turn counter and that is the only parameter that separates the configurations c and c''.

Hence we have shown that, by doing all the modifications above, τ becomes a regular winning strategy for Eloise in the transition game $\mathbf{g}(\mathbf{E}, q, \langle\langle A \rangle\rangle\, \Phi)$. □

Remark 4.11. As discussed in Remark 3.1, the "full notion" of configuration also includes 3-bit flag indicating if it is the first, second or some later round of the game (this parameter is used for showing when state formulae and formulae with X can be verified). When choosing actions for agents in step **iii**, the strategies can also potentially depend on this additional parameter. However, winning strategies can easily be made independent of this parameter: Suppose that τ is a strategy which makes a different choice at a configuration c based on the number of rounds (0, 1 or ≥ 2) which it took for reaching c. We can now simply modify τ so that it always makes the choice at c according to the choice at the latest round number—in which c can be reached with τ. (It is important to remember here that the both the truth function and the state are the same in c regardless when c is reached.)

By Lemma 4.10 and by the remark above, it follows that the choices for coalitions A by a regular strategy *depend only on the pairs* (q, T), *where* q *is the current state and* T *is the current truth function.* Also note that since, by (i), Eloise makes all the valid verifications and falsifications, the truth function T is always determined by the path that has been formed by the transition game.

Regular strategies will play an important role in the next section where we prove the equivalence of GTS and the standard compositional semantics for ATL^+. This is because a regular strategy of Eloise in a transition game for $\langle\!\langle A \rangle\!\rangle\, \Phi$ can be used in a straightforward way for formulating a collective strategy S_A for the coalition A (and vice versa).

5. GTS vs compositional semantics for ATL$^+$

In this section we show that our game-theoretic semantics is equivalent to the standard (perfect-recall) compositional semantics of ATL^+. From the results of the previous section it follows that this equivalence holds for both unbounded GTS and bounded GTS with a stable timer bound.

We begin with some preliminary definitions. We first define a so-called **finite path semantics**, to be used later. (See [2] for a similar definition.) We define the **length** $\mathrm{lgt}(\lambda)$ of a finite path λ as the number of *transitions* in λ (whence the last state of λ is $\lambda[\mathrm{lgt}(\lambda)]$). If λ is a prefix sequence of λ', we write $\lambda \preceq \lambda'$.

Definition 5.1. Let $\mathcal{M}$ be a CGM and $\lambda \in \mathsf{paths}_{\mathrm{fin}}(\mathcal{M})$. **Truth** of a path formula Φ of ATL^+ on λ is defined recursively as follows:

- $\mathcal{M}, \lambda \models \varphi$ iff $\mathcal{M}, \lambda[0] \models \varphi$ (where φ is a state formula).
- $\mathcal{M}, \lambda \models \neg\Phi$ iff $\mathcal{M}, \lambda \not\models \Phi$.
- $\mathcal{M}, \lambda \models \Phi \vee \Psi$ iff $\mathcal{M}, \lambda \models \Phi$ or $\mathcal{M}, \lambda \models \Psi$.
- $\mathcal{M}, \lambda \models \mathsf{X}\,\varphi$ iff $\mathrm{lgt}(\lambda) \geq 1$ and $\mathcal{M}, \lambda[1] \models \varphi$.
- $\mathcal{M}, \lambda \models \varphi\,\mathsf{U}\,\psi$ iff there exists some $i \leq \mathrm{lgt}(\lambda)$ such that $\mathcal{M}, \lambda[i] \models \psi$ and $\mathcal{M}, \lambda[j] \models \varphi$ for all $j < i$.

In the following definition, when $\Lambda \in \mathsf{paths}(\mathcal{M})$, we use the notation $\Lambda[0, i]$ for denoting the finite path $(\Lambda[0], \ldots, \Lambda[i])$.

Definition 5.2. Let $\mathcal{M}$ be a CGM, $\Lambda \in \mathsf{paths}(\mathcal{M})$ and Φ a path formula of ATL^+. An index $i \geq 1$ is a **truth-swap point** of Φ on Λ if either of the following holds:

1. $\mathcal{M}, \Lambda[0, i-1] \not\models \Phi$ and $\mathcal{M}, \Lambda[0, i] \models \Phi$.
2. $\mathcal{M}, \Lambda[0, i-1] \models \Phi$ and $\mathcal{M}, \Lambda[0, i] \not\models \Phi$.

Moreover, we define the **truth-swap number** of Φ on Λ to be

$$TSN(\Phi, \Lambda) := |\{i \mid i \text{ is a truth-swap point of } \Phi \text{ on } \Lambda\}|.$$

The claims of the following lemma are easy to prove. Similar observations have been made in [2].

Lemma 5.3. *Let $\mathcal{M}$ be a CGM, $\Lambda \in \mathsf{paths}(\mathcal{M})$ and Φ a path formula of ATL^+. Now, the following claims hold:*

1. $TSN(\Phi, \Lambda) \leq |\{\Psi \in At(\Phi) \mid \Psi$ *is a temporal subformula*$\}| \leq |At(\Phi)|$.
2. $\mathcal{M}, \Lambda \models \Phi$ *iff there is some* $k \in \mathbb{N}$ *s.t.* $\mathcal{M}, \lambda \models \Phi$ *for every finite* $\lambda \preceq \Lambda$ *for which* $\mathrm{lgt}(\lambda) \geq k$.

We are now ready prove the equivalence between the unbounded GTS and the standard perfect-recall semantics of ATL^+.

Theorem 5.4. *The unbounded* GTS *is equivalent to the standard (perfect-recall) compositional semantics of* ATL^+*.*

Proof. We prove by induction on ATL^+ state formulae φ that for any CGM $\mathcal{M}$ and a state q in $\mathcal{M}$:

$$\mathcal{M}, q \models \varphi \text{ iff Eloise has a winning strategy in } \mathcal{G}(\mathcal{M}, q, \varphi).$$

If φ is a proposition symbol, then the claim holds trivially.

Let $\varphi = \neg\psi$ and suppose first that $\mathcal{M}, q \models \neg\psi$, i.e. $\mathcal{M}, q \not\models \psi$. By the inductive hypothesis Eloise does not have a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Since evaluation games are determined, Abelard has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Thus Eloise has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \neg\psi)$. Suppose then that Eloise has a winning strategy in the evaluation game $\mathcal{G}(\mathcal{M}, q, \neg\psi)$. Then Abelard has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$ and thus Eloise cannot have a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Hence, by the inductive hypothesis, $\mathcal{M}, q \not\models \psi$, i.e. $\mathcal{M}, q \models \neg\psi$.

Let $\varphi = \psi \vee \theta$ and suppose that $\mathcal{M}, q \models \psi \vee \theta$, i.e. $\mathcal{M}, q \models \psi$ or $\mathcal{M}, q \models \theta$. Suppose first that $\mathcal{M}, q \models \psi$, whence by the inductive hypothesis Eloise has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Now Eloise can win $\mathcal{G}(\mathcal{M}, q, \psi \vee \theta)$ by choosing ψ on the first move. The case when $\mathcal{M}, q \models \theta$ is analogous. Suppose then that Eloise has a winning strategy in the evaluation game $\mathcal{G}(\mathcal{M}, q, \psi \vee \theta)$. Let $\chi \in \{\psi, \theta\}$ be the disjunct that Eloise chooses when following her winning strategy. Now Eloise must have a winning strategy in $\mathcal{G}(\mathcal{M}, q, \chi)$ and thus by the inductive hypothesis $\mathcal{M}, q \models \chi$. Therefore $\mathcal{M}, q \models \psi \vee \theta$.

Finally, let $\varphi = \langle\!\langle A \rangle\!\rangle\, \Phi$. By the inductive hypothesis, it suffices to show that Eloise has winning strategy in the (unbounded) transition game $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$ if and only if the coalition A has a (perfect-recall) strategy S_A such that $\mathcal{M}, \Lambda \models \Phi$ for every $\Lambda \in \mathsf{paths}(q, S_A)$. The cases (a) and (b) which follow correspond to the two directions for proving this equivalence.

(a) Suppose first that $\mathbf{E}$ has a winning strategy τ in the transition game $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle \Phi)$. By Lemma 4.10 we may assume that τ is regular. Let $T_{\mathbf{g}}$ be the game tree that is formed by all of those configurations that can be encountered with τ. We define S_A by using the actions given by τ for the coalition A for every *finite path of states* that occurs in consecutive configurations in $T_{\mathbf{g}}$. The actions for all other finite paths are irrelevant.

In order to show that S_A is well-defined this way, let λ, λ' be finite branches of *configurations* in $T_{\mathbf{g}}$ such that the states occurring in the configurations of λ and λ' are in the same order. Let $c = (\mathbf{P}, q, T, n, \mathbf{iii})$ and $c' = (\mathbf{P}', q, T', n', \mathbf{iii})$ be the last configurations in λ and λ', respectively. It suffices to show that τ assigns the same actions for A in both c and c'. Since λ and λ' have visited the same states, by regularity condition (i), we must have $T = T'$. Therefore, by regularity condition (iii), τ assigns the same actions for c and c'.

Let $\Lambda \in \mathsf{paths}(q, S_A)$, whence the states in Λ occur in some infinite tuple of configurations in $T_{\mathbf{g}}$. In the (infinite) play of $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle \Phi)$, that corresponds to Λ, Eloise does only finitely many verifications and cannot stay as the seeker for infinitely many rounds (since τ is a winning strategy). Let $k \in \mathbb{N}$ be such that Eloise neither does any further verifications/falsifications nor becomes the seeker after the state $\Lambda[k]$. We will use Lemma 5.3, for showing that $\mathcal{M}, \Lambda \models \Phi$. Let $\lambda_0 \preceq \Lambda$ be a finite path such that $|\lambda_0| \geq k$.

We show by induction on the formulae in $\mathrm{SUB}_{At}(\Phi)$ that if an exit location of the form $(\mathbf{P}, \lambda_0[l], \Psi, T)$, where $\Psi \in \mathrm{SUB}_{At}(\Phi)$, can be reached by using τ, then the following equivalence holds:

$$\mathcal{M}, \lambda_0 \models \Psi \;\text{ iff } \mathbf{P} = \mathbf{E}.$$

- The cases $\Psi = \varphi$ and $\Psi = \mathsf{X}\,\varphi$ are easy to prove.
- Let $\Psi = \psi \,\mathsf{U}\, \theta$ and suppose first that $\mathbf{P} = \mathbf{E}$. Since τ is a regular winning strategy, there must be $i \leq k$ such that Eloise verifies $\psi \,\mathsf{U}\, \theta$ by claiming that θ is true at $\lambda_0[i]$. If Abelard challenges Eloise's claim, the evaluation game continues from the position $(\mathbf{E}, \lambda_0[i], \theta, T)$. Hence by the (outer)

inductive hypothesis we have $\mathcal{M}, \lambda_0[i] \models \theta$. Let then $j < i$. Now Abelard can attempt to falsify $\psi \mathsf{U} \theta$ by claiming that ψ is true at $\lambda_0[j]$, whence Eloise must challenge Abelard's claim since τ is a regular winning strategy. Then the evaluation game continues from the position $(\mathbf{E}, \lambda_0[j], \psi, T)$ and thus by the (outer) inductive hypothesis $\mathcal{M}, \lambda_0[j] \models \psi$. Thus we have shown that $\mathcal{M}, \lambda_0 \models \psi \mathsf{U} \theta$.

Suppose then that $\mathbf{P} = \mathbf{A}$. We also suppose, for the sake of contradiction, that $\mathcal{M}, \lambda_0 \models \psi \mathsf{U} \theta$. Now there is $i \leq k$ such that $\mathcal{M}, \lambda_0 \models \theta$. If Abelard verifies $\psi \mathsf{U} \theta$ at $\lambda_0[i]$, then Eloise loses by the (outer) inductive hypothesis. Hence Eloise should falsify $\psi \mathsf{U} \theta$ at some state $\lambda_0[j]$, where $j < i$. But then by the (outer) inductive hypothesis we must have $\mathcal{M}, \lambda_0[j] \not\models \psi$, which is a contradiction.

- Suppose that $\Psi = \neg\Theta$. The next position of the evaluation game is $(\overline{\mathbf{P}}, \lambda[l], \Theta, T)$ and thus, by the (inner) inductive hypothesis, we have $\mathcal{M}, \lambda_0 \not\models \Theta$ iff $\overline{\mathbf{P}} = \mathbf{A}$. Hence we have $\mathcal{M}, \lambda_0 \models \neg\Theta$ iff $\mathbf{P} = \mathbf{E}$.

- The case $\Psi = \Theta_1 \vee \Theta_2$ is proven similarly to the previous case.

Note that Abelard is the seeker at the last state $\lambda_0[m]$ of λ_0 and he may attempt to end the transition game at $\lambda_0[m]$ by ending his seeker turn. By our assumption Eloise does not become the seeker and thus the evaluation game is continued from $(\mathbf{E}, \lambda_0[m], \Phi, T)$ for some T. By the induction proof above, we must have $\mathcal{M}, \lambda_0 \models \Phi$. Hence, by Lemma 5.3, we have $\mathcal{M}, \Lambda \models \Phi$.

(b) Suppose then that there is a collective (perfect-recall) strategy S_A for the coalition A such that $\mathcal{M}, \Lambda \models \Phi$ for every $\Lambda \in \mathsf{paths}(q, S_A)$. First, we define a *perfect-recall* strategy τ for Eloise as follows.

Suppose that the transition game is at some configuration c that has been reached with a finite path $\lambda_0 \preceq \Lambda$ (of states) such that q_0 is the last state of λ_0.

- If $\mathcal{M}, q_0 \models \theta$ for some $\psi \mathsf{U} \theta \in At(\Phi)$, then Eloise claims that θ is true.

- If $\mathcal{M}, q_0 \not\models \psi$ for some $\psi \mathsf{U} \theta \in At(\Phi)$, then Eloise claims that ψ is false.

- Suppose that $q_0 = \Lambda[0]$ and $\psi \in At(\Phi)$ is a state formula. If $\mathcal{M}, q_0 \models \psi$, then Eloise claims that ψ is true.

- Suppose that $q_0 = \Lambda[1]$ and $\mathsf{X}\,\psi \in At(\Phi)$. If $\mathcal{M}, q_0 \models \psi$, then Eloise claims that $\mathsf{X}\,\psi$ is true.

- If Abelard makes any claim on the truth of formulae, Eloise always challenges those claims. (Note here that Abelard's claim must be false, according to the compositional truth condition, as otherwise Eloise would already have made the same claim.)

- If Eloise is the seeker in c and $\mathcal{M}, \lambda_0 \models \Phi$, then Eloise decides to end her seeker turn.

- If Abelard ends his seeker turn at c and $\mathcal{M}, \lambda_0 \not\models \Phi$, then Eloise decides to become the seeker. Otherwise, Eloise ends the transition game at c.

- If Eloise needs to choose actions for agents in coalition A at c, she chooses them according to $S_A(\lambda_0)$.

We next show that none of the configurations c, which are reachable by τ, cannot lead to losing exit locations for Eloise.

- Let $c = (\mathbf{S}, q', T, n, \mathbf{i})$. Since all the claims and challenges by τ are made according to the compositional semantics, Eloise has a winning strategy from any possible exit location by the inductive hypothesis.

- Let $c = (\mathbf{S}, q', T, n, \mathbf{ii})$. By the definition of τ, the transition game can end from this type of configuration only when $\mathcal{M}, \lambda_0 \models \Phi$. If the transition game ends, then the corresponding Boolean game is played from the exit location $(\mathbf{E}, q', \Phi, T)$. By simply following the compositional truth condition of Φ on λ_0, Eloise can play in such a way that the following condition holds for any location $(\mathbf{P}, q', \Psi, T)$, that is reached:

$$\mathcal{M}, \lambda_0 \models \Psi \;\text{ iff } \mathbf{P} = \mathbf{E},$$

where $\Psi \in \mathrm{SUB}_{At}(\Phi)$. Eventually, an ending location of the form $(\mathbf{P}, q', \chi, T)$, where $\chi \in At(\Phi)$, is reached. Since the verifications/falsifications by τ are made according to the compositional truth of the relational atoms of Φ, it now follows from the inductive hypothesis that $(\mathbf{P}, q', \chi, T)$ must be a winning location for Eloise.

- The configurations of the form $c = (\mathbf{S}, q', T, n, \mathbf{iii})$ do not lead to any exit locations.

We still need to show that Eloise cannot stay as the Seeker forever when playing with τ. Because Eloise chooses actions for agents in A according to S_A, every path of *states* that is formed with τ is a prefix sequence of some path $\Lambda \in \mathsf{paths}(q, S_A)$. Since $\mathcal{M}, \Lambda \models \Phi$ for every $\Lambda \in \mathsf{paths}(q, S_A)$, by Lemma 5.3, and the definition of τ, Eloise cannot stay as the seeker forever. If Abelard stays as the seeker forever, then Eloise wins. Hence τ is a perfect-recall winning strategy for Eloise.

Since unbounded transition games are positionally determined, there is also a *positional* winning strategy τ' for Eloise in the transition game $\mathbf{g}(\mathbf{E}, q, \langle\langle A \rangle\rangle\, \Phi)$. This concludes the proof for the equivalence of the two semantics. □

By combining Theorem 5.4 with Corollary 4.7, we immediately obtain the following corollary:

Corollary 5.5. *If* $\Gamma \geq \mathsf{RBB}(\mathcal{M})$*, then the* Γ*-bounded* GTS *is equivalent on* $\mathcal{M}$ *with the standard (perfect-recall) compositional semantics of* ATL^+*.*

6. Model checking ATL$^+$ using GTS

Here we apply the GTS to model checking problems for ATL^+ and its fragments.

6.1. Revisiting the PSPACE *upper bound proof*

As mentioned earlier, the PSPACE upper bound proof for the model checking of ATL^+ in [2] contains a flaw. Indeed, the claim of Theorem 4 in [2] is incorrect and a counterexample to it can be extracted from our Example 3.3, where $\mathcal{M}, q_0 \models \varphi$ for $\varphi = \langle\!\langle a_2 \rangle\!\rangle (\mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$. In the notation of [2], since $|St_{\mathcal{M}}| = 3$ and $\mathcal{APF}(\varphi) = 2$, by the claim there must be a 6-witness strategy for the agent 2 for $(\mathcal{M}, q_0, \mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$. However, this is not the case, since the player 1 can choose to play at q_0 four times β, and then α. Then $\mathcal{M}, \Lambda \not\models^6 (\mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$ on any resulting path Λ.

The reason for the problem indicated above is that compositional semantics easily ignores the role and power of the falsifier (Abelard) in the formula evaluation process. Still, using the GTS introduced above, we will demonstrate in a simple way that the upper bound result is indeed correct.

The input to the model checking problem of ATL^+ is an ATL^+ formula φ, a finite CGM $\mathcal{M}$ and a state q in $\mathcal{M}$. We assume that $\mathcal{M}$ is encoded in the standard way (cf. [1, 2]) that provides a full explicit description of the transition function o. Unlike [1, 2], we do not assume any bounds on the number of proposition symbols or agents in the input. We only consider here the semantics of ATL^+ based on perfect information and perfect-recall strategies.

Theorem 6.1 ([2])**.** *The* ATL^+ *model checking problem is* PSPACE*-complete.*

Proof. We get the lower bound directly from [2], so we only prove the upper bound here. By Theorem 5.4 and Proposition 4.3, if $\mathcal{M}$ is a finite CGM, we have $\mathcal{M}, q \models \varphi$ iff Eloise has a positional winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, N)$ with $N = |\,\mathrm{St}\,| \cdot |\varphi|$. It is routine to construct an alternating Turing machine TM that simulates $\mathcal{G}(\mathcal{M}, q, \varphi, N)$ such that the positions for Eloise correspond to existential states of TM and Abelard's positions to universal states. Due to the timer bound N, the machine runs in polynomial time. It is clear that if Eloise has a (positional or not) winning strategy in the evaluation game, then TM accepts. Conversely, if TM accepts, we can read a non-positional winning strategy for Eloise from the the computation tree (with only one successful move for existential states recorded everywhere) which demonstrates that TM accepts. By Proposition 4.1, Eloise thus also has a positional winning strategy in the evaluation game. Since APTIME = PSPACE, the claim follows. □

6.2. *A hierarchy of tractable fragments of* ATL^+

We now identify a natural hierarchy of tractable fragments of ATL^+. Let k be a positive integer. Define ATL^k to be the fragment of ATL^+ where all formulae $\langle\!\langle A \rangle\!\rangle\, \Phi$ have the property that $|At(\Phi)| \leq k$. Note that ATL^1 is essentially the same as ATL (with Release). Note also that the number of non-equivalent formulae of ATL^k is not bounded for any k, *even in the special case where the number of propositions and actions is constant*, because nesting of strategic operators $\langle\!\langle A \rangle\!\rangle$ is not limited. Still, we will show that the model checking problem for ATL^k is PTIME-complete for any fixed k. Again CGMs are encoded explicitly and no restrictions on the number of propositions or actions is assumed. (In fact, a certain implicit encoding of CGMs leads to Δ_3^{P}-completeness [21].)

With the fully developed GTS in place, the following theorem is now actually straightforward to prove. This demonstrates the potential advantages of GTS.

Theorem 6.2. *For any fixed $k \in \mathbb{N}$, the model checking problem for* ATL^k *is* PTIME-*complete.*

Proof. The claim is well-known for ATL (see [1]), so we have the lower bound for free, for any k. One possible proof strategy for the upper bound would involve using alternating LOGSPACE-machines, but here we argue via Büchi-games instead.

Consider a triple $(\mathcal{M}, q, \varphi)$, where $\varphi \in \mathsf{ATL}^k$. By the proof of Proposition 4.2, there exists a Büchi game BG such that Eloise wins the unbounded evaluation game $\mathcal{G}(\mathcal{M}, q, \varphi)$ iff she wins BG from the state of BG that corresponds to the beginning position of the evaluation game. We then observe that since we are considering ATL^k for a *fixed* k, the domain size of each truth function T used in the evaluation game is at most k, and thus the number of positions in $\mathcal{G}(\mathcal{M}, q, \varphi)$ is *polynomial* in the size of the input $(\mathcal{M}, q, \varphi)$. (To thoroughly check that the number of positions is indeed polynomial, recall to consider also the extra information—given in remark 3.1—that should be encoded in positions of evaluation games.) Thereby we conclude that also the size of BG is polynomial in the input size.

We note that, in order to avoid blow-ups, it is essential that the maximum domain size k of truth functions T is fixed. We also note—as mentioned already in [1]—that the number of transitions in $\mathcal{M}$ is not bounded by the square of the number of states of $\mathcal{M}$. In fact, because we impose no limit (other than finiteness) on the number of actions in $\mathcal{M}$, the number of transitions in relation to states is arbitrary. However, this is no problem to us since an explicit encoding of $\mathcal{M}$—which lists all transitions explicitly—is part of the input to the model checking problem. Since Büchi games can be solved in PTIME, the claim follows. □

7. Analysis of memory resources in ATL^+

Strategies with bounded memory in concurrent game models can be naturally defined using *deterministic finite state transducers* (or, Mealy machines). For a transducer-based definition of bounded memory strategies, see e.g. [22], and see [23] for more on this topic. Using such strategies, an agent's moves are determined both by the current state in the model and by the current state ("*memory cell*") of the agent's transducer. Then, transitions take place both in the model and in the state space of the transducer, thus updating the agent's memory. So, such strategies are positional with respect to the product of the two state spaces. In the compositional m-**bounded memory semantics** ($\models^m$) for ATL^+, agents are allowed to use at most m memory cells, i.e., strategies defined by transducers with at most m states.

7.1. An upper bound for the number of memory cells

Recall the (tractable) fragments ATL^k of ATL^+ from Section 6.2. Since the use of the truth function T in our GTS is analogous to the use of memory cells in m-bounded memory semantics, we obtain the following result.

Theorem 7.1. *For ATL^k, the unbounded GTS is equivalent to the m-bounded memory semantics for $m = 3^k - 2^k$.*

Proof. Let $m := 3^k - 2^k$ and $\varphi \in \mathsf{ATL}^k$. We show that

$$\mathcal{M}, q \Vdash \varphi \;\text{ iff }\; \mathcal{M}, q \models^m \varphi.$$

The implication from right to left is immediate by Theorem 5.4. We prove the other direction by induction on φ. The only non-trivial case is when $\varphi = \langle\langle A \rangle\rangle\, \Phi$. Suppose that Eloise has a winning strategy in $\mathbf{g}(\mathbf{E}, q, \langle\langle A \rangle\rangle\, \Phi)$. By Lemma 4.10 and Remark 4.11 we may assume that choices for agents in A, given by τ, depend only on the current state and the current truth function.

We define a memory transducer $\mathcal{T}$ which Eloise can use to define strategies for all agents in A. We fix the set of states C of $\mathcal{T}$ to be the set of all truth functions T for $At(\Phi)$ such that $T(\chi) = \mathsf{open}$ for at least one $\chi \in At(\Phi)$. Since $T(\chi) \in \{\mathsf{open}, \top, \bot\}$, we have $|C| \leq 3^k - 2^k = m$. The initial state of $\mathcal{T}$ is T_0 where $T_0(\chi) = \mathsf{open}$ for every $\chi \in At(\Phi)$. The transitions in $\mathcal{T}$ are defined according to how Eloise updates the truth function T during the transition game. However, when T becomes fully updated (i.e. $T(\chi) \neq \mathsf{open}$ for every $\chi \in At(\Phi)$), then no further transitions are made, because in this case all relative atoms have been verified/falsified and the truth of Φ on the path is fixed.

Now, the strategy for each agent $a \in A$ is defined positionally on $C \times \mathrm{St}$ as follows: At a state T of $\mathcal{T}$ and state $q \in \mathcal{M}$, the agent a follows the action prescribed by Eloise's winning strategy for the corresponding step phase in the transition game. It is now easy to show that $\mathcal{M}, \Lambda \models^m \Phi$ for any path Λ that is consistent with the resulting collective strategy for the coalition A. □

By Theorem 5.4 and Theorem 7.1, we immediately obtain the following corollary.

Corollary 7.2. *For* ATL^k*, the compositional perfect-recall semantics is equivalent to the* $(3^k - 2^k)$*-bounded memory semantics.*

This extends the known fact that positional strategies (using 1 memory cell) suffice for the semantics of ATL (which is essentially the same as ATL^1). Moreover, given a formula, there is no need for the full perfect-recall semantics, as we may equivalently apply the bounded memory semantics with a bound that is based on the structure of the given formula ("the maximum temporal width"). Note also that this upper bound for the amount memory is independent of the structure of the model.

By $\mathsf{ATL}^k_\mathsf{F}$ we denote the fragment of ATL^k where all the relative atoms are of the form $\mathsf{F}\,\varphi$, that is, the temporal objectives Φ are simply Boolean combinations of reachability objectives. For $\mathsf{ATL}^k_\mathsf{F}$ we can strengthen the result of Theorem 7.1.

Theorem 7.3. *For $\mathsf{ATL}^k_\mathsf{F}$, the unbounded GTS is equivalent to the m-bounded memory semantics for $m = 2^k - 1$.*

Proof. In $\mathsf{ATL}^k_\mathsf{F}$ we may modify the rules of the transition games in such a way that relative atoms cannot be falsified by the players (but naturally they can be verified). This is because $\mathsf{F}\,\psi$ is interpreted as $\top\,\mathsf{U}\,\psi$ and $\top$ is never false: if a player tried to falsify $\top\,\mathsf{U}\,\psi$, that player would immediately lose once the other player challenges the claim. With this modification of the rules, there are at most 2^k different truth functions that may appear in the transition games for $\mathsf{ATL}^k_\mathsf{F}$. Moreover, there is only a single truth function that is fully updated. Hence we may define a memory transducer $\mathcal{T}$ with $2^k - 1$ states as in the proof of Theorem 7.1 and prove the rest of the claim analogously. □

In the next subsection we will show that the result of Theorem 7.3 is optimal in the sense that no smaller number of memory cells guarantees an equivalent semantics. Hence, even for $\mathsf{ATL}^k_\mathsf{F}$, the agents may need exponentially many memory cells with respect to the number of relative atoms.

7.2. A lower bound for the number of memory cells

In this section we will investigate the following simple $\mathsf{ATL}^k_\mathsf{F}$-formula:

$$\xi_k := \langle\!\langle a_1 \rangle\!\rangle\, \Phi_k, \qquad \text{where } \Phi_k := \mathsf{F}\,p_1 \wedge \cdots \wedge \mathsf{F}\,p_k.$$

Note that Φ_k is just a conjunction of reachability goals that agent a_1 needs to fulfill (in any order). Since positional strategies suffice for single reachability objectives, it would be intuitive to think that a_1 needs at most $k-1$ memory cells in order to achieve Φ_k. This is because a_1 needs to change its positional strategy only when completing some of the reachability objectives.[9] However,

[9] This can be seen by analyzing our GTS for ATL^+: note that (1) the strategies in transition games may be assumed to be positional with respect to the states and the truth function; and

we will see that the bounded memory strategy of a_1 must potentially use a transducer that has exponentially many states with respect to k. The model that we will use for proving this claim is constructed in the following example.

Example 7.4. Let $[k] := \{1, \ldots, k\}$ and $\mathcal{M}_k := (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a CGM, where

- $\mathbb{A}\mathrm{gt} = \{a_1, a_2\}$, $\Pi = \{p_1, \ldots, p_k\}$;
- $\mathrm{Act} = [k] \cup \{B \mid B \subseteq \mathcal{P}([k]) \setminus \{\emptyset\}\} \cup \{\mathsf{void}\}$;
- $\mathrm{St} = \{q_0\} \cup \{q_i \mid i \in [k]\} \cup \big\{q_B \mid B \in \mathcal{P}([k]) \setminus \{\emptyset, [k]\}\big\}$;
- $v(p_i) = \{q_i\} \cup \{q_B \in \mathrm{St} \mid i \in B\}$ for all $p_i \in \Pi$;
- $d(q_0, a_1) = \{B \mid B \in \mathcal{P}([k]) \setminus \{\emptyset\}\}$, $\quad d(q_0, a_2) = [k]$
  and $d(q, a_i) = \{\mathsf{void}\}$ when $q \in \mathrm{St} \setminus \{q_0\}$ and $i \in \{1, 2\}$;
- $o(q_0, (B, i)) = \begin{cases} q_i & \text{if } i \in B, \\ q_B & \text{else;} \end{cases}$
  $o(q_i, (\mathsf{void}, \mathsf{void})) = q_0$ when $i \in [k]$
  and $o(q_B, (\mathsf{void}, \mathsf{void})) = q_B$ when $B \in \mathcal{P}([k]) \setminus \{\emptyset, [k]\}$.

See the picture of model $\mathcal{M}_k$, in the special case when $k = 3$, In Figure 1.

The model $\mathcal{M}_k$ can be described as follows: At q_0 the agent a_1 gets to "announce" any nonempty set B of (indices of) proposition symbols in Π. Then, depending on the action chosen by the agent a_2, one of the following happens:

1. *Some* proposition symbol p_i, for which $i \in B$, is reached and then the game returns to q_0. This happens when a_2 chooses $i \in B$, whence a transition is made to q_i and then back to q_0.

2. *All* proposition symbols p_i with $i \in B$ are reached, but thereafter no new proposition symbols can be reached. This happens when a_2 chooses some $i \notin B$, whence a transition is made to q_B, where the game will loop forever.

that (2) the truth function for Φ_k can be updated at most most k times during the transition game for Φ_k.

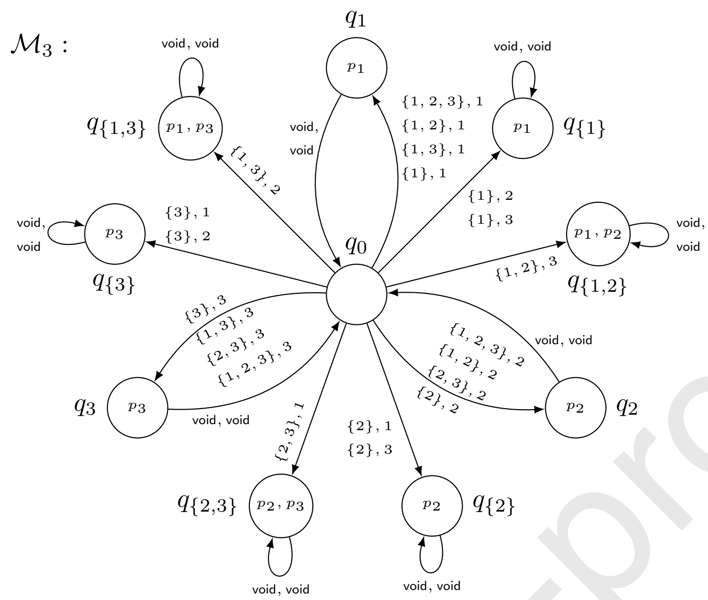


Figure 1: The model $\mathcal{M}_3$ from Example 7.4.

We will show that agent a_1 has a $(2^k - 1)$-bounded memory strategy σ_{a_1} which guarantees the truth of Φ_k on every path in $\mathsf{paths}(q_0, \sigma_{a_1})$. We first define a finite state transducer $\mathcal{T}_k$ as follows:

- The set of states C of $\mathcal{T}_k$ is $\{c_B \mid B \in \mathcal{P}([k]) \setminus \{\emptyset\}\}$. Now $|C| = 2^k - 1$.
- The initial state of $\mathcal{T}_k$ is $c_{[k]}$.
- The transitions of $\mathcal{T}_k$ are define as follows: Suppose that the current state of $\mathcal{T}_k$ is c_B for some $B \in \mathcal{P}([k]) \setminus \{\emptyset\}$ and a state q_j is reached for some $j \in [k]$. Now if $j \in B$ and $B \neq \{j\}$, then $\mathcal{T}_k$ changes its state to $c_{B \setminus \{i\}}$. Else, no transition is made.

See Figure 2 picture for the transducer $\mathcal{T}_k$ in the special case when $k = 3$.

Intuitively, the set B, when it is the index of c_B, denotes the set of indices of those proposition symbols p_i that *have not yet been reached*. We then define the strategy σ_{a_1} simply to select the action B at q_0 when the current state of

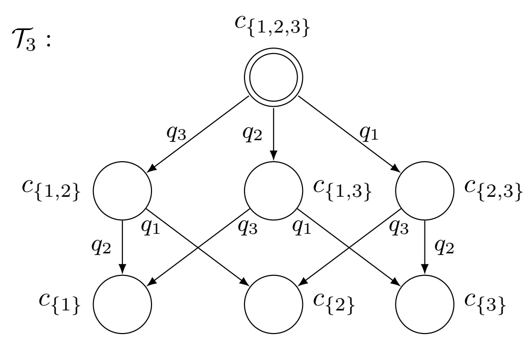


Figure 2: The transducer $\mathcal{T}_3$ from Example 7.4.

$\mathcal{T}_k$ is c_B. (The action void is selected elsewhere.) It is easy to see that σ_{a_1} is a strategy that satisfies Φ_k on every path.

Note that by using $\mathcal{T}_k$, the agent a_1 essentially remembers which *subset* of $\{p_1, \ldots, p_k\}$ of proposition symbols have already been reached. But a_1 does not have to remember in which order these states have been visited; if the order was remembered as well, then the number of states in $\mathcal{T}_k$ would be the number of k-permutations plus the initial state, resulting in $k! + 1$ states.

We prove the following lemma for the model $\mathcal{M}_k$ constructed in Example 7.4.

Lemma 7.5. *$\mathcal{M}_k \not\models^m \xi_k$ when $m < 2^k - 1$.*

Proof. Let σ_{a_1} be a strategy for a_1 using a transducer $\mathcal{T}$ with less than $2^k - 1$ states. We will show that there is a path in $\mathsf{paths}(q_0, \sigma_{a_1})$ on which p_i is not reached for some $i \in [k]$.

We first make the following two observations (i) and (ii):

(i) Suppose a_1 chooses some $B \in \mathcal{P}([k]) \setminus \{\emptyset\}$ at q_0 for which $i \notin B$ for some p_i that *has not yet been reached*. Now the next state may be q_B where it will loop forever. Since $q_B \notin v(p_i)$, the proposition p_i will never be reached.

(ii) Suppose now that a_1 chooses some B at q_0 for which $i \in B$ for some p_i that *has already been reached*. Now the next state may be q_i and thereafter the game returns to q_0. Since p_i is the only proposition symbol that is true at q_i, these transitions did not reach any new proposition symbols.

By the points above, we see that in order to reach all p_i, the agent a_1 has to choose such a set B at q_0 which has the indexes of *exactly those* proposition symbols which have not yet been reached. We denote this behavior of a_1 by $(\star)$.

Since $\mathcal{T}$ has less than $2^k - 1$ states, and $|\mathcal{P}([k]) \setminus \{\emptyset\}| = 2^k - 1$, there must be $B' \in \mathcal{P}([k]) \setminus \{\emptyset\}$ which a_1 never chooses at q_0 when following σ_{a_1}. Supposing that a_1 plays according to $(\star)$, it may happen that exactly those p_i for which $i \in [k] \setminus B$ are reached (by visiting the corresponding states q_i ($i \in [k] \setminus B$) and returning to q_0 after every visit). But, in this situation it is no longer possible for a_1 to follow $(\star)$ and thus impossible to reach all p_i for which $i \in B$. □

By Example 7.4 and Lemma 7.5 we immediately obtain the following corollary.

Corollary 7.6. *The perfect-recall semantics for $\mathsf{ATL}^k_\mathsf{F}$ is not equivalent to m-bounded memory semantics for any $m < 2^k - 1$.*

By this result, agents may need an exponential number of memory cells with respect to the number of relative atoms (in the Boolean combination). Again, this result holds even in the simple case where Φ is just a conjunction of reachability objectives $\mathsf{F}\,p$. Corollary 7.6 also implies that the result of Theorem 7.3 is optimal. We leave it open whether the result of Theorem 7.1 could be improved.

7.3. Some remarks on the amount of memory needed for strategies

There are several ways in which memory resources play a role in strategies. Besides the read-only memory needed to encode a strategy, for the execution of that strategy, one can distinguish different types of measures for the need of memory. These include e.g. the resources needed for

(i) storing any possible input of the strategy,

(ii) computing the value of the strategy function on any given input,

(iii) executing the strategy in any single play (by partially/fully remembering the history of the particular play).

Generally, these can be very different. Usually, the first one is taken as the measure of the memory consumption of a strategy in terms of the required input size (i.e., positional, bounded memory, perfect-recall). The second is usually disregarded and strategies are assumed to be computed by—or even hardwired in—some external devices ("black boxes"). As for the third measure, which involves both the previous two, we are not aware of any explicit consideration of it in the literature. Related to this, we will make some brief comparing remarks on the case of bounded memory strategies considered in this paper.

From Corollary 7.6 we see that agents may need a strategy transducer with $2^k - 1$ memory cells when there are k reachability objectives. This is because a strategy is a *global* plan of action—or a look-up table—that must take into account all possible plays. However, by observing the use of truth functions in transition games, we see that *in every single play of the game*, only $k-1$ memory cells need to be used. That is, the finite state transducer needs to visit only $k-1$ states on every path (cf. Example 7.4 and the transducer $\mathcal{T}_k$). Thus, the state space of the transducer has to be exponential with respect to the number of reachability objectives, but only a linearly large section of the transducer is actually used in every single play. (In fact, the latter is to be expected, in the light of the PTIME complexity of model checking of ATL^k, by Theorem 6.2.) This suggests that the dynamic memory of a computing device could be a better measure of the needed memory resources than the number of states in the transducer encoding the agent's strategy.

Based on the results above, one could argue that agents actually only need to use linear amount of memory in ATL^k, supposing they can manage their memory in a more dynamical ("on-the-fly") way. From an everyday human perspective, it is clear that people can manage to do, say, 10 small tasks by simply remembering what has already been done—that is, by remembering at most 9 pieces of information. In contrast, an exponential amount of memory (1023 memory cells) would be needed by Theorem 7.6 according to the transducer-based approach.

8. Conclusion

In conclusion, we note that the game-theoretic semantics for ATL^+ developed here has both conceptual and technical importance, as it explains better how the memory-based strategies in the compositional semantics can be generated and thus also provides better insight on the algorithmic aspect of that semantics.

We note that a GTS for ATL^+, alternative to the one introduced here, could be obtained via GTS for coalgebraic fixed point logic [24, 25]. However, such a semantics (being designed for more powerful logics) would not directly lead to our GTS that is custom-made for ATL^+ and would thus not directly enable the complexity analysis that we require. Also, that alternative approach would not give a semantics where the construction of finite paths only suffices.

A natural extension of the present work would be to develop GTS for the full ATL^*. Moreover, the correspondence with Büchi games here could be exploited in full.

Acknowledgements

The work of Valentin Goranko was supported by a research grant 2015-04388 of the Swedish Research Council. The work of Antti Kuusisto was partially supported by the ERC grant 647289 "CODA." We thank the anonymous reviewers.

References

References

[1] R. Alur, T. A. Henzinger, O. Kupferman, Alternating-time temporal logic, J. ACM 49 (5) (2002) 672–713.

[2] N. Bulling, W. Jamroga, Verifying agents with memory is harder than it seemed, AI Commun. 23 (4) (2010) 389–403.

[3] V. Goranko, A. Kuusisto, R. Rönnholm, Game-theoretic semantics for alternating-time temporal logic, in: Proc. of AAMAS 2016, IFAAMAS Publ., 2016, pp. 671–679.

[4] V. Goranko, A. Kuusisto, R. Rönnholm, Game-theoretic semantics for alternating-time temporal logic, ACM Trans. Comput. Log. 19 (2018) to appear.

[5] V. Goranko, A. Kuusisto, R. Rönnholm, Game-theoretic semantics for ATL+ with applications to model checking, in: Proc. of AAMAS 2017, IFAAMAS Publ., 2017, pp. 1277–1285.

[6] L. Hella, A. Kuusisto, R. Rönnholm, Bounded game-theoretic semantics for modal mu-calculus, CoRR abs/1706.00753 (2017).

[7] V. Goranko, A. Kuusisto, R. Rönnholm, CTL with finitely bounded semantics, in: 24th International Symposium on Temporal Representation and Reasoning, TIME 2017, October 16-18, 2017, Mons, Belgium, 2017, pp. 14:1–14:19.

[8] V. Goranko, A. Kuusisto, R. Rönnholm, Alternating-time temporal logic atl with finitely bounded semantics, To appear in Theoretical Computer Science.

[9] J. Hintikka, Logic, Language-games and Information: Kantian Themes in the Philosophy of Logic, Clarendon Press, Oxford, 1973.

[10] P. Lorenzen, Ein dialogisches konstruktivitätskriterium, in: A. Mostowski (Ed.), Proceedings of the Symposium on Foundations of Mathematics, Warsaw 1959, Panstwowe wydawnictwo naukowe, Warsaw, 1961, pp. 193–200.

[11] F. Mogavero, A. Murano, L. Sauro, Strategy games: a renewed framework, in: A. L. C. Bazzan, M. N. Huhns, A. Lomuscio, P. Scerri (Eds.), Proc. of AAMAS '14, IFAAMAS/ACM, 2014, pp. 869–876.

[12] P. Cermák, A. Lomuscio, A. Murano, Verifying and synthesising multi-agent systems against one-goal strategy logic specifications, in: B. Bonet, S. Koenig (Eds.), Proc. of AAAI 2015, AAAI Press, 2015, pp. 2038–2044.

[13] E. A. Emerson, A. K. Mok, A. P. Sistla, J. Srinivasan, Quantitative temporal reasoning, Real-Time Systems 4 (4) (1992) 331–352.

[14] R. Alur, T. A. Henzinger, Finitary fairness, ACM Trans. Program. Lang.

Syst. 20 (6) (1998) 1171–1194.

[15] O. Kupferman, N. Piterman, M. Y. Vardi, From liveness to promptness, in: Proc. of CAV 2007, Vol. 4590 of LNCS, Springer, 2007, pp. 406–419.

[16] K. Chatterjee, T. A. Henzinger, F. Horn, Finitary winning in omega-regular games, ACM Trans. Comput. Log. 11 (1) (2009) 1:1–1:27.

[17] J. Hintikka, G. Sandu, Game-theoretical semantics, in: J. van Benthem, A. ter Meulen (Eds.), Handbook of Logic and Language, Elsevier, Amsterdam, 1997, pp. 361–410.

[18] R. Mazala, Infinite games, in: E. Grädel, W. Thomas, T. Wilke (Eds.), Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001], Vol. 2500 of Lecture Notes in Computer Science, Springer, 2001, pp. 23–42.

[19] K. Chatterjee, T. Henzinger, N. Piterman, Algorithms for buchi games, in: 3rd Workshop on Games in Design and Verification, 2006.

[20] E. Grädel, I. Walukiewicz, Positional determinacy of games with infinitely many priorities, Logical Methods in Computer Science 2 (4) (2006).

[21] F. Laroussinie, N. Markey, G. Oreiby, On the expressiveness and complexity of ATL, Logical Methods in Computer Science 4 (2) (2008).

[22] S. Vester, Alternating-time temporal logic with finite-memory strategies, in: Proc, of the Fourth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2013, 2013, pp. 194–207.

[23] T. Brihaye, A. D. C. Lopes, F. Laroussinie, N. Markey, ATL with strategy contexts and bounded memory, in: S. Artëmov, A. Nerode (Eds.), Proc. of LFCS'2009, Springer LNCS 5407, 2009, pp. 92–106.

[24] Y. Venema, Automata and fixed point logic: A coalgebraic perspective, Inf. Comput. 204 (4) (2006) 637–678.

[25] C. Cîrstea, C. Kupke, D. Pattinson, EXPTIME tableaux for the coalgebraic μ-calculus, in: Proc. of CSL 2009, LNCS 5771, Springer-Verlag, Berlin Heidelberg, 2009, pp. 179–193.