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# Non-Parametric Mean Curvature Flow with Prescribed Contact Angle in Riemannian Products

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**Abstract:** Assuming that there exists a translating soliton $u_\infty$ with speed $C$ in a domain $\Omega$ and with prescribed contact angle on $\partial\Omega$, we prove that a graphical solution to the mean curvature flow with the same prescribed contact angle converges to $u_\infty + Ct$ as $t \to \infty$. We also generalize the recent existence result of Gao, Ma, Wang and Weng to non-Euclidean settings under suitable bounds on convexity of $\Omega$ and Ricci curvature in $\Omega$.

**Keywords:** Mean curvature flow; prescribed contact angle; translating graphs

**MSC:** Primary 53C21, 53E10

## 1 Introduction

We study a non-parametric mean curvature flow in a Riemannian product $N \times \mathbb{R}$ represented by graphs

$$M_t := \left\{ \left(x, u(x,t)\right) : x \in \overline{\Omega} \right\} \tag{1.1}$$

with prescribed contact angle with the cylinder $\partial\Omega \times \mathbb{R}$.

We assume that $N$ is a Riemannian manifold and $\Omega \Subset N$ is a relatively compact domain with smooth boundary $\partial\Omega$. We denote by $\gamma$ the inward pointing unit normal vector field to $\partial\Omega$. The boundary condition is determined by a given smooth function $\phi \in C^\infty(\partial\Omega)$, with $|\phi| \le \phi_0 < 1$, and the initial condition by a smooth function $u_0 \in C^\infty(\bar{\Omega})$.

The function $u$ above in (1.1) is a solution to the following evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = W \operatorname{div} \dfrac{\nabla u}{W} & \text{in } \Omega \times [0,\infty), \\ \dfrac{\partial_\gamma u}{W} := \dfrac{\langle \nabla u, \gamma \rangle}{W} = \phi & \text{on } \partial\Omega \times [0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \bar{\Omega}, \end{cases} \tag{1.2}$$

where $W = \sqrt{1 + |\nabla u|^2}$ and $\nabla u$ denotes the gradient of $u$ with respect to the Riemannian metric on $N$ at $x \in \bar{\Omega}$. The boundary condition above can be written as

$$\langle \nu, \gamma \rangle = \phi, \tag{1.3}$$

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where $\nu$ is the downward pointing unit normal to the graph of $u$, i.e.

$$\nu(x) = \frac{\nabla u(x,\cdot) - \partial_t}{\sqrt{1 + |\nabla u(x,\cdot)|^2}}, \ x \in \bar{\Omega}.$$

The longtime existence of the solution $u_t := u(\cdot, t)$ to (1.2) and convergence as $t \to \infty$ have been studied under various conditions on $\Omega$ and $\phi$. Huisken [5] proved the existence of a smooth solution in a $C^{2,\alpha}$-smooth bounded domain $\Omega \subset \mathbb{R}^n$ for $u_0 \in C^{2,\alpha}(\bar{\Omega})$ and $\phi \equiv 0$. Moreover, he showed that $u_t$ converges to a constant function as $t \to \infty$. In [1] Altschuler and Wu complemented Huisken's results for prescribed contact angle in case $\Omega$ is a smooth bounded strictly convex domain in $\mathbb{R}^2$. Guan [4] proved a priori gradient estimates and established longtime existence of solutions in case $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Recently, Zhou [8] studied mean curvature type flows in a Riemannian product $M \times \mathbb{R}$ and proved the longtime existence of the solution for relatively compact smooth domains $\Omega \subset M$. Furthermore, he extended the convergence result of Altschuler and Wu to the case $M$ is a Riemannian surface with nonnegative curvature and $\Omega \subset M$ is a smooth bounded strictly convex domain; see [8, Theorem 1.4].

The key ingredient, and at the same time the main obstacle, for proving the uniform convergence of $u_t$ has been a difficulty to obtain a time-independent gradient estimate. We circumvent this obstacle by modifying the method of Korevaar [6], Guan [4] and Zhou [8] and obtain a uniform gradient estimate in an arbitrary relatively compact smooth domain $\Omega \subset N$ provided there exists a translating soliton with speed $C$ and with the prescribed contact angle condition (1.3).

Towards this end, let $d$ be a smooth bounded function defined in some neighborhood of $\bar{\Omega}$ such that $d(x) = \min_{y \in \partial\Omega} \operatorname{dist}(x, y)$, the distance to the boundary $\partial\Omega$, for points $x \in \Omega$ sufficiently close to $\partial\Omega$. Thus $\gamma = \nabla d$ on $\partial\Omega$. We assume that $0 \le d \le 1$, $|\nabla d| \le 1$ and $|\operatorname{Hess} d| \le C_d$ in $\bar{\Omega}$. We also assume that the function $\phi \in C^\infty(\partial\Omega)$ is extended as a smooth function to the whole $\bar{\Omega}$, satisfying the condition $|\phi| \le \phi_0 < 1$.

Our main theorem is the following:

**Theorem 1.1.** *Suppose that there exists a solution $u_\infty$ to the translating soliton equation*

$$\begin{cases} \operatorname{div} \dfrac{\nabla u_\infty}{\sqrt{1 + |\nabla u_\infty|^2}} = \dfrac{C_\infty}{\sqrt{1 + |\nabla u_\infty|^2}} & \text{in } \Omega, \\ \dfrac{\partial_\gamma u_\infty}{\sqrt{1 + |\nabla u_\infty|^2}} = \phi & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

*where $C_\infty$ is given by*

$$C_\infty = \frac{-\int_{\partial\Omega} \phi \, d\sigma}{\int_\Omega \left(1 + |\nabla u_\infty|^2\right)^{-1/2} dx}. \tag{1.5}$$

*Then the equation* (1.2) *has a smooth solution $u \in C^\infty(\bar{\Omega}, [0,\infty))$ with $W \le C_1$, where $C_1$ is a constant depending on $\phi$, $u_0$, $C_d$, and the Ricci curvature of $\Omega$. Moreover, $u(x,t)$ converges uniformly to $u_\infty(x) + C_\infty t$ as $t \to \infty$.*

Notice that the existence of a solution $u \in C^\infty\left(\bar{\Omega} \times [0,\infty)\right)$ to (1.2) is given by [8, Corollary 4.2].

**Remark 1.2.** Very recently, Gao, Ma, Wang, and Weng [3] proved the existence of such $u_\infty$ and obtained Theorem 1.1 for smooth, bounded, strictly convex domains $\Omega \subset \mathbb{R}^n$ for sufficiently small $|\phi|$; see [3, Theorem 1.1, Theorem 3.1]. It turns out that their proof can be generalized beyond the Euclidean setting under suitable bounds on the convexity of $\Omega$ and the Ricci curvature in $\Omega$.

More precisely, let $\Omega \Subset N$ be a relatively compact, strictly convex domain with smooth boundary admitting a smooth defining function $h$ such that $h < 0$ in $\Omega$, $h = 0$ on $\partial\Omega$,

$$\left(h_{i;j}\right) \ge k_1 \left(\delta_{ij}\right) \tag{1.6}$$

for some constant $k_1 > 0$ and $\sup_\Omega |\nabla h| \le 1$, $h_\gamma = -1$ and $|\nabla h| = 1$ on $\partial\Omega$. Furthermore, by strict convexity of $\Omega$, the second fundamental form of $\partial\Omega$ satisfies

$$\left(\kappa_{ij}\right)_{1 \le i,j \le n-1} \ge \kappa_0 \left(\delta_{ij}\right)_{1 \le i,j \le n-1}, \tag{1.7}$$

where $\kappa_0 > 0$ is the minimal principal curvature of $\partial\Omega$. In the Euclidean case, $N = \mathbb{R}^n$, such functions $h$ are constructed in [2]. We give some simple examples at the end of Section 3.

**Theorem 1.3.** *Let $\Omega \Subset N$ be a smooth, strictly convex, relatively compact domain associated with constants $k_1 > 0$ and $\kappa_0 > 0$ as in* (1.6) *and* (1.7). *Let $\alpha < \min\{\kappa_0, k_1(n-1)/2\}$ and assume that the Ricci curvature in $\Omega$ satisfies $|\,\mathrm{Ric}\,| < \alpha(k_1(n-1)-\alpha)/(n+1)$. Then there exists $\varepsilon_0 > 0$ such that if $\phi =: \cos\theta \in C^3(\bar\Omega)$ satisfies $|\cos\theta| \le \varepsilon_0 \le 1/4$ and $||\nabla\theta||_{C^1(\bar\Omega)} \le \varepsilon_0$ in $\bar\Omega$, there exist a unique constant $C_\infty$ and a solution $u_\infty$ to* (1.4). *Furthermore, $u_\infty$ is unique up to an additive constant.*

We will sketch the proof of Theorem 1.3 in Section 3.

## 2 Proof of Theorem 1.1

Let $u$ be a solution to (1.2) in $\bar\Omega \times \mathbb{R}$. Given a constant $C_\infty \in \mathbb{R}$ we define, following the ideas of Korevaar [6], Guan [4] and Zhou [8], a function $\eta\colon \bar\Omega \times \mathbb{R} \to (0,\infty)$ by setting

$$\eta = e^{K(u-C_\infty t)}\left(Sd + 1 - \frac{\phi}{W}\langle \nabla u, \nabla d\rangle\right), \tag{2.1}$$

where $K$ and $S$ are positive constants to be determined later. We start with a gradient estimate.

**Proposition 2.1.** *Let $u$ be a solution to* (1.2) *and define $\eta$ as in* (2.1). *Then, for a fixed $T > 0$, letting*

$$(W\eta)(x_0, t_0) = \max_{x\in\bar\Omega,\ t\in[0,T]} (W\eta)(x,t),$$

*there exists a constant $C_0$ only depending on $C_d$, $\phi$, $C_\infty$, and the lower bound for the Ricci curvature in $\Omega$ such that $W(x_0, t_0) \le C_0$.*

*Proof.* Let $g = g_{ij}dx^i dx^j$ be the Riemannian metric of $N$. We denote by $(g^{ij})$ the inverse of $(g_{ij})$, $u_j = \partial u/\partial x^j$, and $u_{i;j} = u_{ij} - \Gamma^k_{ij}u_k$. We set

$$a^{ij} = g^{ij} - \frac{u^i u^j}{W^2}$$

and define an operator $L$ by $Lu = a^{ij}u_{i;j} - \partial_t u$. Observe that (1.2) can be rewritten as $Lu = 0$. In all the following, computations will be done at the maximum point $(x_0, t_0)$ of $\eta W$. We first consider the case where $x_0 \in \partial\Omega$. We choose normal coordinates at $x_0$ such that $g_{ij} = g^{ij} = \delta^{ij}$ at $x_0$, $\partial_n = \gamma$,

$$u_1 \ge 0, \quad u_i = 0 \quad \text{for } 2 \le i \le n-1.$$

This implies that

$$d_i = 0 \text{ for } 1 \le i \le n-1,\ d_n = 1, \text{ and } d_{i;n} = 0 \text{ for } 1 \le i \le n.$$

We have

$$\begin{aligned}
0 \ge (W\eta)_n &= W_n\eta + W\eta_n \\
&= e^{K(u-C_\infty t)}\Big(SW_n d + W_n - \frac{\phi W_n}{W}g^{ij}u_i d_j + SWd_n - \frac{W}{W}\phi_n g^{ij}u_i d_j \\
&\quad - \frac{W}{W}\phi g^{ij}(u_{i;n}d_j + u_i d_{j;n}) + W\frac{W_n}{W^2}\phi g^{ij}u_i d_j \\
&\quad + KWu_n(Sd + 1 - \frac{\phi}{W}g^{ij}u_i d_j)\Big) \\
&= e^{K(u-C_\infty t)}\Big(W_n + SW - \phi_n u_n - \phi u_{n;n} + KWu_n(1-\phi^2)\Big).
\end{aligned} \tag{2.2}$$

Using our coordinate system, we get

$$\begin{aligned}0 &\geq \frac{W_n}{W} + S - \frac{\phi_n u_n}{W} - \frac{\phi u_{n;n}}{W} + K u_n(1-\phi^2)\\ &= S - \frac{u_1^2 d_{1;1}}{W^2} + \frac{u_1\phi_1}{W}\left(1 + \frac{2\phi^2}{1-\phi^2}\right) - \frac{\phi u_1}{W} K u_1\\ &\quad - \frac{\phi_n u_n}{W} + K u_n(1-\phi^2)\\ &\geq S - C - \frac{K\phi u_1^2}{W} + K u_n(1-\phi^2)\\ &= S - C - \frac{K\phi}{W} \geq S - C - \frac{K}{W},\end{aligned}$$

for some constant $C$ depending only on $C_d$ and $\phi$. So choosing $S \geq C + 1$, we get that

$$W(x_0, t_0) \leq K. \tag{2.3}$$

Next we assume that $x_0 \in \Omega$ and that $S \geq C+1$, where $C$ is as above. Let us recall from [8, Lemma 3.5] that

$$LW = \frac{2}{W} a^{ij} W_i W_j + \mathrm{Ric}(\nu_N, \nu_N) W + |A|^2 W,$$

where $\nu_N = \nabla u / W$ and $|A|^2 = a^{ij} a^{\ell k} u_{i;k} u_{j;\ell} / W^2$ is the squared norm of the second fundamental form of the graph $M_t$. Since $0 = W_i \eta + W \eta_i$, for every $i = 1, \ldots, n$, we deduce that

$$0 \geq L(W\eta) = WL\eta + \eta\left(LW - 2a^{ij}\frac{W_i W_j}{W}\right) = WL\eta + \eta W\left(|A|^2 + \mathrm{Ric}(\nu_N, \nu_N)\right).$$

This yields to

$$\frac{1}{\eta} L\eta + |A|^2 + \mathrm{Ric}(\nu_N, \nu_N) \leq 0. \tag{2.4}$$

To simplify the notation, we set

$$h = Sd + 1 - \phi u^k d_k / W = Sd + 1 - \phi \nu^k d_k.$$

So we have

$$\frac{1}{\eta} L\eta = K^2 a^{ij} u_i u_j + KL(u - C_\infty t) + \frac{2K}{h} a^{ij} u_i h_j + \frac{1}{h} Lh. \tag{2.5}$$

We can compute $Lh$ as

$$Lh = a^{ij}\left(S d_{i;j} - (\phi d_k)_{i;j} \nu^k - (\phi d_k)_i \nu^k_j - (\phi d_k)_j \nu^k_i - \phi d_k L\nu^k\right) \geq -C - 2a^{ij}(\phi d_k)_i \nu^k_j - \phi d_k L \nu^k.$$

Since, by [8, Lemma 3.5],

$$L\nu^k = \mathrm{Ric}(a^{k\ell}\partial_\ell, \nu_N) - |A|^2 \nu^k$$

and, by Young's inequality for matrices,

$$a^{ij}(\phi d_k)_i \nu^k_j = \frac{1}{W}(\phi d_k)_i a^{ij} a^{\ell k} u_{\ell;j} \leq \frac{|A|^2}{6} + C,$$

we get the estimate

$$Lh \geq -C - |A|^2/3 + \phi d_k \nu^k |A|^2 \tag{2.6}$$

by using the assumption that Ric is bounded.

Next we turn our attention to the other terms in (2.5). We have

$$a^{ij} u_i = \frac{u^j}{W^2} \quad \text{and} \quad a^{ij} u_i u_j = 1 - \frac{1}{W^2}. \tag{2.7}$$

Then we note that by the assumptions, we clearly have

$$KL(u - C_\infty t) = KC_\infty \geq -KC, \tag{2.8}$$

and we are left to consider

$$
\begin{aligned}
a^{ij}u_i h_j = \frac{u^j h_j}{W^2} &= \frac{u^j\left(Sd_j - (\phi d_k)_j \nu^k - \phi d_k \nu^k_j\right)}{W^2} \\
&\geq -C - \frac{\phi d_k u^j \nu^k_j}{W^2} \\
&= -C + \frac{K\phi a^{\ell k} d_k u_\ell}{W} + \frac{\phi}{hW} a^{\ell k} d_k h_\ell \\
&= -C + \frac{K\phi a^{\ell k} d_k u_\ell}{W} \\
&\quad + \frac{S\phi a^{\ell k} d_k d_\ell}{hW} - \frac{\phi a^{\ell k} d_k (\phi d_s)_\ell \nu^s}{hW} - \frac{\phi^2 a^{\ell k} d_k d_s a^{sm} u_{m;\ell}}{hW^2} \\
&\geq -C - \frac{CK}{W^2} - \frac{|A|^2}{3K}.
\end{aligned} \tag{2.9}
$$

Plugging the estimates (2.6), (2.7), (2.8), and (2.9) into (2.5) and using (2.4) with the Ricci lower bound we obtain

$$
\begin{aligned}
0 &\geq K^2\left(1 - \frac{1}{W^2}\right) - CK - \frac{2K}{h}\left(C + \frac{CK}{W} + \frac{CK}{W^2} + \frac{|A|^2}{3K}\right) - \frac{1}{h}\left(C + |A|^2/3 - \phi d_k \nu^k |A|^2\right) + |A|^2 - C \\
&= K^2\left(1 - \frac{1}{W^2} - \frac{C}{hW^2}\right) - KC\left(1 + \frac{1}{h}\right) - \frac{|A|^2}{h} + \frac{\phi d_k \nu^k |A|^2}{h} - \frac{C}{h} + |A|^2 - C.
\end{aligned}
$$

Then collecting the terms including $|A|^2$ and noticing that

$$1 - \frac{1}{h} + \frac{\phi d_k \nu^k}{h} = \frac{Sd}{h} \geq 0$$

we have

$$0 \geq K^2\left(1 - \frac{1}{W^2} - \frac{C}{hW^2}\right) - CK\left(1 + \frac{1}{h}\right) - C.$$

Now choosing $K$ large enough, we obtain $W(x_0, t_0) \leq C_0$, where $C_0$ depends only on $C_\infty$, $d$, $\phi$, the lower bound of the Ricci curvature in $\Omega$, and the dimension of $N$. We notice that the constant $C_0$ is independent of $T$. □

Since

$$e^{K\left(u(\cdot,t) - C_\infty t\right)}(1 - \phi_0) \leq \eta \leq e^{K\left(u(\cdot,t) - C_\infty t\right)}(S + 2),$$

we have

$$
\begin{aligned}
W(x, t) &\leq \frac{(W\eta)(x_0, t_0)}{\eta(x, t)} \\
&\leq \frac{C_0 \eta(x_0, t_0)}{\eta(x, t)} \\
&\leq \frac{C_0(S + 2)}{1 - \phi_0} e^{K\left(u(x_0,t_0) - C_\infty t_0 - u(x,t) + C_\infty t\right)}
\end{aligned} \tag{2.10}
$$

for every $(x, t) \in \bar{\Omega} \times [0, T]$.

We observe that the function $u_\infty(x) + Ct$ solves the equation (1.2) with the initial condition $u_0 = u_\infty$ if $u_\infty$ is a solution to the elliptic equation (1.4) and $C$ is given by (1.5). As in [1, Corollary 2.7], applying a parabolic maximum principle ([7]) we obtain:

**Lemma 2.2.** *Suppose that* (1.4) *admits a solution $u_\infty$ with the unique constant $C$ given by* (1.5). *Let $u$ be a solution to* (1.2). *Then, we have*

$$|u(x, t) - Ct| \leq c_2,$$

*for some constant $c_2$ only depending on $u_0$, $\phi$, and $\Omega$.*

*Proof.* Let $V(x,t) = u(x,t) - u_\infty(x)$, where $u_\infty$ is a solution to (1.4). We see that $V$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t} = \tilde{a}^{ij} V_{i;j} + b^i V_i + C & \text{in } \Omega \times [0,T) \\ \tilde{c}^{ij} V_i \nu_j = 0 & \text{on } \partial\Omega \times [0,T), \end{cases}$$

where $\tilde{a}^{ij}$, $\tilde{c}^{ij}$ are positive definite matrices and $b^i \in \mathbb{R}$. Then the proof of the lemma follows by applying the maximum principle. □

In view of Lemma 2.2, taking $C_\infty = C$, and observing that the constant $C_0$ is independent of $T$, we get from (2.10) a uniform gradient bound.

**Lemma 2.3.** *Suppose that* (1.4) *admits a solution $u_\infty$ with the unique constant $C$ given by* (1.5). *Let $u$ be a solution to* (1.2). *Then $W(x,t) \le C_1$ for all $(x,t) \in \bar{\Omega} \times [0,\infty)$ with a constant $C_1$ depending only on $\phi_0$, $u_0$, and $\Omega$.*

Having a uniform gradient bound in our disposal, applying once more the strong maximum principle for linear uniformly parabolic equations, we obtain:

**Theorem 2.4.** *Suppose that* (1.4) *admits a solution $u_\infty$ with the unique constant $C$ given by* (1.5). *Let $u_1$ and $u_2$ be two solutions of* (1.2) *with the same prescribed contact angle as $u_\infty$. Let $u = u_1 - u_2$. Then $u$ converges to a constant function as $t \to \infty$. In particular, if $C$ is given by* (1.5), *then $u_1(x,t) - u_\infty(x) - Ct$ converges uniformly to a constant as $t \to \infty$.*

*Proof.* The proof is given in [1, p. 109]. We reproduce it for the reader's convenience. One can check that $u$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tilde{a}^{ij} u_{i;j} + b^i u_i & \text{in } \Omega \times [0,\infty) \\ \tilde{c}^{ij} u_i \nu_j = 0 & \text{on } \partial\Omega \times [0,\infty), \end{cases}$$

where $\tilde{a}^{ij}$, $\tilde{c}^{ij}$ are positive definite matrices and $b^i \in \mathbb{R}$. By the strong maximum principle, we get that the function $F_u(t) = \max u(\cdot,t) - \min u(\cdot,t) \ge 0$ is either strictly decreasing or $u$ is constant. Assuming on the contrary that $\lim_{t\to\infty} u$ is not a constant function, setting $u_n(\cdot,t) = u(\cdot, t - t_n)$ for some sequence $t_n \to \infty$, we would get a non-constant solution, say $v$, defined on $\Omega \times (-\infty, +\infty)$ for which $F_v$ would be constant. We get a contradiction with the maximum principle. □

Theorem 1.1 now follows from Lemma 2.3 and Theorem 2.4.

## 3 Proof of Theorem 1.3

Theorem 1.3 is essentially proven in [3, Theorem 2.1, 3.1]. The only extra ingredient we must take into account in our non-flat case is the following Ricci identity for the Hessian $\varphi_{i;j}$ of a smooth function $\varphi$

$$\varphi_{k;ij} = \varphi_{i;kj} = \varphi_{i;jk} + R^{\ell}_{kji}\varphi_\ell. \tag{3.1}$$

For the convenience of the reader, we mostly use the same notations as in [3]. Thus let $h$ be a smooth defining function of $\Omega$ such that $h < 0$ in $\Omega$, $h = 0$ on $\partial\Omega$, $(h_{i;j}) \ge k_1(\delta_{ij})$ for some constant $k_1 > 0$ and $\sup_\Omega |\nabla h| \le 1$, $h_\gamma = -1$ and $|\nabla h| = 1$ on $\partial\Omega$. Furthermore, by strict convexity of $\Omega$, the second fundamental form of $\partial\Omega$ satisfies

$$(\kappa_{ij})_{1\le i,j\le n-1} \ge \kappa_0(\delta_{ij})_{1\le i,j\le n-1},$$

where $\kappa_0 > 0$ is the minimal principal curvature of $\partial\Omega$.

We consider the equation

$$\begin{cases} a^{ij}u_{i;j} := \left(g^{ij} - \frac{u^i u^j}{1+|\nabla u|^2}\right) u_{i;j} = \varepsilon u & \text{in } \Omega \\ \partial_\gamma u = \phi\sqrt{1+|\nabla u|^2} & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

for small $\varepsilon > 0$. Writing $\phi = -\cos\theta$, $v = \sqrt{1+|\nabla u|^2}$ and

$$\Phi(x) = \log w(x) + \alpha h(x),$$

where $w(x) = v - u^\ell h_\ell \cos\theta$ and $\alpha > 0$ is a constant to be determined, we assume that the maximum of $\Phi$ is attained in a point $x_0 \in \bar{\Omega}$. If $x_0 \in \partial\Omega$, we can proceed as in [3, pp. 34-36]. Thus choosing $0 < \alpha < \kappa_0$ and $0 < \varepsilon_0 \le \varepsilon_\alpha < 1$ such that

$$\kappa_0 - \alpha > \frac{\varepsilon_\alpha(M_1+3)}{1-\varepsilon_\alpha^2}, \tag{3.3}$$

where $M_1 = \sup_{\bar{\Omega}} |\nabla^2 h|$, yields an upper bound

$$|\nabla' u(x_0)|^2 \le \frac{\frac{\varepsilon_0(M_1+3)}{1-\varepsilon_0^2} + \alpha}{\kappa_0 - \alpha - \frac{\varepsilon_0(M_1+3)}{1-\varepsilon_0^2}} < \frac{\kappa_0}{\kappa_0 - \alpha - \frac{\varepsilon_\alpha(M_1+3)}{1-\varepsilon_\alpha^2}}$$

for the tangential component of $\nabla u$ on $\partial\Omega$. Combining this with the boundary condition $u_\gamma = -v\cos\theta$ gives an upper bound for $|\nabla u(x_0)|$ and hence for $\Phi(x_0)$.

The only difference to the Euclidean case occurs when $x_0 \in \Omega$, i.e. is an interior point of $\Omega$. At this point we have, using the same notations as in [3, p. 42],

$$0 = \Phi_i(x_0) = \frac{w_i}{w} + \alpha h_i$$

and

$$0 \ge a^{ij}\Phi_{i;j}(x_0) = \frac{a^{ij}w_{i;j}}{w} - \alpha^2 a^{ij}h_i h_j + \alpha a^{ij}h_{i;j} =: I + II + III.$$

We choose normal coordinates at $x_0$ such that $u_1(x_0) = |\nabla u(x_0)|$ and $(u_{i;j}(x_0))_{2\le i,j\le n}$ is diagonal. Then at $x_0$, we have

$$II + III \ge -\alpha^2(1 + 1/v^2) + \alpha k_1(n - 1 + 1/v^2).$$

We denote $J = a^{ij}w_{i;j} = J_1 + \tilde{J}_2 + J_3 + J_4$, where $J_1, J_3$ and $J_4$ are as in [3, (2.19)]. We have, by [3, (2.22)],

$$J_3 + J_4 \ge -C(|\cos\theta| + |\nabla\theta| + |\nabla^2\theta|)u_1 - C(|\cos\theta| + |\nabla\theta|)\sum_{i=2}^{n}|u_{ii}|,$$

where $C$ depends only on $n$, $M_1$ and $\sup_{\bar{\Omega}} |\nabla^3 h|$. Writing $S^\ell = \frac{u_\ell}{v} - h_\ell\cos\theta$ and using the Ricci identity

$$a^{ij}u_{k;ij} = a^{ij}u_{i;jk} + \mathrm{Ric}(\partial_k, \nabla u)$$

(see [8, (2.28)]) and (3.2), we get

$$\begin{aligned} \tilde{J}_2 = a^{ij}\left(\frac{u^k u_{k;ij}}{v} - u_{k;ij}h^k\cos\theta\right) &= S^k a^{ij}u_{i;jk} + S^k\,\mathrm{Ric}(\partial_k, \nabla u) \\ &= -S^k a^{ij}_{;k}u_{i;j} + S^k(\varepsilon u)_k + S^k\,\mathrm{Ric}(\partial_k, \nabla u) \\ &= J_2 + \varepsilon u_1 S^1 + S^k\,\mathrm{Ric}(\partial_k, \partial_1)|\nabla u|, \end{aligned}$$

where $J_2$ is as in [3, (2.19)]. Since $|S^1| \le 2$ and $|S^k| \le 1$ for $k \ge 2$, we obtain

$$\tilde{J}_2 \ge J_2 - (n+1)|\,\mathrm{Ric}_\Omega\,||\nabla u|, \tag{3.4}$$

where $|\operatorname{Ric}_\Omega|$ is the bound for the Ricci curvature in $\Omega$, i.e. $|\operatorname{Ric}(x)| \le |\operatorname{Ric}_\Omega|$ for all unit vectors $x \in T\Omega$. At this point, we can proceed as in [3] to get that

$$J_1 + J_2 \ge \sum_{i=2}^{n} \frac{u_{ii}^2}{2v}.$$

So combining the previous estimates, we find

$$I = \frac{J}{w} \ge -C(|\cos\theta| + |\nabla\theta| + |\nabla^2\theta|) - (n+1)|\operatorname{Ric}_\Omega|.$$

Hence we obtain

$$\begin{aligned} 0 &\ge I + II + III \ge -C(|\cos\theta| + |\nabla\theta| + |\nabla^2\theta|) - (n+1)|\operatorname{Ric}_\Omega| - \alpha^2(1+1/v^2) + \alpha k_1(n-1+1/v^2) \\ &=: C_1 + C_2/v^2, \end{aligned}$$

where

$$C_1 = -C\varepsilon_0 - (n+1)|\operatorname{Ric}_\Omega| + \alpha\big(k_1(n-1) - \alpha\big)$$

and $C_2 = \alpha(k_1 - \alpha)$. If $C_1 > 0$ and $C_2 > 0$, we get a contradiction, and therefore the maximum of $\Phi$ is attained on $\partial\Omega$. If $C_1 > 0$ and $C_2 < 0$, then $v^2 \le -C_2/C_1$ and again we have an upper bound for $\Phi(x_0)$. To have $C_1 > 0$ we need

$$|\operatorname{Ric}_\Omega| < \big(\alpha(k_1(n-1) - \alpha\big) - C\varepsilon_0)/(n+1). \tag{3.5}$$

Fixing $\alpha < \min\{\kappa_0, k_1(n-1)/2\}$ and assuming that

$$|\operatorname{Ric}_\Omega| < \big(\alpha(k_1(n-1) - \alpha\big)/(n+1) \tag{3.6}$$

and, finally, choosing $0 < \varepsilon_0 \le \min\{\varepsilon_\alpha, 1/4\}$ small enough so that (3.5) holds, we end up again with a contradiction, and therefore the maximum of $\Phi$ is attained on $\partial\Omega$. All in all, we have obtained a uniform gradient bound for a solution $u$ to (3.2) that is independent of $\varepsilon$. Once the uniform gradient bound is established the rest of the proof goes as in [1] (or [3]).

In some special cases we get sharper estimates than those above.

**Example 3.1.** As the first example let us consider the hyperbolic space $\mathbb{H}^n$ and a geodesic ball $\Omega = B(o, R)$. Furthermore, we choose

$$h(x) = \frac{r(x)^2}{2R} - \frac{R}{2}$$

as a defining function for $\Omega$. Here $r(\cdot) = d(\cdot, o)$ is the distance to the center $o$. Then $\kappa_0 = \coth R$ and we may choose $k_1 = 1/R$. Since $\operatorname{Ric}(\partial_k, \partial_1) = -(n-1)\delta_{k1}$, (3.4) can be replaced by

$$\tilde{J}_2 \ge J_2 - 2(n-1)|\nabla u|$$

and consequently (3.6) can be replaced by

$$2(n-1) < \alpha\big((n-1)/R - \alpha\big),$$

where $\alpha < \min\{\coth R, \frac{n-1}{2R}\}$. Hence we obtain an upper bound for the radius $R$. For instance, if $n = 2$, then $\alpha < \frac{1}{2R}$ and we need $R < \frac{1}{2\sqrt{2}}$. For all dimensions, $\alpha = 1$ and $R < \frac{n-1}{2n-1}$ will do.

**Example 3.2.** As a second example let $N$ be a Cartan-Hadamard manifold with sectional curvatures bounded from below by $-K^2$, with $K > 0$. Again we choose $\Omega = B(o, R)$ and

$$h(x) = \frac{r(x)^2}{2R} - \frac{R}{2}.$$

Now $1/R \le \kappa_0 \le K\coth(KR)$ and again we may choose $k_1 = 1/R$. This time $\operatorname{Ric}(\partial_1, \partial_1) \ge -(n-1)K^2$ and $\operatorname{Ric}(\partial_k, \partial_1) \ge -\frac{1}{2}(n-1)K^2$ for $k = 2, \dots, n$, and therefore instead of (3.4) and (3.6) we have

$$\tilde{J}_2 \ge J_2 - K^2\big((n+1)^2/2 - 2\big)|\nabla u|$$

and

$$K^2\left((n+1)^2/2-2\right) < \alpha\left((n-1)/R-\alpha\right),$$

where $\alpha < \min\{1/R, \frac{n-1}{2R}\}$. Again we obtain upper bounds for the radius $R$. If $n \geq 3$ we need

$$R < \left(\frac{n-2}{K^2\left((n+1)^2/2-2\right)}\right)^{1/2}$$

whereas for $n = 2$ the bound

$$R < \frac{1}{2\sqrt{2}K}$$

is enough since now $\mathrm{Ric}(\partial_2, \partial_1) = 0$.

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