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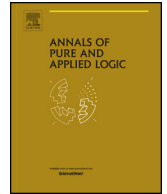
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ABSTRACT

In this paper, we study several propositional team logics that are closed under unions, including propositional inclusion logic. We show that all these logics are expressively complete, and we introduce sound and complete systems of natural deduction for these logics. We also discuss the locality property and its connection with interpolation in these logics.

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1. Introduction

In this paper, we study propositional union closed team logics. These logics are variants of *dependence logic*, which was introduced by Väänänen [34] as a non-classical first-order logic for reasoning about dependencies. This framework extends the classical logic by adding new atomic formulas for characterizing dependence and independence between variables. Examples of such atoms are *dependence atoms* (giving rise to *dependence logic* [34]), *inclusion atoms* (giving rise to *inclusion logic* [13]) and *independence atoms* (giving rise to *independence logic* [18]). Dependence logic and its variants adopts the so-called *team semantics*, which was introduced by Hodges [26,27]. The basic idea of team semantics is that dependency properties can only manifest themselves in *multitudes*. Thus, formulas of these logics are evaluated under *teams*, which, in the propositional context, are *sets* of valuations. In particular, a propositional inclusion atom $p \subseteq q$ is said to be true in a team X , if every truth value of p in X occurs as a truth value of q ; in other words, values of p are “included” in the values of q .

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In logics based on team semantics, also called *team(-based) logics*, two closure properties are of particular interest: the *downwards closure* and the *union closure* property. Dependence logic is *closed downwards*, meaning that the truth of a formula on a team is preserved under taking subteams. In this paper, we focus on propositional team-based logics that are *closed under unions*, meaning that if two teams both satisfy a formula, then their set-theoretic union also satisfies the formula. Inclusion logic is the first studied union closed team logic [13]. First-order dependence and inclusion logic can be translated into existential second-order logic (ESO) [34,13]. More precisely, first-order dependence logic characterizes all downwards closed ESO-team properties [31], whereas some but not all union closed ESO-team properties are definable in first-order inclusion logic [15]. While both dependence and inclusion logic are strictly weaker than ESO on their own, full ESO can be characterized by first-order logic extended with both the downwards closed dependence atoms and the union closed inclusion atoms [13].

Interestingly, the complementary feature of downwards closure and union closure properties is also found on a more basic level in team semantics: Team logics are often defined as conservative extensions of classical logic. The *conservativity* is described through the *flatness* property of classical formulas, which states that every classical formula is satisfied in a team X if and only if every singleton team of a valuation in X satisfies the formula, or equivalently, every valuation in X satisfies the formula in the sense of the usual semantics. The property of flatness can actually be decomposed into downwards closure and union closure, in the sense that a formula is flat if and only if it is both downwards closed and union closed (see Fact 2.1), assuming that the empty team satisfies the formula (which is often the case for many typical team logics).

Understanding the properties of team logics with the downwards closure or union closure property is thus arguably a key theme in the research in team semantics. Compared with the relatively well-understood downwards closed team logics (particularly dependence logic), union closed team logics have received less attention in the literature. Many properties of inclusion logic and other union closed team logics have not yet been well-explored, especially on the propositional level. On the first-order level, in 2013, first-order inclusion logic was shown by Galliani and Hella [15] to be expressively equivalent to positive greatest fixed point logic and thus captures the complexity class NP over finite ordered structures. This breakthrough has sparked increasing interests in inclusion logic and union closed team logics in general in recent years. For instance, model-checking games for first-order inclusion logic were developed in [16,17], first-order consequences of first-order inclusion logic were axiomatized in [36], computational complexity and syntactical fragments of first-order inclusion logic were investigated in [14,20–22,32], a team-based first-order logic characterizing the union closed fragment of existential second-order logic was identified in [28], etc. As for propositional logic, some basic properties of propositional inclusion logic ($\mathbf{CPL}(\subseteq)$) and other union closed team logics were discussed in [38]. The results in [38] are, however, relatively preliminary, compared with the extensive account of propositional downwards closed team logics in the literature (e.g., [7,37]). There are some recent articles on the expressive power and computational complexity properties of modal inclusion logic [23–25] that also cover propositional inclusion logic, but only briefly as a special case. The aim of this paper is to provide a more in-depth account for the logical properties of propositional union closed team logics, including $\mathbf{CPL}(\subseteq)$ and another two logics.

One of the two logics is obtained by extending classical propositional logic with a different type of union closed dependency atoms $\mathbf{p}\Upsilon\mathbf{q}$ (with \mathbf{p}, \mathbf{q} two sequences of propositional variables), called the *anonymity atoms*. These atoms were introduced by Galliani in [12] under the name *non-dependence atoms*, as they state an extreme case of the failure of the functional dependence between \mathbf{p} and \mathbf{q} . Recently Väänänen [35] advocated the anonymity atoms with the motivation of anonymity concerns in data safety (and hence the name): $\mathbf{p}\Upsilon\mathbf{q}$ means “ \mathbf{p} is anonymized with respect to \mathbf{q} ”. Anonymity atoms also correspond exactly to the *afunctional dependencies* studied in database theory (see e.g., [3,4]). A special case $\langle \rangle\Upsilon\mathbf{q}$ of anonymity atoms with the first component being the empty sequence $\langle \rangle$ deserves commenting. Such atoms, denoted also as $\neq(\mathbf{p})$, are also called *inconstancy atoms*, as they state that \mathbf{p} does not have a constant value in the team. First-order logic with inconstancy atoms is known to be equivalent to first-order logic over sentences [13],

and first-order logic with arbitrary anonymity atoms is equivalent to inclusion logic [12]. The propositional logic with these atoms (denoted as $\mathbf{CPL}(\Upsilon)$ and $\mathbf{CPL}(\neq(\cdot))$) has so far not been studied.

The other union closed team logic we consider is obtained by adding to classical propositional logic a new disjunction \vee , called the *relevant disjunction*, which was introduced by Rönnholm [32] as a variant of the standard disjunction \vee in team semantics. The main difference between the two disjunctions is that a team can satisfy a disjunction $\phi \vee \psi$ when only one disjunct is satisfied, whereas the relevant disjunction $\phi \vee \psi$ requires both disjuncts to be satisfied in a non-void manner (and thus both disjuncts are actually “relevant”). The relevant disjunction is studied in the literature also under the name *nonempty disjunction* (e.g., [25,38]). In [25], classical modal logic with \vee was shown to be expressively complete.

The starting point of this paper is the work in [38], where classical propositional logic with relevant disjunction ($\mathbf{CPL}(\vee)$) was shown to be expressively complete. Building on the arguments in [38], we prove in this paper that $\mathbf{CPL}(\Upsilon)$ as well as $\mathbf{CPL}(\neq(\cdot))$ are expressively complete too. It follows essentially from the argument of [25] in the context of modal logic that propositional inclusion logic $\mathbf{CPL}(\subseteq)$ (with actually a weaker version of inclusion atoms than in [25]) is also expressively complete. All of these union closed logics are thus also expressive equivalent, and they all admit certain disjunctive normal form.

We also provide axiomatizations for $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\vee)$, which are lacking in the literature. We introduce sound and complete natural deduction systems for these logics. As with other team logics, these systems do not admit uniform substitution. The completeness theorem is proved by using the disjunctive normal form of the logics.

In union closed team logics, a metalogical property, the *locality* property, deserves particular attention. Locality states that the truth of a formula does not depend on the variables that do not occur in the formula. While this property is often taken for granted in most familiar logics, it is actually a very non-trivial property for team logics, especially for union closed team logics. For example, first-order inclusion logic with the so-called *strict semantics* does not satisfy locality [13]. We give examples to show that under strict semantics, locality fails for propositional inclusion logic $\mathbf{CPL}(\subseteq)$, as well as $\mathbf{CPL}(\vee)$ and $\mathbf{CPL}(\Upsilon)$ too. We also discuss a subtle connection between locality and interpolation. It follows from the work of D’Agostino [8] in the modal team logics context that all expressively complete union closed team logics enjoy uniform interpolation. We highlight the subtle and crucial role that locality plays in the proof of [8], and also give an example to illustrate the failure of Craig’s interpolation in a fragment of $\mathbf{CPL}(\vee)$ under strict semantics that does not satisfy the locality property.

This paper is organized as follows. In Section 2, we recall the basics of team semantics and define the propositional union closed team logics we consider in the paper. In Section 3, we show that these logics are expressively complete (some of the proofs are already known), and obtain a disjunctive normal form for the logics. Making essential use of this disjunctive normal form, in Section 4, we axiomatize $\mathbf{CPL}(\vee)$ and $\mathbf{CPL}(\subseteq)$ as well as a fragment $\mathbf{CPL}(\subseteq_0)$ of $\mathbf{CPL}(\subseteq)$. In Section 5, we revisit the property of locality, and highlight its connection with interpolation. We conclude and discuss further directions in Section 6.

2. Preliminaries

All of the logics we consider in this paper are extensions of classical propositional logic, defined in the team semantics setting. Let us now start by recalling the syntax for *classical propositional logic* (\mathbf{CPL}). Fix a set \mathbf{Prop} of propositional variables. The set of well-formed formulas of \mathbf{CPL} (called *classical formulas*) are given by the grammar:

$$\alpha ::= p \mid \perp \mid \top \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha)$$

where $p \in \mathbf{Prop}$, and \perp and \top are two constants, called *falsum* and *verum*, respectively. Throughout the paper we reserve the first Greek letters $\alpha, \beta, \gamma, \dots$ for classical formulas. As usual, we write $\alpha \rightarrow \beta := \neg\alpha \vee \beta$

Table 1
A team X with $\text{dom}(X) = \{p, q\}$.

	p	q
v_1	0	1
v_2	1	0
v_3	1	1

and $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. We write $\text{Prop}(\alpha)$ for the set of propositional variables occurring in α . We also use the notation $\alpha(\mathbf{N})$ (with $\mathbf{N} \subseteq \text{Prop}$ a set of propositional variables) to indicate that the propositional variables occurring in α are among \mathbf{N} .

Let $\mathbf{N} = \{p_1, \dots, p_n\} \subseteq \text{Prop}$ be a set of propositional variables. In the standard semantics for **CPL**, a classical formula $\alpha(\mathbf{N})$ is evaluated under *valuations*, which are functions $v : \mathbf{N} \cup \{\perp, \top\} \rightarrow \{0, 1\}$ such that $v(\perp) = 0$ and $v(\top) = 1$. Recall that a valuation v extends naturally to all formulas α of **CPL**, and we write $v \models \alpha$ if $v(\alpha) = 1$. In this paper, we adopt *team semantics* for classical formulas, in which a classical formula is evaluated under teams. An (\mathbf{N} -)team X is a set of valuations $v : \mathbf{N} \cup \{\perp, \top\} \rightarrow \{0, 1\}$ with $v(\perp) = 0$ and $v(\top) = 1$. The set \mathbf{N} is called the *domain* of the team X , denoted as $\text{dom}(X)$. In particular, the empty set \emptyset is a team (of an arbitrary domain). We often represent a team as a table. For example, Table 1 represents a team $X = \{v_1, v_2\}$ with $\text{dom}(X) = \{p, q\}$ consisting of two valuations v_1 and v_2 , defined as

$$v_1(p) = 0, v_1(q) = 1, v_2(p) = 1, v_2(q) = 0, \text{ and } v_3(p) = v_3(q) = 1.$$

The notion of a classical formula α being *true* on a team X with $\text{dom}(X) \supseteq \text{Prop}(\alpha)$, denoted by $X \models \alpha$, is defined inductively as follows:

- $X \models p$ iff for all $v \in X$, $v(p) = 1$.
- $X \models \perp$ iff $X = \emptyset$.
- $X \models \top$ always holds.
- $X \models \neg\alpha$ iff for all $v \in X$, $\{v\} \not\models \alpha$.
- $X \models \alpha \wedge \beta$ iff $X \models \alpha$ and $X \models \beta$.
- $X \models \alpha \vee \beta$ iff there exist subteams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \alpha$ and $Z \models \beta$.

For any set $\Gamma \cup \{\alpha\}$ of formulas, we write $\Gamma \models \alpha$ if for all teams X with $\text{dom}(X) \supseteq \bigcup_{\gamma \in \Gamma} \text{Prop}(\gamma) \cup \text{Prop}(\alpha)$, $X \models \gamma$ for all $\gamma \in \Gamma$ implies $X \models \alpha$. We write simply $\alpha \models \beta$ for $\{\alpha\} \models \beta$. If both $\alpha \models \beta$ and $\beta \models \alpha$, we write $\alpha \equiv \beta$ and say that α and β are *semantically equivalent*.

It is easy to verify (by a straightforward induction) that **CPL**-formulas have the *locality property*, *empty team property*, *union closure property* and *downwards closure property*, that is, for any **CPL**-formula α , the following holds:

Empty Team Property: $\emptyset \models \alpha$ holds;

Union Closure: $X \models \alpha$ and $Y \models \alpha$ imply $X \cup Y \models \alpha$;

Downwards Closure: $X \models \alpha$ and $Y \subseteq X$ imply $Y \models \alpha$.

These three properties together are (easily shown to be) equivalent to the *flatness property*:

Flatness: $X \models \alpha$ if and only if $\{v\} \models \alpha$ for all $v \in X$.

Fact 2.1. A formula is flat if and only if it satisfies the empty team property and is both union closed and downwards closed.

Moreover, an easy inductive proof shows that the truth of a classical formula α on singleton teams $\{v\}$ coincides with its truth on the single valuations v in the usual sense, namely,

$$\{v\} \models \alpha \text{ if and only if } v \models \alpha. \tag{1}$$

Putting these observations together, we obtain the following fact, which will also serve as our working team semantics for classical formulas:

Fact 2.2. For any classical formula α , any team X with $\text{dom}(X) \supseteq \text{Prop}(\alpha)$,

$$X \models \alpha \text{ iff } v \models \alpha \text{ for all } v \in X.$$

Intuitively, the above fact (or essentially the flatness of classical formulas) means that the team semantics for classical formulas collapses to the usual (single-valuation) semantics. In this sense, team semantics is conservative over classical formulas. Such a conservativity as reflected in Fact 2.2 also justifies our definition of the team semantics for classical formulas. For instance, our semantic clause “ $X \models \alpha \vee \beta$ ” for a disjunction being true *on the team level* states that the team X can be split into two subteams Y and Z such that each disjunct is true in one of the two subteams. This (by Fact 2.2) is the same as stating that the disjunction $\alpha \vee \beta$ is true *locally* under every valuation v in X , that is, every valuation v in X satisfies at least one of the disjuncts, resulting in a natural split of the team X into two subteams Y and Z with every element in Y making α true and every element in Z making β true. By Fact 2.2 again, this means that α is true at Y on the team level, and β is true at Z on the team level.

Recall that for classical formulas under the usual (single-valuation) semantics, when a formula α is evaluated on a valuation v , the truth of α depends only on how the valuation v evaluates the propositional variables that actually occurs in α , and it is independent of how the other propositional variables are evaluated. That is, for any formula $\alpha(\mathbf{N})$, if v and u are two valuations with $v \upharpoonright \mathbf{N} = u \upharpoonright \mathbf{N}$, then $v \models \alpha$ iff $u \models \alpha$. This is called the *locality property*. While this property is often taken for granted for most logics with the usual semantics, as we will discuss in this paper, in the team semantics setting, the locality property is a nontrivial property. We now give the definition of the property in the team semantics setting.

Definition 2.3. Let $\mathbf{N} \subseteq \text{Prop}$ be a set of propositional variables. A formula $\phi(\mathbf{N})$ is said to satisfy the *locality property* if for any teams X and Y such that $\text{dom}(X), \text{dom}(Y) \supseteq \mathbf{N}$ and $X \upharpoonright \mathbf{N} = Y \upharpoonright \mathbf{N}$, it holds that

$$X \models \phi(\mathbf{N}) \iff Y \models \phi(\mathbf{N}),$$

where for any team Z with $\text{dom}(Z) \supseteq \mathbf{N}$, we write $Z \upharpoonright \mathbf{N} = \{v \upharpoonright \mathbf{N} \mid v \in Z\}$.

Thanks to the flatness of classical formulas, we can, nevertheless, obtain the locality property for classical formulas as an immediate corollary of Fact 2.2.

Proposition 2.4. *Classical formulas satisfy the locality property.*

Proof. Let $\alpha(\mathbf{N})$ be a classical formula. For any teams X and Y with $\text{dom}(X), \text{dom}(Y) \supseteq \mathbf{N}$ and $X \upharpoonright \mathbf{N} = Y \upharpoonright \mathbf{N}$, by Fact 2.2 and the locality property of classical formulas under the usual (single-valuation) semantics, we have that

$$X \models \alpha \iff \forall v \in X : v \models \alpha(\mathbf{N}) \iff \forall u \in Y : u \models \alpha(\mathbf{N}) \iff Y \models \alpha. \quad \square$$

We now extend **CPL** to three non-flat but union closed team-based logics. Consider a new disjunction \vee , called *relevant disjunction*, and atomic formulas of the form $a_1 \dots a_k \subseteq b_1 \dots b_k$ with each $a_i, b_i \in$

$\text{Prop} \cup \{\perp, \top\}$, called *inclusion atoms*, and of the form $p_1 \dots p_k \Upsilon q_1 \dots q_m$ with each $p_i, q_j \in \text{Prop}$, called *anonymity atoms*. Inclusion and anonymity atoms are often represented as $\mathbf{a} \subseteq \mathbf{b}$ and $\mathbf{p}\Upsilon\mathbf{q}$ with letters $\mathbf{a}, \mathbf{b}, \mathbf{p}, \mathbf{q}$ in serif font standing for sequences of propositional variables or constants of certain lengths. Define the logic $\mathbf{CPL}(\vee)$ as the extension of \mathbf{CPL} by adding relevant disjunction \vee , and negation \neg is allowed to occur only in front of classical formulas, that is, formulas of $\mathbf{CPL}(\vee)$ are formed by the grammar:

$$\phi ::= p \mid \perp \mid \top \mid \neg\alpha \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \vee \phi)$$

where α stands for an arbitrary classical formula. Similarly for the extensions $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ of \mathbf{CPL} obtained by adding the inclusion atoms $\mathbf{a} \subseteq \mathbf{b}$ and anonymity atoms $\mathbf{p}\Upsilon\mathbf{q}$, respectively, where, again, negation \neg is allowed to occur only in front of classical formulas.

Define the team semantics of the new connective and atoms as follows:

- $X \models \phi \vee \psi$ iff $X = \emptyset$ or there exist nonempty subteams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$.
- $X \models \mathbf{a} \subseteq \mathbf{b}$ iff for all $v \in X$, there exists $u \in X$ such that $v(\mathbf{a}) = u(\mathbf{b})$.
- $X \models \mathbf{p}\Upsilon\mathbf{q}$ iff for all $v \in X$, there exists $u \in X$ such that $v(\mathbf{p}) = u(\mathbf{p})$ and $v(\mathbf{q}) \neq u(\mathbf{q})$.

It is easy to see that formulas in $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ are not necessarily downwards closed, and thus not necessarily flat. For instance, for the team X from Table 1, we have that $X \models p \vee q$. However, for the subteam $Y = \{v_2\}$ of X , we have that $Y \not\models p \vee q$, since no nonempty subteam of Y makes q true. The reader can also easily verify that for the team $Z = \{v_2, v_3\}$, we have $Z \models p \subseteq q$ and $Z \models \mathbf{p}\Upsilon\mathbf{q}$, whereas for the subteam $Y = \{v_2\}$ of Z , we have $Y \not\models p \subseteq q$ and $Y \not\models \mathbf{p}\Upsilon\mathbf{q}$. On the other hand, formulas in all these three logics satisfy the union closure property, as well as the empty team property and the locality property.

Lemma 2.5. *Formulas in the logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ satisfy the empty team property, the union closure property and the locality property.*

Proof. The lemma is proved by a straightforward induction on the complexity of formulas ϕ in the logics. We only give the proof details for the logic $\mathbf{CPL}(\vee)$ and for the case $\phi = \psi \vee \chi$. By definition, $\emptyset \models \psi \vee \chi$ trivially holds. For union closure, suppose $X \models \psi \vee \chi$ and $Y \models \psi \vee \chi$. If $X = \emptyset$ or $Y = \emptyset$, then $X \cup Y \models \psi \vee \chi$ trivially holds. Assume now $X, Y \neq \emptyset$. Then there are nonempty teams $X_0, X_1 \subseteq X$ and $Y_0, Y_1 \subseteq Y$ such that $X = X_0 \cup X_1$, $Y = Y_0 \cup Y_1$, $X_0 \models \psi$, $Y_0 \models \psi$, $X_1 \models \chi$ and $Y_1 \models \chi$. By induction hypothesis, we obtain $X_0 \cup Y_0 \models \psi$ and $X_1 \cup Y_1 \models \chi$. Clearly, $X_0 \cup Y_0, X_1 \cup Y_1 \neq \emptyset$ and $X \cup Y = (X_0 \cup X_1) \cup (Y_0 \cup Y_1) = (X_0 \cup Y_0) \cup (X_1 \cup Y_1)$. Hence, we conclude that $X \cup Y \models \psi \vee \chi$.

Lastly, we verify the locality property for $\phi(\mathbf{N}) = \psi \vee \chi$. Suppose X, Y are teams with $\text{dom}(X), \text{dom}(Y) \supseteq \mathbf{N}$ and $X \upharpoonright \mathbf{N} = Y \upharpoonright \mathbf{N}$. If $X = \emptyset$, then $Y = \emptyset$, and we have $X \models \psi \vee \chi$ iff $Y \models \psi \vee \chi$ by the empty team property. Now assume that $X \neq \emptyset$, and thus $Y \neq \emptyset$. If $X \models \psi \vee \chi$, then there exist nonempty teams $X_0, X_1 \subseteq X$ such that $X = X_0 \cup X_1$, $X_0 \models \psi$ and $X_1 \models \chi$. Consider

$$Y_0 = \{v \in Y \mid v \upharpoonright \mathbf{N} \in X_0 \upharpoonright \mathbf{N}\} \quad \text{and} \quad Y_1 = \{v \in Y \mid v \upharpoonright \mathbf{N} \in X_1 \upharpoonright \mathbf{N}\}.$$

Since $X_0, X_1 \neq \emptyset$ and $X \upharpoonright \mathbf{N} = Y \upharpoonright \mathbf{N}$, we must have that $Y_0, Y_1 \neq \emptyset$. It is also not hard to verify that

$$Y_0 \upharpoonright \mathbf{N} = X_0 \upharpoonright \mathbf{N}, \quad Y_1 \upharpoonright \mathbf{N} = X_1 \upharpoonright \mathbf{N}, \quad \text{and} \quad Y_0 \cup Y_1 = Y. \quad (2)$$

Now, by induction hypothesis and the first two equations of (2), we obtain $Y_0 \models \psi(\mathbf{N})$ and $Y_1 \models \chi(\mathbf{N})$. Finally, the last equation of (2) then gives $Y \models \psi \vee \chi$. \square

Note the similarity and difference between the semantic clauses of \vee and \vee : They both state that the team in question can be split into two subteams, each satisfying one of the disjuncts. The relevant disjunction \vee requires in addition that the two subteams must be nonempty, as long as the starting team is nonempty. In particular, when applied to classical formulas, a (flat) disjunction $\alpha \vee \beta$ being true on a team X means that either disjunct is true on each valuation v in the team X *locally*, while a relevant disjunction $\alpha \vee \beta$ being true on the same team X requires, in addition to the local truth of the disjuncts, also that each disjunct is actually true on some valuations (and thus both disjuncts are considered “relevant” for the truth of the disjunction). For an illustration of the two different disjunctions, consider the following two sentences in natural language:

- (a) *The coin lands heads or tails.*
- (b) *Either the moon is made of green cheese or it rained today.*

Consider also the team X_a of the records of certain times of coin tossing, and the team X_b of the meteorological and astronomical reports of a certain year. If we interpret the two disjunctions in (a) and (b) using the disjunction \vee from classical logic (i.e., (a) is understood as $h \vee t$ and (b) as $m \vee r$), the two sentences are then both true in the relevant teams X_a and X_b . If we interpret (a) and (b) using the relevant disjunction, as $h \vee t$ and $m \vee r$ instead, then $m \vee r$ will fail in the team X_b , and $h \vee t$ is possible to fail in the team X_a if the coin is a bias one.

Closely related is another disjunction that we shall call the *global disjunction* $\vee\vee$ (also known in the literature by the name *intuitionistic disjunction* or *Boolean disjunction* or *classical disjunction*), defined as

- $X \models \phi \vee\vee \psi$ iff $X \models \phi$ or $X \models \psi$.

The global disjunction states that either disjunct is true *globally* in the team in question. It is easy to verify that $\phi \vee \psi \equiv \phi \vee\vee \psi \vee\vee (\phi \vee \psi)$. The global disjunction does not, however, preserve union closure, as e.g., $p \vee\vee q$ is clearly not closed under unions. We thus do not consider the global disjunction in this paper.

Another related logical constant is the unary operator ∇ , called the *might modality*, whose team semantics is defined as

- $X \models \nabla \phi$ iff $X = \emptyset$ or there exists a nonempty subteam $Y \subseteq X$ such that $Y \models \phi$.

It is easy to verify that the might modality preserves union closure. It was also observed in [25] that the relevant disjunction \vee and the might operator ∇ are inter-definable, as $\nabla \phi \equiv \phi \vee \top$ and $\phi \vee \psi \equiv (\phi \vee \psi) \wedge \nabla \phi \wedge \nabla \psi$. We say that a team-based logic L_1 is expressively weaker than another team-based logic L_2 , denoted as $L_1 \leq L_2$, if for every L_1 -formula ϕ , there exists an L_2 -formula ψ such that $\phi \equiv \psi$. If both $L_1 \leq L_2$ and $L_2 \leq L_1$, then we write $L_1 \equiv L_2$ and say that L_1 and L_2 are *expressively equivalent*. Clearly, $\mathbf{CPL}(\vee) \equiv \mathbf{CPL}(\nabla)$ for the extension $\mathbf{CPL}(\nabla)$ of \mathbf{CPL} with the unary might modality ∇ . In this paper we are more interested in binary connectives and atoms; the might modality is left for future research.

An inclusion atom $\mathbf{a} \subseteq \mathbf{b}$ as we defined can take the two atoms \perp and \top as arguments. It thus has a more relaxed syntax than the standard one in the literature where the arguments a_i, b_i can only be propositional variables. Let us point out that the standard version of inclusion logic $\mathbf{CPL}(\subseteq)$ with this standard syntax of inclusion atoms is strictly weaker in expressive power than our version of $\mathbf{CPL}(\subseteq)$. To see why, consider our inclusion atom $\top \subseteq p$ in one variable. To express this inclusion atom in the standard version of $\mathbf{CPL}(\subseteq)$, by the locality property, it is sufficient to consider formulas in the only variable p . Modulo equivalence, the only such classical formulas are $\top, \perp, p, \neg p$, and the only inclusion atom with merely the propositional variable p is $p \subseteq p$, which is equivalent to \top . All these formulas are flat, and thus are not equivalent to the non-flat inclusion atom $\top \subseteq p$. We will show in this paper that our version of $\mathbf{CPL}(\subseteq)$ is actually

expressively complete. Our proof applies essentially the same argument as in [25] in the context of modal inclusion logic, which has an even more relaxed syntax for inclusion atoms $a \subseteq b$, for which the arguments a_i, b_i are allowed to be arbitrary classical formulas. Such inclusion atoms are known in the literature as the *extended inclusion atom*. It follows from [25] that propositional inclusion logic $\mathbf{CPL}(\subseteq)$ with extended inclusion atoms is expressively complete, and is thus expressively equivalent to our version of $\mathbf{CPL}(\subseteq)$ with the relatively less general inclusion atoms as we defined. Our choice of the syntax of $\mathbf{CPL}(\subseteq)$, or of inclusion atoms, thus enables us to obtain an expressively complete logic with minimal modification to the standard syntax of $\mathbf{CPL}(\subseteq)$.

The anonymity atom $\mathbf{p}\Upsilon\mathbf{q}$, also known as *non-dependence atom*, states an extreme case of the negation of the functional dependence between \mathbf{p} and \mathbf{q} : For every valuation v in the team X in question, there is a witness u in X with respect to v that witnesses the failure of the said functional dependence. Recently Väänänen [35] also used this atom to express the property “ \mathbf{p} is anonymized with respect to \mathbf{q} ”, and hence the name. Such defined anonymity atoms $\mathbf{p}\Upsilon\mathbf{q}$ also correspond exactly to the *afunctional dependencies* studied in database theory (see e.g., [3,4]). We write $\neq(\mathbf{p})$ for the anonymity atom $\langle \rangle\Upsilon\mathbf{p}$ whose left component is the empty sequence $\langle \rangle$, and call such an atom *inconstancy atom*. Clearly, the semantics clause of the inconstancy atom $\neq(\mathbf{p})$ reduces to

- $X \models \neq(\mathbf{p})$ iff either $X = \emptyset$ or there exist $v, u \in X$ such that $v(\mathbf{p}) \neq u(\mathbf{p})$.

Intuitively, $\neq(\mathbf{p})$ states that the sequence \mathbf{p} of propositional variables does not have a constant value in the team in question. It is easy to verify that inconstancy atoms with multiple arguments are definable in terms of those with single arguments:

$$\neq(p_1 \dots p_n) \equiv \neq(p_1) \vee \dots \vee \neq(p_n).$$

In addition, inconstancy atoms with single arguments are easily definable in terms of relevant disjunction: $\neq(p) \equiv p \vee \neg p$.

Recall that an atom of a dual flavor is the *constancy atom* $\equiv(p)$ which states that p has a constant value in the team:

- $X \models \equiv(p)$ iff for all $v, u \in X$, $v(p) = u(p)$.

Constancy atoms are clearly downwards closed. Dually, inconstancy atoms λ are clearly *upwards closed*, meaning that $X \models \lambda$ and $Y \supseteq X$ imply $Y \models \lambda$. Upwards closure clearly implies union closure.

We call the inclusion atoms $x \subseteq a$ with $x_i \in \{\perp, \top\}$ for each i *primitive inclusion atoms*. For instance, $\top \perp \subseteq pq$ and $\perp \top \subseteq \top p$ are primitive inclusion atoms, whereas $p \subseteq q$, $q \subseteq \top$ are not. Interestingly, primitive inclusion atoms are also upwards closed. Denote by $\mathbf{CPL}(\neq(\cdot))$ and $\mathbf{CPL}(\subseteq_0)$ the logics extended from \mathbf{CPL} by adding, respectively, inconstancy atoms with single arguments and primitive inclusion atoms. Arbitrary formulas in these sublogics of $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ are, however, not in general upwards closed, as, e.g., already the propositional variable p is not upward closed.

3. Expressive completeness and normal form

In this section, we study the expressive power of the logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\Upsilon)$ and $\mathbf{CPL}(\neq(\cdot))$ we introduced. The logic $\mathbf{CPL}(\vee)$ was proved in [38] to be expressively complete. Building on this result, we show that the logics $\mathbf{CPL}(\neq(\cdot))$ and $\mathbf{CPL}(\Upsilon)$ are also expressively complete. It was proved in [25] that modal inclusion logic with extended inclusion atoms (which allow arbitrary classical formulas to occur as arguments) is expressively complete, from which it follows essentially that propositional inclusion

logic $\mathbf{CPL}(\subseteq)$ and its variant $\mathbf{CPL}(\subseteq_0)$ are also expressively complete. We recast here the argument of [25] in our propositional setting to give a detailed proof of the expressive completeness of $\mathbf{CPL}(\subseteq_0)$ and $\mathbf{CPL}(\subseteq)$. As an immediate corollary, all these five logics are thus expressively equivalent. From the proof of these expressive completeness results, we will also obtain normal forms for formulas in these logics. The (disjunctive) normal form of $\mathbf{CPL}(\vee)$ has already been introduced in [38]. We will discuss the normal forms for the other logics, as well as normal forms for inclusion atoms and anonymity atoms. The normal forms of these logics will play a crucial role in the axiomatization of the logics in Section 4. The expressive completeness is also required for obtaining the interpolation theorem to be discussed in Section 5.

Let us start by giving formal definitions of the relevant terminologies. A *team property* \mathbb{P} is a set of teams over certain domain $\mathbb{N} \subseteq \mathbf{Prop}$. For any formula $\phi(\mathbb{N})$ in the language of any of the above logics, the set

$$\llbracket \phi \rrbracket_{\mathbb{N}} = \{X \subseteq 2^{\mathbb{N}} : X \models \phi\}$$

of \mathbb{N} -teams that satisfy $\phi(\mathbb{N})$ is a team property (over \mathbb{N}), where $2^{\mathbb{N}}$ stands for the set of all valuations $v : \mathbb{N} \cup \{\perp, \top\} \rightarrow 2 = \{0, 1\}$. Clearly, by locality, for any two formulas $\phi(\mathbb{N})$ and $\psi(\mathbb{N})$, $\llbracket \phi \rrbracket_{\mathbb{N}} = \llbracket \psi \rrbracket_{\mathbb{N}}$ implies $\phi \equiv \psi$. For any formula ϕ in any of the five union closed logics we consider (i.e., $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\Upsilon)$ and $\mathbf{CPL}(\neq(\cdot))$), the set $\llbracket \phi \rrbracket_{\mathbb{N}}$ clearly contains the empty team \emptyset , and is closed under unions, i.e., $X, Y \in \llbracket \phi \rrbracket_{\mathbb{N}}$ implies $X \cup Y \in \llbracket \phi \rrbracket_{\mathbb{N}}$. Let \mathbb{P} be a collection of team properties over some domains. For $\mathbb{N} \subseteq \mathbf{Prop}$, we write

$$\mathbb{P}_{\mathbb{N}} = \{\mathbb{P} \in \mathbb{P} : \mathbb{P} \text{ is a set of } \mathbb{N}\text{-teams}\}$$

for the class of team properties over \mathbb{N} that are in \mathbb{P} . We are now ready to give the definition for the notion of expressive completeness.

Definition 3.1 (*expressive completeness*). We say that a team-based logic \mathbf{L} characterizes \mathbb{P} , or \mathbf{L} is *expressively complete* in \mathbb{P} , if for every set $\mathbb{N} \subseteq \mathbf{Prop}$ of propositional variables,

$$\mathbb{P}_{\mathbb{N}} = \{\llbracket \phi \rrbracket_{\mathbb{N}} : \phi \text{ is an } \mathbf{L}\text{-formula with } \mathbf{Prop}(\phi) = \mathbb{N}\}.$$

That is, if \mathbf{L} is expressively complete in \mathbb{P} , then for every \mathbb{N} , every \mathbf{L} -formula $\phi(\mathbb{N})$ defines a team property $\llbracket \phi \rrbracket_{\mathbb{N}}$ that belongs to $\mathbb{P}_{\mathbb{N}} \subseteq \mathbb{P}$, and conversely, every team property $\mathbb{P} \in \mathbb{P}_{\mathbb{N}}$ is definable by some formula $\phi(\mathbb{N})$ in \mathbf{L} . Clearly, if both \mathbf{L}_1 and \mathbf{L}_2 are expressively complete in some class \mathbb{P} , then \mathbf{L}_1 and \mathbf{L}_2 are expressively equivalent, i.e., $\mathbf{L}_1 \equiv \mathbf{L}_2$, since for every \mathbf{L}_1 -formula $\phi(\mathbb{N})$, the team property $\llbracket \phi \rrbracket_{\mathbb{N}} \in \mathbb{P}_{\mathbb{N}}$ is definable by some \mathbf{L}_2 -formula $\psi(\mathbb{N})$, namely $\llbracket \phi \rrbracket_{\mathbb{N}} = \llbracket \psi \rrbracket_{\mathbb{N}}$ or $\phi \equiv \psi$; and vice versa.

Let \mathbb{F} denote the collection of all flat team properties (i.e., properties \mathbb{P} satisfying $X \in \mathbb{P}$ iff $\{v\} \in \mathbb{P}$ for all $v \in X$). It was proved in [38] that classical propositional logic \mathbf{CPL} is expressively complete in \mathbb{F} . We now recall briefly also the proof of this fact from [38], as a crucial formula Θ_X used in this proof will play an important role in the main theorem (Theorem 3.7) in this section.

Theorem 3.2 ([38]). \mathbf{CPL} is expressively complete in \mathbb{F} .

Proof. We only give a sketch of the proof. For any \mathbf{CPL} -formula $\alpha(\mathbb{N})$, since α is flat, we have $\llbracket \alpha \rrbracket_{\mathbb{N}} \in \mathbb{F}$. Conversely, for any \mathbb{N} -team property $\mathbb{P} \in \mathbb{F}_{\mathbb{N}}$, putting $X = \bigcup \mathbb{P}$, we show that $\llbracket \Theta_X \rrbracket_{\mathbb{N}} = \mathbb{P}$, where

$$\Theta_X := \bigvee_{v \in X} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)})$$

with $v(i)$ short for $v(p_i)$, $p_i^1 := p_i$, $p_i^0 := \neg p_i$ and $\bigvee \emptyset := \perp$. Indeed, each disjunct in the formula Θ_X clearly defines a valuation v in the team X , in the sense that for any \mathbb{N} -team Y ,

$$Y \models p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)} \iff Y \subseteq \{v\}. \quad (3)$$

It then follows that for any \mathbf{N} -team Y ,

$$Y \models \Theta_X \iff Y = \bigcup_{v \in X} Y_v \text{ for some } Y_v \subseteq \{v\} \iff Y \subseteq X. \quad (4)$$

Thus, since \mathbf{P} is flat, we have $Y \models \Theta_X$ iff $Y \subseteq X = \bigcup \mathbf{P} \in \mathbf{P}$ iff $Y \in \mathbf{P}$. \square

The expressive completeness of **CPL** immediately implies the following characterization of classical formulas in the logics we consider in this paper.

Corollary 3.3. *A formula of any of the logics **CPL**(\vee), **CPL**(\subseteq), **CPL**(\subseteq_0), **CPL**(Υ) and **CPL**($\neq(\cdot)$) is flat iff it is equivalent to a classical formula.*

Proof. The right to left direction is obvious. For the other direction, let $\phi(\mathbf{N})$ be a flat formula in any of the five logics. The team property $\llbracket \phi \rrbracket_{\mathbf{N}}$ is clearly also flat. Then, by the expressive completeness of **CPL**, there is a classical formula $\alpha(\mathbf{N})$ such that $\llbracket \phi \rrbracket_{\mathbf{N}} = \llbracket \alpha \rrbracket_{\mathbf{N}}$, which implies $\phi \equiv \alpha$ by locality. \square

Denote by $\mathbb{P}^{\dot{\cup}}$ the collection of all union closed team properties which contain the empty team. Clearly, for any formula $\phi(\mathbf{N})$ in the union closed team logics we introduced, $\llbracket \phi \rrbracket_{\mathbf{N}} \in \mathbb{P}^{\dot{\cup}}$. We will see in this section that the other direction holds for all five union closed team logics we consider in the paper. First, let us recall that this result was proved for **CPL**(\vee) already in [38].

Theorem 3.4 ([38]). ***CPL**(\vee) is expressively complete in $\mathbb{P}^{\dot{\cup}}$*

Proof. See [38] for the detailed proof. We now give a sketch of the proof. It suffices to show that for every team property $\mathbf{P} \in \mathbb{P}^{\dot{\cup}}$, we can find a **CPL**(\vee) formula $\phi(\mathbf{N})$ such that $\mathbf{P} = \llbracket \phi \rrbracket_{\mathbf{N}}$. For any team X (i.e., a set of valuations v) with $\text{dom}(X) = \mathbf{N} = \{p_1, \dots, p_n\}$, define a **CPL**(\vee)-formula

$$\Psi_X := \bigvee_{v \in X} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)}), \quad (5)$$

where, again, $v(i)$ is short for $v(p_i)$, $p_i^1 := p_i$, $p_i^0 = \neg p_i$ and $\bigvee \emptyset := \perp$. Since (3) holds for each v -disjunct in the formula Ψ_X , the formula Ψ_X characterizes the team X modulo the empty team, in the sense that for any \mathbf{N} -team Y

$$Y \models \Psi_X \iff Y = \emptyset, \text{ or } Y = \bigcup_{v \in X} Y_v \text{ and } Y_v = \{v\} \iff Y = \emptyset \text{ or } Y = X. \quad (6)$$

Finally, we show $\mathbf{P} = \llbracket \bigvee_{X \in \mathbf{P}} \Psi_X \rrbracket_{\mathbf{N}}$. The inclusion “ \subseteq ” is immediate. For the other inclusion “ \supseteq ”, for any \mathbf{N} -team Y , if $Y \models \bigvee_{X \in \mathbf{P}} \Psi_X$, then $Y = \bigcup_{X \in \mathbf{P}} Y_X$ for some Y_X satisfying $Y_X \models \Psi_X$. It follows from (6) that each $Y_X = X$ or $Y_X = \emptyset$. Thus, $Y = \bigcup \mathbf{P}'$ for some (possibly empty) subclass $\mathbf{P}' \subseteq \mathbf{P}$. Now, $Y \in \mathbf{P}$ follows from the fact that \mathbf{P} contains the empty team \emptyset and is closed under unions. \square

The proof of the above theorem gives rise to a disjunctive normal form for formulas in **CPL**(\vee).

Corollary 3.5 ([38], Normal Form). *Every **CPL**(\vee) formula $\phi(\mathbf{N})$ is equivalent to a formula for the form $\bigvee_{X \in \mathcal{X}} \Psi_X$ for some collection \mathcal{X} of \mathbf{N} -teams.*

Proof. Put $P = \llbracket \phi \rrbracket_N$. Since $P \in \mathbb{P}_N^{\bar{\cup}}$, by the proof of Theorem 3.4, we have $\llbracket \phi \rrbracket_N = P = \llbracket \bigvee_{X \in P} \Psi_X \rrbracket_N$, and thus $\phi \equiv \bigvee_{X \in P} \Psi_X$ follows from locality. \square

Another immediate corollary of the expressive completeness of $\mathbf{CPL}(\vee)$ (in $\mathbb{P}^{\bar{\cup}}$) is that all union closed team logics with the empty team property (including $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\Upsilon)$ and $\mathbf{CPL}(\neq(\cdot))$) are compact, as $\mathbf{CPL}(\vee)$ was shown in [38] to be compact.

Corollary 3.6 (Compactness). *Let L be a team-based logic that is closed under unions and has the empty team property. Then L is compact, that is, for any set $\Gamma \cup \{\phi\}$ of L -formulas, if $\Gamma \models \phi$, then there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \phi$. In particular, $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\Upsilon)$ and $\mathbf{CPL}(\neq(\cdot))$ are compact.*

Proof. Since L is closed under unions and has the empty team property, for every L -formula $\psi(N) \in \Gamma \cup \{\phi\}$, the N -team property $\llbracket \psi \rrbracket_N$ that ψ defines belongs to $\mathbb{P}_N^{\bar{\cup}}$. Since $\mathbf{CPL}(\vee)$ is expressively complete in $\mathbb{P}^{\bar{\cup}}$, there exists a $\mathbf{CPL}(\vee)$ -formula $\psi^*(N)$ such that $\llbracket \psi^* \rrbracket_N = \llbracket \psi \rrbracket_N$ and thus $\psi \equiv \psi^*$. Since $\Gamma \models \phi$, we have $\Gamma^* \models \phi^*$, where $\Gamma^* = \{\gamma^* \mid \gamma \in \Gamma\}$. Now, since $\mathbf{CPL}(\vee)$ is compact (by [38]), there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0^* \models \phi^*$, which then gives $\Gamma_0 \models \phi$. \square

Building on Theorem 3.4 and its proof, we now show that all the other union closed team logics $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\neq(\cdot))$ and $\mathbf{CPL}(\Upsilon)$ are also expressively complete in $\mathbb{P}^{\bar{\cup}}$. The proofs below for the expressive completeness of $\mathbf{CPL}(\subseteq_0)$ and $\mathbf{CPL}(\subseteq)$ are essentially an adaptation of a similar one for the expressive completeness of modal inclusion logic given in [25].

Theorem 3.7. *The logics $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\neq(\cdot))$ and $\mathbf{CPL}(\Upsilon)$ are all expressively complete in $\mathbb{P}^{\bar{\cup}}$. In particular, $\mathbf{CPL}(\vee) \equiv \mathbf{CPL}(\subseteq_0) \equiv \mathbf{CPL}(\subseteq) \equiv \mathbf{CPL}(\neq(\cdot)) \equiv \mathbf{CPL}(\Upsilon)$.*

Proof. Since $\mathbf{CPL}(\subseteq_0)$ and $\mathbf{CPL}(\neq(\cdot))$ are sublogics of $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$, respectively, we only need to show that $\mathbf{CPL}(\subseteq_0)$ and $\mathbf{CPL}(\neq(\cdot))$ are expressively complete in $\mathbb{P}^{\bar{\cup}}$. That is to show that for every set N of propositional variables, for every N -team property $P \in \mathbb{P}^{\bar{\cup}}$, there is a formula $\phi(N)$ in $\mathbf{CPL}(\subseteq_0)$ and in $\mathbf{CPL}(\neq(\cdot))$ such that $P = \llbracket \phi \rrbracket_N$. Now, by the proof of Theorem 3.4, we know that $P = \llbracket \bigvee_{X \in P} \Psi_X \rrbracket_N$, where each Ψ_X is a $\mathbf{CPL}(\vee)$ -formula. We would thus be done if for every N -team $X \in P$, we can find a $\mathbf{CPL}(\subseteq_0)$ formula $\eta(N)$ and a $\mathbf{CPL}(\neq(\cdot))$ -formula $\psi_X(N)$ such that $\eta_X \equiv \Psi_X \equiv \psi_X$.

We first construct for every N -team X , the formula $\eta_X(N)$ in $\mathbf{CPL}(\subseteq_0)$ such that $\psi_X \equiv \Psi_X$. If $X = \emptyset$, then by definition, $\Psi_X = \perp$ and we can take $\eta_X = \perp$. Now, assume that $X \neq \emptyset$. First, consider the formula

$$\Phi_X := \bigwedge_{v \in X} \underline{v(1)} \dots \underline{v(n)} \subseteq p_1 \dots p_n,$$

where, once again, $v(i)$ is short for $v(p_i)$, $\underline{0} := \perp$ and $\underline{1} := \top$. Each conjunct in Φ_X ensures (modulo the empty team) that the valuation v must belong to the team Y in question, in the sense that for any N -team Y ,

$$Y \models \underline{v(1)} \dots \underline{v(n)} \subseteq p_1 \dots p_n \iff Y = \emptyset \text{ or } v \in Y. \tag{7}$$

To see why (7) holds, note that if $Y = \emptyset$, then $Y \models \underline{v(1)} \dots \underline{v(n)} \subseteq p_1 \dots p_n$. If $Y \neq \emptyset$, then $Y \models \underline{v(1)} \dots \underline{v(n)} \subseteq p_1 \dots p_n$, iff there exists $u \in Y$ such that

$$u(p_i) = u(\underline{v(i)}) = v(p_i) \text{ for all } 1 \leq i \leq n,$$

iff $v \in Y$. Now, by (7), we obtain that

$$Y \models \Phi_X \iff Y = \emptyset, \text{ or } v \in Y \text{ for all } v \in X \iff Y = \emptyset \text{ or } X \subseteq Y. \quad (8)$$

Recall that we have defined a formula Θ_X in the proof of Theorem 3.2, and the equivalence (4) holds for the formula Θ_X . Define now $\eta_X = \Theta_X \wedge \Phi_X$.¹ By (4) and (8), we obtain that for any N-team Y ,

$$Y \models \psi_X \iff Y \subseteq X, \text{ and } Y = \emptyset \text{ or } X \subseteq Y \iff Y = \emptyset \text{ or } Y = X.$$

By (6) and locality, this implies that $\eta_X \equiv \Psi_X$, as we wanted.

Next, for every N-team X , we define a formula ψ_X in $\mathbf{CPL}(\neq(\cdot))$ such that $\psi_X \equiv \Psi_X$. Let $\mathbf{N} = \{p_1, \dots, p_n\}$. We will define, inductively, for each subset $\mathbf{K} = \{p_1, \dots, p_k\}$ ($k \leq n$) of propositional variables from \mathbf{N} a $\mathbf{CPL}(\neq(\cdot))$ -formula $\psi_X^{\mathbf{K}}$ such that $\psi_X^{\mathbf{K}} \equiv \Psi_X^{\mathbf{K}}$, where

$$\Psi_X^{\mathbf{K}} = \bigvee_{v \in X} (p_1^{v(1)} \wedge \dots \wedge p_k^{v(k)}).$$

The above formula $\Psi_X^{\mathbf{K}}$ can be viewed as an approximation of the formula Ψ_X , and obviously $\Psi_X = \Psi_X^{\mathbf{N}}$. The required formula ψ_X can thus be defined as $\psi_X = \psi_X^{\mathbf{N}}$. We now give the definition of $\psi_X^{\mathbf{K}}$ for every N-team X by induction on $|\mathbf{K}|$.

If $\mathbf{K} = \{p_1\}$, we have

$$\Psi_X^{\mathbf{K}} = \left(\bigvee_{v \in X^+} p_1 \right) \vee \left(\bigvee_{v \in X^-} \neg p_1 \right),$$

where $X^+ = \{v \in X \mid v(p_1) = 1\}$ and $X^- = \{v \in X \mid v(p_1) = 0\}$. If $X^+ = \emptyset$, then $\Psi_X^{\mathbf{K}} = \bigvee_{v \in X^-} \neg p_1 \equiv \neg p_1$, and we define $\psi_X^{\mathbf{K}} = \neg p_1$. If $X^- = \emptyset$, then $\Psi_X^{\mathbf{K}} \equiv p_1$ and we define $\psi_X^{\mathbf{K}} = p_1$. If $X^+, X^- \neq \emptyset$, then $\Psi_X^{\mathbf{K}} = p_1 \vee \neg p_1 \equiv (p_1 \vee \neg p_1) \wedge \neq(p_1)$, and we define $\psi_X^{\mathbf{K}} = (p_1 \vee \neg p_1) \wedge \neq(p_1)$.

If $\mathbf{K} = \{p_1, \dots, p_{m+1}\} = \mathbf{K}_0 \cup \{p_{m+1}\}$, let $X^+ = \{v \in X \mid v(p_{m+1}) = 1\}$ and $X^- = \{v \in X \mid v(p_{m+1}) = 0\}$. If $X^+ = \emptyset$, then

$$\begin{aligned} \Psi_X^{\mathbf{K}} &= \bigvee_{v \in X^-} (p_1^{v(1)} \wedge \dots \wedge p_m^{v(m)} \wedge \neg p_{m+1}) \\ &\equiv \left(\bigvee_{v \in X^-} (p_1^{v(1)} \wedge \dots \wedge p_m^{v(m)}) \right) \wedge \neg p_{m+1} \\ &\equiv \psi_{X^-}^{\mathbf{K}_0} \wedge \neg p_{m+1}, \end{aligned} \quad (\text{by induction hypothesis})$$

and we define $\psi_X^{\mathbf{K}} = \psi_{X^-}^{\mathbf{K}_0} \wedge \neg p_{m+1}$. Similarly, if $X^- = \emptyset$, then $\Psi_X^{\mathbf{K}} \equiv \psi_{X^+}^{\mathbf{K}_0} \wedge p_{m+1}$, and we define $\psi_X^{\mathbf{K}} = \psi_{X^+}^{\mathbf{K}_0} \wedge p_{m+1}$. If $X^+, X^- \neq \emptyset$, by induction hypothesis we have that

$$\Psi_X^{\mathbf{K}} \equiv (\psi_{X^+}^{\mathbf{K}_0} \wedge p_{m+1}) \vee (\psi_{X^-}^{\mathbf{K}_0} \wedge \neg p_{m+1}) \equiv ((\psi_{X^+}^{\mathbf{K}_0} \wedge p_{m+1}) \vee (\psi_{X^-}^{\mathbf{K}_0} \wedge \neg p_{m+1})) \wedge \neq(p_{m+1}).$$

Then, we define $\psi_X^{\mathbf{K}} = ((\psi_{X^+}^{\mathbf{K}_0} \wedge p_{m+1}) \vee (\psi_{X^-}^{\mathbf{K}_0} \wedge \neg p_{m+1})) \wedge \neq(p_{m+1})$. \square

The above expressive completeness proof gives rise to normal forms of formulas in the logics $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\neq(\cdot))$ and $\mathbf{CPL}(\Upsilon)$.

¹ This $\mathbf{CPL}(\subseteq_0)$ -formula is essentially adapted from a very similar and slightly more complex modal formula in [25], which uses the more general extended inclusion atoms.

Corollary 3.8 (Normal form).

- (i) Every formula $\phi(\mathbf{N})$ in $\mathbf{CPL}(\subseteq_0)$ and $\mathbf{CPL}(\subseteq)$ is equivalent to a formula of the form $\bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$ for some collection \mathcal{X} of \mathbf{N} -teams.
- (ii) Every formula $\phi(\mathbf{N})$ in $\mathbf{CPL}(\neq(\cdot))$ and $\mathbf{CPL}(\Upsilon)$ is equivalent to a formula of the form $\bigvee_{X \in \mathcal{X}} \psi_X$ for some collection \mathcal{X} of \mathbf{N} -teams.

Let us end this section with some further discussion on the normal forms of the logics $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$, or the normal forms of inclusion atoms and anonymity atoms in the logics in particular. First, note that $\mathbf{CPL}(\subseteq)$ -formulas in the normal form $\bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$ contain primitive inclusion atoms $x \subseteq p$ only. Similarly, $\mathbf{CPL}(\Upsilon)$ -formulas in the normal form $\bigvee_{X \in \mathcal{X}} \psi_X$ contain inconstancy atoms $\neq(p)$ with single arguments only. It then follows that arbitrary (nontrivial) inclusion atoms $\mathbf{b} \subseteq \mathbf{c}$ and anonymity atoms $\mathbf{p}\Upsilon\mathbf{q}$ are definable in terms of regular primitive inclusion atoms $x \subseteq p$ and inconstancy atoms $\neq(p)$ with single arguments, respectively. We provide direct definitions for these atoms in terms of the corresponding simpler atoms in the following.

To simplify notations, we write $v(i)$ for $v(p_i)$. The notation p^1 or p^\top stand for p , and p^0 or p^\perp stand for $\neg p$; similarly, \top^\perp and \perp^\top both stand for \perp . Hereafter, we reserve the letters p, q, r, \dots with or without subscripts for propositional variables, the letters x, y, z, \dots with or without subscripts for the constants \perp and \top , and the letters a, b, c, \dots with or without subscripts for either propositional variables or \perp or \top . The serif font letters $\mathbf{p}, \mathbf{q}, \dots$ will stand for sequences of propositional variables of certain lengths; similarly for $\mathbf{x}, \mathbf{y}, \dots$ in serif font, and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ in serif font. We write $|\mathbf{a}|$ for the length of the sequence \mathbf{a} . For two sequences $\mathbf{a} = \langle a_1, \dots, a_n \rangle$ and $\mathbf{x} = \langle x_1, \dots, x_n \rangle$ with each $a_i \in \mathbf{Prop} \cup \{\top, \perp\}$ and $x_i \in \{\top, \perp\}$, we write $\mathbf{a}^\mathbf{x}$ for $a_1^{x_1} \wedge \dots \wedge a_n^{x_n}$.

Proposition 3.9.

- (i) $\mathbf{p}\Upsilon\mathbf{q}_1 \dots \mathbf{q}_m \equiv \mathbf{p}\Upsilon\mathbf{q}_1 \vee \dots \vee \mathbf{p}\Upsilon\mathbf{q}_m$ and $\mathbf{p}\Upsilon\langle \rangle \equiv \perp$,
- (ii) $\mathbf{p}_1 \dots \mathbf{p}_k \Upsilon \mathbf{q} \equiv \bigvee_{v \in 2^K} (p_1^{v(1)} \wedge \dots \wedge p_k^{v(k)} \wedge \neq(\mathbf{q}))$, where $K = \{p_1, \dots, p_k\}$,
- (iii) $\mathbf{p}\Upsilon\langle \rangle \vdash \perp$

Proof. Easy. \square

Proposition 3.10.

- (i) $\bigwedge_{\mathbf{x} \in \{\top, \perp\}^{|\mathbf{a}|}} (\mathbf{a}^\mathbf{x} \rightarrow \mathbf{x} \subseteq \mathbf{b}) \equiv \mathbf{a} \subseteq \mathbf{b}$.
- (ii) If $|\mathbf{a}| = |\mathbf{c}|$ and $|\mathbf{b}| = |\mathbf{d}|$, then $\mathbf{a} \subseteq \mathbf{a} \equiv \top$,

$$\mathbf{a}\top\mathbf{b} \subseteq \mathbf{c}\top\mathbf{d} \equiv \mathbf{a}\mathbf{b} \subseteq \mathbf{c}\mathbf{d} \equiv \mathbf{a}\perp\mathbf{b} \subseteq \mathbf{c}\perp\mathbf{d} \quad \text{and} \quad \mathbf{a}\top\mathbf{b} \subseteq \mathbf{c}\perp\mathbf{d} \equiv \perp \equiv \mathbf{a}\perp\mathbf{b} \subseteq \mathbf{c}\top\mathbf{d}.$$

Proof. Item (ii) is easy to prove. We only give the detailed proof for item (i). For the left to right direction, suppose $X \models \neg \mathbf{a}^\mathbf{x} \vee \mathbf{x} \subseteq \mathbf{b}$ for all $\mathbf{x} \in \{\top, \perp\}^{|\mathbf{a}|}$. Let $v \in X$ and let $\mathbf{x} \in \{\top, \perp\}^{|\mathbf{a}|}$ be such that $v(\mathbf{x}) = v(\mathbf{a})$. By assumption, there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \neg \mathbf{a}^\mathbf{x}$ and $Z \models \mathbf{x} \subseteq \mathbf{b}$. Clearly, $v \notin Y$, which means $v \in Z$. Then, there exists $u \in Z$ such that $u(\mathbf{b}) = v(\mathbf{x}) = v(\mathbf{a})$. Hence, $X \models \mathbf{a} \subseteq \mathbf{b}$.

Conversely, suppose $X \models \mathbf{a} \subseteq \mathbf{b}$. We show that $X \models \mathbf{a}^\mathbf{x} \rightarrow \mathbf{x} \subseteq \mathbf{b}$ for any $\mathbf{x} \in \{\top, \perp\}^{|\mathbf{a}|}$. Let

$$X_\mathbf{x} = \{v \in X \mid v(\mathbf{a}) \neq v(\mathbf{x})\}.$$

Clearly, $X_x \models \neg a^x$. If $X_x = X$, then $X \models \neg a^x$ and thus $X \models \neg a^x \vee x \subseteq b$ as required. Otherwise, $X \setminus X_x \neq \emptyset$. We show that $X \models x \subseteq b$, which would suffice. Let $u \in X$ and pick any $v \in X \setminus X_x$. We have that $u(x) = v(x) = v(a)$. Since $X \models a \subseteq b$, there exists $w \in X$ such that $w(b) = v(a) = u(x)$, as required. \square

Moreover, we show that primitive inclusion atoms can actually be defined in terms of such atoms of arity 1, i.e., inclusion atoms of the form $\top \subseteq p$ or $\perp \subseteq q$.

Proposition 3.11. $xy \subseteq pq \equiv x \subseteq p \wedge ((y \subseteq q \wedge p^x) \vee \neg p^x)$.

Proof. For the left to right direction, suppose $X \models xy \subseteq pq$. Then clearly $X \models x \subseteq p$ as well. It remains to show $X \models (y \subseteq q \wedge p^x) \vee \neg p^x$. Define

$$Y = \{v \in X \mid v(p) = v(x)\}$$

and $Z = X \setminus Y$. Clearly, $Y \cup Z = X$, $Y \models p^x$ and $Z \models \neg p^x$. We now show that $Y \models y \subseteq q$. For any $v \in Y$, we have $v(p) = v(x)$ by definition. Also, since $Y \subseteq X \models xy \subseteq pq$, there exists $u \in X$ such that $v(x) = u(p)$ and $v(y) = u(q)$. From $u(p) = v(x) = u(x)$, we conclude that $u \in Y$. Hence, $Y \models y \subseteq q$.

Conversely, suppose $X \models x \subseteq p \wedge ((y \subseteq q \wedge p^x) \vee \neg p^x)$. For any $v \in X$, since $X \models x \subseteq p$, there exists $u \in X$ such that $u(p) = v(x) = u(x)$. On the other hand, since $X \models (y \subseteq q \wedge p^x) \vee \neg p^x$, there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models y \subseteq q \wedge p^x$ and $Z \models \neg p^x$. Clearly, $u \notin Z$ and so $u \in Y \neq \emptyset$. Now, since $Y \models y \subseteq q$, there exists $w \in Y$ such that $w(q) = u(y) = v(y)$. From $Y \models p^x$, we also conclude that $w(p) = w(x) = v(x)$. Hence $X \models xy \subseteq pq$. \square

Putting these results together, we obtain that an arbitrary nontrivial inclusion atom $a \subseteq b$ can be decomposed into very simple inclusion atoms of the form $\perp \subseteq p$ and $\top \subseteq p$.

Corollary 3.12. *An arbitrary inclusion atom $a \subseteq b$ is either equivalent to \perp or \top , or it can be expressed in terms of primitive inclusion atoms of arity 1.*

Proof. Given an arbitrary inclusion atom $a \subseteq b$, by applying Proposition 3.10(i), we obtain an equivalent formula ϕ in which all inclusion atoms are primitive and of the form $x \subseteq b$. Next, apply Proposition 3.10(ii) to remove the constants \perp and \top from the sequence b on the right-hand side of all inclusion atoms $x \subseteq b$. Some of the resulting formulas are equivalent to \perp or \top . Finally, we apply Proposition 3.11 exhaustively to turn every primitive inclusion atom $x \subseteq p$ obtained in the previous step into an equivalent formula in which inclusion atoms are all of arity 1. \square

It is interesting to note that Corollary 3.12 also gives rise to a direct definition of inclusion atoms $a \subseteq b$ in terms of relevant disjunction \vee : First transform $a \subseteq b$ to a formula that contains primitive inclusion atoms $\perp \subseteq p$ or $\top \subseteq p$ of arity 1. Then observe that $\perp \subseteq p \equiv \neg p \vee \top$ and $\top \subseteq p \equiv p \vee \top$.

4. Axiomatizations

In this section, we axiomatize the union closed team logics $\mathbf{CPL}(\vee)$ and $\mathbf{CPL}(\subseteq)$ as well as $\mathbf{CPL}(\subseteq_0)$. We define systems of natural deduction for these logics and prove the completeness theorem for these systems. Our argument for the completeness proof makes heavy and essential use of the disjunctive normal form of the logics given in Corollary 3.5 and Corollary 3.8 from the previous section. Such a technique is a generalization of the similar ones developed in [37,38] for propositional team logics. As seen in Corollary 3.8(ii), the disjunctive normal form for the logic $\mathbf{CPL}(\Upsilon)$ is substantially more complex, our approach thus does not suit well for $\mathbf{CPL}(\Upsilon)$. We leave the axiomatization for $\mathbf{CPL}(\Upsilon)$ for future work.

Table 2
Rules for constants and classical connectives.

$\frac{}{\top} \top I$	$\frac{[\alpha] \quad D}{\perp} \neg I$ (1)	$\frac{D_0 \quad D_1 \quad \alpha \quad \neg \alpha}{\phi} \neg E$	$\frac{[\neg \alpha] \quad D}{\perp} RAA$ (1)
$\frac{D_0 \quad D_1 \quad \phi \quad \psi}{\phi \wedge \psi} \wedge I$	$\frac{D \quad \phi \wedge \psi}{\phi} \wedge E$	$\frac{D \quad \phi \wedge \psi}{\psi} \wedge E$	
$\frac{D \quad \phi}{\phi \vee \psi} \vee I$	$\frac{D \quad \phi}{\psi \vee \phi} \vee I$	$\frac{D \quad [\phi] \quad D_0 \quad [\psi] \quad D_1 \quad \chi \quad \chi}{\phi \vee \psi \quad \chi} \vee E$ (2)	

(1) The undischarged assumptions^a in the derivation D contain classical formulas only.
 (2) The undischarged assumptions in the derivations D_0 and D_1 contain classical formulas only.

^a When this rule is applied, the (open) assumption α at the top of the branch of the derivation will be deemed as *closed* and thus discharged (from the set of open assumptions of the derivation). All the remaining assumptions in the derivation are regarded as “undischarged” assumptions.

Table 3
Rules for \vee and interactions.

$\frac{D_0 \quad D_1 \quad \phi \quad \psi}{\phi \vee \psi} \vee I$	$\frac{D_0 \quad D_1 \quad \phi \vee \psi \quad \chi}{\chi \vee \psi} \vee Mon$ (1)
$\frac{D \quad \phi \vee \psi}{\psi \vee \phi} \vee Com$	$\frac{D \quad \phi \vee (\psi \vee \chi)}{(\phi \vee \psi) \vee \chi} \vee Ass$
$\frac{D \quad [\phi] \quad D_0 \quad [\psi] \quad D_1 \quad [\phi \vee \psi] \quad D_2 \quad \chi}{\phi \vee \psi \quad \chi} \vee E_{\vee}$	$\frac{D \quad \phi \vee \psi}{\phi \vee \psi} \vee \vee Tr$
$\frac{D \quad \phi \vee \perp}{\psi} \vee \perp E$	$\frac{D \quad \phi \vee (\psi \vee \chi)}{(\phi \vee \psi) \vee (\phi \vee \chi)} Dstr \vee \vee$

(1) The undischarged assumptions in the derivation D_1 contain classical formulas only.

4.1. CPL(\vee)

In this subsection, we define a system of natural deduction for **CPL**(\vee) and prove the completeness theorem.

Let us first present the system. We adopt the standard conventions of systems of natural deduction; readers who are not familiar with natural deduction systems are referred to, e.g., [33,9]. For example, the letter D (with or without subscripts) in the following definition stands for an arbitrary derivation.

Definition 4.1. The system of **CPL**(\vee) consists of all rules given in Tables 2 and 3, where α ranges over classical formulas only.

We write $\Gamma \vdash_{\mathbf{CPL}(\vee)} \phi$ or simply $\Gamma \vdash \phi$ if ϕ is derivable from the set Γ of formulas by applying the rules of the system of **CPL**(\vee). We write simply $\phi \vdash \psi$ for $\{\phi\} \vdash \psi$. Two formulas ϕ and ψ are said to be *provably equivalent*, written $\phi \dashv\vdash \psi$, if both $\phi \vdash \psi$ and $\psi \vdash \phi$.

Our system does not admit uniform substitution, as, e.g., the rules for negation \neg apply to classical formulas only. When restricted to classical formulas the system coincides with the system of classical propo-

sitional logic. In particular, the disjunction \vee and the negation \neg admit the usual elimination rule $\vee E$, introduction rule $\neg I$ and *reductio ad absurdum* rule RAA, respectively, under the condition that the undischarged assumptions in the derivations involved contain classical formulas only. It is interesting to note that the soundness of the disjunction elimination rule $\vee E$ is a nontrivial feature of the union closed team logics, especially because this same rule is actually not sound for the propositional team logics with the downwards closure property or without any closure property (see [37,38]).

The rules for the relevant disjunction \vee are peculiar. Unsurprisingly, the usual introduction rule $(\phi/\phi \vee \psi)$ is not sound for the relevant disjunction \vee , because, e.g., obviously $\phi \not\models \phi \vee \perp$. While the relevant disjunction introduction rule $\vee I$ we have in the system is considerably weak, the relevant disjunction \vee does admit the usual elimination rule under the same side condition as that for $\vee E$. We will show in the next proposition that such restricted elimination rule is derivable from the monotonicity rule of the relevant disjunction \vee Mon. The rules \vee Com and \vee Ass are added in the system also in order to compensate the weakness of the nonstandard introduction and elimination rule for \vee . The two rules $\vee E_{\vee}$ and $\vee \vee Tr$ together simulate the evident equivalence $\phi \vee \psi \equiv \phi \vee \psi \vee (\phi \vee \psi)$ with $\vee E_{\vee}$ simulating the left to right direction and $\vee \vee Tr$ simulating the right to left direction. The rule $\vee \perp E$ characterizes the fact that each disjunct in a relevant disjunction has to be satisfied by a nonempty team (if the starting team is not empty). The distributive rule $Dstr \vee \vee$ is actually invertible, as we will show in the next proposition that lists also some other useful clauses for our system.

Proposition 4.2.

- (i) Let Δ be a set of classical formulas. If $\Delta, \phi \vdash \chi$ and $\Delta, \psi \vdash \chi$, then $\Delta, \phi \vee \psi \vdash \chi$.
- (ii) $\phi \vee \phi \dashv\vdash \phi$.
- (iii) $\phi \vee (\psi \vee \chi) \dashv\vdash (\phi \vee \psi) \vee (\phi \vee \chi)$.
- (iv) $\perp \vdash \phi$.
- (v) $\phi \vee \alpha, \neg \alpha \vdash \psi$

Proof. For item (i), since $\Delta, \phi \vdash \chi$, we derive by \vee Mon that $\Delta, \phi \vee \psi \vdash \chi \vee \psi$. Similarly, from $\Delta, \psi \vdash \chi$ we derive $\Delta, \chi \vee \psi \vdash \chi \vee \chi$. Thus, $\Delta, \phi \vee \psi \vdash \chi \vee \chi$. By $\vee \vee Tr$ and $\vee E$ we derive $\chi \vee \chi \vdash \chi \vee \chi \vdash \chi$. Hence we conclude $\Delta, \phi \vee \psi \vdash \chi$.

For item (ii), the left to right direction is a special case of item (i), and the right to left direction follows from $\vee I$.

For item (iii), the left to right direction follows from $Dstr \vee \vee$. For the other direction, by $\vee E$ it suffices to prove $\phi \vee \psi \vdash \phi \vee (\psi \vee \chi)$ and $\phi \vee \chi \vdash \phi \vee (\psi \vee \chi)$. But these follow easily from $\vee I$ and \vee Mon.

Item (iv) is proved by the usual argument by applying $\neg I$ and $\neg E$.

For item (v), by $\neg E$, we have $\alpha, \neg \alpha \vdash \perp$, which gives $\phi \vee \alpha, \neg \alpha \vdash \phi \vee \perp$ by \vee Mon. Furthermore, we have $\phi \vee \perp \vdash \psi$ by $\vee \perp E$. Hence $\phi \vee \alpha, \neg \alpha \vdash \psi$. \square

Theorem 4.3 (Soundness). For any set $\Gamma \cup \{\phi\}$ of $CPL(\vee)$ -formulas, we have that $\Gamma \vdash \phi \implies \Gamma \models \phi$.

Proof. The soundness of the rules in Table 2 and the first four rules in Table 3 are easy to verify. The soundness of $\vee E_{\vee}$ and $\vee \vee Tr$ follow from the fact that $X \models \phi \vee \psi$ if and only if $X \models \phi$ or $X \models \psi$ or $X \models \phi \vee \psi$. The rule $\vee \perp E$ is also clearly sound, since the assumption $\phi \vee \perp$ is satisfied only by the empty team, which satisfies every formula ψ . We only verify the soundness of the rule $Dstr \vee \vee$.

Assuming that $X \models \phi \vee (\psi \vee \chi)$ for some nonempty team X we show that $X \models (\phi \vee \psi) \vee (\phi \vee \chi)$. By the assumption, there are nonempty teams $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi \vee \chi$. The latter implies that there are subteams $W, U \subseteq Z$ such that $Z = W \cup U$, $W \models \psi$ and $U \models \chi$. If $W = \emptyset$, then $U \neq \emptyset$ as $Z \neq \emptyset$. In this case $X = Y \cup U \models \phi \vee \chi$ and thus $X \models (\phi \vee \psi) \vee (\phi \vee \chi)$. Symmetrically, if $U = \emptyset$, then

$W \neq \emptyset$ and $X \models (\phi \vee \psi) \vee (\phi \vee \chi)$ as well. Lastly, if $W, U \neq \emptyset$, then $Y \cup W \models \phi \vee \psi$ and $Y \cup U \models \phi \vee \chi$. Thus, we have that $(Y \cup W) \cup (Y \cup U) = X \models (\phi \vee \psi) \vee (\phi \vee \chi)$. \square

The rest of this section is devoted to the proof of the completeness theorem of our system. We will show that every **CPL**(\vee)-formula is provably equivalent to a formula in the normal form $\bigvee_{X \in \mathcal{X}} \Psi_X$ given by Corollary 3.5.

Lemma 4.4. *Let $\mathbf{N} = \{p_1, \dots, p_n\}$. Every **CPL**(\vee)-formula $\phi(\mathbf{N})$ is provably equivalent to a formula of the form*

$$\bigvee_{X \in \mathcal{X}} \Psi_X, \text{ where } \Psi_X = \bigvee_{v \in X} (p_1^{v(1)} \wedge \dots \wedge p_n^{v(n)}), \tag{9}$$

and \mathcal{X} is a finite set of \mathbf{N} -teams.

We shall postpone the technical proof of this above lemma till the end of this section. The completeness then follows from some derivations in the system that uses the specific syntactic shape of the normal form. One important step in this proof is to obtain from the semantic side that the entailment $\bigvee_{X \in \mathcal{X}} \Psi_X \models \bigvee_{Y \in \mathcal{Y}} \Psi_Y$ of two formulas in the disjunctive normal form implies that each team X in \mathcal{X} is identical to the union of all teams from a subcollection of \mathcal{Y} . We now prove this last semantic property and also its converse direction.

Lemma 4.5. *For any nonempty finite sets \mathcal{X} and \mathcal{Y} of \mathbf{N} -teams, the following are equivalent:*

- (i) $\bigvee_{X \in \mathcal{X}} \Psi_X \models \bigvee_{Y \in \mathcal{Y}} \Psi_Y$.
- (ii) For each $X \in \mathcal{X}$, there exists $\mathcal{Y}_X \subseteq \mathcal{Y}$ such that $X = \bigcup \mathcal{Y}_X$.

Proof. (i) \implies (ii): For each $X_0 \in \mathcal{X}$, we have $X_0 \models \Psi_{X_0}$ by Equation (6). Thus $X_0 \models \bigvee_{X \in \mathcal{X}} \Psi_X$, which by (i) implies that $X_0 \models \bigvee_{Y \in \mathcal{Y}} \Psi_Y$. This means that for each $Y \in \mathcal{Y}$, there exists $Z_Y \subseteq X_0$ such that $X_0 = \bigcup_{Y \in \mathcal{Y}} Z_Y$ and each $Z_Y \models \Psi_Y$. The latter implies, by Equation (6) again, that $Z_Y = Y$ or $Z_Y = \emptyset$. Thus we obtain $X_0 = \bigcup_{Y \in \mathcal{Y}_X} Y$ for some $\mathcal{Y}_X \subseteq \mathcal{Y}$.

(ii) \implies (i): Suppose Z is any \mathbf{N} -team satisfying $Z \models \bigvee_{X \in \mathcal{X}} \Psi_X$. Then, by Equation (6), there exists $\mathcal{X}' \subseteq \mathcal{X}$ such that $Z = \bigcup_{X \in \mathcal{X}'} X$. By (ii), for each $X \in \mathcal{X}'$, there exists $\mathcal{Y}_X \subseteq \mathcal{Y}$ such that $X = \bigcup \mathcal{Y}_X$. Thus, we have that $Z = \bigcup_{X \in \mathcal{X}'} \bigcup \mathcal{Y}_X = \bigcup \mathcal{Y}'$, where $\mathcal{Y}' = \bigcup_{X \in \mathcal{X}'} \mathcal{Y}_X \subseteq \mathcal{Y}$. Hence, $Z \models \bigvee_{Y \in \mathcal{Y}'} \Psi_Y$ by Equation (6) again, thereby $Z \models \bigvee_{Y \in \mathcal{Y}} \Psi_Y$. \square

Next, we prove a technical lemma that concerns an interesting interaction between the two disjunctions \vee and \vee .

Lemma 4.6. *Let \mathcal{Y} be a finite set with each $Y \in \mathcal{Y}$ being a finite set of indices. Then*

$$\bigvee_{i \in \bigcup \mathcal{Y}} \phi_i \vdash \bigvee_{Y \in \mathcal{Y}} \bigvee_{i \in Y} \phi_i.$$

Proof. Note that elements in \mathcal{Y} are not necessarily disjoint. We thus first derive by applying $\vee I$, \vee Ass and \vee Com that

$$\bigvee_{i \in \bigcup \mathcal{Y}} \phi_i \vdash \bigvee_{Y \in \mathcal{Y}} \bigvee_{i \in Y} \phi_i.$$

Next, by repeatedly applying $\forall\text{Tr}$ and $\forall\text{E}$, we derive that

$$\bigvee_{Y \in \mathcal{Y}} \bigvee_{i \in Y} \phi_i \vdash \bigvee_{Y \in \mathcal{Y}} \bigvee_{i \in Y} \phi_i,$$

which then implies the desired clause. \square

Now, we give the proof of the completeness theorem of our system.

Theorem 4.7 (Completeness). *For any set $\Gamma \cup \{\phi\}$ of $\mathbf{CPL}(\forall)$ -formulas, we have that $\Gamma \models \phi \iff \Gamma \vdash \phi$.*

Proof. It suffices to prove the left to right direction. Suppose $\Gamma \models \phi$. By the compactness theorem (Corollary 3.6) we may assume that Γ is a finite set. Let $\psi = \bigwedge \Gamma$, and suppose ϕ, ψ are formulas in $\mathbf{N} = \{p_1, \dots, p_n\}$. By Lemma 4.4,

$$\psi \dashv\vdash \bigvee_{X \in \mathcal{X}} \Psi_X \quad \text{and} \quad \phi \dashv\vdash \bigvee_{Y \in \mathcal{Y}} \Psi_Y$$

for some finite sets \mathcal{X} and \mathcal{Y} of \mathbf{N} -teams. The soundness theorem implies that

$$\bigvee_{X \in \mathcal{X}} \Psi_X \models \bigvee_{Y \in \mathcal{Y}} \Psi_Y. \quad (10)$$

If $\mathcal{X} = \emptyset$, then $\psi \dashv\vdash \perp$, and we derive $\psi \vdash \phi$ by Proposition 4.2(iv). If $\mathcal{Y} = \emptyset$, then $\phi \dashv\vdash \perp$. In view of (10), it must be that $\mathcal{X} = \emptyset$ as well. Thus $\psi \dashv\vdash \perp$ giving that $\psi \vdash \phi$.

If $\mathcal{X}, \mathcal{Y} \neq \emptyset$, then by Lemma 4.5, for each $X \in \mathcal{X}$ we have that $X = \bigcup \mathcal{Y}_X$ for some $\mathcal{Y}_X \subseteq \mathcal{Y}$. Thus, we derive $\Psi_X \vdash \bigvee_{Y \in \mathcal{Y}_X} \Psi_Y \vdash \bigvee_{Y \in \mathcal{Y}} \Psi_Y$ by Lemma 4.6 and $\forall\text{I}$. Finally, we obtain $\bigvee_{X \in \mathcal{X}} \Psi_X \vdash \bigvee_{Y \in \mathcal{Y}} \Psi_Y$ by $\forall\text{E}$, thereby $\psi \vdash \phi$. \square

Before we supply the proof of Lemma 4.4, let us first give an example of the applications of our system of $\mathbf{CPL}(\forall)$, in the context of the implication problem of anonymity atoms (or afunctional dependencies). Anonymity atoms $\mathbf{p}\Upsilon\mathbf{q}$ are definable in $\mathbf{CPL}(\forall)$, and recall from Proposition 3.9 the concrete definitions. The implication problem of anonymity atoms (i.e., the problem of whether $\Gamma \models \phi$ for a set $\Gamma \cup \{\phi\}$ of anonymity atoms) is shown in [35] to be completely axiomatized by the rules listed in the next example (read the clauses in the example as rules). We now show that these rules are derivable in the system of $\mathbf{CPL}(\forall)$ (via the translation given in Proposition 3.9).

Example 4.8. Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{p}', \mathbf{q}', \mathbf{r}'$ be sequences of propositional variables.

- (i) $\mathbf{pqr}\Upsilon\mathbf{p}'\mathbf{q}'\mathbf{r}' \vdash \mathbf{qpr}\Upsilon\mathbf{p}'\mathbf{q}'\mathbf{r}' \wedge \mathbf{pqr}\Upsilon\mathbf{q}'\mathbf{p}'\mathbf{r}'$ (permutation)
- (ii) $\mathbf{pq}\Upsilon\mathbf{r} \vdash \mathbf{p}\Upsilon\mathbf{rs}$ (monotonicity)
- (iii) $\mathbf{pq}\Upsilon\mathbf{rq} \vdash \mathbf{pq}\Upsilon\mathbf{r}$ (weakening)
- (iv) $\mathbf{p}\Upsilon\langle \rangle \vdash \perp$

Proof. Items (i) and (iv) are clear. For item (ii), noting that $\mathbf{p}\Upsilon\mathbf{rs} := \mathbf{p}\Upsilon\mathbf{r} \vee \mathbf{p}\Upsilon\mathbf{s}$, by $\forall\text{I}$ it suffices to show $\mathbf{pq}\Upsilon\mathbf{r} \vdash \mathbf{p}\Upsilon\mathbf{r}$. Let $\mathbf{r} = \langle r_1 \dots r_n \rangle$. By $\forall\text{E}$ and $\forall\text{I}$, it further suffices to show that for each $1 \leq i \leq n$, $\mathbf{pq}\Upsilon\mathbf{r}_i \vdash \mathbf{p}\Upsilon\mathbf{r}_i$, which is

$$\bigvee_{v \in 2^K, u \in 2^M} (p_1^{v(1)} \wedge \dots \wedge p_k^{v(k)} \wedge q_1^{u(1)} \wedge \dots \wedge q_m^{u(m)} \wedge \Upsilon\mathbf{r}_i) \vdash \bigvee_{v \in 2^K} (p_1^{v(1)} \wedge \dots \wedge p_k^{v(k)} \wedge \Upsilon\mathbf{r}_i),$$

where $K = \{p_1, \dots, p_k\}$ and $M = \{q_1, \dots, q_m\}$. But this follows easily from $\wedge E$.

For item (iii), we show $\text{pq}\Upsilon r \vee \text{pq}\Upsilon q \vdash \text{pq}\Upsilon r$. By $\vee E$ and Proposition 4.2(iv), it suffices to show that $\text{pq}\Upsilon q \vdash \perp$. Since $\text{pq}\Upsilon q = \text{pq}\Upsilon q_1 \vee \dots \vee \text{pq}\Upsilon q_m$, it suffices to show that $\text{pq}\Upsilon q_i \vdash \perp$ for each $1 \leq i \leq m$, i.e.,

$$\bigvee_{v \in 2^K, u \in 2^M} (p_1^{v(1)} \wedge \dots \wedge p_k^{v(k)} \wedge q_1^{u(1)} \wedge \dots \wedge q_m^{u(m)} \wedge (q_i \vee \neg q_i)) \vdash \perp. \quad (11)$$

Now, by Proposition 4.2(v) we have that $q_i^{u(i)} \wedge (q_i \vee \neg q_i) \vdash \perp$ for each $u \in 2^M$. Thus, in (11) each disjunct of the formula on the left-hand-side of the turnstile implies \perp , from which we conclude that (11) holds by $\vee E$. \square

Finally, we give the proof of the normal form lemma, Lemma 4.4, which requires a few further lemmas. The first one shows that a generalized version of the rule $\vee E_\vee$ with disjunctions of multiple formulas is derivable in our system.

Lemma 4.9. *For any nonempty index set I , we have that*

$$\Gamma, \bigvee_{i \in I} \phi_i \vdash \chi \iff \Gamma, \bigvee_{i \in I_0} \phi_i \vdash \chi \text{ for all nonempty sets } I_0 \subseteq I.$$

Proof. The direction from left to right follows easily from $\vee \vee Tr$ and $\vee I$. We prove the other direction by induction on $|I|$. The case $|I| = 1$ is trivial. Now, if $I = J \cup \{k\}$, then we have that

$$\begin{aligned} & \forall I_0 \subseteq I, I_0 \neq \emptyset : \Gamma, \bigvee_{i \in I_0} \phi_i \vdash \chi \\ \implies & \forall J_0 \subseteq J, J_0 \neq \emptyset : \Gamma, \bigvee_{i \in J_0} \phi_i \vdash \chi \ \& \ \Gamma, \phi_k \vdash \chi \ \& \ \forall J_1 \subseteq J, J_1 \neq \emptyset : \Gamma, \phi_k \vee \bigvee_{j \in J_1} \phi_j \vdash \chi \\ \implies & \forall J_0 \subseteq J, J_0 \neq \emptyset : \Gamma, \bigvee_{i \in J_0} \phi_i \vdash \chi \ \& \ \Gamma, \phi_k \vdash \chi \ \& \ \forall J_1 \subseteq J, J_1 \neq \emptyset : \Gamma, \bigvee_{j \in J_1} (\phi_k \vee \phi_j) \vdash \chi \\ & \quad \text{(since } \bigvee_{j \in J_1} (\phi_k \vee \phi_j) \vdash \phi_k \vee \bigvee_{j \in J_1} \phi_j \text{ by } \vee \text{ Ass, } \vee \text{ Com and Proposition 4.2(ii))} \\ \implies & \Gamma, \bigvee_{i \in J} \phi_i \vdash \chi \ \& \ \Gamma, \phi_k \vdash \chi \ \& \ \Gamma, \bigvee_{j \in J} (\phi_j \vee \phi_k) \vdash \chi \quad \text{(induction hypothesis)} \\ \implies & \Gamma, \bigvee_{i \in J} \phi_i \vdash \chi \ \& \ \Gamma, \phi_k \vdash \chi \ \& \ \Gamma, (\bigvee_{j \in J} \phi_j) \vee \phi_k \vdash \chi \quad \text{(Dstr } \vee \vee) \\ \implies & \Gamma, (\bigvee_{i \in J} \phi_i) \vee \phi_k \vdash \chi \quad \text{(} \vee E_\vee) \\ \implies & \Gamma, \bigvee_{i \in I} \phi_i \vdash \chi. \quad \square \quad \text{(since } I = J \cup \{k\}) \end{aligned}$$

Recall that the formula Ψ_X in the normal form defines the team X modulo the empty team in the sense of Equation (6) from the proof of Theorem 3.4 in Section 3. Therefore for distinct teams X and Y , the two formulas Ψ_X and Ψ_Y are contradictory to each other. We now prove this fact in our system, and the proof of Lemma 4.4 follows.

Lemma 4.10. *If X and Y are two distinct N -teams, then $\Psi_X, \Psi_Y \vdash \phi$.*

Proof. Let $N = \{p_1, \dots, p_n\}$. If $X = \emptyset$ or $Y = \emptyset$, then $\Psi_X = \perp$ or $\Psi_Y = \perp$, and $\perp \vdash \phi$ follows from Proposition 4.2(iv). Now assume that $X, Y \neq \emptyset$. Since $X \neq Y$, there exists (w.l.o.g.) some $v \in X \setminus Y$.

By Proposition 4.2(v), we have $\Psi_{\{v\}} \vee \Psi_{X \setminus \{v\}}, \neg \Psi_{\{v\}} \vdash \phi$, i.e., $\Psi_X, \neg \Psi_{\{v\}} \vdash \phi$. To derive $\Psi_X, \Psi_Y \vdash \phi$ it then suffices to derive $\Psi_Y \vdash \neg \Psi_{\{v\}}$. By Proposition 4.2(i), this reduces to showing that for each $u \in Y$, $\Psi_{\{u\}} \vdash \neg \Psi_{\{v\}}$, which is equivalent (by the usual rules for classical formulas) to

$$p_1^{u(1)}, \dots, p_n^{u(n)} \vdash \neg p_1^{v(1)} \vee \dots \vee \neg p_n^{v(n)}.$$

We have $u \neq v$ by the assumption, thus $p_i^{u(i)} = \neg p_i^{v(i)}$ for some $1 \leq i \leq n$, from which and $\vee I$ the above clause follows. \square

Proof of Lemma 4.4. We prove the lemma by induction on the complexity of ϕ . If $\phi(p_1, \dots, p_n) = p_i$, then we can prove by the usual rules of classical formulas (which are all present or derivable in our system) that

$$\begin{aligned} p_i &\Vdash \bigvee_{v \in 2^{\mathbb{N} \setminus \{p_i\}}} (p_1^{v(1)} \wedge \dots \wedge p_{i-1}^{v(i-1)} \wedge p_i \wedge p_{i+1}^{v(i+1)} \wedge \dots \wedge p_n^{v(n)}) \\ &\Vdash \bigvee_{\{u\} \in \mathcal{X}_i} \Psi_{\{u\}}, \text{ where } \mathcal{X}_i = \{\{u\} \mid u \in 2^{\mathbb{N}}, u(i) = 1\}. \end{aligned}$$

If $\phi = \perp$, then trivially $\perp \Vdash \bigvee \emptyset = \perp$. If $\phi = \top$, we derive similarly by the rules of classical formulas that

$$\top \Vdash \bigvee_{v \in 2^{\mathbb{N}}} (p_1^{v(1)} \wedge \dots \wedge p_n^{v(n)}) \Vdash \bigvee_{\{v\} \in \mathcal{X}_\top} \Psi_{\{v\}}, \text{ where } \mathcal{X}_\top = \{\{v\} \mid v \in 2^{\mathbb{N}}\}.$$

Suppose $\alpha(\mathbb{N})$ is a classical formula, and $\alpha \Vdash \bigvee_{X \in \mathcal{X}} \Psi_X$. We show that $\neg \alpha \Vdash \bigvee_{v \in 2^{\mathbb{N}} \setminus \bigcup \mathcal{X}} \Psi_{\{v\}}$. It is sufficient to prove that $\bigvee_{X \in \mathcal{X}} \Psi_X \Vdash \bigvee_{v \in \bigcup \mathcal{X}} \Psi_{\{v\}}$, which then implies, by the rules of negation \neg and other usual rules of classical formulas, that $\neg \alpha \Vdash \neg \bigvee_{v \in \bigcup \mathcal{X}} \Psi_{\{v\}} \Vdash \bigvee_{v \in 2^{\mathbb{N}} \setminus \bigcup \mathcal{X}} \Psi_{\{v\}}$. Now, we first have by the soundness theorem that $\alpha \Vdash \bigvee_{X \in \mathcal{X}} \Psi_X$ implies that $\alpha \equiv \bigvee_{X \in \mathcal{X}} \Psi_X$. Then, observe that for each $v \in \bigcup \mathcal{X}$, $\{v\} \in \mathcal{X}$. Indeed, by Equation (6) in Section 3, it is easy to see that $\bigcup \mathcal{X} \models \bigvee_{X \in \mathcal{X}} \Psi_X$. Since the classical formula α is flat, we further have that $\{v\} \models \bigvee_{X \in \mathcal{X}} \Psi_X$, which by Equation (6) again implies that $\{v\} = X_0$ for some $X_0 \in \mathcal{X}$, namely $\{v\} \in \mathcal{X}$.

Thus, we derive $\Psi_{\{v\}} \vdash \bigvee_{X \in \mathcal{X}} \Psi_X$ by $\vee I$. Hence we obtain $\bigvee_{v \in \bigcup \mathcal{X}} \Psi_{\{v\}} \vdash \bigvee_{X \in \mathcal{X}} \Psi_X$ by $\vee E$. To prove the other direction, for each $X \in \mathcal{X}$, since $X \subseteq \bigcup \mathcal{X}$, we derive by applying $\vee \vee \text{Tr}$ and $\vee I$ that

$$\Psi_X = \bigvee_{u \in X} \Psi_{\{u\}} \vdash \bigvee_{u \in X} \Psi_{\{u\}} \vdash \bigvee_{v \in \bigcup \mathcal{X}} \Psi_{\{v\}}.$$

Thus, we conclude that $\bigvee_{X \in \mathcal{X}} \Psi_X \vdash \bigvee_{v \in \bigcup \mathcal{X}} \Psi_{\{v\}}$ by applying $\vee E$.

Suppose $\psi(\mathbb{N})$ and $\chi(\mathbb{N})$ satisfy $\psi \Vdash \bigvee_{X \in \mathcal{X}} \Psi_X$ and $\chi \Vdash \bigvee_{Y \in \mathcal{Y}} \Psi_Y$, for some finite sets \mathcal{X} and \mathcal{Y} of \mathbb{N} -teams. The case $\phi = \psi \vee \chi$ is clear. If $\phi = \psi \vee \chi$, and $\mathcal{X} = \emptyset$ or $\mathcal{Y} = \emptyset$, i.e., $\psi \Vdash \perp$ or $\chi \Vdash \perp$, then we derive $\psi \vee \chi \Vdash \perp = \bigvee \emptyset$ by $\vee \text{Mon}$, $\vee \perp E$ and Proposition 4.2(iv). If $\mathcal{X}, \mathcal{Y} \neq \emptyset$, we show that $\psi \vee \chi \Vdash \bigvee_{X \in \mathcal{X}, Y \in \mathcal{Y}} \Psi_{X \cup Y}$. For the left to right direction, we have that

$$\begin{aligned} \psi \vee \chi &\vdash \left(\bigvee_{X \in \mathcal{X}} \Psi_X \right) \vee \left(\bigvee_{Y \in \mathcal{Y}} \Psi_Y \right) && \text{(by induction hypothesis and } \vee \text{Mon)} \\ &\vdash \bigvee_{X \in \mathcal{X}} \left(\Psi_X \vee \left(\bigvee_{Y \in \mathcal{Y}} \Psi_Y \right) \right) && \text{(Dstr } \vee \vee) \\ &\vdash \bigvee_{X \in \mathcal{X}} \bigvee_{Y \in \mathcal{Y}} (\Psi_X \vee \Psi_Y) && \text{(Dstr } \vee \vee e) \end{aligned}$$

$$\vdash \bigvee_{X \in \mathcal{X}, Y \in \mathcal{Y}} \Psi_{X \cup Y}. \quad (\text{apply Prop. 4.2(ii) and } \forall E \text{ for the case } X = Y)$$

The other direction is proved similarly using $\forall I$ and Proposition 4.2(iii).

If $\phi = \psi \wedge \chi$, and $\mathcal{X} = \emptyset$ or $\mathcal{Y} = \emptyset$, i.e., $\psi \dashv\vdash \perp$ or $\chi \dashv\vdash \perp$, then we derive $\psi \wedge \chi \dashv\vdash \perp = \bigvee \emptyset$ by $\wedge E$ and Proposition 4.2(iv). If $\mathcal{X}, \mathcal{Y} \neq \emptyset$, we show that $\psi \wedge \chi \dashv\vdash \bigvee_{Z \in \mathcal{Z}} \Psi_Z$, where

$$\mathcal{Z} = \{ \bigcup \mathcal{X}' \mid \mathcal{X}' \subseteq \mathcal{X} \text{ and } \bigcup \mathcal{X}' = \bigcup \mathcal{Y}' \text{ for some } \mathcal{Y}' \subseteq \mathcal{Y} \}.$$

For the right to left direction, by $\forall E$ it suffices to derive $\Psi_Z \vdash \psi \wedge \chi$ for each $Z = \bigcup \mathcal{X}' = \bigcup \mathcal{Y}' \in \mathcal{Z}$, where $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$. By Lemma 4.6, we have that $\Psi_Z \vdash \bigvee_{X \in \mathcal{X}'} \Psi_X$. Further, by $\forall I$ and the induction hypothesis we derive $\bigvee_{X \in \mathcal{X}'} \Psi_X \vdash \bigvee_{X \in \mathcal{X}} \Psi_X \vdash \psi$. Hence, $\Psi_Z \vdash \psi$. The fact $\Psi_Z \vdash \chi$ is proved similarly.

For the left to right direction, by induction hypothesis and Lemma 4.9 it suffices to prove that for each nonempty $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$,

$$\bigvee_{X \in \mathcal{X}'} \Psi_X, \bigvee_{Y \in \mathcal{Y}'} \Psi_Y \vdash \bigvee_{Z \in \mathcal{Z}} \Psi_Z.$$

Note that elements in \mathcal{X}' and in \mathcal{Y}' may not be disjoint. So by Proposition 4.2(ii) we further reduce showing the above clause to showing $\Psi_{\bigcup \mathcal{X}'}, \Psi_{\bigcup \mathcal{Y}'} \vdash \bigvee_{Z \in \mathcal{Z}} \Psi_Z$. But now, if $\bigcup \mathcal{X}' \neq \bigcup \mathcal{Y}'$, the desired clause follows simply from Lemma 4.10. Otherwise, if $\bigcup \mathcal{X}' = \bigcup \mathcal{Y}' \in \mathcal{Z}$, then we have $\Psi_{\bigcup \mathcal{X}'} \vdash \bigvee_{Z \in \mathcal{Z}} \Psi_Z$ by $\forall I$. \square

4.2. $\mathbf{CPL}(\subseteq_0)$

In this subsection, we axiomatize the sublogic $\mathbf{CPL}(\subseteq_0)$ of $\mathbf{CPL}(\subseteq)$, by introducing a sound and complete system of natural deduction. Recall that $\mathbf{CPL}(\subseteq_0)$ contains inclusion atoms of primitive form $x \subseteq a$ with $x_i \in \{\top, \perp\}$ only. The system of $\mathbf{CPL}(\subseteq)$ will be introduced in the next subsection as an extension of the one for $\mathbf{CPL}(\subseteq_0)$. The proof of the completeness theorem for the system of $\mathbf{CPL}(\subseteq_0)$ applies essentially the same argument (via normal form) as that in the previous subsection for $\mathbf{CPL}(\forall)$. Since the normal form of $\mathbf{CPL}(\subseteq_0)$ (given in Corollary 3.8(i)) is more complex, the proofs in this subsection will involve more steps. Let us start, again, by presenting the deduction system.

Definition 4.11. The system of $\mathbf{CPL}(\subseteq_0)$ consists of the rules for constants and connectives in Table 2 and the rules for inclusion atoms in Table 4, where α ranges over classical formulas only, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ (with or without subscripts) are arbitrary (and possibly empty) sequences of elements in $\text{Prop} \cup \{\top, \perp\}$, \mathbf{x} (with or without subscripts) stands for an arbitrary sequence of constants \top and \perp , and the notation $\alpha(\mathbf{a})$ indicates that the propositional variables and constants occurring in α are among \mathbf{a} .

All the rules except for the last two in Table 4 are actually sound also for arbitrary inclusion atoms (that are not necessarily primitive). It was proved in [5] that the implication problem of inclusion dependencies is completely axiomatized by the rules $\subseteq \text{Id}$ and $\subseteq \text{Trs}$ together with the following projection rule:

$$\frac{a_1 \dots a_k \subseteq b_1 \dots b_k}{a_{i_1} \dots a_{i_m} \subseteq b_{i_1} \dots b_{i_m}} \subseteq \text{Proj} \quad (i_1, \dots, i_m \in \{1, \dots, k\}).$$

This rule $\subseteq \text{Proj}$ is easily shown to be equivalent to the three rules $\subseteq \text{Exc}$, $\subseteq \text{Ctr}$, and $\subseteq \text{Wk}$ together in our system.

The inclusion atom compression rule $\subseteq \text{Cmp}$ is a natural generalization of a similar rule introduced in [19] for first-order inclusion atoms. The primitive inclusion atom extension rule $\subseteq_0 \text{Ext}$ in four different forms is

Table 4
Rules for (primitive) inclusion atoms.

$\frac{D}{\frac{abc \subseteq a'b'c'}{bac \subseteq b'a'c'} \subseteq \text{Exc}}$	$\frac{D}{\frac{ab \subseteq cd}{a \subseteq c} \subseteq \text{Ctr}}$	$\frac{D}{\frac{ab \subseteq cd}{aab \subseteq ccd} \subseteq \text{Wk}}$											
$\frac{D_0 \quad D_1}{\frac{a \subseteq b \quad b \subseteq c}{a \subseteq c} \subseteq \text{Trs}}$	$\frac{}{a \subseteq a} \subseteq \text{Id}$	$\frac{D_0 \quad D_1}{\frac{a \subseteq b \quad \alpha(b)}{\alpha(a)} \subseteq \text{Cmp}}$											
$\frac{D}{\frac{a \subseteq b}{\top a \subseteq \top b} \subseteq_0 \text{Ext}}$	$\frac{D}{\frac{a \subseteq b}{\perp a \subseteq \perp b} \subseteq_0 \text{Ext}}$	$\frac{D_0 \quad D_1 \quad p}{\top a \subseteq pb} \subseteq_0 \text{Ext}$	$\frac{D_0 \quad D_1 \quad \neg p}{\perp a \subseteq pb} \subseteq_0 \text{Ext}$										
<table style="width: 100%; border: none;"> <tr> <td style="text-align: center; padding: 5px;">$[\phi]$</td> <td style="text-align: center; padding: 5px;">$[x \subseteq a]$</td> <td style="text-align: center; padding: 5px;">$[\psi]$</td> <td style="text-align: center; padding: 5px;">$[\phi \vee \psi]$</td> <td style="text-align: center; padding: 5px;">$[x \subseteq a]$</td> </tr> <tr> <td colspan="2" style="text-align: center; padding: 5px;"> $\frac{D_0 \quad \dots \quad D_1}{\chi} \vee_{\subseteq_0} E$ </td> <td style="text-align: center; padding: 5px;">χ</td> <td colspan="2" style="text-align: center; padding: 5px;"> $\frac{D_2 \quad \dots \quad D_k}{\chi} \vee_{\subseteq_0} E$ </td> </tr> </table>				$[\phi]$	$[x \subseteq a]$	$[\psi]$	$[\phi \vee \psi]$	$[x \subseteq a]$	$\frac{D_0 \quad \dots \quad D_1}{\chi} \vee_{\subseteq_0} E$		χ	$\frac{D_2 \quad \dots \quad D_k}{\chi} \vee_{\subseteq_0} E$	
$[\phi]$	$[x \subseteq a]$	$[\psi]$	$[\phi \vee \psi]$	$[x \subseteq a]$									
$\frac{D_0 \quad \dots \quad D_1}{\chi} \vee_{\subseteq_0} E$		χ	$\frac{D_2 \quad \dots \quad D_k}{\chi} \vee_{\subseteq_0} E$										
$\frac{D \quad (\phi \wedge x \subseteq a) \vee \psi}{((\phi \vee a_1^{x_1} \vee \dots \vee a_k^{x_k}) \wedge x_1 \subseteq a_1 \wedge \dots \wedge x_k \subseteq a_k) \vee \psi} \subseteq_0 \text{Dst}$													

evidently sound. Note however that a stronger form of the extension rule $a \subseteq b/pa \subseteq pb$ is easily seen to be *not* sound.

The rule $\vee_{\subseteq_0} E$ simulates the entailment

$$(\phi \wedge x \subseteq a) \vee \psi \models (\phi \wedge x \subseteq a) \vee \psi \vee ((\phi \vee \psi) \wedge x \subseteq a), \quad (12)$$

which highlights the fact that in a team X satisfying the formula $(\phi \wedge x \subseteq a) \vee \psi$, if the left disjunct of the formula is satisfied by a nonempty subteam of X , then the primitive inclusion atom $x \subseteq a$ (being upward closed) is true actually in the whole team X . Note that the converse direction of the entailment (12) does not hold, because the third disjunct $(\phi \vee \psi) \wedge x \subseteq a$ of the formula on the right-hand-side does not necessarily imply $(\phi \wedge x \subseteq a) \vee \psi$. Instead, the formula $(\phi \vee \psi) \wedge x \subseteq a$ implies $((\phi \vee a^x) \wedge x \subseteq a) \vee \psi$, as the rule $\subseteq_0 \text{Dst}$ states.

Theorem 4.12 (Soundness). *For any set $\Gamma \cup \{\phi\}$ of $\text{CPL}(\subseteq_0)$ -formulas, we have that $\Gamma \vdash \phi \implies \Gamma \models \phi$.*

Proof. The soundness of the rule $\vee_{\subseteq_0} E$ follows from the entailment (12), which can be easily checked. We now verify the soundness of the rules $\subseteq \text{Cmp}$ and $\subseteq_0 \text{Dst}$. The other rules are easily seen to be sound (for primitive and also arbitrary inclusion atoms).

For $\subseteq \text{Cmp}$, we will verify its soundness for arbitrary inclusion atoms (that are not necessarily primitive). Suppose $X \models a \subseteq b$ and $X \models \alpha(b)$. By Fact 2.2, to show that $X \models \alpha(a)$ it suffices to show $v \models \alpha(a)$ for any $v \in X$. By the assumption, there exists $u \in X$ such that $u(b) = v(a)$. Since $X \models \alpha(b)$ and $\alpha(b)$ is flat, we have that $u \models \alpha(b)$. Now, if all elements in the sequences \mathbf{a}, \mathbf{b} are propositional variables, then $v \models \alpha(a)$ follows from locality. In case some elements in \mathbf{a}, \mathbf{b} are constants \top or \perp , replace these elements with fresh propositional variables to obtain two new sequences \mathbf{a}', \mathbf{b}' of propositional variables. Let v', u' be valuations for \mathbf{a}', \mathbf{b}' that agree with v, u respectively on all propositional variables from \mathbf{a}, \mathbf{b} , and map the fresh propositional variable corresponding to \top to 1 and the fresh propositional variable corresponding to \perp to 0. Clearly, $u'(b') = u(b) = v(a) = v'(a')$. Thus, by properties of classical propositional logic, we have that

$$v \models \alpha(\mathbf{a}) \text{ iff } v' \models \alpha(\mathbf{a}'), \quad \text{and} \quad u \models \alpha(\mathbf{b}) \text{ iff } u' \models \alpha(\mathbf{b}').$$

Now, the fact that $u \models \alpha(\mathbf{b})$ then implies $u' \models \alpha(\mathbf{b}')$, which further implies, by locality, that $v' \models \alpha(\mathbf{a}')$. Thus, $v \models \alpha(\mathbf{a})$ follows.

For $\subseteq_0\text{Dst}$, suppose that $X \models \phi \vee \psi$ and $X \models \mathbf{x}_1 \subseteq \mathbf{a}_1 \wedge \cdots \wedge \mathbf{x}_k \subseteq \mathbf{a}_k$. The latter implies that there are $v_1, \dots, v_k \in X$ such that $v_1(\mathbf{a}_1) = v_1(\mathbf{x}_1), \dots, v_k(\mathbf{a}_k) = v_k(\mathbf{x}_k)$. Thus, $\{v_1\} \models \mathbf{a}_1^{\mathbf{x}_1}, \dots, \{v_k\} \models \mathbf{a}_k^{\mathbf{x}_k}$. On the other hand, there are $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \phi$ and $Z \models \psi$. Clearly, $Y \cup \{v_1, \dots, v_k\} \models \phi \vee \mathbf{a}_1^{\mathbf{x}_1} \vee \cdots \vee \mathbf{a}_k^{\mathbf{x}_k}$ and $Y \cup \{v_1, \dots, v_k\} \models \mathbf{x}_1 \subseteq \mathbf{a}_1 \wedge \cdots \wedge \mathbf{x}_k \subseteq \mathbf{a}_k$. Hence, we conclude that $Y \cup \{v_1, \dots, v_k\} \cup Z = X \models ((\phi \vee \mathbf{a}_1^{\mathbf{x}_1} \vee \cdots \vee \mathbf{a}_k^{\mathbf{x}_k}) \wedge \mathbf{x}_1 \subseteq \mathbf{a}_1 \wedge \cdots \wedge \mathbf{x}_k \subseteq \mathbf{a}_k) \vee \psi$. \square

The proof of the completeness theorem uses a similar normal form argument to that in the previous subsection. The general structure and key ingredients of the proof are the same as in the completeness proof for the system of $\mathbf{CPL}(\vee)$. The crucial step is to show the following lemma that every formula in $\mathbf{CPL}(\subseteq_0)$ is provably equivalent to a formula in the normal form $\bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$ (see also Corollary 3.8(i)). Since the normal form for $\mathbf{CPL}(\subseteq_0)$ is more complex, the proof of this lemma involves more preparation steps. As in the previous section, we will only give the detailed proof of the lemma after we presented the completeness proof (in Theorem 4.15).

Lemma 4.13. *Let $\mathbf{N} = \{p_1, \dots, p_n\}$. Every $\mathbf{CPL}(\subseteq_0)$ -formula $\phi(\mathbf{N})$ is provably equivalent to a formula of the form $\bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$, where \mathcal{X} is a finite set of \mathbf{N} -teams,*

$$\Theta_X := \bigvee_{v \in X} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)}), \quad \text{and} \quad \Phi_X := \bigwedge_{v \in X} \underline{v(1)} \dots \underline{v(n)} \subseteq p_1 \dots p_n, \quad (13)$$

Recall from the proof of Theorem 3.7 that each disjunct $\Theta_X \wedge \Phi_X$ of the normal form $\bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$ above is equivalent to the $\mathbf{CPL}(\vee)$ -formula Ψ_X . Thus, Lemma 4.5 from the previous subsection with respect to the formulas Ψ_X holds also with $\Theta_X \wedge \Phi_X$ in place of Ψ_X .

Another key lemma for the completeness theorem is the following fact that corresponds to a specific case of Lemma 4.6 in the previous subsection.

Lemma 4.14. *For any finite set \mathcal{Y} of \mathbf{N} -teams, $\Theta_{\bigcup \mathcal{Y}}, \Phi_{\bigcup \mathcal{Y}} \vdash \bigvee_{Y \in \mathcal{Y}} (\Theta_Y \wedge \Phi_Y)$.*

Proof. Let $\mathcal{Y} = \{Y_1, \dots, Y_k\}$. We first derive that

$$\begin{aligned} \Theta_{Y_1 \cup \dots \cup Y_k}, \Phi_{Y_1} &\vdash (\Theta_{Y_1} \vee (\Theta_{Y_2} \vee \cdots \vee \Theta_{Y_k})) \wedge \Phi_{Y_1} && (\vee I) \\ &\vdash ((\Theta_{Y_1} \vee \Theta_{Y_1}) \wedge \Phi_{Y_1}) \vee (\Theta_{Y_2} \vee \cdots \vee \Theta_{Y_k}) && (\subseteq_0 \text{ Dst}) \\ &\vdash (\Theta_{Y_1} \wedge \Phi_{Y_1}) \vee (\Theta_{Y_2} \vee \cdots \vee \Theta_{Y_k}) && (\vee E) \end{aligned}$$

Similarly, we have that

$$(\Theta_{Y_1} \wedge \Phi_{Y_1}) \vee (\Theta_{Y_2} \vee \cdots \vee \Theta_{Y_k}), \Phi_{Y_2} \vdash (\Theta_{Y_2} \wedge \Phi_{Y_2}) \vee ((\Theta_{Y_1} \wedge \Phi_{Y_1}) \vee \Theta_{Y_3} \vee \cdots \vee \Theta_{Y_k})$$

and so on. In the end, putting all these steps together, we obtain that

$$\Theta_{Y_1 \cup \dots \cup Y_k}, \Phi_{Y_1}, \dots, \Phi_{Y_k} \vdash (\Theta_{Y_1} \wedge \Phi_{Y_1}) \vee \cdots \vee (\Theta_{Y_k} \wedge \Phi_{Y_k}). \quad \square$$

Theorem 4.15 (Completeness). *For any set $\Gamma \cup \{\phi\}$ of $\mathbf{CPL}(\subseteq_0)$ -formulas, we have that $\Gamma \models \phi \iff \Gamma \vdash \phi$.*

Proof. The proof follows from the same argument as that for Theorem 4.7 with the $\mathbf{CPL}(\vee)$ -formula Ψ_X now replaced with the $\mathbf{CPL}(\subseteq_0)$ -formula $\Theta_X \wedge \Phi_X$. In the key steps, we now apply Lemma 4.13 and Lemma 4.5 for $\Theta_X \wedge \Phi_X$. Another crucial fact that $\Theta_X \wedge \Phi_X \vdash \bigvee_{Y \in \mathcal{Y}_X} (\Theta_Y \wedge \Phi_Y)$ in case $X = \bigcup \mathcal{Y}_X$ for some $\mathcal{Y}_X \subseteq \mathcal{Y}$ is given by Lemma 4.14. \square

The rest of this subsection is devoted to the proof of Lemma 4.13. We first prove the following technical lemma.

Proposition 4.16.

- (i) $\neg(a_1^{x_1} \wedge \cdots \wedge a_n^{x_n}), x_1 \dots x_n \subseteq a_1 \dots a_n \vdash \perp$.
- (ii) $p_1^{x_1}, \dots, p_n^{x_n} \vdash x_1 \dots x_n \subseteq p_1 \dots p_n$.
- (iii) $\vdash \bigvee_{v \in 2^N} (\Theta_v \wedge \Phi_v)$ for any $N \subseteq \mathbf{Prop}$.

Proof. For item (i), we derive by $\subseteq \mathbf{Cmp}$ that

$$\neg(a_1^{x_1} \wedge \cdots \wedge a_n^{x_n}), x_1 \dots x_n \subseteq a_1 \dots a_n \vdash \neg(x_1^{x_1} \wedge \cdots \wedge x_n^{x_n}) \vdash \neg(\top \wedge \cdots \wedge \top) \vdash \perp.$$

For item (ii), we derive by applying $\subseteq_0 \mathbf{Ext}$ and $\subseteq \mathbf{Ctr}$ that

$$p_1^{x_1}, \top \subseteq \top \vdash x_1 \top \subseteq p_1 \top \vdash x_1 \subseteq p_1.$$

Since $\vdash \top \subseteq \top$ by $\subseteq \mathbf{Id}$, we conclude $p_1^{x_1} \vdash x_1 \subseteq p_1$. By $\subseteq_0 \mathbf{Ext}$ again, we derive $p_2^{x_2}, x_1 \subseteq p_1 \vdash x_2 x_1 \subseteq p_1 p_2$, and thus $p_1^{x_1}, p_2^{x_2} \vdash x_2 x_1 \subseteq p_1 p_2$. Proceed in the same way we obtain $p_1^{x_1}, \dots, p_n^{x_n} \vdash x_1 \dots x_n \subseteq p_1 \dots p_n$ in the end.

For item (iii), we first derive by rules of classical formulas that $\vdash \bigvee_{v \in 2^N} \Theta_v$. For each $v \in 2^N$, by item (ii) we have that $\Theta_v \vdash \Phi_v \vdash \Theta_v \wedge \Phi_v$. Hence we conclude $\vdash \bigvee_{v \in 2^N} (\Theta_v \wedge \Phi_v)$ by applying $\vee \mathbf{I}$ and $\vee \mathbf{E}$. \square

Next, we show that the rule $\vee_{\subseteq_0} \mathbf{E}$ for single primitive inclusion atoms can be generalized to one with multiple primitive inclusion atoms, and further to one with multiple disjunctions.

Lemma 4.17.

(i) Let $x_1 \subseteq a_1, \dots, x_k \subseteq a_k$ be primitive inclusion atoms. If

$$\Gamma, \phi, x_1 \subseteq a_1, \dots, x_k \subseteq a_k \vdash \chi, \quad \Gamma, \psi \vdash \chi, \quad \text{and} \quad \Gamma, \phi \vee \psi, x_1 \subseteq a_1, \dots, x_k \subseteq a_k \vdash \chi,$$

then $\Gamma, (\phi \wedge x_1 \subseteq a_1 \wedge \cdots \wedge x_k \subseteq a_k) \vee \psi \vdash \chi$.

(ii) Let I be a nonempty finite index set. For each $i \in I$, let ι_i be the conjunction of some finitely many primitive inclusion atoms. If for every nonempty $J \subseteq I$,

$$\Gamma, \bigvee_{i \in J} \phi_i, \bigwedge_{i \in J} \iota_i \vdash \chi, \tag{14}$$

then $\Gamma, \bigvee_{i \in I} (\phi_i \wedge \iota_i) \vdash \chi$.

Proof. (i). To show that $\Gamma, ((\phi \wedge x_2 \subseteq a_2 \wedge \cdots \wedge x_k \subseteq a_k) \wedge x_1 \subseteq a_1) \vee \psi \vdash \chi$, by $\vee_{\subseteq_0} \mathbf{E}$ it suffices to show that

$$\Gamma, \phi, x_2 \subseteq a_2, \dots, x_k \subseteq a_k, x_1 \subseteq a_1 \vdash \chi, \quad \Gamma, \psi \vdash \chi, \quad \text{and} \quad \Gamma, (\phi \wedge x_2 \subseteq a_2 \wedge \cdots \wedge x_k \subseteq a_k) \vee \psi, x_1 \subseteq a_1 \vdash \chi.$$

The first two clauses are given already by the assumption. To prove the third clause, by $\vee_{\subseteq_0}E$ again, it suffices to prove that

$$\begin{aligned} \Gamma, \phi, x_3 \subseteq a_3, \dots, x_k \subseteq a_k, x_1 \subseteq a_1, x_2 \subseteq a_2 \vdash \chi, \quad \Gamma, \psi, x_1 \subseteq a_1 \vdash \chi, \\ \text{and } \Gamma, (\phi \wedge x_3 \subseteq a_3 \wedge \dots \wedge x_k \subseteq a_k) \vee \psi, x_1 \subseteq a_1, x_2 \subseteq a_2 \vdash \chi. \end{aligned}$$

Again, the first two clauses follow from the assumption, and the third clause can be reduced to simpler clauses by applying $\vee_{\subseteq_0}E$. Proceed this way, in the end it remains to show that $\Gamma, \phi \vee \psi, x_1 \subseteq a_1, \dots, x_k \subseteq a_k \vdash \chi$. But this is also given by the assumption, and we are then done.

(ii). Suppose (14) holds for all nonempty $J \subseteq I$. We first prove a lemma that for any disjoint $K, L \subseteq I$ with $K \neq \emptyset$,

$$\Gamma, \bigvee_{k \in K} \phi_k \vee \bigvee_{l \in L} (\phi_l \wedge \iota_l), \bigwedge_{k \in K} \iota_k \vdash \chi. \quad (15)$$

We proceed by induction on $|L|$. If $L = \emptyset$, then $\Gamma, \bigvee_{k \in K} \phi_k, \bigwedge_{k \in K} \iota_k \vdash \chi$ is given by assumption (since $K \neq \emptyset$). Suppose the claim holds for L . We show that

$$\Gamma, \bigvee_{k \in K} \phi_k \vee (\bigvee_{l \in L} (\phi_l \wedge \iota_l)) \vee (\phi_0 \wedge \iota_0), \bigwedge_{k \in K} \iota_k \vdash \chi.$$

By item (i), it suffices to show that

$$\begin{aligned} \Gamma, \phi_0, \iota_0, \bigwedge_{k \in K} \iota_k \vdash \chi, \quad \Gamma, \bigvee_{k \in K} \phi_k \vee \bigvee_{l \in L} (\phi_l \wedge \iota_l), \bigwedge_{k \in K} \iota_k \vdash \chi \\ \text{and } \Gamma, \phi_0 \vee \bigvee_{k \in K} \phi_k \vee \bigvee_{l \in L} (\phi_l \wedge \iota_l), \bigwedge_{k \in K} \iota_k, \iota_0 \vdash \chi. \end{aligned}$$

The first clause follows from the assumption that $\Gamma, \phi_0, \iota_0 \vdash \chi$. The last two clauses follow from the induction hypothesis.

Now we prove $\Gamma, \bigvee_{i \in I} (\phi_i \wedge \iota_i) \vdash \chi$ by induction on $|I|$. If $|I| = 1$, the claim trivially holds. Suppose the claim holds for I . We show that the claim holds also for $I \cup \{0\}$, that is $\Gamma, (\bigvee_{i \in I} (\phi_i \wedge \iota_i)) \vee (\phi_0 \wedge \iota_0) \vdash \chi$ holds, assuming that (14) holds for any $J \subseteq I \cup \{0\}$. By item (i), it suffices to show that

$$\Gamma, \bigvee_{i \in I} (\phi_i \wedge \iota_i) \vdash \chi, \quad \Gamma, \phi_0, \iota_0 \vdash \chi, \quad \Gamma, (\bigvee_{i \in I} (\phi_i \wedge \iota_i)) \vee \phi_0, \iota_0 \vdash \chi.$$

The second clause is given by the assumption. The first clause follows from the induction hypothesis, since for every subset $J \subseteq I \subseteq I \cup \{0\}$, (14) holds by assumption. The third clause follows from (15). \square

We call a primitive inclusion atom $x \subseteq p$ *regular* if the p is a sequence of distinct propositional variables. For example, the primitive inclusion atoms $\perp \top \perp \subseteq ppq$ and $\top \perp \perp \subseteq pq\perp$ are not regular. We now show that every nontrivial primitive inclusion atom can be transformed to a regular one.

Lemma 4.18. *Let $x \subseteq a$ be a primitive inclusion atom. Either $x \subseteq a \dashv\vdash \top$, or $x \subseteq a \dashv\vdash \perp$, or $x \subseteq a \dashv\vdash x_0 \subseteq a_0$ for some regular primitive inclusion atom $x_0 \subseteq a_0$.*

Proof. We first eliminate constants \top, \perp one by one from the right side of a primitive inclusion atom $x \subseteq a$. Consider a constant v in a . By $\subseteq \text{Exc}$ we may without loss of generality assume that v occurs at the last position of the sequence a , i.e., the inclusion atom $x \subseteq a$ is $yz \subseteq bv$. If $z = v$, by $\subseteq \text{Ctr}$ and $\subseteq_0 \text{Ext}$, we have that

$$y\top \subseteq b\top \dashv\vdash y \subseteq b \quad \text{and} \quad y\perp \subseteq b\perp \dashv\vdash y \subseteq b.$$

For the special case when y and b are the empty sequence $\langle \rangle$, by $\subseteq \text{Id}$ we have $\top \langle \rangle \subseteq \langle \rangle$. Then we derive that $y\top \subseteq b\top \dashv\vdash \top$ and $y\perp \subseteq b\perp \dashv\vdash \top$ by $\top \text{I}$.

If $z \neq v$, we show that

$$y\top \subseteq b\perp \dashv\vdash \perp \quad \text{and} \quad y\perp \subseteq b\top \dashv\vdash \perp.$$

The right to left direction of the above two clauses follows from Proposition 4.2(iv) (which is true also for the system of $\mathbf{CPL}(\subseteq_0)$). For the other direction, we only give the proof for $y\top \subseteq b\perp \dashv\vdash \perp$, the other case being symmetric. By $\subseteq_0 \text{Ext}$, we have that $y\top \subseteq b\perp \dashv\vdash y\top \perp \subseteq b\perp \perp$. Since $\top \perp \leftrightarrow \perp$ by classical rules, we derive by $\subseteq \text{Cmp}$ that $y\top \subseteq b\perp \dashv\vdash \perp$, as required.

Lastly, we remove repeated propositional variables from the right side of a primitive inclusion atom $x \subseteq a$. By $\subseteq \text{Ctr}$ and $\subseteq \text{Wk}$, we have that

$$y\top\top \subseteq b\text{pp} \dashv\vdash y\top \subseteq b\text{p} \quad \text{and} \quad y\perp\perp \subseteq b\text{pp} \dashv\vdash y\perp \subseteq b\text{p}.$$

For the last case, we show that $y\top\perp \subseteq b\text{pp} \dashv\vdash \perp$. The right to left direction follows from Proposition 4.2(iv). For the other direction, since $\top p \leftrightarrow p$, by $\subseteq \text{Cmp}$ we derive that $y\top\perp \subseteq b\text{pp} \dashv\vdash \perp$. \square

Finally, we are ready give the proof the normal form lemma, Lemma 4.13.

Proof of Lemma 4.13. We prove the lemma by induction on ϕ . If $\phi(p_1, \dots, p_n) = p_i$, then

$$\begin{aligned} p_i &\dashv\vdash \bigvee_{v \in 2^{\mathbb{N} \setminus \{p_i\}}} (p_1^{v(1)} \wedge \dots \wedge p_{i-1}^{v(i-1)} \wedge p_i \wedge p_{i+1}^{v(i+1)} \wedge \dots \wedge p_n^{v(n)}) \\ &\dashv\vdash \bigvee_{v \in 2^{\mathbb{N} \setminus \{p_i\}}} (\Theta_{\{v\}} \wedge p_i \wedge \underline{v(1)} \dots \underline{v(i-1)} \top \underline{v(i+1)} \dots \underline{v(n)} \subseteq p_1 \dots p_n) \quad (\text{Proposition 4.16(ii)}) \\ &\dashv\vdash \bigvee_{\{u\} \in \mathcal{X}_i} (\Theta_{\{u\}} \wedge \Phi_{\{u\}}) \quad \text{where } \mathcal{X}_i = \{\{u\} \mid u \in 2^{\mathbb{N}}, u(i) = 1\}. \end{aligned}$$

If $\phi(p_1, \dots, p_n) = \top$, by Proposition 4.16(iii) we have that

$$\top \dashv\vdash \bigvee_{\{v\} \in \mathcal{X}_\top} (\Theta_{\{v\}} \wedge \Phi_{\{v\}}), \quad \text{where } \mathcal{X}_\top = \{\{v\} \mid v \in 2^{\mathbb{N}}\}.$$

Then, by $\top \text{I}$, we have $\top \dashv\vdash \bigvee_{\{v\} \in \mathcal{X}_\top} (\Theta_{\{v\}} \wedge \Phi_{\{v\}})$. If $\phi(p_1, \dots, p_n) = \perp$, then trivially $\perp \dashv\vdash \bigvee \emptyset = \perp$.

If $\phi(p_1, \dots, p_n)$ is a primitive inclusion atom. By Lemma 4.18 and $\subseteq \text{Exc}$, we may assume that $\phi = \top$ or $\phi = \perp$ or $\phi = x_1 \dots x_k \subseteq p_1 \dots p_k$ ($k \leq n$) is regular. The first two cases reduce to the previous cases. For the last case, we show that $\phi \dashv\vdash \bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$ where

$$\mathcal{X} = \{X \subseteq 2^{\mathbb{N}} \mid \exists v \in X \text{ such that } v(p_1) = x_1, \dots, v(p_k) = x_k\}.$$

For the right to left direction, by $\forall \text{E}$ it suffices to show that $\Theta_X, \Phi_X \vdash x_1 \dots x_k \subseteq p_1 \dots p_k$ for each $X \in \mathcal{X}$. For the valuation $v \in X$ such that $v(p_1) = x_1, \dots, v(p_k) = x_k$, we know that $x_1 \dots x_k \underline{v(k+1)} \dots \underline{v(n)} \subseteq p_1 \dots p_k p_{k+1} \dots p_n$ is a conjunct in Φ_X . Thus, we derive $\Phi_X \vdash x_1 \dots x_k \subseteq p_1 \dots p_k$ by $\subseteq \text{Ctr}$.

Conversely, for the left to right direction, we first have by Proposition 4.16(iii) that $\top \vdash \bigvee_{v \in 2^{\mathbb{N}}} (\Theta_v \wedge \Phi_v)$. Then it suffices to derive $\bigvee_{v \in 2^{\mathbb{N}}} (\Theta_v \wedge \Phi_v), x_1 \dots x_k \subseteq p_1 \dots p_k \vdash \bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$, which by Lemma 4.17(ii) reduces to derive that for each nonempty $Y \subseteq 2^{\mathbb{N}}$,

$$\Theta_Y, \Phi_Y, x_1 \dots x_k \subseteq p_1 \dots p_k \vdash \bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X). \tag{16}$$

Now, if $Y \in \mathcal{X}$, then the above holds by $\forall I$. Otherwise, if $Y \notin \mathcal{X}$, then for each $v \in Y$, $v(p_i) \neq x_i$ for some $1 \leq i \leq k$. Thus,

$$\Theta_v \vdash p_1^{v(1)} \wedge \dots \wedge p_k^{v(k)} \vdash \neg(p_1^{x_1} \wedge \dots \wedge p_k^{x_k}),$$

which implies $\Theta_Y \vdash \neg(p_1^{x_1} \wedge \dots \wedge p_k^{x_k})$ by $\vee E$. By Proposition 4.16(i), we have that $\neg(p_1^{x_1} \wedge \dots \wedge p_k^{x_k}), x_1 \dots x_k \subseteq p_1 \dots p_k \vdash \perp$. Hence, we obtain (16) by Proposition 4.2(iv).

Suppose α is a classical formula, and $\alpha \dashv\vdash \bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$. We show that $\neg\alpha \dashv\vdash \bigvee_{v \in 2^N \setminus \bigcup \mathcal{X}} (\Theta_{\{v\}} \wedge \Phi_{\{v\}})$. It is sufficient to prove that $\bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X) \dashv\vdash \bigvee_{v \in \bigcup \mathcal{X}} \Theta_{\{v\}}$, since we will then have that

$$\begin{aligned} \neg\alpha \dashv\vdash \neg \bigvee_{v \in \bigcup \mathcal{X}} \Theta_{\{v\}} & \quad \text{(by the standard rules for classical formulas)} \\ \dashv\vdash \bigvee_{v \in 2^N \setminus \bigcup \mathcal{X}} \Theta_{\{v\}} & \quad \text{(by the standard rules for classical formulas)} \\ \dashv\vdash \bigvee_{v \in 2^N \setminus \bigcup \mathcal{X}} (\Theta_{\{v\}} \wedge \Phi_{\{v\}}) & \quad \text{(Proposition 4.16(ii) and } \wedge E) \end{aligned}$$

Now, by the same argument as that in the proof of Lemma 4.4, since α is flat, we have that for each $v \in \bigcup \mathcal{X}$, $\{v\} \in \mathcal{X}$. Moreover, by Proposition 4.16(ii), $\Theta_{\{v\}} \vdash \Theta_{\{v\}} \wedge \Phi_{\{v\}}$. Thus, the direction $\bigvee_{v \in \bigcup \mathcal{X}} \Theta_{\{v\}} \vdash \bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X)$ follows from $\forall I$ and $\vee E$. For the other direction, for each $X \in \mathcal{X}$, we derive by $\forall I$ that

$$\Theta_X \wedge \Phi_X \vdash \bigvee_{u \in X} \Theta_{\{u\}} \vdash \bigvee_{v \in \bigcup \mathcal{X}} \Theta_{\{v\}},$$

from which $\bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X) \vdash \bigvee_{v \in \bigcup \mathcal{X}} \Theta_{\{v\}}$ follows by $\vee E$.

Suppose $\psi(N)$ and $\chi(N)$ satisfy

$$\psi \dashv\vdash \bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X) \quad \text{and} \quad \chi \dashv\vdash \bigvee_{Y \in \mathcal{Y}} (\Theta_Y \wedge \Phi_Y), \tag{17}$$

for some finite sets \mathcal{X} and \mathcal{Y} of N -teams. The case $\phi = \psi \vee \chi$ follows from induction hypothesis. If $\phi = \psi \wedge \chi$, we show that $\psi \wedge \chi \dashv\vdash \bigvee_{Z \in \mathcal{Z}} (\Theta_Z \wedge \Phi_Z)$, where

$$\mathcal{Z} = \{ \bigcup \mathcal{X}' \mid \mathcal{X}' \subseteq \mathcal{X} \text{ and } \bigcup \mathcal{X}' = \bigcup \mathcal{Y}' \text{ for some } \mathcal{Y}' \subseteq \mathcal{Y} \}.$$

For the right to left direction, by $\vee E$ it suffices to derive $\Theta_Z, \Phi_Z \vdash \psi \wedge \chi$ for each $Z = \bigcup \mathcal{X}' = \bigcup \mathcal{Y}' \in \mathcal{Z}$, where $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$. By Lemma 4.14 and $\forall I$, we have that $\Theta_Z, \Phi_Z \vdash \bigvee_{X \in \mathcal{X}'} (\Theta_X \wedge \Phi_X) \vdash \bigvee_{X \in \mathcal{X}} (\Theta_X \wedge \Phi_X) \vdash \psi$. Similarly, $\Theta_Z, \Phi_Z \vdash \chi$.

For the left to right direction, by Lemma 4.17(ii) it suffices to prove that for each nonempty $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$,

$$\bigvee_{X \in \mathcal{X}'} \Theta_X, \bigwedge_{X \in \mathcal{X}'} \Phi_X, \bigvee_{Y \in \mathcal{Y}'} \Theta_Y, \bigwedge_{Y \in \mathcal{Y}'} \Phi_Y \vdash \bigvee_{Z \in \mathcal{Z}} (\Theta_Z \wedge \Phi_Z),$$

which reduces to showing that

Table 5
Rules for inclusion atoms.

$\frac{\bigwedge_{x \in \{\top, \perp\}^{ \mathbf{a} }} (a^x \rightarrow x \subseteq b)}{a \subseteq b} \subseteq \text{Ext}$	$\frac{D \quad a \subseteq b}{a^x \rightarrow x \subseteq b} \subseteq \text{Rdt}$
---	--

$$\Theta_{\cup \mathcal{X}'}, \Theta_{\cup \mathcal{X}'}, \Theta_{\cup \mathcal{Y}'}, \Phi_{\cup \mathcal{Y}'} \vdash \bigvee_{Z \in \mathcal{Z}} (\Theta_Z \wedge \Phi_Z) \quad (18)$$

as elements in \mathcal{X}' and \mathcal{Y}' may not be disjoint. Now, if $\cup \mathcal{X}' = \cup \mathcal{Y}' \in \mathcal{Z}$, then the above clause follows easily from $\forall I$. Otherwise, if $\cup \mathcal{X}' \neq \cup \mathcal{Y}'$, assume w.l.o.g. there exists some $v \in \cup \mathcal{X}' \setminus \cup \mathcal{Y}'$. First, we derive $\Phi_v, \neg \Theta_v \vdash \bigvee_{Z \in \mathcal{Z}} (\Theta_Z \wedge \Phi_Z)$ by Proposition 4.16(i). Since $v \notin \cup \mathcal{Y}'$, by the standard rules for classical formulas, we have that $\Theta_{\cup \mathcal{Y}'} \vdash \neg \Theta_v$. Putting these together, we obtain (18). \square

4.3. CPL(\subseteq)

In this section, we extend the system of $\mathbf{CPL}(\subseteq_0)$ to obtain a sound and complete system for propositional inclusion logic $\mathbf{CPL}(\subseteq)$ with arbitrary inclusion atoms. Recall from Proposition 3.10(i) that arbitrary inclusion atoms are definable in terms of primitive ones. Such an interaction between arbitrary and primitive inclusion atoms is characterized by the two rules we add to the system of $\mathbf{CPL}(\subseteq)$, inclusion atom extension $\subseteq \text{Ext}$ and reduction $\subseteq \text{Rdt}$ rule.

Definition 4.19. The system of $\mathbf{CPL}(\subseteq)$ consists of all rules in the system of $\mathbf{CPL}(\subseteq_0)$ together with the rules in Table 5, where \mathbf{a} and \mathbf{b} are arbitrary (and possibly empty) sequences of elements in $\text{Prop} \cup \{\top, \perp\}$, \mathbf{x} stands for an arbitrary sequence of constants from $\{\top, \perp\}$, and $|\mathbf{a}|$ denotes the length of the sequence \mathbf{a} .

By Proposition 3.10(i) the two new rules $\subseteq \text{Ext}$ and $\subseteq \text{Rdt}$ are clearly sound, and thus the system is sound. By applying $\subseteq \text{Ext}$ and $\subseteq \text{Rdt}$, we can easily reduce an arbitrary inclusion atom to a formula with primitive inclusion atoms only:

$$a \subseteq b \dashv\vdash \bigwedge_{x \in \{\top, \perp\}^{|\mathbf{a}|}} (a^x \rightarrow x \subseteq b).$$

From this the completeness of the system of $\mathbf{CPL}(\subseteq)$ follows.

Theorem 4.20 (Completeness). For any set $\Gamma \cup \{\phi\}$ of $\mathbf{CPL}(\subseteq)$ -formulas, we have that $\Gamma \models \phi \iff \Gamma \vdash \phi$.

Let us end this section by illustrating the derivation of the replacement rule for inclusion atoms in the system of $\mathbf{CPL}(\subseteq)$.

Example 4.21.

- (i) $a \leftrightarrow b, ac \subseteq de \vdash bc \subseteq de$.
- (ii) $a \leftrightarrow b, de \subseteq ac \vdash de \subseteq bc$.

Proof. (i) By $\subseteq \text{Ext}$ it suffices to show $a \leftrightarrow b, ac \subseteq de \vdash bc^{xy} \rightarrow xy \subseteq de$ for all $xy \in \{\top, \perp\}^{|\mathbf{y}|+1}$. First, by $\subseteq \text{Rdt}$ we have $ac \subseteq de \vdash ac^{xy} \rightarrow xy \subseteq de$. Next, since

$$a \leftrightarrow b, \neg ac^{xy} \vdash \neg bc^{xy} \vdash \neg bc^{xy} \vee xy \subseteq de$$

and $xy \subseteq de \vdash \neg bc^{xy} \vee xy \subseteq de$, we obtain by applying $\vee E$ that

$$a \leftrightarrow b, \neg ac^{xy} \vee xy \subseteq de \vdash \neg bc^{xy} \vee xc \subseteq de$$

i.e., $a \leftrightarrow b, ac^{xy} \rightarrow xy \subseteq de \vdash bc^{xy} \rightarrow xc \subseteq de$. Hence, we conclude that $a \leftrightarrow b, ac \subseteq de \vdash bc^{xy} \rightarrow xc \subseteq de$ as required.

(ii). By item (i) we derive that $bc \subseteq bc, a \leftrightarrow b \vdash ac \subseteq bc$, which implies $a \leftrightarrow b \vdash ac \subseteq bc$, since $\vdash bc \subseteq bc$ by $\subseteq Id$. By $\subseteq Trs$, we have that $de \subseteq ac, ac \subseteq bc \vdash de \subseteq bc$. Thus, we conclude that $a \leftrightarrow b, de \subseteq ac \vdash de \subseteq bc$, as required. \square

5. Locality revisited and interpolation

Having studied the expressive power and proof theory for our union closed team logics in the previous two sections, let us in this section revisit the fundamental property of these logics, the *locality property*. The locality property states that the propositional variables not occurring in a formula are irrelevant for the evaluation of the formula (see Lemma 2.5). It was observed already in [13] in the context of first-order inclusion logic that such a basic property is in fact nontrivial in the team-based logics (especially in non-downwards closed team logics) and thus cannot be taken for granted. In particular, a counterexample was given in [13] to show that first-order inclusion logic under the so-called *strict semantics* does not any more satisfy locality. In this section, we give counterexamples to show that locality fails for propositional inclusion logic $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\vee)$ and $\mathbf{CPL}(\Upsilon)$ as well if strict semantics is applied.

We also point out a subtle connection between locality and interpolation property, where the interpolation property states that for any entailment $\phi \models \psi$, there exists an interpolant θ in the common language of ϕ and ψ . It follows from recent work [8] by D’Agostino in the modal team logics setting that all of the expressively complete propositional team logics admit uniform interpolation (a stronger property than Craig’s interpolation requiring, in addition, that the interpolants are uniform). Therefore, the expressively complete union closed logics we consider in this paper (e.g., $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$, see Theorems 3.4 and 3.7) all admit uniform interpolation, demonstrating also that our logics have good meta-logical properties. The proof in [8] requires the assumption that the logics in question satisfy the locality property. We reformulate and elaborate this proof in our setting, so as to also highlight the crucial role that the locality property plays in this argument. It is thus natural to ask whether locality is actually a necessary condition or presupposition for interpolation. Roughly speaking, the locality property and the interpolation property both describe the redundant role of irrelevant variables. This similar flavor seems to suggest that these two properties may actually be connected. This problem is particularly relevant for union closed team logics, as locality may fail for these logics under the strict semantics. In line with this discussion, we give, in this section, an example of a non-local fragment of $\mathbf{CPL}(\vee)$ under strict semantics in which Craig’s interpolation actually fails.

As described above, the non-triviality of the locality property in the team semantics setting is (at least partly) due to the fact that there are alternative versions of the semantics. The team semantics we defined in Section 2 is known as the *lax semantics*. Another competing version is called the *strict semantics*, which is otherwise the same as lax semantics except that the semantics for the disjunction is subtly different. The strict semantics for the disjunctions \vee and \vee require the team in question to be split into two disjoint subteams. More precisely, the satisfaction relation \models^s for the disjunctions in strict semantics is defined as:

- $X \models^s \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ such that $Y \cap Z = \emptyset, X = Y \cup Z, Y \models^s \phi$ and $Z \models^s \psi$.
- $X \models^s \phi \vee \psi$ iff $X = \emptyset$ or there exist nonempty subteams $Y, Z \subseteq X$ such that $Y \cap Z = \emptyset, X = Y \cup Z, Y \models \phi$ and $Z \models \psi$.

Clearly, the strict semantics for \vee coincide with the lax semantics in downwards closed logics. The union closed logics we consider in this paper are, however, not downwards closed.

We now present our examples to illustrate that under strict semantics none of the logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ satisfies the locality property.

Example 5.1. Consider the team X over domain $\{p, q, r, s\}$ illustrated in the left table below:

	p	q	r	s
X :	1	0	0	0
	0	1	0	0
	0	1	0	1
	0	0	1	0

	p	q	r
$X \upharpoonright \{p, q, r\}$:	v_1	1	0
	v_2	0	1
	v_3	0	0

Under strict semantics the $\mathbf{CPL}(\vee)$ -formula $(p \vee q) \vee (q \vee r)$ and the $\mathbf{CPL}(\Upsilon)$ -formula $(\neg p \wedge \neq(q)) \vee (\neq(q) \wedge \neg r)$ are both satisfied by X , because X can be split into two disjoint subteams (illustrated by two shaded table fragments of different tones) each satisfying one distinct \vee -disjunct from each formula. But both formulas fail in the restricted team $X' = X \upharpoonright \{p, q, r\}$ under strict semantics, because the full team X' does not satisfy any of the four \vee -disjuncts in the two formulas, and $\{v_1, v_2\}$ is the only nonempty subteam of X' that satisfies $p \vee q$ and $\neq(q) \wedge \neg r$, leaving the reminder subteam $\{v_3\}$ falsifying $q \vee r$ and $\neg p \wedge \neq(q)$.

Consider also the team Y over domain $\{p, q, r, s, t, u, v\}$ defined below:

	p	q	r	s	t	u	v
Y :	0	0	1	1	0	1	0
	1	1	0	0	1	0	1
	1	1	0	0	1	0	0
	0	1	1	0	0	0	0

	p	q	r	s	t	u
Y' :	w_1	0	0	1	1	0
	w_2	1	1	0	0	1
	w_3	0	1	1	0	0

Under strict semantics the $\mathbf{CPL}(\subseteq)$ -formula $pq \subseteq rs \vee tu \subseteq rs$ is satisfied by the team Y , but falsified by the restricted team $Y' = Y \upharpoonright \{p, q, r, s, t, u\}$.

Let us remark that the above example actually only shows that formulas in the logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\Upsilon)$ and $\mathbf{CPL}(\subseteq)$ under strict semantics are not *downwards* local, where we say that a formula $\phi(\mathbf{N})$ is *downwards local* if for any teams X and Y with $\text{dom}(X) \supseteq \text{dom}(Y) \supseteq \mathbf{N}$ and $X \upharpoonright \mathbf{N} = Y \upharpoonright \mathbf{N}$, it holds that

$$X \models \phi \implies Y \models \phi.$$

It is easy to verify that a formula ϕ is local iff it is both downwards and upwards local, where we say that $\phi(\mathbf{N})$ is *upwards local* if for any X and Y as above,

$$X \models \phi \implies Y \models \phi.$$

By a straightforward inductive argument (similar to the proof of Lemma 2.5), one can prove that the logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\Upsilon)$ and $\mathbf{CPL}(\subseteq)$ under strict semantics are nevertheless upwards closed.

Remark 5.2. Consider again the teams X, Y and the three formulas in Example 5.1. It is easy to see that under strict semantics, the first two formulas are satisfied in both $\{v_1, v_2\}$ and $\{v_2, v_3\}$ but not in their union X' ; similarly, the third formula is satisfied in both $\{w_1, w_2\}$ and $\{w_2, w_3\}$ but not in their union Y' . This shows that none of the three logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ is any more union closed when strict semantics is applied. This fact for propositional inclusion logic $\mathbf{CPL}(\subseteq)$ was observed already in [23]. $\mathbf{CPL}(\subseteq)$ behaves differently under strict and lax semantics also in terms of computational properties; the reader is referred to [23,24] for details.

Remark 5.3. Given any union closed N -team property $P \in \mathbb{P}^{\dot{\cup}}$ that contains the empty team. It is not hard to see that the $\mathbf{CPL}(\vee)$ -formula $\bigvee_{X \in P} \Psi_X$ under strict semantics still characterizes P . The corresponding formulas in $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ defined in Theorem 3.7 are still equivalent to $\bigvee_{X \in P} \Psi_X$ under strict semantics. Therefore under strict semantics all properties in $\mathbb{P}^{\dot{\cup}}$ are still definable in the three logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$. But as illustrated in Remark 5.2, under strict semantics these three logics can also define properties that are not union closed. Determining the expressive power of these logics under strict semantics is left as future work. It is worthwhile to mention that first-order inclusion logic is known to have the same expressive power as positive greatest fixed point logic under lax semantics [15], whereas under strict semantics it is so strictly stronger that it is equivalent to existential second-order logic [14].

Let us now define another key notion for this section, namely the (uniform) interpolation property.

Definition 5.4. We say that a logic L enjoys *(Craig’s) interpolation property* if for any pair of L -formulas $\phi(KN)$ and $\psi(MN)$ with K, M, N pairwise disjoint sets of propositional variables, if $\phi \models \psi$, then there exists an L -formula $\theta(N)$ in the common language N (called an *interpolant*) such that $\phi \models \theta$ and $\theta \models \psi$.

Depending on the formula ψ , the interpolant θ may be different. Uniform interpolation property requires the interpolant to be uniform for all such ψ .

Definition 5.5. We say that a logic L enjoys *uniform interpolation property* if for any L -formula $\phi(K)$ and any $N \subseteq K$, there is an L -formula $\theta(N)$ (called a *uniform interpolant*) such that $\phi \models \theta$, and for any L -formula $\psi(M)$ with $K \cap M \subseteq N$, we have that $\phi \models \psi$ implies $\theta \models \psi$.

Clearly, uniform interpolation implies Craig’s interpolation. For more in-depth discussions on interpolation, the reader is referred to, e.g., [11,29]. We now proceed to reformulate the result in [8] that given the locality property, any propositional team-based logic that is expressively complete in some forgetful class of team properties enjoys uniform interpolation, and thus all of the expressively complete union closed team logics from Theorems 3.4 and 3.7 enjoy uniform interpolation. We call a class \mathbb{P} of team properties *forgetful* if for any N -team property $P \in \mathbb{P}$ and any $M \subseteq N$, $P \upharpoonright_M \in \mathbb{P}$, where $P \upharpoonright_M = \{X \upharpoonright M : X \in P\}$. Intuitively, the notion is such termed as the team property $P \upharpoonright_M$ in the definition simply “forgets” the information concerning all propositional variables in the set $N \setminus M$. For example, the collection $\mathbb{P}^{\dot{\cup}}$ of all union closed team properties which contain the empty team is forgetful, so are the collection \mathbb{F} of all flat team properties, and the collection of all downwards closed team properties which contain the empty team (in which propositional dependence logic is expressively complete [37]), etc.

One important lemma in the argument of [8] (formulated in our setting) is the observation that team semantics has the *amalgamation property* in the following sense, where we write simply MN for the union $M \cup N$ of two domains M, N .

Lemma 5.6 (Amalgamation). *For any K -team X and M -team Y such that $X \upharpoonright (K \cap M) = Y \upharpoonright (K \cap M)$, there exists a KM -team Z such that $Z \upharpoonright K = X$ and $Z \upharpoonright M = Y$.*

Proof. Clearly the required KM -team Z can be defined as

$$Z = \{v : KM \cup \{\top, \perp\} \rightarrow \{0, 1\} \mid v \upharpoonright K \in X, v \upharpoonright M \in Y \text{ and } v(\top \perp) = 10\}. \quad \square$$

Now, we are ready to give the proof of the uniform interpolation result for expressively complete propositional team logics that satisfy locality property. The argument is due to [8]. We provide here a detailed proof in which all the steps involving applications of the (upwards and downwards) locality property are explicitly spelled out.

Theorem 5.7 (Uniform interpolation). *Let \mathbf{L} be a team-based propositional logic that has the locality property and is expressively complete in some forgetful class \mathbb{P} of team properties. Then \mathbf{L} enjoys uniform interpolation property.*

Proof. Let $\phi(\mathbf{K})$ be an \mathbf{L} -formula and $\mathbf{N} \subseteq \mathbf{K}$. Since \mathbf{L} is expressively complete in \mathbb{P} , $\llbracket \phi \rrbracket_{\mathbf{K}} \in \mathbb{P}$. As \mathbb{P} is forgetful, $\llbracket \phi \rrbracket_{\mathbf{K} \upharpoonright \mathbf{N}} \in \mathbb{P}$ as well. By the expressive completeness again, we find an \mathbf{L} -formula $\theta(\mathbf{N})$ such that $\llbracket \theta \rrbracket_{\mathbf{N}} = \llbracket \phi \rrbracket_{\mathbf{K} \upharpoonright \mathbf{N}}$. We show that θ is the required uniform interpolant.

To see that $\phi \models \theta$, suppose $X \models \phi$ with $\text{dom}(X) \supseteq \mathbf{K}$. Since \mathbf{L} is (downwards) local, we have also that $X \upharpoonright \mathbf{K} \models \phi(\mathbf{K})$, i.e., $X \upharpoonright \mathbf{K} \in \llbracket \phi \rrbracket_{\mathbf{K}}$. Thus

$$X \upharpoonright \mathbf{N} = (X \upharpoonright \mathbf{K}) \upharpoonright \mathbf{N} \in \llbracket \phi \rrbracket_{\mathbf{K} \upharpoonright \mathbf{N}} = \llbracket \theta \rrbracket_{\mathbf{N}},$$

namely $X \upharpoonright \mathbf{N} \models \theta$. Then, since \mathbf{L} is (upwards) local, we conclude that $X \models \theta$.

Next, assuming that $\psi(\mathbf{M})$ is an \mathbf{L} -formula with $\mathbf{K} \cap \mathbf{M} \subseteq \mathbf{N}$ and $\phi \models \psi$, we show that $\theta \models \psi$. Suppose $X \models \theta$ and $\text{dom}(X) \supseteq \mathbf{M}\mathbf{N}$. Since \mathbf{L} is (downwards) local, we have $X \upharpoonright \mathbf{N} \models \theta(\mathbf{N})$. Thus $X \upharpoonright \mathbf{N} \in \llbracket \theta \rrbracket_{\mathbf{N}} = \llbracket \phi \rrbracket_{\mathbf{K} \upharpoonright \mathbf{N}}$. It follows that there exists a \mathbf{K} -team Y such that $Y \models \phi$ and $Y \upharpoonright \mathbf{N} = X \upharpoonright \mathbf{N}$.

Since $\mathbf{N} \subseteq \mathbf{K}$ and $\mathbf{K} \cap \mathbf{M} \subseteq \mathbf{N}$, we have that

$$\mathbf{N} \subseteq \mathbf{K} \cap \mathbf{M}\mathbf{N} = (\mathbf{K} \cap \mathbf{M}) \cup (\mathbf{K} \cap \mathbf{N}) \subseteq \mathbf{N},$$

thus $\mathbf{K} \cap \mathbf{M}\mathbf{N} = \mathbf{N}$. Now, by Lemma 5.6, there exists a $\mathbf{K}\mathbf{M}\mathbf{N}$ -team Z such that $Z \upharpoonright \mathbf{K} = Y$ and $Z \upharpoonright \mathbf{M}\mathbf{N} = X \upharpoonright \mathbf{M}\mathbf{N}$. Since $Y \models \phi(\mathbf{K})$, we have $Z \upharpoonright \mathbf{K} \models \phi$, which by the (upwards) locality of \mathbf{L} implies $Z \models \phi$. It then follows from assumption $\phi \models \psi$ that $Z \models \psi$. Since \mathbf{L} is (downwards) local, we obtain $Z \upharpoonright \mathbf{M}\mathbf{N} \models \psi$, which implies $X \upharpoonright \mathbf{M}\mathbf{N} \models \psi$. Hence, we conclude $X \models \psi$, as \mathbf{L} is (upwards) local. \square

It then follows from Theorems 3.4 and 3.7 that uniform interpolation holds for all the expressively complete union closed team logics we consider in this paper.

Corollary 5.8. *The logics $\mathbf{CPL}(\vee)$, $\mathbf{CPL}(\subseteq)$, $\mathbf{CPL}(\subseteq_0)$, $\mathbf{CPL}(\Upsilon)$, and $\mathbf{CPL}(\neq \cdot)$ enjoy uniform interpolation property and thus also Craig's interpolation property.*

Let us emphasize again that the proof of Theorem 5.7 makes essential use of the locality property (both upwards and downwards locality, to be more precise). It is not clear whether the locality property is actually a necessary condition for uniform interpolation. Yet let us now demonstrate that the interpolation property can fail for team-based logics without the locality property. Recall from Example 5.1 that $\mathbf{CPL}(\vee)$ with strict semantics is not local, and the counterexample can be built with four propositional variables. We shall consider the restricted language of $\mathbf{CPL}(\vee)$ with four propositional variables p, q, r, s and constants \top, \perp only. This language, denoted as $\mathbf{CPL}(\vee)_4$, is clearly still not local under strict semantics. We now illustrate that $\mathbf{CPL}(\vee)_4$ does not admit (Craig's) interpolation.

Example 5.9. Consider $\mathbf{CPL}(\vee)_4$ with strict semantics and consider the team X from Example 5.1 again. We claim that

$$\Psi_{X'} \wedge ((p \vee q) \vee (q \vee r)) \models^s s \vee \neg s, \quad (19)$$

where $\Psi_{X'} = (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$ is the formula that defines (under lax semantics) the team $X' = X \upharpoonright \{p, q, r\}$ modulo the empty team in the sense of Equation (6) from Section 3. Now, observe that in the entailment (19) the common language of two formulas on two sides of the turnstile (\models) is empty. Hence there is no interpolant for the entailment (19) (the constants \perp and \top are clearly not interpolants).

To see why (19) holds, take any nonempty team Y over the domain $\{p, q, r, s\}$ that satisfies both $\Psi_{X'}$ and $(p \vee q) \vee (q \vee r)$ under strict semantics. It is easy to see that $Y \models^s \Psi_{X'}$ implies that $Y \upharpoonright \{p, q, r\} = X'$. Now, similarly to what we have argued in Example 5.1, in order for the $\{p, q, r, s\}$ -team Y to satisfy $(p \vee q) \vee (q \vee r)$ under strict semantics, the valuation v_2 in X' must extend in Y to two distinct valuations v'_2 and v''_2 . In the language $\mathbf{CPL}(\vee)_4$ with four propositional variables only, this can only be the case if $v'_2(s) = 0$ and $v''_2(s) = 1$. From this we must conclude that $Y \models^s s \vee \neg s$, as desired.

6. Conclusion and further directions

In this paper, we have studied the expressive power, axiomatization problem and locality property for several propositional union closed team logics. Building on the result in [38] that $\mathbf{CPL}(\vee)$ is expressively complete, we proved that $\mathbf{CPL}(\subseteq)$ and $\mathbf{CPL}(\Upsilon)$ as well as their fragments $\mathbf{CPL}(\subseteq_0)$ and $\mathbf{CPL}(\neq(\cdot))$ are also expressively complete. It is interesting to note that our version of the propositional inclusion logic $\mathbf{CPL}(\subseteq)$ allows inclusion atoms $a \subseteq b$ with the constants \top and \perp in the arguments. As we illustrated, the original version of $\mathbf{CPL}(\subseteq)$, the version in which inclusion atoms $p \subseteq q$ can only have propositional variables in the arguments, is actually strictly less expressive, and not expressively complete. Recall that first-order inclusion logic was shown in [15] to be not expressively complete either, since some of the union closed existential second-order team properties cannot be defined in the logic. While the union closed fragment of existential second-order logic was already characterized in [28] by using an involved fragment of inclusion-exclusion logic, it is reasonable to ask whether it is possible to find a simpler expressively complete union closed first-order team-based logic, by extending first-order logic with certain more general inclusion atoms, as done in the present paper on the propositional level.

We have introduced sound and complete natural deduction systems for $\mathbf{CPL}(\vee)$ and $\mathbf{CPL}(\subseteq)$ as well as $\mathbf{CPL}(\subseteq_0)$. How to axiomatize the logic $\mathbf{CPL}(\Upsilon)$ is left as future work. The completeness proofs for the systems of $\mathbf{CPL}(\vee)$ and $\mathbf{CPL}(\subseteq)$ makes heavy use of the disjunctive normal form of the two logics. Since the normal form of $\mathbf{CPL}(\Upsilon)$ (Corollary 3.8(i)) is substantially more involved, in order to obtain an elegant proof system for $\mathbf{CPL}(\Upsilon)$, one may need to take a different approach, or at least to formulate a simpler normal form for the logic. Introducing (cut-free) sequent calculi for all these union closed logics and investigating their proof-theoretic properties are natural further directions. For propositional downwards closed logics, some first steps along this line were taken in [10,30].

We have also analyzed the locality property in union closed team logics. We stressed that this simple property should not be taken for granted in the context of team semantics. We gave examples to illustrate that the union closed team logics considered in this paper under strict semantics actually lack the locality property. We have also briefly discussed that the locality property can actually be decomposed into the upwards and downwards locality. While this distinction between upwards and downwards locality did not lead to new result in this paper, it is our hope that this distinction can inspire further research on locality. We have also discussed the connection between locality and interpolation. We reformulated the proof of the result in [8] that given locality, all expressively complete team-based propositional logics (in some forgetful class) enjoy uniform interpolation (Theorem 5.7). We highlighted the subtle but crucial role that the locality property play in the argument for this result. This then naturally raises the question whether locality is actually a presupposition for interpolation. The example we gave in Example 5.9 is at least consistent with this idea. A thorough investigation into the connection between locality and interpolation is left for future work.

We end by mentioning two other further directions. First is to find applications of union closure team logics in other fields. Propositional downwards closed team logics have natural interpretations in inquisitive semantics (see e.g., [6]). Developing similar connections for union closed team logics in natural language and other contexts would be an important further direction. Along this line, recent work by Aloni [1] used a union closed team-based modal logic with the atom NE to model *free-choice* inferences in natural language, where

NE states that the team in question is nonempty (and thus the resulting logic, studied in [2], does not satisfy the empty team property). Another interesting direction is to consider team-based (propositional) logics with other closure properties. The team-based logics considered in the literature are usually conservative extensions of classical logic. The characteristic property of classical propositional formulas is the flatness property (Corollary 3.3), which is equivalent to the combination of the empty team property, the union closure property and the downwards closure property. In this respect, the union closure and downwards closure property are natural closure properties for team-based logics. In contrast, for instance, the upwards closure property is not very natural, because, as we pointed out already, classical formulas (e.g., already the propositional variable p) are not upwards closed. Nevertheless, there may well be other meaningful ways to decompose the flatness property. For an obvious example, the flatness property is stated as a property of two directions, each of which corresponds to a closure property that has not yet been considered in the literature. The closure properties obtained from this and possibly other decompositions could give rise to other interesting logics.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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