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Growth Expectations and Asset Prices in Production Economies and Labor Market Matching Models

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Declaration

The work presented in this thesis is my own.

Georg Kaltenbrunner

To my family,

my wife, my mother, my father, my brothers, my grandmother,

the people who mean everything to me.

Abstract

This thesis improves the standard production economy model (standard stochastic growth (RBC) model) along several important dimensions. In the first part of the thesis we incorporate versions of the labor market matching framework into standard production economy models and show that the so extended model can alleviate two important shortcomings.

Firstly, it is a known problem that news about future productivity growth cause a contraction in most standard business cycle models, which is counterfactual. We show that a standard business cycle model that incorporates a search and matching model of the labor market can generate an expansion.

Secondly, it has been shown that jointly explaining fluctuations in macroeconomic time series, in particular aggregate employment, and asset prices is difficult. The reason is that households, once allowed to, use their labor-leisure decision to engineer counterfactually smooth consumption profiles, resulting in a counterfactually negative correlation of consumption with employment, and low equity risk premiums. We show that a version of our extended model (combined with habit preferences and capital adjustment costs) where both consumption and labor are endogenous can explain the behavior of asset prices as well as key macroeconomic time series, including aggregate employment.

The standard production economy model without habit preferences has been found to fail markedly at explaining asset prices. In the second part of the thesis we show that by two simple adjustments, we disentangle the coefficient of relative risk aversion from the elasticity of intertemporal substitution and we carefully engineer and calibrate the process for wages and consequentially for dividends, the standard model without habit preferences can actually be enabled to explain asset prices to a remarkable extent. This is important, because in many models that assume habit preferences, in particular in production economy models where simple internal habits are assumed, the risk-free rate is far too volatile, and higher risk premiums are in a sense generated through a too volatile stochastic discount factor.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

This thesis improves the standard production economy model (standard stochastic growth (RBC) model) along several important dimensions. In the first part of the thesis we incorporate versions of the Mortensen and Pissarides (1994) labor market matching framework into standard production economy models and show that the so extended model can alleviate two important shortcomings.

Firstly, it is a known problem that news about future productivity growth cause a contraction in most standard business cycle models, which is counterfactual.[1] We show that a standard business cycle model that incorporates a search and matching model of the labor market can generate an expansion.

Secondly, it has been shown that jointly explaining fluctuations in macroeconomic time series, in particular aggregate employment, and asset prices is difficult.[2] We show that a version of our extended model (combined with habit preferences and capital adjustment costs) where both consumption and labor are endogenous can explain the behavior of asset prices as well as key macroeconomic time series, including aggregate employment.

In the second part of the thesis we show that by two simple adjustments, we disentangle the coefficient of relative risk aversion from the elasticity of intertemporal substitution and we carefully engineer and calibrate the process for wages and consequentially for dividends, the standard model without habit preferences can actually be enabled to explain asset prices to a remarkable extent. This runs counter to what is often taken for granted in the literature.[3]

[1]Beaudry and Portier (2006) provide formal empirical evidence that business cycles are caused by anticipated changes in future productivity. Beaudry and Portier (2005) document that most existing neoclassical models cannot generate Pigou cycles. In a Pigou cycle, output, consumption, investment, and hours worked jointly increase in response to an anticipated increase in productivity and these variables decline when the anticipated increase fails to materialize.

[2]Boldrin, Christiano, Fisher (2001): '[RBC models] have been notoriously unsuccessful in accounting for the joint behavior of asset prices and consumption.'.

[3]Rouwenhorst (1995): '[...] it is more difficult to explain substantial risk premiums in a production

Chapter 2 — Anticipated Growth and Business Cycles in Matching Models

Positive news about future productivity growth cause a contraction in most neo-classical business cycle models, which is counterfactual. In Chapter 2 we show that a business cycle model that incorporates the standard matching framework can generate an expansion. Although the wealth effect of an increase in expected productivity induces workers to reduce their labor supply, the matching friction has the opposite effect leaving labor supply roughly unaffected. Employment increases because the matching friction also induces firms to post more vacancies. This translates into additional resources, which makes it possible for both consumption and investment to increase in response to positive news about future productivity growth before the actual increase in productivity materializes.

Chapter 3 — Asset Pricing in Production Economies — A Matching Model

A large literature explains moments and dynamics of asset prices with models where consumption is an exogenously specified process. Most successful models in that literature assume preferences that are non-standard. Lettau and Uhlig (2000) and Uhlig (2004), amongst others, show that as soon as households have access to a savings technology, that is consumption is endogenous, or leisure enters the households' utility function, that is the labor-leisure choice is endogenous, the most commonly used preference specifications developed for exchange economy models turn out to have very unrealistic implications for choices of aggregate variables such as consumption and employment and consequentially also for asset prices, and we are left with models that can neither explain asset prices nor the behavior of aggregate consumption, investment, and employment.

In Chapter 3 we develop a model that can *jointly* explain important moments and dynamics of asset prices as well as key aspects of the behavior of major macroeconomic time series such as output, consumption, investment, and employment. We build on a one-sector standard production economy model with endogenous consumption and investment and endogenize aggregate employment by means of a state-of-the-art search-theoretical model of the labor market, more specifically, a version of the Mortensen and Pissarides (1994) labor market matching model. Those models assume a labor market search and matching friction. The reason we impose this friction, apart from being closer to reality, is that without a labor market friction households use labor to excessively smooth consumption. Then we rely on insights gained in the finance literature and augment the basic framework with habit preferences and capital adjustment costs so as to enable the

economy, because consumption choices are endogenously determined and become smoother as risk aversion increases.', Cochrane (2005): '[Jermann (1998)] starts with a standard real business cycle (one-sector stochastic growth) model and verifies that its asset-pricing implications are a disaster.'

resulting model to explain the behavior of asset prices.

We show that our model can match key asset pricing moments as well as several important dimensions of asset pricing dynamics. The model can also replicate key moments of macroeconomic time series, including aggregate fluctuations in employment. The correlation of employment and output is, as in the data, positive. Compared to standard matching models, the volatility of employment relative to output is higher due to a feedback channel from financial markets, via the value of firms, on labor markets.

Chapter 4 — Long-Run Risk through Consumption Smoothing

In this chapter we demonstrate how long-run consumption risk arises *endogenously* in standard production economy models and how this additional risk factor can help these models to jointly explain the dynamic behavior of consumption, investment, and asset prices.

We assume that consumers have Epstein and Zin (1989) preferences and dislike negative shocks to future economic growth prospects. Unlike the case of power utility preferences, where risk is only associated with the shock to *realized* consumption growth, investors in this economy also dislike negative shocks to *expected* consumption growth and consequentially demand a premium for holding assets correlated with this shock. We show that even when the log technology process is a random walk, endogenous consumption smoothing increases the price of risk in the production economy model exactly because it increases the amount of long-run risk in the economy. The long-run risk, in turn, arises because consumption smoothing induces highly persistent time-variation in expected consumption growth rates.

In equilibrium, time-varying expected consumption growth turns out to be a small, but highly persistent, fraction of realized consumption growth. Note that this result is of particular interest since it is very difficult to empirically distinguish a small predictable component of consumption growth from i.i.d. consumption growth given the short sample of data we have available. Bansal and Yaron (2004), for instance, *calibrate* a process for consumption growth with a highly persistent trend component and demonstrate that their process can match a number of moments of aggregate consumption growth. In lieu of robust empirical evidence on this matter, the model presented in this chapter provides a theoretical justification for the previously proposed long-run risk dynamics of aggregate consumption growth based on a standard production economy setup.

Chapter 5 — Asset Pricing in Production Economies — Long-Run Risks, Wages, Dividends

In this chapter we show that by two simple adjustments the standard production economy model without habit preferences can be enabled to jointly explain consumption and asset prices to a remarkable extent. This runs counter to what is often taken for granted in the literature.[4]

When we allow the elasticity of intertemporal substitution to be different from the reciprocal of the coefficient of relative risk aversion, that is we use Epstein and Zin (1989) preferences, consumers care about long-run risks. In Chapter 4 we demonstrate that this allows a standard production economy model with Epstein and Zin preferences to easily match the market price of risk and equity Sharpe ratios. There we show that standard production economy models give rise to endogenous long-run risks due to consumption smoothing activities of the representative household. In this chapter we rely on this mechanism in order to enable the standard production economy model to generate realistic equity return Sharpe ratios.

Furthermore, in many standard production economy models the process for wages and consequentially the process for dividends are misspecified. Wages are assumed to be the marginal product of labor and turn out much too volatile and too procyclical relative to their empirical counterpart. This often renders dividends *countercyclical*. We show that by specifying a different wage process, following the search and matching literature in labor economics, and calibrating that process to the data, we can alleviate this problem. The resulting dividend process from our model turns out to be quite close to the data and, importantly, is *procyclical*. This drives up equity risk premiums and allows the model, given that the model can already match the equity Sharpe ratio, to generate both a realistic value of the equity premium as well as realistic equity return volatility.

In many models that rely on habit preferences, in particular in production economy models where simple internal habits are assumed, the risk-free rate is way too volatile, and higher risk premiums are in that sense generated through a too volatile risk-free rate. Since the risk-free rate is the reciprocal of the conditional expected value of the stochastic discount factor, a misspecified risk-free rate implies a misspecified stochastic discount factor. The standard production economy model we calibrate can generate realistic risk premiums *without* excessive risk-free rate volatility and *without* unrealistically high levels of risk aversion.

[4]See, e.g., Rouwenhorst (1995), Jermann (1998), Boldrin, Christiano, Fisher (2001).

Chapter 6 — The Term Structure of Interest Rates in Standard Production Economy Models

In this chapter we examine the term structure of interest rates in standard production economy models with Epstein and Zin (1989) preferences. We find that the average term structure generated by exchange economy and production economy models both with power utility preferences *and* with Epstein and Zin preferences is downward sloping. The jury seems to be still out on whether this contradicts the data or not.[5]

We demonstrate that the dynamic behavior of the term structure generated by the standard production economy model on the other hand clearly fits the data well: during an economic expansion the term structure is flatter or even inverted, while the term structure is steeper and upward sloping during a recession. When we feed shocks to total factor productivity from the data into the model, the correlation between the model implied time series of the term spread and the empirical time series of the term spread is relatively high.

Earlier findings in the literature assert that both standard consumption based exchange economy models and standard production economy models with power utility preferences fail at generating sufficiently volatile term spreads.[6] We show that the same is true for the model with Epstein and Zin (1989) preferences. So, while augmenting the standard production economy model with Epstein and Zin preferences improves the ability of the model to explain equity risk premiums to a substantial degree, as demonstrated in Chapters 4 and 5, the same can, unfortunately, not be said of the standard model's ability to explain the behavior of bond risk premiums.

[5]See Piazzesi and Schneider (2006).
[6]See, e.g., Backus, Gregory, Zin (1989) and Den Haan (1995).

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Chapter 2

Anticipated Growth and Business Cycles in Matching Models*

2.1 Introduction

Economists have long recognized the importance of expectations in explaining economic fluctuations. As early as 1927, Pigou postulated that '*the varying expectations of business men ... constitute the immediate cause and direct causes or antecedents of industrial fluctuations.*'[1] A recent episode where many academic and non-academic observers attribute a key role to expectations is the economic expansion of the 1990s. During the 1990s, economic agents observed an increase in current productivity levels, but also became more optimistic regarding future growth rates of productivity. In fact, there was a strong sense of moving towards a new era, the 'new economy', of higher average productivity growth rates for the foreseeable future. With the benefit of hindsight it is easy to characterize the optimism about future growth rates as 'unrealistic'. At the time, however, the signals about future productivity were in fact remarkable, and the view that a new era was about to begin was shared by many experts, including economic policy makers such as Alan Greenspan.[2] Similarly, the question arises whether the downward adjustment of these high expectations about future growth rates did not at least magnify, if not cause, the economic downturn that took place at the beginning of the new millennium.

More formal empirical evidence that business cycles are caused by anticipated changes

*This chapter is joint work with Prof. Wouter den Haan, University of Amsterdam.

[1] Pigou (1927, p. 29).

[2] See, for example, the following quote in Greenspan (2000): '... there can be little doubt that not only has productivity growth picked up from its rather tepid pace during the preceding quarter-century but that the growth rate has continued to rise, with scant evidence that it is about to crest. In sum, indications ... support a distinct possibility that total productivity growth rates will remain high or even increase further.'

in future productivity is provided by Beaudry and Portier (2006). They use changes in stock prices to identify that fraction of future changes in productivity that is anticipated, and argue that this fraction is actually quite large. They show that innovations in technology are small but initiate substantial future increases in productivity.[3] Moreover, this expectation shock leads to a boom in output, consumption, investment, and hours worked before the anticipated productivity growth actually materializes.

Beaudry and Portier (2005) analyze whether existing neo-classical models can generate Pigou cycles. In a Pigou cycle, output, consumption, investment, and hours worked jointly increase in response to an anticipated increase in productivity and these variables decline when the anticipated increase fails to materialize. They consider a large class of models and show that the answer is no.[4] Instead, the typical response is an increase in consumption but a decrease in investment and hours worked. The reason is that the wealth effect induces agents to increase consumption and leisure. It is not difficult to generate an increase in investment, because the anticipated increase in productivity also causes the expected return on capital to go up.[5] The problem is, however, that higher levels of investment are typically financed by a reduction in consumption, not by an increase in hours worked. The real challenge is therefore to build a model in which hours worked increase in response to anticipated productivity growth.

Perhaps, it should not come as a surprise that an anticipated increase in productivity does not lead quite naturally to a boom in existing models. In most business cycle models, aggregate productivity is an exogenous process, and agents get the increase in productivity 'for free'. As a result, the aggregate economy behaves the way an individual agent behaves if he finds out about a windfall to be received in the near future. He goes on a spending spree (i.e., consumption increases), takes a vacation (i.e., employment decreases), and finances this indulgence by dissaving (i.e., investment decreases).

Recently, some models have been developed where an increase in expected productivity generates a business cycle boom even though productivity improvements still fall like manna from heaven. Exemplary papers are Beaudry and Portier (2004), Beaudry and Portier (2005), Christiano, Motto, Rostagno (2006), and Jaimovich and Rebelo (2006). In Beaudry and Portier (2004), Beaudry and Portier (2005), and Jaimovich and Rebelo, the positive co-movement of investment and consumption is generated by making it too

[3]The standard assumption that productivity follows an AR(1) with an autoregressive coefficient close to or equal to one is not consistent with this empirical evidence. For an AR(1) process, a positive shock implies that expected productivity growth *decreases* when the autoregressive coefficient is less than one and remains unchanged when the autoregressive coefficient is equal to one.

[4]Cochrane (2004) and Danthine, Donaldson, Johnsen (1998) have made the same observation for more specific models.

[5]If the elasticity of intertemporal substitution is high enough, then the substitution effect dominates the wealth effect, and investment increases.

costly for variables to move in the 'wrong' direction. This can be accomplished by complementarities in the production technology or particular forms of capital adjustment costs. Christiano, Motto, Rostagno assume that nominal wages are sticky and show that this implies an expansionary monetary policy when expected future productivity increases. The reason is that the increase in the real wage caused by the expansion brings about a reduction in inflation when nominal wages are sticky. The reduction in inflation in turn leads to a reduction in interest rates when the central bank follows a Taylor rule.

In this chapter, we approach the challenge to build a model that can generate Pigou cycles from a different angle. The key idea is that increases in aggregate productivity are not free, at least not to everybody, and that in order to benefit from the anticipated increase in productivity agents have to invest resources. To show the strength of this argument, we build a model in which the productivity process is still exogenous, and an increase in productivity is free to everybody already engaged in productive activities. The increase in productivity does not come for free, however, to firms and workers that are not already engaged in market production. Instead, forming a productive relationship takes time and requires resources. As a result, firms start investing in new projects immediately and do not delay looking for additional workers when expected productivity growth increases. Similarly, there is an incentive for workers to enter the labor force as soon as expectations about future productivity increase. As employment increases, consumption and investment can increase before actual productivity goes up.

In particular, we incorporate a standard labor market matching framework into a real business cycle (RBC) model.[6] The labor market matching model is becoming (or is) the benchmark model to explain fluctuations in aggregate employment. Because of the matching friction it is well suited to model the idea that not everybody automatically benefits from productivity increases. To see whether this model can generate Pigou cycles, we study the transition from a low-growth-*low*-expectations regime to a low-growth-*high*-expectations regime. This transition does not affect actual productivity, but it does affect the probability of switching to a regime with high productivity growth rates. Expected future productivity, thus, increases. We will show that this increase in expected productivity generates an economic expansion even though productivity levels themselves have not yet gone up.

In the standard matching framework, the mass of workers that is either employed or searching for a job, that is, the total labor force, is fixed. In contrast, in the standard RBC model, labor supply is determined by a labor/leisure decision. An anticipated

[6] Several papers have incorporated the matching framework into a real business cycle model and examined the effect of *realized* increases in productivity. Examples are Merz (1995), Andolfatto (1996), and den Haan, Ramey, Watson (2000).

productivity increase then generates a reduction in employment, because the wealth effect increases the demand for leisure. The standard matching model—by keeping the mass of potential workers fixed—does not allow for this channel to operate and consequently makes it easier to generate Pigou cycles. We show, however, that the model can still generate Pigou cycles if we allow the wealth effect to affect labor supply just as in standard RBC models.[7]

In our model, just as in the standard RBC model, the wealth effect associated with an increase in expected productivity growth has a downward effect on labor supply. Nevertheless, labor supply only displays a very modest decline in our framework. The reason is that—because of the matching friction—the increase in the expected productivity growth rate increases the benefits for workers of being in a productive relationship, just like it increases the benefits for firms. Due to the higher number of vacancies being posted, the small decrease in the labor force goes together with an increase in the employment rate and a reduction in the unemployment rate.

The chapter is structured as follows. In Section 2.2 we discuss the model. In Section 2.3, we discuss standard summary statistics and we explain why our model generates a countercyclical unemployment rate, whereas other models with endogenous labor force participation do not. In Section 2.4, we document that the model can generate a typical expansion in response to an increase in the anticipated productivity growth rate. The last section concludes.

2.2 The Model

There are three types of agents: a representative household, entrepreneurs, and workers. The representative household takes the decision how much to consume, how much to save, and how much labor to supply. Entrepreneurs decide in how many new projects to invest and how much capital to rent for existing projects. In the planning phase, each new project requires a periodic fixed investment until production starts. Starting a new project also entails posting a vacancy. The number of vacancies and the number of workers searching for a job determine—using a standard matching function—the number of new productive relationships. Exogenous separation occurs with probability ρ^x. Productivity is high enough so that endogenous separation does not occur. At the end of any given period, all the agents in the economy distribute their net earnings to the representative household.

[7]In the matching literature, it is more common to model changes in the labor supply by means of endogenous search intensity. The advantage of our approach to endogenize the labor supply is that there is a clear empirical counterpart, which facilitates the calibration of the model.

2.2.1 Production

Production takes place within a relationship consisting of a worker and an entrepreneur. The production technology is given by:

$$y_t = Z_t k_t^{\alpha}, \tag{2.1}$$

where Z_t is an aggregate productivity, y_t firm output, and k_t firm capital. Capital is rented by the firm at rate R_t and the firm pays the worker a wage W_t. Each period the worker and the entrepreneur divide revenues net of capital payments:

$$\overline{p}_t = Z_t k_t^{\alpha} - R_t k_t. \tag{2.2}$$

The wage process is given by:

$$W_t = (1 - \omega_1)\omega_0 E\left[\overline{p}_t\right] + \omega_1 \omega_0 \overline{p}_t, \tag{2.3}$$

where ω_0 and ω_1 are fixed parameters and $E\left[\overline{p}_t\right]$ is the unconditional expectation of $\overline{p}_t$. The parameter ω_1 controls how the wage rate responds to changes in net revenues. If $\omega_1 = 0$, wages are fixed, whereas if $\omega_1 = 1$, wages are proportional to net revenues. The average wage rate, $E[W_t]$, is equal to $\omega_0 E[\overline{p}_t]$. Thus, ω_0 determines the fraction of net revenues the worker receives on average.

The firm chooses the capital stock that maximizes (2.2). Thus:

$$k_t = \left(\frac{Z_t}{R_t}\right)^{1/(\alpha-1)}. \tag{2.4}$$

2.2.2 The Productivity Process

The process of aggregate productivity is given by:

$$\ln Z_t = G_t + \rho \ln Z_{t-1} + \sigma \varepsilon_t. \tag{2.5}$$

The innovation, ε_t, has a standard Normal distribution. The drift term, G_t, can take a low value, G^{low}, and a high value, G^{high}. There are two regimes in which $G_t = G^{low}$. If, in period t, the economy is in the *low-growth-low-expectations* regime (or regime 1), then $G_t = G^{low}$ and it is impossible that $G_{t+1} = G^{high}$. If the economy is in the *low-growth-high-expectations* regime (or regime 2), then $G_t = G^{low}$ but $G_{t+1} = G^{high}$ is possible. Moving from regime 1 to regime 2, therefore, corresponds to a situation where expectations increase but current productivity levels remain unchanged.[8] There is only one regime in which $G_t = G^{high}$. This is the *high-growth* regime (or regime 3). The probability of moving from regime i to regime j is denoted by π_{ij}. The key restriction on the transition probabilities is that $\pi_{23} > \pi_{13}$.

2.2.3 New Projects

Entrepreneurs decide whether they want to start new projects. In the planning phase, projects require an investment equal to ψ each period. If the plan turns out to be successful, production can start. In the planning phase, entrepreneurs also search for a worker. The number of entrepreneurs with projects in the planning phase is determined by the free-entry condition, that is, the cost, ψ, has to equal the value of a successful project times the probability of being successful.

Profits of successful projects, p_t, are equal to net revenues minus wage payments, i.e., $p_t = \overline{p}_t - W_t$. The value of a successful project to the entrepreneur is simply the discounted value of profits, taking into account that the project is subject to the possibility of exogenous destruction in subsequent periods. Thus:

$$V_t = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} (p_{t+1} + (1 - \rho^x) V_{t+1}) \right], \tag{2.6}$$

where $(C_{t+1}/C_t)^{-\gamma}$ is the marginal rate of substitution. The free-entry condition can then be written as:

$$\psi = \lambda_t^f V_t, \tag{2.7}$$

where λ_t^f is the probability that a project in the planning phase is successful and a suitable worker is found. In Fujita (2003) planning of the project and searching for a worker are modeled separately. For parsimony, we adopt here the standard convention, subsume planning and searching under one phase, and assume that the probability λ_t^f describes success on both counts.

2.2.4 The Matching Market

On the matching market, entrepreneurs post vacancies and search for a worker. The number of matches, M_t, is determined by the number of searching workers, i.e., the unemployed, N_t^s, and the number of vacancies, N_t^v, which is equal to the number of projects in the planning phase. The matching process is modeled with the standard

[8] This regime change corresponds to the experiment considered in Beaudry and Portier (2005).

constant returns to scale matching function, unless matches are less than the number of workers searching for a job. That is:

$$M_t = \min\{\mu_0 N_t^s \left(\frac{N_t^v}{N_t^s}\right)^{\mu_1}, N_t^s\}, \tag{2.8}$$

$$\lambda_t^w = \frac{M_t}{N_t^s}, \text{ and } \lambda_t^f = \frac{M_t}{N_t^v}. \tag{2.9}$$

We allow the number of matches to exceed the number of vacancies,which would correspond with a value of λ_t^f bigger than 1.[9] Note that it is logically not impossible that a firm manages to get more than one worker by posting only one vacancy.

2.2.5 The Household

The household chooses consumption, C_t, total labor supply, and next period's beginning-of-period capital stock, K_{t+1}. Labor supply is equal to the sum of employed workers, N_t^w, and workers searching for a job, N_t^s. Capital earns a rate of return R_t and depreciates at rate δ.

Next period's beginning of period employment consists of those workers that have not experienced exogenous separation, $(1-\rho^x)N_t^w$, and those workers that are matched during the current period, $\lambda_t^w N_t^s$. Thus:

$$N_{t+1}^w = \lambda_t^w N_t^s + (1-\rho^x)N_t^w. \tag{2.10}$$

The household trades off the benefits of being engaged in market activity with benefits of activities such as leisure and home production. Searching is assumed to be a full-time activity. Consequently, the time spent on leisure and home production, L_t, is equal to $N^* - N_t^s - N_t^w$. The utility of current-period leisure is given by $\phi L_t^{1-\kappa}/(1-\kappa)$.

[9] For the functional form of the matching function, the equilibrium choice for N^v is bounded away from zero as long as $\psi < V_t$. If one would impose that $M_t \leq N_t^v$ (i.e., $\lambda_t^f \leq 1$), then a sudden jump to $N^v = 0$ would occur if V_t would drop below ψ. By allowing $M_t > N_t^v$ whenever $\psi > V_t$, one avoids this discontinuity. This simplifies the computational procedure considerably even though it is rare that $\psi > V_t$.

The household's maximization problem is as follows:

$$\max_{\left\{C_{t+j},N^s_{t+j},N^w_{t+j+1}\right\}_{j=0}^{\infty}} E_t \sum_{j=0}^{\infty} \left[\beta^j \frac{C_{t+j}^{1-\gamma}-1}{1-\gamma} + \phi \frac{\left(N^* - N^s_{t+j} - N^w_{t+j}\right)^{1-\kappa}}{1-\kappa} \right], \tag{2.11}$$

s.t.

$$N^w_{t+j+1} = \lambda^w_{t+j} N^s_{t+j} + (1-\rho^x) N^w_{t+j}, \tag{2.12}$$

$$C_{t+j} + K_{t+j+1} = W_{t+j} N^w_{t+j} + R_{t+j} K_{t+j} + (1-\delta) K_{t+j} + P_{t+j} - N^v_{t+j}\psi. \tag{2.13}$$

Here, $P_t = p_t N^w_t$ are the profits of the entrepreneurial sector.

This specification of the utility function for the representative agent assumes that there is perfect risk sharing, not only in terms of consumption, but also in terms of leisure.[10] An alternative would be to use the lottery setup of Rogerson (1988), where agents use lotteries to insure consumption against unfavorable labor market outcomes.[11] This approach seems less suitable for a model with labor force participation, since it implies that not only being unemployed, but also not being in the labor force is a random outcome. Moreover, Ravn (2006) shows that in a matching model the implied linear utility function leads to a relationship between aggregate consumption and labor market tightness, N^v_t/N^s_t, that is inconsistent with the empirical properties of smooth aggregate consumption on one hand and volatile tightness on the other for reasonable parameter values. In Appendix C, we show that our specification avoids Ravn's consumption-tightness puzzle.

Let η^m_t be the Lagrange multiplier of the constraint of the law of motion of N^w_t. This multiplier represents the shadow price for a worker of being in a match. The first-order conditions are as follows:

$$C_t^{-\gamma} = E_t \left[\beta C_{t+1}^{-\gamma} \left(r_{t+1} + (1-\delta)\right)\right], \tag{2.14}$$

$$\phi L_t^{-\kappa} = \lambda^w_t \eta^m_t, \tag{2.15}$$

$$\eta^m_t = \beta E \left[W_{t+1} C_{t+1}^{-\gamma} - \phi L_{t+1}^{-\kappa} + (1-\rho^x)\eta^m_{t+1}\right]. \tag{2.16}$$

Equation (2.14) is the standard intertemporal Euler equation. Equation (2.15) is the first-order condition of leisure. The left-hand side of this equation is the disutility of entering the labor market, i.e., the disutility of searching, and the right-hand side is the expected benefit of searching, $\lambda^m_t \eta^m_t$, that is, the worker gets η^m_t with probability λ^m_t. Equation (2.16) specifies the expected benefit of leaving period t employed, η^m_t. First, a

[10] A similar approach is followed by Hornstein (1998), Shi and Wen (1999), and Tripier (2003).

[11] The utility of leisure would then be given by $\phi\left[(N^* - N^s_t - N^w_t) \times 1^{1-\kappa} + (N^s_t + N^w_t) \times 0^{1-\kappa}\right]/(1-\kappa)$, which is equal to $\phi(N^* - N^s_t - N^w_t)/(1-\kappa)$, i.e., utility is linear in leisure.

matched worker obtains a wage payment worth $W_{t+1}C_{t+1}^{-\gamma}$. Second, the worker has to put in effort, generating disutility of $-\phi L_{t+1}^{-\kappa}$. Finally, in case of no separation, the worker gets the expected benefits of leaving period $t+1$ employed, η_{t+1}^{m}.

2.2.6 Recursive Equilibrium

Equilibrium on the market for rental capital requires that total demand for capital is equal to the available aggregate capital stock:

$$N_t^w k_t = K_t. \tag{2.17}$$

The state variables of the model, s_t, consist of Z_t, the growth regime, K_t, and N_t^w. An equilibrium is a set of functions $C(s_t)$, $K'(s_t)$, $N^s(s_t)$, $N^w(s_t)$, $N^v(s_t)$, $V(s_t)$, $\eta_t^m(s)$, $R(s_t)$, $\lambda_t^w(s)$, $\lambda_t^f(s)$, and $k(s_t)$ that are consistent with:

- household optimization, that is, the first-order conditions (2.14), (2.15), and (2.16), the budget constraint (2.13), and the law of motion for matched workers (2.12),
- the free-entry condition (2.7),
- firm optimization, that is, the first-order condition (2.4),
- the value of a successful project to the entrepreneur given by (2.6),
- the definition of the matching probabilities, and
- the capital market clearing condition (2.17).

2.2.7 Allocation Friction

We also consider a version of the model where substitution between consumption and investment is costly. The idea is that different technologies are used to produce consumption and investment, and that moving away from the steady state ratio of consumption to investment is therefore costly. In this modified version of the model, the aggregate budget constraint is given by:

$$\left[\eta_c \left(C_t^{\nu_c}\right)^{\xi} + \eta_i \left(I_t^{\nu_i}\right)^{\xi}\right]^{1/\xi} = W_t N_t^w + R_t K_t + P_t - N_t^v \psi, \tag{2.18}$$

$$K_{t+1} = I_t + (1-\delta)K_t. \tag{2.19}$$

If $\eta_c = \eta_i = \nu_c = \nu_i = \xi = 1$, the model is identical to the benchmark model. As the value of ξ increases above 1, it becomes costlier to substitute consumption and investment. We

Table 2.1: **Calibration**

	Monthly Model	
Parameter	Value	Target / Source
Discount factor, β	0.9966	Standard annual value = 0.96
Relative risk aversion, γ	0.475	Range of values considered
Scaling utility of leisure, ϕ	0.44	$N^s + N^w = 1$
Curvature utility of leisure, κ	2.5	$\sigma\left[\frac{N^s+N^w}{N^*}\right]/\sigma\left[\ln\frac{Y}{N^w}\right] = 0.182$
Time endowment, N^*	1.5938	$\frac{N^s+N^w}{N^*} = 0.6274$
Curvature production function, α	0.315	$\alpha + \omega_0(1-\alpha) = 1/3$
Depreciation rate, δ	0.0084	Standard annual value = 0.10
Drift term, $g^{high} = g^{low}$	0.005/12	Congressional Budget Office
ζ_2	12	Expected duration of staying in regime 2
ζ_3	120	Expected duration of staying in regime 3
Persistence parameter, ρ	0.98	See discussion in the main text
Innovation standard deviation, σ	0.0042	$\sigma[\ln Z_t = 0.95\ln Z_{t-1} + 0.007\varepsilon_t]$
Wage sensitivity, ω_1	0.7547	$\sigma[\ln W]/\sigma\left[\ln\frac{Y}{N^w}\right] = 0.755$
Share of entrepreneur, ω_0	0.9725	$\sigma\left[\frac{N^w}{N^*}\right]/\sigma\left[\ln\frac{Y}{N^w}\right] = 0.437$
Match elasticity, μ_1	0.50	Petrongolo and Pissarides (2001)
Match scaling, μ_0	0.39	$\lambda^w = 45.4\%$
Period entry cost, ψ	0.94	$\lambda^f = 33.8\%$
Exogenous destruction rate, ρ^x	0.027	$\frac{N^s}{N^s+N^w} = 5.7\%$

choose η_c and η_i such that the steady state of the modified model is equal to the steady state of the benchmark model. We choose ν_c and ν_i such that the volatility of consumption and investment are not affected by the magnitude of the allocation costs.

2.2.8 Calibration

Parameter values are either set to standard values or calibrated to match key characteristics of empirical data. The model period is one month. Parameter values are given in Table 2.1. This table also reports either the source for the parameter value or the empirical moment that is most important for the identification of the parameter value. Data sources used to estimate the empirical moments are given in Appendix A.

Preferences

Using a standard annual discount rate of 4% implies for a monthly model a value of β equal to 0.9966. The coefficient of relative risk aversion, γ, plays a key role in the model and we will consider several values. The benchmark value is 0.475. The reason for this choice will become clear in Section 2.4.4. The scaling factor of the utility of leisure, ϕ, is chosen so that the steady state labor force, $N^s + N^w$, is equal to 1. To ensure that labor force participation, $(N^s + N^w)/N^*$, is equal to the observed value of 0.6274, we set $N^* = 1.5938$. The curvature parameter in the utility function of leisure, κ, is chosen to ensure that the model matches the volatility of labor force participation. The calibrated value of κ implies an elasticity of labor supply with respect to the expected benefit of being matched, $\lambda^w \eta^m$, equal to 0.24.[12] This is slightly higher than 0.15, which is the typical value of the Frisch elasticity used in New-Keynesian models.[13]

Production Technology

The standard annual depreciation of 10% corresponds to a value of δ equal to 0.0084 on a monthly basis. The value for α is chosen so that the labor share is equal to the standard value of two thirds. The remaining one third is divided between capital providers, who get a share α of total output, and entrepreneurs, who get $\omega_0(1-\alpha)$. Thus, $\alpha+\omega_0(1-\alpha) = 1/3$. The calibrated value for ω_0 is equal to 0.9725 (see discussion below). Thus, $\alpha = 0.315$. This implies a steady state ratio of physical capital to output, k, equal to 2.22 on an annual basis. The ratio of total capital to output $(N^w k + N^w V)/N^w y$ is equal to 2.27 on an annual basis, which is fairly close to the typical value of 2.5.

Productivity Process

Edge, Laubach, Williams (2004) report real-time estimates of long-run productivity growth and find that estimates of the Congressional Budget office vary from a low of 1.2% in 1996 to a high of 2.7% in 2001. In our monthly model, we set $G^{high} = -G^{low} = \left(\frac{1}{12}\right) 0.5\%$. This means that on an annual basis the difference between the low-growth and the high-growth regime is equal to one percentage point. Such a change seems reasonable, so our results are not driven by unrealistic changes in expectations.

To examine whether our model can generate Pigou cycles, we need to specify the transition matrix such that the expected growth rate in regime 2 exceeds the expected growth rate in regime 1. An easy way to do this is to set $\pi_{13} = 0$. For parsimony, we set $\pi_{21} = \pi_{32} = 0$. Finally, we set the transition probabilities such that the unconditional probability of being in the high-growth regime is equal to 1/2, so that $\mathrm{E}[G_t] = \mathrm{E}[\ln(Z_t)] = 0$. Given these restrictions, the transition matrix Π is fully determined by the expected duration of staying in regime 2, ζ_2, and the expected duration of staying in regime 3, ζ_3,

[12] The elasticity of labor supply with respect to the expected benefit of being matched is equal to $(N^*/(N^s + N^w) - 1)/\kappa$.

[13] See, for example, Ball and Romer (1989).

and Π can be written as:

$$\Pi = \begin{bmatrix} 1-(\zeta_3-\zeta_2)^{-1} & (\zeta_3-\zeta_2)^{-1} & 0 \\ 0 & 1-\zeta_2^{-1} & \zeta_2^{-1} \\ \zeta_3^{-1} & 0 & 1-\zeta_3^{-1} \end{bmatrix}. \tag{2.20}$$

We set ζ_3 equal to 120, so that the expected duration of staying in the high-growth regime is ten years. We set ζ_2 equal to 12 months.

A switch from regime 2 to regime 3 increases the value of productivity, Z_t, in the first period by $G^{high} - G^{low}$ percentage points. In each subsequent period the economy remains in regime 3, the value of Z_t continues to increase. The higher the value of ρ, the more persistent the effects of the regime change on the growth rate. The idea of our regime change is a persistent change in the growth rate so ρ should be high, but for computational reasons we consider values less than 1. In particular, we set ρ equal to 0.98. We have also considered $\rho = 0.99$ and $\rho = 0.95$ and found that the results are qualitatively very similar.

The value for the volatility of the innovation, σ, is chosen as follows. The standard process for quarterly productivity in the real business cycle literature is $\ln(\widetilde{Z}_t) = 0.95\ln(\widetilde{Z}_{t-1}) + 0.007\widetilde{\varepsilon}_t$, where $\widetilde{\varepsilon}_t$ is a standard normal.[14] We set the value of σ so that the volatility of HP-filtered $\widetilde{Z}_t$ is equal to the volatility of our process Z_t, as specified in Equation (2.5), after the monthly observations of Z_t have been transformed into quarterly data and then HP-filtered.

Wage Process

The parameter ω_1 controls the sensitivity of wages to changes in net revenues, $Z_t k_t^\alpha - R_t k_t$. We calibrate ω_1 to match the volatility of wages relative to the volatility of labor productivity. The value of ω_0 represents the share of net revenues that workers receive. A smaller value of $1 - \omega_0$ implies that firm value, V_t, is more responsive to changes in productivity and implies a higher level of employment volatility. We choose the value of ω_0 to match the volatility of the employment ratio, N_t^w/N^*, relative to the volatility of labor productivity, implying a value for ω_0 of 0.9725. This value and our value for α imply that workers obtain 66.67% of value added, providers of capital receive 31.45%, and entrepreneurs receive 1.89%.

[14] See, for example, Cooley and Prescott (1995).

Matching Technology

The matching elasticity with respect to labor market tightness, μ_1, is taken from Petrongolo and Pissarides (2001). The values of μ_0, ψ, and ρ^x are chosen to match (i) a steady state matching probability for the worker equal to the empirical average of 45.4%, (ii) a steady state matching probability for the firm equal to 33.8%, and (iii) a steady state unemployment rate equal to the empirical average of 5.7%.[15]

Model without Labor Force Participation

The parameters for the model without endogenous labor force participation are identical to those of the model with endogenous labor force participation, except that $\phi = 0$, that is, there is no disutility of labor, and the time endowment, N^*, is rescaled so that the mass of the labor force, $N_t^s + N_t^w$, remains equal to the steady state value in the benchmark model, i.e., $N_t^s + N_t^w = 1$.

2.3 Summary Statistics

2.3.1 Standard Business Cycle and Labor Market Statistics

Table 2.2 reports standard business cycle as well as labor market statistics for the model with and without endogenous labor force participation. Generated volatilities for consumption, investment, and output have the standard ordering. That is, consumption is less volatile and investment is more volatile relative to output.[16] HP-filtered output is 42% more volatile than total factor productivity in the model with labor force participation and 28% more volatile in the model without labor force participation, so endogenous labor supply is helpful in magnifying shocks.

Output and labor productivity are not quite as volatile as in the data. Just as in most models, shocks are not sufficiently magnified. However, the volatilities of labor market statistics relative to the volatility of labor productivity, such as the volatility of tightness and the volatilities of the matching probabilities, do look very good. As pointed out by Shimer (2005), standard matching models cannot generate sufficient volatility in those

[15] A monthly matching probability for the firm equal to 33.8% implies that the probability of not being matched within any given quarter is equal to 29%, which corresponds to the value reported in van Ours and Ridder (1992).

[16] The model underestimates the relative volatility of consumption. The same is true for standard business cycle models. See, for example, Cooley and Prescott (1995). Employment in our model fluctuates less than hours in the standard RBC model, which explains why the marginal productivity of aggregate capital and the rental rate are less volatile as well. This in turn explains why investment in our model is somewhat less volatile than in the standard RBC model.

Table 2.2: **Summary Statistics**

U.S. Data are quarterly and HP-filtered using a smoothing parameter of 1600. Model data are monthly and transformed into quarterly data and then HP-filtered. All variables that are not expressed as a rate are logged.

	Data	Model I	Model II	Model III	Model III*
Variable labor force participation		Yes	No	No	No
Allocation costs		No	No	Yes	Yes
Used for calibration of Model I					
$\sigma\left[\frac{N^s+N^w}{N^*}\right]/\sigma\left[\ln\frac{Y}{N^w}\right]$	0.182	0.189	na	na	na
$\sigma\left[\frac{N^w}{N^*}\right]/\sigma\left[\ln\frac{Y}{N^w}\right]$	0.437	0.437	0.282	0.255	0.296
$\sigma\left[\ln W\right]/\sigma\left[\ln\frac{Y}{N^w}\right]$	0.755	0.755	0.755	0.755	0.755
Not used for calibration					
$\sigma\left[\ln Y\right]$	0.016	0.013	0.011	0.011	0.011
$\sigma\left[\ln I\right]/\sigma\left[\ln Y\right]$	4.560	3.733	3.753	1.924	3.757
$\sigma\left[\ln C\right]/\sigma\left[\ln Y\right]$	0.696	0.305	0.301	0.637	0.299
$\sigma\left[\ln\frac{N^w}{N^*}\right]/\sigma\left[\ln Y\right]$	0.466	0.448	0.342	0.317	0.355
$\sigma\left[\ln Y\right]/\sigma\left[\ln Z\right]$		1.420	1.280	1.238	1.281
$\sigma\left[\ln\frac{Y}{N^w}\right]$	0.013	0.008	0.008	0.008	0.008
$\sigma\left[\frac{N^wW}{Y}\right]/\sigma\left[\ln\frac{Y}{N^w}\right]$	0.644	0.163	0.163	0.164	0.163
$\sigma\left[\frac{N^v}{N^s}\right]/\sigma\left[\ln\frac{Y}{N^w}\right]$	18.98	21.49	20.79	19.47	21.23
$\sigma\left[\lambda^w\right]/\sigma\left[\ln\frac{Y}{N^w}\right]$	2.644	3.426	3.342	3.156	3.379
$\sigma\left[\ln Y, \frac{N^s}{N^s+N^w}\right]$	-0.86	-0.80	-0.89	-0.89	-0.88
$\sigma\left[\ln\frac{Y}{N^w}, \frac{N^s}{N^s+N^w}\right]$	-0.33	-0.61	-0.76	-0.78	-0.73
$\sigma\left[\ln N^v, \frac{N^s}{N^s+N^w}\right]$	-0.93	-0.40	-0.72	-0.72	-0.71

statistics. In our model, the share that accrues to the entrepreneur is—as in Hagedorn and Manovskii (2006)— relatively small, inducing volatile profits, which in turn generate sufficiently volatile labor market statistics. One labor market statistic that the model does not fit well is the volatility of the labor share. In the standard RBC model, the labor share does not fluctuate at all and is equal to $1 - \alpha$ in every period. Similarly, in our model the combined share that goes to the worker and the entrepreneur is fixed and equal to $1 - \alpha$. This fixed share, however, is divided in non-constant proportions, so that in our model the labor share does fluctuate. The standard deviation of the labor share, relative to the standard deviation of labor productivity is, however, still only 25% of its empirical counterpart.

A look at the Beveridge Curve, i.e., the co-movement between unemployment and vacancies, reveals a difference between the model with and without labor force participation. The data display a very strong negative correlation. Both models can generate a strong negative correlation, although not as high as the empirical counterpart. In the model with endogenous labor force participation, however, the stronger incentive to enter the labor market when economic conditions improve induces a lower correlation between unemployment and vacancies. In particular, the correlation between unemployment and vacancies is equal to -0.40 in the model with endogenous labor force participation, -0.72 in the model without endogenous labor force participation, and -0.93 in the data.

2.3.2 Countercyclical Unemployment Rate

Tripier (2003) argues that RBC models with both a matching framework and endogenous labor force participation cannot generate a countercyclical unemployment rate. Veracierto (2004) reaches the same conclusion. Our model, however, does generate a countercyclical unemployment rate. In the data the correlation between the unemployment rate and output is equal to -0.86 and we find correlation coefficients equal to -0.80 and -0.89 for the model with and without endogenous labor force participation, respectively. The correlation between unemployment and labor productivity is equal to -0.61 in the model with endogenous labor force participation, even stronger than the empirical counterpart, which is equal to -0.33.

The question arises why a model as simple as ours can generate a countercyclical unemployment rate whereas other models with endogenous labor force participation cannot.[17] The difficulty of generating a countercyclical unemployment rate is related to the challenge for matching models to generate sufficient volatility in tightness, N_t^v/N_t^s. If

[17] Haefke and Reiter (2006) develop a much more intricate model with heterogeneity in home production that also generates a countercyclical unemployment rate.

tightness and therefore the matching probability are not very responsive to a positive productivity shock, then an increase in labor force participation leads to an increase in the unemployment rate. In our model, profits and therefore vacancies do respond strongly to an increase in productivity. The sharp increase in vacancies allows the model to generate an increase in labor force participation *and* a reduction in the unemployment rate at the same time.

2.4 Pigou Cycles: Expectation Driven Real Business Cycles

In this section, we document that our model can generate Pigou cycles. In particular, we document that an increase in the probability of moving to the high-growth regime can generate an increase in output, consumption, investment, and employment, i.e., the features of a typical boom.[18]

2.4.1 Impulse Response Functions

Output Response when Expectations Increase

Figures 2.1 to 2.3 plot the impulse response function of output if the economy moves into the low-growth-high-expectations regime. Since the model is nonlinear, we calculate the impulse response function at different points in the state space.[19] The figures plot the average responses and the responses corresponding to the 10^{th} and 90^{th} percentile.

From Figure 2.1 we can see that output does not change in the month when the economy moves from regime 1 to regime 2. The reason is that capital and employment are predetermined. The figure however demonstrates that output does increase in the second month even though current productivity levels still have not changed. Figure 2.2 plots the response of output net of the investment costs associated with new projects. Initially, the investment in new projects reduces the amount of remaining available resources. After already two months, however, this output measure displays a positive response even when the responses for the 10^{th}-percentile are used, which are below the mean responses.

The figures also make clear the dependence on initial conditions, because the impulse

[18] In terms of our driving process this means a movement from the low-growth-low-expectations regime to the low-growth-high-expectations regime.

[19] The impulse response function measures the effect of moving to regime 2 compared to staying in regime 1. To determine the set of initial points in the state space to consider, we simulate the economy for 100,000 periods, and then use the values of the state variables in the periods when the economy switches from regime 1 to regime 2.

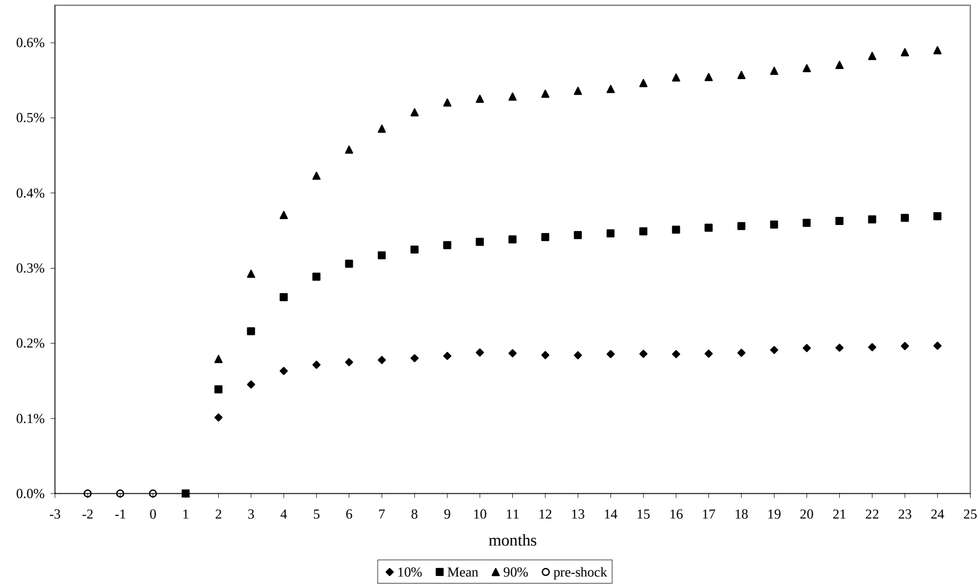

Figure 2.1: **Response to Anticipated Productivity Increase — Output**
This figure plots the responses when the economy moves from regime 1 to regime 2. Because of nonlinearities in the model, responses are calculated at different initial conditions. We plot the mean response as well as the 10th and the 90th percentile.

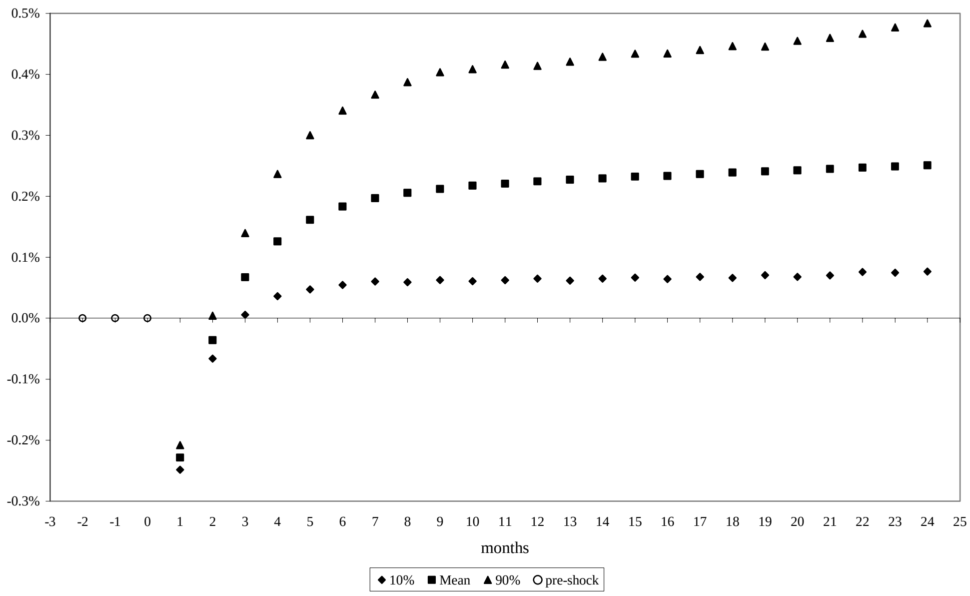

Figure 2.2: **Resp. to Anticip. Prod. Incr. — Output Net of Inv. in New Projects**
This figure plots the responses when the economy moves from regime 1 to regime 2. Because of nonlinearities in the model, responses are calculated at different initial conditions. We plot the mean response as well as the 10th and the 90th percentile.

response function of the 10th percentile lies substantially below the response of the 90th percentile.[20] Although it matters for the quantitative results in which state of the world the economy switches, the nonlinearity does not affect the qualitative results. We will, therefore, for the sake of expositional clarity, from now on only report the mean responses.

Responses of Consumption, Investment, and Hours Worked when Expectations Increase

Figure 2.3 plots the impulse response function of employment and Figure 2.4 plots the responses of consumption, investment, and investment excluding the investment in new projects. Our timing assumption implies that projects started in period t can at best be productive in period $t+1$. Consequently, employment only starts to increase in the period after the shock. As documented in the figure, it takes several periods before the increase in employment settles down.

Because both capital and employment are predetermined in the period that the shock occurs, either consumption or total investment must decrease in the first period. For the chosen value of the elasticity of intertemporal substitution $(1/\gamma)$ the value of consumption increases, thus, total investment decreases in the period of the shock. In the second period, however, when employment and resources are higher, both the consumption and the total investment response are positive.

In the period of the shock, investment in new projects sharply increases by 15%. Since total investment decreases, this means that the investment in existing projects decreases by more than total investment. In the first period, investment in existing projects decreases by 1.4% compared with a 0.35% decrease in total investment. Investment in existing projects quickly recovers, however, and displays a positive response after one quarter.

An Expected Increase in Productivity versus an Actual Increase

Figure 2.5 plots the (mean) impulse response function of output when the economy moves to regime 2 and the impulse response function when the economy moves to regime 3 after having been in regime 2 for 120 months.[21] The figure makes clear that just a change in expectations can generate a response that is initially substantial relative to the actual

[20] The switch to regime 2 has the strongest effects on economic activity when the initial levels of employment, capital, and productivity are the lowest.

[21] The responses when the economy moves to regime 3 are hardly affected by the number of periods the economy has spent in regime 2. Below, we calculate the trend component of the response using a two-sided filter and by letting the economy be in regime 2 for such a long time period we ensure that the trend component of the response of moving to regime 2 is—at least initially—not affected by the subsequent shift to regime 3.

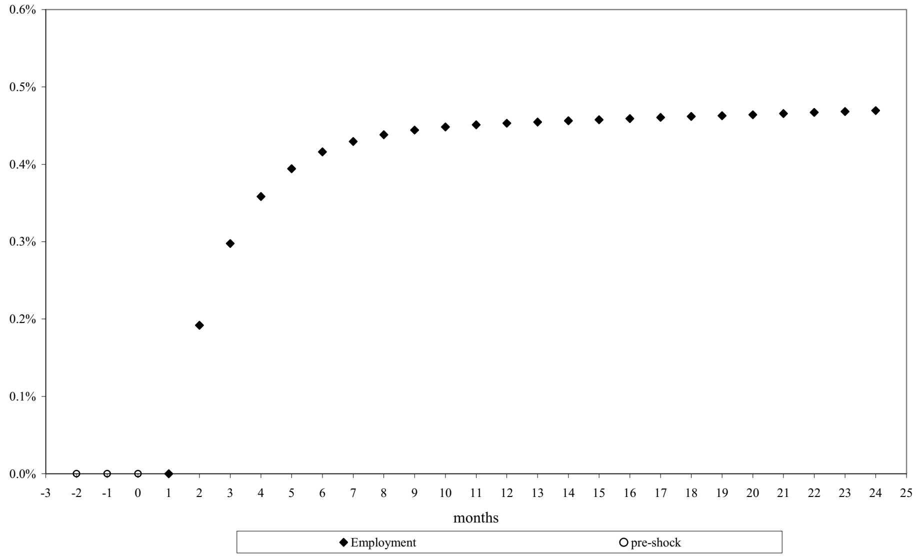

Figure 2.3: **Response to Anticipated Productivity Increase — Hours Worked**
This figure plots the responses when the economy moves from regime 1 to regime 2. Because of nonlinearities in the model, responses are calculated at different initial conditions. We plot the mean responses.

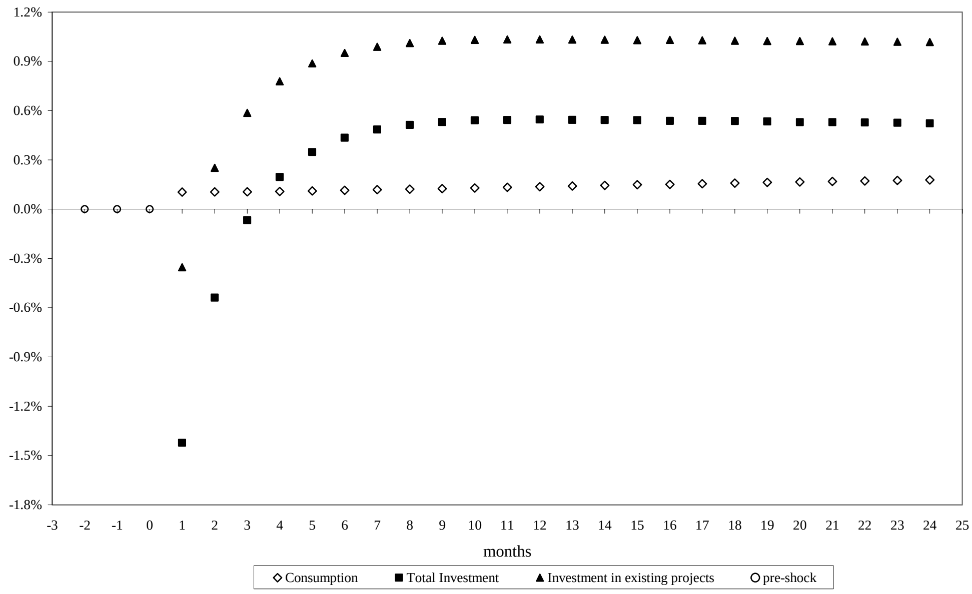

Figure 2.4: **Response to Anticipated Productivity Increase — Cons. and Inv.**
This figure plots the responses when the economy moves from regime 1 to regime 2. Because of nonlinearities in the model, responses are calculated at different initial conditions. We plot the mean responses.

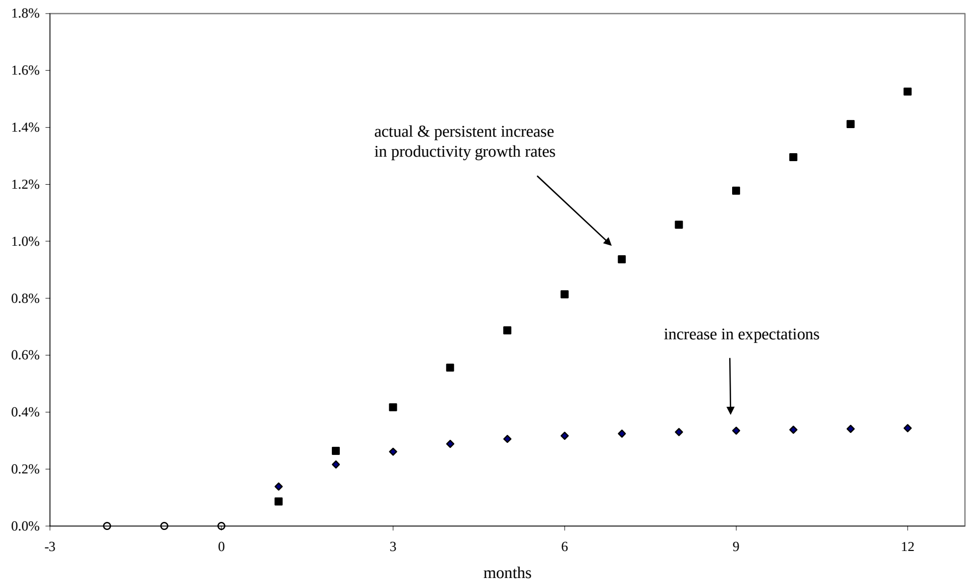

Figure 2.5: **Expected vs. Actual Increase in Prod. — Actual Output Response**
This figure plots the average responses when the economy moves from regime 1 to regime 2 and when the economy moves from regime 2 to regime 3. We calculate the actual responses when the economy moves first into regime 2 and (after 120 periods) into regime 3. Period 1 corresponds to the first period when the actual output response is positive.

response if the growth rate does increase. In particular, the output response when the economy moves into regime 2 relative to the response when the economy moves into regime 3 is equal to 39% and 12% after six and twelve periods, respectively. After some time, however, the increase of output if the economy continues to be in regime 3 naturally overwhelms the response in regime 2, because the growth rate is assumed to be persistent.

Figure 2.6 therefore plots the cyclical component of moving into regime 2 and into regime 3. The figure shows that the cyclical response of output when expected productivity increases is substantial relative to the cyclical response of output when an actual increase in productivity occurs. In fact, in the first 18 months the cyclical response of output when moving to regime 2 exceeds the cyclical response when moving to regime 3. In both cases, the cyclical component is initially negative, which is due to the increase in the trend.[22] The cyclical component becomes positive much quicker when the economy moves into regime 2 than when it moves into regime 3, which is due to a smaller increase in the trend in regime 2. When the economy moves into regime 2, the maximum cyclical response is equal to 42% of the maximum response observed when the economy moves into regime 3.

[22]The trend is calculated using the HP filter, which is a two-sided filter. Higher future values, thus, raise the current trend value.

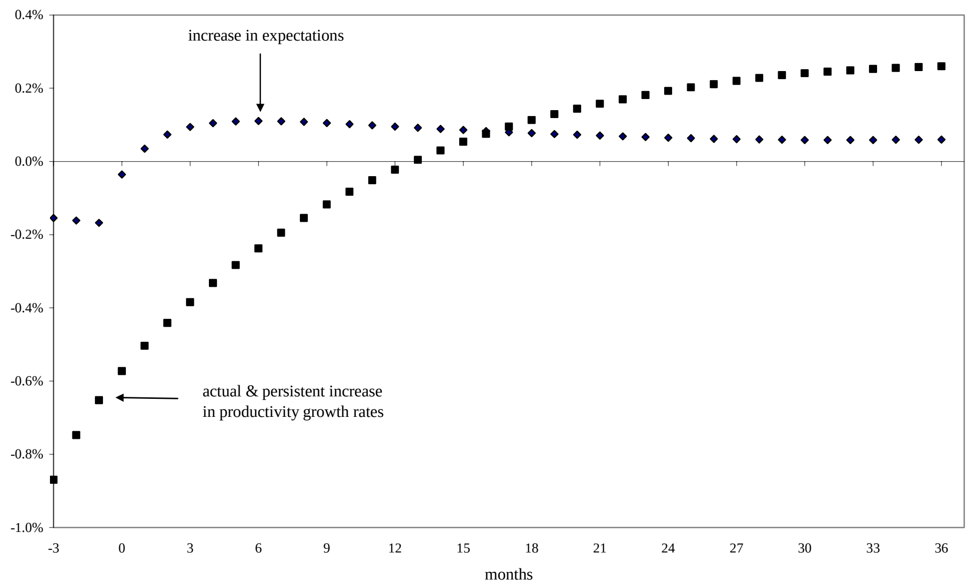

Figure 2.6: **Expected vs. Actual Increase in Prod. — Cyclical Output Response**
This figure plots the average responses when the economy moves from regime 1 to regime 2 and when the economy moves from regime 2 to regime 3. We calculate the corresponding cyclical responses when the HP filter is used to calculate the trend. Period 1 corresponds to the first period when the actual output response is positive.

2.4.2 The Role of Labor Force Participation

Figure 2.7 plots the output response of switching to regime 2 for both the economy with and without endogenous labor force participation. The response is slightly bigger in the economy without endogenous labor force participation.

In standard RBC models, endogenous labor supply is an important reason why the model cannot generate Pigou cycles. In particular, the increase in consumption reduces the marginal benefit of working, resulting in lower labor supply. This channel is operating in our matching model with endogenous labor supply as well. Moreover, since wages are related to current-period profits, wages actually decrease in regime 2, further reducing labor supply.

In our model, in contrast to standard RBC models with endogenous labor supply, there is, however, a force that pushes labor supply up. The anticipated increase in productivity implies an increase in expected future wages and, thus, the benefits of being employed. Workers therefore enter the labor force for the same reason as entrepreneurs start new projects. Moreover, the increase in vacancies increases the matching probability for the worker, further pushing labor force participation up. This channel almost offsets the substitution effect of the current-period wage reduction and the wealth effect. Conse-

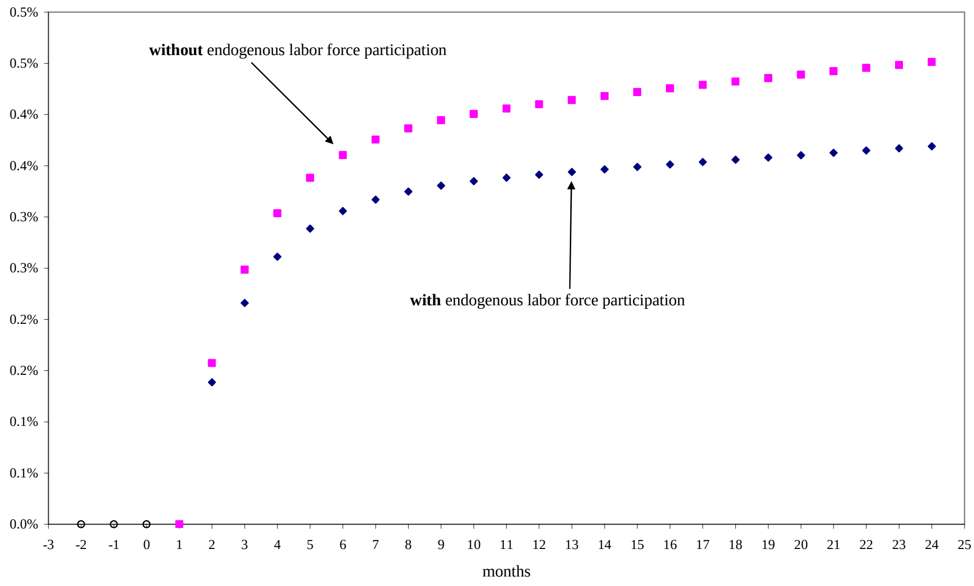

Figure 2.7: **Output Resp. with and without Endog. Labor Force Participation**
This figure plots the responses when the economy moves from regime 1 to regime 2. Because of nonlinearities in the model, responses are calculated at different initial conditions. We plot the mean responses.

quently, the labor force declines very little in regime 2, which explains why the response of employment and output in the economy with endogenous labor force participation are so similar to the responses in the economy without endogenous labor force participation.

2.4.3 Beaudry-Portier Puzzle

As discussed in the introduction, Beaudry and Portier (2005) show that for a large class of neoclassical models it is not possible for consumption, investment, and hours worked to jointly increase if productivity remains constant. There are two parts to this puzzle. First, investment has to increase. Second, the increase of investment must be financed by higher employment not by less consumption. As pointed out by Beaudry and Portier (2005), obtaining the first is easy, but obtaining the second is less straightforward.[23] In our model, however, the forward-looking behavior of firms starting new projects and workers deciding to enter the labor force does lead to an increase in resources. Moreover, for our benchmark parameter values this increase in resources is used for both an increase in consumption and investment. The result that both consumption and investment increase is, however, sensitive to the value of the elasticity of intertemporal substitution, $1/\gamma$,

[23] If agents have an infinite elasticity of intertemporal substitution then the higher expected rental rate of capital induces more investment, which is made possible by lowering consumption.

which will be discussed in the next section.

Beaudry and Portier (2005) use a local approach and investigate whether consumption, investment, and hours worked can simultaneously and *instantaneously* increase in response to positive news. Our model cannot generate such an instantaneous co-movement, because—given our timing assumption—capital and employment are predetermined.[24] The instantaneous or first-period response, however, is not that interesting. What really matters is whether consumption, investment, and hours worked move up together in the periods following the increase in agents' expectations. If the reader insists on a positive co-movement in the first period he can always think of the first period as the first quarter which corresponds to the first three periods in our model.

2.4.4 The Role of the Elasticity of Intertemporal Substitution

So far we have shown that both the model with and the model without endogenous labor force participation can generate a Pigou boom, i.e., consumption, investment, and hours worked jointly increase when expectations about future productivity increase. Although the increase in employment is a robust result, the result that both consumption and investment increase is not and only holds for a small range of values for the intertemporal substitution, $1/\gamma$. In particular, the responses of consumption and investment are both positive only when γ takes on values in the range from 0.45 to 0.5 (0.425 to 0.55) when we consider the responses starting in the second(third) period. Our benchmark value of γ was chosen to be in this range. For smaller values of γ, investment increases, but consumption—at least initially—decreases, whereas for higher values of γ we find that consumption increases, investment in new projects increases, but investment in existing projects decreases. Moreover, as investment in existing projects decreases, at some point the reduction in physical capital more than offsets the increase in employment and output decreases. For example, when $\gamma = 1$, the output response turns negative after 50 months and output minus investment in new projects turns negative after 24 months. In the next section, we show how allocation costs can make the predictions of the model much more robust with respect to changes in γ.

[24] Because capital and employment are predetermined, total resources are initially constant. Thus, if consumption increases then investment must decrease and vice versa. In an earlier version of this chapter, we used an alternative timing assumption where matching takes place at the beginning of the period and the creation of productive relationships can therefore take place within the period. Resources can then increase 'instantaneously'. In the current version, we adopt the standard timing assumption in the literature, because the standard timing makes it easier to calibrate the model. For example, with the alternative timing assumption there are agents who within a given period are both searching, i.e., unemployed, and working.

2.4.5 Allocation Costs

In the benchmark model, the household can costlessly allocate available resources between consumption and investment. Let Y_t be equal to output net of investment in new projects. Consumption, C_t, and investment in existing projects, I_t, have to satisfy:

$$C_t + I_t = Y_t. \tag{2.21}$$

The implicit assumption is that the price of one unit of investment is always equal to one unit of consumption, which is not very realistic. In the version of the model with allocation costs, consumption and investment have to satisfy:

$$\left[\eta_c \left(C_t^{\nu_c}\right)^{\xi} + \eta_i \left(I_t^{\nu_i}\right)^{\xi}\right]^{1/\xi} = Y_t \tag{2.22}$$

with $\xi \geq 1$. By increasing ξ, the allocation friction becomes bigger, and if ξ approaches infinity, then consumption and investment are a fixed fraction of each other. Given a value for ξ, the values of η_c and η_i are chosen to ensure that steady state values are the same as in the model without allocation costs ($\xi = 1$). By varying ν_c and ν_i we can furthermore ensure that the volatilities of consumption and investment are not affected.

Table 2.3 reports the range of values for γ for different values of ξ such that both consumption and total investment increases in regime 2. Since there is an initial decrease in at least one of the two variables, we report the range of values economy when consumption and investment are positive from the second period onwards, from the third period onwards, and when consumption and investment are eventually positive. As one allows for a longer time for the consumption and the investment response to become positive, the range, of course, increases. The table makes clear that as the value of ξ increases, the presence of Pigou cycles becomes a much more robust outcome of the model, that is, consumption and investment jointly increase for a larger set of values of γ. For example, when ξ is equal to 3, then both consumption and investment are positive from the third period onwards when γ is in the range from 0.400 to 1.950. The presence of Pigou cycles becomes even more robust as ξ takes on yet higher values.[25]

For computational convenience, we set $\nu_c = \nu_i = 1$ in the numerical experiments in Table 2.3. Therefore, as ξ increases, the volatilities of consumption and investment change and become more similar. By adjusting ν_c and ν_i we can correct for this change.

[25] As the value of ξ increases, the model becomes more nonlinear and more difficult to solve, so we do not report a systematic set of results for higher values of ξ. We have, however, solved several models with higher values of ξ, and find that the presence of Pigou cycles really does seem to become more robust. For example, when $\xi = 7$ and $\gamma = 3$, then consumption, employment, and total investment are all positive in the second month.

Table 2.3: **Admissible Range for the Coefficient of Relative Risk Aversion**
This table reports the range of values of the coefficient of relative risk aversion (γ) for which both the consumption and the investment response are positive eventually, as well as the range of values this occurs from the second period onwards and the third period onwards.

Allocation costs	$\xi = 1.00$	$\xi = 1.25$	$\xi = 1.50$	$\xi = 3.00$
Using total investment				
Y, C, N^w, I up				
from second period	$[0.450, 0.500]$	$[0.650, 0.825]$	$[0.750, 1.000]$	$[1.025, 1.900]$
from third period	$[0.425, 0.550]$	$[0.575, 0.875]$	$[0.600, 1.100]$	$[0.400, 1.950]$
eventually	$[0.000, 0.675]$	$[0.000, 1.000]$	$[0.000, 1.175]$	$[0.000, 2.000]$
Using only investment in physical capital (by existing projects)				
Y, C, N^w, I up				
from third period	$[0.425, 0.450]$	$[0.575, 0.650]$	$[0.600, 0.700]$	$[0.400, 0.925]$
eventually	$[0.000, 0.550]$	$[0.000, 0.775]$	$[0.000, 0.875]$	$[0.000, 1.175]$

This is documented in the last two columns of Table 2.2. The column under Model III reports summary statistics when $\nu_c = \nu_i = 1$ (and $\xi = 1.25$). The column under Model III* reports summary statistics when ν_c and ν_i are adjusted to obtain the same consumption and investment volatilities as when ξ is equal to 1. When we compare the results under Model III and Model III* we can see that the statistics are very similar, except, of course, for the volatility of consumption and investment. When we compare the results for Model III*, which has $\xi = 1.25$, with Model II, which has $\xi = 1$, then we see that all the summary statistics are very similar. Thus, the introduction of allocation costs can make the results much more robust in the sense that Pigou cycles can be obtained for a much wider range of values of γ, without affecting the other properties of the model.

2.5 Conclusion

The standard RBC model together with the basic matching framework is successful in generating an increase in resources in anticipation of an increase in productivity. It is less successful in generating a Pigou cycle. That is, although there are values of the intertemporal substitution, $1/\gamma$, for which both consumption and investment increase, the range of values is small. We show, however, that by introducing allocation costs, the model can generate Pigou cycles for a wide range of values for γ. There are likely to be other mechanisms that can generate a positive co-movement between consumption and investment when resources increase. For example, if a large enough fraction of the economy are 'rule-of-thumb' consumers that simply consume a fraction of net resources, then, by construction consumption and investment increase if net resources increase.[26]

In our model, existing productive relationships automatically benefit from higher productivity growth. That is, technological progress is assumed to be disembodied. It may very well be the case that in reality existing relationships also have to invest some resources in terms of investment or additional workers in order to benefit from productivity increases. This could reinforce the mechanism highlighted in this chapter.

[26]Recently, rule-of-thumb consumers have also been used in New Keynesian models. See, for example, Gali, López-Salido, Vallés (2004).

2.6 References

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2.7 Appendix A — Data Description

In this section, we discuss the data used for the calibration of the model and for the calculation of the summary statistics in Table 2.2.

National Income Series

- Real gross domestic product, GDPC96
- Real gross private domestic investment, GPDIC96
- Real personal consumption expenditures (nondurable goods), PCNDGC96

These series were downloaded from Federal Reserve Economic Data (FRED).

Job Finding and Separation Probabilities

- Job finding probability
- Job separation probability

The 'probabilities' are obtained from the continuous-time 'rates' using:

$$probability = 1 - \exp(-rate). \tag{2.23}$$

See Shimer (2005) and Shimer's home page for additional details.

Current Population Survey

- Unemployment rate, LNS14000000Q
- Employment population ratio, LNS12300000Q
- Civilian labor force participation rate, LNS11300000Q

Vacancies

- Index of Help Wanted Advertising in Newspapers, HELPWANT

These data were downloaded from Federal Reserve Economic Data (FRED). These series, together with the unemployment rate, are used to construct a measure of labor market tightness.

Productivity and Technology Statistics

- Output, PRS85006043
- Current $ output, PRS85006053
- Employment, PRS85006033
- Nominal compensation, PRS85006063
- Labor share, PRS85006173

These series were downloaded from the Bureau of Labor Statistics. As is standard in the literature, these series are for the non-farm business sector. Real output and employment are used to construct labor productivity. The wage rate was calculated using:

$$wage\ rate = \frac{compensation}{employment} \times \frac{output}{current\ \$\ output}. \tag{2.24}$$

The labor share index was turned into an actual labor share series by rescaling the index so that the observed value in 2002Q3 is equal to 78%, the value reported in Gomme and Rupert (2004).

2.8 Appendix B — Numerical Solution

The model is solved with standard projection techniques; Chebyshev nodes are used to construct the grid, and third-order Chebyshev polynomials are used to approximate the conditional expectations. Integrals are calculated using Hermite-Gaussian quadrature. Details of the numerical algorithm are given in Table 2.4.

One problem with rectangular grids is that they may lead to combinations of the state variables that never actually materialize, and where the approximation, or perhaps even the model itself, is not well defined. We have encountered this problem for combinations of low values of Z_t and K_t and high values of N_t^w. At these points in the state space, the desired amount of leisure combined with the value of employment, which is a state variable, result in negative numbers of agents searching, N_t^s. When simulating, the economy never gets close to these problematic points in the state space. We deal with this problem by simply not using the problematic grid points in the projection step.

Table 2.4: **Appendix — Details of Numerical Solution**

# Chebyshev nodes	
K	4
N^w	4
Z	12
Bounds on grid	
K	$[95, 130]$
N^w	$[0.82, 0.98]$
Z	$[0.89, 1.11]$
Order of polynomial	
K	3
N^w	3
Z	3
# quadrature nodes	3

2.9 Appendix C — Avoiding the Consumption-Tightness Puzzle

2.9.1 Consumption and Tightness in Ravn (2006)

Ravn (2006) highlights a problematic relationship between aggregate consumption and labor market tightness in DSGE models with labor market matching and endogenous labor force participation. Ravn assumes that agents can use financial markets to ensure that consumption does not depend on their labor market outcome, but that their labor market status does affect their utility. The budget set of the agent is not convex, because labor market choices are indivisible. To deal with the indivisibility, Ravn uses lotteries—as in Hansen (1985) and Rogerson (1988)—to determine which agents engage in what kind of activity.[27] This assumption corresponds to a current-period utility function for

[27] The lottery approach seems less convincing with a labor force participation choice, because it implies that a random draw determines whether a worker is searching for a job or not. Another awkward aspect of the lottery approach in a matching framework is that a key aspect of the matching model is workers being fixed in a relationship. The idea that lotteries each period determine whether a worker is searching, working, or unemployed seems inconsistent with the matching friction.

the representative household of the following form:

$$U(C_t, N_t^s, N_t^w) = \frac{C_t^{1-\gamma} - 1}{1-\gamma} + N_t^s H(1-\phi_s) + N_t^w H(1-\phi_w) + (N^* - N_t^s - N_t^w) H(1). \tag{2.25}$$

Here, $H(\cdot)$ is the function that measures the utility of current-period leisure. Its argument is equal to $1-\phi_s$ when searching, $1-\phi_w$ when working, and 1 when not doing either. Since these values are constant, this utility specification can be written in the following linear form:

$$U(C_t, N_t^s, N_t^w) = \frac{C_t^{1-\gamma} - 1}{1-\gamma} - N_t^s \overline{\phi}_s - N_t^w \overline{\phi}_w. \tag{2.26}$$

Ravn shows that the matching framework combined with the linear utility specification and Nash bargaining implies that the volatility of the marginal utility of consumption is proportional to the volatility of labor market tightness. In particular, the following relationship has to hold:

$$\overline{\phi}_s = \frac{\eta_1}{(1-\eta_1)} \theta_t \psi C_t^{-\gamma}, \tag{2.27}$$

where η_1 denotes the bargaining weight of the worker. From this equation we obtain:

$$d \ln \theta_t = \gamma d \ln C_t. \tag{2.28}$$

It follows that tightness should be highly correlated with consumption and that—for reasonable values of γ—tightness cannot be too volatile. Ravn shows that the first property is satisfied in the data, but that the second is not. Since observed tightness is much more volatile than consumption, this relationship can only be satisfied for values of γ that are generally considered implausible.

2.9.2 Consumption and Tightness in Our Model

In our model, the volatility of labor market tightness is not constrained by the volatility of aggregate consumption in this manner. In fact, in our benchmark model the ratio of the standard deviation of labor market tightness relative to the standard deviation of consumption is equal to 42. This is even higher than the empirical counterpart, which is equal to 22.8. In this section, we demonstrate how our model avoids the problematic restriction discovered by Ravn (2006), because we do not use Nash bargaining and because we use a different utility function. First, we analyze the link between tightness and consumption when our wage setting rule is used. Second, we analyze the link between tightness and consumption when our utility function is used.

No Nash Bargaining

Suppose that, as in Ravn (2006), the utility function is given by Equation (2.26). Then the first-order condition is equal to:

$$\overline{\phi}_s = \lambda_t^w E_t \left[\sum_{j=0}^{\infty} \overline{\beta}^j \left\{ \beta \left(W_{t+j+1} C_{t+j+1}^{-\gamma} - \overline{\phi}_w \right) \right\} \right], \tag{2.29}$$

where $\overline{\beta} = \beta(1 - \rho^x)$. This is equivalent to:

$$\overline{\phi}_s + \frac{\lambda_t^w \beta \overline{\phi}_w}{1 - \overline{\beta}} = \lambda_t^w E_t \left[\sum_{j=0}^{\infty} \overline{\beta}^j \left\{ \beta W_{t+j+1} C_{t+j+1}^{-\gamma} \right\} \right]. \tag{2.30}$$

The free-entry condition for the firm can be written as:

$$\psi C_t^{-\gamma} = \lambda_t^f E_t \left[\sum_{j=0}^{\infty} \overline{\beta}^j \beta \left(\overline{p}_{t+j+1} - W_{t+j+1} \right) C_{t+j+1}^{-\gamma} \right]. \tag{2.31}$$

Redefining coefficients, we can write the wage rule as:

$$W_t = \overline{\omega}_0 + \overline{\omega}_1 \overline{p}_t. \tag{2.32}$$

Then Equations (2.30) and (2.31) can be written as:

$$\overline{\phi}_s + \frac{\lambda_t^w \beta \overline{\phi}_w}{1 - \overline{\beta}} = \lambda_t^w \left(\Omega_t + \overline{\omega}_1 E_t \left[\sum_{j=0}^{\infty} \overline{\beta}^j \beta \overline{p}_{t+j+1} C_{t+j+1}^{-\gamma} \right] \right), \text{ and} \tag{2.33}$$

$$\psi C_t^{-\gamma} = \lambda_t^f \left(-\Omega_t + (1 - \overline{\omega}_1) E_t \left[\sum_{j=0}^{\infty} \overline{\beta}^j \beta \overline{p}_{t+j+1} C_{t+j+1}^{-\gamma} \right] \right), \tag{2.34}$$

$$\text{where } \Omega_t = \overline{\omega}_0 E_t \left[\sum_{j=0}^{\infty} \overline{\beta}^j \beta C_{t+j+1}^{-\gamma} \right]. \tag{2.35}$$

Combining Equations (2.33) and (2.34) we obtain:

$$\overline{\phi}_s + \frac{\lambda_t^w \beta \overline{\phi}_w}{1 - \overline{\beta}} = \lambda_t^w \left(\frac{\Omega_t}{1 - \overline{\omega}_1} + \frac{\overline{\omega}_1}{(1 - \overline{\omega}_1)} \frac{\lambda_t^w}{\lambda_t^f} \psi C_t^{-\gamma} \right). \tag{2.36}$$

This equation is the counterpart of Equation (2.27). Just like Equation (2.27) it is based on a utility function that is linear in N_t^s and N_t^w, but is based on our wage rule instead of

Nash bargaining. In addition to consumption and labor market tightness it also contains Ω_t. More importantly, it does not impose such a tight constraint as the one discovered by Ravn (2006) for Nash bargaining. To see the link between the restriction implied by Equation (2.27) and the (lack of a) restriction in Equation (2.36), consider the wage setting rule in which agents simply obtain constant fractions of revenues, that is, $\omega_0 = 0$, and $0 < \omega_1 < 1$.[28] Then Equation (2.36) can be written as:

$$\overline{\phi}_s + \mu_0 \theta_t^{\mu_1} \frac{\beta \overline{\phi}_w}{1 - \overline{\beta}} = \frac{\overline{\omega}_1}{(1 - \overline{\omega}_1)} \theta_t \psi C_t^{-\gamma}. \tag{2.37}$$

This equation is similar to Equation (2.27), but there is one additional time-varying term on the left-hand side. So it no longer is the case that $\ln \theta_t - \gamma \ln C_t$ has to be equal to a constant.

If wages are completely sticky then Equation (2.36) can be written as:

$$\overline{\phi}_s + \frac{\lambda_t^w \overline{\phi}_w}{1 - \beta} = \lambda_t^w \Omega_t. \tag{2.38}$$

Now we have a relationship between tightness and Ω_t.[29] Again this relationship is less problematic than the one derived in Ravn (2006). To see this, first consider the case where $\overline{\phi}_w = 0$, in which case the left-hand side is constant. Then we obtain a relationship similar to the one derived in Ravn, but with the marginal utility replaced by the discounted sum of marginal utilities. Unless changes in the discount factor are quantitatively important, the discounted sum of marginal utilities may very well be less volatile than the current marginal utility. This specification would then also induce the consumption-tightness puzzle. But when $\overline{\phi}_w$ is positive then a change in labor market tightness increases both the left-hand and the right-hand side of Equation (2.38) and a smaller change in Ω_t is therefore needed.

Non-linear Utility of Leisure

More important than the wage setting rule is the utility specification. Our nonlinear specification makes it easier to avoid Ravn's consumption-tightness puzzle. In particular,

[28] This implies that $\Omega_t = 0$.

[29] λ_t^w is a function of tightness only.

for our specification the equivalent of Equation (2.27) would be:

$$\begin{aligned}&\phi L_t^{-\kappa} + \lambda_t^w E_t\left[\sum_{j=0}^{\infty}\overline{\beta}^j \phi\beta L_t^{-\kappa}\right]\\&= \lambda_t^w\left(\frac{\Omega_t}{1-\overline{\omega}_1} + \frac{\overline{\omega}_1}{(1-\overline{\omega}_1)}\frac{\lambda_t^w}{\lambda_t^f}\psi C_t^{-\gamma}\right).\end{aligned} \tag{2.39}$$

There is another time-varying term on the left-hand side. Moreover, if labor force participation is fairly inelastic, i.e., κ is high, then this additional term is quite volatile even if leisure is not.

As in Ravn (2006), we assume that there is complete insurance of the consumption stream, but in our framework leisure is insured as well. With the lottery approach, the unemployed agent is best off, since he receives the same amount of consumption and more leisure. In our model agents are equally well off, either because agents that do not work spend more time on household chores, or because agents truly care about the overall amount of leisure of the joint household.

Both the representative agent of Ravn (2006) and ours are abstractions. Part of the motivation of a particular abstraction lies in the properties of the model it generates. Ravn shows that the linear specification leads to a 'consumption-tightness' puzzle. We have shown that the consumption-tightness puzzle can be avoided by using a nonlinear utility function and an alternative wage setting rule.

Chapter 3

Asset Pricing in Production Economies — A Matching Model

3.1 Introduction

A large literature explains moments and dynamics of asset prices with models where consumption (the endowment process) is an exogenously specified process that resembles observed processes of aggregate consumption. This implies, given a household preference specification, a relationship between aggregate consumption and prices. With standard preferences (power utility) however, those models have great difficulties to generate realistic asset prices (see Mehra and Prescott (1985)). This is the reason why most successful models in that literature assume preferences that are non-standard (for example, Campbell and Cochrane (1999)). The natural next question to ask is how agents with those preferences choose consumption endogenously, once allowed to.

Lettau and Uhlig (2000) and Uhlig (2004), amongst others, show that as soon as households have access to a savings technology, that is consumption is endogenous, or leisure enters the households' utility function, that is the labor-leisure choice is endogenous, the most commonly used preference specifications developed for exchange economy models turn out to have very unrealistic implications for choices of aggregate variables such as consumption and employment. The reason is, in a nutshell, that with those preferences agents are (locally) very risk averse. As a consequence agents typically engineer, once allowed to, extremely smooth consumption profiles, both with their investment decision as well as with their labor-leisure choice. The results are a stark reduction of the volatility of aggregate consumption and countercyclical aggregate employment. The smooth consumption path and countercyclical employment are not only counterfactual, but in turn destroy the favorable asset pricing properties of the models, and we are left with models that can neither explain asset prices nor the behavior of aggregate consumption, investment, and employment.

In this chapter we develop a model that can *jointly* explain important moments and dynamics of asset prices as well as key aspects of the behavior of major macroeconomic time series such as output, consumption, investment, and employment. We build on a one-sector standard production economy model (standard stochastic growth (RBC) model) with endogenous consumption and investment and endogenize aggregate employment by means of a state-of-the-art search-theoretical model of the labor market, more specifically, a version of the Mortensen and Pissarides (1994) labor market matching model.[1] Those models assume a labor market search and matching friction. The reason we impose this friction, apart from being closer to reality, is that without a labor market friction households use labor to excessively smooth consumption. Then we rely on insights gained in the finance literature and augment the basic framework with habit preferences and capital adjustment costs so as to enable the resulting model to explain the behavior of asset prices: (i) consumers have to be rendered sufficiently sensitive to consumption risk, the reason for the adoption of habit preferences, and (ii) households have to be prevented from engaging in excessive consumption smoothing activities via their investment plans, we impose convex capital adjustment costs on the economy to that end.

We show that our model can explain key asset pricing moments as well as several important dimensions of asset pricing dynamics: the model can match the equity premium, the level of the risk-free rate and comes close to match the volatility of the equity return. The model explains the cyclical variation of the price-dividend ratio, the equity premium, the expected equity premium, the conditional volatility of the equity return, and the conditional Sharpe ratio. It goes quite some way in accounting for the predictability of returns as well as the autocorrelation structure of returns. The volatility of the risk-free rate turns out too high which is due to the simple internal habit specification used.

The model can also replicate key moments of macroeconomic time series, including aggregate fluctuations in employment: the model generates realistic fluctuations of investment and consumption relative to output. The correlation of employment and output is, as in the data, positive. Compared to standard matching models, the volatility of employment relative to output is higher due to a feedback channel from financial markets, via the value of firms, on labor markets. Furthermore, the model is able to account for the excess sensitivity of consumption to income. We also examine the internal magnification and propagation mechanisms of the model and conclude that it constitutes a substantial improvement compared to standard production economy models.

[1] Similar models are due to Merz (1995), Andolfatto (1996), and den Haan, Ramey, Watson (2000).

The structure of the chapter is as follows: we start by providing an overview of related literature, mapping this chapter into the existing literature at the same time. Then we develop the model. In Sections 3.4 and 3.5 we calibrate and solve the model, demonstrate and interpret results, and provide intuition. In Section 3.6 we compare the model to the benchmark model in the literature (see Section 3.2 where we provide an overview of the literature). Section 3.7 concludes and gives an outlook.

3.2 Related Literature

Starting out from the puzzling observation made by Mehra and Prescott (1985) that the standard representative agent consumption-based asset pricing model fails at replicating a key asset pricing moment, the equity premium, a substantial part of the finance literature has evolved around the discovery of other failures of the basic model and the development of a multitude of mechanisms in order to alleviate those shortcomings. This research agenda has been very fruitful and empowered the basic model to explain a wide range of asset pricing moments and dynamics. Almost always this literature has focused on the understanding of the behavior of asset prices by means of models with exogenous processes for consumption and leisure, and most of the times research has centered around the design of preference specifications, mostly some sort of habit preferences (a seminal example are preferences with a slow-moving external habit level due to Campbell and Cochrane (1999)).

However, Jermann (1998), Boldrin, Christiano, Fisher (2001), and in particular Lettau and Uhlig (2000), point out that as soon as households have access to a savings technology, that is consumption is endogenous, or leisure enters the households' utility function, that is the labor-leisure choice is endogenous, the habit preferences developed for exchange economy models turn out to have very unrealistic implications for choices of aggregate variables such as consumption and leisure (Lettau and Uhlig demonstrate this for the Campbell and Cochrane (1999) preferences). The reason is, in a nutshell, that with habit preferences agents are (locally) very risk averse. As a consequence agents typically engineer, once allowed to, extremely smooth consumption profiles, both with their investment decision as well as with their labor-leisure choice. The results are a stark reduction of the volatility of aggregate consumption and countercyclical aggregate employment. The smooth consumption path and countercyclical employment are not only counterfactual, but in turn destroy the favorable asset pricing properties of the models, and we are left with models that can neither explain asset prices nor the behavior of aggregate consumption, investment, and employment. This is disconcerting, not least because the question arises of how much faith to put in preference specifications that destroy any model's ability to generate realistic time series for consumption and employment.[2]

The challenge has thus become one of developing frameworks with endogenous consumption and employment that can jointly explain asset prices and the behavior of basic macroeconomic time series. Cochrane (2005) dubs this evolving literature in his 2005 survey article 'production-based asset pricing' and provides a powerful rationale for this ambitious research agenda: 'The centerpiece of dynamic macroeconomic theory is the equation of savings and investment, the equation of marginal rates of substitution with marginal rates of transformation [...]. Asset markets are the mechanisms that does all this equating. [...] If we can learn the marginal value of wealth from asset markets, we have a powerful measurement of the key ingredient of all modern, dynamic, intertemporal macroeconomics. [...] In sum, the program of understanding the real, macroeconomic risks that drive asset prices [...] is not some weird branch of finance; it is the trunk of the tree. As frustatingly slow as progress is, this is the only way to answer the central questions of finance, and a crucial and unavoidable set of uncomfortable measurements for macroeconomics.'

One of the first models is due to Jermann (1998) in his paper 'Asset Pricing in Production Economies'. Jermann starts out by demonstrating that the asset pricing implications of standard production economy models are extremely poor (virtually no equity premium). He then goes on to outline two complementary features that enable those models to match basic asset pricing moments: (i) households have to be sufficiently sensitive to consumption risk — Jermann includes habit preferences, and (ii) households have to be prevented from using their investment decision to rid themselves of most of the consumption risk they might otherwise face — Jermann imposes capital adjustment costs on the economy. Jermann manages to match with his model basic asset pricing moments, such as the equity premium and the volatility of the equity return, as well as basic moments of macroeconomic time series. The model we develop in this chapter is based on the basic Jermann framework. We follow Jermann and adopt habit preferences and capital adjustment costs.

Jermann (1998), however, assumes labor to be fixed. Boldrin, Christiano, Fisher (2001), and Uhlig (2004) show that once the labor-leisure choice is endogenized in the Jermann framework, the good asset pricing properties of the model all but disappear. Uhlig actually manages to demonstrate that this finding holds more general. He combines

[2]Limited stock market participation is another prominent mechanism that has been developed in the finance literature in order to empower the standard representative agent consumption-based asset pricing model to explain the behavior of asset prices. In fact, Uhlig (2004) and Guvenen (2005) show that models with limited stock market participation and *endogenous* consumption or labor-leisure basically suffer from the same shortcomings as models with habit preferences.

the Jermann setup with a very general utility specification which is separable across time but non-separable between consumption and leisure. Uhlig shows that, as long as labor is not fixed, it is not possible to find preferences so as to enable the Jermann framework to jointly match basic asset pricing moments and basic moments of macroeconomic time series. The intuition for this remarkable result is as follows: if agents are very averse to fluctuations in consumption (for example due to habit preferences), they can use their labor-leisure choice to insure themselves against fluctuations in consumption, that is, agents can choose to work hard when aggregate productivity is low, and work less when output is otherwise high, thus buffering the impact of technology shocks on output and consumption. The implications are a reduction in consumption risk and a counterfactual *negative* correlation between output and employment. The solution Uhlig points out is to let employment be determined solely by an exogenous process for wages and the demand for labor, that is by the firm's first-order condition. This step effectively takes the labor-leisure choice away from households. Depending on the exogenous process for wages, this approach can obviously generate many different labor market dynamics. Uhlig comments on this: 'I do not claim that I have good microfoundations [for the exogenous wage process]. Rather, I regard it as a heuristically plausible starting point. [...] I can imagine that a microfoundation [...] could be found, following Hall (2003) [Hall (2005)] or the labor market search literature [...].' Uhlig concludes: 'The key to understanding macroeconomic facts and asset pricing facts jointly may be in understanding labor markets rather than agent heterogeneity.' The model developed in this chapter thus in a sense follows up on Uhlig's suggestion and incorporates a search and matching model of the labor market into the Jermann framework.

The above discussion suggests that we have to combine habit preferences not only with capital adjustment frictions but also with some sort of labor market friction in order to jointly explain asset prices and macroeconomic time series. Boldrin, Christiano, Fisher (2001) do exactly that. They propose a two-sector economy model with one consumption good and one investment good sector. Boldrin, Christiano, Fisher then impose the following set of frictions: (i) both aggregate capital and aggregate labor are predetermined, that is they cannot be adjusted in response to a technology shock and (ii) both capital and labor cannot be reallocated across sectors in response to a technology shock. That is, both production factors cannot be adjusted (neither across sectors nor in the aggregate) in response to a technology shock. Boldrin, Christiano, Fisher show that it is possible for the model to match both basic asset pricing moments and basic moments of macroeconomic time series. Contrary to Jermann, output and aggregate employment are, as in the data, *positively* correlated. Relative to Boldrin, Christiano, Fisher, we develop a model in this chapter where employment can be adjusted in the same period as the technology shock. The friction we impose instead is a search and matching friction, as is standard in modern labor economics. There are two advantages to our approach. Firstly, the friction we impose is less ad-hoc. Secondly, the labor market matching model has many different possible implementations, depending on the particular incarnation of the search-theoretical model incorporated, and thus allows to carefully model fluctuations in aggregate employment and wages. This can be very important for asset prices too, because dividends are the residual payment of output, investment, and *wages*.

Another interesting result for the production-based asset pricing literature is due to Tallarini (2000). The only modification Tallarini makes to the basic production economy model is to replace the standard preferences (power utility) with Epstein and Zin (1989, 1991) preferences, designed to disentangle agents' risk aversion ($CRRA$) from their elasticity of intertemporal substitution (EIS). He demonstrates that the behavior of asset prices in the model is driven by the agents' $CRRA$, while macroeconomic time series are driven by the EIS. With standard preferences, increasing the $CRRA$ so as to match asset pricing moments decreases the EIS one for one, thus destroying the basic model's ability to account for macroeconomic moments, again, because consumption, once endogenized, turns out far too smooth, in turn destroying asset pricing moments. With Epstein and Zin preferences, by increasing the $CRRA$ to very high levels while keeping the EIS unchanged, Tallarini manages to match with the standard model both asset pricing as well as macroeconomic moments, with the correlation between output and employment being positive. However, while Epstein and Zin preferences enable the standard model to match both the risk-free rate as well as the market price of risk, Tallarini is unable to generate any kind of sizeable equity premium. In Chapter 4 and Chapter 5 we remedy some of Tallarini's shortcomings. We broaden the setup Tallarini uses who restricts himself to an EIS of unity. Relative to Tallarini we show in Chapter 4 and Chapter 5 that an EIS greater than unity and careful calibration of the standard model can dramatically improve the model's ability to match asset pricing moments due to endogenous long-run risks in consumption, and without unrealistically high levels of risk aversion. We believe that it would be very interesting to combine the production economy model we develop in Chapter 4 and Chapter 5 (Epstein and Zin preferences and capital adjustment costs) with a search-theoretical model of the labor market as demonstrated in Chapter 2 and in this chapter.

Guvenen (2005) is the first to link the extensive limited stock market participation literature to the standard production economy model with adjustment costs (the Jermann (1998) framework). He introduces two types of agents into the model: stockholders who own the aggregate capital stock, and non-stockholders whose only source of income is labor income and who are restrained from participating in stock markets.

Non-stockholders are thus forced to rely on bond markets for their insurance needs. The following mechanism lies at the heart of the model's ability to generate realistic asset pricing moments: due to a higher degree of risk aversion (heterogeneity in the utility function), non-stockholders prefer a smoother consumption path compared to stockholders. Because non-stockholders are forced to rely on bond markets, stockholders must bear more aggregate risk in equilibrium, as they provide insurance to the non-stockholders via a riskless bond which merely redistributes aggregate risk. As a reward for increased levels of consumption volatility, which they cannot rid themselves of due to capital adjustment costs, stockholders demand a higher equity premium. With this setup, Guvenen matches basic asset pricing moments, an impressive wealth of asset pricing dynamics, as well as basic moments of macroeconomic time series. However, as does Jermann, Guvenen assumes labor to be fixed. In fact, Uhlig (2004) also shows for the Guvenen setup that as soon as the labor-leisure choice is endogenized, aggregate employment and output are, once again, negatively correlated, causing the good performance of the Guvenen model along asset pricing dimensions to collapse. In a very recent response, Guvenen (2006) however demonstrates that some of Uhlig's results seem to be driven by the inaccuracy of the log-linearizations Uhlig relies upon in order to solve the Guvenen (2005) model (see Chapter 3 for a discussion of the importance to solve the Jermann framework with modern non-linear solution techniques). Moreover, Guvenen (2006) highlights that using preferences that are non-separable between consumption and leisure when endogenizing the labor-leisure choice in the Guvenen (2005) model can help the Guvenen (2005) model to preserve its asset pricing properties to a considerable extent. However, aggregate employment is still not, as in the data, strongly procyclical.

3.3 The Model

The model economy consists of a representative household and firms. The representative household comprises a continuum of workers with unit mass and is the owner of the aggregate capital stock. Production takes place within projects. A project consists of one worker and one firm. New projects enter the economy via a matching market where workers and firms are brought together. Projects get destroyed when they receive a low idiosyncratic productivity shock. In order to produce, projects have to rent capital on a capital market from households each period anew. There is a potentially infinite number of firms. In what follows, we successively describe each component of the economy in detail.

3.3.1 Households

The aggregate capital stock and all projects are owned by a representative household. All workers are members of this household and at the end of the period the household receives all wages earned by the workers.[3] Aggregate consumption is chosen with the aim to maximize the expected utility of the representative household:

$$\max_{\{C_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t) \right], \text{ where } u(C_t) = \frac{(C_t - \eta C_{t-1})^{1-\gamma}}{1-\gamma}. \tag{3.1}$$

If $\eta > 0$, household preferences are characterized by habit formation, where ηC_{t-1} corresponds to the household's 'habit stock'. Following Jermann (1998) and Boldrin, Christiano, Fisher (2001), we adopt this internal habit specification, in order to create a sufficiently large desire of the household for a smooth consumption path.

The aggregate capital stock evolves according to the following accumulation equation with capital adjustment costs:

$$K_{t+1} \leq \phi(I_t/K_t) K_t + (1-\delta)K_t, \tag{3.2}$$

where δ denotes the rate of depreciation, and $\phi(\cdot)$ is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it step by step.[4] This specification allows Tobin's q to vary over time, because consumption goods cannot be converted into capital goods one for one. We let:

$$\phi(I_t/K_t) = \frac{\alpha_1}{1-1/\xi} \left(\frac{I_t}{K_t} \right)^{(1-1/\xi)} + \alpha_2, \tag{3.3}$$

with:

$$\alpha_1 = (\exp(\bar{\varepsilon}) - 1 + \delta)^{1/\xi}, \tag{3.4}$$

$$\alpha_2 = \frac{1}{\xi - 1} (1 - \delta - \exp(\bar{\varepsilon})), \tag{3.5}$$

[3] We assume that at the beginning of each period the representative household splits up into a continuum of workers with unit mass. After production, at the end of each period, workers pool their income and aggregate into the representative household once again. That is, we assume perfect risk sharing between all workers, unemployed or not, in the economy. In other words, markets are assumed to be complete with respect to the contingency of unemployment.

[4] This formulation of the capital accumulation equation with adjustment costs and $\phi(\cdot) \geq 0$, $\phi''(\cdot) \leq 0$ has been used by Uzawa (1969), Lucas and Prescott (1971), Hayashi (1982), Baxter and Crucini (1993), Jermann (1998), Boldrin, Christiano, Fisher (1999), Uhlig (2004), Guvenen (2005), amongst many others.

where $\overline{\varepsilon}$ is the trend growth rate of the economy.[5] The parameter ξ is the elasticity of the investment-capital ratio with respect to Tobin's q.[6] If $\xi = \infty$, (3.2) reduces to the capital accumulation equation of the standard growth model without capital adjustment costs.

It is the combination of capital adjustment costs and habit preferences that allows the model to match asset prices: without habit preferences, marginal rates of substitution are not sufficiently varying over time. In other words, agents do not care enough about consumption risk. On the other hand, without adjustment costs, the household uses his savings decision to smooth consumption over time, thus eliminating all consumption risk it might otherwise face.

Aggregate investment I_t is the difference between the aggregate income of the household in period t and aggregate consumption C_t:

$$I_t = N_t w_t + r_t K_t + \Pi_t - \psi_t V_t - C_t, \tag{3.6}$$

where aggregate income is composed out of the following components. Aggregate labor income:

$$N_t w_t, \tag{3.7}$$

the aggregate proceeds from the rental of capital to projects:

$$r_t K_t = r_t N_t k_t, \tag{3.8}$$

and aggregate profits:

$$\Pi_t = N_t \pi_t, \tag{3.9}$$

net of the aggregate posting costs of vacancies:

$$\psi_t V_t. \tag{3.10}$$

In Sections 3.3.2 to 3.3.4 we will discuss how employment, or equivalently the number of active projects, N_t, wages w_t, the project level capital stock k_t, the capital rental rate r_t,

[5] It is straightforward to verify that $\phi(\frac{I_t}{K_t}) > 0$ and $\phi''(\frac{I_t}{K_t}) < 0$ for $\xi > 0$ and $\frac{I_t}{K_t} > 0$. Furthermore, $\phi(\frac{I}{K}) = \frac{I}{K}$ and $\phi'(\frac{I}{K}) = 1$, where $\frac{I}{K} = (\exp(\overline{\varepsilon}) - 1 + \delta)$ is the steady state investment-capital ratio. As a result, the balanced growth path of the economy is unaffected by the adjustment cost parameter ξ.

[6] The elasticity of the investment-capital ratio with respect to Tobin's q is $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \frac{\partial(I_t/K_t)}{\partial q_t} \times \frac{q_t}{(I_t/K_t)} = \left[\frac{\partial q_t}{\partial(I_t/K_t)} \times \frac{(I_t/K_t)}{q_t}\right]^{-1}$. Tobin's q is given by $\frac{1}{\phi'(I_t/K_t)} = \alpha_1 \left(\frac{I_t}{K_t}\right)^{1/\xi}$. It follows that $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \left[(1/\xi)\, \alpha_1 \left(\frac{I_t}{K_t}\right)^{1/\xi - 1} \frac{1}{\alpha_1} \left(\frac{I_t}{K_t}\right)^{1-1/\xi}\right]^{-1} = [1/\xi]^{-1} = \xi$.

profits π_t, and aggregate posting costs $\psi_t V_t$ are determined.

K_{t+1} and C_t are determined by maximization of (3.1) subject to (3.2), for which the following Euler equation is a necessary condition:

$$(C_t - \eta C_{t-1})^{-\gamma} = \beta E_t \left[\begin{array}{c} \left(\begin{array}{c} (C_{t+1} - \eta C_t)^{-\gamma} \\ -\beta\eta E_{t+1}\left[(C_{t+2} - \eta C_{t+1})^{-\gamma}\right] \end{array} \right) \\ \times \left(\phi'\left(\frac{I_t}{K_t}\right) / \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \right) \\ \times \left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\left[r_{t+1} - \frac{I_{t+1}}{K_{t+1}}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array} \right) \\ +\eta\left(C_{t+1} - \eta C_t\right)^{-\gamma} \end{array} \right], \tag{3.11}$$

where:

$$\phi'\left(\frac{I_t}{K_t}\right) = \delta^{1/\xi}\left(\frac{I_t}{K_t}\right)^{-1/\xi}. \tag{3.12}$$

For a detailed derivation please refer to Appendix A.

3.3.2 Projects

Every project consists of one worker and one firm. The project's production technology is specified as:

$$y_t = \theta_t^{1-\alpha} k_t^{\alpha}, \tag{3.13}$$

where total factor productivity, θ_t, develops according to:

$$\ln\theta_t = \ln\theta_{t-1} + \varepsilon_t, \tag{3.14}$$

with:

$$\varepsilon_t \sim N\left(\bar{\varepsilon}, \sigma_\varepsilon^2\right). \tag{3.15}$$

It follows that total factor productivity is a non-stationary process. In Appendix C we show how to normalize this economy so that in the transformed model all variables are stationary.

At the beginning of each period t, projects observe current-period productivity θ_t. Then projects rent capital k_t at the market clearing interest rate r_t and pay a wage w_t to their workers. A project's profit π_t follows as:

$$\pi_t = \theta_t^{1-\alpha} k_t^{\alpha} - w_t - r_t k_t. \tag{3.16}$$

Projects are assumed to maximize profits on behalf of their owners. The amount of capital a project rents, taking the interest rate r_t and wages w_t as given, thus follows from the first-order condition of the project:

$$\frac{\partial \pi_t}{\partial k_t} = 0 : \theta_t^{1-\alpha} \alpha k_t^{\alpha-1} - r_t = 0, \tag{3.17}$$

$$k_t = \theta_t \left(\frac{r_t}{\alpha} \right)^{\frac{1}{\alpha-1}}. \tag{3.18}$$

Note that in a non-stationary (growing) economy wages have to grow together with the aggregate economy, as otherwise the profits of projects would increase *relative* to total output, resulting in ever-increasing profitability. Therefore, we set wages as a constant fraction of aggregate output in the economy:

$$w_t = \overline{w} Y_t, \tag{3.19}$$

where:

$$Y_t = N_t y_t, \tag{3.20}$$

and N_t denotes the number of active projects in period t. With this specification of the wage process, wages are, however, by construction, highly procyclical and very volatile (as volatile as output). Now, because $\left(\theta_t^{1-\alpha} k_t^{\alpha} - r_t k_t\right)$ is procyclical, profits (π_t) are both more procyclical and more volatile the stickier (the less procyclical) wages are. We can specify the following more general process for wages:

$$w_t = \overline{w} Y_t^{\kappa} K_t^{1-\kappa}, \tag{3.21}$$

so that wages are a constant fraction of a composite measure of the size of the aggregate economy. Note that Y_t fluctuates much more around the trend growth rate than does K_t. As a result, wages are less procyclical and less volatile the lower κ, that is the more weight we place on the aggregate capital stock in our composite measure. We will exploit this channel in parts of this chapter in order to render project profits both more procyclical and more volatile. At the same time, a lower value for κ brings the wage process closer to the data, because in the data wages are considerably less volatile than output.

3.3.3 The Matching Market

Projects are productive through discrete time until they get destroyed. Destruction occurs exogenously at the beginning of period with probability ρ^x. We interpret this exogenous destruction shock as a low idiosyncratic productivity shock.[7] New projects enter the economy via a matching market where workers and firms are brought together. In the economy there is a continuum of workers with unit mass. The mass of unmatched workers seeking employment in period t is denoted by U_t.[8] There is a continuum of firms with potentially infinite mass. Firms can decide to enter the matching market by posting a vacancy for a worker with the goal to set up a new project. The mass of firms posting vacancies in period t is denoted by V_t. The matching process within a period takes place after observation of aggregate productivity for that period, but before actual production takes place. The number of successful matches in any given period is determined by a Cobb Douglas specification which is standard in the search-theoretical literature:[9]

$$m_t = \min\left\{\mu\left(U_t^{\nu} V_t^{1-\nu}\right), U_t, V_t\right\}. \tag{3.22}$$

The unmatched workers of the current period re-enter next period's matching market together with the workers whose projects are destroyed at the beginning of next period.

After $\rho^x N_{t-1}$ projects have been destroyed at the beginning of period t, but before the matching process in period t, the number of (still) active projects ready to produce is $(1-\rho^x) N_{t-1}$, where N_{t-1} is the number of active and producing projects in period $(t-1)$. The number of unemployed workers who enter the matching market in period t follows as:

$$U_t = 1 - (1-\rho^x) N_{t-1}. \tag{3.23}$$

Firms can choose freely whether or not to post a vacancy in a particular period at a cost ψ_t, thereby entering the matching market in that period. At the beginning of period t, firms observe aggregate productivity θ_t, and decide whether or not to post a vacancy. The expected net present value of all future profits that can be made with a project is:

$$\begin{aligned} F_t &= E_t\left[\sum_{j=0}^{\infty}\beta^j \frac{u'(C_{t+j})}{u'(C_t)}(1-\rho^x)^j \times \left(\theta_{t+j}^{1-\alpha} k_{t+j}^{\alpha} - w_{t+j} - r_{t+j}k_{t+j}\right)\right] \\ &= \left(\theta_t^{1-\alpha} k_t^{\alpha} - w_t - r_t k_t\right) + E_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}(1-\rho^x)F_{t+1}\right]. \end{aligned} \tag{3.24}$$

[7] Den Haan, Ramey, Watson (2000) endogenize the destruction decision. We abstract here from this endogeneity.

[8] We abstract from the labor force participation decision of workers, and assume instead that workers are either productive within a project or are part of the unemployment pool, searching for new employment.

[9] Note that for small enough values for U_t or V_t, the Cobb Douglas specification $m_t = \mu\left(U_t^{\nu} V_t^{1-\nu}\right)$ by itself can lead to values for m_t such that $m_t > \min(U_t, V_t)$. As the number of matches must not exceed the pool of either unemployed workers or posted vacancies, we specify $m_t = \min\left\{\mu\left(U_t^{\nu} V_t^{1-\nu}\right), U_t, V_t\right\}$ to rule out those cases, as is standard in the literature. In the numerical solution of all the models in this chapter $m_t < \min(U_t, V_t)$ always holds.

In equilibrium, a firm will post a vacancy at time t only if the expected benefit of posting a vacancy, which is the probability to get matched once a vacancy has been posted, λ_t^F, times the expected value of a project conditional on being matched, F_t, are equal to the cost of posting a vacancy, ψ_t:

$$\psi_t = \lambda_t^F \times F_t, \tag{3.25}$$

where the probability to get matched out of the perspective of a firm, λ_t^F, is the number of successful matches in any given period divided by the number of posted vacancies in that period:[10]

$$\lambda_t^F = \frac{m_t}{V_t} = \mu \left(\frac{U_t}{V_t}\right)^{\nu}. \tag{3.26}$$

From the free-entry condition (3.25) together with (3.24) and (3.26), the equilibrium number of vacancies posted, V_t, follows, determining the number of active projects in period t, N_t:

$$\psi_t = \mu \left(\frac{U_t}{V_t}\right)^{\nu} \times F_t, \tag{3.27}$$

where rearranging gives:

$$V_t = U_t \left(\frac{\mu F_t}{\psi_t}\right)^{\frac{1}{\nu}}, \tag{3.28}$$

resulting in the following expression for the law of motion of the number of active projects and thus employment:

$$\begin{aligned} N_t &= (1-\rho^x)N_{t-1} + m_t \\ &= (1-\rho^x)N_{t-1} + \lambda_t^F V_t \\ &= (1-\rho^x)N_{t-1} + \mu U_t \times \left(\frac{\mu F_t}{\psi_t}\right)^{\frac{1}{\nu}-1}. \end{aligned} \tag{3.29}$$

It follows that projects that get created within any given period can already produce within that period.[11]

For the number of projects in a growing economy to be stationary, the costs of posting a vacancy must grow together with the aggregate economy. Otherwise it would become

[10] For the sake of expositional clarity we abstain here from using the full specification of the matching technology ($m_t = \min\left\{\mu\left(U_t^{\nu}V_t^{1-\nu}\right), U_t, V_t\right\}$). Instead, we show equilibrium for $m_t = \mu\left(U_t^{\nu}V_t^{1-\nu}\right)$, implicitly assuming $m_t < \min(U_t, V_t)$ always to hold. In the numerical solution of all models in this chapter $m_t < \min(U_t, V_t)$ always holds.

[11] The usual assumption in the search and matching literature is that firms that get created within any given period start producing at the beginning of the following period (see Chapter 2). We choose an alternative timing assumption to make clear that one of our key results, the procyclicality of aggregate employment, is not driven by the assumption that employment cannot respond in the same period as the technology shock. In our model, employment thus *can* respond in the same period as the technology shock.

more and more profitable over time to post a vacancy (as profits grow with output), the number of vacancies would grow over time, and the economy would inevitably reach full employment and remain there. Therefore, we assume the posting costs to be a constant fraction of the size of the aggregate economy, as measured by the aggregate capital stock of the economy:

$$\psi_t = \overline{\psi} K_t. \tag{3.30}$$

3.3.4 The Capital Market

In period t, N_t projects enter the capital market with the purpose of renting capital from households. The total supply of capital in period t, K_t, is fixed, as it is determined by the savings decision of the representative household in period $(t-1)$, *before* observation of the random shock θ_t. The capital market clears when capital demand equals capital supply:

$$N_t k_t = K_t. \tag{3.31}$$

The market clearing interest rate, r_t, follows from the equilibrium condition (3.31) together with the single project's optimal choice of capital, taking interest rates as given (see equation (3.18)), as:

$$\begin{aligned} r_t &= \theta_t^{1-\alpha} \alpha k_t^{\alpha-1} \\ &= \theta_t^{1-\alpha} \alpha \left(\frac{K_t}{N_t} \right)^{\alpha-1}. \end{aligned} \tag{3.32}$$

3.3.5 Equilibrium

The state variables in this economy are $\theta, K, C_{-1}, N_{-1}$. The recursive equilibrium consists of functions $\theta'(\theta, K, C_{-1}, N_{-1})$, $K'(\theta, K, C_{-1}, N_{-1})$, $C(\theta, K, C_{-1}, N_{-1})$, $N(\theta, K, C_{-1}, N_{-1})$ such that (3.2), (3.6), (3.9), (3.11), (3.13), (3.14), (3.15), (3.16), (3.19), (3.20), (3.23), (3.24), (3.25), (3.26), (3.29), (3.30), (3.31), (3.32) hold simultaneously.

3.3.6 Asset Prices

The Equity Return

The representative household owns the aggregate capital stock as well as all projects in the economy. As such it receives a set of distinct payments. For analytical purposes we split those payments into flowing from three different sources, we refer to them as three different 'corporations', all of which are owned by the representative household. The

Aggregate equity return is then defined as the return of the household's claim on the payments it receives from all three corporations together.

The Capital Leasing Corporation The Capital Leasing Corporation rents the aggregate capital stock out to projects on a quarterly basis and finances aggregate investment out of the rental proceeds. We define the return of the Capital Leasing Corporation as follows:

$$V_t^{lease} = \beta E_t \left[\begin{array}{c} \frac{\left[(C_{t+1}-\eta C_t)^{-\gamma} - \beta\eta E_{t+1}\left[(C_{t+2}-\eta C_{t+1})^{-\gamma}\right]\right]}{(C_t-\eta C_{t-1})^{-\gamma} - \beta\eta E_t\left[(C_{t+1}-\eta C_t)^{-\gamma}\right]} \\ \times \left(D_{t+1}^{lease} + V_{t+1}^{lease}\right) \end{array} \right], \tag{3.33}$$

where:

$$D_t^{lease} = r_t K_t - I_t. \tag{3.34}$$

The definition of the return is immediate:

$$r_{t+1}^{lease} = \frac{D_{t+1}^{lease} + V_{t+1}^{lease}}{V_t^{lease}} - 1. \tag{3.35}$$

The Capital Renting Corporation The Capital Renting Corporation comprises all currently existing projects in the economy. Existing projects rent capital, pay out wages, and pay out the residual as profits (π_t). An existing project gets destroyed with probability ρ^x at the beginning of next period. We can define the return of this corporation in terms of F_t and D_t^{rent}, the value and dividend respectively of *one* existing project (see equation (3.24)), as follows:

$$F_t = D_t^{rent} + E_t \left[\beta \frac{u'(C_{t+1})}{u'(C_t)} (1-\rho^x) F_{t+1} \right], \tag{3.36}$$

and:

$$D_t^{rent} = \pi_t, \tag{3.37}$$

where the return follows as:

$$\begin{aligned} r_{t+1}^{rent} &= \frac{(1-\rho^x) N_t D_{t+1}^{rent} + (1-\rho^x) N_t \left(F_{t+1} - D_{t+1}^{rent}\right)}{N_t \left(F_t - D_t^{rent}\right)} - 1 \\ &= \frac{(1-\rho^x) F_{t+1}}{(F_t - \pi_t)} - 1. \end{aligned} \tag{3.38}$$

The Start-up Corporation The Start-up Corporation is responsible for the creation of new projects. This corporation operates by means of a matching market, producing start-up projects while generating posting costs. The return of the Start-up Corporation is zero:

$$\begin{aligned} r_{t+1}^{startup} &= \frac{m_{t+1}F_{t+1}}{\psi_{t+1}V_{t+1}} - 1 = \frac{\frac{m_{t+1}}{V_{t+1}}F_{t+1}}{\psi_{t+1}} - 1 \\ &= \frac{\lambda_{t+1}^F F_{t+1}}{\psi_{t+1}} - 1 = 1 - 1 = 0. \end{aligned} \tag{3.39}$$

This follows directly from the free-entry condition in equilibrium: $\lambda_t^F \times F_t = \psi_t$ (see Section 3.3.3).

The Aggregate Equity Return The Aggregate equity return is simply defined as the return of a claim on all three corporations:

$$V_t^E = \beta E_t \left[\begin{array}{c} \frac{\left[(C_{t+1}-\eta C_t)^{-\gamma} - \beta\eta E_{t+1}\left[(C_{t+2}-\eta C_{t+1})^{-\gamma}\right]\right]}{(C_t-\eta C_{t-1})^{-\gamma} - \beta\eta E_t\left[(C_{t+1}-\eta C_t)^{-\gamma}\right]} \\ \times \left(D_{t+1}^E + V_{t+1}^E\right) \end{array} \right], \tag{3.40}$$

where:

$$\begin{aligned} D_t^E &= D_t^{lease} + D_t^{rent} + (m_t F_t - \psi_t V_t) \\ &= D_t^{lease} + D_t^{rent} \\ &= (r_t K_t - I_t) + ((1-\rho^x) N_t \pi_t). \end{aligned} \tag{3.41}$$

The definition of the Aggregate equity return is immediate:

$$r_{t+1}^E = \frac{D_{t+1}^E + V_{t+1}^E}{V_t^E} - 1. \tag{3.42}$$

Equivalently, we can write this return as:

$$r_{t+1}^E = \frac{D_{t+1}^E + V_{t+1}^{lease} + (1-\rho^x) N_t (F_{t+1} - \pi_{t+1})}{V_t^{lease} + N_t (F_t - \pi_t)}. \tag{3.43}$$

Interpretation In Appendix B we show that:

$$V_t^{lease} = E_t \left[\sum_{i=1}^{\infty} \beta^i \frac{u'_{t+i}}{u'_t} D_{t+i}^{lease} \right] = \frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)}, \tag{3.44}$$

and that the return of the Capital Leasing Corporation can thus be written as:

$$r_{t+1}^{lease} = \frac{r_{t+1} + \frac{\partial K_{t+2}}{\partial K_{t+1}} P_{K,t+1}}{P_{K,t}} - 1, \tag{3.45}$$

where:

$$\frac{\partial K_{t+2}}{\partial K_{t+1}} = \left[(1-\delta) + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right], \tag{3.46}$$

$$P_{K,t} = \frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}, \tag{3.47}$$

and $P_{K,t}$ denotes the price at time t of one additional unit of capital in terms of the consumption good. One marginal unit of capital increases output by r_{t+1}, measured in consumption goods, and next period's capital stock by $\frac{\partial K_{t+2}}{\partial K_{t+1}}$, keeping I_{t+1} constant, measured in capital goods. It is worthwhile to point out that a very similar expression holds for the Aggregate equity return in the original Jermann (1998) model as well as in the Boldrin, Christiano, Fisher (2001) model (see Appendix D). In all those models the behavior of firm values and equity returns is thus determined by the technology to transform consumption into capital goods. It follows that the NPV of future dividends does not vary independently from the aggregate capital stock of the economy. In other words, an increase in dividend volatility induces offsetting changes in the pricing kernel in general equilibrium so that equation (3.44) continues to hold (that is, the analogous equation in Jermann or Boldrin, Christiano, Fisher). Fluctuations in long-term growth rates or changes in the discount rate for instance have basically no effect on asset prices. Now, in our model *only* the return of the *Capital Leasing Corporation* is determined by the capital transformation technology while the return of the Capital Renting Corporation is not. The value of the Capital Renting Corporation, and consequently the Aggregate equity return which is a weighted average of the returns of both corporations, *are* sensitive to fluctuations in long-term growth rates, discount rates, and in particular the volatility of profits. We exploit this channel in Section 3.5.1 as follows. The return of the Capital Renting Corporation is the return of a claim on profits from all existing projects. Recall that profits of existing projects are defined as:

$$\pi_t = \left(\theta_t^{1-\alpha} k_t^{\alpha} - r_t k_t\right) - w_t. \tag{3.48}$$

Because $\left(\theta_t^{1-\alpha} k_t^{\alpha} - r_t k_t\right)$ is procyclical, profits are both more procyclical and more volatile the stickier (the less procyclical) wages are. By making profits more volatile and more procyclical we induce households to demand a higher premium for holding the Capital Renting Corporation, thus also driving the Aggregate equity premium up.

The Risk-Free Rate

The risk-free rate in this economy is given by:

$$r_{t+1}^F = \frac{1}{\beta} \frac{\begin{array}{c}(C_t - \eta C_{t-1})^{-\gamma} \\ -\beta\eta E_t\left[(C_{t+1} - \eta C_t)^{-\gamma}\right]\end{array}}{E_t\left[\begin{array}{c}(C_{t+1} - \eta C_t)^{-\gamma} \\ -\beta\eta E_{t+1}\left[(C_{t+2} - \eta C_{t+1})^{-\gamma}\right]\end{array}\right]} - 1. \tag{3.49}$$

3.4 Calibration

We choose the following calibration of the model, as laid out in Table 3.1. In order to explain asset prices with production economy models, (i) consumers have to be sufficiently sensitive to consumption risk, and (ii) consumers must be prevented to rid themselves of most consumption risk via the investment technology. In our model, as in Jermann (1998), this is achieved by the combination of a high habit level ($\eta = 0.80$) with substantial capital adjustment costs ($\xi = 0.50$).[12] We are forced to choose a relatively high value for the discount factor ($\beta = 0.99975$) in order to enable the model to account for the level of the risk-free rate.

We set ρ^x to 10%, following den Haan, Ramey, Watson (2000) who rely on results from Hall (1995) and Davis, Haltiwanger, Schuh (1996) to justify a range for ρ^x between 8% and 11%. We use $\overline{w}$ and $\overline{\psi}$ to match the average aggregate labor share and steady state employment. We set the exponent of the matching technology, ν, that is the matching elasticity with respect to labor market tightness, equal to 0.40. Petrongolo and Pissarides (2001) survey the relevant literature and report the 'consensus estimate' of ν to be equal to 0.50. We use a lower value for ν in order to generate some additional movement in employment. The remaining parameter μ, the multiplier of the matching technology, is set so as to jointly match statistics from simulated data to empirical measures of the worker and firm matching probabilities (λ^W and λ_t^F respectively) to the extent possible.

Under the benchmark calibration, the average capital share of the economy is 36%, the average labor share is 63%, and the average profit share is 1%. The average aggregate posting costs amount to about 1% of total output.[13,14] The remaining parameters are set

[12]Jermann (1998) parameterizes $\gamma = 5.00$, $\eta = 0.82$, $\xi = 0.23$. Abel (1980) estimates the elasticity of the investment capital ratio with respect to Tobin's q (ξ) to lie between 0.27 and 0.52.

[13]We compute the shares as follows: Capital share $= \frac{r_t K_t}{(Y_t - V_t\psi_t)}$, labor share $= \frac{N_t w_t}{(Y_t - V_t\psi_t)}$, profit share $= \frac{N_t \pi_t}{(Y_t - V_t\psi_t)}$, share of posting costs $= \frac{-V_t\psi_t}{(Y_t - V_t\psi_t)}$.

[14]Andolfatto (1996) also calibrates his production economy model with a matching market for labor

Table 3.1: **Benchmark Calibration**

Quarterly Model		
Parameter	Description	Value
α	Elasticity of capital	0.36
δ	Depreciation rate of capital	0.025
ξ	Elasticity of $\left(\frac{I_t}{K_t}\right)$ w.r.t. Tobin's q	0.50
β	Discount factor	0.99975
γ	Coefficient of relative risk aversion	3.00
η	Habit level	0.80
μ	Matching technology: multiplier	0.60
ν	Matching technology: exponent	0.40
ρ^x	Exogenous destruction rate	0.10
$\overline{\psi}$	Posting cost: fraction of capital stock	0.50%
$\overline{w}$	Wage rate: fraction of output	67.50%
$\overline{\varepsilon}$	Mean growth rate	0.30%
σ_ε	Standard deviation of shock to ε	1.90%

to standard values for quarterly parameterizations (see, e.g., Boldrin, Christiano, Fisher (2001)).

Relying on den Haan and Marcet (1990), we use the Parameterized Expectations Algorithm (PEA) to solve all models. For details regarding the numerical solution technique please refer to Appendix C. In Appendix D we demonstrate the importance of solving the Jermann (1998) framework with modern non-linear solution techniques, in contrast to Jermann himself who uses linear approximations.

3.5 Model Results

In this section we report results for four different versions of the matching model: a version without habit formation and without capital adjustment costs (henceforth Basic model), a version of the model with habit formation but without capital adjustment costs (henceforth Habit model), a version with capital adjustment costs but without habit formation (henceforth Adjustment Cost model), and finally the model with both habit formation and capital adjustment costs (henceforth Benchmark model). We calibrate the Benchmark model as discussed in Section 3.4. The parameter values for all other versions of the model are the same, unless indicated otherwise in Table 3.2.

We solve two different versions of the Benchmark model. A version with $\kappa = 1.000$,

so that the average aggregate posting costs amount to about 1% of total output.

Table 3.2: **Four Versions of the Matching Model**

Parameter	Basic	Habit	Adj. Cost	Benchmark
η	0	0.80	0	0.80
ξ	∞	∞	0.50	0.50
κ	1.000	1.000	1.000	1.000 / 0.825
σ_ε	0.022	0.022	0.022	0.019

where wages are a constant fraction of aggregate output: $w_t = \overline{w}Y_t$ and as such highly procyclical and as volatile as output, and a version with $\kappa = 0.825$, where the process for wages is given by: $w_t = \overline{w}Y_t^\kappa K_t^{1-\kappa}$ (see Section 3.3.2 for a discussion).[15] With $\kappa = 0.825$, wages are less volatile and less procyclical (stickier) due to the fact that K_t fluctuates much less around the trend growth rate than does Y_t.[16]

As will be demonstrated in Section 3.5.4, versions of the model without both habit formation and capital adjustment costs are lacking the internal magnification mechanism of the Benchmark model. We are therefore forced to increase the standard deviation of the technology shock (σ_ε) for all other versions of the model in order to allow all models to match the empirical value of output volatility.

We proceed by first discussing and interpreting the Benchmark model's ability to explain key asset pricing moments. Then we examine the dynamic behavior of asset prices. In Section 3.5.3 we demonstrate the Benchmark model's ability to explain key moments of macroeconomic time series, and in Section 3.5.4 we discuss some additional properties of the macroeconomic time series generated by the model.

3.5.1 Key Asset Pricing Moments

Table 3.3 contains key asset pricing moments for all versions of the matching model. For the Benchmark model we report the Aggregate equity return (henceforth Aggr) and the return of the Capital Renting Corporation (henceforth Rent) (see Section 3.3.6 for a discussion). The return of the Capital Leasing Corporation turns out very similar to the Aggregate equity return.[17] For all other versions of the matching model (Basic, Habit,

[15]For the version of the model with $\kappa = 0.825$ we recalibrate $\overline{w} = 0.45$ in order to keep the average output shares of capital, labor, profits, and aggregate posting costs the same as for the version with $\kappa = 1.000$.

[16]In the data, the standard deviation of wages relative to output $(\frac{\sigma[hp \ln w_t]}{\sigma[hp \ln y_t]})$ is equal to 0.63. With $\kappa = 1.000$, $\frac{\sigma[hp \ln w_t]}{\sigma[hp \ln y_t]} = 1.00$, with $\kappa = 0.825$, $\frac{\sigma[hp \ln w_t]}{\sigma[hp \ln y_t]} = 0.83$. Thus, lowering κ moves the wage process closer to the data in that sense. We use U.S. data on wages and output from 1952 to 2004 from the Bureau of Labor Statistics (BLS). For a discussion of the data see Chapters 2 and 5.

[17]The behavior of the Aggregate equity return is driven to a relatively large extent by the return of the Capital Leasing Corporation. This is due to the fact that the Leasing Corporation is much larger

Table 3.3: **Results for the Matching Model — Key Asset Pricing Moments**
σ_x denotes the standard deviation of variable x. Rates of return are annualized and in percent. 'Aggr' denotes the Aggregate equity return, 'Rent' denotes the return of the Capital Renting Corporation, as outlined in Section 3.3.6. For the Basic, the Habit, and the Adj. Cost models only the Aggregate equity return is reported. The respective values for the return of the Capital Renting Corporation are very similar. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1892 to 1987. We take those values from Boldrin, Christiano, Fisher (2001), who in turn rely on Cecchetti, Lam, Mark (1993). Results for all models are based on 200 replications of sample size 200 each.

Statistic	Basic	Habit	Adj. Cost	Benchmark Model				Data
				$\kappa = 1.000$		$\kappa = 0.825$		
$\mathrm{E}\left[r_t^F\right]$	3.50	3.36	3.25	2.07		1.81		1.19
								(0.81)
σ_{r^F}	0.73	0.64	0.32	8.47		8.62		5.27
								(0.74)
				Aggr	Rent	Aggr	Rent	
$\mathrm{E}\left[r_t^E - r_t^F\right]$	0.01	0.01	0.46	4.71	0.37	6.13	1.24	6.63
								(1.78)
σ_{r^E}	0.73	0.69	6.72	28.69	9.35	30.61	12.55	19.40
								(1.56)
$\frac{E\left[r_t^E - r_t^F\right]}{\sigma_{r^E}}$	0.01	0.02	0.07	0.16	0.04	0.20	0.10	0.34
								(0.09)

Adj. Cost) only the Aggregate equity return is reported because the return of the Capital Leasing and the Capital Renting Corporation are very similar.

Table 3.3 confirms that the model with both habit preferences and capital adjustment costs can match basic asset pricing moments.[18] The Benchmark model displays relatively realistic values for the equity premium, the volatility of the equity return, therefore for the Sharpe ratio, and for the volatility of the risk-free rate. As will be demonstrated in Section 3.6, the too high volatilities of the equity return and the risk-free rate carry over from the basic Jermann (1998) framework and are due to the simple internal habit specification used.

Recall that project profits are defined as:

$$\pi_t = \left(\theta_t^{1-\alpha} k_t^{\alpha} - r_t k_t\right) - w_t. \tag{3.50}$$

than the Renting Corporation because aggregate proceeds from the rental of capital to projects are much larger than project profits in our calibration.

[18]It is possible to match the level of the risk-free rate also with the Basic, the Habit, and the Adjustment Cost models. We do not change the calibration of those models so as to match the risk-free rate, because the focus of Table 3.3 is on the direct effect of excluding single components from the Benchmark model.

Because $\left(\theta_t^{1-\alpha}k_t^{\alpha} - r_t k_t\right)$ is procyclical, it follows that profits are more procyclical the less volatile and procyclical wages are.[19] If $\kappa = 1.000$, $w_t = \overline{w}Y_t$ and wages are, by construction, perfectly correlated with, and as volatile as, aggregate output. Therefore profits are actually countercyclical for high values of κ. Because the return of the Capital Renting Corporation is basically equivalent to the return of a claim on aggregate profits, the return of the Capital Renting Corporation commands a much lower premium over the risk-free rate compared to the Aggregate equity return, which is a claim on (small) aggregate profits *and* highly procyclical (large) aggregate proceeds from the rental of capital to projects. Wages respond, by construction, *less* to productivity shocks if κ is lower (see Section 3.3.2 for a discussion). As a result, with lower κ project profits are more procyclical, inducing households to demand a higher premium for holding a claim on profits, in other words for holding the Capital Renting Corporation. This in turn also drives the Aggregate equity premium as well as the Aggregate Sharpe ratio *up*: from $\kappa = 1.000$ to $\kappa = 0.825$ the Aggregate equity premium increases from 4.71% to 6.13%.

3.5.2 The Dynamic Behavior of Asset Prices

In this section we compare the dynamic behavior of asset prices in the Benchmark model with their empirical counterparts as well as with the dynamics generated by a standard production economy model (henceforth Standard), with power utility preferences, no capital adjustment costs, and fixed labor supply.[20] We examine the cyclical variation of conditional moments, the predictability of returns, and the autocorrelation structure of returns.

Cyclical Variation of Conditional Moments and the Price-Dividend Ratio

A substantial part of the finance literature is concerned with the cyclical behavior of (conditional) asset pricing moments. Campbell and Shiller (1988), Chou, Engle, Kane (1992), Fama and French (1989), Schwert (1989), and many others, have documented that the price-dividend ratio and the equity premium are procyclical, while the *expected* equity premium, the conditional volatility of the equity return, as well as the conditional Sharpe ratio all display countercyclical variation. Table 3.4 evaluates the performance of the Benchmark model along those dimensions.

[19] This insight is due to Hall (2005). He develops a model with anticyclical wages in order to address a long-standing problem of labor market matching models: too little fluctuation in profits and as a result in firm values, vacancies, and aggregate employment.

[20] We calibrate the standard production economy model (standard stochastic growth model) as follows: $\alpha = 0.36$, $\beta = 0.99975$, $\gamma = 3.00$, $\delta = 0.025$, $\overline{\varepsilon} = 0.0030$, $\sigma_{\varepsilon} = 0.022$. We assume leisure not to enter the utility function, that is the supply of labor is fixed.

Table 3.4: **Variation of Conditional Moments — Cross-Correlations with Output**
All correlation coefficients are computed using annual values. Output is logged and HP-filtered prior to analysis. $p_t - d_t$ denotes the log price-dividend ratio. For the Benchmark model we report values for the Aggregate equity return (see Section 3.3.6 for a discussion). The 'Data' column contains estimates based on U.S. data from 1890 or 1947 to 1991 respectively, as indicated. We take those values from Guvenen (2005), who in turn relies on Campbell (1999). No point estimates are provided for the conditional moments, as no direct empirical measures are available. Results for the models are based on 200 replications of sample size 200 each.

Statistic	Standard	Benchmark		U.S. Data	
		$\kappa = 1.000$	$\kappa = 0.825$	1890-1991	1947-1991
$p_t - d_t$	-0.57	0.17	0.22	0.15	0.42
$r_t^E - r_t^F$	0.09	0.12	0.08	0.22	0.15
$E_t\left[r_{t+1}^E - r_t^F\right]$	0.50	-0.41	-0.41	< 0	< 0
$\sigma_t\left(r_{t+1}^E\right)$	0.50	-0.44	-0.50	< 0	< 0
$\frac{E_t\left[r_{t+1}^E - r_t^F\right]}{\sigma_t\left(r_{t+1}^E\right)}$	0.50	-0.19	0.19	< 0	< 0

From Table 3.4 we conclude that the Benchmark model constitutes a substantial improvement over the standard production economy model. The model manages to match the procyclicality of the price-dividend ratio and the equity premium, as well as the countercyclicality of all conditional moments. For the model with $\kappa = 0.825$, the correlation of the conditional volatility of the equity premium with output turns out slightly too negative relative to the correlation with the expected equity return, thus rendering the cross-correlation of the conditional Sharpe ratio positive.

So what is the crucial difference between the Benchmark model and standard production economies? The answer lies with the habit preferences. As can be seen from Figures 3.1 and 3.2, without habit preferences a permanent income shock translates into permanently higher utility levels for households. On the contrary, with habit preferences this is not the case: higher consumption today and tomorrow does not translate into higher utility today and tomorrow, as the habit level rapidly adjusts. With habit preferences, households thus behave considerably more risk averse in the face of permanent income shocks. Knowing that they can only augment utility levels permanently by means of an increasing consumption path, households increase investment by much more upon impact of the shock compared to standard preferences. Then they gradually lower their savings levels, thus feeding a slowly increasing consumption path (see Figure 3.3 and Figure 3.4). However, the elevated investment levels depress future equity returns, resulting in rapid mean reversion of returns.[21] This mechanism drives the countercyclicality of both the expected equity premium and the conditional volatility.

[21]Equity returns get pulled back towards the mean so rapidly that they actually 'under-shoot'.

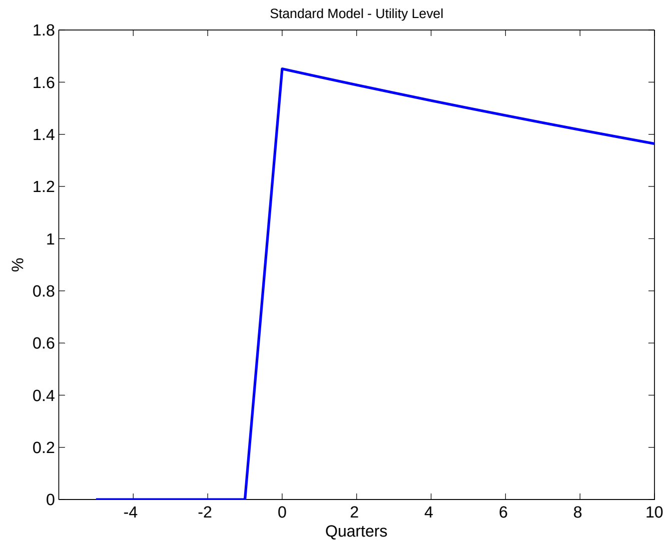

Figure 3.1: **Impulse Response Standard Model — Utility Level**
Effect of a positive technology shock on utility levels in the Standard model. One standard deviation shock to productivity at time t = 0. Without habit preferences, a permanent income shock translates into permanently higher utility levels.

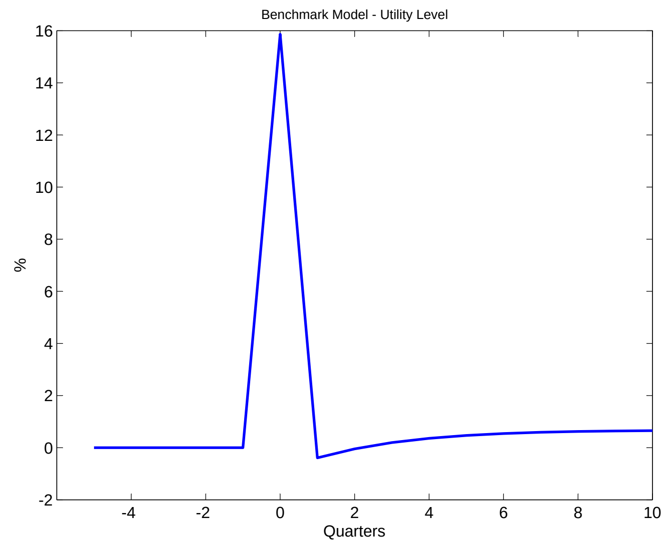

Figure 3.2: **Impulse Response Benchmark Model — Utility Level**
Effect of a positive technology shock on utility levels in the Benchmark model. One standard deviation shock to productivity at time t = 0. Without habit preferences, a permanent income shock translates into permanently higher utility levels. On the contrary, with habit preferences this is not the case: higher consumption today and tomorrow does not translate into higher utility today and tomorrow, as the habit level rapidly adjusts. With habit preferences, households are thus very keen to engineer an as smooth consumption profile as possible.

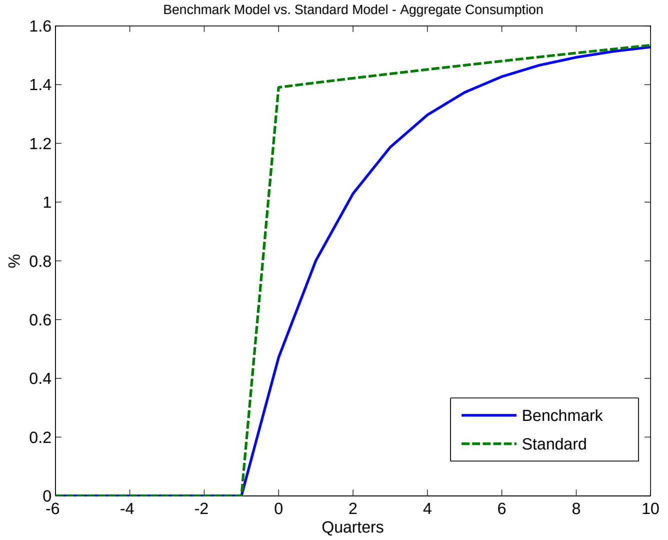

Figure 3.3: **IR Benchmark Model vs. Standard Model — Consumption**
Effect of a positive technology shock on consumption in the Benchmark model and in the Standard model. One standard deviation shock to productivity at time t = 0. Knowing that they can only augment utility levels permanently by means of an increasing consumption path, households crank up investment by more upon impact of the shock compared to standard preferences. Then they gradually lower their savings levels, thus feeding a slowly increasing consumption path.

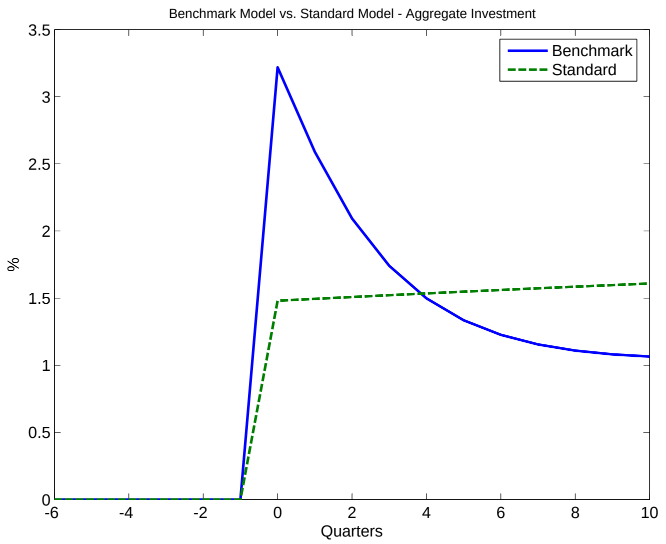

Figure 3.4: **IR Benchmark Model vs. Standard Model — Investment**
Effect of a positive technology shock on investment in the Benchmark model and in the Standard model. One standard deviation shock to productivity at time t = 0. Knowing that they can only augment utility levels permanently by means of an increasing consumption path, households crank up investment by more upon impact of the shock compared to standard preferences. Then they gradually lower their savings levels, thus feeding a slowly increasing consumption path.

Table 3.5: **Long-Horizon Regr. of the Equity Return on the Price-Dividend Ratio** All computations are based on annual values. For the Benchmark model we report values for the Aggregate equity return (see Section 3.3.6 for a discussion). The 'Data' column contains estimates based on U.S. data from 1890 or 1947 to 1991 respectively, as indicated. We take those values from Guvenen (2005), who in turn relies on Campbell (1999). Results for the models are based on 200 replications of sample size 200 each.

$$r^E_{t,t+k} = \alpha + \beta \log\left(P_t/D_{t-1}\right) + \varepsilon_{t,t+k}$$

Horizon	Standard		Benchmark Model				U.S. Data			
			$\kappa = 1.000$		$\kappa = 0.825$		1890-1991		1947-1991	
Years	β	R^2	β	R^2	β	R^2	β	R^2	β	R^2
1	-0.15	0.98	-0.23	0.22	-0.51	0.58	-0.08	0.06	-0.10	0.11
3	-0.38	0.98	-0.27	0.50	-0.67	0.89	-0.16	0.12	-0.18	0.31
5	-0.51	0.98	-0.27	0.70	-0.68	0.95	-0.25	0.21	-0.33	0.51
7	-0.57	0.98	-0.28	0.80	-0.70	0.97	-0.34	0.27	-0.44	0.57
10	-0.56	0.98	-0.28	0.88	-0.71	0.99	-0.36	0.39	-0.64	0.73

A similar explanation holds for the procyclicality of the price-dividend ratio in the Benchmark model: due to high initial investment levels, dividends drop in response to permanent income shocks, while habit preferences render share prices much more sensitive to shocks. As a consequence the price-dividend ratio is, contrary to the standard production economy model, procyclical.

The Predictability of Returns: Long-Horizon Regressions

We now turn our attention to the predictability of equity returns: the seminal long-horizon regressions of the equity return on the price-dividend ratio, where high price-dividend ratios forecast low future returns. Empirically, the (negative) coefficients on the price-dividend ratio are decreasing with the time horizon of the regression while the R^2 values are increasing.

We see from Table 3.5 that the Benchmark model can match the empirical regression results only to some extent. The coefficient on the price-dividend ratio at short horizons is too negative and does not decrease enough with the time horizon. The R^2 values display an increasing pattern similar to the data, however, their overall level is too high. As discussed above, households with habit preferences react very risk averse to permanent income shocks: they crank up savings relative to how they would behave under standard preferences and so speed up the mean reversion of equity returns. It is this rapid mean reversion of equity returns that causes the coefficients on the price-dividend ratio both to be too negative at short horizons as well as not to decrease by much more with the time horizon.

Table 3.6: **The Autocorrelation Structure of Returns**
All correlation coefficients are computed using annual values. $p - d$ denotes the log price-dividend ratio. For the Benchmark model we report values for the Aggregate equity return (see Section 3.3.6 for a discussion). The 'Data' column contains estimates based on U.S. data from 1890 or 1947 to 1991 respectively, as indicated. We take those values from Guvenen (2005), who in turn relies on Campbell (1999). Results for all models are based on 200 replications of sample size 200 each.

	Lag j (years)		1	2	3	5
$p-d$	Benchmark	$\kappa = 1.000$	0.37	0.06	0.01	-0.02
	Benchmark	$\kappa = 0.825$	0.28	0.04	-0.02	-0.02
	Standard		0.88	0.73	0.60	0.38
	1890-1991		0.79	0.59	0.52	0.35
	1947-1991		0.85	0.69	0.60	0.23
$r^E - r^F$	Benchmark	$\kappa = 1.000$	-0.01	-0.03	-0.01	-0.02
	Benchmark	$\kappa = 0.825$	-0.03	-0.03	-0.03	-0.01
	Standard		-0.02	-0.03	-0.02	-0.03
	1890-1991		0.03	-0.22	0.08	-0.14
	1947-1991		-0.08	-0.24	0.19	0.05
r^F	Benchmark	$\kappa = 1.000$	0.38	0.04	-0.02	-0.03
	Benchmark	$\kappa = 0.825$	0.39	0.04	-0.03	-0.04
	Standard		0.88	0.73	0.60	0.38
	1890-1991		0.53	0.36	0.23	0.14
	1947-1991		0.52	0.24	0.36	0.07
$\lvert r^E \rvert$	Benchmark	$\kappa = 1.000$	0.01	-0.03	-0.03	-0.03
	Benchmark	$\kappa = 0.825$	-0.21	-0.07	-0.00	-0.01
	Standard		0.88	0.73	0.60	0.38
	1890-1991		0.13	0.09	0.06	0.14
	1947-1991		0.03	-0.28	0.06	-0.10

The Autocorrelation Structure of Returns

Empirically, returns and the price-dividend ratio display a set of distinct autocorrelation patterns: both the price-dividend ratio and the risk-free rate are relatively persistent, while the equity premium exhibits no discernible autocorrelation (maybe mild mean-reversion). The absolute value of the equity return displays very weak (if at all then positive) autocorrelation. In Table 3.6 we compare the autocorrelation structure of returns from the Benchmark model with the data, as well as with the structure of returns generated by a standard production economy model.

In the Benchmark model both the price-dividend ratio as well as the risk-free rate exhibit substantial positive autocorrelation at the first lag like their empirical counterparts (albeit with a too low value for the price-dividend ratio). However, unlike in the data, autocorrelation dies out rapidly at longer yearly horizons. In other words, both the price-dividend ratio and the risk-free rate are not sufficiently persistent. The reasons are, again, that (i) households speed up the mean reversion of equity returns and the price-dividend ratio by a very conservative savings schedule, and that (ii) utility levels rapidly bounce back towards the mean, pulling with them the risk-free rate. In the model, as in the data, both the equity premium as well as the absolute value of the equity return display no substantial autocorrelation patterns.[22]

As can be seen both from Table 3.6, in the standard production economy model the price-dividend ratio and the risk-free rate are actually highly persistent. However, the standard production economy model is rejected by the data on the grounds that the autocorrelation structures of the price-dividend ratio and the risk-free rate carry over to the absolute value of the equity return. Also, as can be seen from Table 3.6, the price-dividend ratio displays a counterfactual negative cross-correlation with output.

Conclusion — The Dynamic Behavior of Asset Prices

My overall conclusion from the discussion of the dynamic behavior of asset prices in the Benchmark model is that due to the simple internal habit specification the model relies on, consumption habits adjust too rapidly. Campbell and Cochrane (1999) show in a seminal paper that a slow-moving external consumption habit allows their model, an endowment economy, to do an excellent job in explaining the dynamic behavior of asset prices.

3.5.3 Key Moments of Macroeconomic Time Series

Table 3.7 contains key moments of macroeconomic time series for all versions of the matching model. We can see that the Benchmark model displays a reasonable ability to match the volatility of output, the volatilities of consumption and investment, as well as all cross-correlations with output. Weaker points are the empirical labor market flows (λ^F, λ^W) which the model matches only to some extent.[23]

Note that the correlation of output and employment, a key feature of business cycles,

[22] In the Benchmark model with $\kappa = 0.825$ the absolute value of the equity return displays a (counterfactual) mild negative autocorrelation at the first lag.

[23] In general, it is possible for labor market matching models to replicate those probabilities. Den Haan, Ramey, Watson (2000) replicate the flows exactly with a model that incorporates endogenous destruction of employment relationships. In Chapter 2 we also calibrate a standard stochastic growth model that incorporates a search and matching model of the labor market such that the model matches both λ^F and λ^W. The reason we do not match both λ^F and λ^W with our calibration in this chapter is that our calibration strategy here is to match the exogenous destruction rate (ρ^x) to the data (see Section 3.4) instead of treating it as a free parameter as in Chapter 2.

Table 3.7: **Results for the Matching Model — Key Macroeconomic Moments**
σ_x denotes the standard deviation of variable x. $\rho_{x,y}$ denotes the correlation between variables x and y. All variables (apart from probabilities) are logged and HP-filtered prior to analysis. The statistic σ_Y is reported in percent. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1964 to 1987. We take those values from Boldrin, Christiano, Fisher (2001). The values for the matching probabilities for the firm (λ^F) and the worker (λ^W) are taken from den Haan, Ramey, Watson (2000). Results for all models are based on 200 replications of sample size 200 each. All values reported in the table are quarterly.

Statistic	Basic	Habit	Adj. Cost	Benchmark		Data
				$\kappa = 1.000$	$\kappa = 0.825$	
σ_Y	1.80	1.83	1.80	1.67	1.85	1.89
						(0.21)
σ_C/σ_Y	0.98	0.59	1.18	0.78	0.77	0.40
						(0.04)
σ_I/σ_Y	1.06	2.54	0.61	2.13	2.11	2.39
						(0.06)
σ_N/σ_Y	0.01	0.02	0.01	0.10	0.23	0.80
						(0.05)
$\rho_{Y,C}$	1.00	0.71	1.00	0.81	0.81	0.76
$\rho_{Y,I}$	1.00	0.93	1.00	0.87	0.88	0.96
$\rho_{Y,N}$	0.99	0.96	0.99	0.82	0.96	0.78
λ^F	0.61	0.61	0.61	0.61	0.63	0.71
λ^W	0.59	0.59	0.59	0.59	0.57	0.45

is *positive*. However, as Boldrin, Christiano, Fisher (2001) and Uhlig (2004) demonstrate, both the original Jermann (1998) framework as well as the recent Guvenen (2005) model generate a *negative* correlation between output and employment once the labor-leisure choice is endogenized, thereby destroying the favorable asset pricing properties of both models.[24] The reason is that households use labor to insure themselves against fluctuations in aggregate consumption and output.[25]

So why is the correlation in the Benchmark model positive? The key is that the value of a project, as measured by the expected net present value of all future profits, F_t (see equation (3.24)), is procyclical. From Figures 3.5 and 3.6 we can see the following: whenever there is a positive shock to productivity, project values (F_t) shoot up. Firms react by posting more vacancies, finally translating into higher aggregate employment.[26]

Another striking observation from Table 3.7 is the fact that the volatilities of both output and employment increase in the Benchmark model compared to all other version of the model: the relative volatility of employment displays a 10-fold increase in the Benchmark model with $\kappa = 1.000$ relative to the Basic model and a more than 20-fold increase with $\kappa = 0.825$.[27]

So where does this internal magnification mechanism stem from? The major difference between the Benchmark model with both habit preferences and capital adjustment costs, compared to models with only habit preferences or only capital adjustment costs, is the volatility of the pricing kernel of the households.[28] The reason is that without capital adjustment costs, households use their investment decision to smooth consumption to such an extent that there is not much variation over time in the pricing kernel. Without

[24]In a very recent response, Guvenen (2006) points out that some of the Uhlig (2004) results seem to be driven by the inaccuracy of the log-linearizations Uhlig (2004) relies upon in order to solve the Guvenen (2005) model. Moreover, Guvenen (2006) highlights that using preferences that are non-separable between consumption and leisure when endogenizing the labor-leisure choice in the Guvenen (2005) model can help the Guvenen (2005) model to preserve its asset pricing properties to a considerable extent. However, aggregate employment is still not, as in the data, strongly procyclical.

[25]Agents can choose to work hard when aggregate productivity is low, and work less when output is otherwise high, thus buffering the impact of technology shocks on output and consumption. The implications are a reduction in consumption risk, thus destroying the model's ability to generate realistic asset prices, and a negative correlation between output and employment.

[26]As demonstrated in Section 3.3.3, from the free-entry condition $\psi_t = \lambda_t^M \times F_t$ together with $\lambda_t^M = \mu \left(\frac{U_t}{V_t}\right)^{\nu}$, we can derive $V_t = U_t \left(\frac{\mu F_t}{\psi_t}\right)^{\frac{1}{\nu}}$ (see equation (3.28)). Given that both μ and ν are positive, $\frac{\partial V_t}{\partial F_t} > 0$. Because $N_t = (1-\rho^x)N_{t-1} + m_t$ and $\frac{\partial m_t}{\partial V_t} \geq 0$, $\frac{\partial N_t}{\partial F_t} \geq 0$ follows.

[27]Note that the Basic model, the Habit model, and the Adj. Cost model can match output volatility only because the technology shock is considerably more volatile than in the Benchmark model. In fact, if we solve those models with the same volatility for the technology shock as in the Benchmark model, output volatility turns out far too low at about 1.52%.

[28]We refer to the marginal rate of substitution as the 'pricing kernel': $\frac{u'(C_{t+1})}{u'(C_t)} = \frac{\left((C_{t+1}-\eta C_t)^{-\gamma} - \beta E_{t+1}\left[\eta(C_{t+2}-\eta C_{t+1})^{-\gamma}\right]\right)}{\left((C_t-\eta C_{t-1})^{-\gamma} - \beta E_t\left[\eta(C_{t+1}-\eta C_t)^{-\gamma}\right]\right)}$.

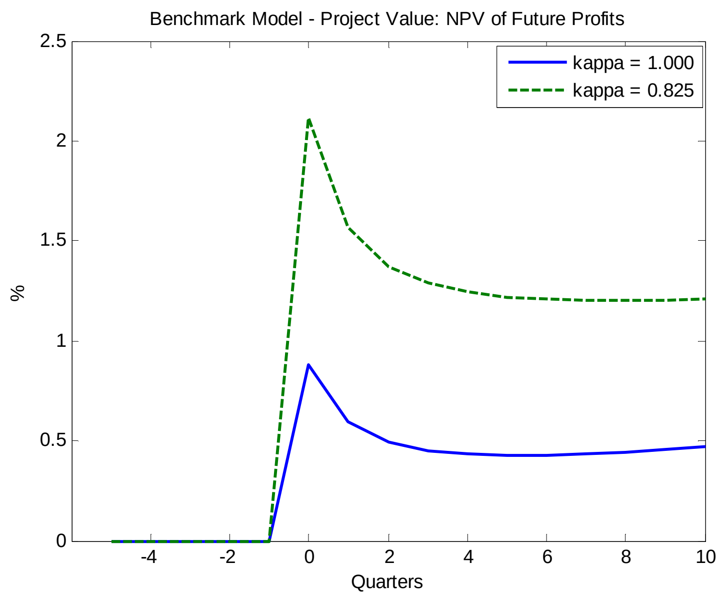

Figure 3.5: **Impulse Response Benchmark Model — NPV of Future Profits**
One standard deviation shock to productivity at time t = 0. The net present value of future profits shoots up. With kappa = 0.825 wages respond less to productivity shocks. As a result, profits and thus NPVs respond by more.

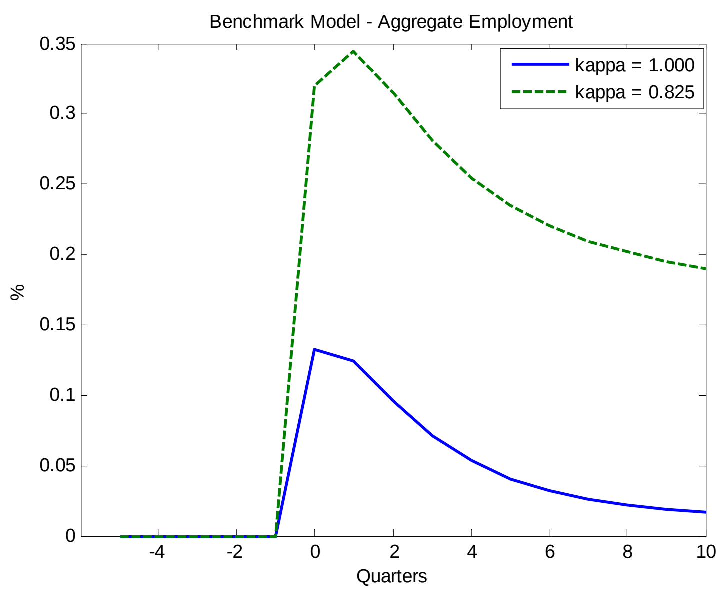

Figure 3.6: **Impulse Response Benchmark Model — Aggregate Employment**
One standard deviation shock to productivity at time t = 0. The net present value of future profits shoots up. Firms react by posting more vacancies, translating into an increase in aggregate employment.

consumption habits on the other hand, the pricing kernel is by itself not sufficiently sensitive to fluctuations in consumption, thus not displaying much variation.[29] It is this time variation in the pricing kernel that allows the Benchmark model to generate a sizeable equity premium, as well as much more realistic volatilities of the equity return and the risk-free rate. Through the matching market the high volatility of the pricing kernel has an additional effect: in order to compute the expected NPV of future profits (F_t), firms discount the expected dividend stream with the pricing kernel of their projects' owners, the households:

$$\pi_t = \theta_t^{1-\alpha} k_t^{\alpha} - w_t - r_t k_t \tag{3.51}$$

$$F_t = E_t \left[\sum_{j=0}^{\infty} \beta^j \frac{u'(C_{t+j})}{u'(C_t)} (1-\rho^x)^j \pi_{t+j} \right]. \tag{3.52}$$

Thus, a more volatile pricing kernel induces more volatile project values, which feed through to the volatility of vacancies and ultimately employment. More (procyclical) movement in the labor market in turn increases the volatility of aggregate output. Given that it is a seminal problem of labor market matching models to generate sufficient movement in employment, this is an interesting result: because of feedback effects from financial markets, through the value of firms, the volatility of employment increases.[30] There is an important channel from financial markets through firm values and labor markets on the real economy.

We furthermore observe from Table 3.7 that with $\kappa = 0.825$ employment volatility almost *doubles* compared to the model with $\kappa = 1.000$. Wages respond, by construction, *less* to productivity shocks if κ is lower (see Section 3.3.2 for a discussion). As a result, project profits are more procyclical and more volatile, and project NPVs respond by *more* to shocks. If NPVs of (future) projects shoot up by more, firms react by posting more vacancies, translating into higher employment (see Figures 3.5 and 3.6). Therefore, the *less* procyclical wages are, the *more* aggregate employment and thus aggregate output respond to productivity shocks, thereby also increasing consumption risk.

3.5.4 Additional Properties of Macroeconomic Time Series

In this section we assess the Benchmark model along some additional macroeconomic dimensions. We start by evaluating the model's internal magnification and propagation

[29] The standard deviations (in percent) of the pricing kernel in the different models are: Basic: 4.15, Habit: 4.56, Adj. Cost: 4.93, Benchmark: 9.86 ($\kappa = 1.000$), 10.52 ($\kappa = 0.825$).

[30] For accounts of (and solutions for) the lack of movement in employment within the standard Mortensen and Pissarides (1994) labor market matching framework, see for example den Haan, Ramey, Watson (2000), Hall (2005), Hornstein, Krusell, Violante (2005), and Shimer (2005), amongst others.

Table 3.8: **Magnification and Propagation**

Impact Magnification is the immediate response of output to a one standard deviation shock to technology. Total Magnification is the ratio of the (unconditional) standard deviation of output to the (unconditional) standard deviation of technology shocks. All variables are logged and HP-filtered prior to analysis. Results for all models are based on 200 replications of sample size 200 each.

		Impact Magnification	Total Magnification
Standard		0.6081	0.6388
Basic		0.6113	0.6436
Benchmark	$\kappa = 1.000$	0.6615	0.6918
Benchmark	$\kappa = 0.825$	0.7159	0.7641

mechanisms. Then we examine whether the Benchmark model can account for the excess sensitivity of consumption to income, a feature of macroeconomic time series Boldrin, Christiano, Fisher (2001) document their model to match.

Magnification and Propagation

Cochrane (1994) argues that none of the popular candidates for external shocks (fiscal policy, technology, or oil price shocks) can account for the bulk of economic fluctuations. The goal has therefore become to develop macroeconomic models with internal magnification and propagation mechanisms that can generate sufficiently volatile and persistent output series given that we feed them with realistic external shocks.

We now assess the degree to which technology shocks are magnified and propagated in the Benchmark model. Following den Haan, Ramey, Watson (2000), we distinguish between impact magnification and total magnification. We refer to impact magnification as the degree to which the impact of a technology shock on output is magnified upon impact, that is within the same period as the shock. We refer to total magnification as the overall effect of technology shocks on output, that is the magnitude of the response of output to technology shocks including the dynamic response. Propagation, that is the persistence technology shocks generate in macroeconomic time series, is measured as the difference between total magnification and impact magnification. Impact magnification is computed by standardizing the response of output to a one standard deviation technology shock by that one standard deviation. Total magnification is measured as the ratio of the (unconditional) standard deviation of output as generated by the model to the (unconditional) standard deviation of technology shocks.

We conclude from Table 3.8 that the Benchmark model constitutes a substantial improvement in terms of magnification compared to both the standard production economy model as well as the Basic model. The reason is that employment is both procyclical and sufficiently volatile in the Benchmark model, thus considerably magnifying the impact of technology shocks on output. The Benchmark model displays some propagation due to some persistence in the response of employment to technology shocks (see Figure 3.6).[31]

Excess Sensitivity of Consumption to Income

Applying instrumental variable techniques, Campbell and Mankiw (1989, 1991) estimate the following model:

$$\Delta c_t = \beta_1 + \beta_2 \Delta y_t + \beta_3 r^F_{t-1} + \varepsilon_t, \tag{3.53}$$

where Δc_t and Δy_t are the first differences of log-consumption and log-income respectively. Campbell and Mankiw (1989, 1991) replace all variables by their fitted values from a regression on a set of instruments. Then they estimate the model via an OLS regression. Campbell and Mankiw argue that standard dynamic stochastic equilibrium models in consumption theory predict the coefficient on income growth (β_2) to be zero and the coefficient on the interest rate (β_3) to be substantially above zero (equal to the elasticity of intertemporal substitution). However, Campbell and Mankiw find that in the data, β_2 is substantially larger than zero, and β_3 is basically equal to zero. They conclude that this empirical evidence constitutes a refutation of the standard representative agent optimizing framework.[32] Boldrin, Christiano, Fisher (2001) take up this challenge. They manage to demonstrate that the two-sector model they propose is able to move the standard production economy model much closer to the data: running the Campbell and Mankiw regressions over data generated by their model, they find that, similar to empirical data, the coefficient on income growth is large and positive, while the coefficient on the interest rate is close to zero.

In Table 3.9 we report the original Campbell and Mankiw (1989) empirical estimates, as well as estimates from data generated by three different models: the standard production economy model Boldrin, Christiano, Fisher (2001) report in their paper (henceforth BCF Stand), the two-sector model they propose (henceforth BCF Model), and the Bench-

[31] Den Haan, Ramey, Watson (2000) incorporate a matching market for labor into a production economy model and show that the model's internal propagation mechanisms strongly improve as soon as the decision to break up a worker-firm relationship is endogenous. In the Benchmark model this break up occurs exogenously. It could thus be interesting to endogenize the decision to break up. Maybe this would also help to improve the persistence of the price-dividend ratio and the risk-free rate (see Section 3.5.2).

[32] They instead propose an alternative model in which a fraction of households behaves as postulated by the standard dynamic stochastic framework with power utility, while another fraction of households behaves according to a 'rule of thumb' and simply consumes current income.

Table 3.9: **Excess Sensitivity of Consumption to Income**
$\Delta x_t = \log(X_t) - \log(X_{t-1})$. The model is estimated via OLS after replacing all variables with their fitted values from a regression on the following set of instruments: $\{\Delta c_{t-2}, \Delta c_{t-3}, \Delta c_{t-4}, r^F_{t-3}, r^F_{t-4}, r^F_{t-5}\}$. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1953 to 1986. We take those values from Campbell and Mankiw (1989). Results for the Benchmark model are based on 200 replications of sample size 200 each.

$$\Delta c_t = \beta_1 + \beta_2 \Delta y_t + \beta_3 r^F_{t-1} + \varepsilon_t$$

	BCF Stand	BCF Model	Benchmark		Data
			$\kappa = 1.000$	$\kappa = 0.825$	
β_2	0.52	1.01	0.40	0.42	0.47
					(0.15)
β_3	0.92	0.05	-0.14	-0.15	0.09
					(0.11)

mark model.[33,34]

We conclude that the Benchmark model constitutes a move much closer to the data: the coefficient on income growth matches its empirical counterpart, while the coefficient on the interest rate is close to zero, albeit negative.[35] The rationale behind the remarkable performance of both the BCF model as well as the Benchmark model in terms of their ability to generate a very low coefficient on the interest rate is that households in both models are assumed to have habit preferences: for high habit levels the elasticity of intertemporal substitution is very low. So, intuitively, households' savings and thus consumption decisions are very insensitive to changes in real interest rates.

3.6 Comparison to Jermann (1998)

In this section we compare results from the Benchmark model with two versions of the Jermann (1998) model: to a version with fixed labor supply (as in Jermann (1998), henceforth Jermann model), and to a version of the model where leisure enters the households' utility function, that is the labor-leisure choice is endogenous (henceforth Labor-Leisure model).

[33]Following Campbell and Mankiw (1989) and Boldrin, Christiano, Fisher (2001), we regress on the following list of instruments: Δc_{t-2}, Δc_{t-3}, Δc_{t-4}, r^F_{t-3}, r^F_{t-4}, r^F_{t-5}. This instrument list is representative for instruments typically used in this literature.

[34]The BCF Stand model is parameterized as follows: $\alpha = 0.36$, $\beta = 0.99999$, $\delta = 0.021$, $\bar{\varepsilon} = 0.0040$, $\sigma_\varepsilon = 0.018$. We have managed to replicated the exact values Boldrin, Christiano, Fisher (2001) report in their paper.

[35]Note that a negative value for the coefficient on the interest rate is within one standard deviation of the point estimate from the data. In fact, Campbell and Mankiw (1991) find the coefficient on the interest rate to be slightly negative for a row of countries apart from the U.S.

3.6.1 The Models

The Original Jermann (1998) Model

This model is exactly the same as the original Jermann (1998) model.[36] The representative household maximizes:

$$\max_{\{C_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{(C_t - \eta C_{t-1})^{1-\gamma}}{1-\gamma} \right], \tag{3.54}$$

subject to the capital accumulation equation:

$$K_{t+1} \leq \phi\left(I_t/K_t\right) K_t + (1-\delta) K_t, \tag{3.55}$$

where $I_t = Y_t - C_t$, with the production technology:

$$Y_t = \theta_t^{1-\alpha} K_t^{\alpha}. \tag{3.56}$$

The Euler equation follows as:

$$\begin{aligned}
& (C_t - \eta C_{t-1})^{-\gamma} \\
= \quad & \beta E_t \left[\begin{array}{c} \left(\begin{array}{c} (C_{t+1} - \eta C_t)^{-\gamma} \\ -\beta \eta E_{t+1}\left[(C_{t+2} - \eta C_{t+1})^{-\gamma} \right] \end{array} \right) \\ \times \frac{\phi'\left(\frac{I_t}{K_t}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \\ \times \left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times \left[(\alpha - 1) K_{t+1}^{\alpha-1} \theta_{t+1}^{1-\alpha} + C_{t+1}/K_{t+1} \right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array} \right) \\ +\eta\left(C_{t+1} - \eta C_t\right)^{-\gamma} \end{array} \right].
\end{aligned} \tag{3.57}$$

For a detailed derivation of the Euler equation please refer to Appendix A.

The Jermann (1998) Model with Flexible Labor-Leisure Decision

The representative household maximizes:

$$\max_{\{C_t, N_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t, N_t) \right], \tag{3.58}$$

[36] Jermann (1998) uses the following process for productivity: $\ln \theta_t = \rho \ln \theta_{t-1} + \varepsilon_t$. While we set $\rho = 1$, Jermann (1998) sets $\rho = 0.99$.

where:

$$u(C_t, N_t) = \frac{\left[(C_t - \eta C_{t-1})(1 - N_t)^{\tau}\right]^{1-\gamma}}{1-\gamma}, \tag{3.59}$$

subject to:

$$K_{t+1} \leq \phi\left(I_t/K_t\right) K_t + (1-\delta) K_t, \tag{3.60}$$

where $I_t = Y_t - C_t$, with the production technology:

$$Y_t = K_t^{\alpha}\left(\theta_t N_t\right)^{1-\alpha}. \tag{3.61}$$

The Euler equations for consumption and labor are:

$$\begin{aligned} & (C_t - \eta C_{t-1})^{-\gamma}\left[(1-N_t)^{\tau}\right]^{1-\gamma} \\ = \quad & \beta E_t \left[\begin{array}{c} \left(\begin{array}{c} (C_{t+1} - \eta C_t)^{-\gamma} \\ \times \left[(1-N_{t+1})^{\tau}\right]^{1-\gamma} \\ -\beta\eta E_{t+1} \left[\begin{array}{c} (C_{t+2} - \eta C_{t+1})^{-\gamma} \\ \times \left[(1-N_{t+2})^{\tau}\right]^{1-\gamma} \end{array} \right] \end{array} \right) \\ \times \frac{\phi'\left(\frac{I_t}{K_t}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \\ \times \left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times \left[\begin{array}{c} (\alpha-1) K_{t+1}^{\alpha-1}\left(\theta_{t+1} N_{t+1}\right)^{1-\alpha} \\ + C_{t+1}/K_{t+1} \end{array} \right] \\ + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array} \right) \\ + \eta\left(C_{t+1} - \eta C_t\right)^{-\gamma}\left[(1-N_{t+1})^{\tau}\right]^{1-\gamma} \end{array} \right], \end{aligned} \tag{3.62}$$

and:

$$\begin{aligned} & (1-\alpha) K_t^{\alpha} \theta_t^{1-\alpha} N_t^{-\alpha} \\ = \quad & \frac{\left[\begin{array}{c} \left[(C_t - \eta C_{t-1})(1-N_t)^{\tau}\right]^{-\gamma} \\ \times (C_t - \eta C_{t-1})\, \tau\, (1-N_t)^{\tau-1} \end{array} \right]}{\left[\begin{array}{c} (C_t - \eta C_{t-1})^{-\gamma}\left[(1-N_t)^{\tau}\right]^{1-\gamma} \\ -\beta\eta E_t\left[(C_{t+1} - \eta C_t)^{-\gamma}\left[(1-N_{t+1})^{\tau}\right]^{1-\gamma}\right] \end{array} \right]}. \end{aligned} \tag{3.63}$$

For a detailed derivation of these equations please refer to Appendix A.

Table 3.10: **Comparison Across Models — Key Asset Pricing Moments**
σ_x denotes the standard deviation of variable x. Rates of return are annualized and in percent. For the Benchmark model the Aggregate equity return is reported (for a discussion see Section 3.3.6). The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1892 to 1987. We take those values from Boldrin, Christiano, Fisher (2001), who in turn rely on Cecchetti, Lam, Mark (1993). Results for all models are based on 200 replications of sample size 200 each.

Statistic	Jermann	Labor-Leisure	Benchmark		Data
			$\kappa = 1.000$	$\kappa = 0.825$	
$\mathrm{E}\left[r_t^F\right]$	2.50	3.00	2.07	1.81	1.19
					(0.81)
σ_{r^F}	8.10	2.18	8.47	8.62	5.27
					(0.74)
$\mathrm{E}\left[r_t^E - r_t^F\right]$	4.37	1.32	4.71	6.13	6.63
					(1.78)
σ_{r^E}	27.65	12.80	28.69	30.61	19.40
					(1.56)
$\frac{E\left[r_t^E - r_t^F\right]}{\sigma_{r^E}}$	0.16	0.10	0.16	0.20	0.34
					(0.09)

3.6.2 Calibration

For both the Jermann model as well as the Labor-Leisure model we use exactly the same parameter values as for the Benchmark model (see Section 3.4), where applicable.[37]

3.6.3 Results of the Comparison

Table 3.10 compares key asset pricing moments across the Benchmark model, the Jermann model, and the Labor-Leisure model. From Table 3.10 it becomes clear that the Jermann (1998) framework breaks down as soon as leisure enters the households' utility function: households use their labor-leisure choice as a smoothing device for their consumption profile. Whenever there is a positive (negative) shock to productivity, and output increases (decreases), leading to a higher (lower) consumption level, households react by working less (more), thus dampening the effect of the technology shock on output and consumption. Here the extent to which households manage to rid themselves of consumption risk they might otherwise have faced becomes clear: the equity premium displays a steep reduction from 4.37% down to 1.32%.

We conclude that the basic asset pricing moments are similar across the Benchmark model and the Jermann model with fixed labor supply. In particular, the volatilities

[37]For the Labor-Leisure model we set $\tau = 2.18$, in order to fix the steady state value of N at 0.30.

of the equity return and the risk-free rate turn out too high in both models. In the Benchmark model, in particular with $\kappa = 0.825$, equity premium and Sharpe ratio are higher compared to the Jermann model, reflecting higher consumption risk households face due to procyclical and relatively volatile aggregate employment. Wages respond less to productivity shocks if κ is lower, inducing less countercyclical project profits and a higher premium for holding the Capital Renting Corporation and therefore a higher Aggregate equity premium and Sharpe ratio (see Section 3.5.1).

The dynamics of asset prices (not reported) in the Jermann model also turn out quite similar compared to the Benchmark model: the Jermann model does slightly worse in terms of the cyclical variation of the price-dividend ratio and the equity premium, while the correlation of the conditional moments is, as in the data and as in the Benchmark model, negative.[38] Both the Jermann model's performance in the long-horizon regressions of the equity return on the price-dividend ratio as well as the autocorrelation structure of returns is basically the same as the Benchmark model's. We conclude that the dynamic behavior of asset prices in both models is largely driven by the high internal habit levels.

Table 3.11 compares key moments of macroeconomic time series across the three different models. Again, we observe the breakdown of the Jermann (1998) framework as soon as the labor-leisure choice is endogenous. As a consequence of the households' consumption smoothing activities, output and aggregate employment display a counterfactual *negative* correlation. We get a sense of how extensive households use this channel when we focus on the steep reduction of output volatility in the Labor-Leisure model relative to the Jermann model (output volatility drops from 1.58% to 1.00%), and on the extreme volatility of employment.

When we compare the non-labor market macroeconomic moments between the Benchmark model and the Jermann model, we conclude that they are quite similar. When we furthermore compare the additional properties of the macroeconomic time series from Section 3.5.4 across the two models (not reported), the result is the same: while the Benchmark model displays, due to the labor market, a much better performance in terms of magnification of technology shocks compared to the Jermann model, the results for the instrumental variable regressions of consumption growth on income growth and the real interest rate (excess sensitivity puzzle) are very similar.

[38] Jermann model: $corr(p_t - d_t, y_t) = 0.10$, $corr(r_t^E - r_t^F, y_t) = 0.09$. See Section 3.5.2, Table 3.4 for the respective values for the Benchmark model as well as for U.S. data.

Table 3.11: **Comparison Across Models — Key Macroeconomic Moments**

σ_x denotes the standard deviation of variable x. $\rho_{x,y}$ denotes the correlation between variables x and y. All variables are logged and HP-filtered prior to analysis. The statistic σ_Y is reported in percent. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1964 to 1987. We take those values from Boldrin, Christiano, Fisher (2001). Results for all models are based on 200 replications of sample size 200 each.

Statistic	Jermann	Labor-Leisure	Benchmark		Data
			$\kappa = 1.000$	$\kappa = 0.825$	
σ_Y	1.58	1.00	1.67	1.85	1.89
					(0.21)
σ_C/σ_Y	0.80	0.90	0.78	0.77	0.40
					(0.04)
σ_I/σ_Y	2.10	1.81	2.13	2.11	2.39
					(0.06)
σ_N/σ_Y	na	1.11	0.10	0.23	0.80
					(0.05)
$\rho_{Y,C}$	0.82	0.87	0.81	0.81	0.76
					(0.05)
$\rho_{Y,I}$	0.86	0.82	0.87	0.88	0.96
					(0.01)
$\rho_{Y,N}$	na	-0.66	0.82	0.96	0.78
					(0.05)

3.7 Conclusion

A substantial part of the finance literature has successfully focused on the development of mechanisms with the purpose of empowering the standard representative agent consumption-based asset pricing model so that the model can explain key moments and dynamics of asset prices. Seminal examples of such mechanisms are limited stock market participation or preferences with a slow-moving external habit level (Campbell and Cochrane (1999)). This literature has almost always imposed exogenous processes for consumption and labor. However, as soon as consumption is endogenous or leisure enters the households' utility function, that is the labor-leisure choice is endogenous, those mechanisms cease to function, and the resulting models cannot explain asset prices and have counterfactual implications for aggregate consumption and aggregate employment. The challenge has therefore become to develop models that can explain the behavior of asset prices and key macroeconomic series at the same time. We propose a model that does just that. We combine habit preferences with capital adjustment costs and a search-theoretical model of the labor market. We show that our model can explain key asset pricing moments and several important dimensions of asset pricing dynamics as well as key moments of macroeconomic time series.

The framework we develop has many different possible implementations, depending on the particular incarnation of the search and matching model incorporated. The framework thus allows to carefully model fluctuations in aggregate employment and wages. This is potentially very important for asset prices too, because dividends are the residual of output, investment, and *wages*. We show in Chapter 5 the importance of carefully modeling wages and thus dividends, opening up interesting avenues for future research based on the framework developed in this chapter.

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3.9 Appendix A — Derivation of Euler Equations

3.9.1 Benchmark Model

The Lagrangian is given by:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \frac{(C_t - \eta C_{t-1})^{1-\gamma}}{1-\gamma} \\ +\lambda_t \left[\begin{array}{c} \phi\left(\frac{N_t w_t + r_t K_t + \Pi_t - C_t - \psi_t V_t}{K_t}\right) K_t \\ +(1-\delta)K_t - K_{t+1} \end{array} \right] \end{array} \right]. \tag{3.64}$$

First-order conditions follow as:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= 0 : (C_t - \eta C_{t-1})^{-\gamma} - \beta\eta E_t\left[(C_{t+1} - \eta C_t)^{-\gamma}\right] - \lambda_t \phi'\left(\frac{I_t}{K_t}\right) \\ &= 0, \end{aligned} \tag{3.65}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= 0 : -\lambda_t + \beta E_t\left[\lambda_{t+1}\left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\left[\frac{C_{t+1} + \psi_{t+1}V_{t+1} - N_{t+1}w_{t+1} - \Pi_{t+1}}{K_{t+1}}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array}\right)\right] \\ &= -\lambda_t + \beta E_t\left[\lambda_{t+1}\left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\left[\frac{r_{t+1}K_{t+1} - I_{t+1}}{K_{t+1}}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array}\right)\right] \\ &= -\lambda_t + \beta E_t\left[\lambda_{t+1}\left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\left[r_{t+1} - \frac{I_{t+1}}{K_{t+1}}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array}\right)\right] = 0, \end{aligned} \tag{3.66}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \geq 0 : \phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta)K_t - K_{t+1} \geq 0. \tag{3.67}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t}\lambda_t = 0. \tag{3.68}$$

Re-arranging (3.65) and (3.66) gives the Euler equation:

$$(C_t - \eta C_{t-1})^{-\gamma} = \beta E_t \left[\begin{array}{c} \left(\begin{array}{c} (C_{t+1} - \eta C_t)^{-\gamma} \\ -\beta\eta E_{t+1}\left[(C_{t+2} - \eta C_{t+1})^{-\gamma}\right] \end{array}\right) \\ \times\left(\phi'\left(\frac{I_t}{K_t}\right) / \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\right) \\ \times\left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times\left[r_{t+1} - \frac{I_{t+1}}{K_{t+1}}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array}\right) \\ +\eta\left(C_{t+1} - \eta C_t\right)^{-\gamma} \end{array} \right]. \tag{3.69}$$

3.9.2 Jermann Model

The Lagrangian is given by:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \frac{(C_t - \eta C_{t-1})^{1-\gamma}}{1-\gamma} \\ +\lambda_t \left[\phi\left(\frac{K_t^{\alpha}\theta_t^{1-\alpha} - C_t}{K_t}\right) K_t + (1-\delta)K_t - K_{t+1} \right] \end{array} \right]. \tag{3.70}$$

First-order conditions follow as:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= 0 : (C_t - \eta C_{t-1})^{-\gamma} - \beta \eta E_t \left[(C_{t+1} - \eta C_t)^{-\gamma} \right] - \lambda_t \phi' \left(\frac{I_t}{K_t} \right) \\ &= 0, \end{aligned} \tag{3.71}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= 0 : -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \left[\begin{array}{c} (\alpha - 1) K_{t+1}^{\alpha-1} \theta_{t+1}^{1-\alpha} \\ +C_{t+1}/K_{t+1} \end{array} \right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array} \right) \right] \\ &= 0, \end{aligned} \tag{3.72}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \geq 0 : \phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta)K_t - K_{t+1} \geq 0. \tag{3.73}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t}\lambda_t = 0. \tag{3.74}$$

Re-arranging (3.71) and (3.72) gives the Euler equation:

$$\begin{aligned} & (C_t - \eta C_{t-1})^{-\gamma} \\ = \ & \beta E_t \left[\begin{array}{c} \left(\begin{array}{c} (C_{t+1} - \eta C_t)^{-\gamma} \\ -\beta \eta E_{t+1} \left[(C_{t+2} - \eta C_{t+1})^{-\gamma} \right] \end{array} \right) \\ \times \frac{\phi'\left(\frac{I_t}{K_t}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \\ \times \left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times \left[(\alpha-1) K_{t+1}^{\alpha-1}\theta_{t+1}^{1-\alpha} + C_{t+1}/K_{t+1} \right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array} \right) \\ +\eta \left(C_{t+1} - \eta C_t \right)^{-\gamma} \end{array} \right]. \end{aligned} \tag{3.75}$$

3.9.3 Labor-Leisure Model

The Lagrangian is given by:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \frac{\left[(C_t - \eta C_{t-1})(1-N_t)^{\tau}\right]^{1-\gamma}}{1-\gamma} \\ +\lambda_t \left[\phi\left(\frac{K_t^{\alpha}(\theta_t N_t)^{1-\alpha} - C_t}{K_t}\right) K_t + (1-\delta)K_t - K_{t+1} \right] \end{array} \right]. \tag{3.76}$$

First-order conditions follow as:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial C_t} &= 0 : (C_t - \eta C_{t-1})^{-\gamma} \left[(1-N_t)^{\tau}\right]^{1-\gamma} \\ &\quad -\beta\eta E_t \left[(C_{t+1} - \eta C_t)^{-\gamma} \left[(1-N_{t+1})^{\tau}\right]^{1-\gamma}\right] - \lambda_t \phi' \left(\frac{I_t}{K_t}\right) \\ &= 0, \end{aligned} \tag{3.77}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial N_t} &= 0 : -\left[(C_t - \eta C_{t-1})(1-N_t)^{\kappa}\right]^{-\gamma} (C_t - \eta C_{t-1})\, \kappa\, (1-N_t)^{\kappa-1} \\ &\quad +\lambda_t \phi' \left(\frac{I_t}{K_t}\right) K_t \times (1-\alpha)\, K_t^{\alpha-1} \theta_t^{1-\alpha} N_t^{-\alpha} \\ &= -\left[(C_t - \eta C_{t-1})(1-N_t)^{\kappa}\right]^{-\gamma} \times (C_t - \eta C_{t-1})\, \kappa\, (1-N_t)^{\kappa-1} \\ &\quad +\lambda_t \phi' \left(\frac{I_t}{K_t}\right) (1-\alpha)\, K_t^{\alpha} \theta_t^{1-\alpha} N_t^{-\alpha} \\ &= 0, \end{aligned} \tag{3.78}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= 0 : -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times \left[\begin{array}{c} (\alpha-1)\, K_{t+1}^{\alpha-1} (\theta_{t+1} N_{t+1})^{1-\alpha} \\ +C_{t+1}/K_{t+1} \end{array}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array}\right)\right] \\ &= 0, \end{aligned} \tag{3.79}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \geq 0 : \phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta) K_t - K_{t+1} \geq 0. \tag{3.80}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \lambda_t = 0. \tag{3.81}$$

Re-arranging (3.77), (3.78) and (3.79) gives the Euler equations:

$$\begin{aligned} & \left(C_t - \eta C_{t-1}\right)^{-\gamma}\left[(1-N_t)^{\tau}\right]^{1-\gamma} \\ = \; & \beta E_t \left[\begin{array}{c} \left(\begin{array}{c} \left(C_{t+1} - \eta C_t\right)^{-\gamma}\left[(1-N_{t+1})^{\tau}\right]^{1-\gamma} \\ -\beta\eta E_{t+1}\left[\begin{array}{c} \left(C_{t+2} - \eta C_{t+1}\right)^{-\gamma} \\ \times\left[(1-N_{t+2})^{\tau}\right]^{1-\gamma} \end{array} \right] \end{array} \right) \\ \times \frac{\phi'\left(\frac{I_t}{K_t}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \\ \times \left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times \left[\begin{array}{c} (\alpha-1) K_{t+1}^{\alpha-1}\left(\theta_{t+1}N_{t+1}\right)^{1-\alpha} \\ + C_{t+1}/K_{t+1} \end{array} \right] \\ + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array} \right) \\ + \eta\left(C_{t+1} - \eta C_t\right)^{-\gamma}\left[(1-N_{t+1})^{\tau}\right]^{1-\gamma} \end{array} \right], \end{aligned} \tag{3.82}$$

and:

$$\begin{aligned} & (1-\alpha) K_t^{\alpha}\theta_t^{1-\alpha}N_t^{-\alpha} \\ = \; & \frac{\left[\begin{array}{c} \left[\left(C_t - \eta C_{t-1}\right)(1-N_t)^{\tau}\right]^{-\gamma} \\ \times \left(C_t - \eta C_{t-1}\right)\tau(1-N_t)^{\tau-1} \end{array} \right]}{\left[\begin{array}{c} \left(C_t - \eta C_{t-1}\right)^{-\gamma}\left[(1-N_t)^{\tau}\right]^{1-\gamma} \\ -\beta\eta E_t\left[\left(C_{t+1} - \eta C_t\right)^{-\gamma}\left[(1-N_{t+1})^{\tau}\right]^{1-\gamma}\right] \end{array} \right]}. \end{aligned} \tag{3.83}$$

3.10 Appendix B — Proof: Return of the Capital Leasing Corporation

The return of the Capital Leasing Corporation is defined as follows:

$$r_{t+1}^{lease} = \frac{D_{t+1}^{lease} + V_{t+1}^{lease}}{V_t^{lease}} - 1, \tag{3.84}$$

where:

$$D_t^{lease} = r_t K_t - I_t, \tag{3.85}$$

and:

$$V_t^{lease} = E_t\left[\sum_{j=1}^{\infty}\beta^j \frac{u'_{t+j}}{u'_t} D_{t+j}^{lease}\right]. \tag{3.86}$$

We show that $V_{t+j}^{lease} = \frac{K_{t+j+1}}{\phi'\left(\frac{I_{t+j}}{K_{t+j}}\right)}$, and that therefore the return of the Capital Leasing

Corporation can be written as:

$$r_{t+1}^{lease} = \frac{r_{t+1} + \frac{\partial K_{t+2}}{\partial K_{t+1}} P_{K,t+1}}{P_{K,t}} - 1, \tag{3.87}$$

where:

$$\frac{\partial K_{t+2}}{\partial K_{t+1}} = \left[(1-\delta) + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}} \right], \tag{3.88}$$

$$P_{K,t} = \frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}. \tag{3.89}$$

$P_{K,t}$ denotes the price at time t of one additional unit of capital in terms of the consumption good. One marginal unit of capital increases income by r_{t+1}, measured in consumption goods, and next period's capital stock by $\frac{\partial K_{t+2}}{\partial K_{t+1}}$, measured in capital goods. From the Euler equation given by:

$$u'_{t+j} = E_{t+j}\left[\beta u'_{t+j+1} \times \left(\frac{\phi'\left(\frac{I_{t+j}}{K_{t+j}}\right)}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)} \times \left(\begin{array}{c} \phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right) \\ \times \left[r_{t+j+1} - \frac{I_{t+j+1}}{K_{t+j+1}} \right] \\ +\phi\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right) + (1-\delta) \end{array} \right) \right)\right], \tag{3.90}$$

we obtain:

$$\begin{aligned} \frac{K_{t+j+1}}{\phi'\left(\frac{I_{t+j}}{K_{t+j}}\right)} &= E_{t+j}\left[\beta \frac{u'_{t+j+1}}{u'_{t+j}} \left(\begin{array}{c} [r_{t+j+1}K_{t+j+1} - I_{t+j+1}] \\ + \frac{\left(\phi\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)K_{t+j+1} + (1-\delta)K_{t+j+1}\right)}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)} \end{array} \right)\right] \\ &= E_{t+j}\left[\beta \frac{u'_{t+j+1}}{u'_{t+j}} \times \left(\begin{array}{c} [r_{t+j+1}K_{t+j+1} - I_{t+j+1}] \\ + \frac{K_{t+j+2}}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)} \end{array} \right)\right], \end{aligned} \tag{3.91}$$

and finally:

$$\begin{aligned} & E_{t+j}\left[\beta \frac{u'_{t+j+1}}{u'_{t+j}} [r_{t+j+1}K_{t+j+1} - I_{t+j+1}]\right] \\ = & \frac{K_{t+j+1}}{\phi'\left(\frac{I_{t+j}}{K_{t+j}}\right)} - E_{t+j}\left[\beta \frac{u'_{t+j+1}}{u'_{t+j}} \frac{K_{t+j+2}}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)}\right]. \end{aligned} \tag{3.92}$$

We use this result to simplify the expression for V_t^{lease} from the definition of the return of the Capital Leasing Corporation:

$$\begin{aligned}
V_t^{lease} &= E_t\left[\sum_{j=1}^{\infty}\beta^j \frac{u'_{t+j}}{u'_t} D_{t+j}^{lease}\right] \\
&= E_t\left[\beta\frac{u'_{t+1}}{u'_t}D_{t+1}^{lease} + \beta^2\frac{u'_{t+2}}{u'_t}D_{t+2}^{lease} + \ldots\right] \\
&= E_t\left[\begin{array}{c}\beta\frac{u'_{t+1}}{u'_t}\left[r_{t+1}K_{t+1} - I_{t+1}\right] \\ +\beta\frac{u'_{t+1}}{u'_t}\left(\beta\frac{u'_{t+2}}{u'_{t+1}}\left[r_{t+2}K_{t+2} - I_{t+2}\right]\right) + \ldots\end{array}\right] \\
&= E_t\left[\begin{array}{c}\frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)} - \beta\frac{u'_{t+1}}{u'_t}\frac{K_{t+2}}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \\ +\beta\frac{u'_{t+1}}{u'_t}\left(\frac{K_{t+2}}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} - \beta\frac{u'_{t+2}}{u'_{t+1}}\frac{K_{t+3}}{\phi'\left(\frac{I_{t+2}}{K_{t+2}}\right)}\right) + \ldots\end{array}\right] \\
&= E_t\left[\frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)} - \beta^2\frac{u'_{t+2}}{u'_t}\frac{K_{t+3}}{\phi'\left(\frac{I_{t+2}}{K_{t+2}}\right)} + \ldots\right] \\
&= \frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)}.
\end{aligned} \tag{3.93}$$

Using this expression for V_t^{lease}, we rearrange the definition of the return:

$$\begin{aligned}
r_{t+1}^{lease} &= \frac{D_{t+1}^{lease} + V_{t+1}^{lease}}{V_t^{lease}} - 1 = \frac{(r_{t+1}K_{t+1} - I_{t+1}) + V_{t+1}^{lease}}{V_t^{lease}} - 1 \\
&= \frac{(r_{t+1}K_{t+1} - I_{t+1}) + \frac{K_{t+2}}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)}}{\frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)}} - 1 \\
&= \frac{r_{t+1} - \frac{I_{t+1}}{K_{t+1}} + \frac{1}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)}\frac{K_{t+2}}{K_{t+1}}}{\frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}} - 1 \\
&= \frac{r_{t+1} + \left[\begin{array}{c}(1-\delta) + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ -\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\end{array}\right]\frac{1}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)}}{\frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}} - 1 \\
&= \frac{r_{t+1} + \frac{\partial K_{t+2}}{\partial K_{t+1}}P_{K,t+1}}{P_{K,t}} - 1.
\end{aligned} \tag{3.94}$$

This concludes the proof.

3.11 Appendix C — Model Solution

3.11.1 Normalization

Since the process for productivity is not stationary, we need to normalize the economy by θ_t, in order to be able to numerically solve the model. To be precise, we let $\widehat{K}_t = \frac{K_t}{\theta_t}, \widehat{C}_t = \frac{C_t}{\theta_t}, \widehat{I}_t = \frac{I_t}{\theta_t}, \widehat{F}_t = \frac{F_t}{\theta_t}, \widehat{\Pi}_t = \frac{\Pi_t}{\theta_t}, \widehat{k}_t = \frac{k_t}{\theta_t}, \widehat{\pi}_t = \frac{\pi_t}{\theta_t}$, and substitute. We need not normalize the following variables, as they are already stationary in the non-stationary economy: $N_t, U_t, V_t, \lambda_t^F, r_t$.[39] In the so transformed model all variables are stationary. We can work directly on the same (appropriately normalized) set of equations as spelt out in Section 3.3. The state variables are now $\widehat{K}$, $\widehat{C}_{-1}$, N_{-1}.[40] We show the normalization of four key equations. The Euler equation becomes:

$$\left(\widehat{C}_t - \eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-\gamma} = \beta E_t \left[\begin{array}{c} \left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \\ \times \left(\begin{array}{c} \left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \\ -\beta E_{t+1} \left[\begin{array}{c} \eta \left(e^{\varepsilon_{t+2}}\right)^{-\gamma} \\ \times \left(\widehat{C}_{t+2} - \eta e^{-\varepsilon_{t+2}}\widehat{C}_{t+1}\right)^{-\gamma} \end{array} \right] \end{array} \right) \\ \times \left(\phi'\left(\frac{\widehat{I}_t}{\widehat{K}_t}\right) / \phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right)\right) \\ \times \left(\begin{array}{c} \phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right)\left[r_{t+1} - \frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right] \\ +\phi\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right) + (1-\delta) \end{array} \right) \\ + \left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \eta \left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \end{array} \right], \tag{3.95}$$

where ε_t is the i.i.d. shock to the growth rate of productivity, as laid out in Section 3.3.2. The capital accumulation equation becomes:

$$\widehat{K}_{t+1}e^{\varepsilon_{t+1}} = \phi\left(\widehat{I}_t/\widehat{K}_t\right)\widehat{K}_t + (1-\delta)\widehat{K}_t. \tag{3.96}$$

[39] Note that N_t, U_t, V_t are stationary if λ_t^F is stationary. Because $\lambda_t^F = \frac{\psi_t}{F_t}$, and ψ_t and F_t grow at the same rate, λ_t^F is stationary.

[40] Note that θ is *not* a state variable of the normalized model. This is due to the fact that we assume the autoregressive coefficient of the process for productivity $\ln\theta_{t+1} = \rho\ln\theta_t + \varepsilon_{t+1}$ to be unity: $\rho = 1$. As a consequence, $\Delta\theta$ is serially uncorrelated.

The NPV of all future profits becomes:

$$\widehat{F}_t = \left(\widehat{k}_t^{\alpha} - \overline{w}\widehat{Y}_t^{\kappa}\widehat{K}_t^{1-\kappa} - r_t\widehat{k}_t\right) + E_t\left[\beta\frac{\left(\begin{array}{c}\left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \\ -\beta E_{t+1}\left[\begin{array}{c}\eta\left(e^{\varepsilon_{t+2}}\right)^{-\gamma} \\ \times\left(\widehat{C}_{t+2} - \eta e^{-\varepsilon_{t+2}}\widehat{C}_{t+1}\right)^{-\gamma}\end{array}\right]\end{array}\right)}{\left(\begin{array}{c}\left(\widehat{C}_t - \eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-\gamma} \\ -\beta E_t\left[\begin{array}{c}\eta\left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \\ \times\left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma}\end{array}\right]\end{array}\right)} \times\left(e^{\varepsilon_{t+1}}\right)^{1-\gamma}(1-\rho^x)\widehat{F}_{t+1}\right], \tag{3.97}$$

and finally the free-entry condition:

$$\overline{\psi}\widehat{K}_t = \lambda_t^F \times \widehat{F}_t. \tag{3.98}$$

3.11.2 Numerical Solution

Relying on den Haan and Marcet (1990), we use the Parameterized Expectations Algorithm (PEA) to solve all models. For the solution of the benchmark model we use $7th$ order Chebyshev orthogonal polynomials and a $8 \times 8 \times 8$ Chebyshev grid for the state variables $\widehat{K}$, $\widehat{C}_{-1}$, and N_{-1}. Gauss-Hermite quadrature with 5 nodes is used to approximate the expectations operators. To check for accuracy, we use a very fine grid ($100 \times 100 \times 100$), and at each grid point compare the polynomial approximation of the Euler equation with the 'true' expectation, computed using Gauss-Hermite quadrature with 10 nodes. The maximum absolute percentage difference between parameterized expectation and true expectation is smaller than 0.015% for all models considered in the chapter.[41]

[41] We express the maximum absolute percentage difference between the polynomial approximation of the Euler equation and the 'true' expectation in terms of the consumption good.

We solve the following system of equations:

$$\widehat{C} = \Psi^A \left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}, N_{-1}\right)^{-1/\gamma} + \eta e^{-\varepsilon}\widehat{C}_{-1}, \tag{3.99}$$

$$\widehat{F} = \Psi^B \left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}, N_{-1}\right), \tag{3.100}$$

$$N = (1-\rho^x)N_{-1} + \mu\left[1-(1-\rho^x)N_{-1}\right] \times \left(\frac{\mu\widehat{F}}{\overline{\psi}}\right)^{\frac{1}{\nu}-1}, \tag{3.101}$$

$$\widehat{Y} = N\left(\frac{\widehat{K}}{N}\right)^{\alpha}, \tag{3.102}$$

$$\widehat{K}_{+1}e^{\varepsilon_{+1}} = \phi\left(\frac{\widehat{Y}-\widehat{C}}{\widehat{K}}\right)\widehat{K} + (1-\delta)\widehat{K}, \tag{3.103}$$

where the functions Ψ^A and Ψ^B are the parameterized conditional expectations as spelt out in equation (3.95) and equation (3.97).

3.12 Appendix D — Accuracy of Results in the Literature for the Jermann Model

This appendix tries to replicate two sets of results that have been reported in the literature so far for the Jermann (1998) framework: the results Jermann (1994) reports himself in a more extensive working paper version of his 1998 JME paper, and the results Boldrin, Christiano, Fisher (1999) report in a more extensive working paper version of their 2001 AER paper.

Relying on den Haan and Marcet (1990), we apply the Parameterized Expectations Algorithm (PEA), a state-of-the art non-linear numerical solution technique. This allows us to evaluate the accuracy of the loglinear-lognormal solution techniques Jermann (1994) and Jermann (1998) apply. While we have conducted our evaluation mainly in order to judge whether the benefits of applying non-linear solution techniques outweigh the costs in the case of the Jermann framework, our results are also of some general interest, given that a substantial fraction of the finance and economics literature relies on linear solution techniques to approximate solutions to a variety of non-linear models. From our results we can get a sense of how economically significant the inevitable inaccuracies of solutions obtained with linear approximations *can* turn out.

We manage to replicate the results Jermann (1994) and Boldrin, Christiano, Fisher (1999) report, with some minor exceptions. We conclude that for the more interesting versions of the Jermann framework linear solution techniques generate quite inaccurate solutions that result in economically significant errors along several important dimensions. In the examples we consider, equity premiums, the volatility of returns, and the volatility of aggregate investment turn out substantially too low. We therefore recommend to apply modern non-linear solution techniques such as PEA in order to solve versions of the Jermann framework.

3.12.1 Boldrin, Christiano, Fisher (1999)

Boldrin, Christiano, Fisher (1999), a working paper version of the 2001 AER paper, solve a version of the Jermann (1998) model with a flexible labor-leisure decision, and use it as a benchmark for the two-sector specification they develop in their paper. The goal of this section is to replicate the results Boldrin, Christiano, Fisher report for their version of the Jermann model.

The Model

Aggregate consumption is chosen with the aim to maximize the expected utility of the representative household:

$$\max_{\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \left[\log\left(C_t - \eta C_{t-1}\right) - N_t \right] \right], \tag{3.104}$$

subject to the following accumulation equation of the aggregate capital stock with adjustment costs:

$$K_{t+1} \leq \phi\left(I_t/K_t\right) K_t + (1-\delta) K_t, \tag{3.105}$$

where $N_t \geq 0$ denotes labor chosen at time t, $I_t = Y_t - C_t$, and $\phi(\cdot)$ is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it step by step.[42] I follow Jermann (1998) and Boldrin, Christiano, Fisher (1999) and specify:

$$\phi\left(I_t/K_t\right) = \frac{\alpha_1}{1 - 1/\xi} \left(\frac{I_t}{K_t} \right)^{(1-1/\xi)} + \alpha_2, \tag{3.106}$$

[42] This formulation of the capital accumulation equation with adjustment costs and $\phi(\cdot) \geq 0$, $\phi''(\cdot) \leq 0$ has been used by Uzawa (1969), Lucas and Prescott (1971), Hayashi (1982), Baxter and Crucini (1993), amongst many others.

with:

$$\alpha_1 = \left(\exp(\overline{\varepsilon}) - 1 + \delta\right)^{1/\xi}, \tag{3.107}$$

$$\alpha_2 = \frac{1}{\xi - 1}\left(1 - \delta - \exp(\overline{\varepsilon})\right), \tag{3.108}$$

where $\overline{\varepsilon}$ is the trend growth rate of the economy.[43] The parameter ξ is the elasticity of the investment-capital ratio with respect to Tobin's q.[44] If $\xi = \infty$, (3.105) reduces to the capital accumulation equation of the standard growth model. The aggregate production technology is specified as:

$$Y_t = K_t^{\alpha}\left(\theta_t N_t\right)^{1-\alpha}. \tag{3.109}$$

The process for aggregate productivity (θ_t) is given by:

$$\ln\theta_t = \ln\theta_{t-1} + \varepsilon_t, \tag{3.110}$$

where:

$$\varepsilon_t \sim N\left(\overline{\varepsilon}, \sigma_{\varepsilon}^2\right). \tag{3.111}$$

It follows that aggregate productivity is non-stationary. Below, we normalize this economy so that in the transformed model all variables are stationary.

First, we set up the Lagrangian and derive first-order conditions:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \log\left(C_t - \eta C_{t-1}\right) - N_t \\ +\lambda_t\left[\phi\left(\frac{K_t^{\alpha}(\theta_t N_t)^{1-\alpha} - C_t}{K_t}\right) K_t + (1-\delta)K_t - K_{t+1}\right] \end{array}\right], \tag{3.112}$$

[43] It is straightforward to verify that $\phi(\frac{I_t}{K_t}) > 0$ and $\phi''(\frac{I_t}{K_t}) < 0$ for $\xi > 0$ and $\frac{I_t}{K_t} > 0$. Furthermore, $\phi(\frac{I}{K}) = \frac{I}{K}$ and $\phi'(\frac{I}{K}) = 1$, where $\frac{I}{K} = (\exp(\overline{\varepsilon}) - 1 + \delta)$ is the steady state investment-capital ratio. As a result, the balanced growth path of the economy is unaffected by the adjustment cost parameter ξ.

[44] The elasticity of the investment-capital ratio with respect to Tobin's q is $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \frac{\partial(I_t/K_t)}{\partial q_t} \times \frac{q_t}{(I_t/K_t)} = \left[\frac{\partial q_t}{\partial(I_t/K_t)} \times \frac{(I_t/K_t)}{q_t}\right]^{-1}$. Tobin's q is given by $\frac{1}{\phi'(I_t/K_t)} = \alpha_1\left(\frac{I_t}{K_t}\right)^{1/\xi}$. It follows that $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \left[(1/\xi)\,\alpha_1\left(\frac{I_t}{K_t}\right)^{1/\xi - 1} \frac{1}{\alpha_1}\left(\frac{I_t}{K_t}\right)^{1-1/\xi}\right]^{-1} = [1/\xi]^{-1} = \xi$.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_t} &= 0 : (C_t - \eta C_{t-1})^{-1} - \beta\eta E_t\left[(C_{t+1} - \eta C_t)^{-1}\right] \\
&\quad -\lambda_t \phi'\left(\frac{I_t}{K_t}\right) \\
&= 0, && (3.113) \\
\frac{\partial \mathcal{L}}{\partial N_t} &= 0 : -1 + \lambda_t \phi'\left(\frac{I_t}{K_t}\right) K_t \times \left[(1-\alpha)\,\theta_t^{1-\alpha} K_t^{\alpha-1} N_t^{-\alpha}\right] \\
&= -1 + \lambda_t \phi'\left(\frac{I_t}{K_t}\right)\left[(1-\alpha)\,\theta_t^{1-\alpha} K_t^{\alpha} N_t^{-\alpha}\right] = 0, && (3.114) \\
\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= 0 : -\lambda_t \\
&\quad +\beta E_t\left[\lambda_{t+1}\left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times\left[\begin{array}{c} (\alpha-1)\,K_{t+1}^{\alpha-1}\left(\theta_{t+1}N_{t+1}\right)^{1-\alpha} \\ +C_{t+1}/K_{t+1} \end{array}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array}\right)\right] \\
&= 0, && (3.115) \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &\geq 0 : \phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta)K_t - K_{t+1} \geq 0, && (3.116) \\
\frac{\partial \mathcal{L}}{\partial \lambda_t}\lambda_t &= 0. && (3.117)
\end{aligned}$$

The Euler equations follow as:

$$\begin{aligned}
&(C_t - \eta C_{t-1})^{-1} \\
&= \beta E_t\left[\begin{array}{c} \left((C_{t+1} - \eta C_t)^{-1} - \beta\eta E_{t+1}\left[(C_{t+2} - \eta C_{t+1})^{-1}\right]\right) \\ \times\frac{\phi'\left(\frac{I_t}{K_t}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \\ \times\left(\begin{array}{c} \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ \times\left[\begin{array}{c} (\alpha-1)\,K_{t+1}^{\alpha-1}\left(\theta_{t+1}N_{t+1}\right)^{1-\alpha} \\ +C_{t+1}/K_{t+1} \end{array}\right] \\ +\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) + (1-\delta) \end{array}\right) \\ +\eta\left(C_{t+1} - \eta C_t\right)^{-1} \end{array}\right], \qquad (3.118)
\end{aligned}$$

and:

$$N_t = \left[(C_t - \eta C_{t-1})^{-1} - \beta\eta E_t\left[(C_{t+1} - \eta C_t)^{-1}\right](1-\alpha)\,\theta_t^{1-\alpha} K_t^{\alpha}\right]^{1/\alpha}. \qquad (3.119)$$

Boldrin, Christiano, Fisher (2001) use the following definition of the equity return:

$$r_{t+1}^{E} = \frac{D_{t+1} + \frac{\partial K_{t+2}}{\partial K_{t+1}} P_{K,t+1}}{P_{K,t}} - 1, \tag{3.120}$$

where:

$$D_{t+1} = \alpha K_{t+1}^{\alpha-1} \left(\theta_{t+1} N_{t+1}\right)^{1-\alpha}, \tag{3.121}$$

$$\frac{\partial K_{t+2}}{\partial K_{t+1}} = \left[(1-\delta) + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\right], \tag{3.122}$$

$$P_{K,t} = \frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}. \tag{3.123}$$

$P_{K,t}$ denotes the price at time t of one additional unit of capital in terms of the consumption good. One marginal unit of capital increases output by D_{t+1}, measured in consumption goods, and next period's capital stock by $\frac{\partial K_{t+2}}{\partial K_{t+1}}$, keeping I_{t+1} constant, measured in capital goods.

The risk-free rate is given by:

$$r_{t+1}^{F} = \frac{1}{\beta} \frac{\left(C_t - \eta C_{t-1}\right)^{-1} - \beta\eta E_t\left[\left(C_{t+1} - \eta C_t\right)^{-1}\right]}{E_t\left[\left(C_{t+1} - \eta C_t\right)^{-1} - \beta\eta E_{t+1}\left[\left(C_{t+2} - \eta C_{t+1}\right)^{-1}\right]\right]} - 1. \tag{3.124}$$

Since the process for productivity is non-stationary, we need to normalize the economy by θ_t, in order to be able to numerically solve the model. To be precise, we let $\widehat{K}_t = \frac{K_t}{\theta_t}, \widehat{C}_t = \frac{C_t}{\theta_t}, \widehat{I}_t = \frac{I_t}{\theta_t}$, and substitute. We need not normalize N_t, as this variable is already stationary in the non-stationary economy.[45] In the so transformed model all variables are stationary. The state variables of the normalized model are $\widehat{K}$ and $e^{-\varepsilon}\widehat{C}_{-1}$.[46] We can work directly on the same (appropriately normalized) set of equations as spelt out above.

[45] This result is due to King, Plosser, Rebelo (1988). With labor augmenting technological progress, labor (N) is stationary if the income and substitution effects of permanent increases in real wages balance. King, Plosser, Rebelo show for which set of utility specifications those two effects balance.

[46] Note that θ is *not* a state variable of the normalized model. This is due to the fact that we assume, as do Boldrin, Christiano, Fisher (1999), that the autoregressive coefficient of the process for productivity $\ln\theta_{t+1} = \rho\ln\theta_t + \varepsilon_{t+1}$ is unity: $\rho = 1$. As a consequence, $\Delta\theta$ is serially uncorrelated.

We show the normalization of four key equations. The Euler equations become:

$$\begin{aligned}
&\left(\widehat{C}_t-\eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-1}\\
&=\beta E_t\left[\begin{array}{c}
e^{-\varepsilon_{t+1}}\left(\begin{array}{c}\left(\widehat{C}_{t+1}-\eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-1}\\ -\beta\eta E_{t+1}\left[e^{-\varepsilon_{t+2}}\left(\widehat{C}_{t+2}-\eta e^{-\varepsilon_{t+2}}\widehat{C}_{t+1}\right)^{-1}\right]\end{array}\right)\\
\times\frac{\phi'\left(\frac{\widehat{I}_t}{\widehat{K}_t}\right)}{\phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right)}\\
\times\left(\begin{array}{c}\phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right)\\ \times\left[(\alpha-1)\widehat{K}_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha}+\widehat{C}_{t+1}/\widehat{K}_{t+1}\right]\\ +\phi\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right)+(1-\delta)\end{array}\right)\\
+\eta e^{-\varepsilon_{t+1}}\left(\widehat{C}_{t+1}-\eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-1}
\end{array}\right],
\end{aligned}\tag{3.125}$$

and:

$$N_t=\left[\left(\begin{array}{c}\left(\widehat{C}_t-\eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-1}\\ -\beta\eta E_t\left[e^{-\varepsilon_{t+1}}\left(\widehat{C}_{t+1}-\eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-1}\right]\end{array}\right)(1-\alpha)\widehat{K}_t^{\alpha}\right]^{1/\alpha}.\tag{3.126}$$

The capital accumulation equation:

$$\widehat{K}_{t+1}e^{\varepsilon_{t+1}}=\phi\left(\widehat{I}_t/\widehat{K}_t\right)\widehat{K}_t+(1-\delta)\widehat{K}_t,\tag{3.127}$$

and finally the risk-free rate:

$$r_{t+1}^F=\frac{1}{\beta}\frac{\begin{array}{c}\left(\widehat{C}_t-\eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-1}\\ -\beta\eta E_t\left[\left(e^{\varepsilon_{t+1}}\right)^{-1}\left(\widehat{C}_{t+1}-\eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-1}\right]\end{array}}{E_t\left[\begin{array}{c}\left(e^{\varepsilon_{t+1}}\right)^{-1}\\ \left(\begin{array}{c}\left(\widehat{C}_{t+1}-\eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-1}\\ -\beta\eta E_{t+1}\left[\left(e^{\varepsilon_{t+2}}\right)^{-1}\left(\widehat{C}_{t+2}-\eta e^{-\varepsilon_{t+2}}\widehat{C}_{t+1}\right)^{-1}\right]\end{array}\right)\end{array}\right]}-1.\tag{3.128}$$

Calibration

Boldrin, Christiano, Fisher (1999) and Boldrin, Christiano, Fisher (2001) solve three different parameterizations of the above discussed model. We solve exactly the same three parameterizations. The parameter values reported in Table 3.12 are the same for

Table 3.12: **Appendix — Calibration (Boldrin, Christiano, Fisher)**

Quarterly Model		
Parameter	Description	Value
α	Elasticity of capital	0.36
δ	Depreciation rate of capital	0.021
β	Discount factor	0.99999
$\overline{\varepsilon}$	Mean growth rate	0.0040
σ_{ε}	Standard deviation of shock to ε	0.018

Table 3.13: **Appendix — Three Versions of the Model**

Parameter	Description	Standard	Habit	Jermann
η	Habit level	0	0.90	0.90
ξ	Elasticity of $\left(\frac{I_t}{K_t}\right)$ w.r.t. Tobin's q	∞	∞	0.23

all three versions of the model. The parameter values reported in Table 3.13 characterize the three different versions of the model (henceforth Standard, Habit, Jermann model respectively).

Boldrin, Christiano, Fisher (2001) report results for both the Standard as well as the Habit model in their 2001 AER paper. For the Jermann model they report results only in their more extensive 1999 working paper version.

Numerical Solution

Relying on den Haan and Marcet (1990), Boldrin, Christiano, Fisher (1999) apply the Parameterized Expectations Algorithm (PEA). We also use PEA in order to solve all models. We use 5th order Chebyshev orthogonal polynomials and a 6×6 Chebyshev grid for the state variables $\widehat{K}$ and $e^{-\varepsilon}\widehat{C}_{-1}$. Gauss-Hermite quadrature with 5 nodes is used to approximate the expectations operators.

To check for accuracy, we use a very fine grid (100×100), and at each grid point compare the polynomial approximation of the Euler equation with the 'true' expectation, computed using Gauss-Hermite quadrature with 100 nodes. The maximum absolute percentage difference between parameterized expectation and true expectation is smaller than 0.00001% for all models.[47]

[47] We express the maximum absolute percentage difference in terms of the consumption good or employment respectively.

We solve the following system of equations:

$$\widehat{C} = \Psi^A\left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}\right)^{-1} + \eta e^{-\varepsilon}\widehat{C}_{-1}, \tag{3.129}$$

$$N = \left[\Psi^B\left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}\right)(1-\alpha)\widehat{K}^{\alpha}\right]^{1/\alpha}, \tag{3.130}$$

$$\widehat{Y} = \widehat{K}^{\alpha}N^{1-\alpha}, \tag{3.131}$$

$$\widehat{K}_{+1}e^{\varepsilon_{+1}} = \phi\left(\frac{\widehat{Y}-\widehat{C}}{\widehat{K}}\right)\widehat{K} + (1-\delta)\widehat{K}, \tag{3.132}$$

where the functions Ψ^A and Ψ^B are the following parameterized conditional expectations:

$$\begin{aligned}\Psi^A\left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}\right) &= \left(\widehat{C} - \eta e^{-\varepsilon}\widehat{C}_{-1}\right)^{-1} \\ &= \beta E\left[\begin{array}{c} e^{-\varepsilon_{+1}} \times \left(\begin{array}{c}\left(\widehat{C}_{+1} - \eta e^{-\varepsilon_{+1}}\widehat{C}\right)^{-1} \\ -\beta\eta E_{+1}\left[e^{-\varepsilon_{+2}}\left(\widehat{C}_{+2} - \eta e^{-\varepsilon_{+2}}\widehat{C}_{+1}\right)^{-1}\right]\end{array}\right) \\ \times \frac{\phi'\left(\frac{\widehat{I}}{\widehat{K}}\right)}{\phi'\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right)} \\ \times \left(\begin{array}{c}\phi'\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right) \\ \times\left[(\alpha-1)\widehat{K}_{+1}^{\alpha-1}N_{+1}^{1-\alpha} + \widehat{C}_{+1}/\widehat{K}_{+1}\right] \\ +\phi\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right) + (1-\delta)\end{array}\right) \\ +\eta e^{-\varepsilon_{+1}}\left(\widehat{C}_{+1} - \eta e^{-\varepsilon_{+1}}\widehat{C}\right)^{-1}\end{array}\right],\end{aligned} \tag{3.133}$$

$$\begin{aligned}\Psi^B\left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}\right) &= \left[\left(\widehat{C} - \eta e^{-\varepsilon}\widehat{C}_{-1}\right)^{-1} - \beta\eta E\left[e^{-\varepsilon_{+1}}\left(\widehat{C}_{+1} - \eta e^{-\varepsilon_{+1}}\widehat{C}\right)^{-1}\right]\right] \\ &= \beta E\left[\begin{array}{c} e^{-\varepsilon_{+1}} \times \left(\begin{array}{c}\left(\widehat{C}_{+1} - \eta e^{-\varepsilon_{+1}}\widehat{C}\right)^{-1} \\ -\beta\eta E_{+1}\left[e^{-\varepsilon_{+2}}\left(\widehat{C}_{+2} - \eta e^{-\varepsilon_{+2}}\widehat{C}_{+1}\right)^{-1}\right]\end{array}\right) \\ \times \frac{\phi'\left(\frac{\widehat{I}}{\widehat{K}}\right)}{\phi'\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right)} \times \left(\begin{array}{c}\phi'\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right) \\ \times\left[(\alpha-1)\widehat{K}_{+1}^{\alpha-1}N_{+1}^{1-\alpha} + \widehat{C}_{+1}/\widehat{K}_{+1}\right] \\ +\phi\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right) + (1-\delta)\end{array}\right)\end{array}\right].\end{aligned} \tag{3.134}$$

Results

Table 3.14 contains moments of macroeconomic time series for all three models, as well as the values Boldrin, Christiano, Fisher (1999) (henceforth BCF) report, in order to facilitate comparison. As demonstrated in Table 3.14, we manage to replicate all results

Table 3.14: **Appendix — Comparison: Macroeconomic Moments**
σ_x denotes the standard deviation of variable x. $\rho_{x,y}$ denotes the correlation between variables x and y. All variables are logged and HP-filtered prior to analysis. The statistic σ_Y is reported in percent. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1964 to 1987. We take those values from Boldrin, Christiano, Fisher (2001). Results for all models are based on 200 replications of sample size 200 each.

Statistic	Standard	BCF	Habit	BCF	Jermann	BCF	Data
σ_Y	2.10	2.11	1.78	1.79	0.60	0.60	1.89
							(0.21)
σ_C/σ_Y	0.53	0.53	0.30	0.30	1.03	1.02	0.40
							(0.04)
σ_I/σ_Y	1.86	1.86	2.59	2.58	1.34	1.35	2.39
							(0.06)
σ_N/σ_Y	0.48	0.48	0.27	0.27	2.72	2.73	0.80
							(0.05)
$\rho_{Y,C}$	0.99	0.99	0.48	0.48	0.91	0.91	0.76
							(0.05)
$\rho_{Y,I}$	1.00	0.99	0.98	0.98	0.83	0.83	0.96
							(0.01)
$\rho_{Y,N}$	0.99	0.99	0.99	0.99	-0.59	-0.60	0.78
							(0.05)

Boldrin, Christiano, Fisher report.

Table 3.15 displays asset pricing moments for all models. As can be seen from Table 3.15, we manage to replicate almost all asset pricing moments Boldrin, Christiano, Fisher (1999) report. However, for the Jermann model we find a discrepancy in the equity premium as well as the standard deviation of the equity return. For two reasons we are not too concerned. First of all, and as demonstrated above, we can replicate almost exactly all other results Boldrin, Christiano, Fisher report for the Standard, the Habit, as well as the Jermann model. Finally, Boldrin, Christiano, Fisher report the two moments we were not able to replicate only in the 1999 working paper version and not in the 2001 AER version.

Some Additional Results

Boldrin, Christiano, Fisher (2001) document that their preferred two-sector model can account for two features of macroeconomic time series: (i) the excess sensitivity of consumption to income and (ii) the inverted leading indicator property of interest rates. Boldrin, Christiano, Fisher also report results for the Standard and the Habit model. We replicate those results below, as well as the autocorrelation coefficients of income and

Table 3.15: **Appendix — Comparison: Asset Pricing Moments**
σ_x denotes the standard deviation of variable x. $\rho_{x,y}$ denotes the correlation between variables x and y. Rates of return are annualized. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1892 to 1987. We take those values from Boldrin, Christiano, Fisher (2001), who in turn rely on Cecchetti, Lam, Mark (1993). Results for all models are based on 200 replications of sample size 200 each.

Statistic	Standard	BCF	Habit	BCF	Jermann	BCF	Data
$\mathrm{E}[r_t^F]$	1.58	1.58	1.58	1.58	1.58	1.55	1.19
							(0.81)
$\mathrm{E}[r_t^E - r_t^F]$	0.001	0.001	0.001	0.001	**0.57**	**0.15**	6.63
							(1.78)
σ_{r^E}	0.50	0.48	0.42	0.40	**11.31**	**5.53**	19.40
							(1.56)
σ_{r^F}	0.48	0.46	0.40	0.38	0.33	0.30	5.27
							(0.74)
$\frac{E[r_t^E - r_t^F]}{\sigma_{r^E}}$	0.002	0.002	0.002	0.002	0.05	0.03	0.34
							(0.09)
σ_{P_K}	0	0	0	0	3.51	3.59	8.56
							(0.85)
ρ_{Y,P_K}	na	na	na	na	0.43	0.43	0.30
							(0.08)

consumption growth for the Standard, the Habit, and the Jermann model.

Excess Sensitivity of Consumption to Income Applying instrumental variable techniques, Boldrin, Christiano, Fisher (2001) estimate the following model:

$$\Delta c_t = \beta_1 + \beta_2 \Delta y_t + \beta_3 r_{t-1}^F + \varepsilon_t,$$

where Δc_t and Δy_t are the first differences of log-consumption and log-income respectively. Boldrin, Christiano, Fisher (2001) replace all variables by their fitted values from a regression on a set of instruments.[48] Then they estimate the model via an OLS regression, using data generated by different theoretical models.[49]

In Table 3.16 we report the original Campbell and Mankiw (1989, 1991) empirical estimates and estimates from data generated by both the Standard and the Habit model, as well as the values Boldrin, Christiano, Fisher (2001) report in their paper, in order to facilitate comparison.

[48]Boldrin, Christiano, Fisher (2001) regress on the following list of instruments: Δc_{t-2}, Δc_{t-3}, Δc_{t-4}, r_{t-3}^F, r_{t-4}^F, r_{t-5}^F.

[49]The results Boldrin, Christiano, Fisher (2001) report are based on 500 replications of sample size 200 each.

Table 3.16: **Appendix — Excess Sensitivity of Consumption to Income**
$\Delta x_t = \log(X_t) - \log(X_{t-1})$. The model is estimated via OLS after replacing all variables with their fitted values from a regression on the following set of instruments: $\{\Delta c_{t-2}, \Delta c_{t-3}, \Delta c_{t-4}, r^F_{t-3}, r^F_{t-4}, r^F_{t-5}\}$. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1953 to 1986. We take those values from Campbell and Mankiw (1989). Results for all models are based on 200 replications of sample size 200 each.

$$\Delta c_t = \beta_1 + \beta_2 \Delta y_t + \beta_3 r^F_{t-1} + \varepsilon_t$$

	Stand	BCF	Habit	BCF	Data
β_2	0.52	0.52	0.19	0.18	0.47
					(0.15)
β_3	0.92	0.92	2.58	2.58	0.09
					(0.11)

Table 3.17: **Appendix — Inverted Leading Indicator Phenomenon**
$\rho_{x,y}$ denotes the correlation between variables x and y. All variables are logged and HP-filtered prior to analysis. The 'Data' column contains estimates based on U.S. data from 1964 to 1987. We take those values from Boldrin, Christiano, Fisher (2001). Results for all models are based on 200 replications of sample size 200 each.

	Stand	BCF	Habit	BCF	Data
$\rho\left(r^F_{t-2}, Y_t\right)$	0.51	0.51	0.53	0.54	-0.35
$\rho\left(r^F_{t}, Y_t\right)$	0.98	0.99	0.97	0.98	0.00
$\rho\left(r^F_{t+2}, Y_t\right)$	0.37	0.38	0.34	0.35	0.16

Inverted Leading Indicator Phenomenon The inverted leading indicator phenomenon denotes the empirical fact that the correlation between output and past (future) real and nominal interest rates is negative (positive), while the contemporaneous correlation is basically zero.

Again, Boldrin, Christiano, Fisher (2001) report those correlations for the Standard and the Habit model. In Table 3.17 we report statistics from data generated by both the Standard and the Habit model, as well as the values Boldrin, Christiano, Fisher report in their paper.

Autocorrelation Coefficient of Income and Consumption Growth Table 3.18 reports the autocorrelation coefficient of income and consumption growth from data generated by the Standard, the Habit, and the Jermann model, as well as the values Boldrin, Christiano, Fisher (2001) report in their 1999 working paper version.

Table 3.18: **Appendix — Autocorrelation of Income and Cons. Growth**
$\rho(\Delta x)$ denotes the autocorrelation of $\log x_t - \log x_{t-1}$. The 'Data' column contains estimates (standard errors in parenthesis) based on U.S. data from 1964 to 1987. We take those values from Boldrin, Christiano, and Fisher (2001). Results for all models are based on 200 replications of sample size 200 each.

Statistic	Stand	BCF	Habit	BCF	Jermann	BCF	Data
$\rho(\Delta Y)$	0.002	0.002	0.01	0.02	0.40	0.38	0.34
							(0.07)
$\rho(\Delta C)$	0.05	0.05	0.90	0.90	0.87	0.86	0.24
							(0.09)

3.12.2 Jermann (1994)

In this section, we set up and solve the original Jermann (1998) model. As in Jermann (1994) and in Jermann (1998), we solve two different versions of the model (henceforth Model I and Model II) characterized by different driving processes for productivity. The goal is to (i) replicate results Jermann (1994) reports in a more extensive working paper version of his 1998 JME paper, and to (ii) assess the accuracy of the loglinear-lognormal approximations Jermann (1994) and Jermann (1998) apply in order to solve a variety of parameterizations of their model.

Jermann assumes labor to be fixed, that is leisure does not enter the households' utility function. This is in contrast to the version of the Jermann model Boldrin, Christiano, Fisher (2001) consider, where leisure does enter the households' utility function (see section about Boldrin, Christiano, Fisher in this appendix).

Model I

Aggregate consumption is chosen with the aim to maximize the expected utility of the representative household:

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{(C_t - \eta C_{t-1})^{1-\gamma}}{1-\gamma} \right], \tag{3.135}$$

subject to the following accumulation equation of the aggregate capital stock with adjustment costs:

$$K_{t+1} \leq \phi\left(I_t/K_t\right) K_t + (1-\delta) K_t, \tag{3.136}$$

where $I_t = Y_t - C_t$, and $\phi(\cdot)$ is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it step by step.[50]

The aggregate production technology is specified as:

$$Y_t = e^{\bar{\varepsilon}t(1-\alpha)}\theta_t K_t^{\alpha}, \tag{3.137}$$

where θ_t evolves according to:

$$\ln\theta_t = \rho\ln\theta_{t-1} + \varepsilon_t, \tag{3.138}$$

with:

$$\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right). \tag{3.139}$$

For the autoregressive coefficient $\rho < 1$, this driving process of aggregate productivity ($e^{\bar{\varepsilon}t(1-\alpha)}\theta_t$) constitutes an AR(1) that fluctuates around a deterministic trend ($e^{\bar{\varepsilon}t}$). Since this process is non-stationary, we need to normalize the economy by the non-stationary component of the driving process, that is $e^{\bar{\varepsilon}t}$, in order to be able to numerically solve the model. To be precise, we let $\widehat{K}_t = \frac{K_t}{e^{\bar{\varepsilon}t}}, \widehat{C}_t = \frac{C_t}{e^{\bar{\varepsilon}t}}, \widehat{I}_t = \frac{I_t}{e^{\bar{\varepsilon}t}}$, and substitute. In the so transformed model all variables are stationary. Note that the resulting model has *three* state variables: $\widehat{K}$, $\widehat{C}_{-1}$, and θ.

It is straightforward (see above section about Boldrin, Christiano, Fisher in this appendix for analogous steps) to derive the Euler equation from the Lagrangian. Below, we report the main equations of the stationary model.

The Euler equation:

$$\begin{aligned} &\left(\widehat{C}_t - e^{-\bar{\varepsilon}}\eta\widehat{C}_{t-1}\right)^{-\gamma} \\ = \;& \beta e^{-\gamma\bar{\varepsilon}} E_t \left[\begin{array}{c} \left(\begin{array}{c} \left(\widehat{C}_{t+1} - e^{-\bar{\varepsilon}}\eta\widehat{C}_t\right)^{-\gamma} \\ -\beta e^{-\gamma\bar{\varepsilon}}\eta E_{t+1}\left[\left(\widehat{C}_{t+2} - e^{-\bar{\varepsilon}}\eta\widehat{C}_{t+1}\right)^{-\gamma}\right] \end{array}\right) \\ \times \frac{\phi'\left(\frac{\widehat{I}_t}{\widehat{K}_t}\right)}{\phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right)} \\ \times \left(\begin{array}{c} \phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right) \\ \times\left[\theta_{t+1}(\alpha-1)\widehat{K}_{t+1}^{\alpha-1} + \widehat{C}_{t+1}/\widehat{K}_{t+1}\right] \\ +\phi\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right) + (1-\delta) \end{array}\right) \\ +\eta\left(\widehat{C}_t - e^{-\bar{\varepsilon}}\eta\widehat{C}_{t-1}\right)^{-\gamma} \end{array}\right], \end{aligned} \tag{3.140}$$

[50]See above section about Boldrin, Christiano, Fisher in this appendix for a detailed discussion of the adjustment cost technology.

the capital accumulation equation:

$$\widehat{K}_{t+1}e^{\overline{\varepsilon}} = \phi\left(\widehat{I}_t/\widehat{K}_t\right)\widehat{K}_t + (1-\delta)\widehat{K}_t, \tag{3.141}$$

the production technology:

$$\widehat{Y}_t = \widehat{K}_t^{\alpha} - \widehat{C}_t, \tag{3.142}$$

and the risk-free rate:

$$r^F_{t+1} = \frac{1}{\beta e^{-\gamma\overline{\varepsilon}}}\frac{\begin{array}{c}\left(\widehat{C}_t - e^{-\overline{\varepsilon}}\eta\widehat{C}_{t-1}\right)^{-\gamma}\\ -\beta e^{-\gamma\overline{\varepsilon}}\eta E_t\left[\left(\widehat{C}_{t+1} - e^{-\overline{\varepsilon}}\eta\widehat{C}_t\right)^{-\gamma}\right]\end{array}}{E_t\left[\begin{array}{c}\left(\widehat{C}_{t+1} - e^{-\overline{\varepsilon}}\eta\widehat{C}_t\right)^{-\gamma}\\ -\beta e^{-\gamma\overline{\varepsilon}}\eta E_{t+1}\left[\left(\widehat{C}_{t+2} - \eta e^{-\overline{\varepsilon}}\widehat{C}_{t+1}\right)^{-\gamma}\right]\end{array}\right]} - 1. \tag{3.143}$$

Jermann (1998) uses the following definition of the equity return:

$$r^E_{t+1} = \frac{V_{t+1} + d_{t+1}}{V_t} - 1, \tag{3.144}$$

where:

$$\begin{aligned} V_t &= \beta E_t\left[\frac{\left[(C_{t+1}-\eta C_t)^{-\gamma} - \beta\eta E_{t+1}\left[(C_{t+2}-\eta C_{t+1})^{-\gamma}\right]\right]}{(C_t-\eta C_{t-1})^{-\gamma} - \beta\eta E_t\left[(C_{t+1}-\eta C_t)^{-\gamma}\right]}(d_{t+1}+V_{t+1})\right] \\ &= \beta E_t\left[\begin{array}{c}\frac{\left[(C_{t+1}-\eta C_t)^{-\gamma} - \beta\eta E_{t+1}\left[(C_{t+2}-\eta C_{t+1})^{-\gamma}\right]\right]}{(C_t-\eta C_{t-1})^{-\gamma} - \beta\eta E_t\left[(C_{t+1}-\eta C_t)^{-\gamma}\right]}\\ \times(Y_{t+1} - w_{t+1} - I_{t+1} + V_{t+1})\end{array}\right] \\ &= \beta E_t\left[\begin{array}{c}\frac{\left[(C_{t+1}-\eta C_t)^{-\gamma} - \beta\eta E_{t+1}\left[(C_{t+2}-\eta C_{t+1})^{-\gamma}\right]\right]}{(C_t-\eta C_{t-1})^{-\gamma} - \beta\eta E_t\left[(C_{t+1}-\eta C_t)^{-\gamma}\right]}\\ \times(Y_{t+1} - (1-\alpha)Y_{t+1} - (Y_{t+1} - C_{t+1}) + V_{t+1})\end{array}\right] \\ &= \beta E_t\left[\begin{array}{c}\frac{\left[(C_{t+1}-\eta C_t)^{-\gamma} - \beta\eta E_{t+1}\left[(C_{t+2}-\eta C_{t+1})^{-\gamma}\right]\right]}{(C_t-\eta C_{t-1})^{-\gamma} - \beta\eta E_t\left[(C_{t+1}-\eta C_t)^{-\gamma}\right]}\\ \times(C_{t+1} - (1-\alpha)Y_{t+1} + V_{t+1})\end{array}\right]. \end{aligned} \tag{3.145}$$

Appropriately normalized:

$$r^E_{t+1} = e^{\overline{\varepsilon}}\frac{\widehat{V}_{t+1} + \widehat{d}_{t+1}}{\widehat{V}_t} - 1, \tag{3.146}$$

$$\widehat{V}_t = \beta e^{(1-\gamma)\overline{\varepsilon}}E_t\left[\begin{array}{c}\frac{\left[\left(\widehat{C}_{t+1}-e^{-\overline{\varepsilon}}\eta\widehat{C}_t\right)^{-\gamma} - \beta e^{-\gamma\overline{\varepsilon}}\eta E_{t+1}\left[\left(\widehat{C}_{t+2}-e^{-\overline{\varepsilon}}\eta\widehat{C}_{t+1}\right)^{-\gamma}\right]\right]}{\left(\widehat{C}_t-e^{-\overline{\varepsilon}}\eta\widehat{C}_{t-1}\right)^{-\gamma} - \beta e^{-\gamma\overline{\varepsilon}}\eta E_t\left[\left(\widehat{C}_{t+1}-e^{-\overline{\varepsilon}}\eta\widehat{C}_t\right)^{-\gamma}\right]}\\ \times\left(\widehat{C}_{t+1} - (1-\alpha)\widehat{Y}_{t+1} + \widehat{V}_{t+1}\right)\end{array}\right]. \tag{3.147}$$

Model II

This specification of the model has exactly the same components as Model I, apart from the aggregate production technology:

$$Y_t = \theta_t^{1-\alpha} K_t^{\alpha}, \tag{3.148}$$

where θ_t evolves according to:

$$\ln \theta_t = \ln \theta_{t-1} + \varepsilon_t, \tag{3.149}$$

with:

$$\varepsilon_t \sim N\left(\bar{\varepsilon}, \sigma_\varepsilon^2\right). \tag{3.150}$$

Since this process for aggregate productivity (θ) is non-stationary, we need to normalize the economy by θ_t, in order to be able to numerically solve the model. To be precise, we let $\widehat{K}_t = \frac{K_t}{\theta_t}, \widehat{C}_t = \frac{C_t}{\theta_t}, \widehat{I}_t = \frac{I_t}{\theta_t}$, and substitute. In the so transformed model all variables are stationary. The state variables of the normalized model are $\widehat{K}$ and $e^{-\varepsilon}\widehat{C}_{-1}$.[51] We can work directly on the same (appropriately normalized) set of equations as spelt out below. The Euler equation:

$$\begin{aligned}
&\left(\widehat{C}_t - \eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-\gamma} \\
= \;& \beta E_t \left[\begin{array}{c}
\times \left(\begin{array}{c}
\left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \\
\left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \\
-\beta\eta E_{t+1}\left[\begin{array}{c}
\left(e^{\varepsilon_{t+2}}\right)^{-\gamma} \\
\times \left(\widehat{C}_{t+2} - \eta e^{-\varepsilon_{t+2}}\widehat{C}_{t+1}\right)^{-\gamma}
\end{array}\right]
\end{array}\right) \\
\times \dfrac{\phi'\left(\frac{\widehat{I}_t}{\widehat{K}_t}\right)}{\phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right)} \\
\times \left(\begin{array}{c}
\phi'\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right) \\
\times \left[(\alpha-1)\widehat{K}_{t+1}^{\alpha-1} + \widehat{C}_{t+1}/\widehat{K}_{t+1}\right] \\
+\phi\left(\frac{\widehat{I}_{t+1}}{\widehat{K}_{t+1}}\right) + (1-\delta)
\end{array}\right) \\
+\eta\left(e^{\varepsilon_{t+1}}\right)^{-\gamma}\left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma}
\end{array}\right],
\end{aligned} \tag{3.151}$$

the capital accumulation equation:

$$\widehat{K}_{t+1}e^{\varepsilon_{t+1}} = \phi\left(\widehat{I}_t/\widehat{K}_t\right)\widehat{K}_t + (1-\delta)\widehat{K}_t, \tag{3.152}$$

[51] Note that θ is *not* a state variable of the normalized model. This is due to the fact that we assume, as do Boldrin, Christiano, Fisher (2001), that the autoregressive coefficient of the process for productivity $\ln\theta_{t+1} = \rho\ln\theta_t + \varepsilon_{t+1}$ is unity: $\rho = 1$. As a consequence, $\Delta\theta$ is serially uncorrelated.

the risk-free rate:

$$r^F_{t+1} = \frac{1}{\beta} \frac{\left[\begin{array}{c} \left(\widehat{C}_t - \eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-\gamma} \\ -\beta\eta E_t \left[\left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \right] \end{array} \right]}{E_t \left[\begin{array}{c} \left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \\ -\beta\eta \left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \\ E_{t+1} \left[\left(e^{\varepsilon_{t+2}}\right)^{-\gamma} \left(\widehat{C}_{t+2} - \eta e^{-\varepsilon_{t+2}}\widehat{C}_{t+1}\right)^{-\gamma} \right] \end{array} \right]} - 1, \tag{3.153}$$

and the equity return:

$$r^E_{t+1} = e^{\varepsilon_{t+1}} \times \frac{\widehat{V}_{t+1} + \widehat{d}_{t+1}}{\widehat{V}_t} - 1, \tag{3.154}$$

$$\widehat{V}_t = \beta E_t \left[\begin{array}{c} \left(e^{\varepsilon_{t+1}}\right)^{1-\gamma} \dfrac{\left(\begin{array}{c} \left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \\ -\beta E_{t+1} \left[\eta \left(e^{\varepsilon_{t+2}}\right)^{-\gamma} \left(\widehat{C}_{t+2} - \eta e^{-\varepsilon_{t+2}}\widehat{C}_{t+1}\right)^{-\gamma} \right] \end{array} \right)}{\left(\begin{array}{c} \left(\widehat{C}_t - \eta e^{-\varepsilon_t}\widehat{C}_{t-1}\right)^{-\gamma} \\ -\beta E_t \left[\eta \left(e^{\varepsilon_{t+1}}\right)^{-\gamma} \left(\widehat{C}_{t+1} - \eta e^{-\varepsilon_{t+1}}\widehat{C}_t\right)^{-\gamma} \right] \end{array} \right)} \\ \times \left(\widehat{C}_{t+1} - (1-\alpha)\widehat{Y}_{t+1} + \widehat{V}_{t+1}\right) \end{array} \right]. \tag{3.155}$$

Calibration

We solve several parameterizations of both versions of the Jermann (1998) model. The parameterizations are taken from Jermann (1994), the working paper version of Jermann (1998). Jermann (1994) solves and reports exactly the same two model versions as Jermann (1998), however, Jermann (1994) uses somewhat different parameter values and contains many more parameterizations. We follow Jermann (1994) and calibrate as in Table 3.19.

We solve both versions of the model (Model I and Model II), characterized by the driving process of productivity ($\rho = 0.95$ or $\rho = 1.00$), for several combinations of the parameter values for ξ, γ, and η, as indicated in Table 3.19.[52] Following Jermann (1994), we set the discount factor (β) so that the steady state risk-free rate is 6.50% on an annual

[52]For Model II, where productivity is assumed to follow a random walk ($\ln\theta_t = \ln\theta_{t-1} + \varepsilon_t$), we assume the aggregate production technology to be $Y_t = \theta_t^{1-\alpha} K_t^\alpha$ instead of $Y_t = e^{\bar{\varepsilon}t(1-\alpha)}\theta_t K_t^\alpha$ (for Model I), in order to be able to normalize the economy by θ_t. Thus, because θ_t enters the aggregate production technology in Model II to the power of $(1-\alpha)$, we have to recalibrate the standard deviation of the shock to aggregate productivity (σ_ε), so as to match output volatility also with Model II. For Model II, we set $\sigma_\varepsilon = 0.0135$.

Table 3.19: **Appendix — Calibration (Jermann)**

Quarterly Model		
Parameter	Description	Value
α	Elasticity of capital	0.37
δ	Depreciation rate of capital	0.025
ξ	Elasticity of $\left(\frac{I_t}{K_t}\right)$ w.r.t. Tobin's q	13.0 / 0.33
r_{ss}^F	Steady state risk-free rate (annual)	6.50%
γ	Coefficient of relative risk aversion	2 / 10
η	Habit level	0.00 / 0.60
ρ	Autoregressive coefficient	0.95 / 1.00
$\overline{\varepsilon}$	Mean growth rate	0.0040
σ_ε	Standard deviation of shock to ε	0.0085

basis.[53]

Numerical Solution

Jermann (1994) and Jermann (1998) use loglinear-lognormal approximations to solve all models.[54] Relying on den Haan and Marcet (1990), we use the Parameterized Expectations Algorithm (PEA) to solve all models. We use up to $5th$ order Chebyshev orthogonal polynomials and a 6×6 Chebyshev grid for the state variables $\widehat{K}$, $\widehat{C}_{-1}$ (and θ). Gauss-Hermite quadrature with 5 nodes is used to approximate the expectations operators.

To check for accuracy, we use a very fine grid (100×100), and at each grid point compare the polynomial approximation of the Euler equation with the 'true' expectation, computed using Gauss-Hermite quadrature with 100 nodes. The maximum absolute percentage difference between parameterized expectation and true expectation is smaller than 0.001% for all models.[55]

We solve the following system of equations:[56]

$$\widehat{C} = \Psi\left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}\right)^{-1/\gamma} + \eta e^{-\varepsilon}\widehat{C}_{-1}, \tag{3.156}$$

$$\widehat{Y} = \widehat{K}^{\alpha}, \tag{3.157}$$

$$\widehat{K}_{+1}e^{\varepsilon+1} = \phi\left(\frac{\widehat{Y}-\widehat{C}}{\widehat{K}}\right)\widehat{K} + (1-\delta)\widehat{K}, \tag{3.158}$$

[53]In the steady state, the risk-free rate on an annual basis is $r_{ss}^F = \frac{1}{(\beta e^{-\gamma\overline{\varepsilon}})^4} - 1$. Thus, we set $\beta = \frac{(1.0650)^{-1/4}}{e^{-\gamma\overline{\varepsilon}}}$.

[54]Jermann combines lognormal asset pricing formulae with a system of loglinear equations for the macroeconomic variables from the standard production economy model.

[55]We express the maximum absolute percentage difference in terms of the consumption good.

[56]We show here the system of equations we solve for Model II. The system for Model I is analogous.

where the function Ψ is the following parameterized conditional expectation (the Euler equation):

$$\Psi\left(\widehat{K}, e^{-\varepsilon}\widehat{C}_{-1}\right) = \left(\widehat{C} - \eta e^{-\varepsilon}\widehat{C}_{-1}\right)^{-\gamma}$$
$$= \beta E \left[\begin{array}{c} \left(e^{\varepsilon_{+1}}\right)^{-\gamma} \\ \times \left(\begin{array}{c} \left(\widehat{C}_{+1} - \eta e^{-\varepsilon_{+1}}\widehat{C}\right)^{-\gamma} \\ -\beta\eta E_{+1}\left[\left(e^{\varepsilon_{+2}}\right)^{-\gamma}\left(\widehat{C}_{+2} - \eta e^{-\varepsilon_{+2}}\widehat{C}_{+1}\right)^{-\gamma}\right] \end{array} \right) \\ \times \frac{\phi'\left(\frac{\widehat{I}}{\widehat{K}}\right)}{\phi'\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right)} \\ \times \left(\begin{array}{c} \phi'\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right)\left[(\alpha-1)\widehat{K}_{+1}^{\alpha-1} + \widehat{C}_{+1}/\widehat{K}_{+1}\right] \\ +\phi\left(\frac{\widehat{I}_{+1}}{\widehat{K}_{+1}}\right) + (1-\delta) \end{array} \right) \\ +\eta\left(e^{\varepsilon_{+1}}\right)^{-\gamma}\left(\widehat{C}_{+1} - \eta e^{-\varepsilon_{+1}}\widehat{C}\right)^{-\gamma} \end{array} \right]. \quad (3.159)$$

Results

Table 3.20 contains moments of macroeconomic time series for all models we solve (indicated by Parameterized Expectations Algorithm (PEA)), as well as the values Jermann (1994) reports (indicated by loglinear-lognormal approximation (LIN)).

The difference we find for Model Ic seems to be due to a small error ('typo') in the Jermann (1994) working paper version: Jermann (wrongly) reports the same values for Model Ic and Model IIc. So the only discrepancy we find is for Model IIf: investment is more volatile in our solution compared to Jermann. Above we demonstrate that approximating Jermann models with a low-order polynomial seems to result in investment being not volatile enough. We therefore ascribe the difference in Table 3.20 to the inaccuracy of the Jermann linear approximation.

Table 3.21 contains asset pricing moments for all models. We find striking discrepancies between our solution and Jermann (1994): for Models Ib, Ic, Ie, If, IIb, IIc, IIe, IIf we find our equity premium to be around *double* as high as the one reported by Jermann. Below we solve the Jermann model with low-order polynomial approximations and demonstrate that discrepancies such as the ones we observe in Table 3.21 can easily arise due to inaccuracies. In particular, we demonstrate that model solutions obtained with low-order approximations seem to generate an equity premium which is too low.

Table 3.20: **Appendix — Comparison: Macroeconomic Moments**

σ_x denotes the standard deviation of variable x. ρ denotes the autoregressive coefficient of the productivity process, η the habit level, γ the coefficient of relative risk aversion, ξ the elasticity of the capital-investment ratio with respect to Tobin's q. 'PEA' stands for Parameterized Expectations Algorithm, 'LIN' stands for loglinear-lognormal approximation. All variables are logged and first-differenced prior to analysis. The statistic σ_Y is reported in percent. The 'Data' row contains estimates based on U.S. data from 1954 to 1989. We take those values from Jermann (1998). Results for all models are based on 200 replications of sample size 200 each.

	σ_Y		σ_C/σ_Y		σ_I/σ_Y	
Data	0.98		0.51		2.65	
Model I: $\rho = 0.95$	PEA	LIN	PEA	LIN	PEA	LIN
Model Ia: $\eta = 0.00$, $\xi = 13.0$, $\gamma = 2$	0.86	0.86	0.46	0.46	2.54	2.54
Model Ib: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 2$	0.86	0.86	1.12	1.12	0.65	0.65
Model Ic: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 10$	0.86	0.85	**0.78**	**1.23**	**1.60**	**0.34**
Model Id: $\eta = 0.60$, $\xi = 13.0$, $\gamma = 2$	0.85	0.86	0.26	0.26	3.29	3.28
Model Ie: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 2$	0.86	0.86	0.79	0.79	1.84	1.85
Model If: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 10$	0.85	0.86	0.44	0.44	2.78	2.82
Model II: $\rho = 1.00$	PEA	LIN	PEA	LIN	PEA	LIN
Model IIa: $\eta = 0.00$, $\xi = 13.0$, $\gamma = 2$	0.85	0.85	0.87	0.87	1.36	1.36
Model IIb: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 2$	0.84	0.85	1.18	1.17	0.51	0.51
Model IIc: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 10$	0.85	0.85	1.26	**1.23**	0.36	**0.34**
Model IId: $\eta = 0.60$, $\xi = 13.0$, $\gamma = 2$	0.85	0.85	0.49	0.49	2.79	2.80
Model IIe: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 2$	0.85	0.85	0.85	0.85	1.71	1.71
Model IIf: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 10$	0.85	0.85	0.73	0.73	**2.04**	**2.26**

Table 3.21: **Appendix — Comparison: Asset Pricing Moments**

σ_x denotes the standard deviation of variable x. ρ denotes the autoregressive coefficient of the productivity process, η the habit level, γ the coefficient of relative risk aversion, ξ the elasticity of the capital-investment ratio with respect to Tobin's q. 'PEA' stands for Parameterized Expectations Algorithm, 'LIN' stands for loglinear-lognormal approximation. Rates of return are annualized and reported in percentage terms. The 'Data' row contains estimates based on U.S. data from 1892 to 1987. We take those values from Boldrin, Christiano, Fisher (2001), who in turn rely on Cecchetti, Lam, Mark (1993). Results for all models are based on 200 replications of sample size 200 each.

	$\mathrm{E}[r_t^F]$		$\mathrm{E}[r_t^E - r_t^F]$		σ_{r^E}	σ_{r^F}
Data	1.19		6.63		19.40	5.27
Model I: $\rho = 0.95$	PEA	LIN	PEA	LIN	PEA	PEA
Model Ia: $\eta = 0.00$, $\xi = 13.0$, $\gamma = 2$	6.49	6.48	0.02	0.01	0.90	0.29
Model Ib: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 2$	6.41	6.43	**0.33**	**0.14**	7.41	1.04
Model Ic: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 10$	5.58	5.61	**2.26**	**1.16**	18.26	2.98
Model Id: $\eta = 0.60$, $\xi = 13.0$, $\gamma = 2$	6.49	6.48	0.01	0.01	1.08	0.25
Model Ie: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 2$	6.42	6.15	**2.01**	**0.86**	21.13	8.85
Model If: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 10$	5.17	4.67	**5.54**	**2.87**	32.71	9.10
Model II: $\rho = 1.00$	PEA	LIN	PEA	LIN	PEA	PEA
Model IIa: $\eta = 0.00$, $\xi = 13.0$, $\gamma = 2$	6.49	6.46	0.01	0.01	0.67	0.45
Model IIb: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 2$	6.41	6.42	**0.20**	**0.11**	5.64	0.12
Model IIc: $\eta = 0.00$, $\xi = 0.33$, $\gamma = 10$	3.99	4.28	**0.45**	**0.10**	3.93	0.35
Model IId: $\eta = 0.60$, $\xi = 13.0$, $\gamma = 2$	6.47	6.45	0.01	0.01	0.96	0.39
Model IIe: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 2$	6.34	6.12	**1.78**	**0.75**	19.29	8.23
Model IIf: $\eta = 0.60$, $\xi = 0.33$, $\gamma = 10$	2.27	2.00	**4.52**	**2.85**	23.04	9.16

Two Definitions of the Equity Return

Jermann (1998) uses the following definition of the equity return:[57]

$$r_{t+1}^{EA} = \frac{d_{t+1} + V_{t+1}}{V_t} - 1, \tag{3.160}$$

where:

$$\begin{aligned} d_t &= Y_t - w_t - I_t \\ &= \theta_t^{1-\alpha} K_t^{\alpha} - (1-\alpha)\,\theta_t^{1-\alpha} K_t^{\alpha} - I_t \\ &= \alpha \theta_t^{1-\alpha} K_t^{\alpha} - I_t, \end{aligned} \tag{3.161}$$

and:

$$V_t = E_t \left[\sum_{i=1}^{\infty} \beta^i \frac{u'_{t+i}}{u'_t} d_{t+i} \right]. \tag{3.162}$$

An alternative definition, following Boldrin, Christiano, Fisher (2001) and many others in the literature, takes the equity return as the marginal return on the aggregate capital stock:

$$r_{t+1}^{EB} = \frac{D_{t+1} + \frac{\partial K_{t+2}}{\partial K_{t+1}} P_{K,t+1}}{P_{K,t}} - 1, \tag{3.163}$$

where:

$$D_{t+1} = \alpha \theta_t^{\alpha-1} K_{t+1}^{\alpha-1}, \tag{3.164}$$

$$\frac{\partial K_{t+2}}{\partial K_{t+1}} = \left[(1-\delta) + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) - \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right) \frac{I_{t+1}}{K_{t+1}} \right], \tag{3.165}$$

$$P_{K,t} = \frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}. \tag{3.166}$$

$P_{K,t}$ denotes the price at time t of one additional unit of capital in terms of the consumption good. One marginal unit of capital increases output by D_{t+1}, measured in consumption goods, and next period's capital stock by $\frac{\partial K_{t+2}}{\partial K_{t+1}}$, keeping I_{t+1} constant, measured in capital goods.

In this section, we set out to formally show that the two above definitions of the equity return are in fact equivalent.

[57] Jermann (1998) implicitly assumes the following production technology: $Y_t = \theta_t^{1-\alpha} K_t^{\alpha} N_t^{1-\alpha}$, where $N_t = 1$ because leisure is assumed not to enter the households' utility function.

From the Euler equation given by:

$$u'_{t+j} = E_{t+j}\left[\beta u'_{t+j+1} \times \left(\begin{array}{c} \frac{\phi'\left(\frac{I_{t+j}}{K_{t+j}}\right)}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)} \\ \times \left(\begin{array}{c} \phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right) \\ \times \left[\begin{array}{c} (\alpha-1)\,\theta_{t+j+1}^{1-\alpha} K_{t+j+1}^{\alpha-1} \\ +C_{t+j+1}/K_{t+j+1} \end{array}\right] \\ +\phi\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right) + (1-\delta) \end{array}\right) \end{array}\right)\right], \tag{3.167}$$

we obtain:

$$\begin{aligned} \frac{K_{t+j+1}}{\phi'\left(\frac{I_{t+j}}{K_{t+j}}\right)} &= E_{t+j}\left[\begin{array}{c} \beta\frac{u'_{t+j+1}}{u'_{t+j}} \\ \times\left(\begin{array}{c} \left[(\alpha-1)\,\theta_{t+j+1}^{1-\alpha} K_{t+j+1}^{\alpha} + C_{t+j+1}\right] \\ +\frac{\left(\phi\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)K_{t+j+1}+(1-\delta)K_{t+j+1}\right)}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)} \end{array}\right) \end{array}\right] \\ &= E_{t+j}\left[\begin{array}{c} \beta\frac{u'_{t+j+1}}{u'_{t+j}} \\ \times\left(\begin{array}{c} \left[\alpha\theta_{t+j+1}^{1-\alpha} K_{t+j+1}^{\alpha} - I_{t+j+1}\right] \\ +\frac{K_{t+j+2}}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)} \end{array}\right) \end{array}\right], \end{aligned} \tag{3.168}$$

and finally:

$$\begin{aligned} & E_{t+j}\left[\beta\frac{u'_{t+j+1}}{u'_{t+j}}\left[\alpha\theta_{t+j+1}^{1-\alpha}K_{t+j+1}^{\alpha} - I_{t+j+1}\right]\right] \\ = & \frac{K_{t+j+1}}{\phi'\left(\frac{I_{t+j}}{K_{t+j}}\right)} - E_{t+j}\left[\beta\frac{u'_{t+j+1}}{u'_{t+j}}\frac{K_{t+j+2}}{\phi'\left(\frac{I_{t+j+1}}{K_{t+j+1}}\right)}\right]. \end{aligned} \tag{3.169}$$

We use this result to simplify the expression for V_t from the Jermann (1998) definition of the equity return:

$$\begin{aligned}
V_t &= E_t\left[\sum_{i=1}^{\infty}\beta^i\frac{u'_{t+i}}{u'_t}d_{t+i}\right] \\
&= E_t\left[\beta\frac{u'_{t+1}}{u'_t}d_{t+1}+\beta^2\frac{u'_{t+2}}{u'_t}d_{t+2}+...\right] \\
&= E_t\left[\begin{array}{c}\beta\frac{u'_{t+1}}{u'_t}\left[\alpha\theta_{t+1}^{1-\alpha}K_{t+1}^{\alpha}-I_{t+1}\right] \\ +\beta\frac{u'_{t+1}}{u'_t}\left(\beta\frac{u'_{t+2}}{u'_{t+1}}\left[\alpha\theta_{t+2}^{1-\alpha}K_{t+2}^{\alpha}-I_{t+2}\right]\right)+...\end{array}\right] \\
&= E_t\left[\begin{array}{c}\frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)}-\beta\frac{u'_{t+1}}{u'_t}\frac{K_{t+2}}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \\ +\beta\frac{u'_{t+1}}{u'_t}\left(\frac{K_{t+2}}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)}-\beta\frac{u'_{t+2}}{u'_{t+1}}\frac{K_{t+3}}{\phi'\left(\frac{I_{t+2}}{K_{t+2}}\right)}\right)+...\end{array}\right] \\
&= E_t\left[\frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)}-\beta^2\frac{u'_{t+2}}{u'_t}\frac{K_{t+3}}{\phi'\left(\frac{I_{t+2}}{K_{t+2}}\right)}+...\right] \\
&= \frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)}. \qquad (3.170)
\end{aligned}$$

Using this expression for V_t, we rearrange the Jermann (1998) definition of the equity return:

$$\begin{aligned}
r_{t+1}^{EA} &= \frac{d_{t+1}+V_{t+1}}{V_t}-1 \\
&= \frac{\left(\alpha\theta_t^{1-\alpha}K_t^{\alpha}-I_t\right)+V_{t+1}}{V_t}-1 \\
&= \frac{\alpha\theta_{t+1}^{1-\alpha}K_{t+1}^{\alpha}-I_{t+1}+\frac{K_{t+2}}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)}}{\frac{K_{t+1}}{\phi'\left(\frac{I_t}{K_t}\right)}}-1 \\
&= \frac{\alpha\theta_{t+1}^{1-\alpha}K_{t+1}^{\alpha-1}-\frac{I_{t+1}}{K_{t+1}}+\frac{1}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)}\frac{K_{t+2}}{K_{t+1}}}{\frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}}-1 \\
&= \frac{\alpha\theta_{t+1}^{1-\alpha}K_{t+1}^{\alpha-1}+\left[\begin{array}{c}(1-\delta)+\phi\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ -\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}}\end{array}\right]\frac{1}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)}}{\frac{1}{\phi'\left(\frac{I_t}{K_t}\right)}}-1 \\
&= r_{t+1}^{EB}. \qquad (3.171)
\end{aligned}$$

This concludes the proof.

Table 3.22: **Appendix — Accuracy: Difference Param. Exp. and 'True' Exp.**
We use a very fine grid (100x100), and at each grid point compare the polynomial approximation of the Euler equation with the 'true' expectation, computed using Gauss-Hermite quadrature with 100 nodes. We report the maximum absolute percentage difference between parameterized expectation and true expectation in terms of the consumption good. ρ denotes the autoregressive coefficient of the productivity process, η the habit level, γ the coefficient of relative risk aversion, ξ the elasticity of the investment-capital ratio with respect to Tobin's q. '1st', '2nd', '3rd', '5th' denote the order of the Chebyshev polynomial used for the parameterization of the expectations operator.

Model IIa: $\rho = 1.00$, $\eta = 0.00$, $\xi = 13.0$, $\gamma = 2$	3rd	2nd	1st
Maximum Absolute Percentage Difference	<0.001%	<0.001%	0.004%
Model IIc: $\rho = 1.00$, $\eta = 0.00$, $\xi = 0.33$, $\gamma = 10$	3rd	2nd	1st
Maximum Absolute Percentage Difference	<0.001%	<0.001%	0.016%
Model IId: $\rho = 1.00$, $\eta = 0.60$, $\xi = 13.0$, $\gamma = 2$	3rd	2nd	1st
Maximum Absolute Percentage Difference	<0.001%	0.014%	**0.101%**
Model IIf: $\rho = 1.00$, $\eta = 0.60$, $\xi = 0.33$, $\gamma = 10$	5th	3rd	2nd
Maximum Absolute Percentage Difference	<0.001%	0.030%	**0.251%**

Accuracy

We try to assess the accuracy of solutions for the Jermann (1998) model obtained with low-order approximations. To that end we apply PEA and compare several properties of solutions obtained with different orders of the Chebyshev polynomial employed for the parameterization of the expectations operator. We first compare by means of a (formal) accuracy test, then we compare macroeconomic and asset pricing moments generated by solutions obtained with different orders of the polynomial.

Table 3.22 reports results from an accuracy test for selected models, where the accuracy of the parameterization of the expectations operator is evaluated and compared across solutions that employ different orders of the Chebyshev polynomial for the parameterization. The accuracy test uses a very fine grid, and at each grid point compares the polynomial approximation of the Euler equation with the 'true' expectation, computed using Gauss-Hermite quadrature with a large number of nodes. We report the maximum absolute percentage difference between parameterized expectation and true expectation in terms of the consumption good.

For Models IId and IIf a low-order approximation turns out quite inaccurate. For Model IIf we need to employ a $5th$ order polynomial approximation in order to obtain sufficiently accurate solutions. Note that Model IIf represents exactly the kind of parameterization (high habit levels combined with substantial capital adjustment costs) we are

Table 3.23: **Appendix — Accuracy: Macroeconomic Moments**

σ_x denotes the standard deviation of variable x. ρ denotes the autoregressive coefficient of the productivity process, η the habit level, γ the coeff. of relative risk aversion, ξ the elasticity of the investment-capital ratio with respect to Tobin's q. '1st', '2nd', '3rd', '5th' denote the order of the Chebyshev polynomial used for the parameterization of the expectations operator. All variables are logged and first-differenced prior to analysis. The statistic σ_Y is reported in percent. Results for all models are based on 200 replications of sample size 200 each.

	σ_Y			σ_C/σ_Y			σ_I/σ_Y		
Model IIa: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.00$, $\xi = 13.0$,	0.85	0.85	0.85	0.87	0.87	0.87	1.36	1.36	1.36
$\gamma = 2$									
Model IIc: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.00$, $\xi = 0.33$,	0.85	0.84	0.85	1.26	1.28	1.27	0.36	0.35	0.39
$\gamma = 10$									
Model IId: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.60$, $\xi = 13.0$,	0.85	0.84	0.84	0.49	0.49	0.49	**2.79**	**2.78**	**2.73**
$\gamma = 2$									
Model IIf: $\rho = 1.00$	5th	3rd	2nd	5th	3rd	2nd	5th	3rd	2nd
$\eta = 0.60$, $\xi = 0.33$,	0.85	0.85	0.85	0.73	0.74	0.76	**2.04**	**2.05**	**1.89**
$\gamma = 10$									

interested in when we work with the Jermann (1998) framework.

In the following three tables we try to assess whether numerical solution techniques that rely on low-order (in particular linear) approximations result in model solutions that generate different macroeconomic or asset pricing moments compared to more accurate numerical model solutions. To that end we compare moments of macroeconomic time series (Table 3.23) and asset pricing moments (Table 3.24 and Table 3.25) generated by model solutions obtained with PEA employing different orders of the Chebyshev polynomial used for the parameterization of the expectations operator.

Inspecting Table 3.23 we conclude that there can indeed be sizeable differences between macroeconomic moments generated by different solutions. Model solutions obtained with low-order polynomial approximations can display too low volatility of investment. This finding is in accordance with our comparison above of our (more accurate) numerical solution with the original Jermann (1994) loglinear-lognormal approximation. There we found that for Model IIf the volatility of our investment series was higher than the volatility reported by Jermann.

Table 3.24 and Table 3.25 assess the accuracy of different model solutions in terms

Table 3.24: **Appendix — Accuracy: Asset Pricing Moments I**
σ_x denotes the standard deviation of variable x. ρ denotes the autoregressive coefficient of the productivity process, η the habit level, γ the coefficient of relative risk aversion, ξ the elasticity of the investment-capital ratio with respect to Tobin's q. '1st', '2nd', '3rd', '5th' denote the order of the Chebyshev polynomial used for the parameterization of the expectations operator. Rates of return are annualized and reported in percentage terms. Results for all models are based on 200 replications of sample size 200 each.

	$\mathrm{E}[r_t^F]$			$\mathrm{E}[r_t^{EA} - r_t^F]$			$\mathrm{E}[r_t^{EB} - r_t^F]$		
Model IIa: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.00$, $\xi = 13.0$,	6.49	6.42	6.44	0.01	0.01	0.03	0.01	0.01	0.03
$\gamma = 2$									
Model IIc: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.00$, $\xi = 0.33$,	3.99	4.04	4.15	**0.45**	**0.46**	**0.20**	**0.45**	**0.45**	**0.02**
$\gamma = 10$									
Model IId: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.60$, $\xi = 13.0$,	6.47	6.45	6.40	**0.01**	**0.02**	**-0.16**	**0.01**	**0.02**	**-0.41**
$\gamma = 2$									
Model IIf: $\rho = 1.00$	5th	3rd	2nd	5th	3rd	2nd	5th	3rd	2nd
$\eta = 0.60$, $\xi = 0.33$,	2.27	2.18	2.47	**4.52**	**4.79**	**4.53**	**4.52**	**4.75**	**2.65**
$\gamma = 10$									

of their ability to generate asset pricing moments. We report both definitions of the equity return (see above for a discussion). Recall from above that the two definitions are equivalent. Therefore an accurate numerical solution should generate the same value of the equity return for both definitions.

We find that solutions obtained with low-order polynomial approximations generate economically significant inaccuracies in asset pricing moments, in particular for the kind of parameterizations (high habit levels combined with substantial capital adjustment costs) we are interested in when working with versions of the Jermann (1998) framework. For Model IIf even a 2*nd* order approximation generates a substantial difference between the two definitions of the equity return. Furthermore, and in accordance with our findings from the comparison with the results of the loglinear-lognormal approximation reported by Jermann (1994), solutions obtained with low-order approximations seem to have a tendency to generate equity returns which are too low.

From Table 3.25 we conclude that solutions obtained with low-order polynomials also fail in terms of generating accurate return volatilities.

Table 3.25: **Appendix — Accuracy: Asset Pricing Moments II**
σ_x denotes the standard deviation of variable x. ρ denotes the autoregressive coefficient of the productivity process, η the habit level, γ the coefficient of relative risk aversion, ξ the elasticity of the investment-capital ratio with respect to Tobin's q. '1st', '2nd', '3rd', '5th' denote the order of the Chebyshev polynomial used for the parameterization of the expectations operator. Rates of return are annualized and reported in percentage terms. Results for all models are based on 200 replications of sample size 200 each.

	σ_{r^F}			$\sigma_{r^{EA}}$			$\sigma_{r^{EB}}$		
Model IIa: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.00$, $\xi = 13.0$,	0.45	0.47	0.48	0.67	0.69	0.70	0.67	0.69	0.70
$\gamma = 2$									
Model IIc: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.00$, $\xi = 0.33$,	0.35	0.37	0.36	3.93	3.91	3.91	3.93	3.88	4.27
$\gamma = 10$									
Model IId: $\rho = 1.00$	3rd	2nd	1st	3rd	2nd	1st	3rd	2nd	1st
$\eta = 0.60$, $\xi = 13.0$,	0.39	0.38	0.38	0.96	1.05	0.95	0.96	0.95	0.92
$\gamma = 2$									
Model IIf: $\rho = 1.00$	5th	3rd	2nd	5th	3rd	2nd	5th	3rd	2nd
$\eta = 0.60$, $\xi = 0.33$,	**9.16**	**9.38**	**8.87**	**23.04**	**23.40**	**25.96**	**23.04**	**23.01**	**19.96**
$\gamma = 10$									

3.12.3 Conclusion

We conclude that we have largely succeeded in replicating two sets of results that have been reported in the literature so far for the Jermann (1998) framework: the results Jermann (1994) reports himself in a more extensive working paper version of his 1998 JME paper, and the results Boldrin, Christiano, Fisher (1999) report in a more extensive working paper version of their 2001 AER paper.

We find that relying on solutions obtained with low-order polynomial approximations (such as the loglinear-lognormal solution techniques Jermann (1994) and Jermann (1998) apply) can result in economically significant inaccuracies. In particular the economically more interesting parameterizations (high habit levels combined with substantial capital adjustment costs) of the Jermann framework require numerical solutions obtained with high-order polynomial approximations. In the examples we consider, low-order polynomial approximations generate time series for investment that are not volatile enough, too low equity premiums, as well as too little return volatility. We therefore recommend to apply modern non-linear solution techniques when working with (interesting) versions of the Jermann model.

Even though we have evaluated the accuracy of linear solution techniques by means of only one model-class example, we believe that we are in a position to recommend a generally heightened level of caution when working with linear solution techniques: for our example linear solutions fall well short of more accurate solutions. Researchers who choose, for this or that reason, to rely on linear solution techniques should therefore generally check the critical solutions so obtained for robustness by applying non-linear solution techniques.

Chapter 4

Long-Run Risk through Consumption Smoothing*

4.1 Introduction

Long-run consumption risk has recently been proposed as a mechanism for explaining important asset price moments such as the Sharpe ratio of equity market returns, the equity premium, the level and volatility of the risk-free rate, and the cross-section of stock returns (see Bansal and Yaron (2004), Hansen, Heaton, Li (2005), and Parker and Julliard (2005)). In this chapter, we demonstrate how long-run consumption risk arises *endogenously* in a standard production economy framework and how this additional risk factor can help these models to jointly explain the dynamic behavior of consumption, investment, and asset prices.[1]

We assume that consumers have Epstein and Zin (1989) preferences and dislike negative shocks to future economic growth prospects. Unlike the case of power utility preferences, where risk is only associated with the shock to *realized* consumption growth, investors in this economy also dislike negative shocks to *expected* consumption growth and consequentially demand a premium for holding assets correlated with this shock.[2] The latter source of risk has been labeled 'long-run risk' in previous literature (Bansal and Yaron (2004)). We show that even when the log technology process is a random walk,

*This chapter is joint work with Dr. Lars Lochstoer, London Business School.

[1]For extensive discussions of the poor performance of standard production economy models in terms of jointly explaining asset prices and macroeconomic moments, refer to Rouwenhorst (1995), Lettau and Uhlig (2000), Uhlig (2004), and Cochrane (2005), amongst others.

[2]Epstein and Zin (1989) preferences provide a convenient separation of the elasticity of intertemporal substitution (ψ) from the coefficient of relative risk aversion (γ), which are restricted to $\gamma = \frac{1}{\psi}$ in the power utility case. If $\gamma > \frac{1}{\psi}$, investors prefer early resolution of uncertainty (Duffie and Epstein (1992)) and, thus, are averse to shocks to expected consumption growth (Bansal and Yaron (2004)).

endogenous consumption smoothing increases the price of risk in the production economy model exactly because it increases the amount of long-run risk in the economy. The long-run risk, in turn, arises because consumption smoothing induces highly persistent time-variation in expected consumption growth rates.

Why does the consumer optimally choose a consumption process that leads to a high price of risk? The price of risk is related to risk across states, while the agent maximizes the level of expected utility which also is a function of substitution across time. The agent thus trades off the benefit of shifting consumption across time with the cost of higher volatility of marginal utility across states. Asset prices in the production economy reflect the optimal outcome of this trade-off. A higher elasticity of intertemporal substitution results in more substitution across time at the expense of additional risk across states, and thus a higher price of risk, higher Sharpe ratios, and a lower and less volatile risk-free rate.

In equilibrium, time-varying expected consumption growth turns out to be a small, but highly persistent, fraction of realized consumption growth. When the model is calibrated to fit standard macroeconomic moments, the *endogenous* expected consumption growth rate process is quantitatively very close to the exogenous processes that have been specified in the recent asset pricing literature (see, e.g., Bansal and Yaron (2004)). Note that this result is of particular interest since it is very difficult to empirically distinguish a small predictable component of consumption growth from i.i.d. consumption growth given the short sample of data we have available (see Harvey and Shepard (1990), and Hansen, Heaton, Li (2005), amongst others). Bansal and Yaron (2004), for instance, *calibrate* a process for consumption growth with a highly persistent trend component and demonstrate that their process can match a number of moments of aggregate consumption growth. In lieu of robust empirical evidence on this matter, the model presented in this chapter provides a theoretical justification for the previously proposed long-run risk dynamics of aggregate consumption growth based on a standard production economy setup. We conclude that simple consumption smoothing in an economy with i.i.d. technology growth naturally induces long-run consumption risk. Long-run consumption risk is therefore not an esoteric assumption for aggregate consumption dynamics. On the contrary, it is the natural assumption, given our standard theoretical models, for exogenous consumption growth processes in exchange economy models.

The persistence of the technology shocks is crucial for the asset pricing implications of long-run risk in the model. In short, permanent shocks lead to time-varying expected consumption growth that increases the price of risk in the economy, while transitory shocks lead to time-varying expected consumption growth that decreases the price of risk. The intuition for this is as follows. A permanent positive shock to productivity implies a permanently higher optimal level of capital. As a result, investors increase investment in order to build up a higher capital stock. High investment today implies low current consumption, but high future consumption. Thus, expected consumption growth is high. The higher investors' elasticity of intertemporal substitution, the more willing investors are to substitute consumption today for higher consumption in the future, and the stronger this effect is. Since agents in this economy dislike negative shocks to future economic growth prospects, both shocks to expected consumption growth and realized consumption growth are risk factors. Furthermore, the shocks are positively correlated and thus reinforce each other. In this case, endogenous consumption smoothing increases the price of risk in the economy. On the other hand, if shocks to technology are transitory, the endogenous long-run risk *decreases* the price of risk in the economy. A transitory, positive shock to technology implies that technology is expected to revert back to its long-run trend. Thus, if realized consumption growth is high, expected future long-run consumption growth is low as consumption also reverts to the long-run trend. The shock to expected future consumption growth is in this case negatively correlated with the shock to realized consumption growth, so the long-run risk component acts as a hedge for shocks to current consumption.[3] The overall price of risk in the economy is then decreasing in the magnitude of long-run risk.[4]

We evaluate the quantitative effects of transitory versus permanent technology shocks on aggregate macroeconomic and financial moments with calibrated versions of our model. If technology shocks are permanent, the model can match the high historical equity Sharpe ratio and the level and volatility of the risk-free rate with a low coefficient of relative risk aversion. The equity premium, however, is still too low in the baseline model, which is a well-known problem of standard production economy models (see, e.g., Jermann (1998)). We address this problem in Chapter 5 by calibrating the wage process of the model to the data. This brings the endogenous dividend process closer to the data, and as a result the equity premium as well as the equity return volatility increase by an order of magnitude to levels close to empirical values. Thus, the standard real business cycle model (without habit preferences) has the clear potential to jointly explain asset prices and macroeconomic time series.

[3]This description is intentionally loose to emphasize the intuition. The consumption response to transitory technology shocks is actually hump-shaped. Thus, a positive shock to realized consumption growth is followed by high expected consumption growth in the near term, but lower expected consumption growth in the long term - the negative correlation arises at lower frequencies. The low frequency effect dominates for standard values of the time-discounting parameter and leads to a lower price of risk unless the transitory shocks are extremely persistent.

[4]If, on the other hand, agents *like* long-run risk, endogenous long-run risk would increase the price of risk when technology shocks are transitory and decrease the price of risk when technology shocks are permanent.

The production economy model relates the aggregate level of technology (total factor productivity), consumption, and investment to the dynamic behavior of aggregate consumption growth. We use this link to derive new testable implications. Our model implies that the ratio of total factor productivity to consumption is a good proxy for the otherwise hard to measure expected consumption growth rate. We find empirical support for this by showing that the ratio of log total factor productivity to consumption forecasts future consumption growth over long horizons. We furthermore test a linear approximation of the model on the cross-section of stock returns and show, using the above proxy, that shocks to *expected* consumption growth are a priced risk factor that substantially improves the ability of the Consumption CAPM to explain the cross-section of stock returns.

We proceed as follows. We start by providing an overview of related literature. Then we give a preview of our results and develop and interpret the model. In Section 4.4 we calibrate and solve the model, demonstrate and interpret results, and provide intuition. In Section 4.5 we test some empirical implications of our model. Section 4.6 concludes.

4.2 Related Literature

This chapter is mainly related to three strands of the literature: the literature on consumption smoothing, the literature on long-run risk, and the literature that aims to jointly explain macroeconomic aggregates and asset prices.

It is well-known that (risk averse) agents want to smooth consumption over time. The permanent income hypothesis of Friedman (1957) is the classic reference. Hall (1978) is a seminal empirical investigation of this hypothesis. Hall shows that consumption should approximately follow a random walk and finds support for this in the data. The results in this chapter are consistent with Hall: we also find that consumption should be very close to a random walk. But, different from Hall, we emphasize that consumption growth has a small, highly persistent, time-varying component. Time-variation in expected growth rates, arising from consumption smoothing in production economy models, has been pointed out before. For example, den Haan (1995) demonstrates that the risk-free rate in production economy models is highly persistent (close to a random walk) even when the level of technology is i.i.d.

Bansal and Yaron (2004) show that a small, persistent component of consumption growth can have quantitatively important implications for asset prices if the representative agent has Epstein and Zin (1989) preferences. Bansal and Yaron term this source of risk 'long-run risk' and show that it can explain many aspects of asset prices. They specify exogenous processes for dividends and consumption with a slow-moving expected growth rate component and demonstrate that the ensuing long-run consumption risk greatly improves their model's performance with respect to asset prices without having to rely on, e.g., habit formation and the high relative risk aversion such preferences imply. We show that the process for consumption Bansal and Yaron assume as exogenous can be generated endogenously in a standard production economy model with Epstein and Zin preferences and the same preference parameters Bansal and Yaron use. Since it is very difficult to empirically distinguish between i.i.d. consumption growth and consumption growth with a very small, highly persistent time-varying component, this result is of particular importance for the Bansal and Yaron framework. Hansen, Heaton, Li (2005) emphasize this point in their study of the impact of long-run risk on the cross-section of stock returns. We also consider the implications for aggregate investment and output, which Bansal and Yaron abstract from, and we endogenize the aggregate dividend process. A recent paper that generates interesting consumption dynamics is due to Panageas and Yu (2006). These authors focus on the impact of major technological innovations and real options on consumption and the cross-section of asset prices. They assume, as do we, the technology process to be i.i.d. The major technological innovations, however, are assumed to occur at a very low frequency (about 20 years), and are shown to carry over into a small, highly persistent component of aggregate consumption. In that sense, Panageas and Yu assume, contrary to us, the low frequency of the predictable component of consumption growth. Moreover, time-variation in expected consumption growth (long-run risk) is not itself a priced risk factor in the Panageas and Yu model because the representative agent does not have Epstein and Zin preferences, but external ratio-habit as in Abel (1990). Finally, since investment in their model means paying a 'gardener' to plant a tree, their model does not have a clear separation of investment and labor income. Parker and Julliard (2005) find that the CCAPM can explain the cross-section of stock returns only when consumption growth is measured over longer horizons. This is consistent both with frictions to consumption adjustment and the presence of long-run risks.

There are quite a few papers before Bansal and Yaron (2004) that emphasize a small, highly persistent component in the pricing kernel. An early example is Backus and Zin (1994) who use the yield curve to reverse-engineer the stochastic discount factor and find that it has high conditional volatility and a persistent, time-varying conditional mean with very low volatility. These dynamics are also highlighted in Cochrane and Hansen (1992). This is exactly the dynamic behavior generated endogenously by the models considered in this chapter, and as such the chapter complements the above earlier studies. The use of Epstein and Zin (1989) preferences provides a justification for why the small, slow-moving time-variation in expected consumption growth generates high volatility of the stochastic discount factor. These preferences have become increasingly popular in the asset pricing literature. By providing a convenient separation between the coefficient of relative risk aversion and the elasticity of intertemporal substitution, they help to jointly explain asset market data and aggregate consumption dynamics. An early implementation is Epstein and Zin (1991), while Malloy, Moskowitz, Vissing-Jorgensen (2005) and Yogo (2006) are more recent, successful examples.

This chapter is also part of the strand of the asset pricing literature that tries to jointly explain asset prices and aggregate consumption. The first models are due to Jermann (1998) and Boldrin, Christiano, Fisher (1999, 2001). Both models rely on two complementary features that enable them to match basic asset pricing moments: (i) households have to be sufficiently sensitive to consumption risk - both Jermann as well as Boldrin, Christiano, Fisher use habit preferences, and (ii) households have to be prevented from using their investment decision to rid themselves of most of the consumption risk they might otherwise face - Jermann imposes capital adjustment costs on the economy, while Boldrin, Christiano, Fisher propose a two-sector economy and assume that capital cannot be reallocated across sectors in response to a technology shock. Both Jermann and Boldrin, Christiano, Fisher manage to match with their models most of the basic asset pricing moments, such as the equity premium and the equity return volatility, as well as basic moments of macroeconomic time series. However, the models suffer from the usual drawbacks of habit preferences: A much too volatile risk-free rate, and (implicitly) very high levels of relative risk aversion. The model we propose is better in the sense that we can match Sharpe ratios without the risk-free rate being counterfactually volatile and without excessive assumptions on preference parameters. On the other hand, in our model the level of the equity premium turns out too low because equity returns are not volatile enough. We address this shortcoming in Chapter 5.

Tallarini (2000) proposes a model that is closely related to our setup. In essence, Tallarini restricts himself to a special case of our model with the elasticity of intertemporal substitution fixed at unity and no capital adjustment costs. By increasing the coefficient of relative risk aversion to very high levels Tallarini manages to match some asset pricing moments such as the market price of risk (Sharpe ratio) as well as the level of the risk-free rate, while equity premium and return volatilities in his model remain basically zero. We differ from Tallarini in that our focus is on changing the elasticity of intertemporal substitution and the implications for the pricing and existence of long-run risk. Relative to the Tallarini setup we show that (moderate) capital adjustment costs together with an elasticity of intertemporal substitution greater than unity can dramatically improve the model's ability to match asset pricing moments. We confirm Tallarini's conclusion that the behavior of macroeconomic time series is driven by the elasticity of intertemporal substitution and largely unaffected by the coefficient of relative risk aversion. However, we do not confirm a 'separation theorem' of quantity and price dynamics. When we change the elasticity of substitution in our model, both macroeconomic quantity and asset price dynamics are greatly affected.

4.3 The Model

The model is a standard real business cycle model (Kydland and Prescott (1982), and Long and Plosser (1983)). There is a representative firm with Cobb Douglas production technology and capital adjustment costs, and a representative agent with Epstein and Zin (1989) preferences. Our objective is to demonstrate how standard production economy models endogenously give rise to long-run consumption risk and that this long-run risk can improve the performance of these models in replicating important moments of asset prices. To that end we keep both production technology as well as the process for total factor productivity as simple and as standard as possible. In particular, we do not assume any propagation mechanisms such as time-to-build. We describe the key components of our model in turn.

4.3.1 The Representative Agent

We assume a representative household whose preferences are in the recursive utility class of Epstein and Zin (1989):

$$U_t\left(C_t\right) = \left\{ \left(1-\beta\right) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}, \tag{4.1}$$

where E_t denotes the expectation operator, C_t denotes aggregate consumption, β the discount factor, and $\theta = \frac{1-\gamma}{1-1/\psi}$. Epstein and Zin show that γ governs the coefficient of relative risk aversion and ψ the elasticity of intertemporal substitution. These preferences thus have the useful property that it is possible to separate the agent's relative risk aversion from the elasticity of intertemporal substitution, unlike the standard power utility case where $\gamma = \frac{1}{\psi}$. If $\gamma \neq \frac{1}{\psi}$, the utility function is no longer time-additive and agents care about the temporal distribution of risk - a feature that is central to our analysis. We focus on the case where $\gamma > \frac{1}{\psi}$. In this case investors have a preference for early resolution of uncertainty. As a result, investors dislike fluctuations in future economic growth prospects (i.e., fluctuations in expected consumption growth).[5] We discuss this property and its implications in more detail below.

4.3.2 The Stochastic Discount Factor and Risk

The stochastic discount factor, M_{t+1}, is the ratio of the representative agent's marginal utility between today and tomorrow: $M_{t+1} = \frac{U'(C_{t+1})}{U'(C_t)}$. Using a recursive argument, Epstein and Zin (1989) show that:

$$\ln M_{t+1} \equiv m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\, r_{a,t+1}, \tag{4.2}$$

where $\Delta c_{t+1} \equiv \ln \frac{C_{t+1}}{C_t}$ and $r_{a,t+1} \equiv \ln \frac{A_{t+1}+C_{t+1}}{A_t}$ is the return on the total wealth portfolio with A_t denoting total wealth at time t.[6] If $\gamma = \frac{1}{\psi}$, $\theta = \frac{1-\gamma}{1-1/\psi} = 1$, and the stochastic discount factor collapses to the familiar power utility case, where shocks to realized consumption growth are the only source of risk in the economy. However, if $\gamma \neq \frac{1}{\psi}$, the return on the wealth portfolio appears as a risk factor. Persistent time-variation in expected consumption growth (the expected 'dividends' on the total wealth portfolio) induces higher volatility of asset returns (Barsky and DeLong (1993)). Thus, the return on any asset is a function of the dynamic behavior of realized *and* expected consumption growth (Bansal and Yaron (2004)). Depending on the sign of θ and the covariance between realized consumption growth and the return on the total wealth portfolio, the volatility of the stochastic discount factor (i.e., the price of risk in the economy) can be higher or lower relative to the benchmark power utility case. We show later how this covariance, and thus the amount of long-run risk due to endogenous consumption smoothing, changes with the persistence of the technology shock.

We focus on the case where investors prefer early resolution of uncertainty ($\gamma > \frac{1}{\psi}$) and therefore dislike fluctuations in future economic growth prospects. In the appendix, we explain in more detail how a preference for early resolution of uncertainty translates into aversion of time-varying expected consumption growth. We will refer to the volatility of expected future consumption growth rates as 'long-run risk'.

[5] See the appendix for a discussion of the difference and implications of a preference for early vs. late resolution of uncertainty.

[6] Note that our representative household's total wealth portfolio is composed of the present value of future labor income in addition to the value of the firm.

4.3.3 Technology

There is a representative firm with a Cobb Douglas production technology:

$$Y_t = (Z_t H_t)^{1-\alpha} K_t^{\alpha}, \tag{4.3}$$

where Y_t denotes output, K_t the firm's capital stock, H_t the number of hours worked, and Z_t denotes the (stochastic) level of aggregate technology. This constant returns to scale and decreasing marginal returns production technology is standard in the macroeconomic literature. We assume households to supply a constant amount of hours worked (following, e.g., Jermann (1998)) and normalize $H_t = 1$.[7] The productivity of capital and labor depends on the level of technology, Z_t, which is the exogenous driving process of the economy. We model log technology, $z \equiv \ln(Z)$, both as a random walk with drift, and as an AR(1) with a time trend:

$$z_{t+1} = \mu + z_t + \sigma_\varepsilon \varepsilon_{t+1}, \tag{4.4}$$

$$\varepsilon_t \sim N(0,1), \tag{4.5}$$

or:

$$z_{t+1} = \mu t + \varphi z_t + \sigma_\varepsilon \varepsilon_{t+1}, \tag{4.6}$$

$$\varepsilon_t \sim N(0,1) \;\; , \;\; |\varphi| < 1. \tag{4.7}$$

Thus, (4.4) implies that technology shocks are permanent whereas (4.6) implies that technology shocks are transitory. Both specifications are commonly used in the literature.[8] We discuss the two specifications separately.

4.3.4 Capital Accumulation and Adjustment Costs

The agent can shift consumption from today to tomorrow by investing in capital. The firm accumulates capital according to the following law of motion:

$$K_{t+1} = (1-\delta) K_t + \phi\left(\frac{I_t}{K_t}\right) K_t, \tag{4.8}$$

[7] Assuming that households supply a constant amount of labor amounts to assuming that households incur no disutility from working, which is the case for our representative agent.

[8] See, for example, Campbell (1994), who considers permanent and transitory, Cooley and Prescott (1995), transitory, Jermann (1998), permanent and transitory, Prescott (1986), permanent, Rouwenhorst (1995), permanent and transitory.

where I_t is aggregate investment and $\phi(\cdot)$ is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it step by step. We follow Jermann (1998) and Boldrin, Christiano, Fisher (1999) and specify:

$$\phi\left(I_t/K_t\right) = \frac{\alpha_1}{1-1/\xi}\left(\frac{I_t}{K_t}\right)^{(1-1/\xi)} + \alpha_2, \tag{4.9}$$

where α_1, α_2 are constants and $\alpha_1 > 0$.[9] The parameter ξ is the elasticity of the investment-capital ratio with respect to Tobin's q. If $\xi = \infty$ the capital accumulation equation reduces to the standard growth model accumulation equation without capital adjustment costs.

Each period the firm's output, Y_t, can be used for either consumption or investment. Investment increases the firm's capital stock, which in turn increases future output. High investment, however, means the agent must forego some consumption today, as can been from the accounting identity $C_t = Y_t - I_t$.

4.3.5 The Return of Investment and the Firm's Problem

Let $\Pi\left(K_t, Z_t; W_t\right)$ be the operating profit function of the firm, where W_t are equilibrium wages.[10] Firm dividends, D_t, equal operating profits minus investment:

$$D_t = \Pi\left(K_t, Z_t; W_t\right) - I_t. \tag{4.10}$$

The firm maximizes firm value. Let $M_{t,t+1}$ denote the stochastic discount factor. The firm's problem is then:

$$\max_{\{I_t, K_{t+1}, H_t\}_{t=0}^{T}} E_0 \sum_{t=0}^{T} M_{0,t} D_t, \tag{4.11}$$

where E_t denotes the expectation operator conditioning on information available up to time t. In the appendix, we demonstrate that the return on investment can be written as:

$$R_{t+1}^{I} = \phi'\left(\frac{I_t}{K_t}\right)\left(\begin{array}{c} \Pi_K\left(K_{t+1}, Z_{t+1}; W_{t+1}\right) \\ +\frac{1-\delta+\phi\left(\frac{I_{t+1}}{K_{t+1}}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} - \frac{I_{t+1}}{K_{t+1}} \end{array}\right). \tag{4.12}$$

[9] In particular, we set $\alpha_1 = \left(\exp(\mu) - 1 + \delta\right)^{1/\xi}$ and $\alpha_2 = \frac{1}{\xi-1}\left(1 - \delta - \exp(\mu)\right)$. It is straightforward to verify that $\phi(\frac{I_t}{K_t}) > 0$ and $\phi''(\frac{I_t}{K_t}) < 0$ for $\xi > 0$ and $\frac{I_t}{K_t} > 0$. Furthermore, $\phi(\frac{I}{K}) = \frac{I}{K}$ and $\phi'(\frac{I}{K}) = 1$, where $\frac{I}{K} = (\exp(\mu) - 1 + \delta)$ is the steady state investment-capital ratio.

[10] Wages are in this chapter assumed to be the marginal productivity of labor: $W_t = (1-\alpha)\, Y_t$. Since $C_t = D_t + W_t$, we have in this case that $D_t = \alpha Y_t - I_t$.

This return to the firm's investment is equivalent to the firm's equity return in equilibrium, $R^E_{t+1} \equiv \frac{D_{t+1}+P_{t+1}}{P_t}$, where P_t denotes the net present value of a claim on all future dividends (see, e.g., Restoy and Rockinger (1994), and Zhang (2005)).

4.4 Results

The model generates macroeconomic time series such as output, investment, and consumption, as well as aggregate asset prices. In the first part of our analysis, we present a baseline calibration of the model compared to calibrations based on power utility preferences. This illustrates how endogenous long-run risk can improve the ability of the standard production-based model to jointly explain macroeconomic time series and asset prices and motivates the subsequent analysis of the mechanisms within the model that generate long-run risk. We then investigate the model's implications for both macroeconomic time series and asset prices more generally. Our discussion is centered around different values of the elasticity of intertemporal substitution and the two specifications of technology (permanent vs. transitory). We solve the model numerically by means of the value function iteration algorithm. Please refer to the appendix for a detailed discussion of our solution technique.

4.4.1 Calibration

We report calibrated values of model parameters that are constant across models in Table 4.1. The capital share (α), the depreciation rate (δ), the mean technology growth rate (μ), and the persistence of the transitory technology shocks (φ), are set to standard values for quarterly parameterizations (see, e.g., Boldrin, Christiano, Fisher (2001)). We set the coefficient of relative risk aversion (γ) to 5 and the capital adjustment costs (ξ), which denotes the elasticity of the investment to capital ratio with respect to Tobin's q, to 22 across all models. The former is in the middle of the range of reasonable coefficients of relative risk aversion, as suggested by Mehra and Prescott (1985), while the latter implies only moderate capital adjustment costs. We choose this level of capital adjustment costs to match the macroeconomic moments with our Baseline Model. We vary the elasticity of intertemporal substitution (ψ), the rate of time-discounting preference (β), and also the volatility of shocks to technology (σ_z), across models. We will discuss the choice of specific parameter values for these variables as we go along.

Table 4.1: **Calibration**
Calibrated values of parameters that are constant across models.

Quarterly Model Calibration		
Parameter	Description	Value
α	Elasticity of capital	0.34
δ	Depreciation rate of capital	0.021
γ	Coefficient of relative risk aversion	5
μ	Mean technology growth rate	0.4%
φ	Persistence of AR(1) technology	0.90

4.4.2 Results of the Baseline Model

Table 4.2 shows the Baseline calibration of our model. Panel A shows the moments the model is calibrated to match. Technology shocks are permanent, and the volatility of technology shocks (σ_z) is calibrated to match the volatility of output. The time preference parameter (β) is set to 0.998 in order to match the level of the risk-free rate, the elasticity of intertemporal substitution (ψ) is set to 1.5 to match the volatility of consumption growth, while the risk aversion (γ) is set to 5 to match the Sharpe ratio of equity returns. The model matches all of these moments simultaneously, which is a significant achievement for this class of models. As highlighted by, amongst others, Rouwenhorst (1995), Jermann (1998), and Boldrin, Christiano, Fisher (2001), the standard production economy model with power utility preferences cannot jointly explain the dynamic behavior of macroeconomic variables and asset prices.[11] For comparison, Table 4.2 also shows two calibrations of the power utility model, which restricts $\gamma = \frac{1}{\psi}$. In Power Utility Model I, we use the same EIS parameter, $\psi = 1.5$, which implies $\gamma = 2/3$. This model can match the volatility of output and consumption, but generates a Sharpe ratio that is an order of magnitude too low compared to the empirical value. Power Utility Model II uses the same coefficient of relative risk aversion as in the Baseline Model, which implies $\psi = 1/5$. The low EIS leads to a too high level of the risk-free rate. The Sharpe ratio is now 0.26 versus 0.33 in the data. However, the higher Sharpe ratio is achieved with a consumption growth volatility that is twice as high as both in the data and in the Baseline Model. But why is it that the Baseline Model yields a higher Sharpe ratio with the same coefficient of risk aversion and only half the consumption volatility? When $\gamma > \frac{1}{\psi}$, consumers have a preference for early resolution of uncertainty, which creates a role for long-run risk (see appendix). The dynamic behavior of the optimal, endogenous consumption choice gives rise to such long-run consumption risk, which is the reason the equity Sharpe ratio is

[11] Boldrin, Christiano, Fisher (2001): '[RBC models] have been notoriously unsuccessful in accounting for the joint behavior of asset prices and consumption.'. See also Cochrane (2005).

Table 4.2: **Asset Pricing Moments — Adjusted Model versus a Standard Model**
This table reports annual asset pricing moments for two calibrations of the standard stochastic growth model where the representative agent has power utility preferences, as well as the Baseline Model presented in this chapter. All models have permanent technology shocks. The parameters are the same across the models ($\beta = 0.998$ and $\xi = 22$), except the coefficient of relative risk aversion (γ) and the elasticity of intertemporal substitution (ψ). The volatility of shocks to technology, σ_z, is calibrated so that the models fit the volatility of output growth. The equity returns in both models are for an unlevered claim on the endogenous, aggregate dividends. The equity premium due to short-run risk is defined as $\gamma cov(\Delta c_t, R_t^E - R_{f,t})$.

Statistic	U.S. Data 1929 – 1998	Power Utility Model I $\psi = 1.5, \gamma = \frac{1}{1.5}$	Power Utility Model II $\psi = \frac{1}{5}, \gamma = 5$	Baseline Model $\psi = 1.5, \gamma = 5$
Panel A - Matched Moments				
Volatility of Consumption Growth				
$\sigma[\Delta c]$ (%)	2.72	2.72	5.48	2.72
Relative Volatility of Consumption and Output (GDP)				
$\sigma[\Delta c] / \sigma[\Delta y]$	0.52	0.52	1.05	0.52
Level of Risk-free Rate				
$E[R_f]$ (%)	0.86	1.83	5.07	0.86
Sharpe ratio of Equity Returns				
$E\left[R^E - R_f\right] / \sigma\left[R^E - R_f\right]$	0.33	0.02	0.26	0.33
Panel B - Other Moments				
Volatility of the Risk-free Rate				
$\sigma[R_f]$ (%)	0.97	0.45	1.00	0.43
Equity Returns				
$E\left[R^E - R_f\right]$ (%)	6.33	0.01	0.10	0.19
$\sigma\left[R^E - R_f\right]$ (%)	19.42	0.61	0.38	0.57
Decomposing the Equity Premium (%)				
	Short-Run Risk	0.01 (100%)	0.10 (100%)	0.07 (39%)
	Long-Run Risk	0.00 (0%)	0.00 (0%)	0.12 (61%)

higher in the Baseline Model although the volatility of consumption growth is lower.

Panel B shows financial moments the Baseline Model was not calibrated to fit. The risk-free rate has low volatility, as in the data, despite the high price of risk. This feature is an important improvement over production economy models with habit preferences, which can match the high price of risk, but generate much too volatile risk-free rates (see, e.g., Jermann (1998), and Boldrin, Christiano, Fisher (2001)). Since the reciprocal of the risk-free rate is the conditional expectation of the stochastic discount factor, mismatching the risk-free rate volatility implies that the dynamic behavior of the stochastic discount factor is also mismatched. The equity claim is defined as the (unlevered) claim to aggregate dividends. The equity return volatility is quite low in all the models, but the equity premium in the Baseline Model is more than an order of magnitude higher than for Power Utility Model I and twice as high as for Power Utility Model II. As was the case for the Sharpe ratio, this is both due to a higher coefficient of relative risk aversion and the presence of long-run risk. In fact, Panel B reports that 61% of the risk premium in the Baseline Model is due to long-run risk, where short-run risk is defined as $\gamma \times cov\left(R_t^E - R_{f,t}, \Delta c_t\right)$.

At 0.19% per year, however, the equity premium is still more than an order of magnitude too low compared to historical values. This is typical for production economies, as the equity claim is not volatile enough. One standard remedy for this problem is to assume a stochastic depreciation rate (see, e.g., Storesletten, Telmer, Yaron (2005), and Gomes and Michaelides (2005)). In Chapter 5, we propose an alternative remedy. We show that if we calibrate the wage process to the data, the ensuing dividend process is also closer to the data. As a result, the risk premium increases by an order of magnitude to levels close to historical values.

In the following, we explain how endogenous long-run risk arises in the model, how it affects asset prices and macroeconomic moments, and, in particular, the dynamic behavior of consumption.

4.4.3 The Endogenous Consumption Choice and The Price of Risk

Before we report moments from different calibrations of the model, it is useful to provide general intuition for the endogenous consumption choice and how it is related to the persistence of the technology shocks and the price of risk in the economy. From the stochastic discount factor (see (4.2)), we can see that there are two sources of risk in this economy. The first is the shock to realized consumption growth, which is the usual risk factor in the Consumption CAPM. The second risk factor is the shock to the return on

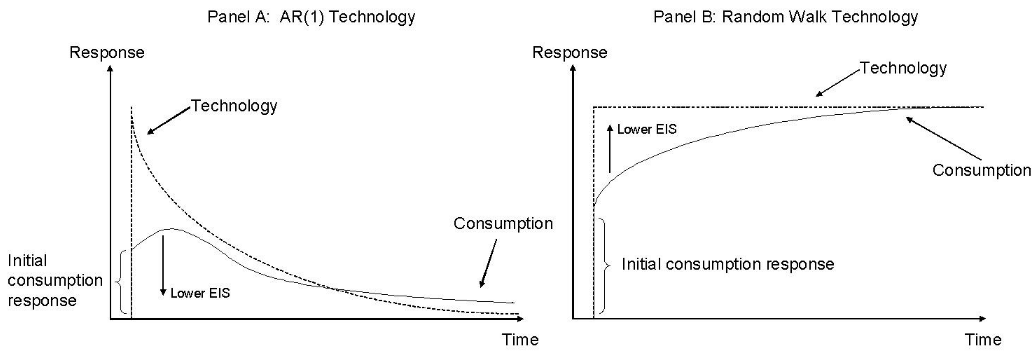

Figure 4.1: **Transitory and Permanent Shocks — IR for Tech. and Cons.** Panel A shows the impulse response of technology and consumption to a transitory technology shock. Panel B shows the impulse response of technology and consumption to a permanent technology shock. The arrows show the direction in which the optimal consumption response changes if the desire for a smoother consumption path increases (i.e., the elasticity of intertemporal substitution decreases).

total wealth. Total wealth is the sum of human and financial capital, and the dividend to total wealth is consumption. Assume for the moment that future expected consumption growth and returns are constant. Total wealth, A_t, is then given by:

$$A_t = \frac{C_t}{r_a - g_c}, \tag{4.13}$$

where r_a is the expected return to wealth and g_c is expected consumption growth: total wealth is a function of both current and future expected consumption. Therefore, shocks to both realized and expected consumption growth translate into shocks to the realized return to wealth. This example illustrates how we can think of shocks to expected consumption growth as the second risk factor instead of the return to wealth.[12] Understanding the dynamic behavior of consumption growth is thus necessary in order to understand the asset pricing properties of the production economy model with Epstein and Zin (1989) preferences. In the following, we consider the consumption response to both transitory and permanent technology shocks.[13]

[12]Following Bansal and Yaron (2004), we explicitly show this in the appendix through a log-linear approximation of the return to wealth.

[13]We make a strong distinction between transitory and permanent shocks in this section to provide clear intuition. As $\varphi \to 1$, the transitory shock specification (4.6) approaches the permanent shock specification (4.4). The dynamics of the model are in that case very similar for both specifications, so there is actually no discontinuity at $\varphi = 1$ in terms of the model's asset pricing implications. However, the transitory shocks need to be extremely persistent for the transitory and permanent cases to be similar. At $\varphi = 0.9$, which is the case we consider in our calibration, the dynamic behavior of the model with permanent shocks is very different from the model with transitory shocks. The reader could therefore think of 'transitory vs. permanent' shocks as 'not extremely persistent vs. extremely persistent' shocks.

Transitory Technology Shocks

Panel A of Figure 4.1 shows the impulse response functions of technology and consumption to a transitory technology shock. Agents in this economy want to take advantage of the temporary increase in the productivity of capital due to the temporarily high level of technology. To do so, they invest immediately in capital at the expense of current consumption. As a result, the consumption response is hump-shaped. This figure illustrates how time-varying expected consumption growth arises endogenously in the production economy model: a *positive* shock to realized consumption growth (the initial consumption response) is associated with positive short-run expected consumption growth, but *negative* long-run expected consumption growth as consumption reverts back to the steady state. Thus, the shock to long-run expected consumption growth has the opposite sign of the shock to realized consumption growth, implying that shocks to realized consumption growth are hedged by shocks to the expected long-run consumption growth rate. As a consequence, long-run risk *decreases* the price of risk in the economy with transitory technology shocks.

Permanent Technology Shocks

With permanent technology shocks, long-run consumption risk has the opposite effect. Panel B of Figure 4.1 shows the impulse response functions of technology and consumption to a permanent technology shock. Technology adjusts immediately to the new steady state, and the permanently higher productivity of capital implies that the optimal long-run levels of both capital and consumption are also higher. Agents invest immediately in order to build up capital at the expense of current consumption, and consumption gradually increases towards the new steady state after the initial shock. Thus, a positive shock to realized consumption growth (the initial consumption response) is associated with *positive* long-run expected consumption growth. In this case, long-run risk increases the price of risk in the economy because a positive technology shock induces positive shocks to *both* realized consumption growth and long-run expected consumption growth.

The Elasticity of Intertemporal Substitution

The elasticity of intertemporal substitution (EIS) is an important determinant of the dynamic behavior of consumption growth. A low EIS translates into a strong desire for intertemporally smooth consumption paths. In other words, agents strive to minimize the difference between their level of consumption today (after the shock) and future expected consumption levels. The arrows in Figure 4.1 indicate the directions in which the initial optimal consumption responses change if the desire for a smoother consumption path increases. As the elasticity of intertemporal substitution decreases, agents desire a 'flatter' response curve. From the figure, we can conjecture that a lower EIS decreases the volatility of expected future consumption growth. A high EIS, on the other hand, implies a higher willingness to substitute consumption today for higher future consumption levels. Therefore, the higher the EIS, the higher the volatility of expected consumption growth and the higher the levels of long-run risk in the economy. A high EIS thus decreases the price of risk if technology shocks are transitory, but increases the price of risk if technology shocks are permanent.

Capital Adjustment Costs

Capital adjustment costs (CAC) make it more costly for firms to adjust investment. Therefore, higher CAC induce lower investment volatility. We can therefore use CAC to, as far as possible, match the empirical relative volatilities of consumption, investment, and output with each model.

4.4.4 Results of Calibrated Models

We confirm the intuition from the impulse responses in Figure 4.1 by reporting relevant macroeconomic moments and the equilibrium price of risk for different model calibrations. In particular, Table 4.3 reports relevant macroeconomic moments and consumption dynamics for models with either transitory or permanent technology shocks and different levels of the elasticity of intertemporal substitution ($\psi = 1/\gamma, 0.5, 1.5$). We match the U.S. output volatility over the period 1929 to 1998 with all models by setting the volatility of the technology shocks, σ_ε, appropriately. We re-calibrate the discount factor (β) for each model so as to jointly match the values for (C/Y), (I/Y), (D/Y), that is aggregate average consumption, investment, and dividends relative to output, with each model. This is quite important, both since these are first-order moments and because we compare the volatility of growth rates across models. Capital adjustment costs (ξ) are the same across models and the value of ξ is set in order to match the relative volatility of consumption to output with the Baseline Model. The coefficient of relative risk aversion (γ) is constant across models. We show in the appendix, confirming Tallarini (2000), that the level of γ has only second-order effects on the time series behavior of the macroeconomic variables.

Table 4.3: **Macroeconomic Moments and Consumption Dynamics**

This table reports relevant macroeconomic moments and consumption dynamics for models with either transitory ($\varphi = 0.90$) or permanent technology shocks and different levels of the elasticity of intertemporal substitution (ψ). The coefficient of relative risk aversion (γ) is 5 across all models. We re-calibrate the discount factor (β) for Models 1 to 5 so as to jointly match the values for (C/Y), (I/Y), (D/Y), with each model. In the Baseline Model, $\beta = 0.998$, allowing the model to match the level of the risk-free rate. Capital adjustment costs (ξ) are 22 in order to match the relative volatility of consumption to output with the Baseline Model. We estimate the following process for the consumption dynamics: $\Delta c_{t+1} = \mu + x_t + \eta_{t+1}$, $x_{t+1} = \rho x_t + e_{t+1}$. $\Delta x = \log(X_t) - \log(X_{t-1})$, and $\sigma[X]$ denotes the standard deviation of variable X. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis. The sample is the same as in Bansal and Yaron (2004).

		Model 1	Model 2	Model 3	Model 4	Model 5	**Baseline**
		Transitory Shocks			Permanent Shocks		
		$z_{t+1} = \mu t + \varphi z_t + \sigma_\varepsilon \varepsilon_{t+1}$			$z_{t+1} = \mu + z_t + \sigma_\varepsilon \varepsilon_{t+1}$		
Statistic		$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$
Panel A: Macroeconomic Moments (Quarterly)							
	U.S. Data 1929-1998						
$\sigma[\Delta y]$ (%)	2.62	2.62	2.62	2.62	2.62	2.62	2.62
$\sigma[\Delta c]/\sigma[\Delta y]$	0.52	0.29	0.34	0.40	1.01	0.84	0.52
$\sigma[\Delta i]/\sigma[\Delta y]$	3.32	3.97	3.68	3.49	0.97	1.65	1.90
Panel B: Consumption Dynamics: $\Delta c_{t+1} = \mu + x_t + \eta_{t+1}, \quad x_{t+1} = \rho x_t + e_{t+1}.$							
	Bansal, Yaron Calibration						
$\sigma[\Delta c]$ (%)	1.360	0.760	0.891	1.048	2.646	2.201	1.362
ρ	0.938	0.992	0.982	0.954	0.983	0.972	0.969
$\sigma[x]$ (%)	0.172	0.067	0.097	0.158	0.116	0.205	0.329
Panel C: The Price of Risk and the Sharpe ratio of the Equity Return (Annual)							
$\sigma[M]/E[M]$	n/a	0.074	0.062	0.054	0.255	0.270	0.337
$SR[R^E]$	0.33	0.069	0.059	0.051	0.253	0.266	0.331

The Volatility of Realized Consumption Growth

The volatility of realized consumption growth is the standard risk factor in consumption-based asset pricing models, where a higher volatility of consumption growth leads to a higher price of risk. This is not necessarily true in this model.

In Models 1 to 3, technology shocks are transitory and the EIS is increasing across models from the power utility case ($\psi = 1/\gamma = 0.2$, Model 1) to 1.5 (Model 3). Panel A of Table 4.3 shows that consumption volatility is increasing with EIS. Agents with higher EIS take advantage of a temporarily high technology level by consuming relatively more today and less in the future as technology reverts back to its long-run trend. As a result, the level of risk associated with shocks to realized consumption growth is increasing with the EIS in the model with transitory shocks. The overall price of risk in the economy, however, is decreasing in the EIS (Panel C in Table 4.3), due to long-run risks.

In Models $4, 5$, and the Baseline Model, technology shocks are permanent. Here the consumption growth volatility is decreasing with the EIS. Consider a positive shock to technology. Since the shock is permanent, agents with a high EIS want to increase the capital stock to its new optimal level as quickly as possible for consumption to grow faster towards its new, permanently higher level. To that end they need to invest more today, implying a smaller initial consumption response. Thus, the level of risk associated with shocks to realized consumption growth is decreasing with the EIS in the model with permanent shocks. With respect to this standard risk factor, a higher EIS therefore reduces risk in the permanent shock model. Nevertheless, Panel C of Table 4.3 shows that the price of risk in this case is, again, due to long-run risks increasing in the EIS.

Thus, the models imply a surprising *inverse* relation between the volatility of realized consumption growth and the price of risk. The magnitude of this relation is large. In the transitory shock models, the relative consumption volatility increases by 40% from Model 1 to Model 3, while the price of risk decreases by 30%. In the permanent shock models, the relative consumption volatility decreases by 50% from Model 4 to the Baseline Model, while the price of risk increases by 30%.

The Volatility of Expected Consumption Growth (Long-run Risk)

The above results are due to the varying degree and effect of long-run risk in the models because shocks to *expected* consumption growth are also a risk factor. In Panel B of Table 4.3, we report both the volatility of consumption growth, the volatility of conditional expected consumption growth (x_t), and the latter's first order autocorrelation (ρ). To a first order, these statistics summarize the magnitude and nature of long-run risk in the models.[14] The implied system for consumption growth is:

$$\Delta c_{t+1} = \mu + x_t + \eta_{t+1}, \tag{4.14}$$

$$x_{t+1} = \rho x_t + e_{t+1}, \tag{4.15}$$

$$E_t\left[\eta_{t+1}\right] = E_t\left[e_{t+1}\right] = 0, \tag{4.16}$$

which is similar to that assumed in the exchange economy of Bansal and Yaron (2004). For comparison, Panel B also gives the parameters that Bansal and Yaron use in their calibration. The relative magnitudes of the volatility of realized and expected consumption growth show that the time-varying growth component is very small. The implied average R^2 across models is around $1 - 2\%$. Note however that the persistence of the expected consumption growth rate (ρ) is very high, which is important if risk associated with a small time-varying expected consumption growth rate component is to have quantitatively interesting asset pricing implications. As expected from the discussion in Section 4.4.3, the volatility of expected consumption growth, $\sigma\left[x\right]$, is increasing in the elasticity of intertemporal substitution. Whether shocks to expected consumption growth increase or decrease the price of risk in the economy, however, depends on their effect on the return to total wealth and its correlation with realized consumption growth. The negative correlation between shocks to realized and expected consumption growth induced by transitory technology shocks yields a price of risk that is decreasing in the amount of long-run risk. For permanent technology shocks, this correlation is positive and the price of risk in the economy is increasing in the amount of long-run risk.

4.4.5 Asset Pricing Implications

Table 4.4 presents key financial moments. We calibrate the volatility of aggregate consumption growth to its empirical value for each model we report in Table 4.4 by adjusting the volatility of technology growth. Keeping the volatility of aggregate consumption growth constant across models allows us to compare asset prices while holding this traditional measure of risk constant. This approach highlights the impact of long-run risk, with the caveat that the volatility of output is mismatched for all models but the Baseline Model. We use the coefficient of relative risk aversion (γ), the discount factor (β), and adjustment costs (ξ) to respectively match the equity Sharpe ratio, the level of the risk-free rate, as well as the relative volatility of consumption to output with the Baseline

[14] In the appendix, we show that these moments indeed capture most of the dynamics of consumption growth generated by the models and as such are meaningful moments to consider. There is some heteroscedasticity in both shocks to expected and realized consumption growth, but these effects are second order.

Table 4.4: **Financial Moments**

This table reports relevant financial moments and consumption dynamics for models with either transitory ($\varphi = 0.90$) or permanent technology shocks and different levels of the elasticity of intertemporal substitution (ψ). The coefficient of relative risk aversion (γ) is 5 across all models, while the discount factor (β) is 0.998 and capital adjustment costs (ξ) are 22, in order to match the equity Sharpe ratio, the level of the risk-free rate, and the relative volatility of consumption to output with the Baseline Model. We re-calibrate σ_ε in order to match the volatility of consumption growth with each model. We estimate the following process for the consumption dynamics: $\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}$, $x_{t+1} = \rho x_t + \sigma_e e_{t+1}$. $\sigma[X]$ denotes the standard deviation of variable X. The data are taken from Bansal and Yaron (2004) who use annual U.S. data from 1929 to 1998.

		Model 7	Model 8	Model 9	Model 10	Model 11	**Baseline**
		Transitory Shocks			Permanent Shocks		
		$z_{t+1} = \mu t + \varphi z_t + \sigma_\varepsilon \varepsilon_{t+1}$			$z_{t+1} = \mu + z_t + \sigma_\varepsilon \varepsilon_{t+1}$		
Statistic	Data	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$
Panel A: The Price of Risk and Consumption Dynamics (Annual)							
$\sigma[\Delta c]$ (%)	2.72	2.72	2.72	2.72	2.72	2.72	2.72
$\sigma[x]$ (%)	n/a	0.12	0.14	0.19	0.06	0.10	0.33
$\sigma[M]/E[M]$	n/a	0.13	0.08	0.02	0.13	0.18	0.34
$SR[R^A]$	n/a	0.12	0.08	0.02	0.13	0.17	0.34
$SR[R^E]$	0.33	0.11	0.08	0.02	0.13	0.17	0.33
Panel B: Financial Moments (Annual)							
$E[R_f]$ (%)	0.86	7.63	3.70	1.80	7.68	3.45	0.86
$\sigma[R_f]$ (%)	0.97	1.14	0.56	0.25	0.58	0.41	0.43
The Consumption Claim							
$E[R^A - R_f]$ (%)	n/a	0.78	0.36	0.05	0.03	0.19	1.46
$\sigma[R^A - R_f]$ (%)	n/a	6.74	4.71	1.85	0.17	1.14	4.35
The Dividend Claim							
$E[R^E - R_f]$ (%)	6.33	0.25	0.13	0.03	0.04	0.05	0.19
$\sigma[R^E - R_f]$ (%)	19.42	2.37	1.78	1.29	0.22	0.28	0.58

Model, which is the model that best fits the macroeconomic moments. We keep these parameters (γ, β, ξ) constant across models in order to examine the effect of the EIS and technology specification on endogenous long-run risk and asset prices. The empirical moments are taken from Bansal and Yaron (2004), who use annual U.S. data from 1929 to 1998.

The Price of Risk and Sharpe Ratios

Panel A of Table 4.4 reconfirms that long-run risk, as measured by the volatility of the expected consumption growth rate, $\sigma[x]$, increases substantially for both the case of permanent and transitory shocks as we increase the EIS, and that the price of risk $(\sigma[M]/E[M])$, as well as the Sharpe ratio of the return on wealth and the return on equity, are *decreasing* with the EIS for transitory shocks and *increasing* with the EIS for permanent shocks. In the case of transitory shocks, the Sharpe ratio of equity returns drops more than five-fold from 0.11 to 0.02 as the EIS increases from $0.2\,(= 1/\gamma)$ to 1.5. In the case of permanent shocks, the Sharpe ratio of equity returns increases from 0.13 to 0.33 as the EIS increases from 0.2 to 1.5. Comparing Model 10 (power utility) with the Baseline Model $(\gamma > \frac{1}{\psi})$, endogenous long-run risk combined with a preference for early resolution of uncertainty almost triples the price of risk in the economy.

The Risk-free Rate

Panel B of Table 4.4 shows that the risk-free rate is decreasing in the EIS, as expected. A higher EIS increases the intertemporal substitution effect (see (4.52)), and the volatility of the risk-free rate is low. The time-variation in expected consumption growth rates does not induce too volatile risk-free rates, because the growth shocks are very persistent and not very volatile. This is an improvement over habit formation models like in Jermann (1998) or Boldrin, Christiano, Fisher (2001), where time-variation in the state variable 'surplus consumption' induces much too volatile risk-free rates when the models are calibrated to match empirical proxies for the price of risk (e.g., the equity Sharpe ratio).

Since the risk-free rate is the reciprocal of the conditional expected value of the stochastic discount factor, a misspecified risk-free rate implies a misspecified stochastic discount factor. Therefore, it is important to note that the production economy model we consider in this chapter, in conjunction with permanent technology shocks, can match both the level and volatility of the risk-free rate, as well as empirical measures of the price of risk.

The Level and the Volatility of Excess Returns

For the models with permanent shocks, the average excess returns on both total wealth and equities are strongly increasing in the EIS (Panel B of Table 4.4). From the power utility model with $\gamma = 5$, $\psi = \frac{1}{\gamma}$ to the case with $\gamma = 5$, $\psi = 1.5$ the equity premium increases six-fold from 0.04% to 0.19%, and equity return volatility increases from 0.22% to 0.58%. While these are substantial relative increases, the equity premium is still roughly an order of magnitude too low due to the low volatility of equity returns. Note, however, that the premium and volatility of returns to the wealth portfolio are much more sensitive to increases in the EIS. The return premium on the consumption claim increases almost two orders of magnitude, from 0.03% to 1.46%.[15]

The production economy model generates dividends endogenously and the endogenous dividend process differs from the endogenous consumption process along important dimensions: while equity dividends are given by $D_t^E = \alpha Y_t - I_t$, dividends to the wealth portfolio are given by $D_t^A = C_t = Y_t - I_t$. Consider a permanent, positive shock to technology. If investors have higher EIS, this results in higher investment volatility and higher expected future consumption growth. Both equity dividends as well as dividends to the wealth portfolio now respond less to a positive shock. However, equity dividends are much more sensitive to this effect, and may even decrease in response to a shock, implying a negative correlation between dividend growth and expected consumption growth. So, while the price of the equity claim increases, the current dividend decreases, which dampens the total equity return response to technology shocks. The result is that the equity return volatility, and thus the equity premium, increase by less with the EIS relative to the total asset return.

In an exchange economy, it is possible to exploit the fact that the claims to total wealth and equity have different dividend processes (i.e., consumption and dividends), and use this as a degree of freedom to fit the asset pricing moments. Bansal and Yaron (2004), for instance, exogenously specify the dividend process such that expected dividend growth is very sensitive to shocks to expected consumption growth, which makes the equity claim risky and volatile. That way they are able to fit the equity volatility, and thus the equity premium, with roughly the same (exogenous) consumption process and preference parameters as in the Baseline Model. The production economy model, on the other hand, restricts the joint dynamic behavior of aggregate consumption and dividends. Thus, while the general equilibrium framework considered so far in this chap-

[15] Many papers define dividends as a levered claim to the consumption stream, in order to fit the volatility of dividend growth, the high equity return volatility and the equity risk premium. With a leverage factor of about 3 the 'equity' return premium for the Baseline Model would be around 4% with a return volatility of about 12%.

ter provides a theoretical justification for a consumption process with long-run risk, it imposes constraints on dividends that are unfavorable in terms of matching the volatility of equity returns. We take a closer look at those constraints in Chapter 5.

4.4.6 The Case of $\gamma < \frac{1}{\psi}$

In the previous discussions our focus was on the case of $\gamma > \frac{1}{\psi}$, where agents prefer early resolution of uncertainty and dislike fluctuations in expected consumption growth rates. With transitory shocks and $\gamma > \frac{1}{\psi}$, this second risk factor acts as a hedge for shocks to realized consumption growth and therefore reduces the price of risk. This raises the possibility that if agents *like* fluctuations in expected consumption growth rates, that is when $\gamma < \frac{1}{\psi}$, consumption smoothing increases the price of risk when technology shocks are transitory. The question is whether this channel can give rise to long-run risks that help in explaining asset prices with a *low* elasticity of intertemporal substitution. The short answer is yes. However, a low EIS unfortunately gives rise to a risk-free rate puzzle (Weil (1989)). A detailed analysis of this case is given in the appendix.

4.5 Expected Consumption Growth and the Cross-Section of Stock Returns

We test key predictions of the model for the time series of technology (total factor productivity) and consumption growth, as well as for the cross-section of stock returns. In particular, we test whether proxies for expected consumption growth suggested by our model actually forecast long-horizon consumption growth or not, whether shocks to expected consumption growth are a priced risk factor or not, and whether the price of risk is positive or negative.

In a recent paper, Bansal, Kiku, Yaron (2006) test the exchange economy version of the model in this chapter and show that consumption growth is indeed predictable using forecasting variables such as lagged consumption growth, the default spread, and the market price-dividend ratio. Furthermore, they show, using the cross-section of stock returns, that shocks to expected consumption growth are indeed a positively priced risk factor. We therefore confine our empirical analysis to test restrictions that are particular to the production economy. We consider an instrument Bansal, Kiku, and Yaron do not use and which is related to the level of technology - the driving process of the production economy model.

The consumption data and data on Total Factor Productivity (TFP; the equivalent

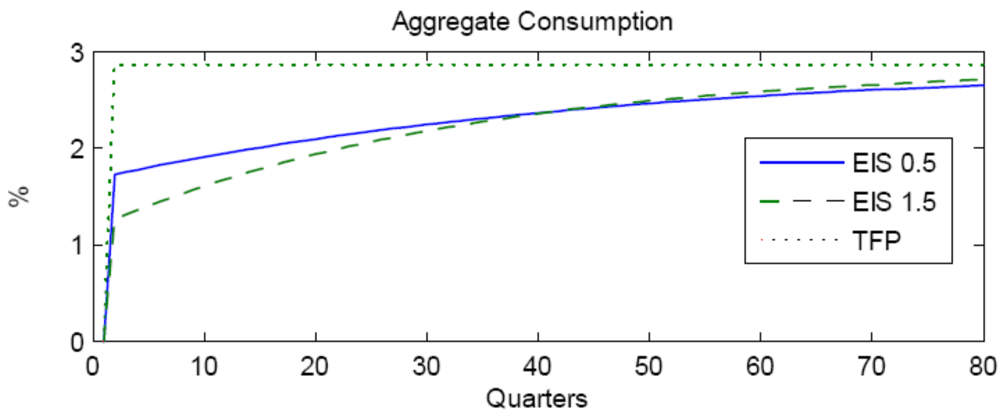

Figure 4.2: **Impulse Response of Consumption**
Impulse responses of consumption to a one standard deviation positive and permanent shock to technology for different levels of the EIS. The impulse responses are for Model 5 (EIS = 0.5) and Model 6 (EIS = 1.5), respectively.

to 'technology' in our model) are obtained from the Bureau of Economic Analysis and the Bureau of Labor Statistics, respectively. The return data are from Kenneth French.

4.5.1 Expected Consumption Growth

As highlighted by Harvey and Shepard (1990), Bansal and Yaron (2004) and Hansen, Heaton, Li (2005), amongst others, it is difficult to estimate long-run consumption growth dynamics from the relatively short samples of data we have available. In our model, slow-moving expected consumption growth dynamics arise due to endogenous consumption smoothing, and the production economy model therefore identifies observable proxies for the otherwise unobservable expected consumption growth rate. In particular, the ratio of the level of technology to the level of consumption forecasts future consumption growth with a positive sign when technology shocks are permanent. This intuition is confirmed in Figure 4.2, which shows the impulse response of consumption to a one standard deviation permanent shock to technology (total factor productivity) for high and low levels of the EIS. Investors respond to a technology shock by increasing investment in order to build up higher levels of capital. Thus, while technology immediately adjusts to its new permanent level, consumption only slowly grows to a permanently higher level as capital needs to be built up to support the new steady state consumption level.

In particular, define:

$$zc_t \equiv \ln\left(\frac{Z_t}{C_t}\right). \tag{4.17}$$

If zc_t is high, consumption is more likely to increase towards a new steady-state level. Our model thus implies that zc_t is a good instrument for the expected consumption growth rate. Our model predicts that zc_t is stationary even though both Z_t and C_t are non-stationary ($I(1)$), because C_t evolves around the stochastic trend Z_t: if the production technology is specified as:

$$Y_t = Z_t^{1-\alpha} K_t^{\alpha} N_t^{1-\alpha}, \tag{4.18}$$

as is the case in our model, all endogenous variables in the economy evolve around the stochastic trend Z_t (see Appendix B). We get data on Z_t (TFP) from the Bureau of Labor Statistics (BLS). The BLS computes TFP as follows. First it collects data on Y_t (output), on K_t (capital input), and on N_t (labor input). Then the BLS estimates a value for the parameter α and computes TFP as the Solow residual:

$$\ln \widetilde{Z}_t = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln N_t. \tag{4.19}$$

Note that the BLS specifies the following production technology:

$$Y_t = \widetilde{Z}_t K_t^{\alpha} N_t^{1-\alpha}. \tag{4.20}$$

It follows that we need to normalize:

$$Z_t = \widetilde{Z}_t^{1/(1-\alpha)}. \tag{4.21}$$

We take as the value for α the value we use in our model ($\alpha = 0.34$). We check our results for robustness by assuming different values for $\alpha \in [0.30, 0.40]$, and find that our results are robust with respect to the choice of α.

The Baseline Model, which has permanent technology shocks, suggests the following forecasting relationship:

$$\Delta c_{t,t+j} = \alpha + \beta z c_t + \varepsilon_{t,t+j}, \tag{4.22}$$

where $\beta > 0$. In the model, the relation is not exactly linear, but when simulating data from our model (Baseline Model) we find that zc_t accounts for more than 99% of the variation in expected consumption growth in a linear regression. We test this forecasting relationship both on data from 1948 to 2005 and on data generated by our model (Baseline Model). In particular, in Table 4.5 we report results from forecasting regressions of annual log nondurable- and services consumption growth on the lagged log TFP to consumption ratio, our measure of expected consumption growth.

Panel A shows that consumption growth is forecastable by the $zc-ratio$ using simulated data from our model. The regression coefficient is increasing with the horizon up to

Table 4.5: **Estimating Expected Consumption Growth**
This table reports forecasting regressions of annual log nondurable- and services consumption growth on a lagged measure of expected consumption growth, the log TFP to Consumption ratio. The consumption and TFP data are from the Bureau of Economic Analysis and the Bureau of Labor Statistics respectively. We use annual data from 1948 to 2005, resulting in 58 - j observations for a regression with a j year forecasting horizon: Multi-year forecasting regressions are overlapping at an annual frequency. The standard error estimates (in parenthesis) are corrected for heteroscedasticity and overlapping observations using Hodrick (1992) standard errors. Results for the model are based on 10,000 replications of sample size 58 × 4 each. Numbers in bold indicate significance at the 5% level or more in a two-tailed t-test, while an asterisk indicates significance at the 10% level.

Regression: $\Delta c_{t,t+j} = \alpha + \beta zc_t + \varepsilon_{t,t+j}$

Panel A: Model implied (j denotes forecasting horizon in years)

j	1	2	3	4	5	7	10
β	0.077	0.135	0.178	0.208	0.226	0.233	0.192
	(0.051)	(0.101)	(0.151)	(0.201)	(0.251)	(0.345)	(0.468)
R^2_{adj}	9.2%	10.2%	10.9%	10.8%	10.3%	9.3%	8.3%
	(0.092)	(0.118)	(0.128)	(0.131)	(0.131)	(0.128)	(0.124)

Panel B: Historical estimates (j denotes forecasting horizon in years)

j	1	2	3	4	5	7	10
β	0.021*	0.041*	0.060*	0.084*	0.107*	0.147*	0.233*
	(0.012)	(0.024)	(0.036)	(0.048)	(0.060)	(0.085)	(0.127)
R^2_{adj}	3.3%	5.5%	7.6%	11.1%	13.8%	17.6%	30.0%

7 years. The forecasting regression coefficients are found by simulating 10,000 samples of length 58 years, running the regression on each sample, and computing the average regression coefficient. The sample errors are the sample standard deviation of each β-estimate. Interestingly, the regression coefficient is not significant in any of the regressions using data simulated from the model. The variation in expected consumption growth is too slow-moving for the regressions to on average uncover the forecasting relationship over the relatively short sample period.

Panel B shows the results from the forecasting regressions using real data. Here both the regression coefficients and the R^2's are increasing with the forecasting horizon. The coefficients are significant at the 10% level for all regressions using Hodrick (1992) standard errors, which have relatively good small sample properties for overlapping regressions. The coefficients are overall lower than those estimated using simulated data. This could be because there is less variation in expected consumption growth in the data or because the empirical $zc - ratio$ is measured with noise.[16]

We conclude that the log TFP to consumption ratio, a measure of expected consumption growth implied by our theoretical model, forecasts future consumption growth and that the level of measured variation in expected consumption growth is similar in magnitude to that implied by our model with $\gamma = 5$ and $\psi = 1.5$.

4.5.2 The Cross-Section of Stock Returns

The model in this chapter implies that the shock to expected consumption growth is a priced risk factor as long as $\gamma \neq \frac{1}{\psi}$, i.e. as long as agents care about the temporal resolution of risk. The cross-section of stock returns can tell us both whether shocks to expected consumption growth are a priced risk factor and whether the price of risk on this factor is positive or negative, which in turn depends on whether the relative risk aversion of the representative agent is smaller or larger then the reciprocal of the elasticity of substitution and on the persistence of the technology shocks. To relate the consumption dynamics directly to the stochastic discount factor, we assume that the dynamic behavior

[16]The R^2's are higher for the long-horizon regressions than predicted by the model, although Valkanov (2003) cautions that R^2's are badly behaved in small samples where the fraction of overlapping observations relative to the total sample length is large. Thus, the sample R^2's are likely to overstate somewhat the true R^2's.

of consumption growth generated by the model follows:

$$\Delta c_{t+1} = \mu + x_t + \eta_{t+1}, \tag{4.23}$$

$$x_{t+1} = \rho x_t + e_{t+1}, \tag{4.24}$$

$$\sigma_{\eta,e} = corr\left(e_{t+1}, \eta_{t+1}\right). \tag{4.25}$$

We verify in the appendix that this parsimonious specification captures the true behavior of consumption growth from our model well. Given the consumption dynamics in (4.23) and log-linearizing the return on the wealth portfolio around the steady state ratio of wealth to aggregate consumption, the stochastic discount factor can be written:

$$m_{t+1} \approx a - b_1 \Delta c_{t+1} - b_2 e_{t+1} - b_3 x_t, \tag{4.26}$$

where Δc_{t+1} denotes realized consumption growth, x_t is the current level of expected consumption growth, e_{t+1} is the shock to expected consumption growth, and $b_1 = \gamma$, $b_2 = (1-\theta) A_1 \kappa_1$, $b_3 = (\theta - 1) A_1 (1 - \kappa_1 \rho)$ (see the appendix for a detailed derivation and definitions of the constants $A_1, \kappa_1 > 0$. If $\gamma > \frac{1}{\psi}$, the coefficients $b_1, b_2 > 0$ and $b_3 < 0$. We will test these restrictions using the cross-section of stock returns.

We use the log ratio of TFP to consumption, zc_t, to obtain measures of x_t and e_{t+1} from the following regressions:

$$\Delta c_{t+1} = \hat{k}_0 + \hat{k}_1 zc_t + \hat{\eta}_{t+1}, \tag{4.27}$$

$$\widehat{x}_t = \hat{k}_1 \left(zc_t - E_T\left[zc_t\right]\right), \tag{4.28}$$

$$zc_{t+1} = \hat{k}_3 + \hat{k}_4 zc_t + \hat{\nu}_{t+1}, \tag{4.29}$$

$$\widehat{e}_{t+1} = \hat{k}_1 \left[zc_{t+1} - \hat{k}_3 - \hat{k}_4 zc_t\right], \tag{4.30}$$

where $E_T[\cdot]$ denotes the sample mean and $\hat{k}_i$ is an OLS regression coefficient. The permanent shock model predicts that shocks to realized ($\hat{\eta}$) and expected ($\hat{e}$) consumption are positively correlated, which we confirm is the case in our sample.

By applying a standard log-linear approximation of the stochastic discount factor (see appendix), we arrive at the linear factor model:

$$\begin{aligned} E\left[R_{i,t+1} - R_{0,t+1}\right] = {} & b_1 Cov\left(\Delta c_{t+1}, R_{i,t+1} - R_{0,t+1}\right) + b_2 Cov\left(e_{t+1}, R_{i,t+1} - R_{0,t+1}\right) \\ & + b_3 Cov\left(x_t, R_{i,t+1} - R_{0,t+1}\right). \end{aligned} \tag{4.31}$$

The standard consumption-based asset pricing model with power utility implies that $b_1 = \gamma = \frac{1}{\psi}$, while $b_2 = b_3 = 0$. As noted above, if $\gamma > \frac{1}{\psi}$, however, $b_2 > 0$ and $b_3 < 0$.

Table 4.6: **The Price of Long-Run Risk from Cross-Sectional Regressions**
This table reports the estimated loadings on the factors of the Consumption CAPM and the long-run risk production-based model developed in this chapter (Prod.CAPM). Test assets are the 25 Fama-French portfolios sorted by size and book-to-market equity ratios. All variables are annual. There are 57 observations from 1949 to 2005. Estimation is by two-pass regression, where the standard errors are corrected for generated regressors (Shanken (1992)). P-values are reported for each variable, where the null hypothesis is that the estimate is zero. Numbers in bold indicate significance at the 10% level or more in a two-tailed t-test.

Tests of Factor Significance:	$m_{t+1} = a - b_1 \Delta c_{t+1} - b_2 e_{t+1} - b_3 x_t$	
Factor loading b	*Cons.CAPM*	*Prod.CAPM*
Realized Cons. Growth (b_1)	> 0	< 0
(p-value)	(0.24)	(0.82)
Shock to Exp. Cons. Growth (b_2)		$>$ **0**
(p-value)		(**0.09**)
Expected Cons. Growth (b_3)		< 0
(p-value)		(0.11)
Joint test: $b_2 = b_3 = 0$		*reject*
(p-value)		(**0.06**)
R^2_{adj}	17.4%	46.7%

Because TFP data are only available on an annual basis from the Bureau of Economic Analysis, we use annual returns on the 25 Fama-French portfolios as test assets along with the risk-free rate (U.S. t-Bill). The sample consists of 57 observations from 1948 to 2005. Table 4.6 displays results for the benchmark Consumption CAPM ($b_2 = b_3 = 0$) and the three-factor model of this chapter.

Table 4.6 displays the sign of the estimated quantities with p-values in parentheses. The factor loading on realized consumption growth risk is insignificant for the standard Consumption CAPM model and the adjusted cross-sectional R^2 is 17.4%. The three-factor model including measures of the level and the shock to expected consumption growth increases the adjusted R^2 to 46.6%. The factor loading on the shock to expected consumption growth carries a positive sign and is significant at the 10% level. The sign on the coefficient on the measure of expected consumption growth (b_3) is negative as predicted, but not quite significant. The power utility benchmark implies that $b_2 = b_3 = 0$. A test of this joint hypothesis yields a p-value of 0.06. Thus, the statistical evidence is not very strong, but significant at the 10% level. On the other hand, we are relying on a noisy proxy and the sample is fairly small.

We conclude that the linear three-factor model derived from our theoretical model outperforms the benchmark Consumption CAPM. We reject the null hypothesis that long-run risk is not important relative to the standard Consumption CAPM for the cross-section of stock returns. The signs on factor loadings b_2 and b_3 are consistent with a model where agents prefer early resolution of uncertainty ($\gamma > \frac{1}{\psi}$).

4.6 Conclusion

We analyze a standard stochastic growth model where agents have Epstein and Zin (1989) preferences. We show that long-run risk arises endogenously as a consequence of consumption smoothing, even though log technology follows a random walk. When the coefficient of relative risk aversion is greater than the reciprocal of the elasticity of intertemporal substitution, agents dislike negative shocks to future economic growth prospects and shocks to expected consumption growth appear as a risk factor. The presence of long-run risk in this case decreases the market price of risk if technology shocks are transitory, while it increases the market price of risk if technology shocks are permanent. The model therefore provides a theoretical justification for a long-run risk component in aggregate consumption growth. This result is of particular interest since it is very difficult to empirically distinguish a small predictable component of consumption growth from i.i.d. consumption growth given the short sample of data we have available.

We calibrate the model to key aggregate macroeconomic moments and show that we can match the level and volatility of the risk-free rate and the unconditional equity Sharpe ratio. The model achieves this with a low level of relative risk aversion, unlike habit formation models where typical implementations also generate too much volatility in the risk-free rate. The elasticity of intertemporal substitution, which strongly affects the dynamics of the macroeconomic variables, also strongly affects the price of risk and the Sharpe ratio of equity in the model. Thus, there is a tight link between quantity dynamics and asset prices in our implementation of the standard stochastic growth model.

The production economy model identifies the ratio of technology to consumption as a proxy for the otherwise hard to estimate expected consumption growth. We test this link in the time-series of consumption growth and in the cross-section of stock returns. We find support for both tests. In particular, the production-based CAPM outperforms the standard CCAPM in a cross-sectional test. The parameter estimates obtained from the cross-sectional analysis are consistent with a model where technology shocks are permanent and agents have a preference for early resolution of uncertainty.

4.7 References

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4.8 Appendix A — Model Solution

The Return to Investment and the Firm's Problem The firm maximizes firm value. Let $M_{t,t+1}$ denote the stochastic discount factor. The firm's problem is then:

$$\max_{\{I_t,K_{t+1},H_t\}_{t=0}^{T}} E_0\left[\sum_{t=0}^{T} M_{0,t}\left\{\begin{array}{c}(Y_t - W_t H_t - I_t) - \\ q_t\left(K_{t+1} - (1-\delta)K_t - \phi\left(\frac{I_t}{K_t}\right)K_t\right)\end{array}\right\}\right], \tag{4.32}$$

where q_t denotes the shadow price of the capital accumulation constraint, equivalent to marginal q: the expected present value of one marginal unit of capital. Maximizing over labor we obtain $(1-\alpha)Z_t^{1-\alpha}K_t^{\alpha}H_t^{-\alpha} = W_t$ and $H_t = (1-\alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}} K_t$. In other words, we assume an exogenous wage process such that it is optimal for the firm to always hire at full capacity ($H_t = 1$), which is the same amount of labor as the representative agent is assumed to supply. In this case, total wages $W_t H_t = W_t = (1-\alpha)Y_t$, so wages are pro-cyclical and have the same growth rate volatility as total output. The operating profit function of the firm follows as:

$$\begin{aligned}
\Pi(K_t, Z_t; W_t) &= Z_t^{1-\alpha}\left[(1-\alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}} K_t\right]^{1-\alpha} K_t^{\alpha} - W_t (1-\alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}} K_t \\
&= Z_t^{1-\alpha}\left[(1-\alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{-\frac{1}{\alpha}}\right]^{1-\alpha} K_t - (1-\alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}} K_t \\
&= \left((1-\alpha)^{\frac{1}{\alpha}-1} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}} - (1-\alpha)^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}}\right) K_t \\
&= \left(\alpha(1-\alpha)^{\frac{1}{\alpha}-1} Z_t^{\frac{1}{\alpha}-1} W_t^{1-\frac{1}{\alpha}}\right) K_t. \qquad (4.33)
\end{aligned}$$

The operating profit function of the firm is thus linearly homogenous in capital. Substituting out equilibrium wages we obtain $\Pi(K_t, Z_t; W_t) = \alpha Y_t$. We re-state the firm's problem:

$$\max_{\{I_t,K_{t+1}\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty} M_{0,t}\left\{\begin{array}{c}\Pi(\cdot) - I_t - \\ q_t\left(K_{t+1} - (1-\delta)K_t - \phi\left(\frac{I_t}{K_t}\right)K_t\right)\end{array}\right\}\right]. \tag{4.34}$$

Each period in time the firm decides how much to invest, taking marginal q as given. The first order conditions with respect to I_t and K_{t+1} are immediate:

$$0 = -1 + q_t \phi' \left(\frac{I_t}{K_t} \right), \tag{4.35}$$

and

$$0 = -q_t + E_t \left[M_{t+1} \left\{ \Pi_K(\cdot) + q_{t+1} \begin{pmatrix} (1-\delta) + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right) \\ -\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}} \end{pmatrix} \right\} \right]. \tag{4.36}$$

Substituting out q_t and q_{t+1} in (4.36) yields:

$$\frac{1}{\phi'\left(\frac{I_t}{K_t}\right)} = E_t \left[M_{t+1} \left\{ \begin{array}{c} \Pi_K(\cdot) \\ + \frac{(1-\delta) - \phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)\frac{I_{t+1}}{K_{t+1}} + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} \end{array} \right\} \right], \tag{4.37}$$

$$1 = E_t \left[M_{t+1} \left\{ \phi'\left(\frac{I_t}{K_t}\right) \begin{pmatrix} \Pi_K(\cdot) \\ + \frac{1-\delta+\phi\left(\frac{I_{t+1}}{K_{t+1}}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} - \frac{I_{t+1}}{K_{t+1}} \end{pmatrix} \right\} \right], \tag{4.38}$$

$$1 = E_t \left[M_{t+1} R^I_{t+1} \right]. \tag{4.39}$$

Equation (4.39) is the familiar law of one price, with the firm's return to investment:

$$R^I_{t+1} = \phi'\left(\frac{I_t}{K_t}\right) \begin{pmatrix} \Pi_K(K_{t+1}, Z_{t+1}; W_{t+1}) \\ + \frac{1-\delta+\phi\left(\frac{I_{t+1}}{K_{t+1}}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} - \frac{I_{t+1}}{K_{t+1}} \end{pmatrix}. \tag{4.40}$$

4.9 Appendix B — Numerical Solution

Solution Algorithm We solve the following model:

$$V(K_t, Z_t) = \max_{C_t, K_{t+1}} \left\{ \left[\begin{array}{c} (1-\beta) C_t^{\frac{1-\gamma}{\theta}} \\ +\beta \left(E_t \left[V(K_{t+1}, Z_{t+1})^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \end{array} \right]^{\frac{\theta}{1-\gamma}} \right\}, \tag{4.41}$$

$$K_{t+1} = (1-\delta) K_t + \phi\left(\frac{I_t}{K_t}\right) K_t, \tag{4.42}$$

$$I_t = Y_t - C_t, \tag{4.43}$$

$$Y_t = Z_t^{(1-\alpha)} K_t^{\alpha}, \tag{4.44}$$

$$\ln Z_{t+1} = \varphi \ln Z_t + \varepsilon_{t+1}, \tag{4.45}$$

$$\varepsilon_t \sim N(\mu, \sigma_\varepsilon). \tag{4.46}$$

We focus in this appendix on the case where $\varphi = 1$. Since then the process for productivity is non-stationary, we need to normalize the economy by Z_t, in order to be able to numerically solve the model. To be precise, we let $\widehat{K}_t = \frac{K_t}{Z_t}, \widehat{C}_t = \frac{C_t}{Z_t}, \widehat{I}_t = \frac{I_t}{Z_t}$, and substitute. In the so transformed model all variables are stationary. The only state variable of the normalized model is $\widehat{K}$.[17] We can work directly on the appropriately normalized set of equations and then re-normalize after having solved the model.[18]

The value function is given by:

$$\widehat{V}\left(\widehat{K}_t\right) = \max_{\widehat{C}_t, \widehat{K}_{t+1}} \left\{ \left[\begin{array}{c} (1-\beta)\,\widehat{C}_t^{\frac{1-\gamma}{\theta}} \\ +\beta\left(E_t\left[\left(e^{\varepsilon_{t+1}}\right)^{1-\gamma}\left(\widehat{V}\left(\widehat{K}_{t+1}\right)\right)^{1-\gamma}\right]\right)^{\frac{1}{\theta}} \end{array} \right]^{\frac{\theta}{1-\gamma}} \right\}. \tag{4.47}$$

We parameterize the value function with a $5th$ order Chebyshev orthogonal polynomial over a 6×1 Chebyshev grid for the state variable $\widehat{K}$:

$$\Psi^A\left(\widehat{K}\right) = \widehat{V}\left(\widehat{K}\right). \tag{4.48}$$

We use the value function iteration algorithm. At each grid point for the state $\widehat{K}$, given a polynomial for the value function $\Psi_i^A\left(\widehat{K}\right)$, we use a numerical optimizer to find the policy $(\widehat{C}^*)$ that maximizes the value function:

$$\widehat{K}_{t+1}^* e^{\varepsilon_{t+1}} = \widehat{Y}_t - \widehat{C}_t^* + (1-\delta)\widehat{K}_t, \tag{4.49}$$

$$\widehat{V}^*\left(\widehat{K}_t\right) = \left[\begin{array}{c} (1-\beta)\left(\widehat{C}_t^*\right)^{\frac{1-\gamma}{\theta}} \\ +\beta\left(E_t\left[\left(e^{\varepsilon_{t+1}}\right)^{1-\gamma}\left(\Psi_i^A\left(\widehat{K}_{t+1}^*\right)\right)^{1-\gamma}\right]\right)^{\frac{1}{\theta}} \end{array} \right]^{\frac{\theta}{1-\gamma}}, \tag{4.50}$$

where Gauss-Hermite quadrature with 5 nodes is used to approximate the expectations operator. We use a regression of $\widehat{V}^*$ on $\widehat{K}$ in order to update the coefficients of the polynomial for the value function and so obtain $\Psi_{i+1}^A\left(\widehat{K}\right)$.

[17] Note that Z is *not* a state variable of the normalized model. This is due to the fact that we assume the autoregressive coefficient of the process for productivity $\ln Z_{t+1} = \rho \ln Z_t + \varepsilon_{t+1}$ to be unity: $\rho = 1$. As a consequence, ΔZ is serially uncorrelated.

[18] In this chapter we also report results for models where $\rho < 1$. In this case we work directly on the above non-normalized set of equations. The state variables are then K and Z. The solution algorithm is identical to the case where $\rho = 1$.

4.10 Appendix C — Risk and the Dynamic Behavior of Consumption

Epstein and Zin (1989) preferences have been used with increasing success in the asset pricing literature over the last years (e.g., Bansal and Yaron (2004), Hansen, Heaton, Li (2005), Malloy, Moskowitz, Vissing-Jorgensen (2005), Yogo (2006)). This is both due to their recursive nature, which allows time-varying growth rates to increase the volatility of the stochastic discount factor through the return on the wealth portfolio, as well as the fact that these preferences allow a convenient separation of the elasticity of intertemporal substitution from the coefficient of relative risk aversion.

Departing from time-separable power utility preferences with $\gamma = \frac{1}{\psi}$ means agents care about the temporal distribution of risk. In particular, Epstein and Zin (1989) show that $\gamma > \frac{1}{\psi}$ implies a preference for early resolution of uncertainty. This is a key assumption of our analysis, because it is precisely this departure from the classic preference structure that renders time-varying expected consumption growth rates induced by optimal consumption smoothing behavior a priced risk factor in the economy.

4.10.1 Early Resolution of Uncertainty and Aversion to Time-Varying Growth Rates

To gain some intuition for why a preference for early resolution of uncertainty implies aversion to time-varying growth rates, we revisit an example put forward in Duffie and Epstein (1992). Consider a world where each period of time consumption can be either high or low. Next, the consumer is given a choice between two consumption gambles, A and B. Gamble A entails eating $C_0 \equiv \frac{1}{2}C^H + \frac{1}{2}C^L$ today, where C^H is a high consumption level and C^L is a low consumption level. Tomorrow you flip a fair coin. If the coin comes up heads, you will get C^H each period forever. If the coin comes up tails, you will get C^L each period forever. Gamble B entails eating C_0 today, and then flip a fair coin *each* subsequent period t. If the coin comes up heads at time t, you get C^H at time t, and if it comes out tails, you get C^L at time t. Thus, in the first case uncertainty about future consumption is resolved early, while in the second case uncertainty is resolved gradually (late). If $\gamma = \frac{1}{\psi}$ (power utility), the consumer is indifferent with respect to the timing of the resolution of uncertainty and thus indifferent between the two gambles. However, an agent who prefers early resolution of uncertainty (i.e., she likes to plan), prefers gamble A.

We can now also phrase this discussion in terms of *growth rates*. From this perspective, gamble A has constant expected consumption growth, while gamble B has a mean-reverting process for expected consumption growth. Thus, a preference for early resolution of uncertainty translates into an aversion of time-varying expected consumption growth.

Another, more mechanical, way to see this is by directly looking at the stochastic discount factor. It is well known, e.g. Rubinstein (1976), that the stochastic discount factor, M_{t+1}, is the ratio of the representative agent's marginal utility between today and tomorrow: $M_{t+1} = \frac{U'(C_{t+1})}{U'(C_t)}$. Using a recursive argument, Epstein and Zin (1989) show that:

$$\ln M_{t+1} \equiv m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\, r_{a,t+1}, \tag{4.51}$$

where $\Delta c_{t+1} \equiv \ln \frac{C_{t+1}}{C_t}$ and $r_{a,t+1} \equiv \ln \frac{C_{t+1}+A_{t+1}}{A_t}$ is the return on the total wealth portfolio with A_t denoting total wealth at time t.[19] If $\gamma = \frac{1}{\psi}$, $\theta = \frac{1-\gamma}{1-1/\psi} = 1$, and the stochastic discount factor collapses to the familiar power utility case. However, if the agent prefers early resolution of uncertainty, the return on the wealth portfolio appears as a risk factor. More time-variation in expected consumption growth (the expected 'dividends' on the total wealth portfolio) induces higher volatility of asset returns, in turn resulting in a more volatile stochastic discount factor and thus a higher price of risk in the economy.[20]

The effect on the equity premium can be understood by considering a log-linear approximation (see Campbell (1999)) of returns and the pricing kernel, yielding the following expressions for the risk-free rate and the equity premium:

$$r_{f,t+1} \approx -\log \beta + \frac{1}{\psi} E_t \left[\triangle c_{t+1}\right] - \frac{\theta}{2\psi^2}\sigma^2_{t,c} + \frac{(\theta-1)}{2}\sigma^2_{t,r^A}, \tag{4.52}$$

$$E_t\left[r^E_{t+1}\right] - r_{f,t+1} \approx \frac{\theta}{\psi}\sigma_{t,r^E c} + (1-\theta)\,\sigma_{t,r^E r^A} - \frac{\sigma^2_{t,r^E}}{2}, \tag{4.53}$$

where $E_t\left[\triangle c_{t+1}\right]$ is expected log consumption growth, $\sigma_{t,c}$, σ_{t,r^A}, σ_{t,r^E}, are the conditional standard deviations of log consumption growth, the log return on the total wealth portfolio, and the log equity return, and $\sigma_{t,r^E c}$ and $\sigma_{t,r^E r^A}$ are the conditional covariances of the log equity return with log consumption growth and the log return on the total wealth portfolio respectively. We can see how the level of the equity premium depends directly on the covariance of equity returns with returns on the wealth portfolio.

[19] Note that our representative household's total wealth portfolio is composed of the present value of future labor income in addition to the value of the firm.

[20] This assumes that the correlation between the return on the wealth portfolio and consumption growth is non-negative, which it is for all parameter values we consider in this chapter (and many more).

4.10.2 Technology and Risk Aversion

Standard production technologies do not allow agents to hedge the technology shock. Agents must in the aggregate hold the claim to the firm's dividends. Therefore, the only action available to agents at time t in terms of hedging the shock at time $t+1$, is to increase savings in order to increase wealth for time $t+1$. The shock will still hit the agents at time $t+1$ though, no matter what. Wealth levels may be higher if a bad realization of the technology shock hits the agents, but wealth is also higher if a good realization of the technology shock occurs. The *difference* between the agents' utility for a good realization of the technology shock in period $t+1$ relative to their utility for a bad realization of the shock is thus (almost) unaffected. However, it is this utility *difference* the agents care about in terms of their risk aversion. Now, because the agents' utility function is concave, this is not quite true. A higher wealth level in both states of the world does decrease the difference between utility levels. Agents thus respond by building up what is referred to as 'buffer-stock-savings'. This is, however, a second-order effect. As a result, the dynamic behavior of consumption growth is largely unaffected by changing agents' coefficient of relative risk aversion. The fundamental consumption risk in the economy remains therefore (almost) the same when we increase risk aversion (γ) while holding the EIS (ψ) constant. Asset prices, of course, respond as usual to higher levels of risk aversion. Table 4.7 confirms this result. We report Model 3 and the Baseline Model from Table 4.3 with a coefficient of relative risk aversion (γ) of 5, as well as versions of the models with a higher level of risk aversion ($\gamma = 25$).

4.11 Appendix D — The Case of $\gamma < \frac{1}{\psi}$

In the previous discussions our focus was on the case of $\gamma > \frac{1}{\psi}$, where agents prefer early resolution of uncertainty and dislike fluctuations in expected consumption growth rates. With transitory shocks and $\gamma > \frac{1}{\psi}$, this second risk factor acts as a hedge for shocks to realized consumption growth and therefore reduces the price of risk. This raises the possibility that if agents *like* fluctuations in expected consumption growth rates, that is when $\gamma < \frac{1}{\psi}$, consumption smoothing increases the price of risk when technology shocks are transitory. In this section, we investigate whether this channel can give rise to long-run risks that help in explaining asset prices with a *low* elasticity of intertemporal substitution. The short answer is yes. However, a low EIS unfortunately gives rise to a risk-free rate puzzle (Weil (1989)).

Table 4.8 shows calibrated macroeconomic and financial moments from models with $\gamma = 5$, and $\psi = \frac{1}{5}$ and $\psi = 0.1$. From Model 7 to Model 15 we increase capital adjustment

Table 4.7: **Appendix — Effect of Risk Aversion on Macroeconomic Time Series**
This table reports relevant macroeconomic moments and consumption dynamics for models with either transitory ($\varphi = 0.90$) or permanent technology shocks and different levels of the coefficient of relative risk aversion. The elasticity of intertemporal substitution (ψ) is 1.5 across all models. We re-calibrate the discount factor (β) for each model so as to jointly match the values for (C/Y), (I/Y), (D/Y), with each model. Capital adjustment costs (ξ) are 22 in order to match the relative volatility of consumption to output with the Baseline Model. Model 3 and Model 13 share the same parameter values apart from the coefficient of relative risk aversion (γ). The same is true for the Baseline Model and Model 14. We estimate the following process for the consumption dynamics: $\Delta c_{t+1} = \mu + x_t + \eta_{t+1}$, $x_{t+1} = \rho x_t + e_{t+1}$. $\Delta x = \log(X_t) - \log(X_{t-1})$, and $\sigma[X]$ denotes the standard deviation of variable X. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis. The sample is the same as in Bansal and Yaron (2004). Under Panel B we report the calibration of the exogenous consumption process Bansal and Yaron use. All values reported in the table are quarterly.

		Model 3	Model 13	**Baseline**	Model 14
		Transitory Shocks		Permanent Shocks	
		$z_{t+1} = \mu t + \varphi z_t + \sigma_\varepsilon \varepsilon_{t+1}$		$z_{t+1} = \mu + z_t + \sigma_\varepsilon \varepsilon_{t+1}$	
Statistic		$\gamma = 5$	$\gamma = 25$	$\gamma = 5$	$\gamma = 25$
Panel A: Macroeconomic Moments (Quarterly)					
	U.S. Data 1929-1998				
$\sigma[\Delta y]$ (%)	2.62	2.62	2.62	2.62	2.62
$\sigma[\Delta c]/\sigma[\Delta y]$	0.52	0.40	0.40	0.52	0.45
$\sigma[\Delta i]/\sigma[\Delta y]$	3.32	3.49	3.48	1.90	1.61
Panel B: Consumption Dynamics: $\Delta c_{t+1} = \mu + x_t + \eta_{t+1}, \quad x_{t+1} = \rho x_t + e_{t+1}.$					
	Bansal, Yaron Calibration				
$\sigma[\Delta c]$ (%)	1.360	1.048	1.035	1.362	1.179
$\sigma[x]$ (%)	0.172	0.158	0.157	0.329	0.325
ρ	0.938	0.954	0.953	0.969	0.973

Table 4.8: **Appendix — The Case of $\gamma < \frac{1}{\psi}$: Asset Pricing Implications**
This table reports relevant macroeconomic moments, consumption dynamics, and financial moments for models with transitory ($\varphi = 0.90$) technology shocks and different levels of the elasticity of intertemporal substitution. The coefficient of relative risk aversion (γ) is 5 and β is 0.998 across all models. We re-calibrate σ_ε in order to match the volatility of consumption growth with each model. We estimate the following process for the consumption dynamics: $\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}$, $x_{t+1} = \rho x_t + \sigma_e e_{t+1}$. $\Delta x = \log(X_t) - \log(X_{t-1})$, and $\sigma[X]$ denotes the standard deviation of variable X. We use annual U.S. data from 1929 to 1998 from the Bureau of Economic Analysis and from Bansal and Yaron (2004). Under Panel B we report the calibration of the exogenous consumption process Bansal and Yaron use.

		Model 7	Model 15	Model 16
			Transitory Shocks	
			$z_{t+1} = \mu t + \varphi z_t + \sigma_\varepsilon \varepsilon_{t+1}$	
		$\psi = \frac{1}{\gamma}$	$\psi = \frac{1}{\gamma}$	$\psi = 0.10$
Statistic		$\xi = 22$	$\xi = 2$	$\xi = 2$
Panel A: Macroeconomic Moments (Quarterly)				
	U.S. Data 1929-1998			
$\sigma[\Delta y]$ (%)	2.62	4.72	2.62	2.62
$\sigma[\Delta c]/\sigma[\Delta y]$	0.52	0.29	0.52	0.52
$\sigma[\Delta i]/\sigma[\Delta y]$	3.32	4.83	2.88	4.03
Panel B: Consumption Dynamics (Quarterly): $\Delta c_{t+1} = \mu + x_t + \eta_{t+1}$, $x_{t+1} = \rho x_t + e_{t+1}$.				
	Bansal, Yaron Calibration			
$\sigma[\Delta c]$ (%)	1.360	1.360	1.360	1.360
$\sigma[x]$ (%)	0.172	0.123	0.206	0.165
ρ	0.938	0.934	0.917	0.924
Panel B: Financial Moments (Annual)				
$\sigma[M]/E[M]$	n/a	0.13	0.13	0.19
$SR\left[R^E\right]$	0.33	0.11	0.13	0.19
$E[R_f]$ (%)	0.86	7.63	7.61	14.92
$\sigma[R_f]$ (%)	0.97	1.14	1.96	3.30
$E\left[R^E - R_f\right]$ (%)	6.33	0.25	1.05	2.06
$\sigma\left[R^E - R_f\right]$ (%)	19.42	2.37	7.76	10.82

costs in order to fit the relative volatility of consumption to output. Higher capital adjustment costs increase the equity return volatility, as marginal q is now more volatile. From Model 15 to Model 16 we decrease the EIS relative to the benchmark power utility model. A lower EIS induces a higher equity return Sharpe ratio and a higher price of risk due to the preference for time-varying expected consumption growth, which is negatively correlated with shocks to realized consumption growth. In other words, the effect of long-run risk on the price of risk is now *decreasing* in the EIS, as opposed to the case of permanent technology shocks. Long-run risk with a low EIS is quite different from the intuition in Bansal and Yaron (2004) and may seem surprising given that a low EIS implies that consumers strive to make the consumption path smooth over time and therefore minimize the volatility of expected consumption growth. However, the fact that we are increasing capital adjustment costs to match the macroeconomic moments renders consumption smoothing more costly. Therefore, in equilibrium, a substantial amount of long-run risk remains in the economy even with a low EIS. As a result, the price of risk almost doubles relative to the power utility benchmark model. The equity return volatility also increases due to high capital adjustment costs. The net effect is a substantial increase in the equity premium.

Unfortunately, decreasing the elasticity of intertemporal substitution also leads to a risk-free rate puzzle. The average annual risk-free rate is with 7.63% already much too high in the benchmark power utility model and increases further to 14.92% as we decrease the EIS. Thus, endogenous long-run risk can substantially improve the asset pricing properties of production economy models even with a very low EIS. However, the risk-free rate is too high given an average annual consumption growth rate of 1.6% as assumed in this chapter. If one is willing to assume that the average real consumption growth rate is close to zero, it would be possible to also fit the average risk-free rate.

4.12 Appendix E — Accuracy of the Approximation of the Consumption Process

In Section 4.4.4 we propose the following approximation for the dynamics of the endogenous process for consumption:

$$\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}, \tag{4.54}$$

$$x_{t+1} = \rho x_t + \sigma_e e_{t+1}, \tag{4.55}$$

$$\sigma_{\eta,e} = corr\left(\eta_{t+1}, e_{t+1}\right). \tag{4.56}$$

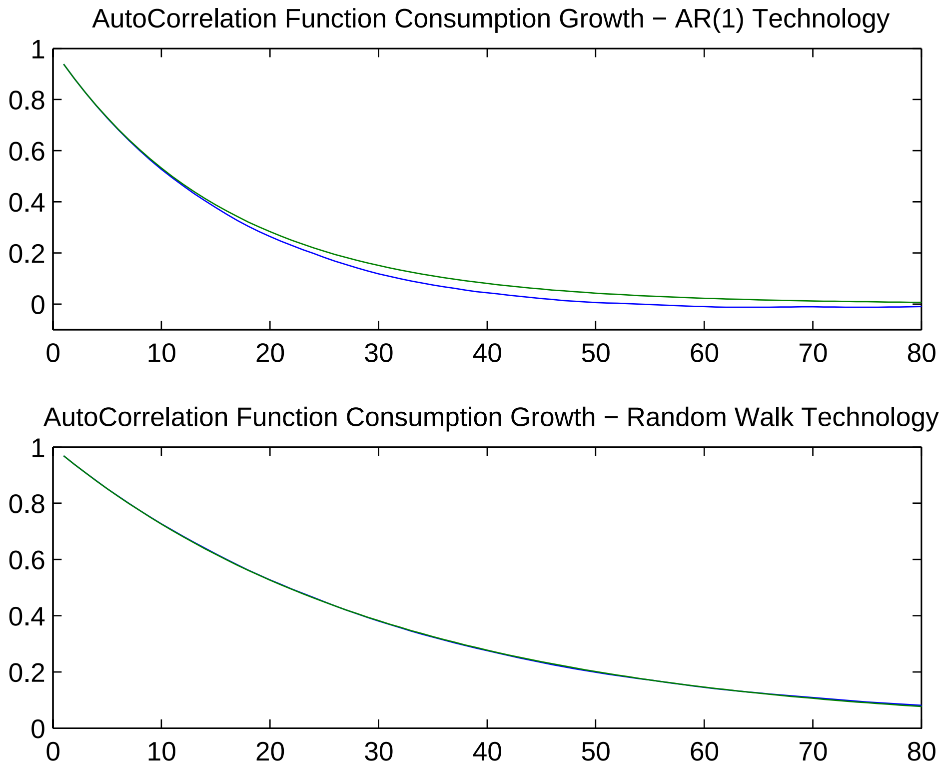

Figure 4.3: **Autocorrelation Functions Consumption Growth**
Comparison of the autocorrelation function obtained directly from simulated data of Model 3 and the Baseline Model to the autocorrelation function implied by the postulated process for expected consumption growth which we have estimated from the same simulated data of Model 3 and the Baseline Model respectively.

Here Δc_{t+1} is log realized consumption growth, x_t is the time-varying component of expected consumption growth, and η_t, e_t are zero mean, unit variance, and normally distributed disturbance terms with correlation $\sigma_{\eta,e}$. This functional form for log consumption growth is identical to the one assumed by Bansal and Yaron (2004) as driving process of their exchange economy model. Our results therefore provide a theoretical justification for their particular exogenous consumption growth process assumption. To evaluate whether the above specified process is a good approximation of the true consumption growth dynamics we first estimate the process from simulated data for a whole range of different model calibrations both with random walk- as well as with AR(1) technology processes. Then we compare the autocorrelation function obtained directly from the simulated data to the one implied by the above specified process which we have imposed on the data.

For the random walk technology the autocorrelation functions are virtually indistinguishable in all cases we have examined. Figure 4.3 shows this for the Baseline Model.[21] For the AR(1) technology the approximation turns out to get worse the lower the persistence of the driving process. Figure 4.3 shows the autocorrelation functions for Model 3. A look at Figure 4.1 makes clear why the above specified approximation for the dynamics of the endogenous process for consumption is worse for the case where technology shocks are transitory, because the impulse response of consumption to technology shocks is 'hump-shaped'. We therefore conclude that our postulated process is a good representation of the endogenous consumption growth dynamics for models with highly persistent technology shocks.[22]

4.13 Appendix F — The Linear Factor Model

The log stochastic discount factor is:

$$m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\, r_{a,t+1}, \tag{4.57}$$

where $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. The process for consumption growth is:

$$\Delta c_{t+1} = \mu + x_t + \sigma_\eta \eta_{t+1}, \tag{4.58}$$

$$x_{t+1} = \rho x_t + \sigma_e e_{t+1}, \tag{4.59}$$

$$\sigma_{\eta,e} = corr\left(\varepsilon_{t+1}, \eta_{t+1}\right). \tag{4.60}$$

For convenience, the shocks are normalized to have unit variance here, unlike in the main part of the chapter. Linearizing the wealth-consumption ratio around it's steady state, we obtain (see Campbell (1999) for a detailed derivation):

$$r_{a,t+1} \approx \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1}, \tag{4.61}$$

where pc_t is the log wealth-consumption ratio, $\kappa_1 = \frac{\exp(\overline{pc_t})}{1+\exp(\overline{pc_t})} \approx 0.96$, and $\overline{pc_t}$ is the steady state log wealth-consumption ratio. Assuming log aggregate consumption growth Δc_{t+1} to follow (4.58), Bansal and Yaron (2004) show that the log wealth-consumption ratio

[21] We assume the disturbance terms η and e to be i.i.d. normally distributed. The shocks we obtain when we estimate our postulated process for consumption growth from simulated data turn out to be very close to normal. They display mild heteroscedasticity.

[22] This conclusion relies on the assumption that the consumption process is covariance-stationary, which it is since the production function is constant returns to scale and preferences are homothetic. The autocorrelation function is then one of the fundamental time series representations. See, e.g., Hamilton (1994).

can be written as:

$$pc_{t+1} \approx A_0 + A_1 x_{t+1}, \tag{4.62}$$

where

$$A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}. \tag{4.63}$$

Since $0 < \rho < 1$, $0 < \kappa_1 \rho < 1$. Thus, $A_1 > (<)\ 0$ if $\psi > (<)\ 1$. We substitute for $r_{a,t+1}$ in the log stochastic discount factor (4.57):

$$\begin{aligned}
m_{t+1} &\approx \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\,\kappa_0 \\
&- (1-\theta)\,\kappa_1 pc_{t+1} + (1-\theta)\,pc_t - (1-\theta)\,\Delta c_{t+1} \\
&\approx \theta \ln \beta - (1-\theta)\,\kappa_0 - \left(1 - \theta + \frac{\theta}{\psi}\right) \Delta c_{t+1} \\
&- (1-\theta)\left[\kappa_1 A_0 + \kappa_1 A_1 x_{t+1} - A_0 - A_1 x_t\right] \\
&= \theta \ln \beta - (1-\theta)\,\kappa_0 - \left(1 - \theta + \frac{\theta}{\psi}\right) \Delta c_{t+1} - (1-\theta)\,A_0 \kappa_1 + (1-\theta)\,A_0 \\
&- (1-\theta)\left[\kappa_1 A_1 \left(\rho x_t + \sigma_e e_{t+1}\right) - A_1 x_t\right] \\
&= \theta \ln \beta - (1-\theta)\,\kappa_0 - (1-\theta)\,A_0 \kappa_1 + (1-\theta)\,A_0 - \left(1 - \theta + \frac{\theta}{\psi}\right) \Delta c_{t+1} \\
&- (1-\theta)\,A_1 \kappa_1 \rho x_t - (1-\theta)\,A_1 \kappa_1 \sigma_e e_{t+1} + (1-\theta)\,A_1 x_t.
\end{aligned} \tag{4.64}$$

Let:

$$\alpha = \theta \ln \beta - (1-\theta)\,\kappa_0 - (1-\theta)\,A_0 \kappa_1 + (1-\theta)\,A_0. \tag{4.65}$$

Then:

$$\begin{aligned}
m_{t+1} &\approx \alpha - \gamma \Delta c_{t+1} - (1-\theta)\,A_1 \kappa_1 \rho x_t - (1-\theta)\,A_1 \kappa_1 \sigma_e e_{t+1} + (1-\theta)\,A_1 x_t \\
&= \alpha - \gamma \Delta c_{t+1} + (1-\theta)\,A_1 \left(1 - \kappa_1 \rho\right) x_t - (1-\theta)\,A_1 \kappa_1 \sigma_e e_{t+1}.
\end{aligned} \tag{4.66}$$

Write this as:

$$m_{t+1} \approx a - b_1 \Delta c_{t+1} - b_2 e_{t+1} - b_3 x_t, \tag{4.67}$$

where $b_1 = \gamma$, $b_2 = (1-\theta)\,A_1 \kappa_1 \sigma_e > 0$, $b_3 = -\,(1-\theta)\,A_1 \left(1 - \kappa_1 \rho\right) < 0$, since $(1-\theta)\,A_1 = \frac{\gamma - \frac{1}{\psi}}{1 - \kappa_1 \rho}$. Thus, if $\gamma > \frac{1}{\psi}$, then $(1-\theta)\,A_1 > 0$. By applying a standard log-linear first-order approximation (see, e.g., Yogo (2006) for a similar application), the (not log) stochastic discount factor can be written as:

$$\frac{M_t}{E\left[M_t\right]} \approx 1 + m_t - E\left[m_t\right]. \tag{4.68}$$

This in turn implies a linear unconditional factor model (see Cochrane (2001)):

$$E\left[R_{i,t+1}-R_{0,t+1}\right]=b_1Cov\left(\Delta c_{t+1},R_{i,t+1}-R_{0,t+1}\right)+b_2Cov\left(e_{t+1},R_{i,t+1}-R_{0,t+1}\right)$$
$$+b_3Cov\left(x_t,R_{i,t+1}-R_{0,t+1}\right), \tag{4.69}$$

where $R_{i,t}$ denotes the time t gross return on asset i, and $R_{0,t}$ denotes the time t gross return on a reference asset (the risk-free rate). Recall that $b_1, b_2 > 0$, $b_3 < 0$. The sign of the price of risk of each factor depends on the covariance matrix of the factors. The permanent shock model predicts that $cov\left(\eta_{t+1}, e_{t+1}\right) > 0$, which we confirm in the data.

Chapter 5

Asset Pricing in Production Economies — Long-Run Risks, Wages, Dividends

5.1 Introduction

A large literature successfully explains moments and dynamics of asset prices by means of exchange economy models. Those models assume the process for consumption to be exogenous. It is clear that it would be very desirable to design models that can *jointly* explain the process for aggregate consumption and asset prices. However, the standard production economy model, while being able to generate realistic processes for consumption, is usually perceived to fail markedly at explaining asset prices.[1] We show in this chapter that a closer look is warranted. We show that by simply adjusting two usual assumptions the standard production economy model can be enabled to jointly explain the process for aggregate consumption as well as the most prominent asset pricing moments to an astonishing extent, and, importantly, without high levels of risk aversion: (i) the standard production economy model assumes wages that are far too volatile and far too procyclical, that way also misspecifying the process for aggregate dividends, and (ii) the standard production economy model furthermore assumes the elasticity of intertemporal substitution to be the reciprocal of the coefficient of relative risk aversion, that way

[1] Rouwenhorst (1995): '[...] it is more difficult to explain substantial risk premiums in a production economy, because consumption choices are endogenously determined and become smoother as risk aversion increases.', Boldrin, Christiano, Fisher (2001): '[RBC models] have been notoriously unsuccessful in accounting for the joint behavior of asset prices and consumption.', Cochrane (2005): '[Jermann (1998)] starts with a standard real business cycle (one-sector stochastic growth) model and verifies that its asset-pricing implications are a disaster.'

making consumers indifferent to endogenous long-run risks.

When we allow the elasticity of intertemporal substitution to be different from the reciprocal of the coefficient of relative risk aversion, that is we use Epstein and Zin (1989) preferences, consumers care about long-run risks. Bansal and Yaron (2004) have demonstrated that this allows a standard exchange economy model with Epstein and Zin preferences and exogenously assumed long-run consumption risks to easily match the market price of risk and equity Sharpe ratios. In Chapter 4 we demonstrate that the same is true for production economy models. There we show that standard production economy models give rise to endogenous long-run risks due to consumption smoothing activities of the representative household. In this chapter we rely on this mechanism in order to enable the standard production economy model to generate realistic equity return Sharpe ratios. Furthermore, in many standard production economy models the process for wages and consequentially the process for dividends are misspecified.[2] Wages are assumed to be the marginal product of labor and turn out much too volatile and too procyclical relative to their empirical counterpart. This renders dividends often *countercyclical*. We show that by specifying a different wage process, following the search and matching literature in labor economics, and calibrating that process to the data, we can alleviate this problem. The resulting dividend process from our model turns out to be quite close to the data and, importantly, is *procyclical*. This drives up equity risk premiums and allows the model, given that the model can already match the equity Sharpe ratio, to generate both a realistic value of the equity premium as well as realistic equity return volatility. Another way of understanding this is that by bringing wages closer to the data, that is by making them 'stickier', we increase the operating leverage of the firm, in turn driving up return premiums and return volatility.

Paralleling the history of explaining asset prices by means of exchange economy models, researchers have augmented the basic production economy model with habit preferences in order to remedy its shortcomings. Also paralleling recent and important research on exchange economy models (see Bansal and Yaron (2004)) we show that by carefully calibrating certain key processes (the process for aggregate consumption in the case of exchange economy models and the process for wages in the case of production economy models) and preference parameters (decoupling the elasticity of intertemporal substitution from the coefficient of relative risk aversion in both cases), habit preferences are actually not necessarily required. This can turn out advantageous, because in many models that assume habit preferences, in particular in production economy models where simple internal habits are assumed, the risk-free rate is way too volatile, and higher risk

[2]Dividends are defined as operating profits less investment, where operating profits are firm revenues less wage payments.

premiums are in that sense generated through a too volatile risk-free rate. Since the risk-free rate is the reciprocal of the conditional expected value of the stochastic discount factor, a misspecified risk-free rate implies a misspecified stochastic discount factor. The standard production economy model we propose can generate realistic risk premiums *without* excessive risk-free rate volatility and *without* unrealistically high levels of risk aversion.

This chapter is structured as follows. In Section 5.2 we give an overview of related literature. In Sections 5.3 and 5.4 we develop and calibrate the model. Section 5.5 reports and discusses results. In Section 5.6 we conclude.

5.2 Related Literature

This chapter makes a contribution to a literature Cochrane (2005) terms 'production-based asset pricing'. This literature tries to jointly explain the behavior of macroeconomic time series, in particular aggregate consumption, and asset prices. The starting point of this literature is the standard production economy model (standard stochastic growth model) and the observation that this model, while being able to generate realistic processes for consumption and investment, fails markedly at explaining asset prices.

Both Jermann (1998) and Boldrin, Christiano, Fisher (2001) augment the basic production economy framework with habit preferences in order to remedy its shortcomings. Boldrin, Christiano, Fisher also assume a two-sector economy with adjustment frictions across sectors and across time. Boldrin, Christiano, Fisher furthermore endogenize the labor-leisure decision, they assume however that labor cannot be adjusted immediately in response to technology shocks. Jermann and in particular Boldrin, Christiano, Fisher succeed to a considerable extent in jointly explaining with their models macroeconomic time series and asset prices. However, the price both models pay, typical for simple internal habit specifications, is excessive volatility of the risk-free rate. In a sense, their internal habits buy volatility in equity returns with volatility in risk-free rates. Since the risk-free rate is the reciprocal of the conditional expected value of the stochastic discount factor, a misspecified risk-free rate implies a misspecified stochastic discount factor. Relative to Jermann and Boldrin, Christiano, Fisher our contribution is to demonstrate that the standard production economy model without habit preferences (and without capital adjustment frictions) can actually, once appropriately adjusted and calibrated, jointly explain basic macroeconomic time series as well as asset prices *without* excessive risk-free rate volatility and high levels of risk aversion.

Jermann (1998) assumes labor to be fixed. Boldrin, Christiano, Fisher (2001) and

Uhlig (2004) show that once the labor-leisure choice is endogenized in the Jermann framework, the good asset pricing properties of the model disappear. The reason is that agents can use their labor-leisure choice to insure themselves against fluctuations in consumption, that is, agents can choose to work hard when aggregate productivity is low and work less when output is otherwise high, thus buffering the impact of technology shocks on output and consumption. The implications are a reduction in consumption risk and a counterfactual negative correlation between output and employment. Boldrin, Christiano, Fisher circumvent this problem by simply assuming that labor cannot respond immediately to a technology shock. The solution Uhlig points out is to let employment be determined solely by an exogenous process for wages and the demand for labor, that is by the firm's first-order condition: 'I do not claim that I have good microfoundations [for the exogenous wage process]. Rather, I regard it as a heuristically plausible starting point. [...] I can imagine that a microfoundation [...] could be found, following Hall (2003) [Hall (2005)] or the labor market search literature [...].' Uhlig concludes: 'The key to understanding macroeconomic facts and asset pricing facts jointly may be in understanding labor markets rather than agent heterogeneity.' In this chapter we abstract from the household's and the firm's labor-leisure decision. We assume instead that labor is fixed. However, we do something similar to Uhlig because we also specify a wage process that we calibrate to the data. In Chapter 3 we incorporate a search and matching model into a version of the Jermann model (with habit preferences) and show that this model can jointly explain macroeconomic time series and important moments and dynamics of asset prices. An important next step is to endogenize labor in the framework developed in this chapter, for example via a search and matching model like in Chapter 3.

This chapter is also related to the finance literature on long-run risks. Bansal and Yaron (2004) show in a standard exchange economy setup that a small but highly persistent and empirically barely distinguishable component of aggregate consumption growth can have quantitatively very important implications for asset prices if the representative agent cares about long-run consumption risks, that is if the representative agent has Epstein and Zin (1989) preferences. The point Bansal and Yaron make, similar to the point we make in this chapter for standard production economy models, is that standard exchange economy models without habit preferences can, contrary to Mehra and Prescott's (1985) seminal finding, generate realistic asset pricing moments and dynamics, with only minor adjustments to the driving process and preferences. In this chapter we show that similar minor adjustments can also enable the standard production economy model to jointly explain aggregate consumption and asset prices, contrary to what is often taken for granted in the literature. Like Bansal and Yaron we also assume Epstein and Zin preferences. While Bansal and Yaron amend the driving process, we adjust the process for wages.

For a more extensive discussion of the literature that tries to jointly explain macroeconomic time series and asset prices, please refer to Chapter 3. For a more extensive discussion of the finance literature about long-run risks, please refer to Chapter 4.

5.3 The Model

Our model is a standard production economy model (one-sector stochastic growth model) with capital adjustment costs and a representative household with Epstein and Zin (1989) preferences. We keep both production technology as well as the process for total factor productivity as standard as possible. We describe the key components of our model in turn.

5.3.1 Technology

We assume the following accumulation equation for the aggregate capital stock with capital adjustment costs:

$$K_{t+1} \leq \phi\left(I_t/K_t\right) K_t + (1-\delta)K_t, \tag{5.1}$$

where δ denotes the rate of depreciation, and $\phi(\cdot)$ is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it step by step.[3] We let:

$$\phi\left(I_t/K_t\right) = \frac{\alpha_1}{1-1/\xi}\left(\frac{I_t}{K_t}\right)^{(1-1/\xi)} + \alpha_2, \tag{5.2}$$

with:

$$\alpha_1 = \left(\exp(\bar{\varepsilon}) - 1 + \delta\right)^{1/\xi}, \tag{5.3}$$

$$\alpha_2 = \frac{1}{\xi - 1}\left(1 - \delta - \exp(\bar{\varepsilon})\right), \tag{5.4}$$

[3] This formulation of the capital accumulation equation with adjustment costs and $\phi(\cdot) \geq 0$, $\phi''(\cdot) \leq 0$ has been used by Uzawa (1969), Lucas and Prescott (1971), Hayashi (1982), Baxter and Crucini (1993), Jermann (1998), Boldrin, Christiano, Fisher (1999), Uhlig (2004), Guvenen (2005), amongst many others.

where $\overline{\varepsilon}$ is the trend growth rate of the economy.[4] The parameter ξ is the elasticity of the investment-capital ratio with respect to Tobin's q.[5] If $\xi = \infty$, (6.1) reduces to the capital accumulation equation without capital adjustment costs. We introduce capital adjustment costs in order to allow the model to fit a key moment of the aggregate consumption process: the volatility of consumption relative to the volatility of output. We show below that the assumption of capital adjustment costs is not in any way crucial to obtain our results. This is, to the best of our knowledge, in stark contrast to all other production economy models that can jointly explain asset prices and aggregate consumption (see Jermann (1998), Boldrin, Christiano, Fisher (2001), Guvenen (2005)).

The aggregate production technology is specified as:

$$Y_t = K_t^{\alpha} \left(Z_t N_t\right)^{1-\alpha}, \tag{5.5}$$

where N_t denotes aggregate employment and the process for aggregate productivity (Z_t) is given by:

$$\ln Z_t = \ln Z_{t-1} + \varepsilon_t, \tag{5.6}$$

with:

$$\varepsilon_t \sim N\left(\overline{\varepsilon}, \sigma_{\varepsilon}^2\right). \tag{5.7}$$

5.3.2 Preferences and Long-Run Risks

We assume a representative household with Epstein and Zin (1989) preferences:

$$U_t\left(C_t\right) = \left\{\left(1-\beta\right) C_t^{\frac{1-\gamma}{\theta}} + \beta\left(E_t\left[U_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right\}^{\frac{\theta}{1-\gamma}}, \tag{5.8}$$

where E_t denotes the expectation operator, C_t denotes aggregate consumption, β the discount factor, and $\theta = \frac{1-\gamma}{1-1/\psi}$. The Epstein and Zin preferences allow us to disentangle the coefficient of relative risk aversion (γ) from the elasticity of intertemporal substitution (ψ). The usual assumption of power utility preferences amounts to a special case of Epstein and Zin preferences with $\gamma = \frac{1}{\psi}$. Bansal and Yaron (2004) show that if consumers dislike fluctuations in expected consumption growth this can induce a high price of risk in

[4]It is straightforward to verify that $\phi(\frac{I_t}{K_t}) > 0$ and $\phi''(\frac{I_t}{K_t}) < 0$ for $\xi > 0$ and $\frac{I_t}{K_t} > 0$. Furthermore, $\phi(\frac{I}{K}) = \frac{I}{K}$ and $\phi'(\frac{I}{K}) = 1$, where $\frac{I}{K} = (\exp(\overline{\varepsilon}) - 1 + \delta)$ is the steady state investment-capital ratio. As a result, the balanced growth path of the economy is unaffected by the adjustment cost parameter ξ.

[5]The elasticity of the investment-capital ratio with respect to Tobin's q is $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \frac{\partial(I_t/K_t)}{\partial q_t} \times \frac{q_t}{(I_t/K_t)} = \left[\frac{\partial q_t}{\partial(I_t/K_t)} \times \frac{(I_t/K_t)}{q_t}\right]^{-1}$. Tobin's q is given by $\frac{1}{\phi'(I_t/K_t)} = \alpha_1 \left(\frac{I_t}{K_t}\right)^{1/\xi}$. It follows that $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \left[(1/\xi)\,\alpha_1 \left(\frac{I_t}{K_t}\right)^{1/\xi - 1} \frac{1}{\alpha_1} \left(\frac{I_t}{K_t}\right)^{1-1/\xi}\right]^{-1} = [1/\xi]^{-1} = \xi$.

the economy in the presence of long-run risks. As demonstrated in Chapter 4, consumption smoothing in a standard production economy model gives rise to a time-varying and slow-moving trend component in the endogenous consumption growth process, in other words, the production economy model we assume in this chapter endogenously generates long-run consumption risks. Following Bansal and Yaron and our findings in Chapter 4 we rely on endogenous long-run consumption risks in order to generate the high equity Sharpe ratios we observe in the data. We therefore calibrate the representative household to dislike fluctuations in future economic growth prospects ($\gamma > \frac{1}{\psi}$). For a more detailed discussion of the Epstein and Zin preferences and their relation to long-run consumption risks please refer to Chapter 4.

5.3.3 Employment, Wages, and Dividends

We assume households to supply a constant amount of hours worked and normalize $N_t = 1$. A very common assumption in the literature for the wage process in production economy models is that wages are equal to the marginal product of labor:[6]

$$\begin{aligned} \frac{\partial Y_t}{\partial N_t} &= W_t \\ &= (1-\alpha)\, K_t^{\alpha} Z_t^{1-\alpha} N_t^{-\alpha} \\ &= (1-\alpha)\, K_t^{\alpha} Z_t^{1-\alpha} \\ &= (1-\alpha)\, Y_t. \end{aligned} \tag{5.9}$$

The process for aggregate dividends follows:

$$\begin{aligned} D_t &= Y_t - W_t - I_t \\ &= Y_t - (1-\alpha)\, Y_t - I_t \\ &= \alpha Y_t - I_t. \end{aligned} \tag{5.10}$$

Because both wages as defined above and investment are both very procyclical, dividends often turn out *countercyclical*. As can be seen from equation (5.9), assuming wages to equal the marginal product of labor amounts to assuming that wages are equally volatile and perfectly correlated with output. While investment is very procyclical in the data, wages are not. Empirically, wages are about half as volatile as output and not very procyclical.

In the recent labor market search literature, making wages less volatile and less pro-

[6] This assumption ensures that the firm chooses $N_t = 1$. For the purpose of this chapter we *assume* that the firm chooses $N_t = 1$.

cyclical, that is in other words calibrating the wage process to the data in that sense, has been identified as an important avenue for making operating profits and consequentially fluctuations in firm values and ultimately employment more volatile and more procyclical. In that literature, the counterfactually low volatility of employment, induced by too low volatility of firm values, has been a long-standing puzzle, dubbed the 'Shimer puzzle'.[7] Making wages 'stickier' has been promoted as an important contribution to the resolution of the Shimer puzzle by Hall (2005).[8] We essentially propose the same in order to resolve the equity premium puzzle in production economy models. Instead of assuming that labor is paid its marginal product we postulate a different wage process and calibrate that process to the data.[9] As a result, firm operating profits and firm dividends are rendered more volatile and more procyclical compared to the original model. This in turn leads to more volatile firm values and equity returns and to a higher equity risk premium. One way of viewing this is that we increase the operating leverage of the firm by introducing a less volatile and less procyclical 'fixed-cost-component'.

We specify the following wage process:

$$W_t^{adj} = \omega_0 Y_t^{\omega_1} K_t^{1-\omega_1},$$

so that wages are a constant fraction of a composite measure of the size of the aggregate economy. Note that Y_t fluctuates much more around the trend growth rate than does K_t. As a result, wages are less volatile the lower ω_1.

5.4 Calibration

To be sure that our results don't depend on particular calibration choices we calibrate two different model versions to two different sets of aggregate macroeconomic data. In the first calibration (Model 1) we follow our calibration in Chapter 2 and use U.S. data from 1952 to 2004. The second calibration (Model 2) follows our calibration in Chapter 4 and Bansal and Yaron (2004) and calibrates our model to U.S. data from 1929 to 1998. Following Bansal and Yaron we calibrate both Model 1 and Model 2 to U.S. financial

[7]For accounts of (and solutions for) the lack of movement in employment within the standard Mortensen and Pissarides (1994) labor market matching framework, see for example den Haan, Ramey, Watson (2000), Hall (2005), Hornstein, Krusell, Violante (2005), Shimer (2005), and Chapters 2 and 4.

[8]In Chapters 2 and 4 we develop models that rely on more realistic wage processes in order to render operating profits, firm values, and ultimately employment more volatile.

[9]In the presence of labor market frictions, for instance a search and matching friction, there is no apparent reason to assume that labor is paid its marginal product. The search and matching literature assumes instead that wages are set as an outcome of a bargaining process between firm and worker. The wage process we propose is quite similar to the sticky wage rule Hall (2005) proposes in his seminal paper.

Table 5.1: **Calibration**

This table reports calibrated values of model parameters as well as the target moments from the data. We calibrate two different model versions to two different sets of aggregate macroeconomic data. In the first calibration (Model 1) we follow our calibration in Chapter 2 and use U.S. data from 1952 to 2004. We take logs and HP-filter the data using a smoothing parameter of 1600. The second calibration (Model 2) follows our calibration in Chapter 4 and Bansal and Yaron (2004) and calibrates our model to U.S. data from 1929 to 1998. To make our results comparable to Bansal and Yaron as well as to our results in Chapter 4 we take logs and the first difference only. Following Bansal and Yaron we calibrate both Model 1 and Model 2 to U.S. financial markets data from 1929 to 1998. We calibrate the wage process for both models to U.S. data from 1952 to 2004. For a discussion of the data used please refer to the appendix.

Quarterly Model						
Parameter	Description	Value			Target Statistic	
		Model 1a	Model 2a	Model 2b	Model 1	Model 2
		$\psi = 1.5$	$\psi = 1.5$	$\psi = 15$		
β	Discount factor	0.9991	0.9981	0.9950	$E\left[R_f\right] = 0.86\%$	
γ	CRRA	9.50	5.00	5.00	$SR[R^E] = 0.33$	
ξ	Cap. adj. costs	4.50	22.00	3.00	$\frac{\sigma[hp \ln c]}{\sigma[hp \ln y]} = 0.70$	$\frac{\sigma[\triangle \ln c]}{\sigma[\triangle \ln y]} = 0.52$
σ_ε	Std. tech. shock	1.19%	2.61%	2.61%	$\sigma\left[hp \ln y\right] = 1.58\%$	$\sigma\left[\triangle \ln y\right] = 2.51\%$
ω_0	Wage process	0.19	0.19	0.19	$E\left[\frac{D}{Y}\right] = 13\%$	
ω_1	Wage process	0.63	0.63	0.63	$\frac{\sigma[hp \ln w]}{\sigma[hp \ln y]} = 0.63$	

markets data from 1929 to 1998. We calibrate the wage process for both models to U.S. data from 1952 to 2004. For a discussion of the data used please refer to the appendix. For both calibrations we set the capital share (α), the depreciation rate (δ), and the mean productivity growth rate ($\overline{\varepsilon}$) to standard values used in the literature.[10] We follow our calibration in Chapter 4 and set the elasticity of intertemporal substitution (ψ) to 1.5 for Model 1a and Model 2a. In order to match moments from financial markets as well as possible we set ψ to 15 for Model 2b.[11] Table 5.1 contains the remaining model parameters, their calibrated values, as well as the target moments from the data.

5.5 Results

We use the value function iteration algorithm to solve all models. For more details regarding the numerical solution technique please refer to the appendix. Table 5.2 reports asset pricing moments for all models. We report both the original equity return, that is

[10]In particular, we set $\alpha = 0.34$, $\delta = 0.021$, $\overline{\varepsilon} = 0.4\%$ (see, e.g., Boldrin, Christiano, Fisher (2001))

[11]A value for the elasticity of intertemporal substitution (EIS) of 1.5 is not unreasonable. Hansen and Singleton (1982) as well as Attanasio and Weber (1989) estimate the EIS to be even higher than 1.5. Vissing-Jorgensen (2002) and Guvenen (2006) also argue that the EIS is higher than unity, while Hall (1988) and Campbell (1999) estimate the EIS to be well below unity. Of course, a value for the EIS of 15 is unreasonable.

the return of a claim on the original dividend process:

$$D_t = \alpha Y_t - I_t, \tag{5.11}$$

as well as the adjusted equity return, that is a claim on the new dividend process:

$$D_t^{adj} = Y_t - W_t^{adj} - I_t, \tag{5.12}$$
$$W_t^{adj} = \omega_0 Y_t^{\omega_1} K_t^{1-\omega_1}. \tag{5.13}$$

From Table 5.2 we can see that the standard production economy model does actually have the potential to match both the level as well as the volatility of the equity premium. As soon as we correct the process for wages and therefore dividends the premium of the unlevered equity return increases from 0.19% to 1.41% for Model 2a and from 1.29% to 2.39% for Model 2b. Because the equity return from the data is the return on a levered equity claim we add financial leverage to our model (see Appendix C for more details). With financial leverage the model is able to generate an equity premium of up to 4.31% with an equity return volatility of 14.39% and thus comes very close to generate realistic values for the equity premium as well as for the equity return volatility. Importantly, the model generates volatility in the equity return not through excessive volatility of the risk-free rate. If anything, the volatility of the risk-free rate in our model is too low. This is in contrast to many models that rely on habit preferences, in particular the production economy models due to Jermann (1998) and Boldrin, Christiano, Fisher (2001). Note also that the model does not rely on capital adjustment frictions to generate volatility in returns. In Section 5.5.1 we show that the production economy model can still generate high return premiums and volatility in the absence of any capital adjustment frictions.

From Table 5.2 we see that the model underestimates the volatility of dividend growth, while it is better at matching the relatively low correlation of dividend growth with consumption growth.[12] But given that we calibrate the wage process and *not* the process for dividends, the model does generate remarkably realistic dividend processes. Higher dividend volatility would, everything else equal of course, further increase the equity return premium and volatility.

Table 5.3 reports macroeconomic moments for all models. We choose the capital adjustment costs for each model so as to match the relative volatility of consumption to output with each model. We can see that all models underestimate the relative volatility of investment with output as a result. This is unsatisfactory but not the topic of this

[12]We take U.S. data from 1929 to 1998 on unlevered dividends from NIPA (personal income receipts on assets = personal interest income + personal dividend income).

Table 5.2: **Results — Asset Pricing Moments**

This table reports asset pricing moments for all models. The coefficient of relative risk aversion (γ) is 9.5 for Model 1 and 5 for Model 2, in order to match with both models the equity Sharpe ratio. We adjust the discount factor (β) across models in order to match the level of the risk-free rate with each model. The elasticity of intertemporal substitution (ψ) is 1.5 for Model 1a and Model 2a and 15 for Model 2b. We report both the original equity return, that is the return of a claim on the original dividend process, as well as the adjusted equity return, that is a claim on the adjusted dividend process. B/V denotes the leverage ratio, that is the value of debt (B) relative to the total firm value (V). The data are taken from Bansal and Yaron (2004) who use U.S. financial markets data from 1929 to 1998. We take U.S. data from 1929 to 1998 on unlevered dividends from NIPA (personal interest income plus personal dividend income). All values reported in the table are annual.

Statistic	Data	Model 1a	Model 2a	Model 2b
$\sigma[M]/E[M]$	na	0.33	0.34	0.35
$E[R_f]$ (%)	0.86	0.89	0.86	0.86
$\sigma[R_f]$ (%)	0.97	0.15	0.43	0.04
$SR[R^E]$	0.33	0.33	0.33	0.35
$E[R^E - R_f]$ (%)	6.33	0.30	0.19	1.29
$\sigma[R^E - R_f]$ (%)	19.42	0.90	0.58	3.69
$SR[R^{Eadj}]$	0.33	0.33	0.34	0.35
$E[R^{Eadj} - R_f]$ (%)	6.33	0.76	1.41	2.39
$\sigma[R^{Eadj} - R_f]$ (%)	19.42	2.29	4.21	6.84
$\sigma[\triangle \ln d^{adj}]$ (%)	5.27	2.05	0.99	2.12
$\sigma\left[\triangle \ln d^{adj}, \triangle \ln c\right]$	0.45	1.00	0.45	0.16
Financial Leverage (Model 2a)				
	$B/V = 0.00$	$B/V = 0.33$	$B/V = 0.50$	
$E[R^{Eadj} - R_f]$ (%)	1.41	2.12	2.67	
$\sigma[R^{Eadj} - R_f]$ (%)	4.21	6.56	8.59	
Financial Leverage (Model 2b)				
	$B/V = 0.00$	$B/V = 0.33$	$B/V = 0.50$	
$E[R^{Eadj} - R_f]$ (%)	2.39	3.51	4.31	
$\sigma[R^{Eadj} - R_f]$ (%)	6.84	10.74	14.39	

Table 5.3: **Results — Macroeconomic Moments**
This table reports macroeconomic moments for all models. We calibrate two different model versions to two different sets of aggregate macroeconomic data. In the first calibration (Model 1) we follow our calibration in Chapter 2 and use U.S. data from 1952 to 2004. We take logs and HP-filter the data using a smoothing parameter of 1600. The second calibration (Model 2) follows our calibration in Chapter 4 and Bansal and Yaron (2004) and calibrates our model to U.S. data from 1929 to 1998. To make our results comparable to Bansal and Yaron as well as to our results in Chapter 4 we take logs and the first difference only. Capital adjustment costs (ξ) are adjusted for each model so as to match the relative volatility of consumption to output. The standard deviation of the technology shock (σ_ε) is set to match the volatility of output. The elasticity of intertemporal substitution (ψ) is 1.5 for Model 1a and Model 2a and 15 for Model 2b. All values reported in the table are quarterly.

Statistic	Data 1	Model 1a	Statistic	Data 2	Model 2a	Model 2b
$\sigma[hp \ln y]$ (%)	1.58	1.57	$\sigma[\triangle \ln y]$ (%)	2.51	2.52	2.52
$\sigma[hp \ln c] / \sigma[hp \ln y]$	0.70	0.69	$\sigma[\triangle \ln c] / \sigma[\triangle \ln y]$	0.52	0.52	0.52
$\sigma[hp \ln i] / \sigma[hp \ln y]$	4.56	1.58	$\sigma[\triangle \ln i] / \sigma[\triangle \ln y]$	3.32	1.90	2.03
$\sigma[hp \ln c, hp \ln y]$	0.78	1.00	$\sigma[\triangle \ln c, \triangle \ln y]$	0.66	0.99	0.98
$\sigma[hp \ln i, hp \ln y]$	0.88	1.00	$\sigma[\triangle \ln i, \triangle \ln y]$	0.22	0.99	1.00

chapter. Usually in the real business cycle literature models are calibrated so as to underestimate the relative volatility of consumption to output, thereby increasing the relative volatility of investment.[13] We focus our calibration on the relative volatility of consumption to output, because we want the model to generate an as realistic as possible endogenous process for consumption, as we compare our production economy model to exchange economy models that assume an exogenous and realistic process for consumption.

5.5.1 The Model without Capital Adjustment Costs

We haves used capital adjustment costs solely as a free parameter in order to match the relative volatility of consumption to output with all models. In this section we show that the standard production economy model, even in the absence of any capital adjustment frictions, is still able to generate realistic asset prices. We assume the standard capital accumulation without capital adjustment costs:

$$K_{t+1} \leq I_t + (1-\delta)K_t, \tag{5.14}$$

$$K_{t+1} \leq Y_t - C_t + (1-\delta)K_t. \tag{5.15}$$

[13] See, for example, Cooley and Prescott (1995).

In Table 5.4 we show results for Model 1c and for Model 2c, where we keep all parameter values (apart from the capital adjustment cost parameter ξ) the same as for Models 1a and 2a respectively.

We conclude that the standard production economy model, even in the absence of any capital adjustment frictions, is still able to generate realistic asset prices. The equity premium decreases somewhat (for example from 1.41% to 1.31% from Model 2a to Model 2c), due to more volatile investment and as a consequence less procyclical dividends ($D_t = Y_t - W_t - I_t$).

5.6 Conclusion

We show that by relatively simple adjustments the standard production economy model can be enabled to jointly explain the process for aggregate consumption as well as the most prominent asset pricing moments to a considerable extent. The adjustments we propose are to preferences, we assume Epstein and Zin (1989) preferences, and to the process for wages and therefore dividends, we move away from the counterfactual assumption of wages being equal to the marginal product of labor and assume instead a process for wages that we calibrate to the data. We refute the notion that the standard production economy model is not able to generate realistic asset prices. Our standard production economy model comes very close to match both level and volatility of the equity premium. Contrary to many models in the literature that build habit preferences into the standard production economy model, our model generates a low volatility of the risk-free rate. Also, our calibration does not assume excessively high levels of risk aversion, and we do not rely on any capital adjustment frictions.

In this chapter we assume labor to be fixed. Of course, an important next step is to endogenize labor within the framework developed in this chapter. Chapters 2 and 4, where we incorporate search and matching models of the labor market into the standard production economy model, show the way. In this chapter we confine ourselves to demonstrating the potential that the standard production economy model has, contrary to what is often asserted.

Table 5.4: **Results — The Model without Capital Adjustment Costs**
This table reports macroeconomic moments and asset pricing moments for models without capital adjustment costs. The coefficient of relative risk aversion (γ) is 9.5 for Model 1a and 5 for Model 2a, in order to match with both models the equity Sharpe ratio. We adjust the discount factor (β) in order to match the level of the risk-free rate with Model 1a and Model 2a. For Model 1c and Model 2c we keep all parameter values (apart from the capital adjustment cost parameter (ξ)) the same as for Model 1a and for Model 2a respectively. The elasticity of intertemporal substitution (ψ) is 1.5 for all models. In the first calibration (Model 1) we follow our calibration in Chapter 2 and use U.S. data from 1952 to 2004 for the macroeconomic series (output, consumption, investment). We take logs and HP-filter the data using a smoothing parameter of 1600. The second calibration (Model 2) follows our calibration in Chapter 4 and Bansal and Yaron (2004) and calibrates our model to U.S. data from 1929 to 1998. To make our results comparable to Bansal and Yaron as well as to our results in Chapter 4 we take logs and the first difference only. For all models, we report both the original equity return, that is the return of a claim on the original dividend process, as well as the adjusted equity return, that is a claim on the adjusted dividend process. The financial markets data are taken from Bansal and Yaron (2004) who use U.S. financial markets data from 1929 to 1998. All values reported in the table are annual.

Statistic	Data	Model 1a $\xi = 4.50$	Model 1c $\xi = \infty$	Model 2a $\xi = 22$	Model 2c $\xi = \infty$
$\sigma[\ln y]$ (%)	3.16 / 5.02	3.14	3.14	5.04	5.04
$\sigma[\ln c]/\sigma[\ln y]$	0.70 / 0.52	0.69	0.48	0.52	0.47
$\sigma[\ln i]/\sigma[\ln y]$	4.56 / 3.32	1.58	1.96	1.90	2.00
$\sigma[M]/E[M]$	na	0.33	0.33	0.34	0.33
$E[R_f]$ (%)	0.86	0.89	1.01	0.86	0.94
$\sigma[R_f]$ (%)	0.97	0.15	0.21	0.43	0.47
$SR[R^E]$	0.33	0.33	0.33	0.33	0.32
$E[R^E - R_f]$ (%)	6.33	0.30	0.02	0.19	0.04
$\sigma[R^E - R_f]$ (%)	19.42	0.90	0.06	0.58	0.12
$SR[R^{Eadj}]$	0.33	0.33	0.33	0.34	0.33
$E[R^{Eadj} - R_f]$ (%)	6.33	0.76	0.63	1.41	1.31
$\sigma[R^{Eadj} - R_f]$ (%)	19.42	2.29	1.93	4.21	3.95

5.7 References

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5.8 Appendix A — Numerical Solution

We solve the following model:

$$V\left(K_t, Z_t\right) = \max_{C_t, K_{t+1}} \left\{ \left[(1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[V(K_{t+1}, Z_{t+1})^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \right\}, \tag{5.16}$$

$$K_{t+1} = (1-\delta) K_t + \phi \left(\frac{I_t}{K_t} \right) K_t, \tag{5.17}$$

$$I_t = Y_t - C_t, \tag{5.18}$$

$$Y_t = Z_t^{(1-\alpha)} K_t^{\alpha}, \tag{5.19}$$

$$\ln Z_{t+1} = \ln Z_t + \varepsilon_{t+1}, \tag{5.20}$$

$$\varepsilon_t \sim N\left(\mu, \sigma_\varepsilon\right). \tag{5.21}$$

Since the process for productivity is non-stationary, we need to normalize the economy by Z_t, in order to be able to numerically solve the model. To be precise, we let $\widehat{K}_t = \frac{K_t}{Z_t}, \widehat{C}_t = \frac{C_t}{Z_t}, \widehat{I}_t = \frac{I_t}{Z_t}$, and substitute. In the so transformed model all variables are stationary. The only state variable of the normalized model is $\widehat{K}$.[14] We can work directly on the appropriately normalized set of equations and then re-normalize after having solved the model.

The value function is given by:

$$\widehat{V}\left(\widehat{K}_t\right) = \max_{\widehat{C}_t, \widehat{K}_{t+1}} \left\{ \left[\begin{array}{c} (1-\beta) \widehat{C}_t^{\frac{1-\gamma}{\theta}} \\ +\beta \left(E_t \left[\left(e^{\varepsilon_{t+1}}\right)^{1-\gamma} \left(\widehat{V}\left(\widehat{K}_{t+1}\right) \right)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \end{array} \right]^{\frac{\theta}{1-\gamma}} \right\}. \tag{5.22}$$

We parameterize the value function with a $10th$ order Chebyshev orthogonal polynomial over a 11×1 Chebyshev grid for the state variable $\widehat{K}$:

$$\Psi^A\left(\widehat{K}\right) = \widehat{V}\left(\widehat{K}\right). \tag{5.23}$$

We use the value function iteration algorithm. At each grid point for the state $\widehat{K}$, given

[14] Note that Z is *not* a state variable of the normalized model. This is due to the fact that we assume the autoregressive coefficient of the process for productivity $\ln Z_{t+1} = \rho \ln Z_t + \varepsilon_{t+1}$ to be unity: $\rho = 1$. As a consequence, ΔZ is serially uncorrelated.

a polynomial for the value function $\Psi_i^A\left(\widehat{K}\right)$, we use a numerical optimizer to find the policy $(\widehat{C}^*)$ that maximizes the value function:

$$\widehat{K}_{t+1}^* e^{\varepsilon_{t+1}} = \widehat{Y}_t - \widehat{C}_t^* + (1-\delta)\widehat{K}_t, \tag{5.24}$$

$$\widehat{V}^*\left(\widehat{K}_t\right) = \left[(1-\beta)\left(\widehat{C}_t^*\right)^{\frac{1-\gamma}{\theta}} + \beta\left(E_t\left[\left(e^{\varepsilon_{t+1}}\right)^{1-\gamma}\left(\Psi_i^A\left(\widehat{K}_{t+1}^*\right)\right)^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \tag{5.25}$$

where Gauss-Hermite quadrature with 5 nodes is used to approximate the expectations operator. We use a regression of $\widehat{V}^*$ on $\widehat{K}$ in order to update the coefficients of the polynomial for the value function and so obtain $\Psi_{i+1}^A\left(\widehat{K}\right)$.

5.9 Appendix B — Data Description

U.S. Data from 1952 to 2004

GDP, Investment, Consumption

- Real gross domestic product, GDPC96
- Real gross private domestic investment, GPDIC96
- Real personal consumption expenditures (nondurable goods), PCNDGC96

These series were downloaded from Federal Reserve Economic Data (FRED).

Wages

- Output, PRS85006043
- Current $ output, PRS85006053
- Employment, PRS85006033
- Nominal compensation, PRS85006063

These series were downloaded from the Bureau of Labor Statistics. As is standard in the literature, these series are for the non-farm business sector. The wage rate was calculated using:

$$wage\ rate = \frac{Compensation}{Employment} \times \frac{Output}{Current\ \$\ Output}. \tag{5.26}$$

U.S. Data from 1929 to 1998

GDP, Investment, Consumption

- Real gross domestic product
- Real fixed investment
- Real personal consumption expenditures (nondurable and services)

These series were downloaded from the Bureau of Economic Analysis (BEA), National Income and Product Accounts (NIPA).

Dividends

- Real personal income receipts on assets (personal interest income plus personal dividend income)

These series were downloaded from the Bureau of Economic Analysis (BEA), National Income and Product Accounts (NIPA).

Financial Markets Data We take the financial markets data directly from Bansal and Yaron (2004).

5.10 Appendix C — Leverage

Let there be a term structure of risk-free corporate bonds, that is we abstract from default. Following Jermann (1998), each period the firm issues j-period discount bonds for a fixed fraction v/j of its capital stock, K_{t+1}, and pays back its debt that comes to maturity, $(v/j)\, K_{t-(j-1)}$. The dividends to shareholders after debt payments follow as:

$$D_t = \Pi_t - I_t - \frac{v}{j}\left(K_{t-(j-1)} - \frac{K_{t+1}}{1 + R_{f,t\to t+j}}\right), \tag{5.27}$$

where $R_{f,t\to t+j}$ denotes the total discount on a zero coupon bond maturing in j periods, and Π_t denotes operating profits ($\Pi_t = Y_t - W_t$). To calculate equity returns, we use the well-known relation:

$$R^A_{t+1} = \frac{P^E_t}{P^A_t} R^E_{t+1} + \frac{P^D_t}{P^A_t} R^D_{t+1}, \tag{5.28}$$

where P_t^E is the market value of equity, P_t^D is the market value of debt, $P_t^A = P_t^E + P_t^D$, R_{t+1}^E is the equity return, and $R_{t+1}^D = R_{f,t+1}$ since debt is assumed to be risk-free. Thus:

$$R_{t+1}^A - R_{f,t+1} = \frac{P_t^A - P_t^D}{P_t^A}\left(R_{t+1}^E - R_{f,t+1}\right), \tag{5.29}$$

$$R_{t+1}^E - R_{f,t+1} = \frac{P_t^A}{P_t^A - P_t^D}\left(R_{t+1}^A - R_{f,t+1}\right). \tag{5.30}$$

The market value of debt is:

$$P_t^D = \left(\frac{v}{j}\right)\sum_{i=0}^{j-1}\frac{K_{t+1-i}}{1 + R_{f,t\to t+j-i}}. \tag{5.31}$$

We assume $j = 40$, that is a maturity of 10 years. Table 5.2 shows that realistic average leverage levels almost double both the equity premium and the equity return volatility.

Chapter 6

The Term Structure of Interest Rates in Standard Production Economy Models

6.1 Introduction

In this chapter we examine the term structure of interest rates in standard production economy models with Epstein and Zin (1989) preferences. We find that the average term structure generated by exchange economy and production economy models both with power utility preferences *and* with Epstein and Zin preferences is downward sloping. The jury seems to be still out on whether this contradicts the data or not. While simple corrections for (expected) inflation, like in this chapter, typically suggest that the average real term structure slopes upward, evidence from indexed bonds suggests that the real term structure actually slopes downward on average.[1]

The dynamic behavior of the term structure generated by the standard production economy model on the other hand clearly fits the data well: during an economic expansion the term structure is flatter or even inverted, while the term structure is steeper and upward sloping during a recession. When we feed shocks to total factor productivity from the data into the model, the correlation between the model implied time series of the term spread and the empirical time series of the term spread is relatively high.

Earlier findings in the literature assert that both standard consumption based exchange economy models and standard production economy models with power utility preferences fail at generating sufficiently volatile term spreads.[2] We show that the same

[1] See Piazzesi and Schneider (2006).

[2] See, e.g., Backus, Gregory, Zin (1989) and Den Haan (1995).

is true for the model with Epstein and Zin (1989) preferences. So, while augmenting the standard production economy model with Epstein and Zin preferences improves the ability of the model to explain equity risk premiums to a substantial degree, as demonstrated in Chapters 4 and 5, the same can, unfortunately, not be said of the standard model's ability to explain the behavior of bond risk premiums.

Recent literature in finance has demonstrated that small changes to driving processes can have enormous impacts on the asset prices generated by exchange economy models (see, e.g., Bansal and Yaron (2004) or Rodriguez (2006)). We examine whether the same is true for production economy models. Following the literature in finance we consider a model where the driving process (total factor productivity) is assumed to have a time-varying and slow-moving trend component, and where the volatility of the technology shock is allowed to change over time. We show that for production economy models those changes to the driving process barely impact the asset prices generated by the model: both the term structure but also the equity returns remain almost unchanged. The reason is that, contrary to exchange economy models, the production economy framework allows agents to optimally respond with their consumption and savings decision to the additional shocks.

We proceed as follows: we start by providing an overview of related literature. Then we present and calibrate the model. In Sections 6.5 and 6.6 we discuss and interpret results. Section 6.7 concludes.

6.2 Related Literature

For a discussion of the literature that builds upon the production economy model in order to jointly explain asset prices and aggregate consumption, as well as the literature that considers small changes to driving processes and demonstrates that those minor adjustments can improve the exchange economy model's performance in terms of explaining the behavior of asset prices, please refer to Chapters 3, 4, and 5. Our paper is furthermore related to the literature that examines consumption based models of the term structure of interest rates.

Backus, Gregory, Zin (1989) and Backus and Zin (1994) are, to the best of our knowledge, the first to point out the failure of the standard consumption based exchange economy model with power utility preferences to explain important aspects of the term structure: 'We show that the model can account for neither the sign nor the magnitude of average risk premiums in forward prices and holding-period returns. The economy is also incapable of generating enough variation in risk premiums [...].' Den Haan (1995) documents the same problem for the standard production economy model with power utility preferences. In Chapters 4 and 5 we have demonstrated that augmenting the standard production economy model with Epstein and Zin (1989) preferences improves the ability of the model to explain equity risk premiums to a substantial degree. In this chapter we document that the same can, unfortunately, not be said of the standard model's ability to explain the behavior of bond risk premiums.

Wachter (2006) proposes a consumption based exchange economy model that can explain important aspects of the term structure. She builds her model around the Campbell and Cochrane (1999) external habit preference specification. She specifies the preferences such that the risk-free rate is negatively correlated with surplus consumption, the state variable, contrary to power utility preferences where the risk-free rate and aggregate consumption, the state variable there, are positively correlated. As a result, bond prices and therefore bond returns and surplus consumption are also positively correlated, again, contrary to the case with power utility preferences. This induces a positive risk premium on bonds and thus a term structure that is on average upward sloping. With external habit preferences the price of risk is furthermore both countercyclical and in particular strongly varying through time. This helps Wachter to explain the high average term spreads and the high volatility of the term spreads. Lettau and Uhlig (2000), however, demonstrate that assuming the Campbell and Cochrane external habit specification in a production economy framework is problematic because those preferences have counterfactual implications once embedded into production economy models. In this chapter we show that a standard production economy model with Epstein and Zin preferences can, unfortunately, not explain the relatively high volatility of the term spreads. We conclude that, while Wachter's model may be able to remedy that shortcoming within the standard exchange economy framework, the literature is still lacking a production economy model that can explain the empirical volatility of the term spreads.

6.3 The Model

As in Chapters 4 and 5, our model is a standard production economy model (one sector stochastic growth model) with capital adjustment costs and a representative household with Epstein and Zin (1989) preferences. We keep both production technology as well as the process for total factor productivity as standard as possible. We describe the key components of our model in turn.

6.3.1 Technology

We assume the following accumulation equation for the aggregate capital stock with capital adjustment costs:

$$K_{t+1} \leq \phi\left(I_t/K_t\right) K_t + (1-\delta)K_t, \tag{6.1}$$

where δ denotes the rate of depreciation, and $\phi(\cdot)$ is a positive, concave function, capturing the notion that adjusting the capital stock rapidly by a large amount is more costly than adjusting it step by step.[3] We let:

$$\phi\left(I_t/K_t\right) = \frac{\alpha_1}{1-1/\xi}\left(\frac{I_t}{K_t}\right)^{(1-1/\xi)} + \alpha_2, \tag{6.2}$$

with:

$$\alpha_1 = \left(\exp(\overline{\mu}) - 1 + \delta\right)^{1/\xi}, \tag{6.3}$$

$$\alpha_2 = \frac{1}{\xi - 1}\left(1 - \delta - \exp(\overline{\mu})\right), \tag{6.4}$$

where $\overline{\mu}$ is the trend growth rate of the economy.[4] The parameter ξ is the elasticity of the investment-capital ratio with respect to Tobin's q.[5] If $\xi = \infty$, (6.1) reduces to the capital accumulation equation without capital adjustment costs. We introduce capital adjustment costs in order to allow the model to fit a key moment of the aggregate consumption process: the volatility of consumption relative to the volatility of output.

The aggregate production technology is specified as:

$$Y_t = K_t^{\alpha}\left(Z_t L_t\right)^{1-\alpha}, \tag{6.5}$$

where L_t denotes aggregate employment. For the purpose of this chapter we assume households to supply a constant amount of hours worked and normalize $L_t = 1$.[6] Z_t

[3] This formulation of the capital accumulation equation with adjustment costs and $\phi(\cdot) \geq 0$, $\phi''(\cdot) \leq 0$ has been used by Uzawa (1969), Lucas and Prescott (1971), Hayashi (1982), Baxter and Crucini (1993), Jermann (1998), Boldrin, Christiano, Fisher (2001), Uhlig (2004), Guvenen (2005), amongst many others.

[4] It is straightforward to verify that $\phi(\frac{I_t}{K_t}) > 0$ and $\phi''(\frac{I_t}{K_t}) < 0$ for $\xi > 0$ and $\frac{I_t}{K_t} > 0$. Furthermore, $\phi(\frac{I}{K}) = \frac{I}{K}$ and $\phi'(\frac{I}{K}) = 1$, where $\frac{I}{K} = (\exp(\overline{\mu}) - 1 + \delta)$ is the steady state investment-capital ratio. As a result, the balanced growth path of the economy is unaffected by the adjustment cost parameter ξ.

[5] The elasticity of the investment-capital ratio with respect to Tobin's q is $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \frac{\partial(I_t/K_t)}{\partial q_t} \times \frac{q_t}{(I_t/K_t)} = \left[\frac{\partial q_t}{\partial(I_t/K_t)} \times \frac{(I_t/K_t)}{q_t}\right]^{-1}$. Tobin's q is given by $\frac{1}{\phi'(I_t/K_t)} = \alpha_1\left(\frac{I_t}{K_t}\right)^{1/\xi}$. It follows that $\frac{\partial(I_t/K_t)}{(I_t/K_t)} / \frac{\partial q_t}{q_t} = \left[(1/\xi)\,\alpha_1\left(\frac{I_t}{K_t}\right)^{1/\xi-1}\frac{1}{\alpha_1}\left(\frac{I_t}{K_t}\right)^{1-1/\xi}\right]^{-1} = [1/\xi]^{-1} = \xi$.

[6] Assuming that households supply a constant amount of labor amounts to assuming that households

denotes the process for aggregate productivity. We model log technology both as a random walk with drift, and as an AR(1) with a time trend:

$$\ln Z_t = \overline{\mu} + \ln Z_{t-1} + \sigma_\varepsilon \varepsilon_t, \tag{6.6}$$

$$\varepsilon_t \sim N(0,1), \tag{6.7}$$

or:

$$\ln Z_t = \overline{\mu} t + \rho \ln Z_{t-1} + \sigma_\varepsilon \varepsilon_t, \tag{6.8}$$

$$\varepsilon_t \sim N(0,1). \tag{6.9}$$

Thus, (6.6) implies that technology shocks are permanent whereas (6.8) implies that technology shocks are transitory. In Section 6.6 we generalize the process for log technology and introduce a time-varying trend growth rate (μ_t), as well as a time-varying volatility of the technology shock ($\sigma_{\varepsilon,t}$).

6.3.2 Preferences

We assume a representative household with Epstein and Zin (1989) preferences:

$$U_t(C_t) = \left\{ (1-\beta) C_t^{\frac{1-\gamma}{\theta}} + \beta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\gamma}}, \tag{6.10}$$

where E_t denotes the expectation operator, C_t denotes aggregate consumption, β the discount factor, and $\theta = \frac{1-\gamma}{1-1/\psi}$. The Epstein and Zin preferences allow us to disentangle the coefficient of relative risk aversion (γ) from the elasticity of intertemporal substitution (ψ). The usual assumption of power utility preferences amounts to a special case of Epstein and Zin preferences with $\gamma = \frac{1}{\psi}$. Bansal and Yaron (2004) show that if consumers dislike fluctuations in expected consumption growth, this can induce a high price of risk in the economy in the presence of long-run risks. As demonstrated in Chapter 4, consumption smoothing in a standard production economy model gives rise to a time-varying and slow-moving trend component in the endogenous consumption growth process, which helps the model to explain equity returns to a considerable extent. In this chapter we examine whether Epstein and Zin preferences can also improve the standard production economy model's ability to explain bond returns. If ($\gamma > \frac{1}{\psi}$) the representative household dislikes fluctuations in future economic growth prospects. For a more detailed discussion of the Epstein and Zin preferences and their relation to long-run consumption risks please

incur no dis-utility from working.

refer to Chapter 4.

6.4 Calibration and Macroeconomic Moments

Following our calibrations in Chapters 4 and 5 we set the capital share (α), the depreciation rate (δ), and the mean productivity growth rate ($\overline{\mu}$) to standard values used in the literature.[7] As in Chapters 4 and 5 we set the coefficient of relative risk aversion (γ) to 5 and the discount factor (β) to 0.998 across all models.[8] We vary the elasticity of intertemporal substitution (ψ) across models and we report results for the two different specifications of the technology process (transitory shocks ($\rho = 0.90$) and permanent shocks). Our Baseline Model is, following Chapters 4 and 5, the model with permanent technology shocks and an elasticity of intertemporal substitution (ψ) of 1.5. As demonstrated in Chapters 4 and 5, in that model endogenous long-run risks help the model to explain the high equity Sharpe ratio in the data. We calibrate the model to U.S. data from 1950Q1 to 2006Q3 (for a description of the data used please refer to the appendix) by lowering the technology shock (σ_ε) sufficiently, relative to our calibrations in Chapters 4 and 5, so that all models in this chapter are able to match the lower consumption volatility in post-war data (in Chapters 4 and 5 we calibrate the model to U.S. data from 1928 to 1998). Because we use the same level of capital adjustment costs (ξ) as in Chapters 4 and 5, our Baseline Model slightly overestimates the relative volatility of consumption to output. In Table 6.1 we report major moments from macroeconomic time series for all models.

6.5 Results

In this section we report results for the Baseline Model as well as several other calibrations, as discussed above. We first discuss summary statistics of the term structure of interest rates. Then we examine the cyclicality of the term structure. We solve all models using the value function iteration algorithm. For details regarding the numerical solution technique please refer to the appendix and to Chapters 4 and 5.

[7] In particular, we set $\alpha = 0.34$, $\delta = 0.021$, $\overline{\mu} = 0.4\%$ (see, e.g., Boldrin, Christiano, Fisher (2001))

[8] We keep the value for the discount factor (β) the same as for our Baseline calibration in Chapters 4 and 5, in order to be able to compare the models across chapters as good as possible. However, with this value for the discount factor (β) the Baseline Model in Chapter 6 overestimates the level of the risk-free rate somewhat. In Chapter 4 the Baseline Model is able to match the level of the risk-free rate because we calibrate the model to data from 1929 to 1998 where the consumption volatility and therefore the volatility of the technology shock are considerably higher. The ensuing buffer stock savings help to drive down the risk-free rate.

Table 6.1: **Calibration and Macroeconomic Moments**

Our Baseline Model is, following Chapters 4 and 5, the model with permanent technology shocks and an elasticity of intertemporal substitution (ψ) of 1.5. We set capital adjustment costs (ξ) to 22 across all models. We adjust the standard deviation of the technology shock (σ_ε) for each model in order to fit the volatility of consumption with all models. Δc denotes the cyclical component of consumption, Δy denotes the cyclical component of aggregate output. All series are logged and HP-filtered with a smoothing parameter of 1600. We calibrate the models to U.S. data from 1950Q1 to 2006Q3 (for a description of the data used please refer to the appendix). All values reported in the table are annual. Results for all models are based on 200 replications of sample size 227 each.

		Model 1	Model 2	Model 3	Model 4	Model 5	**Baseline**
		Transitory Shocks			Permanent Shocks		
	U.S. Data	$z_{t+1} = \overline{\mu}t + \rho z_t + \sigma_\varepsilon \varepsilon_{t+1}$			$z_{t+1} = \overline{\mu} + z_t + \sigma_\varepsilon \varepsilon_{t+1}$		
Statistic	1950-2006	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$
$\sigma[\Delta c]$ (%)	1.57	1.57	1.57	1.57	1.57	1.57	1.57
$\sigma[\Delta c]/\sigma[\Delta y]$	0.49	0.30	0.28	0.31	1.01	0.87	0.52
$\sigma[\Delta i]/\sigma[\Delta y]$	4.49	3.89	2.97	2.44	0.97	1.33	1.96

6.5.1 The Average Term Structure

In Table 6.2 we report for all models summary statistics of the average term structure: the average term spreads at different maturities, the standard deviation of the term spreads, as well as their autocorrelation coefficients.

From Table 6.2 we can see that none of the models is able to explain the average term structure. The models fail along two important dimensions: (i) the average term structure generated by all models is downward sloping, while the average term structure in the data is upward sloping, and (ii) the yield spreads generated by all models are not sufficiently volatile. Those findings are in accordance with Den Haan (1995) who documents the same shortcomings for the standard production economy model with power utility. While Epstein and Zin (1989) preferences can help the standard production economy model to explain equity returns, as demonstrated in Chapters 4 and 5, they do not improve the model's performance with respect to bond returns.

The reason for the downward sloping term structure in the standard model is the strongly procyclical risk-free rate. A positive (negative) shock drives the risk-free rate up (down). As a result, bond prices decrease (increase), inducing negative (positive) bond returns. Bond returns are therefore countercyclical and act as an insurance for consumption risk: they command a negative average risk premium. The reason that the yield spreads are not sufficiently volatile is that the market price of risk displays very little volatility in the standard production economy model, both with power utility preferences as well as with Epstein and Zin (1989) preferences.

However, Piazzesi and Schneider (2006) point out that while simple corrections for

Table 6.2: **The Average Term Structure — Summary Statistics**

In this table we report important summary statistics of the average term structure: the average term spreads at different maturities, the standard deviation of the term spreads, as well as the autocorrelation coefficients. We report results for models with either transitory ($\rho = 0.90$) or permanent technology shocks and different levels of the elasticity of intertemporal substitution (ψ). The coefficient of relative risk aversion (γ) is 5 and the discount factor (β) is 0.998 across all models. Our Baseline Model is, following Chapters 4 and 5, the model with permanent technology shocks and an elasticity of intertemporal substitution (ψ) of 1.5. We set capital adjustment costs (ξ) to 22 across all models. R_f denotes the real yield of 3 month U.S. Government T-Bills, Y^X denotes the real yield of X year maturity U.S. Government bonds. We use quarterly U.S. data from 1950Q1 to 2006Q3 (for a description of all data series please refer to the appendix). All values reported in the table are annual.

		Model 1	Model 2	Model 3	Model 4	Model 5	**Baseline**
		Transitory Shocks			Permanent Shocks		
	U.S. Data	$z_{t+1} = \overline{\mu}t + \rho z_t + \sigma_\varepsilon \varepsilon_{t+1}$			$z_{t+1} = \overline{\mu} + z_t + \sigma_\varepsilon \varepsilon_{t+1}$		
Statistic	1950-2006	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$
$E[R_f]$ (%)	1.082	8.693	4.058	1.876	8.718	4.017	1.660
$E[Y^1 - R_f]$ (%)	0.553	−0.016	−0.007	−0.004	−0.018	−0.011	−0.018
$E[Y^3 - R_f]$ (%)	0.971	−0.036	−0.015	−0.010	−0.052	−0.032	−0.049
$E[Y^5 - R_f]$ (%)	1.134	−0.046	−0.019	−0.015	−0.084	−0.051	−0.075
$\sigma[R_f]$ (%)	1.515	0.440	0.215	0.109	0.315	0.183	0.181
$\sigma[Y^1 - R_f]$ (%)	0.240	0.062	0.040	0.024	0.011	0.008	0.012
$\sigma[Y^3 - R_f]$ (%)	0.382	0.139	0.090	0.053	0.030	0.023	0.032
$\sigma[Y^5 - R_f]$ (%)	0.467	0.182	0.116	0.068	0.048	0.036	0.049
$AC[R_f]$ (%)	0.405	0.979	0.961	0.930	0.963	0.957	0.947
$AC[Y^1 - R_f]$ (%)	0.575	0.868	0.864	0.859	0.963	0.957	0.947
$AC[Y^3 - R_f]$ (%)	0.714	0.870	0.865	0.858	0.963	0.957	0.947
$AC[Y^5 - R_f]$ (%)	0.769	0.871	0.866	0.858	0.963	0.957	0.947

(expected) inflation, like in this chapter, typically suggest that the average real term structure slopes upward, evidence from indexed bonds suggests that the real term structure actually slopes downward on average. Piazzesi and Schneider conclude that 'it is difficult to assess the plausibility of this property of the model [the on average downward sloping real term structure] without a long sample on real yields for the United States. In the United Kingdom, where indexed bonds have been trading for a long time, the real yield curve seems to be downward sloping'.

As pointed out in Chapter 4, in the standard production economy model endogenous consumption smoothing induces a slow-moving, time-varying component in the endogenous consumption process. This small and persistent component mechanically carries over to the marginal rate of substitution (the pricing kernel). As a result, the risk-free rate and the term spreads are highly persistent.[9]

6.5.2 Cyclicality of the Term Structure

We now examine the dynamic behavior of the term structure. Table 6.3 reports the correlation of the term spread at different maturities with both output and consumption.

From Table 6.3 we can see that the term spreads generated by the model are, as in the data, countercyclical. The dynamic behavior of the term structure generated by the model fits the data well: during an economic expansion the term structure is flatter or even inverted, while the term structure is steeper and upward sloping during a recession. To illustrate this, Figure 6.1 plots the average term structure during an expansion and during a recession for our Baseline Model.

As demonstrated in Chapter 4, the endogenous consumption process generated by a standard production economy model can be very well described by the following process:

$$\Delta c_{t+1} = \overline{\mu} + x_t + \eta_{t+1}, \tag{6.11}$$

$$x_{t+1} = \varphi x_t + e_{t+1}, \tag{6.12}$$

where x_t is a small and slowly time-varying trend component. Expected consumption growth (x_t) is highly persistent but stationary ($\varphi < 1$). During a recession, when *current* expected consumption growth is low, *future* expected consumption growth is higher as x_t is expected to revert back to its mean. Higher expected consumption growth in the future implies higher interest rates in the future because higher expected consumption growth means that consumers have less incentives to save and want to borrow more instead. As

[9]We assume that total factor productivity (TFP) is highly persistent. Den Haan (1995) shows that even when TFP is i.i.d., the risk-free rate and the term spreads generated by standard production economy models are still highly persistent.

Table 6.3: **Term Structure Dynamics — Correlations with Output and Cons.** In this table we report the correlation of the term spread at different maturities with both output and consumption. We report results for models with either transitory ($\rho = 0.90$) or permanent technology shocks and different levels of the elasticity of intertemporal substitution (ψ). The coefficient of relative risk aversion (γ) is 5 and the discount factor (β) is 0.998 across all models. Our Baseline Model is, following Chapters 4 and 5, the model with permanent technology shocks and an elasticity of intertemporal substitution (ψ) of 1.5. We set capital adjustment costs (ξ) to 22 across all models. Y^X denotes the real yield of X year maturity U.S. Government bonds. Δc denotes the cyclical component of consumption, Δy denotes the cyclical component of aggregate output. We apply the HP-filter with a smoothing parameter of 1600 to extract the cyclical component. $\sigma(a, b)$ denotes the correlation between a and b. We use quarterly U.S. data from 1950Q1 to 2006Q3 (for a description of all data series please refer to the appendix).

		Model 1	Model 2	Model 3	Model 4	Model 5	**Baseline**
		Transitory Shocks			Permanent Shocks		
	U.S. Data	$z_{t+1} = \overline{\mu}t + \rho z_t + \sigma_\varepsilon \varepsilon_{t+1}$			$z_{t+1} = \overline{\mu} + z_t + \sigma_\varepsilon \varepsilon_{t+1}$		
Statistic	1950-2006	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$	$\psi = 1/\gamma$	$\psi = 0.5$	$\psi = 1.5$
$\sigma(Y^1 - R_f, \Delta c)$	-0.15	-0.68	-0.68	-0.68	-0.43	-0.47	-0.50
$\sigma(Y^3 - R_f, \Delta c)$	-0.35	-0.68	-0.68	-0.68	-0.43	-0.47	-0.50
$\sigma(Y^5 - R_f, \Delta c)$	-0.38	-0.68	-0.68	-0.68	-0.43	-0.47	-0.50
$\sigma(Y^1 - R_f, \Delta y)$	-0.07	-0.72	-0.73	-0.75	-0.43	-0.47	-0.51
$\sigma(Y^3 - R_f, \Delta y)$	-0.36	-0.72	-0.74	-0.76	-0.43	-0.47	-0.51
$\sigma(Y^5 - R_f, \Delta y)$	-0.41	-0.72	-0.74	-0.76	-0.43	-0.47	-0.51

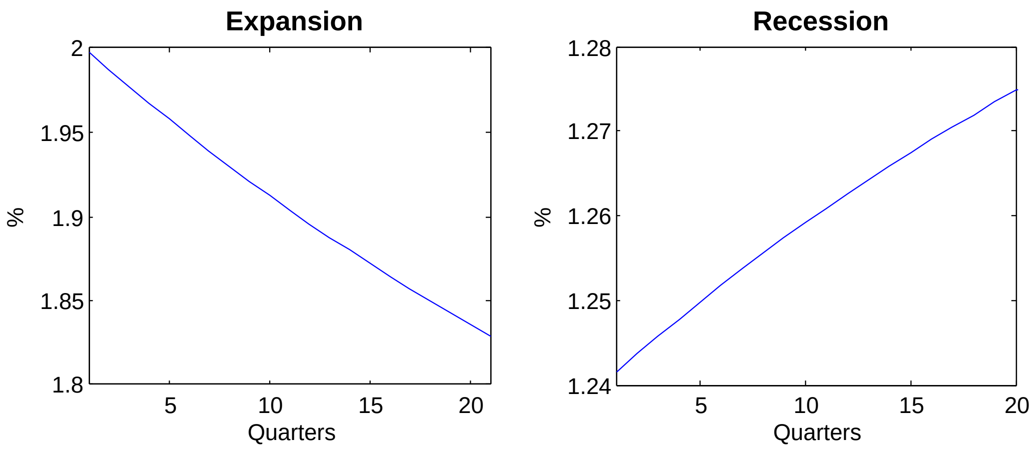

Figure 6.1: **Average Term Structure During Expansions and Recessions** This figure plots the average term structure during an expansion and during a recession for our Baseline Model. We define an expansion (recession) as a period where the cyclical component of aggregate consumption is at least two standard deviations higher (lower) than the mean.

a result, the term structure of interest rates slopes upward as future interest rates are expected to be higher than today's interest rates. The reverse argument applies during an economic expansion when the term structure slopes downward as future expected consumption growth is lower.

Wachter (2006) proposes a consumption based exchange economy model that can explain important aspects of the term structure. She builds her model around the Campbell and Cochrane (1999) external habit preference specification. Wachter feeds empirical consumption data into her consumption based model of the term structure and then compares the model implied term spreads with the term spreads from the data. Wachter manages to match the empirical fluctuations to a considerable extent. She reports a correlation between the 5 year yield spread implied by the model and that in the data of 0.34. We perform the analogous experiment for the production economy model by feeding the Baseline Model with a time series of total factor productivity (TFP) from the data.[10,11] As discussed in Section 6.5.1 and contrary to Wachter, the standard production economy model cannot explain the empirical volatility of the term spread. However, with a correlation between the 5 year yield spread implied by our model and that in the data of 0.45, the model does capture the dynamic behavior of the term structure at least as well as Wachter in that sense.

6.5.3 Predictability of Bond Returns

Campbell and Shiller (1991) run the following seminal regressions:

$$y_{t+1}^{n-1} - y_t^n = \alpha + \beta \frac{1}{n-1} \left(y_t^n - y_t^1 \right) + \varepsilon_{t+1}, \tag{6.13}$$

where

$$y_t^n = -\frac{1}{n} \ln P_t^n, \tag{6.14}$$

and P_t^n denotes the price of a n period maturity bond. Campbell and Shiller show that if $\beta = 1$, the generalized expectations hypothesis holds and bond risk premiums are constant over time (please refer to the appendix for a derivation of this result). Campbell and Shiller (and many others) find however that $\beta < 1$, which is commonly

[10] We construct a quarterly time series for total factor productivity (TFP) in the U.S. from 1950 to 2002, leaving us with 211 quarterly observations of $\Delta \ln Z_t$. We rely on data from the U.S. Department of Labor (Bureau of Labor Statistics). Because capital input is only available on an annual basis, we use cubic splines to quarterize capital input. For a discussion of how we construct the TFP series please refer to the appendix.

[11] The driving process of the Baseline Model is $\ln Z_t = \overline{\mu} + \ln Z_{t-1} + \sigma_\varepsilon \varepsilon_t$. We resolve the Baseline Model with the values for $\overline{\mu}$ and σ_ε we estimate from our empirical TFP series. Both values are somewhat lower in the data relative to our Baseline calibration ($\overline{\mu}$ from 0.40% to 0.33%, σ_ε from 1.18% to 0.95%).

interpreted as evidence against the expectations hypothesis and in favour of time-varying and procyclical risk premiums. This is equivalent to the statement that bond returns are predictable. Wachter (2006) can explain that predictability with a strongly time-varying and countercyclical market price of risk induced by the external habit preferences she assumes. In our model with Epstein and Zin (1989) preferences on the other hand, there is little variation in the market price of risk and the market price of risk is, unfortunately, procyclical.[12] We therefore find $\beta > 1$.

6.6 A Time-Varying Trend Growth Rate

One of the problems of the standard production economy model is, as demonstrated above, that it does not generate sufficiently volatile term spreads because the stochastic discount factor is not volatile enough. Given a realistic consumption process, there is little hope of remedying this shortcoming with power utility preferences. With Epstein and Zin (1989) preferences on the other hand, the stochastic discount factor is given by:

$$M_{t,t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} \left(R^A_{t+1}\right)^{\theta-1}. \tag{6.15}$$

The asset return (R^A_{t+1}), that is the return of a claim on aggregate consumption, now enters the pricing kernel. Bansal and Yaron (2004) exploit this. They assume a small and slowly time-varying component in the process for aggregate consumption. While this change barely affects the current consumption growth rate ($\frac{C_{t+1}}{C_t}$), it can have an enormous impact on the *return* of a claim on the consumption process (R^A_{t+1}). In this section we examine whether the same is true for the production economy model. In other words, can we make the stochastic discount factor more volatile by assuming a small and slowly time-varying component in the driving process (total factor productivity)? The short answer is no. And the reason is that consumers, once given the choice, smooth out the shock to the trend growth rate. When a positive shock to the trend growth rate hits the economy, consumers, in anticipation of the good times ahead, crank up their consumption expenditures at the expense of aggregate investment. High current levels of consumption and low levels of investment in turn depress the effect of the growth rate shock on asset returns.[13]

[12] In the Baseline Model, the correlation between the market price of risk and the cyclical component of consumption is 0.41, where the cyclical component is extracted by means of an HP-filter with a smoothing parameter of 1600.

[13] It is actually a known puzzle that positive news about future productivity growth cause a contraction in most production economy (RBC) models (see, e.g., Cochrane (1994), Beaudry and Portier (2005), Jaimovich and Rebelo (2006)). In Chapter 2 we develop a production economy model that incorporates

We consider the following process for total factor productivity (TFP):

$$\ln Z_t = g_{t-1} + \ln Z_{t-1} + \sigma_\varepsilon \varepsilon_t, \tag{6.16}$$

$$g_t = (1 - \rho_g)\,\overline{\mu} + \rho_g g_{t-1} + \sigma_\omega \omega_t, \tag{6.17}$$

$$\varepsilon_t \sim N\,(0, 1)\,, \tag{6.18}$$

$$\omega_t \sim N\,(0, 1)\,. \tag{6.19}$$

The current level of productivity growth is modeled as a white noise process, while the trend growth rate is modeled as an AR(1), so that shocks to the current level of productivity growth are transitory, while shocks to the trend growth rate are highly persistent. If σ_ω is small relative to σ_ε, then a shock to the trend growth rate has only a very small impact on the current levels of productivity, while at the same time having a massive effect on future levels of productivity, because the growth rate of productivity has changed.

The process as specified by equations (6.16) to (6.19) constitutes an unobserved-component model, where only productivity (Z) is observable, while the trend growth rate (g) is an unobservable state variable. We parameterize $\rho_g = 0.99$, $\sigma_\omega = 0.03\%$ and demonstrate in the appendix that those parameter values are realistic by estimating the model via maximum likelihood based on Harvey's (1981) error decomposition, applying state space techniques (Kalman filter).

When we compare the results from our Baseline Model with the results from the Baseline Model extended with a process for total factor productivity that has a slowly time varying trend growth rate, we find that the asset pricing results are barely affected. As already discussed above, consumers, once given the choice, smooth out the shock to the trend growth rate. As can be seen from Figure 6.2, when a positive shock to the trend growth rate hits the economy, consumers lower their savings and increase their consumption expenditures. High current levels of consumption and low levels of aggregate investment in turn depress the effect of the growth rate shock on equity returns, asset returns, and the stochastic discount factor. This effect is so strong that the equity return actually goes down in response to the growth rate shock.

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the standard labor market matching framework and show that this model can generate an expansion in response to a shock to the trend growth rate.

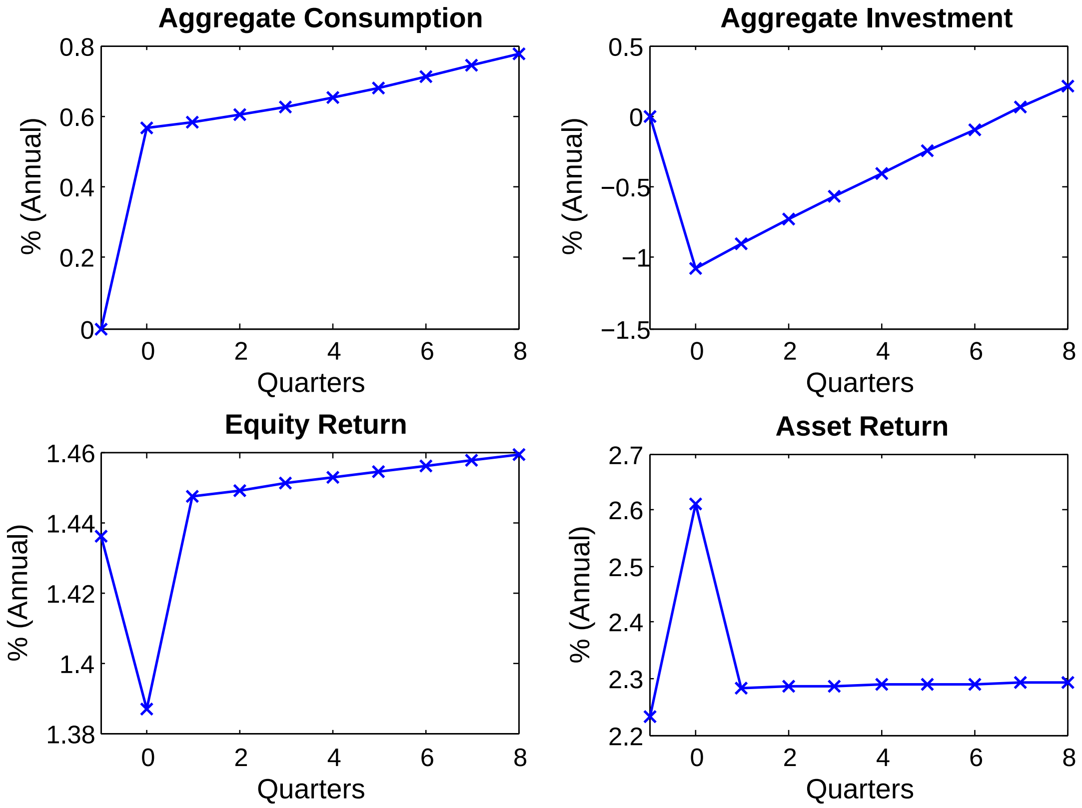

Figure 6.2: **Impulse Responses — Shock to the Trend Growth Rate**
We plot impulse responses to a one standard deviation shock to the trend growth rate of total factor productivity at time $t = 0$. Consumers lower their savings and increase their consumption expenditures. High current levels of consumption and low levels of aggregate investment in turn depress the positive impact of the growth rate shock on equity returns and asset returns. This effect is so strong that the equity return actually goes down in response to the growth rate shock.

6.6.1 Time-Varying Volatility

Following Bansal and Yaron (2004), we also consider a driving process where the volatility of the shock is varying over time:

$$\ln Z_t = \overline{\mu} + \ln Z_{t-1} + \sigma_{\varepsilon,t}\varepsilon_t, \tag{6.20}$$

$$\ln \sigma_{\varepsilon,t} = (1-\rho_\sigma)\ln \sigma_\varepsilon + \rho_\sigma \ln \sigma_{\varepsilon,t-1} + \sigma_\eta \eta_t, \tag{6.21}$$

$$\varepsilon_t \sim N(0,1), \tag{6.22}$$

$$\eta_t \sim N(0,1), \tag{6.23}$$

where $\sigma_{\varepsilon,t}$ is the slowly time-varying volatility of the technology shock. Bansal and Yaron assume both the volatility of the shock to consumption growth and the volatility of the shock to expected consumption growth to share the same time-varying process. In our model we cannot do that because consumption is endogenous and it is not clear whether and how time-varying volatility of the technology shock carries over to time-varying volatility of the shocks to consumption growth and expected consumption growth. However, as it turns out, if we choose the same value for the persistence of the volatility shock ($\rho_\sigma = 0.987$) like Bansal and Yaron and set the magnitude of the volatility shock (σ_η) so that the endogenous volatility of the shock to realized consumption growth generated by our model varies as much as in Bansal and Yaron ($var(\sigma_{c,t}) = 0.32\%$), the endogenous volatility of the shock to expected consumption growth from our model varies by a similar magnitude relative to Bansal and Yaron ($var(\sigma_{x,t}) = 0.014\%$ in Bansal and Yaron vs. $var(\sigma_{x,t}) = 0.009\%$ in our model).[14]

So what is the effect of the time-varying volatility? The effect on equity returns is very similar to Bansal and Yaron (2004). Both equity return premium and equity return volatility increase, albeit not by much. However, the effect on the term structure is minimal. The reason is that time-varying volatility does not affect a basic feature of the standard production economy model: the strongly procyclical risk-free rate.[15]

6.7 Conclusion

The ability of the standard production economy model with Epstein and Zin (1989) preferences to explain the term structure of interest rates is mixed. While the model can

[14]The time-varying volatility of the technology shock we assume carries over to a very substantial extent to the time-varying volatility of the endogenous shock to realized consumption growth generated by the model.

[15]We have also allowed the shock to technology (ε_t) and the shock to the volatility of the technology shock (η_t) to be correlated, and found that the results remain basically the same.

explain the dynamic behavior of the term structure, it fails at generating yield spreads that are sufficiently volatile. The term spreads generated by the model are on average negative, where the jury is still out on whether this is in accordance with the data or not. We conclude that Epstein and Zin preferences can help the standard production economy model to explain equity risk premiums, but they cannot help the model to generate sufficiently volatile term spreads.

In contrast to exchange economy models, the shortcomings of the production economy model cannot simply be remedied by reverse-engineering the driving process. In exchange economy models the driving process, consumption, directly impacts the stochastic discount factor and thus asset prices. In production economy models the impact of the driving process is indirect via the agents' consumption and savings decision.

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6.9 Appendix A — Numerical Solution

We use the value function iteration algorithm to solve the model. For details regarding the solution algorithm please refer to Chapters 4 and 5. Following Wachter (2006) we solve for real bonds as follows. Given a solution of the model, the price of a 3 month bond is given by:

$$P_t^1 = E_t \left[M_{t,t+1} \times 1 \right], \tag{6.24}$$

where $M_{t,t+1}$ is the stochastic discount factor given by:

$$M_{t,t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} \left(R_{t+1}^A \right)^{\theta - 1}. \tag{6.25}$$

The price of a 6 month bond follows as:

$$\begin{aligned} P_t^2 &= E_t \left[M_{t,t+1} M_{t+1,t+2} \times 1 \right] \\ &= E_t \left[M_{t,t+1} E_{t+1} \left[M_{t+1,t+2} \times 1 \right] \right] \\ &= E_t \left[M_{t,t+1} P_{t+1}^1 \right]. \end{aligned} \tag{6.26}$$

It follows that real bond prices can be determined recursively by the investor's Euler equation:

$$P_t^q = E_t \left[M_{t,t+1} P_{t+1}^{q-1} \right]. \tag{6.27}$$

P^1 is a function of the state variables of the economy (Z and K). We specify a polynomial for $P^1 = \Psi^1(Z, K)$ over the state variables and obtain the polynomial coefficients by running a regression of $E_t [M_{t,t+1} \times 1]$ on a grid of the state variables. Given a model solution, $E_t [M_{t,t+1} \times 1]$ is obtained by numerical integration. From $P^1 = \Psi^1(Z, K)$ we obtain P^2 by numerically computing $P_t^2 = E_t \left[M_{t,t+1} P_{t+1}^1 \right] = E_t \left[M_{t,t+1} \Psi^1 (Z_{t+1}, K_{t+1}) \right]$ and regressing it on the grid, and so on.

The real return on a q month real bond is given by:

$$R_t^q = \frac{P_{t+1}^{q-1}}{P_t^q} - 1,$$

the real yield is given by:

$$Y_t^q = \left(P_t^q \right)^{-\frac{1}{q}} - 1.$$

6.10 Appendix B — Data

U.S. Data from 1950Q1 to 2006Q3 — GDP, Consumption, Investment

- Real per capita gross domestic product
- Real per capita gross private domestic investment
- Real per capita personal consumption expenditures (nondurable goods and services)

These series were downloaded from the Bureau of Economic Analysis (BEA), National Income and Product Accounts (NIPA).

U.S. Data from 1950Q1 to 2006Q3 — Term Structure

- R_f^π: U.S. Government 90 day T-Bills Secondary Market (nominal)
- Y_1^π: U.S. Government 1 year Constant Maturity Note Yield (nominal)
- Y_3^π: U.S. Government 3 year Constant Maturity Note Yield (nominal)
- Y_5^π: U.S. Government 5 year Constant Maturity Note Yield (nominal)
- Π: U.S. Consumer Price Index (CPI)

These series were downloaded from Global Financial Data. We obtain real yields (R_f, Y_1, Y_2, Y_3) by correcting the nominal yields for inflation with the U.S. Consumer Price Index (CPI).

U.S. Data from 1950Q1 to 2002Q4 — Total Factor Productivity

We construct a quarterly time series for total factor productivity (TFP) in the U.S. from 1950 to 2002, leaving us with 211 quarterly observations of productivity growth ($\Delta \ln Z_t$). Throughout, we rely on data from the U.S. Department of Labor (Bureau of Labor Statistics).

We take a standard Cobb-Douglas production function:

$$Y_t = Z_t K_t^\alpha L_t^{1-\alpha}, \tag{6.28}$$

where Y_t is quarterly output, Z_t is TFP, K_t is capital input, and L_t is labor input (hours). From the input factors (K_t, L_t) and output (Y_t), we compute TFP using the standard Solow residual formula:

$$\ln Z_t = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln L_t, \tag{6.29}$$

where we use as the value for the elasticity of output with respect to capital (α) the average value of the capital share over the period from 1950 to 2002 in the Bureau of Labor Statistics multifactor productivity dataset, which is 0.31.

While output (Y_t) and labor input (L_t) are available from the Bureau of Labor Statistics on a quarterly basis, capital services (K_t) are only available on an annual basis. We therefore convert capital services to a quarterly frequency, using cubic splines. This entails to fit T polynomials of order 3 ($p_t(t) = \alpha + bt + ct^2 + dt^3$) through the annual observations of a flow variable (capital services), where T is the number of annual observations, in our case 53. It follows that we need $4 \times T$ conditions in order to determine the coefficients of all T polynomials. We use the following set of conditions:

$$p_t(t + 0.125) + p_t(t + 0.375) + p_t(t + 0.625) + p_t(t + 0.875) = 4K_t, \tag{6.30}$$

$$p_t(t) = p_{t-1}(t), \tag{6.31}$$

$$p'_t(t) = p'_{t-1}(t), \tag{6.32}$$

$$p''_t(t) = p''_{t-1}(t), \tag{6.33}$$

$$t \in \{1, ..., T\}, \tag{6.34}$$

where K_t is the annual observation of capital services that corresponds to the t^{th} polynomial. That is, we set the polynomials equal to each other at the transition from one year to another, as well as their first and second derivatives. Finally, the 4 interpolated quarterly observations have to sum up to the one observed annual observation. This leaves us with $4 \times T - 3$ conditions and $4 \times T$ unknown coefficients, implying that we need 3 'initial' conditions. We choose the following conditions:

$$p''_1(1) = 0, \tag{6.35}$$

$$p'''_1(1) = 0, \tag{6.36}$$

$$p''_T(T + 1) = 0. \tag{6.37}$$

Now we are left with a system of $4 \times T$ equations and $4 \times T$ unknown coefficients. We solve this system, evaluate each polynomial t at $(t + 0.125)$, $(t + 0.375)$, etc. and so construct a quarterly series of capital services. The resulting quarterly series is depicted in Figure 6.3.

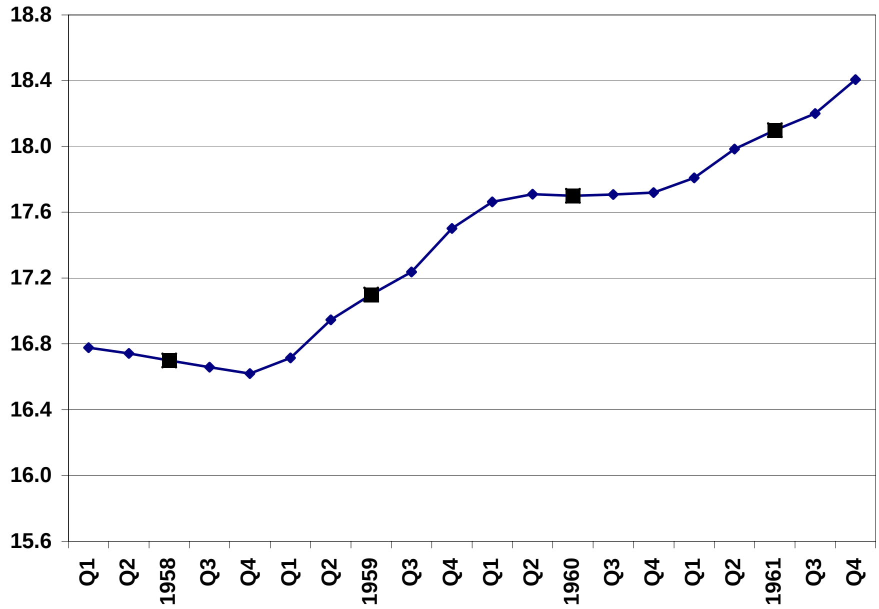

Figure 6.3: **Quarterly Series for Capital Services**
We convert Capital Services available from the Bureau of Labor Statistics on an annual basis to a quarterly frequency, using cubic splines.

6.11 Appendix C — Campbell and Shiller (1991) Regressions

Campbell and Shiller (1991) run the following regression:

$$y_{t+1}^{n-1} - y_t^n = \alpha + \beta \frac{1}{n-1} \left(y_t^n - y_t^1 \right) + \varepsilon_{t+1}, \tag{6.38}$$

where

$$y_t^n = -\frac{1}{n} \ln P_t^n, \tag{6.39}$$

and P_t^n denotes the price of a n period maturity bond. This equation follows directly from the generalized expectations hypothesis. The generalized expectations hypothesis states that longer term interest rates are equal to expected future short term interest rates plus a risk premium that varies with the maturity of the bond but not with time:

$$y_t^n = \frac{1}{n} E_t \left[\sum_{i=0}^{n-1} \left(y_{t+i}^1 \right) \right] + c^n. \tag{6.40}$$

The expected bond return follows as:

$$\begin{aligned} E_t\left[y_{t+1}^{n-1}\right]-y_t^n &= \frac{1}{n-1}E_t\left[\sum_{i=0}^{n-2}\left(y_{t+1+i}^1\right)\right]-\frac{1}{n}E_t\left[\sum_{i=0}^{n-1}\left(y_{t+i}^1\right)\right]+\left(c^{n-1}-c^n\right) \\ &= -\frac{1}{n}y_t^1+\frac{1}{n\,(n-1)}E_t\left[\sum_{i=0}^{n-2}\left(y_{t+1+i}^1\right)\right]+\left(c^{n-1}-c^n\right). \end{aligned} \tag{6.41}$$

We define the term spread as:

$$\begin{aligned} s_t^n &= \frac{1}{n-1}\left(y_t^n-y_t^1\right) \\ &= \frac{1}{n-1}\left(\frac{1}{n}E_t\left[\sum_{i=0}^{n-1}\left(y_{t+i}^1\right)\right]-y_t^1\right)+\frac{1}{n-1}c^n \\ &= \frac{1}{n-1}\left(\frac{1}{n}E_t\left[\sum_{i=1}^{n-1}\left(y_{t+i}^1\right)\right]+\frac{1}{n}y_t^1-y_t^1\right)+\frac{1}{n-1}c^n \\ &= \frac{1}{n-1}\left(\frac{1}{n}E_t\left[\sum_{i=1}^{n-1}\left(y_{t+i}^1\right)\right]+\frac{1-n}{n}y_t^1\right)+\frac{1}{n-1}c^n \\ &= -\frac{1}{n}y_t^1+\frac{1}{n\,(n-1)}E_t\left[\sum_{i=0}^{n-2}\left(y_{t+1+i}^1\right)\right]+\frac{1}{n-1}c^n. \end{aligned} \tag{6.42}$$

It follows that:

$$\begin{aligned} E_t\left[y_{t+1}^{n-1}\right]-y_t^n &= s_t^n-\frac{1}{n-1}c^n+\left(c^{n-1}-c^n\right) \\ &= s_t^n+\left(\frac{n}{n-1}c^n+c^{n-1}\right) \\ &= \alpha+\frac{1}{n-1}\left(y_t^n-y_t^1\right). \end{aligned} \tag{6.43}$$

Therefore, if expectations are rational, and if we run the following regression:

$$y_{t+1}^{n-1}-y_t^n=\alpha+\beta\frac{1}{n-1}\left(y_t^n-y_t^1\right)+\varepsilon_{t+1}, \tag{6.44}$$

then the generalized expectations hypothesis predicts that $\beta=1$.

6.12 Appendix D — Estimation of a Latent-Variable Process for TFP

6.12.1 Introduction

While relatively large quarter-to-quarter swings in productivity growth are common in the data, shifts in the long-run productivity growth rate seem to be infrequent and highly persistent. In this appendix we specify and estimate a latent-variable driving process for total factor productivity growth that disentangles transitory shocks to the *level* of productivity growth from shocks to the underlying persistent *growth rate* of productivity.

In specifying this model we follow a strand of empirical literature that models the growth component of a time series as a unit root (from Clark (1987) over Stock and Watson (1998) to Roberts (2001), Gordon (2003), Morley, Nelson and Zivot (2003), Laubach and Williams (2003), and Edge, Laubach and Williams (2004), and others). We would go along with the assessment of Edge, Laubach and Williams (2004), and do not consider our model as the ultimate DGP, but rather as 'a parsimonious and tractable approximation to a variety of data generating processes for highly persistent shifts in the trend growth rate'.

6.12.2 The Model

We consider the following model:

$$\Delta \ln Z_t = g_t + \sigma_\varepsilon \varepsilon_t, \tag{6.45}$$

$$g_t = (1 - \rho_g)\,\overline{\mu} + \rho_g g_{t-1} + \sigma_\omega \omega_t, \tag{6.46}$$

$$\varepsilon_t \sim N\,(0, 1)\,, \tag{6.47}$$

$$\omega_t \sim N\,(0, 1)\,. \tag{6.48}$$

Here, the disturbance ε_t corresponds to the shock to the current level of productivity growth, while the disturbance ω_t corresponds to the shock to the trend growth rate of productivity. The current level of productivity growth is modeled as a white noise process, while the trend growth rate is modeled as an AR(1), so that shocks to the current level of productivity growth are transitory, while shocks to the trend growth rate are highly persistent. If σ_ω^2 is small relative to σ_ε^2, then a standard deviation shock to the trend growth rate has only a very small impact on the current level of productivity, while at the same time having a massive effect on future levels of productivity, because the growth rate of productivity has changed.

Table 6.4: **Appendix — Estimation Results TFP Process**
We estimate five different models characterized by the autoregressive coefficient of the trend growth rate, ρ_g.

Model	σ_ε	σ_ω
$\rho_g = 1.000$	0.84%	0.030%
	(17.06)	(1.39)
$\rho_g = 0.990$	0.84%	0.029%
	(17.03)	(1.46)
$\rho_g = 0.975$	0.84%	0.026%
	(17.60)	(1.17)
$\rho_g = 0.950$	0.84%	0.012%
	(18.60)	(0.19)
$\rho_g = 0.900$	0.84%	0.000%
	(17.91)	(0.00)

6.12.3 Estimation Results

The process as specified by equations (6.45) to (6.48) constitutes an unobserved-component model, where only productivity (Z) is observable, while the trend growth rate (g) is an unobservable state variable. The parameters of this model can be estimated via maximum likelihood based on Harvey's (1981) error decomposition, applying state space techniques (Kalman filter). Our approach is to fix the parameters $\overline{\mu}$ and ρ_g and to estimate the standard deviations of both disturbances, σ_ε and σ_ω. We estimate five different models characterized by the autoregressive coefficient of the trend growth rate, ρ_g. For all models we set $\overline{\mu} = \overline{\Delta \ln Z_t}$, so that the unconditional mean productivity growth is equal to the sample mean of $\Delta \ln Z_t$ (denoted by $\overline{\Delta \ln Z_t}$). For a detailed outline of the estimation procedure please refer to the next section. We use a quarterly time series for total factor productivity (TFP) in the U.S. from 1958 to 2002, leaving us with 179 quarterly observations of $\Delta \ln Z_t$ (for more details regarding the construction of the TFP series please refer to Appendix B).[16] The results of the estimation are reported in Table 6.4.

As can be seen, the standard deviation of the shock to the trend growth rate of productivity is small and not very significant. However, Stock and Watson (1998) show that maximum-likelihood estimates of the standard deviation of small shocks to trend growth rates are likely to be biased towards zero. The reason is the so called 'pile-up' problem, which refers to a large point mass of the maximum likelihood estimator of σ_ω at zero. Stock and Watson (1998) also propose an alternative estimator to maximum likelihood,

[16]The estimate of σ_ω is not very robust to variations in the number of productivity growth observations ($\Delta \ln Z_t$) used in the estimation.

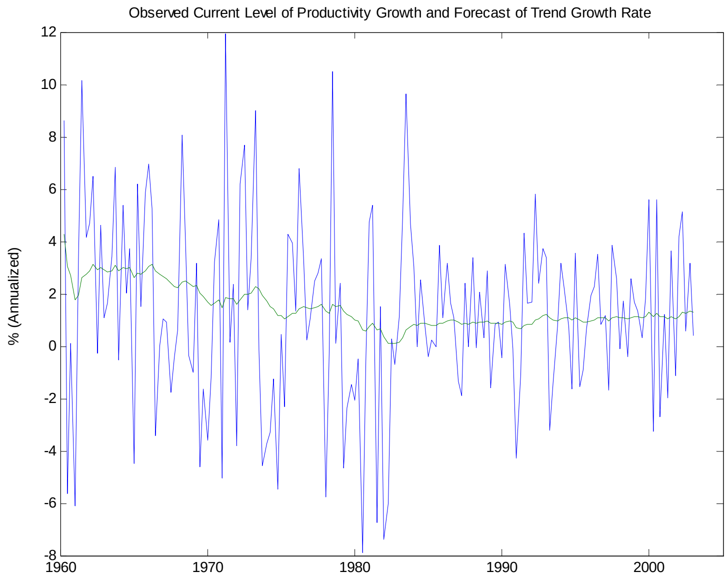

Figure 6.4: **Observed TFP and Forecast of the Trend Growth Rate**
We plot the observed current level of productivity growth and the forecast of the trend growth rate, given the parameter estimates of our benchmark model.

the 'median unbiased estimator'. We have decided to abstain here from employing the median unbiased estimator.[17] We also conclude that our results are robust to a decrease of ρ_g from 1 down to 0.95. However, as we decrease ρ_g further to 0.90, our estimate of σ_ω basically collapses to zero.

Figures 6.4 and 6.5 plot the Kalman estimate of the trend growth rate, given our parameter estimates and the data, applying the Kalman filter and the Kalman smoothing procedure.

6.12.4 The Kalman Filter

We consider the following unobserved-component model:

$$\Delta \ln Z_t = g_t + \sigma_\varepsilon \varepsilon_t, \tag{6.49}$$

$$g_t = (1 - \rho_g)\,\overline{\mu} + \rho_g g_{t-1} + \sigma_\omega \omega_t, \tag{6.50}$$

$$\varepsilon_t \sim N(0,1)\,, \tag{6.51}$$

$$\omega_t \sim N(0,1)\,, \tag{6.52}$$

[17]The Roberts (2001) value for σ_ω ($\sigma_\omega = 0.04\%$) constitutes to the best of our knowledge the upper bound found for σ_ω in the literature.

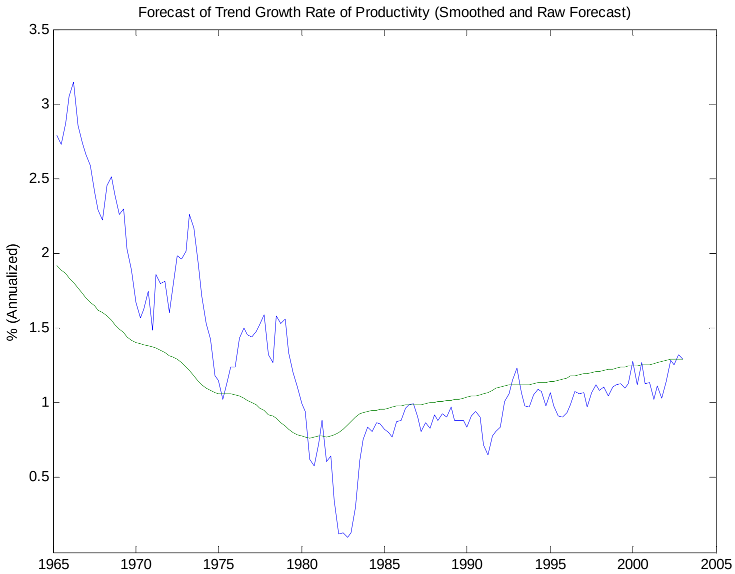

Figure 6.5: **Forecast of the Trend Growth Rate (Smoothed and Raw)**
We plot the forecast of the trend growth rate, given the parameter estimates of our benchmark model. We apply the Kalman filter and the Kalman smoothing procedure.

where only productivity (Z) is observable, and the trend growth rate (g) is an unobservable state variable. We want to estimate the standard deviation of the shock to the level of productivity growth (σ_ε) and the standard deviation of the shock to the trend growth rate of productivity (σ_ω). To do so, we employ the Kalman filter and maximum likelihood estimation based on Harvey's (1981) error decomposition formula. The following equations, which we apply recursively, constitute the Kalman filter. In our exposition we follow Hamilton (1994) and in particular Kim and Nelson (1999). The forecast of g_t given information up to time τ, denoted by $g_{t|\tau}$, follows from (6.50):

$$g_{t|t-1} = (1 - \rho_g)\,\overline{\mu} + \rho_g g_{t-1|t-1}. \tag{6.53}$$

The variance of the forecast of g_t given information up to time τ, denoted by $P_{t|\tau}$, follows from (6.50):

$$P_{t|t-1} = \rho_g^2 P_{t-1|t-1} + \sigma_\omega^2. \tag{6.54}$$

The forecast error, given information up to time $(t-1)$, of $\Delta \ln Z_t$ is:

$$\eta_{t|t-1} = \Delta \ln Z_t - \Delta \ln Z_{t|t-1} = \Delta \ln Z_t - g_{t|t-1}. \tag{6.55}$$

The variance of this forecast error, denoted by $f_{t|t-1}$, is:

$$f_{t|t-1} = P_{t|t-1} + \sigma_{\varepsilon}^{2}. \tag{6.56}$$

Finally, the 'heart' of the recursive filter, the updating, which initiates another round of recursion, is given by the following two equations:

$$g_{t|t} = g_{t|t-1} + P_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1}, \tag{6.57}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} f_{t|t-1}^{-1} P_{t|t-1}. \tag{6.58}$$

These equations can easily be derived given the normality assumption for the shock processes. For a nice treatment, please refer to Kim and Nelson (1999). Note that we need starting values $g_{0|0}$ and $P_{0|0}$ in order to get the recursion going. We set $g_{0|0} = mean(\Delta \ln Z_t)$ and assign a large value to $P_{0|0}$. Intuitively, this amounts to assigning a very high uncertainty to our knowledge of the starting value. We examine robustness of our results to the choice of starting values in the next section. Harvey's (1981) error decomposition formula finally allows us to compute the likelihood function, which we then go on to maximize numerically over (σ_{ε}) and (σ_{ω}):

$$-\sum_{t=1}^{T} \left(\frac{1}{2} \ln \left[(2\pi) \left| f_{t|t-1} \right| \right] - \frac{1}{2} \eta_{t|t-1}' f_{t|t-1}^{-1} \eta_{t|t-1} \right). \tag{6.59}$$

We further minimize the effect of the choice of the starting values for $g_{0|0}$ and $P_{0|0}$ on the estimation results by iterating the basic filter as outlined above from $t = 1$ while evaluating the log-likelihood function only from $t = 16$ onwards, that is we drop the first 4 years before maximizing over the likelihood function.

In order to plot the estimate of the trend growth rate given the parameter estimates $(\sigma_{\varepsilon},\ \sigma_{\omega})$ of our Benchmark model, we first use the Kalman smoothing procedure. The Kalman smoothing procedure uses more information than the basic filter. It basically applies the Kalman filter backwards, using information about future realizations of the observable variable $(\Delta \ln Z)$, thus revising the original estimate of the trend growth rate. The following two equations are iterated backwards from $t = T - 1$ down to $t = 1$ in order to obtain the smoothed estimate of the trend growth rate as well as its variance:

$$g_{t|T} = g_{t|t} + P_{t|t} \rho_g P_{t+1|t}^{-1} \left(g_{t+1|T} - \rho_g g_{t|t} - \overline{\mu} \right), \tag{6.60}$$

$$P_{t|T} = P_{t|t} + P_{t|t} \rho_g P_{t+1|t}^{-1} \left(P_{t+1|T} - P_{t+1|t} \right) P_{t+1|t}^{-1} \rho_g P_{t|t}, \tag{6.61}$$

where the starting values $g_{T|T}$ and $P_{T|T}$ are taken as the values from the last iteration of the basic filter.

6.12.5 Robustness — Starting Values for $g_{0|0}$ and $P_{0|0}$

As discussed above, we need starting values $g_{0|0}$ and $P_{0|0}$ in order to initiate the recursion which constitutes the Kalman filter. For the estimation of all models in this chapter we set $g_{0|0} = mean(\Delta \ln Z_t)$ and assign a large value to $P_{0|0}$. Intuitively, this amounts to assigning a very high uncertainty to our knowledge of the starting value. Not surprisingly, when $P_{0|0}$ is large, we find that our estimates are completely insensitive to the value of $g_{0|0}$.

In the remainder of this section we examine robustness of our results to the starting value for $P_{0|0}$. To that end, note that $P_{t|t}$ converges as $t \to \infty$. Let $P^* = \lim_{t\to\infty} P_{t|t}$.[18] We undertake the following experiment: We keep $g_{0|0} = mean(\Delta \ln Z_t)$ and decrease $P_{0|0}$ from its benchmark value $100 \times P^*$ towards $1 \times P^*$ in several steps and estimate.[19] If we set $P_{0|0} = P^*$, this essentially amounts to assigning the same level of uncertainty to all forecasts of $g_{t|t}$, that is we are equally certain about our initial forecast of the latent variable ($g_{0|0}$) as we are about our final forecast ($g_{T|T}$). The results of the estimation are reported in Table 6.5.

We can see that it matters whether or not we assign a high degree of uncertainty to our initial forecast of the trend growth rate ($g_{0|0}$). Our results are not overturned in the sense that the estimate of σ_ω does not collapse to zero, but the magnitude of the estimate of σ_ω is sensitive to the choice of $P_{0|0}$. Also not surprisingly, if we choose a smaller value for $P_{0|0}$ (from $P_{0|0} = 5 \times P^*$ onwards), our results become sensitive to the choice of $g_{0|0}$. For example, if we set $P_{0|0} = 2 \times P^*$, we estimate the following models, reported in Table 6.6.

[18] As can be seen from equations (6.54), (6.56), and (6.58), the numerical value of P^* depends on the values of σ_ε and σ_ω. We take as P^* the limit of $P_{t|t}$ when $\sigma_\varepsilon = 0.84\%$ and $\sigma_\omega = 0.030\%$ (estimation of model with $\rho_g = 1$).

[19] It turns out that P^* is sufficiently insensitive to such changes in σ_ω that we consider in this experiment. As a result, we can keep (multiples of) P^* from our benchmark model as starting values for the estimation of all other models reported in Table 6.5 for the purpose of our experiment.

Table 6.5: **Appendix — Robustness: Different Starting Values for $P_{0|0}$**
We keep $g_{0|0} = mean(\Delta \ln Z_t)$ and decrease $P_{0|0}$ from its benchmark value $100 \times P^*$ towards $1 \times P^*$ in several steps and estimate. If we set $P_{0|0} = P^*$, this essentially amounts to assigning the same level of uncertainty to all forecasts of $g_{t|t}$, that is we are equally certain about our initial forecast of the latent variable ($g_{0|0}$) as we are about our final forecast ($g_{T|T}$).

Model	σ_ε	σ_ω
$P_{0\|0} = 100 \times P^*$	0.84%	0.030%
	(17.06)	(1.39)
$P_{0\|0} = 10 \times P^*$	0.84%	0.027%
	(17.26)	(1.31)
$P_{0\|0} = 5 \times P^*$	0.84%	0.025%
	(17.06)	(1.22)
$P_{0\|0} = 2 \times P^*$	0.84%	0.020%
	(17.91)	(1.04)
$P_{0\|0} = 1 \times P^*$	0.84%	0.015%
	(17.69)	(0.82)

Table 6.6: **Appendix — Robustness: Different Starting Values for $g_{0|0}$**
We choose a small value for $P_{0|0}$ ($P_{0|0} = 2 \times P^*$). Then we estimate several models with different starting values $g_{0|0}$.

Model	σ_ε	σ_ω
$g_{0\|0} = 1 \times mean(\Delta \ln Z_t)$	0.84%	0.020%
	(17.91)	(1.04)
$g_{0\|0} = \frac{1}{2} \times mean(\Delta \ln Z_t)$	0.85%	0.012%
	(17.54)	(0.48)
$g_{0\|0} = 2 \times mean(\Delta \ln Z_t)$	0.84%	0.032%
	(18.01)	(1.77)