Measuring variations in streetscape skeletons under zoning regulations of the building coverage ratio: a theoretical approach

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Abstract

The arrangement of buildings along roads is one of the most fundamental patterns of three-dimensional streetscape skeletons, defined as a set of building heights and setbacks in a district. Under zoning regulations, building heights and setbacks are indirectly controlled by the building coverage ratio (BCR) and the floor area ratio (FAR). In particular, variations in the BCR result in variations in building heights and setbacks. Thus, understanding the relationship among variations in the BCR, building heights and setbacks is important to harmonise streetscape skeletons. However, this relationship has yet to be theoretically investigated due to its complexity. To this end, we formulate the relationship between variations in building heights and setbacks as the function of the variance of the BCR. We show that as the variance of the BCR increases, the increase in the variance of building heights is greater than that in the variance of setbacks. This finding can contribute to controlling variations in streetscape skeletons.

Keyword: building height, setbacks, streetscape, building coverage ratio, stochastic approach

Introduction

The arrangement of buildings along roads constitutes the three-dimensional streetscape (Harvey et al., 2017; Oliveira, 2016). Streetscapes comprise a skeleton and a skin. While the skin can be characterised by the materials and architectural style of each building, the skeleton, called a streetscape skeleton, can be interpreted as the product of the size and arrangement of buildings. In particular, a building's height and setback are the main components of streetscape skeletons. Thus, a streetscape skeleton is defined as a set of building heights and setbacks in a district.

There are two general ways to regulate a building's height and setback in a plot. In European countries, every building's height and setback are directly regulated, called *direct regulation*, by considering the relation to the building's front road width in order to create harmonious streetscapes (Berghauser Pont and Haupt, 2009; Marshall, 2005). On the other hand, in the United States of America (USA) and Japan, building heights and setbacks are indirectly controlled by the building coverage ratio (BCR), defined as the ratio of a building area to its plot area, and the floor area ratio (FAR), defined as the ratio of the total floor area of a building to its plot size (Batty, 2018; Bertaud, 2018; Bertaud and Brueckner, 2005), collectively called *zoning regulations*.

As long as zoning regulations are adopted, variations in building heights and setbacks are unavoidable. This is because variations in plot sizes, shapes and the BCR generate variations in building heights and setbacks, which not only result in inharmonious streetscape skeletons but also prevent us from predicting them in

advance. Thus, understanding the relationship among these variations is important to harmonise streetscape skeletons. However, the relationship among variations in the BCR, building heights and setbacks has yet to be investigated theoretically due to their complexity.

Therefore, the objective of this paper is to formulate the relationship between variations in building heights and setbacks as the function of the variance of the BCR through the following steps. In the second section, the relationship between the BCR, building height and setback in each plot is investigated schematically and the motivation for adopting a stochastic approach for investigating this complex relationship among buildings is explained. In the third section, the stochastic model of variations in building heights developed in the literature is briefly revisited and then a stochastic model of variations in setbacks is developed by considering variations in plot depths and the BCR. In the fourth section, the relationship among variations in the BCR, building heights and setbacks is investigated and important properties are suggested for harmonising streetscape skeletons. In the final section, concluding remarks and future research are provided.

How does the BCR determine a building's height and setback?

A plot of area s, frontage F and depth D is considered. To make a mathematical investigation tractable, assuming that (1) the plot shape is rectangle and (2) a polygon representing a building of frontage F_B and depth D_B and its plot are similar to one another and these centroids are coincident, the following equations are satisfied: $F_B D_B = \theta F D = \theta s$ and $D_B / F_B = D/F = \gamma$, where θ and γ denote the BCR and the ratio of D to F (equivalent to the ratio of D_B to F_B), respectively. As the value of γ increases, the plot shape becomes narrower and deeper. Thus, γ can be regarded as the indicator of discrepancy from a square shape ($\gamma = 1$).

For given θ and γ values, D and D_B can be formulated as the function of s: $D_B = \sqrt{\theta}D = \sqrt{\gamma\theta s}$, $D = \sqrt{\gamma s}$, $\theta \in (0,1]$. Then, the allowance of a setback of a building is defined as:

$$D_{free}(\theta) \equiv D - D_B = (1 - \sqrt{\theta})D = (1 - \sqrt{\theta})\sqrt{\gamma s}, \qquad \theta \in (0,1]. \qquad (1)$$

Equation (1) suggests that: (1) the larger the plot size, the greater D_{free}; (2) the greater γ, the greater D_{free}; and (3) the smaller θ, the greater D_{free}. Variations in s and γ can be merged into the variation in D. Even if there is no variation in D, the greater the variation in θ, the greater the variation in D_{free}. Given that variations in the BCR, shapes and plot sizes can be observed, the variation in D_{free} can be regarded as the composite of the variations in θ, γ and s (or θ and D).

Moreover, assuming that all buildings' floor shapes are congruent, the building height can be computed as follows:

$$h(\theta) = z\varphi s/\theta s = z\varphi/\theta, \theta \in (0,1], \qquad (2)$$

where φ denotes the FAR and z the average floor height. Equation (2) suggests that for a given φ, the smaller θ, the greater h. The number of building storeys can be computed by φ/θ.

It can be observed that: (1) if θ ranges from 0.2 to 1, $D_{\text{free}}(\theta)$ is an approximately linearly decreasing function of θ; and (2) as θ decreases, $h(\theta)$ does not increase linearly but super-linearly. Therefore, the allowance of a building setback can be obtained at the expense of the greater allowance of building's height. This finding is important in considering how to harmonise vertical and horizontal streetscape skeletons. Given that buildings and their plots have a variety of BCRs, variations in h depend not only on variations in φ but also on variations in θ. Even if φ is small, the potential variations in h are greater as θ decreases. Furthermore, from Equations (1) and (2), although D and φ affect D_{free} and h, respectively, θ affects both D_{free} and h.

This property implies that controlling θ is more important than controlling φ for reducing the potential variations in both D_{free} and h. Therefore, how to control the variation in the BCR should be primarily considered for harmonising vertical and horizontal streetscape skeletons. By considering only one building shape, it is possible to follow these complex relationships in a deterministic manner; however, this is not true when many building shapes with varied plot sizes, shapes and BCRs are considered simultaneously. To overcome this difficulty, these complex relationships should be explicitly investigated by modelling variations in plot sizes, shapes and the BCR based on a stochastic approach rather than a deterministic approach.

Statistical distribution of building heights and setbacks

In the literature, stochastic models of variations in the BCR and building heights have been proposed (Usui, 2021). Each plot owner determines their plot's BCR, whose domain ranges from 0 to 1. As a result of each owner's choice, variations in the BCR emerge as a statistical distribution. Given these characteristics, the statistical distribution of the BCR is modelled as a beta distribution. Then, the probability density function of building heights, denoted by $g(h)$, can be derived as follows:

$$g(h) = \frac{1}{B(\hat{a},\hat{b})} \cdot \frac{\varphi z}{h^2} \cdot \left(\frac{\varphi z}{h}\right)^{\left\{\frac{E[\theta](1-E[\theta])}{V[\theta]}-1\right\}E[\theta]-1} \left(1 - \frac{\varphi z}{h}\right)^{\left\{\frac{E[\theta](1-E[\theta])}{V[\theta]}-1\right\}(1-E[\theta])-1}. \tag{3}$$

where (1) $\varphi z < h$ and (2) $E[\theta]$ and $V[\theta]$ can be estimated from empirical data (Usui, 2021). This condition, $\varphi z < h$, means that h has the lower limit, which is given as the product of the legal FAR and the average floor height.

As mentioned in the second section using a simple schematic model, variations in setbacks depend on the difference between plot and building depth. This means that the variation in setbacks in a district depends both on the variation in plot depth D and the variation in the BCR. A previous study has found that: (1) plot depth follows a log-normal distribution, which can be mainly estimated from road network density (total road network length per unit area); and (2) for any fixed θ, assuming that half of D_{free} is assigned as the setback, setbacks also follow a log-normal distribution (Usui, 2019).

In this paper, setback distribution is theoretically derived by considering the variation in the BCR directly and assuming that D and θ are stochastic variables that are independent from one another. Moreover, the ratio of the setback u to $D_{\text{free}}(\theta)$ is denoted by $\beta \in [0,1]$. By multiplying β with both sides of Equation (1),

$$u \equiv \beta D_{\text{free}}(\theta) = \beta(1 - \sqrt{\theta})D, \qquad \theta \in (0,1]. \tag{4}$$

By taking the logarithm of both sides of Equation (4),

$$\ln u = \ln \beta + \ln(1 - \sqrt{\theta}) + \ln D, \qquad \theta \in (0,1]. \tag{5}$$

Given that the domains of θ and β range from 0 to 1, respectively, $\ln(1 - \sqrt{\theta})$ and $\ln \beta$ are equal to or smaller than 0. As mentioned above, D follows a log-normal distribution, which is equivalent to $\ln D$ following a normal distribution. Considering the reproductive property of a normal distribution, if $\ln(1 - \sqrt{\theta}) < 0$ and $\ln \beta < 0$ follow a normal distribution, then $\ln u$ follows a normal distribution, which is equivalent to u following a log-normal distribution. The hypothesis of the log-normality of $\ln(1 - \sqrt{\theta})$ can be statistically tested by using empirical data of the BCR. The results show that although the hypothesis is not always accepted, $1 - \sqrt{\theta}$ approximately follows a log-normal distribution.

In this paper, to make it tractable to investigate the relationships among u, θ, D and β, it is assumed that: (1) $\ln \beta$ follows a normal distribution; and (2) θ, D and β are independently and identically distributed among one another. Then, the statistical distribution of $\ln u$ can be estimated from the parameters of $\ln D$, $\ln(1 - \sqrt{\theta})$ and $\ln \beta$ as follows:

$$\mu_u = \mu_D + \mu_{1-\sqrt{\theta}} + \mu_\beta, \tag{6}$$

$$\sigma_u^2 = \sigma_D^2 + \sigma_{1-\sqrt{\theta}}^2 + \sigma_\beta^2, \tag{7}$$

where μ_u and σ_u are the mean and the standard deviation of $\ln u$. The approximation of log-normality of $1 - \sqrt{\theta}$ and the aforementioned assumptions enable us to model the statistical distribution of setbacks as a log-normal distribution and to analyse the relationship between the standard deviation of the BCR and variations in building heights and setbacks in a consistent way.

According to Usui (2019), the mean and the variance of $\ln D$, denoted by μ_D and σ_D , respectively, can be mainly estimated from road network density as follows:

$$\mu_D = \ln \frac{\sqrt{\eta_F^2+1}}{\sqrt{\eta^2+1}} \cdot \frac{\alpha \kappa(w,\lambda)}{\lambda(2-w\lambda)}, \qquad \sigma_D^2 = \ln \frac{\eta^2+1}{\eta_F^2+1}, \qquad \text{where } \eta_F < \eta. \tag{8}$$

In Equation (8), η and η_F denote the coefficient of variation of the plot size and frontage, respectively; $\lambda = \Lambda/A$ provides the road network density, where Λ and A denote the total road network length and the area of a district; w is the mean road width; α denotes the number of frontages to that of plots and $\kappa(w, \lambda) \approx 1 - w\lambda$ denotes the ratio of total plot sizes to A.

In theory, the mean and the variance of $\ln(1 - \sqrt{\theta})$, denoted by $\mu_{1-\sqrt{\theta}}$ and $\sigma_{1-\sqrt{\theta}}$, respectively, can be estimated from the mean and the variance of $1 - \sqrt{\theta}$, denoted by $E[1 - \sqrt{\theta}]$ and $V[1 - \sqrt{\theta}]$, respectively. As mentioned above, θ tends to follow a beta distribution. In this case, the mean and the variance of $\sqrt{\theta}$ cannot be analytically derived as the function of $E[\theta]$ and $V[\theta]$. Thus, the following linear approximation is considered:

$$1 - \sqrt{\theta} \approx \frac{1-\sqrt{c}}{1-c}(1 - \theta). \tag{9}$$

where $c \in (0,1)$ and $\theta \in (c,1)$. Then, $\mu_{1-\sqrt{\theta}}$ and $\sigma_{1-\sqrt{\theta}}^2$ can be estimated from $E[\theta]$, $V[\theta]$ and c as follows:

$$\mu_{1-\sqrt{\theta}} \approx \frac{1}{2}\ln\left\{\frac{\left(\frac{1-\sqrt{c}}{1-c}\right)^2(1-E[\theta])^2}{\left(\frac{\sqrt{V[\theta]}}{1-E[\theta]}\right)^2+1}\right\}, \qquad \sigma_{1-\sqrt{\theta}}^2 \approx \ln\left\{\left(\frac{\sqrt{V[\theta]}}{1-E[\theta]}\right)^2+1\right\} \tag{10}$$

The mean and the variance of $\ln\beta$, denoted by μ_β and σ_β, respectively, can be estimated from the mean and the variance of β, denoted by $E[\beta]$ and $V[\beta]$, respectively, as follows:

$$\mu_\beta = \frac{1}{2}\ln\left(\frac{E[\beta]^4}{V[\beta]+E[\beta]^2}\right) = \frac{1}{2}\ln\left(\frac{E[\beta]^2}{\eta_\beta^2+1}\right), \qquad \sigma_\beta^2 = \ln\left(\frac{V[\beta]}{E[\beta]^2}+1\right) = \ln(\eta_\beta^2+1) \tag{11}$$

where η_β denotes the coefficient of the variation of β. From Equations (6), (7), (8), (10) and (11), μ_u and σ_u can be estimated as follows:

$$\mu_u \approx \frac{1}{2}\ln\left[\left(\frac{\eta_F^2+1}{\eta^2+1}\right)\left\{\frac{\alpha\kappa}{\lambda(2-\bar{w}\lambda)}\right\}^2\left\{\frac{\left(\frac{1-\sqrt{c}}{1-c}\right)^2(1-E[\theta])^2}{\left(\frac{\sqrt{V[\theta]}}{1-E[\theta]}\right)^2+1}\right\}\left(\frac{E[\beta]^2}{\eta_\beta^2+1}\right)\right], \tag{12}$$

$$\sigma_u^2 \approx \ln\left[\left(\frac{\eta^2+1}{\eta_F^2+1}\right)\left\{\left(\frac{\sqrt{V[\theta]}}{1-E[\theta]}\right)^2+1\right\}(\eta_\beta^2+1)\right], \tag{13}$$

where $\eta_F < \eta$. Therefore, the probability density function of setbacks can be derived as a log-normal distribution as follows:

$$f(u) = \frac{1}{u\sqrt{2\pi\sigma_u^2}}e^{-\frac{(\ln u - \mu_u)^2}{2\sigma_u^2}}, \qquad u > 0. \tag{14}$$

Relationship among variations in the BCR, building heights and setbacks

The previous sections demonstrated how the statistical distribution of building heights and setbacks can be estimated from common parameters, $E[\theta]$ and $\sqrt{V[\theta]}$. This enables us to investigate the relationship between these two functions and $\sqrt{V[\theta]}$. Given that η, η_F and α tend to be in the region of 1.0, 0.6 and 1.2, respectively (Usui, 2019), these values are given as constants irrespective of the district. Furthermore, in Sumida ward in Tokyo, the typical values of road network density and road width are 0.04 and 8 metres, respectively. Thus, given the values of the other parameters except for $\sqrt{V[\theta]}$, namely $E[\theta]$ = 0.6, $E[\beta]$ =

0.5, η_β = 0.1, c = 0.3, η = 1.0, η_F = 0.6, λ = 0.04, w = 8 [m] and α = 1.2, it is possible to investigate how changes in $\sqrt{V[\theta]}$ affect the shapes of $f(u)$ and $g(h)$.

Figure 1 (left) shows $f(u)$. The shape is right-skewed. It can be observed that as $\sqrt{V[\theta]}$ increases, the mode of u decreases but the range of u increases. In particular, if $\sqrt{V[\theta]}$ = 0.05, the mode of u is approximately 1 metre and the maximum u is at most 4 metres. However, if $\sqrt{V[\theta]}$ increases to 0.25, the mode of u decreases to 0.5 metres but the maximum u increases to 6 metres. More importantly, even if $\sqrt{V[\theta]}$ = 0.05, variations in u are inevitable due to variations in D, θ and β. Hence, unless plot depths are not uniform, variations in setbacks are inevitable even if θ is strictly regulated.

Figure 1 (right) shows $g(h)$ (φ = 1, 2, 3, 4 and 5, respectively), which also exhibits a right-skewed distribution. It can be observed that as $\sqrt{V[\theta]}$ increases, the mode of h decreases but the range of h increases. In particular, if $\sqrt{V[\theta]}$ = 0.05, $g(h)$ converges in the neighbourhood of the mode. However, if $\sqrt{V[\theta]}$ increases to 0.25, the mode of h decreases but the range of h increases. Variations in building heights owe solely to variations in the BCR, because z and φ are constant. This tendency becomes more pronounced as φ increases. Therefore, as φ increases, it is inevitable that the range and the variation in building heights will also increase, making it difficult to harmonise building heights. More importantly, as $\sqrt{V[\theta]}$ increases, the increase in the range of h is greater than those of D_free and u. In particular, the greater φ, the more pronounced this tendency becomes. In fact, if φ = 1 (one of the most restricted cases applied for a residential zone) and $\sqrt{V[\theta]}$ increases to 0.25, even though the potential variation in u is at most 6 metres, the potential variation in h can be as much as 21 metres.

From these theoretical investigations, increasing the allowance of the BCR and setbacks accompanies variations in building heights, which are greater than variations in setbacks. Thus, unless a lower boundary regarding the BCR is not set, it is inevitable that as the BCR decreases, the potential variations in building heights and setbacks increase too. In particular, the potential variations in building heights are more pronounced than those in building setbacks. As a result, streetscape skeletons are inharmonious.

Conclusions

Increasing the allowance of the BCR and setbacks accompanies variations in building heights, which are greater than those in setbacks. Thus, it is inevitable that as the BCR decreases, the potential variations in building heights and setbacks increase too. In particular, the potential variations in building heights are more pronounced than those in building setbacks. These findings guide us through the following steps for harmonising streetscape skeletons: (1) controlling variations in the BCR is the only way of harmonising building heights and should be prioritised; and (2) variations in setbacks should be controlled considering variations in the BCR and plot depth.
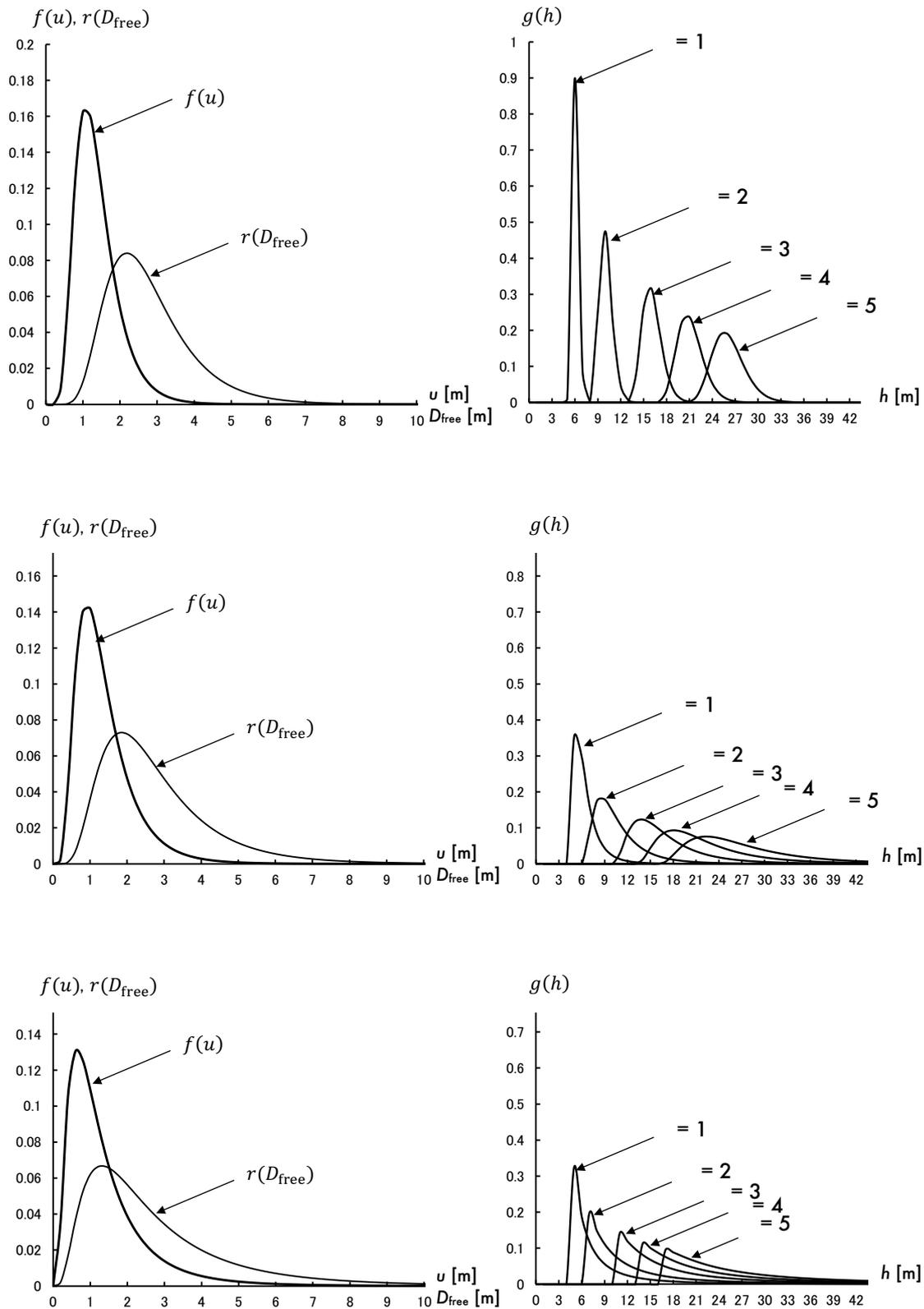
Figure 1. Relationship between the probability density functions of setbacks $f(u)$ (left) and heights $g(h)$ (right) and the standard deviation of BCR (top: $\sqrt{V[\theta]}$ = 0.05, centre: $\sqrt{V[\theta]}$ = 0.15, bottom: $\sqrt{V[\theta]}$ = 0.25). ($E[\vartheta]$ = 0.6, $E[\beta]$ = 0.5, η_β = 0.1, c = 0.3, η = 1.0, η_F = 0.6, λ = 0.04, w = 8 [m], α = 1.2, $\kappa(w, \lambda)$ = 0.68).

This is because given the value of φ, variations in building heights primarily depend on variations in θ in the district and are completely independent of variations in plot sizes and shapes. However, variations in setbacks depend on variations in D, θ and β in the district. Although variations in D depend on the tessellation of the two-dimensional urban plane by road networks and subdivisions in urban blocks, variations in θ and β depend on each plot owner's choice of θ following the maximum BCR regulation and that of β considering the plot size and the owner's preference for the area of open space along the adjacent road.

Future research should empirically test the validation for assuming that the ratio of the setback to the allowance of the setback of a building β follows a log-normal distribution. Moreover, variations in the FAR should be considered in order to refine the proposed models.

Acknowledgements

References

1. Batty, M. (2018) *Inventing future cities* (MIT Press, Cambridge, MA).

2. Berghauser Pont, M. and Haupt, P. (2009) *Space, density and urban form*, doctoral thesis, Delft University of Technology, Delft.

3. Bertaud, A. (2018) *Order without design: how markets shape cities* (MIT Press, Cambridge, MA).

4. Bertaud, A. and Brueckner, J. K. (2005) 'Analyzing building-height restrictions: predicted impacts and welfare costs', *Regional Science and Urban Economics* 35(2), 109-125

5. Harvey, C., Aultman-Hall, L., Troy, A. and Hurley, S. E. (2017) 'Streetscape skeleton measurement and classification', *Environment and Planning B: Urban Analytics and City Science* 44(4), 668-692.

6. Marshall, S. (2005) *Streets and patterns* (Routledge, London).

7. Oliveira, V. (2016) *Urban morphology: an introduction to the study of the physical form of cities* (Springer, Berlin).

8. Usui, H. (2019) 'Statistical distribution of building lot depth: theoretical and empirical investigation of downtown districts in Tokyo', *Environment and Planning B: Urban Analytics and City Science* 46(8), 1499-1516.

9. Usui, H. (2021) 'Variation in building heights under zoning regulations of building coverage ratio and floor area ratio: theoretical and empirical investigation of downtown districts in Tokyo', *ISUF 2020 Virtual Conference Proceedings* (Vol. 1).