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The cover time of a (multiple) Markov chain with rational transition probabilities is rational

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ABSTRACT

The cover time of a Markov chain on a finite state space is the expected time until all states are visited. We show that if the cover time of a discrete-time Markov chain with rational transitions probabilities is bounded, then it is a rational number. The result is proved by relating the cover time of the original chain to the hitting time of a set in another higher dimensional chain. We prove this result in a more general setting where $k \geq 1$ independent copies of a Markov chain are run simultaneously on the same state space.

1. Introduction and results

Let $(X_t)_{t\geq 0}$ be a discrete-time Markov chain with transition matrix $\mathbf{P}$ on a state space Ω, see Aldous and Fill (2002), Levin and Peres (2017) for background. We say a chain is *rational* if all its transition probabilities are rational numbers, i.e. $\mathbf{P}(x,y) \in \mathbb{Q}$ for all $x, y \in \Omega$. The stopping time τ_{cov} is the first time all states are visited, that is

$$\tau_{\mathrm{cov}} := \inf\left\{ t \geq 0 : \bigcup_{k=0}^{t} \{X_k\} = \Omega \right\}.$$

For $x \in \Omega$, let $\mathbb{E}_x[\,\tau_{\mathrm{cov}}\,] = \mathbb{E}[\,\tau_{\mathrm{cov}} \mid X_0 = x\,]$ be the *cover time* from x, that is, the expected time for the chain to visit all states when started from $x \in \Omega$.

Along with mixing and hitting times, the cover time is one of the most natural and well studied stopping times for a Markov chain and has found applications in the analysis of algorithms, see for example Alon et al. (2011), Aldous and Fill (2002, Ch. 6.8) and Levin and Peres (2017, Ch. 11). It is clear that the stopping time τ_{cov} is a natural number, however it is not so clear whether the cover time $\mathbb{E}_x[\,\tau_{\mathrm{cov}}\,]$ is rational, even if the transition probabilities are rational. Our main result shows that, under some natural assumptions, the cover time of a rational Markov chain is rational.

Theorem 1. *Let $(X_t)_{t\geq 0}$ be a discrete-time rational Markov chain on a finite state space Ω. Then, for any $x \in \Omega$ such that $\mathbb{E}_x[\tau_{\mathrm{cov}}] < \infty$, we have $\mathbb{E}_x[\tau_{\mathrm{cov}}] \in \mathbb{Q}$.*

The assumption that Ω is finite is necessary to ensure the cover time is bounded. Recall that a Markov chain is *irreducible* if for every $x, y \in \Omega$ there exists some $t \geq 0$ such that $\mathbf{P}^t(x,y) > 0$, where $\mathbf{P}^t(x,y)$ denotes the probability a chain started at x is at state y after $t \geq 1$ steps. Theorem 1 does not require irreducibility, just that the cover time from the given start vertex is bounded. An example of a non-irreducible Markov chain to which we can apply Theorem 1

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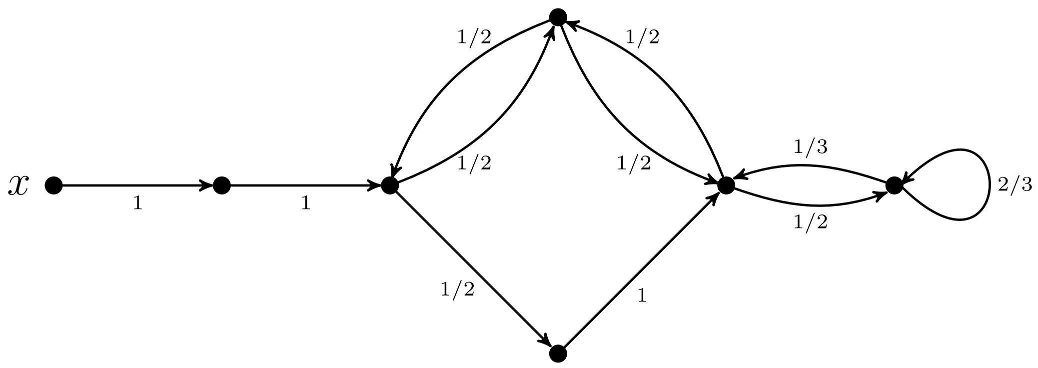

Fig. 1. Example of a non-irreducible Markov chain on seven states where the cover time from x is finite and from any other vertex the cover time is unbounded/undefined.

is given in Fig. 1. In this example the cover time from x is bounded however, the cover time from any other vertex is unbounded/undefined, as if a walk starts from any other vertex, then x (and possibly also the vertex immediately right of x) cannot be reached.

For a concrete example of why rational transition probabilities are necessary in Theorem 1, if one fixes any real number $r \geq 1$ then the two state chain with transition matrix given by

$$\mathbf{P} = \begin{pmatrix} 1 - 1/r & 1/r \\ 1/r & 1 - 1/r \end{pmatrix}, \tag{1}$$

has cover time r. It is well known, see for example Levin and Peres (2017, Lemma 1.13), that the cover time of finite irreducible Markov chain from any start vertex is bounded. This fact, and restricting the example given by (1) to $r \in \mathbb{Q}$, implies the following corollary to Theorem 1.

Corollary 2. *The set of cover times attainable by finite discrete-time irreducible rational Markov chains is* $(\mathbb{Q} \cap [1, \infty)) \cup \{0\}$.

We now introduce multiple Markov chains, which have been studied for their applications to parallelising algorithms driven by random walks, see Alon et al. (2011) and subsequent papers citing it. For any $k \geq 1$, let $\mathbf{X}_t = \left(X_t^{(1)}, \dots, X_t^{(k)}\right)$ be the *k-multiple* of a Markov chain $\mathbf{P}$ where each $X_t^{(i)}$ is an independent copy of the chain $\mathbf{P}$ run simultaneously on the same state space Ω. The k-multiple of $\mathbf{P}$ is itself a Markov chain (with transition matrix $\mathbf{K}$) on Ω^k with transition probabilities

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{k} \mathbf{P}(x^{(j)}, y^{(j)}), \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \Omega^k.$$

As before, we denote the conditional expectation $\mathbb{E}_{(x^{(1)},\dots,x^{(k)})}[\,\cdot\,] := \mathbb{E}\left[\,\cdot\,\middle|\mathbf{X}_0 = (x^{(1)}, \dots, x^{(k)})\right]$, where $X_0^{(i)} = x^{(i)} \in \Omega$ is the start state of the i^{th} walk for each $1 \leq i \leq k$. We let the stopping time $\tau_{\text{cov}}^{(k)} = \inf\{t : \bigcup_{i=0}^{t}\{X_i^{(1)}, \dots, X_i^{(k)}\} = \Omega\}$ be the first time every state in Ω (not Ω^k) has been visited by some walk $X_t^{(i)}$. We then let $\mathbb{E}_{\mathbf{x}}\left[\,\tau_{\text{cov}}^{(k)}\,\right]$ denote the k-walk stopping time from $\mathbf{x} \in \Omega^k$. Note that this is not simply the cover time of the chain $\mathbf{K}$. The multiple walk cover time can have subtle dependences on k and the host underlying Markov chain, see Alon et al. (2011).

We show that Theorem 1 also holds in the more general setting of k-multiple Markov chains.

Theorem 3. *Let $k \geq 1$ and $(\mathbf{X}_t)_{t\geq 0}$ be the k-multiple of a discrete-time rational Markov chain on a finite state space Ω. Then for any $\mathbf{x} \in \Omega^k$ such that $\mathbb{E}_{\mathbf{x}}\left[\tau_{\text{cov}}^{(k)}\right] < \infty$ we have $\mathbb{E}_{\mathbf{x}}\left[\tau_{\text{cov}}^{(k)}\right] \in \mathbb{Q}$.*

Theorem 1 is the special case $k = 1$ of Theorem 3, thus it suffices to prove Theorem 3.

2. Proofs

In this section we shall prove Theorem 3. The first part of the proof (covered in Section 2.1) is to show the expected time to first visit any set of states (hitting time) in a rational Markov chain is rational. Then, in Section 2.2, we show for any $k \geq 1$ and $\mathbf{P}$, the multiple walk with transition matrix $\mathbf{P}$ can be coupled with a higher dimensional Markov chain $\mathbf{Q}$ on a state space V where $|V| \leq |\Omega|^k \cdot 2^{|\Omega|}$. The coupling shows that the first time all states in Ω have been visited by at least one of the k walks has the same distribution as the first visit time a specific set $C \subset V$ is visited in $\mathbf{Q}$.

2.1. Rationality of hitting times

For $S \subseteq \Omega$, a subset of the state space of a Markov chain $\mathbf{P}$, let the stopping time

$$\tau_S := \inf\{t \geq 0 : X_t \in S\},$$

be the first time S is visited. If $S = \{s\}$ is a singleton set we abuse notation slightly by taking τ_s to mean $\tau_{\{s\}}$. For $x \in \Omega$, let $\mathbb{E}_x[\,\tau_S\,]$ be the expected *hitting time* of $S \subseteq \Omega$ for a chain started from x. The next result is the hitting time analogue of Theorem 1.

Proposition 4. *Let $\mathbf{P}$ be a discrete-time rational Markov chain on a finite state space Ω. For a non-empty set $S \subseteq \Omega$ let $B(S) = \{x \in \Omega \,:\, \mathbb{E}_x[\,\tau_S\,] < \infty\}$. Then for any $S \subseteq \Omega$ and $x \in B(S)$ we have $\mathbb{E}_x[\tau_S] \in \mathbb{Q}$.*

Observe that if $\mathbf{P}$ is irreducible then $B(S) = \Omega$ for any $S \subseteq \Omega$ by Levin and Peres (2017, Lemma 1.13). Before proving Proposition 4 we give some definitions and prove an elementary lemma.

For a field $\mathbf{F}$ and integers $n, m \geq 1$ let $\mathbf{F}^n$ and $\mathbf{F}^{m\times n}$ denote the set of n-dimensional vectors and $m \times n$-dimension matrices respectively. Let $\mathbf{I}_n$ denote the $n \times n$ identity matrix.

Lemma 5. *Let $\mathbf{A} \in \mathbb{Q}^{n\times n}$ be non-singular and $\mathbf{b} \in \mathbb{Q}^n$. Then there exists a unique vector $\mathbf{x} \in \mathbb{Q}^n$ such that $\mathbf{Ax} = \mathbf{b}$.*

Proof. Since $\mathbf{A}$ is non-singular there exists a unique solution $\mathbf{x} \in \mathbb{R}^n$ to the linear system given by $\mathbf{Ax} = \mathbf{b}$. Also, again since $\mathbf{A}$ is non-singular, we can compute $\mathbf{A}^{-1}$ by Gaussian elimination. Since all entries of $\mathbf{A}$ are rational, all multiplications preformed during the Gaussian elimination will be rational. Thus, as there are only finitely many row additions and multiplications, $\mathbf{A}^{-1} \in \mathbb{Q}^{n\times n}$. Since $\mathbf{b} \in \mathbb{Q}^n$, we conclude that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \in \mathbb{Q}^n$. □

We now use this lemma to prove Proposition 4.

Proof of Proposition 4. Observe that $B(S) \neq \emptyset$ since $S \subseteq B(S)$ and $\mathbb{E}_s[\,\tau_S\,] = 0$ for all $s \in S$. Let $b := |B(S)|$. Now, each entry of the vector $\mathbf{h} := (\mathbb{E}_x[\tau_S])_{x\in B(S)}$ is bounded and $\mathbf{h}$ is a solution to the following set of linear equations

$$\mathbb{E}_x[\tau_S] = \begin{cases} 1 + \sum_y \mathbf{P}(x,y)\cdot \mathbb{E}_y[\tau_S] & \text{if } x \notin S \\ 0 & \text{if } x \in S. \end{cases}$$

This can be expressed as $\mathbf{Ah} = \mathbf{b}$ where $\mathbf{b} \in \{0,1\}^b$ and $\mathbf{A} := (\mathbf{I}_b - \mathbf{M}) \in \mathbb{Q}^{b\times b}$ for $\mathbf{M} \in \mathbb{Q}^{b\times b}$ given by $\mathbf{M}(i,j) = \mathbf{P}(i,j)$ if $i, j \notin S$ and 0 otherwise. We shall show that

(i) all rows i satisfy $|\mathbf{A}(i,i)| \geq \sum_{j\neq i} |\mathbf{A}(i,j)|$, and

(ii) for each row r_0, there exists a finite sequence of rows $r_0, r_1 \ldots, r_t$ such that $\mathbf{A}(r_{i-1}, r_i) \neq 0$ for all $1 \leq i \leq t$ and $|\mathbf{A}(r_t, r_t)| > \sum_{j\neq r_t} |\mathbf{A}(r_t, j)|$.

Observe that Condition (i) holds since $\mathbf{M}$ is a sub-matrix of $\mathbf{P}$.

For Condition (ii), note that for every row $s \in S$ we have $\sum_j \mathbf{M}(s,j) = 0$. Thus $|\mathbf{A}(s,s)| > \sum_{j\neq s} |\mathbf{A}(s,j)|$ for any row $s \in S$. The fact that each row r_0 corresponds to a state in $B(S)$ implies that, for any row r_0, there exists some $r_t \in S$ and a sequence of states/rows $r_0, r_1 \ldots, r_t$ such that $\mathbf{A}(r_{i-1}, r_i) = -\mathbf{P}(r_{i-1}, r_i) \neq 0$ for each $1 \leq i \leq t$, thus Condition (ii) is satisfied.

Since $\mathbf{A}$ satisfies (i) and (ii) it is weakly chained diagonally dominant, thus by Azimzadeh and Forsyth (2016, Lemma 3.2) $\mathbf{A}$ is non-singular. Thus, by Lemma 5, $\mathbf{h} \in \mathbb{Q}^b$. □

2.2. Encoding cover times as hitting times

Let $\mathbf{P}$ be a Markov chain on a state space Ω with transition matrix $\mathbf{P} = (\mathbf{P}(x,y))_{x,y\in\Omega}$ and $\mathcal{P}(\Omega) = \{S \subseteq \Omega\}$ be the power-set of Ω. For $k \geq 1$ independent walks with transition matrix $\mathbf{P}$ on the same state space Ω we define the *k-walk auxiliary chain* $\mathbf{Q} := \mathbf{Q}(\mathbf{P}, k)$ to be the Markov Chain on state space $V := V(\Omega, k)$ given by

$$V = \left\{((x_1,\ldots,x_k), S) : S \subseteq \Omega,\ x_i \in S \text{ for all } 1 \leq i \leq k\right\} \subseteq \Omega^k \times \mathcal{P}(\Omega),$$

with transition matrix specified by

$$\mathbf{Q}\left((\mathbf{x}, S), (\mathbf{y}, S \cup \{y^{(1)}, \ldots, y^{(k)}\})\right) = \mathbf{P}(x^{(1)}, y^{(1)}) \cdots \mathbf{P}(x^{(k)}, y^{(k)}), \tag{2}$$

for any $S \subseteq \Omega$ and $\mathbf{x}, \mathbf{y} \in \Omega^k$ where $\mathbf{x} = (x^{(1)}, \ldots, x^{(k)})$ and $\mathbf{y} = (y^{(1)}, \ldots, y^{(k)})$.

Fig. 2 shows an example of the auxiliary chain $\mathbf{Q}$ of a single Markov chain $\mathbf{P}$ on three states, that is the case $k = 1$. Staying within the confines of $k = 1$ case for simplicity, one may think of $\mathbf{Q}$ as inducing a directed graph consisting of many 'layers', where each layer is a copy of $\mathbf{P}$ restricted to a subset of Ω. These layers are linked by directed edges which are crossed when a new state not in the current layer is first visited. Thus, since a sequence $x_0, x_1, \ldots$ in the first

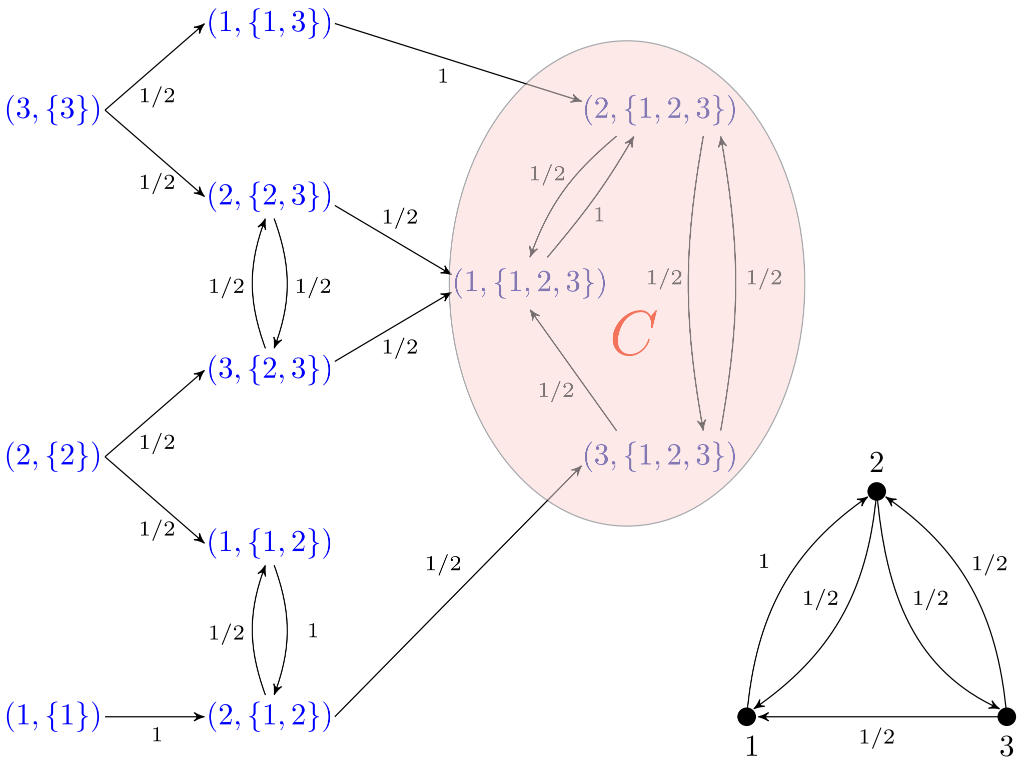

Fig. 2. This figure shows an example of a Markov chain **P** on three states (bottom right) and its associated auxiliary chain **Q**(**P**, 1), where the set C from Lemma 6 is shown in the red shaded ellipse.

component of V evolves according to **P** by (2), each layer encodes which states of chain have been visited so far by a trajectory in **P**.

Similar constructions to **Q**(**P**, 1) were used by the author and co-authors in the study of the Choice and ε-TB random walks, which are walks where a controller can influence which vertices are visited. In particular they were used to show that there exist optimal strategies for covering a graph by these walks which are time invariant in a certain sense (Georgakopoulos et al., 2022) and to show the computational problem of finding optimal strategies to cover a graph by these walks is in PSPACE (Haslegrave et al., 2022).

The next result equates the cover time by $k \geq 1$ multiple Markov chain with transition matrix **P** to the hitting time of a specific set in the auxiliary chain **Q**(**P**, k). For clarity we use the notation $\mathbb{E}^{\mathbf{P}}_{\cdot}[\,\cdot\,]$ to highlight the chain, in this case **P**, in which the expectation is taken.

Lemma 6. *Let* **P** *be a Markov chain on* Ω*, and let* $k \geq 1$ *be an integer. Let* $\mathbf{Q} := \mathbf{Q}(\mathbf{P}, k)$ *be the associated* k*-walk auxiliary chain with state space* $V := V(\Omega, k)$*, and set* $C = \{(\mathbf{u}, \Omega) : \mathbf{u} \in \Omega^k\} \subset W$*. Then, for any* $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Omega^k$ *and real number* a*, we have*

$$\mathbb{P}^{\mathbf{P}}_{\mathbf{x}}\left[\,\tau^{(k)}_{\mathsf{cov}} \geq a\,\right] = \mathbb{P}^{\mathbf{Q}}_{(\mathbf{x},\{x^{(1)},\dots,x^{(k)}\})}\left[\,\tau_C \geq a\,\right].$$

Consequently, $\mathbb{E}^{\mathbf{P}}_{\mathbf{x}}\left[\,\tau^{(k)}_{\mathsf{cov}}\,\right] = \mathbb{E}^{\mathbf{Q}}_{(\mathbf{x},\{x^{(1)},\dots,x^{(k)}\})}\left[\,\tau_C\,\right]$*, for any* $\mathbf{x} \in \Omega^k$*.*

We must introduce some notation before proving Lemma 6. For real valued random variables X, Y we say that Y *stochastically dominates* X if $\mathbb{P}[\,Y \geq a\,] \geq \mathbb{P}[\,X \geq a\,]$ for all real a, and we denote this by $X \preceq Y$. Thus, if $X \preceq Y$ and $Y \preceq X$, then X and Y are equidistributed.

Proof of Lemma 6. We first show how any trajectory $(\mathbf{X}_t)_{t\geq 0}$ of a k-multiple of the Markov chain **P** can be coupled with a trajectory $(Y_t)_{t\geq 0}$ of the auxiliary Markov chain **Q**(**P**, k) given by (2). To begin, given any start vector $\mathbf{X}_0 = \mathbf{x}_0 \in \Omega^k$, where $\mathbf{x}_0 = (x_0^{(1)}, \dots, x_0^{(k)})$, we set $Y_0 = (\mathbf{x}_0, \{x_0^{(1)}, \dots, x_0^{(k)}\}) \in V$. Then, given a trajectory $(\mathbf{X}_t)_{t=0}^T = (\mathbf{x}_t)_{t=0}^T$ we set

$Y_t = \left(\mathbf{x}_t, \bigcup_{i=0}^{t}\bigcup_{j=1}^{k}\left\{x_i^{(j)}\right\}\right)$ for each $0 \le t \le T$. Now by (2),

$$\prod_{i=0}^{t-1}\prod_{j=1}^{k}\mathbf{P}(x_i^{(j)}, x_{i+1}^{(j)}) = \mathbf{Q}\left(\left(\mathbf{x}_0, \bigcup_{j=1}^{k}\left\{x_0^{(j)}\right\}\right), \left(\mathbf{x}_1, \bigcup_{i=0}^{1}\bigcup_{j=1}^{k}\left\{x_i^{(j)}\right\}\right)\right)\cdots \cdot \mathbf{Q}\left(\left(\mathbf{x}_{t-1}, \bigcup_{i=0}^{t-1}\bigcup_{j=1}^{k}\left\{x_i^{(j)}\right\}\right), \left(\mathbf{x}_t, \bigcup_{i=0}^{t}\bigcup_{j=1}^{k}\left\{x_i^{(j)}\right\}\right)\right). \tag{3}$$

Thus given any trajectory $(\mathbf{X}_t)_{t\ge 0}$ of $\mathbf{P}$ we can find a trajectory $(Y_t)_{t\ge 0}$ of $\mathbf{Q}(\mathbf{P}, k)$ with the same measure. To couple a given trajectory $(Y_t)_{t\ge 0}$ of $\mathbf{Q}(\mathbf{P}, k)$ to a trajectory $(\mathbf{X}_t)_{t\ge 0}$ of $\mathbf{P}$ is even simpler; given $\mathbf{Y}_t = (\mathbf{y}_t, S_t)$ we simply 'forget' the second component of $\mathbf{Y}_t$ and set $\mathbf{X}_t = \mathbf{y}_t \in \Omega^k$ for each $t \ge 0$. Again the measure is preserved by (3).

Recall the set $C = \{(\mathbf{u}, \Omega) : \mathbf{u} \in \Omega^k\}$ which is a subset of the state space $V(\Omega, k)$ of the auxiliary chain $\mathbf{Q}(\mathbf{P}, k)$. To complete the proof we show that, for any $\mathbf{x} \in \Omega^k$, the times τ_{cov} and τ_C in the coupled chains $\mathbf{X}_t$ and $\mathbf{Y}_t$, started from $\mathbf{x}$ and $(\mathbf{x}, \{x^1, \dots, x^{(k)}\})$ respectively, are equidistributed.

Suppose we take any trajectory $(\mathbf{X}_t)_{t=0}^{T}$ of length $T \ge 0$ such that $\cup_{i=0}^{T}\cup_{j=1}^{k}\{X_i^{(j)}\} = \Omega$. Then by the coupling above, we have $\mathbf{Y}_T = \left(\mathbf{x}_T, \cup_{i=0}^{T}\cup_{j=1}^{k}\{X_i^{(j)}\}\right) = (\mathbf{x}_T, \Omega) \in C$. Since this holds for any trajectory and any time T such that $\cup_{j=0}^{T}\{X_j\} = \Omega$, we can assume that T is the first such time. That is, we can take $T = \tau_{\text{cov}}$ and then it follows that $\tau_C \preceq \tau_{\text{cov}}$.

Conversely, let $(Y_t)_{t=0}^{T}$ be any trajectory in $\mathbf{Q}$ where $Y_0 = (\mathbf{y}_0, \cup_{j=1}^{k}\{y_0^{(j)}\})$, for some $\mathbf{y}_0 = (y_0^{(1)}, \dots, y_0^{(k)}) \in \Omega^k$ and $Y_T \in C$. Since the only transitions supported by $\mathbf{Q}$ are from $(\mathbf{y}, S)$ to $\left(\mathbf{z}, S \cup \left(\cup_{j=1}^{k}\{z^{(j)}\}\right)\right)$ where $\prod_{j=1}^{k}\mathbf{P}(y^{(j)}, z^{(j)}) > 0$, and $\mathbf{Y}_0 = (\mathbf{y}_0, \cup_{j=1}^{k}\{y_0^{(j)}\})$, it follows that $\cup_{t=0}^{T}\cup_{j=1}^{k}\{y_t^{(j)}\} = \Omega$. Thus, by the coupling above, $\cup_{t=0}^{T}\cup_{j=1}^{k}\{X_t^{(j)}\} = \Omega$. Similarly, since we can take $T = \tau_C$ to be minimal, we have $\tau_{\text{cov}} \preceq \tau_C$.

Thus for any pair of coupled trajectories with fixed start vertices $\mathbf{x}$ and $(\mathbf{x}, \cup_{j=1}^{k}\{x_0^{(j)}\})$ the times τ_{cov} and τ_C are the same. The final statement then follows by taking expectation. □

Lemma 6 equates the cover time of any Markov chain $\mathbf{P}$ on Ω (not just rational chains) to a hitting time in a higher dimensional chain $\mathbf{Q}$ on V. This result may be useful for studying the cover time of an arbitrary Markov chain $\mathbf{P}$ on Ω. However, one drawback of this approach is that for many chains $|V|$ is exponential in $|\Omega|$.

Having established Proposition 4 and Lemma 6 the proof of Theorem 3 is simple.

Proof of Theorem 3. Let $\mathbf{Q} := \mathbf{Q}(\mathbf{P}, k)$ be the auxiliary chain associated with the k-multiple Markov chain with transition matrix $\mathbf{P}$. Then $\mathbb{E}_{\mathbf{x}}^{\mathbf{P}}\left[\tau_{\text{cov}}^{(k)}\right] = \mathbb{E}_{(\mathbf{x},\{x^{(1)},\dots,x^{(k)}\})}^{\mathbf{Q}}\left[\tau_C\right]$ for any $\mathbf{x} = (x^{(1)}, \dots, x^{(k)}) \in \Omega^k$ by Lemma 6, where $C = \{(\mathbf{y}, \Omega) \mid \mathbf{y} \in \Omega^k\}$. By assumption we have $\mathbb{E}_{\mathbf{x}}^{\mathbf{P}}\left[\tau_{\text{cov}}\right] < \infty$ and so $\mathbf{x} \in B(C)$. It follows from Proposition 4 that $\mathbb{E}_{(\mathbf{x},\{x^{(1)},\dots,x^{(k)}\})}^{\mathbf{Q}}\left[\tau_C\right] \in \mathbb{Q}$ and so $\mathbb{E}_{\mathbf{x}}^{\mathbf{P}}\left[\tau_{\text{cov}}\right] \in \mathbb{Q}$ as claimed. □

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