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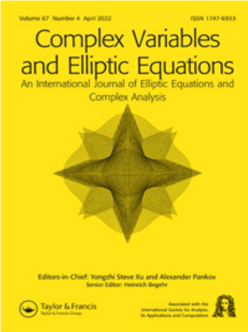

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Dominating sets and reverse Carleson measures on exponentially weighted Bergman spaces

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ABSTRACT

We study the dominating sets and reverse Carleson measures on exponentially weighted Bergman spaces A^p_ω under a new metric. Then, we give some applications of reverse Carleson measure and a generalization of Theorem 1 in Luecking [Sampling measures for Bergman spaces on the unit disk. Math Ann. 2000;316:659–679].

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1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. For $a, z \in \mathbb{D}$, let $\rho(a,z) = |a-z|/|1-\bar{a}z|$ denote the pseudo-hyperbolic metric and $\Delta(z,r) = \{a \in \mathbb{D} : \rho(a,z) < r\}$ be the pseudo-hyperbolic disk. Let $\sigma(a,z) = |a-z|/|1-\bar{a}z|^2$ and $D(z,r) = \{a \in \mathbb{D} : \sigma(a,z) < r\}$, where the metric σ is introduced by Cho and Park [1]. A weight is a positive function $\omega \in L^1(\mathbb{D}, \mathrm{d}A)$, where $\mathrm{d}A(z) = \frac{\mathrm{d}x\,\mathrm{d}y}{\pi}$ is the normalized area measure on $\mathbb{D}$. For a Borel measurable set $E \subset \mathbb{D}$, we define

$$\omega(E) = \int_E \omega(z)\,\mathrm{d}A(z).$$

It is obvious that $\omega(\mathbb{D}) < \infty$. For $0 < p < \infty$, the weighted Bergman space A^p_ω consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{A^p_\omega} = \left(\int_{\mathbb{D}} |f(z)|^p \omega(z)^{p/2}\,\mathrm{d}A(z)\right)^{1/p} < \infty.$$

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We are going to study the dominating sets and reverse Carleson measures on exponentially weighted Bergman spaces A^p_ω, for a certain class $\mathcal{E}$ of radial rapidly decreasing weights. The class $\mathcal{W}$, considered previously in [2–5], consists of the radial decreasing weights of the form $\omega(z) = \mathrm{e}^{-2\varphi(z)}$, where $\varphi \in C^2(\mathbb{D})$ is a radial function such that $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$, for some radial positive function $\tau(z)$ that decreases to 0 as $|z| \to 1^-$ and satisfies $\lim_{r\to 1^-} \tau'(r) = 0$. Here, Δ denotes the standard Laplace operator. Furthermore, we assume that there either exists a constant $C > 0$ such that $\tau(r)(1-r)^{-C}$ increases for r close to 1 or

$$\lim_{r\to 1^-} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

A positive function τ on $\mathbb{D}$ is said to be of class $\mathcal{L}$ if it satisfies the two properties:

(A) there is a constant c_1 such that $\tau(z) \le c_1\,(1-|z|)$ for all $z \in \mathbb{D}$;
(B) there is a constant c_2 such that $|\tau(z) - \tau(\zeta)| \le c_2|z-\zeta|$ for all $z, \zeta \in \mathbb{D}$.

We also use the notation

$$m_\tau := \frac{\min(1, c_1^{-1}, c_2^{-1})}{4},$$

where c_1 and c_2 are the constants appearing in the previous definition. For $a \in \mathbb{D}$ and $\delta > 0$, we use $D_\delta(a)$ to denote the Euclidean disk centered at a and having radius $\delta\tau(a)$. It is easy to see from conditions (A) and (B) (see [4, Lemma 2.1]) that if $\tau \in \mathcal{L}$ and $z \in D(\delta\tau(a))$, then

$$\tfrac{1}{2}\tau(a) \le \tau(z) \le 2\tau(a), \tag{1}$$

for sufficiently small $\delta > 0$, that is, for $\delta \in (0, m_\tau)$.

Definition 1.1: We say that a weight ω is of class $\mathcal{L}^*$ if it is of the form $\omega = e^{-2\varphi}$, where $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, and $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$ with τ being a function in the class $\mathcal{L}$. Here Δ denotes the classical Laplace operator.

It is straightforward to see that $\mathcal{W} \subset \mathcal{L}^*$. Now, we consider the class $\mathcal{E}$ that consists of the weights $\omega \in \mathcal{W}$ satisfying

$$C_1\omega(z) \le \omega(a) \le C_2\omega(z), \quad \text{for } z \in D_{\delta,r}(a), \tag{2}$$

where $D_{\delta,r}(a) := D(\delta\tau(a)) \cup D(a,r)$ and C_1 and C_2 are positive constants. The exponential type weights

$$\omega_\beta(z) := \omega_{\gamma,\sigma,\beta}(z) = (1-|z|^2)^\gamma \exp\left(\frac{-\beta}{(1-|z|^2)^\sigma}\right), \quad \gamma \ge 0, \sigma > 0,\ \beta > 0,$$

are in the class $\mathcal{W}$ with associated subharmonic function

$$\varphi_{\gamma,\sigma,\beta}(z) = -\gamma \log(1-|z|^2) + \beta(1-|z|^2)^{-\sigma}.$$

We have

$$(\Delta\varphi_{\gamma,\sigma,\beta}(z))^{-1} \asymp \tau(z)^2 = (1-|z|^2)^{2+\sigma},$$

and it is easy to see that $\tau(z)$ satisfies the conditions in the definition of the class $\mathcal{W}$ and $\omega_{\gamma,\sigma}$ belongs to $\mathcal{E}$, see Lemma 2.5 in [1] and Lemma 2.1.

For a measurable subset G of $\mathbb{D}$, we say that G is a dominating set for A^p_ω if there exists a constant $C > 0$ (depending on G) such that

$$\|f\|^p_{A^p_\omega} \leq C \int_G |f(z)|^p \omega(z)\, \mathrm{d}A(z), \quad \text{for any } f \in A^p_\omega.$$

Our first main result on the dominating sets for A^p_ω reads as follows.

Theorem 1.2: *Suppose $\omega \in \mathcal{E}$ and $p > 0$. Let G be a measurable subset of $\mathbb{D}$. Then G is a dominating set of A^p_ω if and only if there exist constants $\delta > 0$ and $r \in (0,1)$ such that*

$$\omega(G \cap D(z,r)) > \delta\omega(D(z,r)) \tag{3}$$

for all $z \in \mathbb{D}$.

Let $\omega \in \mathcal{E}$ and $0 < p, q < \infty$. A positive Borel measure μ is a q-Carleson measure for A^p_ω if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^q\, \mathrm{d}\mu(z) \leq C\|f\|^p_{A^p_\omega}, \quad \text{for } f \in A^p_\omega.$$

That means the inclusion $I_\mu : A^p_\omega \to L^q(\mathbb{D}, \mathrm{d}\mu)$ is bounded.

In contrast, for a positive Borel measure μ, we say that μ is a q-reverse Carleson measure for A^p_ω if there exists a constant $C > 0$ such that

$$\|f\|^p_{A^p_\omega} \leq C \int_{\mathbb{D}} |f(z)|^q\, \mathrm{d}\mu(z).$$

The concept of Carleson measures was first introduced by L. Carleson in order to study interpolating sequences and the corona problem [6] on the algebra H^∞ of all bounded analytic functions on the unit disk. It quickly became a powerful tool for the study of function spaces and operators acting on them. The Bergman Carleson measures were first studied by Hastings [7] and further pursued by Oleinik [8], Luecking [9, 10], Cima and Wogen [11], and many others see, for instance, [12–14].

The reverse Carleson inequality on the classical Bergman spaces was firstly raised by Luecking. Luecking [15] studied the dominating sets and the reverse Carleson inequality for the classical Bergman spaces. It was generalized to the Bergman spaces on the unit ball in [16] and in Hardy spaces, we refer the reader to [17]. Moreover, the closed range of the restriction operators was studied based on the characterization of the reverse Carleson inequality. The reverse Carleson inequality for the derivatives of Bergman functions was studied by Luecking [10]. Recently, Korhonen and Rättyä [18] characterized the dominating set for the Bergman spaces with radial doubling weights and gave a necessary condition of the dominating sets. Inspired by the results above, we study the reverse inequality for the exponentially weighted Bergman spaces.

In this paper, we give some sufficient conditions for a p-Carleson measure to be a p-reverse Carleson measure for the Bergman space A^p_ω, see Theorem 1.3 and Theorem 1.4 below.

For any $z \in \mathbb{D}$ and $r \in (0, 1/4)$, we consider

$$k_r(z) = \frac{\mu(D(z,r))}{\omega(D(z,r))} \quad \text{and} \quad \|\mu\|_* = \sup_{z\in\mathbb{D}} k_{\frac{1}{4}}(z).$$

The next theorem describes a condition sufficient to guarantee that a positive Borel measure μ is a p-reverse Carleson measure for A^p_ω.

Theorem 1.3: *Suppose $\omega \in \mathcal{E}$. Let $p > 0$, $\epsilon > 0$ and $\delta > 0$. Let μ be a positive Borel measure such that $\|\mu\|_* < \infty$. If there exists a constant $r \in (0, 1/4)$ such that the set $G = \{z \in \mathbb{D} : k_r(z) > \epsilon\|\mu\|_*\}$ satisfies (1), then μ is a p-reverse Carleson measure for A^p_ω.*

The third result of our findings gives some sufficient conditions for a positive Borel measure μ to satisfy a reverse Carleson inequality for derivatives of functions belonging to A^p_ω.

Theorem 1.4: *Let $\omega \in \mathcal{E}$ and $p > 1$. Let μ be a positive Borel measure satisfying*

(1) *there exists a constant $c > 0$ such that $\mu(D(z,t)) \le c\omega(D(z,t))$ for all $z \in \mathbb{D}$ and $t \in (0, 1/4)$;*
(2) *there exist constants $\delta > 0$ and $r \in (0, 1/4)$ such that $\mu(D(z,r)) > \delta\omega(D(z,r))$ for all $z \in \mathbb{D}$.*

Then there exists a natural number n_0 and a positive constant C such that

$$\left(\int_{\mathbb{D}} |f(z)|^p \omega(z)\,\mathrm{d}A(z)\right)^{\frac{1}{p}} \le C \sum_{j=0}^{n} \left[\int_{\mathbb{D}} \left|\frac{f^{(j)}(z)}{j!}\right|^p (1-|z|^2)^{jp}\,\mathrm{d}\mu(z)\right]^{\frac{1}{p}},$$

for all $f \in A^p_\omega$ and each natural number $n \ge n_0$.

Next, we close our study of reverse Carleson measures with the following theorem that is a generalization of [19, Theorem 1], but before that let us give a definition: Let $\{\mu_n\}$ be a sequence of measures on $\mathbb{D}$. We say that μ_n converges weakly to a measure μ, denoted by $\mu_n \rightharpoonup \mu$, if

$$\int_{\mathbb{D}} h(z)\,\mathrm{d}\mu_n(z) \to \int_{\mathbb{D}} h(z)\,\mathrm{d}\mu(z),$$

for all h in the class $C_c(\mathbb{D})$ of non-negative continuous compactly supported functions in $\mathbb{D}$.

Theorem 1.5: *Let $\omega \in \mathcal{E}$ and $0 < p < \infty$. Let $\{\mu_n\}$ be a sequence of p-Carleson measures for A^p_ω such that*

$$\Lambda = \sup_n \sup_{\xi\in\mathbb{D}} \left(\frac{1}{\tau(\xi)^2}\int_{D(\delta\tau(\xi)/2)} \omega(z)^{-1}\,\mathrm{d}\mu_n(z)\right) < \infty.$$

Then, $\{\mu_n\}$ has a weakly convergent subsequence.

Further, if $\mu_n \rightharpoonup \mu$, then

$$\lim_{n\to\infty} \int_{\mathbb{D}} h(z)\,\mathrm{d}\mu_n(z) = \int_{\mathbb{D}} h(z)\,\mathrm{d}\mu(z), \quad h \in A^p_\omega, \tag{4}$$

and μ is a p-Carleson measure for A^p_ω. Furthermore, if μ_n are p-reverse Carleson measures for A^p_ω, then μ is also a p-reverse Carleson measure for A^p_ω.

The paper is organized as follow: In Section 2, we recall some notations and preliminary results which will be used later. In Section 3, we give some key lemmas that will play an important role to prove the main results of this paper. Sections 4, 5 and 6 are devoted to the proofs of our findings.

2. Preliminaries

In this section, we collect some preliminary results that we shall need in the rest of the paper. We give some useful estimates.

Lemma 2.1: *For any $r > 0$ sufficiently small and $z \in \mathbb{D}$, there exists a constant $C = C(r) > 0$ such that*

$$C_r^{-1} \le \frac{1-|a|^2}{1-|z|^2} \le C_r$$

and

$$C_r^{-1} \le \frac{1-|a|^2}{|1-a\bar{z}|} \le C_r$$

for any $z \in \mathbb{D}$ and all $a \in D(z,r)$.

Lemma 2.1 can be found in [1, 20–22].

Lemma 2.2: *For any $r \in (0,1/2)$ and $z \in \mathbb{D}$, there exists a constant $C > 0$ such that*

$$C^{-1} r^2 (1-|z|^2)^4 \le A(D(z,r)) \le C r^2 (1-|z|^2)^4.$$

Moreover, for $\omega \in \mathcal{E}$, there exist positive constants C_1 and C_2 such that

$$C_1(1-|z|^2)^4 \omega(z) \le \omega(D(z,r)) \le C_2 (1-|z|^2)^4 \omega(z).$$

Proof: Given any $r \in (0,1/2)$ and $z \in \mathbb{D}$, the first chain of inequalities follows from Lemma 2.3 of [1]. We next prove the second chain. Since

$$\omega(D(z,r)) = \int_{D(z,r)} \omega(a)\,\mathrm{d}A(a),$$

and by (2), we have

$$C_1 \omega(z) A(D(z,r)) \le \omega(D(z,r)) \le C_2 \omega(z) A(D(z,r)).$$

The desired result follows from the first inequality of Lemma 2.2. ■

Lemma 2.3: *Suppose $\omega \in \mathcal{E}$ and $0 < r_1, r_2, r_3 < 1/2$. Then there exist constants c and C such that*

$$c \leq \frac{\omega(D(z, r_1))}{\omega(D(a, r_2)} \leq C$$

for any z and a in $\mathbb{D}$ with $\sigma(a, z) \leq r_3$.

Proof: The desired result follows from Lemmas 2.1 and 2.2. ■

The following lemma is a generalized sub-mean value theorem.

Lemma 2.4: *Let $\omega \in \mathcal{E}$, $r \in (0, 1/2)$ and $p > 0$. Then there exists constants $C_0 > 0$ and $C > 0$ such that*

$$|f(z)|^p \leq \frac{C_0}{\omega(D(z, r))} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a)$$

and

$$|f^{(n)}(z)|^p \leq \frac{C}{(1 - |z|^2)^{2np} \omega(D(z, r))} \int_{D(z,r)} |f(\xi)|^p\, \mathrm{d}A(\xi), \tag{5}$$

for any $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

Proof: For any $f \in A^p_\omega$, by Cauchy integral formula together with subharmonicity, there exists a constant $c = c(r) > 0$ such that

$$|f^{(n)}(z)|^p \leq \frac{c}{(1 - |z|^2)^{2np+4}} \int_{D(z,r)} |f(\xi)|^p\, \mathrm{d}A(\xi).$$

By (2), we obtain

$$|f^{(n)}(z)|^p \leq \frac{C}{(1 - |z|^2)^{2np+4} \omega(z)} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a).$$

By Lemma 2.2, we have

$$|f^{(n)}(z)|^p \leq \frac{C'}{(1 - |z|^2)^{2np} \omega(D(z, r))} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a),$$

which completes the proof. ■

The following lemma gives comparable property of the exponential type weight ω_β in $D_\delta(z)$.

Lemma 2.5: *Let $\delta > 0$ be small enough and $\beta > 0$. Then, there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 \omega_{\beta_1}(z) \leq \omega_\beta(\xi) \leq C_2 \omega_{\beta_2}(z), \quad \xi \in D(\delta\tau(z)),$$

where $\beta_1 = 2^t \beta$ and $\beta_2 = 2^{-t} \beta$ with $t = \frac{2\sigma}{2+\sigma}$.

Proof: Recall that

$$\omega_\beta(z) := \omega_{\gamma,\sigma}(z) = (1-|z|^2)^\gamma \exp\left(\frac{-\beta}{(1-|z|^2)^\sigma}\right), \quad \gamma \geq 0, \sigma > 0,\ \beta > 0,$$

and its associated function τ has the following expression:

$$\tau(z) = (1-|z|^2)^{1+\sigma/2}, \quad z \in \mathbb{D},$$

see [3, p 12]. By (1), we have

$$\frac{1}{2^t(1-|z|^2)^\sigma} \leq \frac{1}{(1-|\xi|^2)^\sigma} \leq \frac{2^t}{(1-|z|^2)^\sigma}, \quad \xi \in D(\delta\tau(z)). \tag{6}$$

Then,

$$\exp\left(\frac{-\beta_1}{(1-|z|^2)^\sigma}\right) \leq \exp\left(\frac{-\beta}{(1-|\xi|^2)^\sigma}\right) \leq \exp\left(\frac{-\beta_2}{(1-|z|^2)^\sigma}\right).$$

Using (6), we get the desired result. ■

The following lemma on coverings is due to Oleinik, see [8]. One can also find a similar result in [23].

Lemma 2.6: *Let X be an open subset of $\mathbb{D}$ and let τ be a positive function on $\mathbb{D}$ as defined in above. Let $\delta > 0$ be small enough. Then there exists a sequence of points $\{a_n\} \subset \mathbb{D}$ such that the following conditions are satisfied:*

(i) $a_n \notin D(\frac{\delta}{4}\tau(a_k))$, $n \neq k$.
(ii) $X \subset \bigcup_n D(\delta\tau(a_n))$
(iii) $\widetilde{D}(\delta\tau(a_n)) \subset D(3\delta\tau(a_n))$, *where* $\widetilde{D}(\delta\tau(a_n)) = \bigcup_{z\in D(\delta\tau(a_n))} D(\delta\tau(z))$ $n = 1, 2, 3, \ldots$.
(iv) $\{D(3\delta\tau(a_n))\}$ *is a covering of X of finite multiplicity N.*

The multiplicity N in the previous lemma is independent of δ, and it is easy to see that one can take, for example, $N = 256$. Any sequence satisfying the conditions in Lemma 2.6 will be called a (δ, τ)-lattice.

3. Key lemmas

In this section, we are going to give some key lemmas that will play an important role in the proofs of our main results.

Suppose $f \in H(\mathbb{D})$, $\lambda \in (0, 1)$, $r \in (0, 1)$ and $p > 0$, we define a set

$$E_{\lambda,r}(z) = E_{\lambda,r}(f, z) = \{a \in D(z, r) : |f(a)| > \lambda|f(z)|\}$$

and the operator

$$B_{p,\lambda}f(z) = \frac{1}{\omega(E_{\lambda,r}(z))} \int_{E_{\lambda,r}(z)} |f(a)|^p \omega(a)\, dA(a),$$

where $z \in \mathbb{D}$. To prove the sufficiency of Theorem 1.2, we need the following three lemmas.

Lemma 3.1: *Let $r \in (0, 1/2)$, $\lambda \in (0, 1)$ and $p > 0$. Then there exists a constant $C_1 = C_1(r, \alpha, \beta) > 1$ such that*

$$\log \frac{1}{\lambda^p} + \left(\frac{1}{C_1} - 1\right) \log |f(z)|^p \leq \frac{\omega(E_\lambda(z))}{\omega(D(z,r))} \left(\log \frac{1}{\lambda^p} + \log \frac{B_\lambda f(z)}{|f(z)|^p}\right)$$

holds for any $f \in A^p_\omega$ and $z \in \mathbb{D}$.

Proof: Since $\log |f|^p$ is subharmonic, by Lemma 2.2 and (2), we obtain

$$\begin{aligned} \log |f(z)|^p &\leq \frac{C}{(1-|z|^2)^2} \int_{D(z,r)} \log |f(a)|^p \, \mathrm{d}A(a) \\ &\leq \frac{C'}{(1-|z|^2)^2 \omega(z)} \int_{D(z,r)} \log |f(a)|^p \omega(a) \, \mathrm{d}A(a). \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned} \log |f(z)|^p &\leq \frac{C_1}{\omega(D(z,r))} \int_{D(z,r)} \log |f(a)|^p \omega(a) \, \mathrm{d}A(a) \\ &\leq \frac{C_1}{\omega(D(z,r))} \left(\int_{D(z,r)\setminus E_{\lambda,r}(z)} + \int_{E_{\lambda,r}(z)}\right) \log |f(a)|^p \omega(a) \, \mathrm{d}A(a). \end{aligned} \quad (7)$$

On the one hand, the definition of $E_{\lambda,r}$ implies

$$\frac{1}{\omega(D(z,r))} \int_{D(z,r)\setminus E_{\lambda,r}(z)} \log |f(a)|^p \omega(a) \, \mathrm{d}A(a) \leq \left(1 - \frac{\omega(E_{\lambda,r}(z))}{\omega(D(z,r))}\right) \log \lambda^p |f(z)|^p. \quad (8)$$

On the other hand, by Jensen's inequality for the concave function log, we have

$$\frac{1}{\omega(D(z,r))} \int_{E_{\lambda,r}(z)} \log |f(a)|^p \omega(a) \, \mathrm{d}A(a) \leq \frac{\omega(E_{\lambda,r}(z))}{\omega(D(z,r))} \log B_{p,\lambda} f(z).$$

Plugging this and (8) into (7), we conclude that

$$\left(\frac{1}{C_1} - 1\right) \log |f(z)|^p \leq \log \lambda^p + \frac{\omega(E_{\lambda,r}(z))}{\omega(D(z,r))} \left(\log \frac{B_{p,\lambda} f(z)}{|f(z)|^p} - \log \lambda^p\right),$$

which gives the desired result. ■

Lemma 3.2: *Let $f \in A^p_\omega$, $r \in (0, 1)$, $p > 0$ and $\epsilon > 0$, we define a set*

$$S_r = \left\{ z \in \mathbb{D} : |f(z)|^p < \frac{\epsilon^p}{\omega(D(z,r))} \int_{D(z,r)} |f(\xi)|^p \omega(\xi) \, \mathrm{d}A(\xi) \right\}.$$

Then there exists a constant $C = C(r, \alpha, \beta) > 0$ such that

$$\int_{S_r} |f(\xi)|^p \omega(\xi) \, \mathrm{d}A(\xi) \leq C\epsilon^p \int_{\mathbb{D}} |f(\xi)|^p \omega(\xi) \, \mathrm{d}A(\xi).$$

Proof: For $z \in S_r$, we have

$$|f(z)|^p \le \frac{\epsilon^p}{\omega(D(z,r))} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a),$$

Integrating over $z \in S_r$ on both sides of the inequality above and using Fubini's theorem, we obtain

$$\begin{aligned}\int_{S_r} |f(z)|^p \omega(z)\, \mathrm{d}A(z) &\le \epsilon^p \int_S \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a) \omega(z)\, \mathrm{d}A(z) \\ &\le \epsilon^p \int_D |f(a)|^p \omega(a) \int_S \frac{1}{\omega(D(z,r))} \chi_{D(a,r)}(z) \omega(z)\, \mathrm{d}A(z)\, \mathrm{d}A(a) \\ &\le C\epsilon^p \int_{\mathbb{D}} |f(a)|^p \omega(a)\, \mathrm{d}A(a),\end{aligned}$$

where the last inequality follows from Lemma 2.3 and the fact $\chi_{D(z,r)}(a) = \chi_{D(a,r)}(z)$. ■

Lemma 3.3: *Let $f \in A^p_\omega$, $r \in (0, 1/2)$, $p > 0$ and $\epsilon \in (0, 1)$, we define a set*

$$T_{\lambda,\epsilon} = \{z \in \mathbb{D} : |f(z)|^p < \epsilon^{p+2} B_{p,\lambda} f(z)\}.$$

Then there exists a constant $C_2 = C_2(p, r, \alpha, \beta) > 0$ such that

$$\int_{T_{\lambda,\epsilon}} |f(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi) \le C_2 \epsilon^p \int_{\mathbb{D}} |f(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi).$$

Proof: Let S_r be the same as in Lemma 3.2, we have

$$\int_{T_{\lambda,\epsilon}} |f(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi) = \int_{T_{\lambda,\epsilon} \cap S_r} |f(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi) + \int_{T_{\lambda,\epsilon} \setminus S_r} |f(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi).$$

The first integral has been estimated in Lemma 3.2. Next, we estimate the second integral. By Fubini's theorem, we obtain

$$\begin{aligned}&\int_{T_{\lambda,\epsilon} \setminus S_r} |f(z)|^p \omega(z)\, \mathrm{d}A(z) \\ &\le \epsilon^{p+2} \int_{T_{\lambda,\epsilon} \setminus S_r} \frac{1}{\omega(E_{\lambda,r}(z))} \left(\int_{E_{\lambda,r}(z)} |f(a)|^p\, \omega(a)\, \mathrm{d}A(a) \right) \omega(z)\, \mathrm{d}A(z) \\ &\le \epsilon^{p+2} \int_{\mathbb{D}} |f(a)|^p\, \omega(a) \left(\int_{T_{\lambda,\epsilon} \setminus S_r} \frac{\chi_{D(a,r)}(z)}{\omega(E_{\lambda,r}(z))} \omega(z)\, \mathrm{d}A(z) \right) \mathrm{d}A(a).\end{aligned}$$

We claim that there exists a positive constant $C > 0$ such that

$$\int_{T_{\lambda,\epsilon}\setminus S_r} \frac{\chi_{D(a,r)}(z)}{\omega(E_{\lambda,r}(z))}\omega(z)\,\mathrm{d}A(z) \leq C/\epsilon^2, \quad \text{for any } a \in \mathbb{D} \text{ and } \epsilon \in (0,1). \tag{9}$$

Assuming the claim, we have

$$\int_{T_{\lambda,\epsilon}\setminus S_r} |f(z)|^p \omega(z)\,\mathrm{d}A(z) \leq C_2\epsilon^p \int_{\mathbb{D}} |f(a)|^p \omega(a)\,\mathrm{d}A(a).$$

This finishes the proof. Now, it remains to prove (9). To obtain this, we need first to find an upper bound of the quotient $\frac{\omega(D(z,r))}{\omega(E_{\lambda,r}(z))}$ that is proportional to ϵ^{-2}. Indeed, by Lemmas 2.4 and 2.1, we can find a constant $c_1 = c_1(r,\beta,\alpha) > 0$ such that

$$\begin{aligned}|f(z) - f(a)| &\leq \frac{c_r|z-a|}{(1-|z|^2)^2}\left(\frac{1}{\omega(D(z,r))}\int_{D(z,r)} |f(\xi)|^p\,\omega(\xi)\,\mathrm{d}A(\xi)\right)^{\frac{1}{p}} \\ &\leq \frac{c_1|z-a|}{|1-\bar{z}a|^2}M_f,\end{aligned}$$

for all $a \in D(z,r)$, where

$$M_f := \left(\frac{1}{\omega(D(z,r))}\int_{D(z,r)} |f(\xi)|^p\omega(\xi)\,\mathrm{d}A(\xi)\right)^{\frac{1}{p}}.$$

Letting $\epsilon < 2c_1 r$ and taking $|z-a| < \frac{\epsilon}{2c_1}|1-\bar{z}a|^2$, we have

$$|f(z) - f(a)| \leq \frac{\epsilon}{2}M_f.$$

For any $z \notin S_r$, we have $|f(z)| \geq \epsilon M_f$. Hence,

$$|f(a)| \geq |f(z)| - \frac{\epsilon}{2}M_f \geq \frac{1}{2}|f(z)|.$$

Let $\lambda < 1/2$, we have

$$|f(a)| \geq \lambda|f(z)|,$$

and hence, $a \in E_{\lambda,r}(z)$. Therefore, for any $z \notin S$, $\epsilon < 2c_1 r$ and $\lambda < 1/2$, we have $D(z, \frac{\epsilon}{2c_1}) \subset E_{\lambda,r}(z)$. From this and Lemma 2.4, we can find a positive constant $c_2 = c_2(r,\alpha,\beta)$ such that

$$\frac{\omega(D(z,r))}{\omega(E_{\lambda,r}(z))} \leq \frac{\omega(D(z,r))}{\omega(D(z,\frac{\epsilon}{2c_1}))} \leq c\frac{r^2(1-|z|^2)^4}{\frac{\epsilon^2}{4c_1^2}(1-|z|^2)^4} = \frac{c_2}{\epsilon^2}.$$

Therefore, by lemma 2.3, there exists a constant $c_3 = c_3(r,\alpha,\beta) > 0$ such that

$$\int_{T_{\lambda,\epsilon}\setminus S_r} \frac{\chi_{D(a,r)}(z)}{\omega(E_{\lambda,r}(z))}\omega(z)\,\mathrm{d}A(z) \leq \frac{c_2c_3}{\epsilon^2}.$$

This completes the proof. ∎

The following lemma will be used to prove Theorem 1.3.

Lemma 3.4: *Let $p > 0$ and $0 < r < 1/4$. Let μ and ν be positive Borel measure on $\mathbb{D}$ such that $\mu(D(z,1/4)) \le C_3\omega(D(z,1/4))$ and $\nu(D(z,r)) \le C_4\,\omega(D(z,r))$ for any $z \in \mathbb{D}$. Then there exists a constant $C = C(p,\alpha,\beta) > 0$ such that*

$$\int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(z) - f(a)|^p \,\mathrm{d}\nu(a)\,\mathrm{d}\mu(z) \le r^p C C_3 C_4 \|f\|_{\omega}^p,$$

for any $f \in A^p_\omega$.

Proof: For any $z \in \mathbb{D}$ and $r \in (0,1/4)$, by Lemma 2.1, we can find a sufficiently large $c_1 > 0$ (independent of r) such that

$$\frac{1}{c_1} \le \frac{1-|a|}{|1-\bar{a}z|} \le c_1$$

whenever $z \in D(a,r)$. Similarly, for a sufficient small r, by Lemma 2.4, we can find a sufficiently large $c_2 > 0$ (independent of r) such that

$$|f(z) - f(a)|^p \le \frac{c_2|z-a|^p}{(1-|a|^2)^{2p+4}} \int_{D(a,r)} |f(\xi)|^p \,\mathrm{d}A(\xi),$$

whenever $z \in D(a,r)$. Multiplying both sides by $\chi_{D(a,r)}(z)/\omega(D(a,r))$, integrating with respect to ν in the variable z, we have

$$\begin{aligned}
&\frac{1}{\omega(D(a,r))} \int_{D(a,r)} |f(z) - f(a)|^p \,\mathrm{d}\nu(z)\\
&\le c_2 r^p \frac{1}{\omega(D(a,r))} \int_{D(a,r)} \frac{|1-\bar{a}z|^{2p}}{(1-|a|^2)^{2p+4}} \left(\int_{D(a,\frac{1}{2})} |f(\xi)|^p \,\mathrm{d}A(\xi) \right) \mathrm{d}\nu(z)\\
&\le c_1^{2p} c_2 c_3 r^p \int_{D(a,\frac{1}{2})} \frac{|f(\xi)|^p}{(1-|\xi|^2)^4} \,\mathrm{d}A(\xi) \frac{\nu(D(a,r))}{\omega(D(a,r)}\\
&\le c\,C_4 r^p \int_{D(a,\frac{1}{2})} \frac{|f(\xi)|^p}{(1-|\xi|^2)^4} \,\mathrm{d}A(\xi),
\end{aligned}$$

the penultimate inequality follows from Fubini's theorem and Lemma 2.1, and the last inequality follows from the hypothesis on ν, where the positive constant c is independent of r. Note the fact $\chi_{D(\xi,1/2)}(a) = \chi_{D(a,1/2)}(\xi)$. Integrating both sides with μ in the variable a, by Fubini's theorem, Lemma 2.2 and the hypothesis on μ, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \left(\frac{1}{\omega(D(a,r))} \int_{D(a,r)} |f(z) - f(a)|^p \,\mathrm{d}\nu(z) \right) \mathrm{d}\mu(a)\\
&\le C C_4 r^p \int_{\mathbb{D}} \left(\int_{D(a,\frac{1}{2})} \frac{|f(\xi)|^p}{(1-|\xi|^2)^4} \,\mathrm{d}A(\xi) \right) \mathrm{d}\mu(a)\\
&\le C C_4 r^p \int_{\mathbb{D}} |f(\xi)|^p \frac{\mu(D(\xi,\frac{1}{2})}{(1-|\xi|^2)^4 \omega(\xi)} \omega(\xi) \,\mathrm{d}A(\xi)
\end{aligned}$$

$$\leq CC_4 r^p \int_{\mathbb{D}} |f(\xi)|^p \frac{\mu(D(\xi, \frac{1}{2})}{\omega(D(\xi, \frac{1}{2}))} \omega(\xi)\, \mathrm{d}A(\xi)$$

$$\leq CC_3 C_4 r^p \int_{\mathbb{D}} |f(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi).$$

This completes the proof. ■

We shall use the following lemma in the proof of Theorem 1.4.

Lemma 3.5: *Let $q \geq p > 1$, $n \in \mathbb{N}$ and μ be a positive Borel measure on $\mathbb{D}$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |f^{(n)}(z)|^q \,\mathrm{d}\mu(z) \leq C \left(\int_{\mathbb{D}} |f(z)|^p \omega(z)\, \mathrm{d}A(z) \right)^{\frac{q}{p}} \tag{10}$$

for any $f \in \mathbb{D}$ if there exists a constant $C'' > 0$ such that

$$\mu(D(z,r)) \leq C''(\omega(D(z,r))^{\frac{q}{p}} (1 - |z|^2))^{nq} \tag{11}$$

for some (or equivalently any) $r \in (0, 1)$.

Proof: We take (5), raise it to the q/p—power, and integrate with respect to μ. By Lemmas 2.1 and 2.3, there exists a constant $c_1 = c_1(\alpha, \beta, r, p, n) > 0$ such that

$$int_{\mathbb{D}} |f^{(n)}(z)|^q \,\mathrm{d}\mu(z)$$

$$\leq \int_{\mathbb{D}} \left(\frac{C'}{(1-|z|^2)^{2np} \omega(D(z,r))} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a) \right)^{\frac{q}{p}} \mathrm{d}\mu(z)$$

$$\leq c_1 C'^{\frac{q}{p}} \int_{\mathbb{D}} \left(\int_{D(z,r)} \frac{1}{(1-|a|^2)^{2np} \omega(D(a,r))} |f(a)|^p \omega(a)\, \mathrm{d}A(a) \right)^{\frac{q}{p}} \mathrm{d}\mu(z),$$

where C' is the one defined in Lemma 2.4. Applying Minkowski's inequality to the right-hand side and using the fact $\chi_{D(z,r)}(a) = \chi_{D(a,r)}(z)$, we obtain

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p \,\mathrm{d}\mu$$

$$\leq c_1 C'^{\frac{q}{p}} \left[\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{\chi_{D(a,r)}(z)}{(1-|a|^2)^{2nq} \omega(D(a,r))^{\frac{q}{p}}} |f(a)|^q \omega(a)^{\frac{q}{p}} \,\mathrm{d}\mu(z) \right)^{\frac{p}{q}} \mathrm{d}A(a) \right]^{\frac{q}{p}}$$

$$= c_1 C'^{\frac{q}{p}} \left[\int_{\mathbb{D}} |f(a)|^p \omega(a) \frac{(\mu(D(a,r)))^{\frac{p}{q}}}{\omega(D(a,r))(1-|a|^2)^{2np}} \,\mathrm{d}A(a) \right]^{\frac{q}{p}}$$

$$\leq C \left(\int_{\mathbb{D}} |f(a)|^p \omega(a)\, \mathrm{d}A(a) \right)^{\frac{q}{p}},$$

the last inequality follows from (16). The proof is complete. ■

As is well known, maximal functions play a crucial role in the real-variable theory of Hardy spaces [24]. In this paper, we establish a maximal function characterization for the Bergman spaces. To this end, let X be a measurable subset of $\mathbb{D}$. We define for each sufficiently small $\delta, r > 0$ and $f \in H(X)$:

$$M_\delta(f)(z) = \sup_{\xi \in D(\delta\tau(z))} |f(\xi)|, \quad \text{for } z \in \mathbb{D}.$$

The following result is used in the proof of the Theorem 1.5, but is of independent interest.

Theorem 3.6: *Let $\delta > 0$ be small enough, $0 < p < \infty$. Let X be a measurable connected open subset of $\mathbb{D}$. Then, there exist positive constants C_1 and C_2 such that*

$$C_1 \|M_\delta(f)\|_{L^p(X,\omega)} \le \|f\|_{L^p(X,\omega)} \le C_2 \|M_\delta(f)\|_{L^p(X,\omega)},$$

for any $f \in H(X)$.

Proof: Let $\{a_n\}$ be a (δ, τ)-lattice on X as defined in Lemma 2.6. By applying Lemma A in [2], we have

$$\begin{aligned}
\|M_\delta(f)\|^p_{L^p(X,\omega)} &= \int_X \sup_{\xi \in D(\frac{\delta}{3}\tau(z))} |f(\xi)|^p \omega(z)\,\mathrm{d}A(z) \\
&\le \sum_{n=1}^{\infty} \int_{D(\delta\tau(a_n))} \sup_{\xi \in D(\frac{\delta}{3}\tau(z))} |f(\xi)|^p \omega(z)\,\mathrm{d}A(z) \\
&\le C \sum_{n=1}^{\infty} \int_{D(\delta\tau(a_n))} K_{\delta,\omega}(z)\omega(z)\,\mathrm{d}A(z),
\end{aligned}$$

where

$$K_{\delta,\omega}(z) = \sup_{\xi \in D(\frac{\delta}{3}\tau(z))} \frac{1}{\omega(\xi)\tau(\xi)^2} \int_{D(\frac{\delta}{3}\tau(\xi))} |f(s)|^p \omega(s)\,\mathrm{d}A(s).$$

By Lemma 2.6, (2), (1), and the fact that

$$\begin{aligned}
|s - a_n| &\le |s - \xi| + |\xi - z| + |z - a_n| \le \frac{\delta}{3}\tau(\xi) + \frac{\delta}{3}\tau(z) + \delta\tau(a_n) \\
&\le \frac{2\delta}{3}\tau(z) + \frac{\delta}{3}\tau(z) + \delta\tau(a_n) \le 3\delta\tau(a_n),
\end{aligned}$$

we obtain

$$\int_{D(\delta\tau(a_n))} K_{\delta,\omega}(z)\omega(z)\,\mathrm{d}A(z) \lesssim \int_{D(3\delta\tau(a_n))} |f(s)|^p \omega(s)\,\mathrm{d}A(s).$$

Then,

$$\|M_\delta(f)\|^p_{L^p(X,\omega)} \le C \sum_{n=1}^{\infty} \int_{D(3\delta\tau(a_n))} |f(s)|^p \omega(s)\,\mathrm{d}A(s) \le C \|f\|^p_{L^p(X,\omega)}.$$

In addition, by Lemma A in [2] with $\beta = 0$, the definition of the maximal function $M_\delta(f)$, (2) and (1), we get the other inequality. This completes the proof. ■

4. Proof of Theorem 1.2

Proof: We first prove the necessity. We set $f_a = F_a/\|F_a\|_\omega$ for any $a \in \mathbb{D}$, then $\|f_a\|_\omega = 1$. By Lemma 2.1, we can find $b_1 > 0$ such that

$$\frac{1}{b_1} \le \frac{1-|a|^2}{|1-\bar{z}a|} \le b_1, \tag{12}$$

whenever $z \in D(a,r)$. It follows that

$$\int_{D(a,r)} |f_a(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi) \ge \frac{1}{b_1^2 \|F_a\|_\omega^p}.$$

Then

$$\begin{aligned}\int_{G\setminus D(a,r)} |f_z(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi) &\le \left(\int_{\mathbb{D}} - \int_{D(a,r)}\right) |f_a(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi)\\ &\le 1 - \frac{1}{b_1^2 \|F_a\|_\omega^p}.\end{aligned}$$

Since G is a dominating set of A^p_ω, then there exists a constant $b_2 > 0$ such that

$$\int_{\mathbb{D}} |f_a(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi) \le b_2 \int_G |f_a(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi).$$

By (12), we have

$$\int_{G\cap D(a,r)} |f_a(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi) \le \frac{b_1^2}{\|F_a\|_\omega^p} \frac{\omega(G\cap D(a,r))}{\omega(D(a,r))},$$

Since $G \cap D(a,r) = G \setminus (G \setminus D(a,r))$, we have

$$\begin{aligned}\frac{\omega(G\cap D(a,r))}{\omega(D(a,r))} &\ge \frac{\|F_a\|_\omega^p}{b_1^2}\left(\int_G - \int_{G\setminus D(a,r)}\right) |f_a(\xi)|^p \omega(\xi)\, \mathrm{d}A(\xi)\\ &\ge \frac{\|F_a\|_\omega^p}{b_1^2}\left(\frac{1}{b_2} - \left(1 - \frac{1}{b_1^2\|F_a\|_\omega^p}\right)\right)\\ &= \left(\frac{1}{b_2} - 1\right)\frac{\|F_a\|_\omega^p}{b_1^2} + \frac{1}{b_1^2}\\ &\ge \left(\frac{1}{b_2} - 1\right)\frac{\int_{D(a,r)} \frac{(1-|a|^2)^2}{|1-\bar{z}a|^2}\frac{1}{\omega(D(a,r))}\omega(z)\, \mathrm{d}A(z)}{b_1^2} + \frac{1}{b_1^4}\\ &\ge \frac{1}{b_1^4 b_2}.\end{aligned}$$

The last inequality follows from (12). From this, the necessity follows by taking $\delta = b_1^{-4} b_2^{-1}$.

It remains to prove the sufficiency. We first prove $f \in A^p_\omega$ with $\|f\|_\infty \le 1$. For $\epsilon > 0$, let

$$E = \mathbb{D} \setminus T_{\lambda,\epsilon} = \{z \in \mathbb{D} : |f(z)|^p \ge \epsilon^{p+2} B_\lambda f(z)\},$$

where $T_{\lambda,\epsilon}$ is the one defined in Lemma 3.4. By using this lemma, there exists a constant $C_1 > 1$ such that

$$\begin{aligned}
\int_{\mathbb{D}} |f(\xi)|^p \, \omega(\xi) \, \mathrm{d}A(\xi) &= \left(\int_E + \int_{T_{\lambda,\epsilon}} \right) |f(\xi)|^p \, \omega(\xi) \, \mathrm{d}A(\xi) \\
&\le \int_E |f(\xi)|^p \omega(\xi) \, \mathrm{d}A(\xi) + C_1 \epsilon^p \int_{\mathbb{D}} |f(\xi)|^p \omega(\xi) \, \mathrm{d}A(\xi).
\end{aligned}$$

Choosing ϵ small enough such that $\epsilon^p C_1 < 1/2$, we obtain

$$\int_{\mathbb{D}} |f(\xi)|^p \omega(\xi) \, \mathrm{d}A(\xi) < 2 \int_E |f(\xi)|^p \omega(\xi) \, \mathrm{d}A(\xi). \tag{13}$$

By Lemma 2.4 and the definition of $E_{\lambda,r}$, we can obtain

$$\begin{aligned}
|f(z)|^p &\le C_0 \frac{\int_{E_{\lambda,r}(z)} |f(a)|^p \omega(a) \, \mathrm{d}A(a)}{\omega(D(z,r))} \frac{\int_{D(z,r)} |f(a)|^p \omega(a) \, \mathrm{d}A(a)}{\int_{E_{\lambda,r}(z)} |f(a)|^p \omega(a) \, \mathrm{d}A(a)} \\
&= C_0 \frac{\int_{E_{\lambda,r}(z)} |f(a)|^p \omega(a) \, \mathrm{d}A(a)}{\omega(D(z,r))} \frac{\left(\int_{D(z,r) \setminus E_{\lambda,r}(z)} + \int_{E_{\lambda,r}(z)} \right) |f(a)|^p \omega(a) \, \mathrm{d}A(a)}{\int_{E_{\lambda,r}(z)} |f(a)|^p \omega(a) \, \mathrm{d}A(a)} \\
&\le C_0 \frac{\int_{E_{\lambda,r}(z)} |f(a)|^p \omega(a) \, \mathrm{d}A(a)}{\omega(D(z,r))} \left(1 + \frac{\int_{D(z,r) \setminus E_{\lambda,r}(z)} \omega(a) \, \mathrm{d}A(a)}{\int_{E_{\lambda,r}(z)} \omega(a) \, \mathrm{d}A(a)} \right) \\
&= \frac{C_0}{\omega(E_{\lambda,r}(z))} \int_{E_{\lambda,r}(z)} |f(a)|^p \omega(a) \, \mathrm{d}A(a) \\
&= C_0 B_\lambda f(z),
\end{aligned}$$

Take

$$\lambda^p < \min \left\{ \frac{1}{C_0}, \epsilon^{\frac{2p+4}{\delta}} \right\}.$$

Since $\|f\|_\infty \le 1$ and $C_1 > 1$, we have $(C_1^{-1} - 1) \log |f(z)|^p \ge 0$ for any $z \in \mathbb{D}$. Combining this with (13) and Lemma 3.1, we obtain

$$\begin{aligned}
\frac{\omega(E_{\lambda,r}(z))}{\omega(D(z,r))} &\ge \frac{\log \frac{1}{\lambda^p} + (\frac{1}{C_1} - 1) \log |f(z)|^p}{\log \frac{1}{\lambda^p} + \log \frac{B_\lambda f(z)}{|f(z)|^p}} \\
&\ge \frac{\frac{2}{\delta} \log \frac{1}{\epsilon^{p+2}}}{\frac{2}{\delta} \log \frac{1}{\epsilon^{p+2}} + \log \frac{1}{\epsilon^{p+2}}} \\
&\ge 1 - \frac{\delta}{2},
\end{aligned}$$

for any $z \in E$. Thus,

$$\omega(E_{\lambda,r}(z)) > \omega(D(z,r) \setminus E_{\lambda,r}(z)) + \omega(E_{\lambda,r}(z)) - \frac{\delta}{2}\omega(D(z,r)).$$

Then

$$\omega(D(z,r) \setminus E_{\lambda,r}(z)) \leq \frac{\delta}{2}\omega(D(z,r)).$$

By (3), we obtain

$$\begin{aligned}
\omega(G \cap E_{\lambda,r}(z)) &= \omega(G \cap [D(z,r) \setminus (D(z,r) \setminus E_{\lambda,r}(z))]) \\
&= \omega(G \cap D(z,r)) - \omega(G \cap (D(z,r) \setminus E_{\lambda,r}(z))) \\
&> \delta\omega(D(z,r)) - \frac{\delta}{2}\omega(D(z,r)) \\
&= \frac{\delta}{2}\omega(D(z,r)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{\omega(D(z,r))}\int_G \chi_{D(z,r)}(a)|f(a)|^p\omega(a)\,\mathrm{d}A(a) &\geq \frac{1}{\omega(D(z,r))}\int_G \chi_{E_{\lambda,r}(z)}(a)|f(a)|^p\omega(a)\,\mathrm{d}A(a) \\
&\geq \lambda^p|f(z)|^p\frac{\omega(G \cap E_{\lambda,r}(z))}{\omega(D(z,r))} \\
&\geq \frac{\delta\lambda^p}{2}|f(z)|^p. \qquad (14)
\end{aligned}$$

Integrating both sides over E, by Funini's theorem and (14), we obtain

$$\begin{aligned}
&\int_E \frac{1}{\omega(D(z,r))}\int_G \chi_{D(z,r)}(a)|f(a)|^p\omega(a)\,\mathrm{d}A(a)\omega(z)\,\mathrm{d}A(z) \\
&= \int_G |f(a)|^p\omega(a)\left(\int_E \frac{1}{\omega(D(z,r))}\chi_{D(a,r)}(z)\omega(z)\,\mathrm{d}A(z)\right)\mathrm{d}A(a) \\
&> \frac{\delta\lambda^p}{2}\int_E |f(z)|^p\omega(z)\,\mathrm{d}A(z) \\
&> \frac{\delta\lambda^p}{4}\int_{\mathbb{D}} |f(z)|^p\omega(z)\,\mathrm{d}A(z). \qquad (15)
\end{aligned}$$

By Lemma 2.3, there exists a constant $C = C(\alpha,\beta,r) > 0$ such that

$$\int_E \frac{1}{\omega(D(z,r))}\chi_{D(a,r)}(z)\omega(z)\,\mathrm{d}A(z) \leq C\frac{\omega(E \cap D(a,r))}{\omega(D(a,r))} \leq C.$$

By (15), we obtain

$$\int_G |f(\xi)|^p\omega(\xi)\,\mathrm{d}A(\xi) \geq \frac{C\delta}{4}\lambda^p\int_{\mathbb{D}} |f(\xi)|^p\omega(\xi)\,\mathrm{d}A(\xi).$$

This finishes the proof of the case $f \in A^p_\omega$ with $\|f\|_\infty \leq 1$.

For more general $f \in A^p_\omega \cap H^\infty$, we need only to take $g = f/\|f\|_\infty$. For $f \in A^p_\omega$, it follows from Proposition 1.2 in [3] that polynomials are dense in A^p_ω whenever ω is radial weight. Therefore, we can approximate f in the A^p_ω-norm by a polynomial sequence $\{p_n\}$. Since

$$\int_{\mathbb{D}} |p_n(z)|^p \omega(z)\, \mathrm{d}A(z) \le C \int_G |p_n(z)|^p \omega(z)\, \mathrm{d}A(z)$$

and taking $n \to \infty$, we obtain the desired result. This completes the proof. ■

As an application of Theorem 1.2, we characterize invertible Toeplitz operators T_h on A^2_ω, where h is a bounded measurable function on $\mathbb{D}$. Recall that the Toeplitz operator $T_h : A^2_\omega \to A^2_\omega$ is defined by $T_h(f) = P(hf)$, where P is the Bergman projection from L^2_ω onto A^2_ω. Toeplitz operators are studied intensively during the past decades. Interested readers can refer [3, 23, 25–27] and the references therein.

Corollary 4.1: *Let $p > 0$ and h be a bounded measurable function on $\mathbb{D}$. Let $\omega \in \mathcal{W}$ such that the polynomials are dense in A^p_ω. Then the following are equivalent.*

(1) *T_h is invertible on A^2_ω.*
(2) *There exists $t > 0$ such that the set $G_t = \{z \in \mathbb{D} : |h(z)| > t\}$ satisfies (3).*
(3) *There exists a constant $\eta > 0$ such that*

$$\int_{\mathbb{D}} |h(z)f(z)|^p \omega(z)\, \mathrm{d}A(z) \ge \eta \int_{\mathbb{D}} |f(z)|^p \omega(z)\, \mathrm{d}A(z), \tag{16}$$

for any $f \in A^p_\omega$.

Proof: The proof of the statement (1) is equivalent to (2) is similar to the one obtained in [15, Corollary 3], we omit the details.

Now we assume (3) and prove (2). By (16), there exists a constant $\eta > 0$ such that

$$\begin{aligned}\eta \int_{\mathbb{D}} |f(z)|^p \omega(z)\, \mathrm{d}A(z) &\le \int_{\mathbb{D}} |h(z)f(z)|^p \omega(z)\, \mathrm{d}A(z)\\ &\le \int_{G_t} |h(z)f(z)|^p \omega(z)\, \mathrm{d}A(z) + t^p \int_{|h|\le t} |f(z)|^p \omega(z)\, \mathrm{d}A(z),\end{aligned}$$

for any $f \in A^p_\omega$. Since h is a bounded measurable function on $\mathbb{D}$, then there exists a constant $M > 0$ such that $|h(z)| \le M$, for every $z \in \mathbb{D}$. Taking $t^p < \eta$, we have

$$\begin{aligned}(\eta - t^p) \int_{\mathbb{D}} |f(z)|^p \omega(z)\, \mathrm{d}A(z) &\le \int_{G_t} |h(z)f(z)|^p \omega(z)\, \mathrm{d}A(z)\\ &\le M \int_{G_t} |f(z)|^p \omega(z)\, \mathrm{d}A(z).\end{aligned}$$

That is, G_t is a dominating set. The set G_t satisfies (3), by Theorem 1.2, which gives the desired result.

Conversely, by our hypothesis and Theorem 1.2, there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^p \omega(z)\, \mathrm{d}A(z) \le C \int_{G_t} |f(z)|^p \omega(z)\, \mathrm{d}A(z),$$

for any $f \in A^p_\omega$. It follows that

$$\int_{\mathbb{D}} |h(z)f(z)|^p \omega(z)\, \mathrm{d}A(z) \ge \int_{G_t} |h(z)f(z)|^p \omega(z)\, \mathrm{d}A(z) \ge \frac{t^p}{C} \int_{\mathbb{D}} |f(z)|^p \omega(z)\, \mathrm{d}A(z).$$

By taking $\eta = t^p/C$, the statement (2) is proven. This completes the proof. ■

5. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3: Applying Lemma 3.4 in the case $\mathrm{d}\nu = \omega\, \mathrm{d}A$, then there exists a constant $C = C(p, \alpha, \beta, \mu) > 0$ such that

$$\int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(z) - f(a)|^p \omega(a)\, \mathrm{d}A(a)\, \mathrm{d}\mu(z) \le Cr^p \|f\|^p_\omega, \tag{17}$$

where $0 < r < 1/4$ and $f \in A^p_\omega$. We first prove the case $1 < p < \infty$. Rasing the $1/p$-power to the inequality above and using Mincowski's inequality to the left-hand side, we obtain

$$\begin{aligned}&\left(\int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a)\, \mathrm{d}\mu(z)\right)^{\frac{1}{p}} \\ &\quad - \left(\int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(z)|^p \omega(a)\, \mathrm{d}A(a)\, \mathrm{d}\mu(z)\right)^{\frac{1}{p}} \le C^{\frac{1}{p}} r \|f\|_\omega.\end{aligned} \tag{18}$$

On the one hand, since $0 < r < 1/4$, by Fubini's theorem, Lemma 2.3 and the definition of G, we can find a constant $c_1 > 0$ independent of r such that

$$\begin{aligned}&\int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(a)|^p \omega(a)\, \mathrm{d}A(a)\, \mathrm{d}\mu(z) \\ &= \int_{\mathbb{D}} |f(a)|^p \omega(a) \left(\int_{\mathbb{D}} \frac{\chi_{D(a,r)}(z)}{\omega(D(z,r))}\, \mathrm{d}\mu(z)\right) \mathrm{d}A(a) \\ &\ge c_1 \int_{\mathbb{D}} |f(a)|^p \omega(a) k_r(a)\, \mathrm{d}A(a) \\ &\ge c_1 \epsilon \int_{\mathbb{D}} |f(a)|^p \omega(a) \|\mu\|_* \chi_G(a)\, \mathrm{d}A(a) \\ &= c_1 \epsilon \|\mu\|_* \int_G |f(a)|^p \omega(a)\, \mathrm{d}A(a).\end{aligned}$$

On the other hand, we have

$$\int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(z)|^p \omega(a)\,\mathrm{d}A(a)\,\mathrm{d}\mu(z) = \int_{\mathbb{D}} |f(z)|^p \frac{\int_{D(z,r)} \omega(a)\,\mathrm{d}A(a)}{\omega(D(z,r))}\,\mathrm{d}\mu(z)$$
$$= \int_{\mathbb{D}} |f(z)|^p\,\mathrm{d}\mu(z).$$

By (18), we obtain

$$\left(c_1\epsilon\|\mu\|_* \int_G |f(a)|^p \omega(a)\,\mathrm{d}A(a)\right)^{\frac{1}{p}} - \left(\int_{\mathbb{D}} |f(z)|^p\,\mathrm{d}\mu(z)\right)^{\frac{1}{p}} \leq C^{\frac{1}{p}} r \|f\|_{\omega}.$$

Since G satisfies (3), by Theorem 1.2, there exists a constant $c_2 > 0$ such that

$$\int_{\mathbb{D}} |f|^p \omega\,\mathrm{d}A \leq c_2 \int_G |f|^p \omega\,\mathrm{d}A$$

for all $f \in A^p_\omega$. Choosing r small enough such that $Cr^p \leq c_1\epsilon\|\mu\|_*/c_2$, we obtain

$$\left(\int_{\mathbb{D}} |f|^p \omega\,\mathrm{d}A\right)^{\frac{1}{p}} \leq \frac{1}{\left(\frac{c_1\epsilon\|\mu\|_*}{c_2}\right)^{\frac{1}{p}} - (Cr^p)^{\frac{1}{p}}} \left(\int_{\mathbb{D}} |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}},$$

which proves the case $1 < p < \infty$.

Now we study the case $0 < p \leq 1$. Applying the inequality $|a-b|^p \geq |a|^p - |b|^p$ to the left-hand side of (17), we obtain

$$\int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} f(a)|^p \omega(a)\,\mathrm{d}A(a)\,\mathrm{d}\mu(z)$$
$$- \int_{\mathbb{D}} \frac{1}{\omega(D(z,r))} \int_{D(z,r)} |f(z)|^p \omega(a)\,\mathrm{d}A(a)\,\mathrm{d}\mu(z) \leq Cr^p \|f\|^p_\omega.$$

By an argument similar to that in the case $1 < p < \infty$, we obtain

$$\int_{\mathbb{D}} |f|^p \omega\,\mathrm{d}A \leq \frac{1}{\frac{c_1\epsilon\|\mu\|_*}{c_2} - Cr^p} \int_{\mathbb{D}} |f|^p\,\mathrm{d}\mu.$$

This completes the proof. ∎

Proof of Theorem 1.4: Let ϵ and t be small positive numbers whose exact value will be specified later, and let

$$G = \left\{a \in \mathbb{D} : \frac{\mu(D(a,t))}{\omega(D(a,t))} > \epsilon\right\}.$$

We first use Theorem 1.2 to prove that condition (2) implies that (3) holds for $D(z,2r)$ and some choice of $\epsilon > 0$ and $\delta > 0$, where $r \in (0,1/4)$ is from condition (2). Indeed, on the one hand, if

$$\omega(G \cap D(z,2r)) \leq \delta\omega(D(z,2r))$$

for any ϵ and δ, then we consider the set

$$K = D \setminus G = \left\{ z \in \mathbb{D} : \frac{\mu(D(a,t))}{\omega(D(a,t))} \leq \epsilon \right\}.$$

We would have

$$\begin{aligned}
\omega(K \cap D(z,2r)) &= \omega(D(z,2r) \setminus (G \setminus D(z,2r))) \\
&= \omega(D(z,2r)) - \omega(G \setminus D(z,2r)) \\
&\geq \omega(D(z,2r)) - \delta\omega(D(z,2r)) \\
&= (1-\delta)\omega(D(z,2r)),
\end{aligned}$$

that is,

$$\delta\omega(D(z,2r)) \geq \omega(D(z,2r)) - \omega(K \cap D(z,2r)) = \omega(D(z,2r) \setminus K). \tag{19}$$

Let $t \in (0,r]$. It is easy to see that

$$\omega(D(z,2r) \cap D(w,t)) \geq \chi_{D(z,r)}(w)\omega(D(w,t)) \tag{20}$$

for all $w \in \mathbb{D}$. For any ϵ and δ, it follows form (19), (20) and Lemma 2.3 that there exists a constant $c_2 = c_2(\alpha, \beta, t, r) > 0$ such that

$$\begin{aligned}
\epsilon + \delta \sup_{a \in \mathbb{D}} \frac{\mu(D(a,t))}{\omega(D(a,t))} &\geq \epsilon \frac{\omega(K \cap D(z,2r))}{\omega(D(z,2r))} + \delta \frac{\omega(D(z,2r))}{\omega(D(z,2r))} \sup_{a \in \mathbb{D}} \frac{\mu(D(a,t))}{\omega(D(a,t))} \\
&\geq \frac{1}{\omega(D(z,2r))} \int_{K \cap D(z,2r)} \frac{\mu(D(a,t))}{\omega(D(a,t))} \omega(a)\, \mathrm{d}A(a) \\
&\quad + \frac{1}{\omega(D(z,2r))} \int_{D(z,2r)-K} \frac{\mu(D(a,t))}{\omega(D(a,t))} \omega(a)\, \mathrm{d}A(a) \\
&= \frac{1}{\omega(D(z,2r))} \int_{D(z,2r)} \frac{\mu(D(a,t))}{\omega(D(a,t))} \omega(a)\, \mathrm{d}A(a) \\
&= \frac{1}{\omega(D(z,2r))} \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{\chi_{D(z,2r)}(a)\chi_{D(w,t)}(a)}{\omega(D(a,t))} \omega(a)\, \mathrm{d}A(a) \right) \mathrm{d}\mu(w) \\
&\geq \frac{c}{\omega(D(z,2r))} \int_{\mathbb{D}} \frac{\omega(D(2z,r) \cap D(w,t))}{\omega(D(w,t))}\, \mathrm{d}\mu(w) \\
&\geq \frac{c}{\omega(D(z,2r))} \int_{\mathbb{D}} \frac{\chi_{D(z,r)}(w)\omega(D(w,t))}{\omega(D(w,t))}\, \mathrm{d}\mu(w) \\
&\geq c \frac{\mu(D(z,r))}{\omega(D(z,2r))} > c_2 \frac{\mu(D(z,r))}{\omega(D(z,r))} \\
&> c_2 s > 0.
\end{aligned}$$

It follows from condition (1) that there exists a constant $M > 0$ such that

$$\sup_{a\in\mathbb{D}} \frac{\mu(D(a,t))}{\omega(D(a,t))} < M.$$

For fixed $s > 0$, we can choose some ϵ and δ such that

$$\epsilon + \delta M \le c_2 s.$$

This would be in contradiction with our earlier assumption. Thus, we have shown that (1) holds for $D(z, 2r)$ and some choice of $\epsilon > 0$ and $\delta > 0$.

Write $\mathbb{D} = G \cup K$. It follows from the assertion (2) of Theorem 1.2 that there exists a constant $c_3 > 0$ such that

$$\begin{aligned}\int_{\mathbb{D}} |f(a)|^p \frac{\mu(D(a,t))}{\omega(D(a,t))}\omega(a)\,\mathrm{d}A(a) &\ge \int_G |f(a)|^p \frac{\mu(D(a,t))}{\omega(D(a,t))}\omega(a)\,\mathrm{d}A(a)\\ &\ge \epsilon \int_G |f(a)|^p \omega(a)\,\mathrm{d}A(a)\\ &\ge c_3\epsilon \int_{\mathbb{D}} |f(a)|^p \omega(a)\,\mathrm{d}A(a).\end{aligned}$$

On the other hand, it follows from Lemma 2.3 and Fubini's theorem that there exists a constant $c_4 = c_4(\alpha, \beta, t) > 0$ such that

$$\begin{aligned}\int_{\mathbb{D}} |f(a)|^p \frac{\mu(D(a,t))}{\omega(D(a,t))}\omega(a)\,\mathrm{d}A(a) &= \int_{\mathbb{D}} \frac{|f(a)|^p}{\omega(D(a,t))} \int_{\mathbb{D}} \chi_{D(a,t)}(z)\,\mathrm{d}\mu(z)\omega(a)\,\mathrm{d}A(a)\\ &= \int_{\mathbb{D}} \int_{D(z,t)} \frac{|f(a)|^p}{\omega(D(a,t))}\omega(a)\,\mathrm{d}A(a)\,\mathrm{d}\mu(z)\\ &\le c_4 \int_{\mathbb{D}} \frac{1}{\omega(D(z,t))} \int_{D(z,t)} |f(a)|^p \omega(a)\,\mathrm{d}A(a)\,\mathrm{d}\mu(z).\end{aligned}$$

Therefore, we have established the inequality

$$\left(\int_{\mathbb{D}} |f(\xi)|^p \omega(\xi)\,\mathrm{d}A(\xi)\right)^{\frac{1}{p}} \le C_t \left[\int_{\mathbb{D}} \left(\frac{\int_{D(z,t)} |f(a)|^p \omega(a)\,\mathrm{d}A(a)}{\omega(D(z,t))}\right) \mathrm{d}\mu(z)\right]^{\frac{1}{p}}, \tag{21}$$

for $t > 0$, where $C_t = C_t(\alpha, \beta, \epsilon, p, t) > 0$.

We use Taylor formula to expand $f(a)$ on the right-hand side of (21) and then use Minkowski's inequality, the result is

$$\begin{aligned}&C_t \left[\int_{\mathbb{D}} \left(\frac{\int_{D(z,t)} |f(a)|^p \omega(a)\,\mathrm{d}A(a)}{\omega(D(z,t))}\right) \mathrm{d}\mu(z)\right]^{\frac{1}{p}}\\ &\le C_t \sum_{j=0}^{n} \left[\int_{\mathbb{D}} \frac{1}{\omega(D(z,t))} \int_{D(z,t)} \left|\frac{f^{(j)}(z)(a-z)^j}{j!}\right|^p \omega(a)\,\mathrm{d}A(a)\,\mathrm{d}\mu(z)\right]^{\frac{1}{p}}\end{aligned}$$

$$+ C_t \left[\int_{\mathbb{D}} \frac{1}{\omega(D(z,t))} \int_{D(z,r)} \frac{1}{n!} \int_z^a (a-\zeta)^n f^{(n+1)}(\zeta)\, d\zeta\, |^p \omega(a)\, dA(a)\, d\mu(z) \right]^{\frac{1}{p}}. \tag{22}$$

The sum on j is easily estimated as

$$\sum_{j=0}^{n} \left(\int_{\mathbb{D}} \frac{1}{\omega(D(z,t))} \int_{D(z,t)} \left| \frac{f^{(j)}(z)(a-z)^j}{j!} \right|^p \omega(a)\, dA(a)\, d\mu(z) \right)^{\frac{1}{p}}$$

$$\leq c_5^n \sum_{j=0}^{n} \left(\int_{\mathbb{D}} \left| \frac{f^{(j)}(z)(1-|z|)^j}{j!} \right|^p d\mu(z) \right)^{\frac{1}{p}}, \tag{23}$$

the inequality above follows by the fact that $|a-z| \leq c_5(1-|z|)$ whenever $a \in D(z,t)$, where $c_5 = c_5(t) > 1$. To estimate the second part of (21), we note that $|a-\zeta| \leq c_5(1-|z|)$ and $|a-z| \leq c_5(1-|z|)$. From Lemma 3.5 and the arguments on page 102 of [10], combining (21), (22) and (23), we obtain

$$\left(\int_{\mathbb{D}} |f(\xi)|^p \omega(\xi)\, dA(\xi) \right)^{\frac{1}{p}} \leq C_t c_5^n \left(\sum_{j=0}^{n} \left(\int_{\mathbb{D}} \left| \frac{f^{(j)}(z)(1-|z|)^j}{j!} \right|^p d\mu(z) \right) \right)^{\frac{1}{p}}$$

$$+ C_t C (C't)^n \left(\int_{\mathbb{D}} |f(\xi)|^p \omega(\xi)\, dA(\xi) \right)^{\frac{1}{p}}, \tag{24}$$

where C' and C are two positive constants independent of n and t. We first choose t such that $C't < 1$ and C_t is fixed. Then we choose positive integer n_0 such that $C_t C(C't)^{n_0} < 1$. The desired result now follows by moving the second term on the right-hand side of (24) to the left-hand side. This completes the proof of the theorem. ■

6. Proof of Theorem 1.5

The first statement can be established by following the same proof of [19, Theorem 1].

Now, we prove (4). Let $f \in A^p_\omega$ and $h \in C_c(\mathbb{D})$ satisfying $h(z) \leq 1$ for all $z \in \mathbb{D}$. On the one hand, by Fatouś lemma and since $\mu_n \rightharpoonup \mu$, we have

$$\liminf_{n\to\infty} \int_{\mathbb{D}} |f(z)|^p\, d\mu_n(z) \geq \lim_{n\to\infty} \int_{\mathbb{D}} h(z)|f(z)|^p\, d\mu_n(z) = \int_{\mathbb{D}} h(z)|f(z)|^p\, d\mu(z).$$

Since we may let such h increase to 1 on the whole unit disk, we have

$$\liminf_{n\to\infty} \int_{\mathbb{D}} |f(z)|^p\, d\mu_n(z) \geq \int_{\mathbb{D}} |f(z)|^p\, d\mu(z). \tag{25}$$

On the other hand, let $\varepsilon > 0$ and take r close enough to 1 such that

$$\int_{\mathbb{D}\setminus D(0,r)} |f(z)|^p \omega(z)\, dA(z) \leq \int_{\mathbb{D}\setminus D(0,r_1)} |f(z)|^p \omega(z)\, dA(z) < \varepsilon, \tag{26}$$

where $r_1 = r - 2(1-r)$. Let $h \in C_c(\mathbb{D})$ such that $h(z) \leq 1$ for all $z \in \mathbb{D}$ and $h = 1$ on $\overline{D(0,r)}$. Then,

$$\int_{\mathbb{D}} |f(z)|^p \,\mathrm{d}\mu_n(z) \leq \int_{\mathbb{D}} h(z)\, |f(z)|^p \,\mathrm{d}\mu_n(z) + \int_{\mathbb{D}\setminus\overline{D(0,r)}} |f(z)|^p \,\mathrm{d}\mu_n(z). \tag{27}$$

We first handle the second integral on right-hand side and consider the open sets $E = \mathbb{D} \setminus \overline{D(0,r)}$ and $\widetilde{E} = \mathbb{D} \setminus \overline{D(0,r_1)}$. It is clear that $E \subset \widetilde{E}$. Take $X = \widetilde{E}$ in Lemma 2.6. From (A) of the definition of τ, we have

$$\tau(z) \leq c_1(1-|z|) \quad \text{for } z \in \mathbb{D} \text{ and } \delta c_1 < \tfrac{1}{4}.$$

Then, for sufficiently small $0 < \delta < 1$, $z \in E$ and $\xi \in D(\delta\tau(z))$, we have

$$|\xi| \geq |z| - |\delta\tau(z)| \geq r - \delta c_1(1-r) \geq r - 2(1-r) = r_1,$$

which implies that $\xi \in \widetilde{E}$. Therefore, by Lemma A in [2], Fubini's theorem and (1), we have

$$\begin{aligned}\int_E |f(z)|^p \,\mathrm{d}\mu_n(z) &\leq C \int_E \frac{1}{\tau(z)^2\omega(z)} \int_{D(\frac{\delta}{4}\tau(z))} |f(\xi)|^p \omega(\xi)\,\mathrm{d}A(\xi)\,\mathrm{d}\mu_n(z)\\ &\leq C \int_{\widetilde{E}} M_\delta(f)(\xi)\omega(\xi)\left(\frac{1}{\tau(\xi)^2}\int_{D(\delta\tau(\xi)/2)} \frac{\mathrm{d}\mu_n(z)}{\omega(z)}\right)\mathrm{d}A(\xi).\end{aligned}$$

Since μ_n is p-Carleson measure for A^p_ω,

$$K^\delta_\omega(\mu_n) = \sup_{\xi\in\mathbb{D}}\left(\frac{1}{\tau(\xi)^2}\int_{D(\delta\tau(\xi)/2)} \omega(z)^{-1}\,\mathrm{d}\mu_n(z)\right) < \infty,$$

see Theorem 1 in [4]. Thus, by Theorem 3.6 and (26), we obtain

$$\begin{aligned}\int_E |f(z)|^p \,\mathrm{d}\mu_n(z) &\leq CK^\delta_\omega(\mu_n)\int_{\widetilde{E}} M_\delta(f)(\xi)\omega(\xi)\,\mathrm{d}A(\xi)\\ &\leq CK^\delta_{\omega_{\alpha,\beta_1}}(\mu_n)\int_{\widetilde{E}} |f(\xi)|\omega(\xi)\,\mathrm{d}A(\xi)\\ &\leq C\Lambda\varepsilon.\end{aligned}$$

Therefore, by taking the limit superior of (27), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{D}} |f(z)|^p \,\mathrm{d}\mu_n(z) &\leq \int_{\mathbb{D}} h(z)|f(z)|^p\,\mathrm{d}\mu(z) + C\Lambda\varepsilon\\ &\leq \int_{\mathbb{D}} |f(z)|^p\,\mathrm{d}\mu(z) + C\Lambda\varepsilon\\ &\leq \int_{\mathbb{D}} |f(z)|^p\,\mathrm{d}\mu(z),\end{aligned}$$

since ε is arbitrary. By combining this with (25), we deduce (4).

Moreover, since μ_n are p-Carleson measures for A^p_ω, we have

$$\int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}\mu_n(z) \leq \Lambda_1 \|f\|^p_{A^p_\omega}, \quad f \in A^p_\omega \text{ and } n \in \mathbb{N},$$

where $\Lambda_1 = \sup_n \|I_{\mu_n}\|^p$. By identity (4), we may pass to the limit to obtain

$$\int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}\mu(z) \leq \Lambda_1 \|f\|^p_{A^p_\omega}, \quad f \in A^p_\omega.$$

Thus, μ is a p-Carleson measure for A^p_ω. In the case of reverse Carleson measures, the lower inequality follows in a manner similar to above. Details of this step are omitted.

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