Data Science in Finance and Economics

http://www.aimspress.com/journal/dsfe

DSFE, 1(4): 313–326.
DOI: 10.3934/DSFE.2021017
Received: 10 November 2021
Accepted: 16 December 2021
Published: 23 December 2021

Research article

A modification term for Black-Scholes model based on discrepancy calibrated with real market data

Xiaozheng Lin[1], **Meiqing Wang**[1,*] **and Choi-Hong Lai**[2]

[1] School of Mathematics and Statistics, Fuzhou University, No.2. Wulongjiang North Avenue, Fuzhou, China
[2] School of Computing and Mathematical Sciences, University of Greenwich, SE10 9LS,London, UK

* **Correspondence:** Email: mqwang@fzu.edu.cn.

Abstract: The Black-Scholes option pricing model (B-S model) generally requires the assumption that the volatility of the underlying asset be a piecewise constant. However, empirical analysis shows that there are discrepancies between the option prices obtained from the B-S model and the market prices. Most current modifications to the B-S model rely on modelling the implied volatility or interest rate. In contrast to the existing modifications to the Black-Scholes model, this paper proposes the concept of including a modification term to the B-S model itself. Using the actual discrepancies of the results of the Black-Scholes model and the market prices, the modification term related to the implied volatility is derived. Experimental results show that the modified model produces a better option pricing results when compare to market data.

Keywords: option pricing; black-scholes model; modification term; disturbance term; higher-order algorithm; second-order derivatives

JEL Codes: C63, C02

1. Introduction

Financial derivatives refer to contracts whose values depend on the value of the underlying asset, and are an important part of financial innovation tools (Wu et al., 2016). These contracts can be standardized or non-standardized. With the vigorous development of the financial derivatives trading market, finance and related economic fields are also facing more and more complicated theoretical and practical questions (Smith Jr, 1976). The main question remains to be answered is how to determine the prices of derivatives. Among many financial derivatives, options seems to be the most extensively studied topic by scholars (Chesney and Scott, 1989).

As the core of options trading, reasonable and scientific option pricing can well simulate the optimal pricing of corporate debt and help the market avoid arbitrage. On the other hand, the suitable option pricing model can perform risk assessment on secured loans and pension insurance (Wu et al., 2016).

In 1973, American mathematicians Fischer Black and Myron Scholes derived the famous Black-Scholes option pricing model under a series of strict assumptions (Black and Scholes, 1973). It is currently the most widely used option pricing model in theoretical and actual market transactions, and can effectively calculate option prices. However, it is practically impossible to execute the strict assumption of B-S model in real markets which are usually not perfectly liquid (Lai, 2019) and doesn't conform to the actual situation of option transactions (Wiese et al., 2019). Many scholars have found that there are discrepancies or errors between the option prices obtained from the B-S model and the market prices. The modification and expansion of the Black-Scholes model have important practical significance for the construction of a practical and scientific sound option pricing method, and at the same time can provide an important theoretical basis for the management, avoidance and control of financial risks (Wu et al., 2016).

The current forms of modifications are mainly based on revise and expand the B-S model from the basic model and price parameters. From the perspective of the basic model, Merton.RC and Bollerslev (Bollerslev, 1986) respectively proposed the jump diffusion model and the GARCH model. Scholars such as Menn & Rachev have proposed a double exponential jump diffusion (Menn and Rachev, 2005) and other pricing models (Kou, 2002) that conform to changes in the underlying asset based on further improvements to the jump diffusion model and the GARCH model. As for modifications based on price parameters, most of them are related to the risk-free interest rate and volatility. Excellent methods such as stochastic volatility model, partial volatility model and stochastic interest rate model (Dupire, 1994; Grabbe, 1983; Derman and Kani, 1998; Derman et al., 1996) are still playing an important role in the current field of option pricing.

In contrast to the existing modifications to the Black-Scholes model, the objective of this paper is to calibrate the Black-Scholes model by proposing the concept of a modification term and using a second derivative structure of the option price with respect to the underlying asset to approximate it. In this process, the relationship between the modification term and the implied volatility can be simultaneously characterized. The experimental results show that modification term can not only flexibly fit and predict the actual discrepancies of the B-S equation, but also have a better simulation effect to the actual option prices.

The paper is organised as follow. Section 2 provides a brief overview of the Black-Scholes option pricing model and implied volatility, proposes the concept of a modification term for B-S equation. Section 3 discusses a modified method based on the construction of the modification term. Section 4 gives the corresponding numerical experiments and analysis of the propsed method. Finally, conclusions are drawn.

2. Black-Scholes option pricing model

2.1. Background

In the 1970s, Fischer Black and Myron Scholes proposed the first complete option pricing model (Black and Scholes, 1973). Under a series of strict assumptions, the relationship between the option price, underlying asset price and time to maturity can be described by the following famous Black-Scholes partial differential equation (Black and Scholes, 1973):

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \tag{1}$$

subject to suitable terminal and boundary conditions, provides deterministic price of many options and derivatives. Here S denotes the underlying asset price, V is the option price, r is the risk-free interest, t means time and σ is the volatility of the underlying asset. The Black-Scholes model assumes that the volatility of the underlying asset σ is constant.

Note that in the Black-Scholes option pricing model, the volatilities of the underlying asset cannot be directly obtained from the market, whereas all the other parameters and variables can be given by actual market conditions or option contracts (Wu et al., 2016). According to the Itô integral, the analytical solution of European call options can be derived from Equation (1)

$$\begin{gathered}\mathrm{V}=S\times N(d_1)-e^{-r(T-t)}K\times N(d_2),\\ d_1=\frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},\\ d_2=d_1-\sigma\sqrt{T-t},\end{gathered} \tag{2}$$

among them, T represents the maturity date of the option contract, K is the strike price, and $N(\bullet)$ is the probability distribution function of the standard normal distribution.

2.2. The discrepancies of the Black-Scholes equation for the market

It should be noted from Equation (1) that the right hand side of it after substituting the actual market option prices must be zero. However, a large number of numerical experimental studies have found that the rhs is non-zero (Lai, 2019). This shows that the B-S model under strict assumptions does not reflect the actual market conditions well. The non-zero rhs value in essence is the discrepancies/errors of the Black-Scholes equation relative to the actual option prices. Let L_{BS} denotes the calculation of the B-S equation using the actual market price, and V_{mkt} means the market option price

$$\frac{\partial V_{mkt}}{\partial t}+rS\frac{\partial V_{mkt}}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_{mkt}}{\partial S^2}-rV_{mkt}\triangleq L_{BS}(V_{mkt}). \tag{3}$$

Taking the 2019 SSE 50ETF purchase of December 2950 option as an example, the errors obtained by $L_{BS}(V_{mkt})$ are depicted in Figure 1. It is obvious that the rhs values of B-S equation are non-zero

$$L_{BS}(V_{mkt})\neq 0. \tag{4}$$

2.3. Implied volatility and its surface

Many researchers agreed that a large part of the errors of the Black-Scholes model come from the assumption that the underlying asset prices are constant (Fengler et al., 2002; Heynen et al., 1994)and proposed different volatility models to calibrate the implied volatility surface. This paper considers constructing the error function from the perspective of implied volatilities, and then adding it back to the B-S equation as a modification.

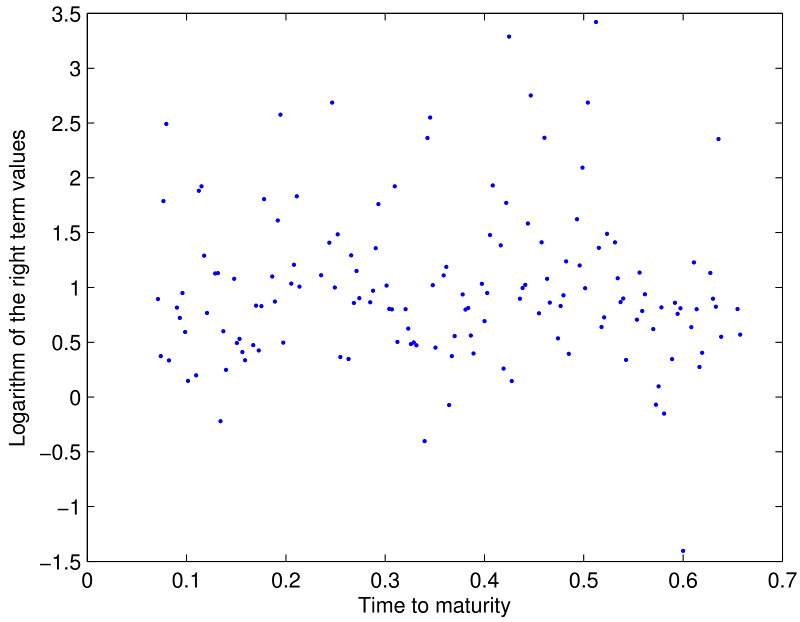


Figure 1. Logarithm of the results of $L_{BS}(V_{mkt})$(In order to show the existence of the errors more prominently, the actual errors are shown in logarithm here).

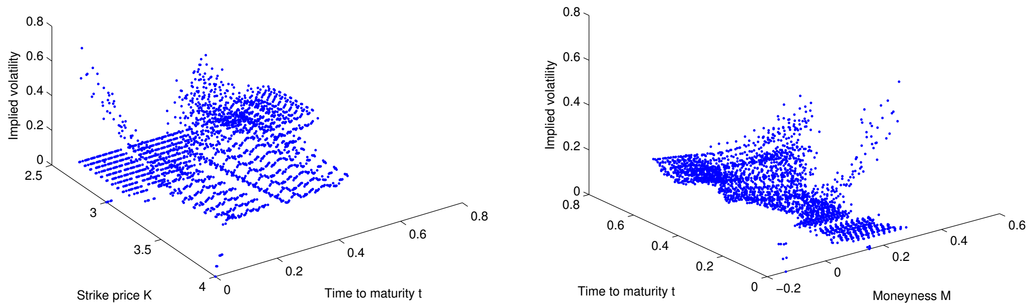
In the Black-Scholes option pricing model, the volatility σ of the underlying asset is a very important parameter (Wu et al., 2016), and its diversification has a significant impact on the outcome of the option prices.

The volatility of the underlying asset calculated reversely from a single option price is called the implied volatility ($\hat{\sigma}$). In essence the actual implied volatility of the transaction market is a non-constant parameter that makes the rhs of the B-S equation zero. The B-S equation using the actual option price V_{mkt} and implied volatility $\hat{\sigma}$ can be written as

$$\frac{\partial V_{mkt}}{\partial t} + rS\frac{\partial V_{mkt}}{\partial S} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V_{mkt}}{\partial S^2} - rV_{mkt} = 0. \tag{5}$$

A large number of empirical studies have found that the implied volatility has a smile, a smirk, or a skew delivery structure with regard to the strike price, and also has a certain term structure with regard to maturity date of an option.

In order to facilitate the visual display and exploration of the relationship between the implied volatilities, the strike prices and the remaining times (or maturity date), they will usually be represented in a three-dimensional coordinate system as a surface. For example, the relationship among the actual implied volatilities, the strike prices and the remaining times of the SSE 50ETF 2950 option (European call) can be shown in Figure 2a. As shown in Figure 2a, the surface is not a flat but a curved surface which means the implied volatilities of B-S model are not constants. And the surface is also called the implied volatility surface(Smith Jr, 1976).



(a) (b)

Figure 2. Implied volatility surface.

To avoid the implied volatilities of the option being overly dependent on the specific strike prices and the underlying asset prices trend, Cassese and Guidolin proposed the concept of "Moneyness" (Cassese and Guidolin, 2006)

$$M = \ln\left(\frac{S}{K}\right), \tag{6}$$

and used moneyness M to replace the strike prices. The relationship among the actual implied volatilities, the moneyness and the remaining times of the SSE 50ETF 2950 option (European call) can be shown in Figure 2b.

The implied volatility surface contains a lot of financial derivatives transaction market information and is an important reference information for option pricing. As shown in Figure 2b, the subsequent implied volatility surfaces in this paper all refer to the surfaces related to moneyness M and the remaining time t.

3. Modelling the discrepancy of Black-Scholes model

3.1. The effect of discrepancy and its modelling

There are inevitable errors/discrepancies of Black-Scholes equation compare to the market values as discussed in Section 2.2. Define the actual errors of Black-Scholes model as

$$L_{BS}\left(V_{mkt}\right) = e(t, S) \neq 0, \tag{7}$$

here $e(t, S)$ is a set of discrepancy data related to the underlying asset prices S where $t \in (0, T), S \in (0, +\infty)$. If the errors/discrepancies of B-S equation $e(t, S)$ can be modelled, then the B-S equation can be modified by using the knowledge of this errors/discrepancies. This paper defines the errors/discrepancies as the modification term of B-S model and try to construct a suitable model for the modification term for the original B-S model.

It should be noted that many researchers considered the errors of the Black-Scholes equation come from the assumption that the volatitlity of the underlying asset is a constant (Windcliff et al., 2004). From Equation (1), it is not difficult to find that the change of volatilities are related to the second derivative of the option prices to the underlying asset prices.

Therefore, this paper proposes to model the modification term as the second derivative of the option price to the underlying asset prices, i.e.

$$L_{BS}(V_{mkt}) = e(t, S) = \epsilon S^2 \frac{\partial^2 V}{\partial S^2}. \tag{8}$$

Here ϵ denotes a model parameter which is used to control the effect of the second derivative. As introduced above, since the errors of the Black-Scholes equation are related to the underlying asset prices and the time, ϵ should be also related to both

$$L_{BS}(V_{mkt}) = e(t, S) = \epsilon(t, S) S^2 \frac{\partial^2 V}{\partial S^2}. \tag{9}$$

Add the modification term model $e(t, S)$ into the Black-Scholes equation, a modified B-S equation can be represented as

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = \epsilon(t, S) S^2 \frac{\partial^2 V}{\partial S^2}. \tag{10}$$

Therefore, Black-Scholes equation can be rewritten as

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + (\frac{1}{2}\sigma^2 - \epsilon(t, S))S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{11}$$

It is natural to consider that $\epsilon(t, S)$ can be regarded as a disturbance/perturbation term added to the constant volatility of the classical B-S model. If the volatility disturbance term $\epsilon(\tau, S)$ can be modelled reasonably well, the B-S model for option pricing should provide better results when compared to market data.

Based on the definition of the implied volatility and equation (5), the following Black-Scholes equation can be obtained

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{12}$$

Comparing Equations (11) and (12), the relationship between the volatility disturbance term and the implied volatility can be expressed in the following form

$$\frac{1}{2}\hat{\sigma}^2 = \frac{1}{2}\sigma^2 - \epsilon(t, S), \tag{13}$$

it is easy to see that the implied volatility of the B-S model and can be expressed as

$$\hat{\sigma} = \sqrt{\sigma^2 - 2\epsilon(t, S)}. \tag{14}$$

Although the modified B-S model (10) deduced in this paper is similar to the Frey and Patie volatility model (Frey and Patie, 2002), the derivation process of the model (10) is based on the small disturbance term function for volatility and modification term model for B-S equation to establish the actual error function.

3.2. *A model for the volatility disturbance term*

From the second-order derivative relationship as shown in Equation (10), the following relationship can be obtained

$$\epsilon(t,S) = \frac{L_{BS}(V_{mkt})}{S^2\frac{\partial^2 V}{\partial S^2}} = \frac{e(t,S)}{S^2\frac{\partial^2 V}{\partial S^2}}. \tag{15}$$

Since the volatility disturbance term is related to the underlying asset price S and the remaining term t, it does not contain any discontinuity and is intuitively sensible to model it by means of a polynomial function

$$\epsilon(t,S) = a_0 + a_1 t + a_2 S + \alpha(t,S), \tag{16}$$

$\alpha(t,S)$ is the error between the equation (16) and the actual volatility disturbance term values obtained from (15). The parameters a_0, a_1 and a_2 can be determined by the least squares method or weighted least squares method based on the actual transaction data(Wu et al., 2016). From equations (9) and (16), the proposed modification term in this paper may be written as

$$e(t,S) = \epsilon(t,S)S^2\frac{\partial^2 V}{\partial S^2} \approx (a_0 + a_1 t + a_2 S + \alpha(t,S))S^2\frac{\partial^2 V}{\partial S^2}. \tag{17}$$

Combining Equation (14), the implied volatility can be rewritten as

$$\hat{\sigma}(t,S) = \sqrt{\sigma^2 - 2\epsilon(t,S)} = \sqrt{\sigma^2 - 2(a_0 + a_1 t + a_2 S + \alpha(t,S))}. \tag{18}$$

This is equivalent to finding implied volatility by modelling the discrepancy modification term and adding it back to the Black-Scholes equation, in order to achieve the purpose of modifying the model.

4. Numerical experiments

4.1. *The errors calculated by fourth-order Padé approximation*

Finite difference method is used to obtain a numerical solution for Equation (1). In modelling and simulation problems described by Equation (1), it is often desirable to use compact higher-order algorithms to calculate numerical solutions accurately and efficiently (Liao and Zhu, 2009; Roul and Prasad Goura, 2021). The Padé approximation is an effective way to construct fourth-order algorithms using a standard 3-point finite difference stencil normally used by the second-order algorithms (Liao and Zhu, 2009).

For the ease of exposition of the finite difference method, the domain has a uniformly distributed rectangular mesh with mesh point (S_i, t_n), where $S_i = i\Delta S, i = 1, 2, \ldots, M$, such that $\Delta S = S_{max}/M$, and $t_n = n\Delta t, n = 1, 2, \ldots, N$, such that $\Delta t = T/N$. Here M and N respectively refer to the number of grid points in the space and time direction.

Let $S = e^z$, here S is the original variable representing the underlying asset price while z is the new variable. The equation (1) can be rewritten as

$$\frac{\partial V}{\partial t} + (r - \frac{\sigma^2}{2})\frac{\partial V}{\partial z} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial z^2} - rV = 0, (z,t) \in (-\infty, +\infty) \times [0,T]. \tag{19}$$

Here, introduce a new function $U = \frac{\partial V}{\partial z}$ to transform Equation (19) to the following,

$$\frac{\partial V}{\partial t} + (r - \frac{\sigma^2}{2})U + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial z^2} - rV = 0. \tag{20}$$

This results to the fourth-order compact scheme using the Padé approximation to Equation (20) at a typical nodal point (S_i, t_n) as below (Liao and Zhu, 2009),

$$
\begin{aligned}
L_{BS}(V_i^n) = &\left(\left(1+\frac{r\Delta t}{2}\right)\left(1+\frac{1}{12}\delta_z^2\right)-\frac{\sigma^2\Delta t}{4h^2}\delta_z^2\right)V_i^n-\left(\left(1-\frac{r\Delta t}{2}\right)\left(1+\frac{1}{12}\delta_z^2\right)+\frac{\sigma^2\Delta t}{4h^2}\delta_z^2\right)V_i^{n+1}\\
&-\Delta t\left(r-\frac{\sigma^2}{2}\right)\left(1+\frac{1}{12}\delta_z^2\right)(U_i^n+U_i^{n+1})=e(t,S),
\end{aligned}
\tag{21}
$$

where δ^2 is a central finite difference operator defined as $\delta^2 V_{i+1}^n = V_{i+1}^n - 2V_i^n + V_{i-1}^n$ (Liao and Zhu, 2009) and $V_i^n \approx V(S_i, t_n)$.

Taking the 2015-2020 SSE 50 ETF 2950 European options as examples, using the actual option prices to calculate $L_{BS}(V_{mkt})$, the calculated errors are shown in Figure 3.

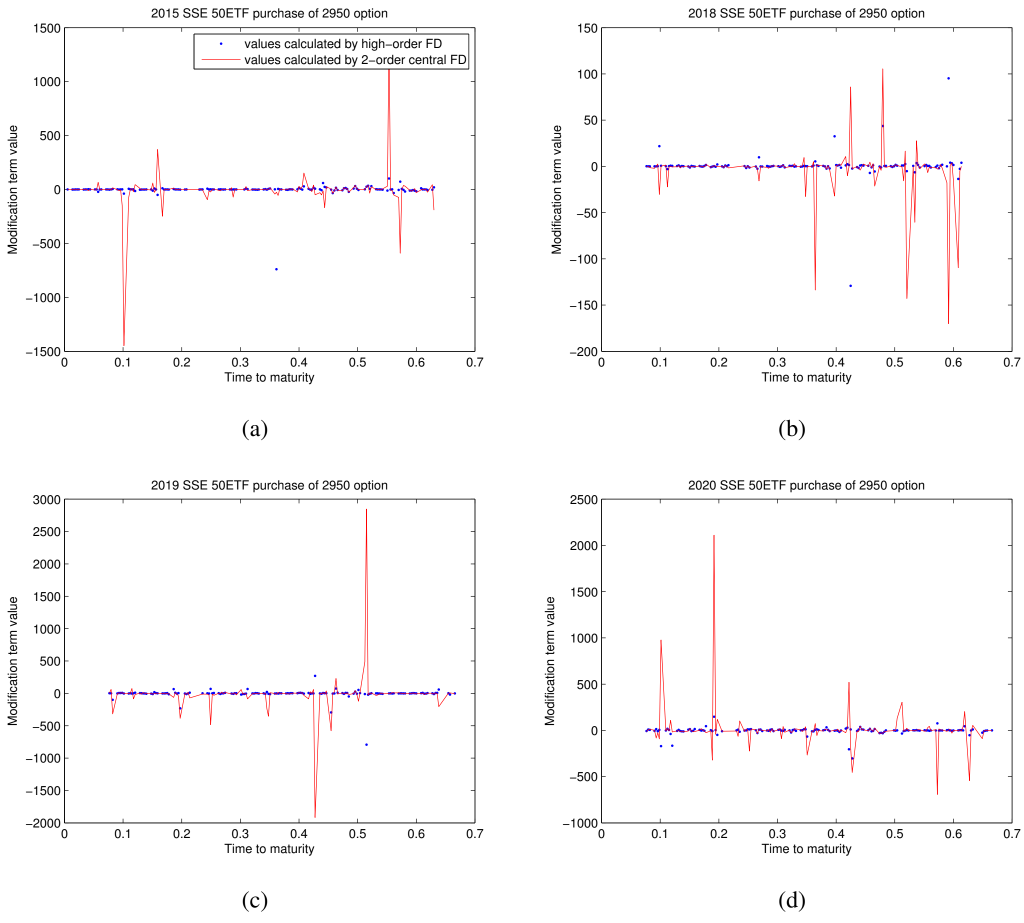


Figure 3. The modification term values calculated by 4-order Padé approximation (There are no transaction records of this option from 2016 to 2017).

It can be seen from Figure 3 that the errors obtained by the fourth-order finite difference method using Padé approximation are more stable than that of the second order finite difference method and there are no large-scale oscillation.

4.2. Accuracy of the volatility disturbance term and the modification term

In order to demonstrate the use of the polynomial function in modelling the disturbance term, numerical experiments were desgined to calculate errors between the results of Equation (16) and the actual disturbance terms. Taking the 2015-2020 SSE 50 ETF 2950 European call options as examples, Table 1 shows the errors of polynomial disturbance function. Here all errors are calculated using L_2 norm.

Table 1. The estimated errors of the volatility disturbance term.

	L_2 norm of error
SSE ETF 2950 call option of 2015	1.7473×10^{-4}
SSE ETF 2950 call option of 2018	1.8514×10^{-4}
SSE ETF 2950 call option of 2019	3.5307×10^{-4}
SSE ETF 2950 call option of 2020	5.3361×10^{-4}

Observing the errors in Table 1, the use of polynomial function to model the volatility disturbance term $e(t, S)$ seems to achieve some good accuracy.

For the same examples, the errors between the values calculated by model (17) and the actual errors $e(t, S)$ are listed below Table 2.

Table 2. The estimated errors by using the modification term.

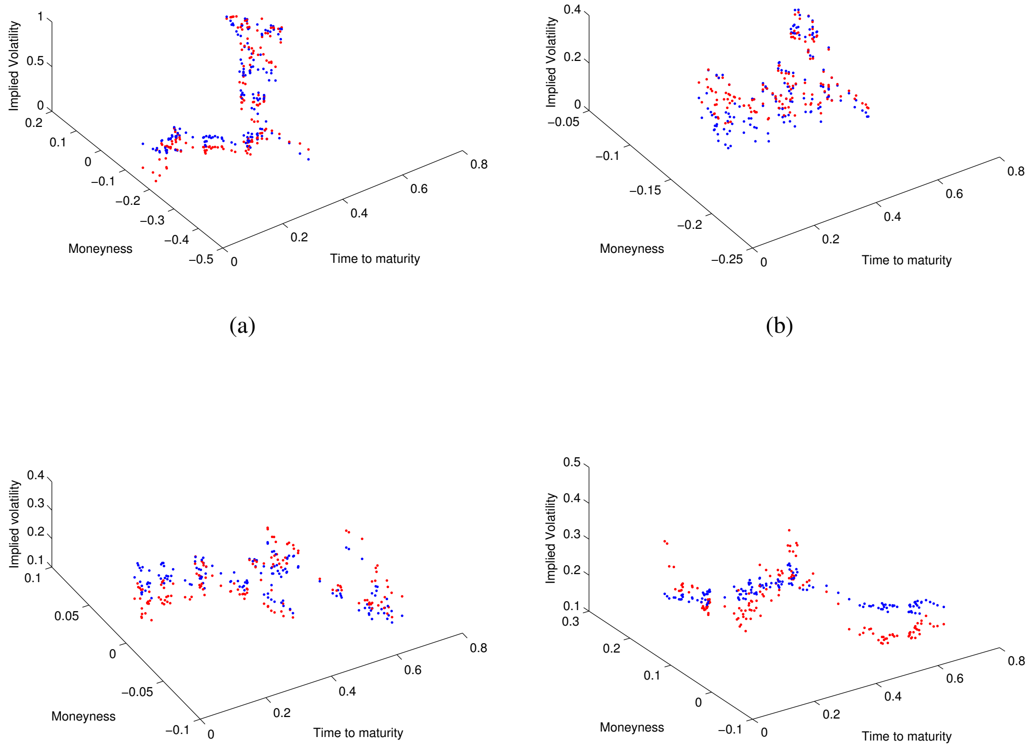
	L_2 norm of error
SSE ETF 2950 call option of 2015	0.0137
SSE ETF 2950 call option of 2018	0.0084
SSE ETF 2950 call option of 2019	0.0018
SSE ETF 2950 call option of 2020	0.0011

It can be found from Table 2 that the model (17) of the modification term can fit the actual values $e(t, S)$ well. Therefore, whatever the model of volatility disturbance term $\epsilon(t, S)$ or modification term $e(t, S)$ can do well in simulating the actual values and transaction markets.

4.3. The implied volatilities and option prices calculated by modification term

After modelling the volatility disturbance term $\epsilon(t, S)$, the implied volatilities can be obtained by the model (18). Taking the 2015-2020 SSE 50 ETF 2950 European call options as examples. In the empirical analysis, the implied volatilities calculated by the model (18) are compared with the actual implied volatilities. In more details, a–d in Figure 4 respectively show the implied volatility surfaces of the SSE 50 ETF for the four years of 2015 and 2018–2020.

It can be seen from Figure 4 a–d that the implied volatilities calculated by the model (18) have a consistent trend with the actual implied volatilities. In order to more conveniently and intuitively show the accuracy of the implied volatilities calculated by Equation (18), Table 3 summarizes the errors between the calculated values of the model (18) and the actual values. The data in Table 3 show that the average errors of implied volatilities calculated by the model (18) are basically maintained at about 10^{-2} , which means model (18) can fit the actual implied volatilities well.



(a) (b)

(c) (d)

Figure 4. The implied volatility surfaces (The red points in the figure represent the actual implied volatilities of the underlying asset, and the blue points represent the implied volatilities calculated by the ours model).

Table 3. The estimated errors of the implied volatilities.

	Euclidean distance(Errors)
SSE ETF 2950 call option of 2015	0.0754
SSE ETF 2950 call option of 2018	0.0347
SSE ETF 2950 call option of 2019	0.0415
SSE ETF 2950 call option of 2020	0.0500

Both Figure 4 and Table 3 indicate that the construction method of the implied volatility model based on modification term proposed in this paper is reasonable.

More importantly, the estimated option prices can be derived through the modification term model (17) and modified Equation (10). For the 2015–2020 SSE 50 ETF 2950 European call options, option pricing results show as Figure 5. At the same time, in order to facilitate the comparison with the traditional Black-Scholes model, the option prices calculated by the traditional B-S model and the actual prices are also respectively output in Figure 5.

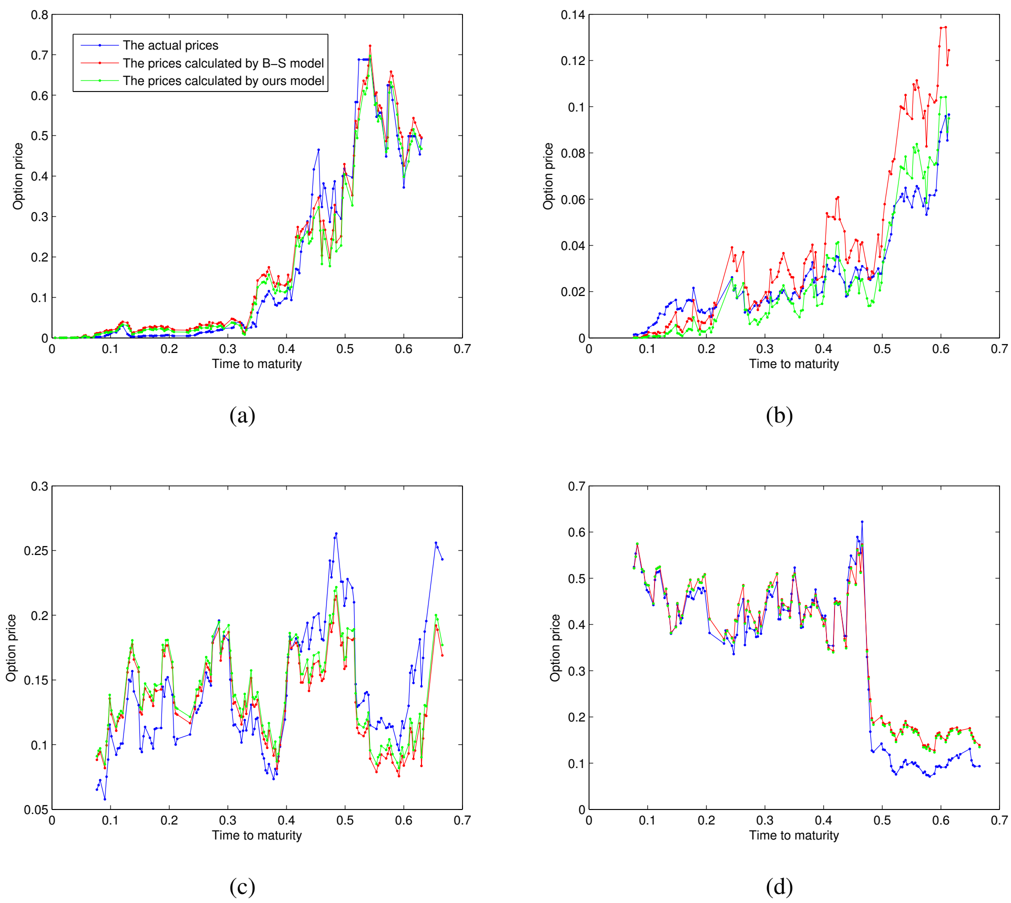


Figure 5. The option pricing results.

From the feedback of the four option pricing results in Figure 5, most of the time the modified Black-Scholes model calculation results are closer to the actual prices than the traditional model. In order to demonstrate and compare the results more intuitively, the errors of the modified model in

Equation (10) and the traditional Black-Scholes model are listed in Table 4. The analytical solution of the B-S model was used to calculate the option prices in the comparison with the modified model.

Table 4. The error of option pricing obtained by means the methods below.

	Black-Scholes model	The modified B-S model
SSE ETF 2950 call option of 2015	0.0316	0.0294
SSE ETF 2950 call option of 2018	0.0141	0.0069
SSE ETF 2950 call option of 2019	0.0257	0.0251
SSE ETF 2950 call option of 2020	0.0314	0.0299

Comparing the option pricing results of two models in Table 4, results of the modified B-S model are show better accuracy. This means that the modified method proposed in this paper is effective in simulating the actual transaction accurately.

5. Conclusion and future research

This paper examines the concept of a modification term of the B-S model based on the discrepancy observed through the use of market data. The idea is based on quantifying the actual discrepancies of the numerical results of the Black-Scholes equation and the actual market data. In the present study a model related to the second derivative of the option price is derived. The implied volatility is computed according to the modified model. Numerical experiments show that the proposed model produces encouraging corrective effect on the traditional B-S model.

Note that the modified method proposed in this paper for the Black-Scholes model is novel, and can be combined with data analytic methods to achieve the expected modification purpose. The method is an early attempt to include correction effect into B-S model. It is certainly not enough to only consider the relationship between the discrepancies/errors and the second derivative of the option price. As part of the current and future work, the authors are planning to add the influence of possible randomness or other high-order terms into the modification term leading to an effective correction for the B-S model.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (61771141, 62172098, 11771084).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

Black F, Scholes M (1973) The pricing of options and corporate liabilities. *J Polit Econ* 81: 637–654.

Bollerslev T (1986) Generalized autoregressive conditional heteroskedasticity. *J Econom* 31: 307–327.

Cassese G, Guidolin M (2006) Modelling the implied volatility surface: Does market efficiency matter?: An application to mib30 index options. *Int Rev Financ Anal* 15: 145–178.

Chesney M, Scott L (1989) Pricing european currency options: A comparison of the modified black-scholes model and a random variance model. *J Financ Quant Anal* 24: 267–284.

Dar AA, Anuradha N (2018) An application of taguchi l9 method in black scholes model for european call option. *Int J Entrep* 22: 1–13.

Derman E, Kani I (1998) Stochastic implied trees: Arbitrage pricing with stochastic term and strike structure of volatility. *Int J Theor Appl Financ* 1: 61–110.

Derman E, Kani I, Zou JZ (1996) The local volatility surface: Unlocking the information in index option prices. *Financ Anal J* 52: 25–36.

Dupire B (1994) Pricing with a smile. *Risk* 7: 18–20.

Düring B, Pitkin A (2019) High-order compact finite difference scheme for option pricing in stochastic volatility jump models. *J Comput Appl Math* 355: 201–217.

Edeki SO, Ugbebor OO, Owoloko EA (2015) Analytical solutions of the black–scholes pricing model for european option valuation via a projected differential transformation method. *Entropy* 17: 7510–7521.

Fengler M, Härdle W, Schmidt P (2002) Common factors governing vdax movements and the maximum loss. *Financ Mark Portf Manage* 16: 16–29.

Frey R, Patie P (2002) Risk management for derivatives in illiquid markets: A simulation study. In Sandmann K, Schönbucher PJ (Eds.), *Advances in finance and stochastics*, Springer, 137–159.

Grabbe JO (1983). The pricing of call and put options on foreign exchange. *J Int Money Financ* 2: 239–253.

Gulen S, Popescu C, Sari M (2019) A new approach for the black–scholes model with linear and nonlinear volatilities. *Mathematics* 7: 760.

Heynen R, Kemna A, Vorst T (1994) Analysis of the term structure of implied volatilities. *J Financ Quant Anal* 29: 31–56.

Kou SG (2002) A jump-diffusion model for option pricing. *Manage Sci* 48: 1086–1101.

Lai CH (2019) Modification terms to the black–scholes model in a realistic hedging strategy with discrete temporal steps. *Int J Comput Math* 96: 2201–2208.

Liao W, Zhu J (2009) An accurate and efficient numerical method for solving black-scholes equation in option pricing. *Int J Math Oper Res* 1: 191–210.

Menn C, Rachev ST (2005) A garch option pricing model with α-stable innovations. *Eur J Oper Res* 163: 201–209.

Rao SCS, Manisha (2018) Numerical solution of generalized black–scholes model. *Appl Math Comput* 321: 401–421.

Roul P, Prasad Goura VMK (2021) A compact finite difference scheme for fractional black-scholes option pricing model. *Appl Numer Math* 166: 40–60.

Smith Jr CW (1976) Option pricing: A review. *J Financ Econ* 3: 3–51.

Wiese M, Bai L, Wood B, et al. (2019) Deep hedging: learning to simulate equity option markets. arXiv preprint arXiv. Available from: https://arxiv.org/abs/1911.01700.

Windcliff H, Forsyth PA, Vetzal KR (2004) Analysis of the stability of the linear boundary condition for the black-scholes equation. *J Comput Financ* 8: 65–92.

Wu X, Wang M, Zhuang Y (2016) Implied volatility model with index parameter. *J Anhui Univ Technol Nat Sci* 2016: 04.