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THE CHARGE 2 MONOPOLE VIA THE ADHMN CONSTRUCTION

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(WITH AN APPENDIX BY DAVID E. BRADEN, PETER BRADEN AND H.W. BRADEN)

Abstract. Recently we have shown how one may use use integrable systems techniques to implement the ADHMN construction and obtain general analytic formulae for the charge n $su(2)$ Euclidean monopole. Here we do this for the case of charge 2 giving the first analytic expressions (for general $x \in \mathbb{R}^3$) for the gauge invariant $\operatorname{Tr}\Phi^2$, where Φ is the Higgs field, and the energy density. A comparison with known results and other approaches is made and new results presented.

Contents

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1. Introduction

This paper describes the exact solution of the gauge and Higgs fields for charge two $su(2)$ Euclidean monopoles. Despite BPS monopoles having been studied for over 35 years, and having uncovered extraordinarily beautiful structures, such analytic reconstruction has (with the exception of some partial results that will later be recalled) proved too hard. We often know more about the moduli space of these solutions than we do the actual fields. This is particularly true in the charge two setting: the Atiyah-Hitchin manifold, the moduli space of the centred charge two monopoles, is a well-studied and rich object and yet the analytic solution of the fields has proved elusive. Recently a general program for reconstructing the gauge theory data for $su(2)$ Euclidean monopoles of general charge has been given, circumventing a number of previously intractable steps. This lowest charge case is a useful testing ground and will produce a number of new results. (The spherically symmetric case for charge one and coincident charge n monopoles is amenable to other approaches.) We will compare our results with some of the numerical studies that have been undertaken. Although constructing exact solutions – be they of gravity or gauge theory – is often viewed as a rather recondite area of research analytic solutions give at the very least some control over numerical results.

The algebro-geometric construction of $su(2)$ Euclidean monopoles described here is built upon the substantive work of a number of authors. Particularly relevant (with more detail following) are:

(i) Nahm's modification of the ADHM construction of instantons [37, 38]. This introduces $n \times n$ matrices $T_i(z)$ $(j = 1, \ldots, 4)$ that satisfy a system of ordinary differential equations (Nahm's equations) and an operator Δ.
(ii) Nahm's equations may be written as a Lax pair $\dot{L} = [L, M]$. Here there is a spectral parameter $L = L(\zeta)$, $\zeta \in \mathbb{P}^1$, and the characteristic equation $P(\eta, \zeta) := \det(\eta - L(\zeta)) = 0$ defines a spectral curve $\mathcal{C} \subset T\mathbb{P}^1$ where the mini-twistor space $T\mathbb{P}^1$ is the geometric setting for Hitchin's description of monopoles [40, 24].
(iii) The gauge and Higgs fields are then constructed from integrals (over z) of bilinears involving the two normalizable solutions to the Weyl equation $\Delta^\dagger \boldsymbol{v} = 0$. Hitchin proved [25] that the regularity of these fields placed certain constraints on the curve $\mathcal{C}$. We shall describe a curve satisfying Hitchin's constraints as a *monopole spectral curve*.
(iv) These integrals may in fact be performed using formulae of Panagopoulos [43, 8].
(v) Ercolani and Sinha showed how one could use integrable systems techniques to solve for a gauge transform of the Nahm data in terms of a Baker-Akhiezer function Φ_{BA} associated to $\mathcal{C}$ [16]. Here one of Hitchin's constraints on the curve is reexpressed in terms of the direction $\mathbf{U}$ of flow on the Jacobian $\mathrm{Jac}(\mathcal{C})$. The Ercolani-Sinha vector $\mathbf{U}$ is a half-period [26, 7].
(vi) Using a lesser known ansatz of Nahm the authors showed how one might solve for $\boldsymbol{v}$ in terms of the Baker-Akhiezer function Φ_{BA} and the same (unkown) gauge transformation employed by Ercolani and Sinha [40, 8].
(vii) Finally it has been shown how to eliminate the unknown gauge transformation to reconstruct the gauge and Higgs fields [8].

At this stage one has a way of analytically constructing the gauge and Higgs fields given a monopole spectral curve. Several remarks are however in order. The number of known monopole spectral curves is few: although Hitchin's constraints on a curve are algebro-geometric in nature a constructive solution is still lacking [9]. The construction outlined does not yet provide a solution to the Nahm equations in standard ($T_4 = 0$) gauge. Notwithstanding such questions we may now in principle analytically construct solutions.

To provide the context to the contents and new results of this paper we must first recall the various analytic approaches to studying BPS monopoles.

1.1. **Three Analytic Approaches.** There have been three approaches to constructing analytic solutions of the $su(2)$ monopole equations on $\mathbb{R}^3$: via the $\mathcal{A}_k$ ansatz of Atiyah-Ward; via an ansatz of Forgács, Horváth and Palla that emerged from their study of axially symmetric monopoles and the Ernst equation; and via Nahm's modification of the ADHM construction of instantons. We shall briefly describe these. In all three approaches the spectral curve of the monopole appears and the importance of this curve was gradually elucidated. Further, most authors focussed on calculating the Higgs field and the gauge invariant quantity $\frac{1}{2}\operatorname{Tr}\Phi^2$. With appropriate choices points on the spatial axes are related to points on the $n = 2$ spectral curve by biquadratic equations rather than the more general quartic equation and this meant the Higgs field on the coordinate axes was more amenable to study. One early result [11][(7.2)] was that the Higgs field at the origin gave (in units described in the sequel)

$$-\frac{1}{2}\operatorname{Tr}\Phi^2\big|_{(0,0,0)} = \frac{(K(1+k'^2)-2E)^2}{K^2k^4}. \tag{1.1}$$

One of the simplifying features of monopoles is that the energy density $\mathcal{E}(\boldsymbol{x})$ is related to $\frac{1}{2}\operatorname{Tr}\Phi^2$ via Ward's formula [48]

$$\mathcal{E}(\boldsymbol{x}) = -\frac{1}{2}\nabla^2\operatorname{Tr}\Phi^2.$$

Once one could calculate $\frac{1}{2}\operatorname{Tr}\Phi^2$ in any of these approaches it was possible to numerically calculate the Laplacian and subsequently the energy density: the culmination of these (amalgams of analytic and numerical) studies were plots and a video using an early supercomputer (see below).

1.1.1. $\mathcal{A}_k$ *ansatz.* Based on Ward's identification [47] of self-dual solutions to the Yang-Mills equations and appropriate vector bundles over twistor space, Atiyah and Ward [2] developed a series of ansätze, the $\mathcal{A}_k$ ansatz, that reduced the construction of $su(2)$ instantons to constructing patching functions g for gauge bundles. In terms of this data Corrigan, Fairlie, Yates and Goddard [13] showed how to reconstruct the gauge fields making connection with Yang's study of $su(2)$ instantons [52] and Yang's equation.

Now Manton in [33] had noted that the field equations for BPS monopoles corresponded to the equations of static self-duality and Ward in [48] described how to modify the patching function data to reproduce such solutions. Ward's initial ansatz produced axially symmetric[1] charge 2 monopoles and for a particular choice of constant he saw regular solutions. Prasad and Rossi [44] then produced the appropriate Atiyah-Ward patching function for the axially symmetric charge n monopole. Ward [49] subsequently generalized his ansatz to account for separated charge 2 monopoles and Corrigan and Goddard [14] extended this to the general charge n monopole with $4n-1$ degrees of freedom. One shortcoming with this approach was that the regularity of the gauge fields was left unproven: although the spectral curve of the monopole makes its appearance the full conditions for regularity were not obtained until Hitchin's work [25].

Ward concludes in [49]: "It seems likely that the expressions for general n-monopole solutions, as functions of x, y and z are so complicated that there would be little point in trying to write them out. Of course, since we have explicit formulae, the fields could be computed numerically to any desired degree of accuracy. One attraction of the technique presented here is that the matrices g are relatively simple, even when the corresponding space-time fields A_μ are extremely complicated. So one can deduce much about instantons and monopoles (such as their existence!) without having to write down space-time expressions for them."

There have been a few works that have sought to apply the Atiyah-Ward construction. Brown, Prasad and Rossi [12] explored the uniqueness and assumptions of [49, 14]; their results differed in cases of non-regular monopoles. In [41] O'Raifeartaigh, Rouhani and Singh looked at solving the Corrigan-Goddard constraints for n monopoles while in [42] they studied the $n=2$ monopole in detail. This latter work presents the Higgs field in terms of various infinite sums and their derivatives: their 'very complicated' expression was evaluated numerically for the axis joining the monopoles where the zero was found to be 'very close' and 'barely distinguishable' from $\pm kK(k)/2$ (in our later notation) [42, §6,

[1]One of the surprises discovered about BPS monopoles was that an axial symmetric monopole corresponded to coincident charges [28].

§9]. They write that they "cannot guess a 'natural' analytic expression" describing this position. Brown [10] later evaluates these infinite sums in terms of elliptic functions. Brown in fact evaluates the Higgs field on each of the axes using the Corrigan-Fairlie-Goddard-Yates formalism reproducing for one axis the earlier result of Brown, Prasad and Panagopoulos [11] (see below) obtained via the Nahm equations with the corresponding value of $\frac{1}{2}\operatorname{Tr}\Phi^2$ at the origin (1.1). Without denying the importance of the Brown's work we believe that there are errors in his formulae describing behaviour on the other axes, in particular his values of the Higgs field at the origin of his y and z axes differ from (1.1).

1.1.2. *The Forgács, Horváth and Palla Ansatz.* Again in [33] Manton introduced an ansatz for axially symmetric BPS monopoles that he was unable to solve. In a series of papers Forgács, Horváth and Palla [17, 18, 21] used the Ernst equation to study such monopoles separate to the developments of the Atiyah-Ward construction. In [19] they obtained a suitable Bäcklund transformation reproducing[2] Ward's results while in [20] they look at $n = 2,\ 3,\ 5$ giving determinantal expressions for $\operatorname{Tr}\Phi^2$ and from this plots for the energy density evaluated numerically.

Forgács, Horváth and Palla subsequently generalized their ansatz [21, 22, 23] to account for separated monopoles; this also made connection to Yang's equation. In [22, 23] their ansatz gives the Higgs field, and from this they numerically calculate the energy density plotting this for the (in our conventions) $x_2 = 0$ plane. Based on the numerical evaluation of their ansatz Forgács, Horváth and Palla [22, (21)] gave the zeros of the Higgs field to be (in our units) $\pm kK(k)/2$ while in their later work[3] [23, §6] they expressed that their earlier result was to be viewed as a very good approximation of the zeros.

Using the Forgács, Horváth and Palla ansatz for the $n = 2$ Higgs field Hey, Merlin, Ricketts, Vaughn and Williams [45, 35] made use of a very early supercomputer to determine the Higgs field and consequently (numerically) the energy density over a region of $\mathbb{R}^3$ for various monopole separations. Together with Atiyah and Hitchin this was used to produce a video describing monopole collisions [1].

1.1.3. *Nahm's modification of the ADHM construction.* Nahm's modification of the ADHM construction was developed in [37, 38, 39] and in [40] he described the algebraic geometry underlying this together with his "lesser known" ansatz. Brown, Prasad and Panagopoulos [11] used Nahm's formalism to explicitly solve for the Higgs field on a portion of the axis joining two separated monopoles. This was possible because Nahm's 4×4 matrix equation $\Delta^\dagger v = 0$ (see below) actually factorizes into two 2×2 matrix equations. We shall show that this holds true for each axis and indeed the same Lamé equation results with appropriate shifts for each axis. A significant early step in tying Nahm's work with integrable systems was then made by Ercolani and Sinha [16] who first made connection with the Baker-Akhiezer function; Houghton, Manton and Romão [26] revisted this making connection with the Corrigan-Goddard constraints [14] and the $\mathcal{A}_k$ ansatz. In a number of works culminating in [8] the authors have shown how given a spectral curve one may solve for the monopole gauge data; this paper will, amongst other things, do this for the $n = 2$ case.

1.1.4. *Spectral Curves.* As noted above a spectral curve underlies each of the analytic approaches just described. Hitchin [24][Theorem 7.6] shows this curve determines the bundle described by the $\mathcal{A}_k$ ansatz and in [25] that it is the spectral curve of Nahm's integrable

[2] Compare (8), (9) of [48] with (22), (23) of [19].

[3] This followed two analytic works: the already noted [42, §6, §9] where the zero was numerically found to be very close to $\pm kK(k)/2$; and in [11] expansions for the zeros of the Higgs field were given for k near 0 and 1, the latter being situated near $\pm kK(k)/2$.

system. Also in [25] Hitchin gives the necessary and sufficient conditions for a spectral curve to yield a nonsingular monopole. Hurtubise [29] then evaluated these constraints to produce the $n = 2$ spectral curve.

1.2. **Overview and Principal Results.** While it has been known for a long time then that the spectral curve fully determines a monopole it has remained less clear how to implement this. Our approach here is to follow Nahm's construction: in [8] we have described this for general n and here we will do this concretely for $n = 2$. We will review this approach in Section 2. Whatever approach is adopted one needs an understanding of the spectral curve and the integrals of certain meromorphic forms on it. Sections 3-6 will determine many of the basic properties of the $n = 2$ curve, its parameterizations and needed integrals. A given point $\boldsymbol{x} \in \mathbb{R}^3$ corresponds to (generically) 4 points on the curve (by what we describe below as the Atiyah-Ward constraint). We uncover a number of new special addition theorems for θ-functions whose arguments are the Abelian images of these points as well new relations for sums of non-complete first and second kind integrals. The explicit answers and derivations for the charge two monopole depend significantly on these. The results of these sections will enable us to make contact with earlier results. (Appendix A will relate the different forms of this curve used by workers over the years.) As remarked upon above, the coordinate axes (under appropriate choices) have a number of simplifying properties and these are described in 5. Section 6 describes spatial points whose twistor lines are bitangent to the spectral curve: these points will also be distinguished in various ways described in the sequel.

Only in sections 7-9 do we come to the data in Nahm's modification of the ADHM construction: section 7 describes a fundamental matrix W of solutions to a matrix first order differential equation $\Delta W = 0$ in terms of the function theory on the curve (in particular the Baker-Akhiezer function); section 8 describes the adjoint of this equation $\Delta^\dagger V = 0$; and section 9 describes the projector from V to two normalizable solutions $\Delta^\dagger \boldsymbol{v} = 0$. (Both Δ and its adjoint $\Delta^\dagger$ are describe more fully below.) From $\boldsymbol{v}$ one may construct the gauge and Higgs field. We illustrate this by constructing the Higgs field in the simpler setting for the x_2 axis in section 10 recovering (1.1). In Appendix E we show this yields the result of Brown, Prasad and Panagopoulos [11] obtained via Lamé's equation. In section 11 we turn to general formulae for the Higgs field and energy density where we obtain the new result for the energy density at the origin in Proposition 11.5,

$$\mathcal{E}_{\boldsymbol{x}=0}(k) = \frac{32}{k^8 {k'}^2 K^4}\left[k^2(K^2{k'}^2 + E^2 - 4EK + 2K^2 + k^2) - 2(E-K)^2\right]^2,$$

with the known limiting values $\mathcal{E}_{\boldsymbol{x}=0}(0) = \frac{8}{\pi^4}(\pi^2 - 8)^2$ at $k = 0$ (coincident monopoles) and $\mathcal{E}_{\boldsymbol{x}=0}(1) = 0$ at $k = 1$. Section 12 evaluates the general formulae for the Higgs field on the coordinate axes; here we are able to give the equation (12.6) describing the zero of the Higgs field. (Again in Appendix E we obtain these solutions via Lamé's equation.) Figure 1 illustrates these results for different scales. Finally in section 13 we take the $k = 0$ limit of our results reproducing Ward's expressions [48] amongst others. Throughout the text we will defer a significant number of proofs and computations to the five Appendices.

We conclude this introduction by comparing our analytic results with numeric computations. Figures 2 and 3 compare our results with the numerical results underlying[4] the charge 2 results of [34] for $-\frac{1}{2}\operatorname{Tr}\Phi^2$ and the energy density respectively. The clear lesson is how well these results agree. From Figure 2 the values of $-\frac{1}{2}\operatorname{Tr}\Phi^2$ are essentially indistinguishable for $x_\star < 5$; for larger $x_\star$ one sees a divergence (attributable to the large intermediate quantities

[4]We thank Paul Sutcliffe for making these available for comparison.

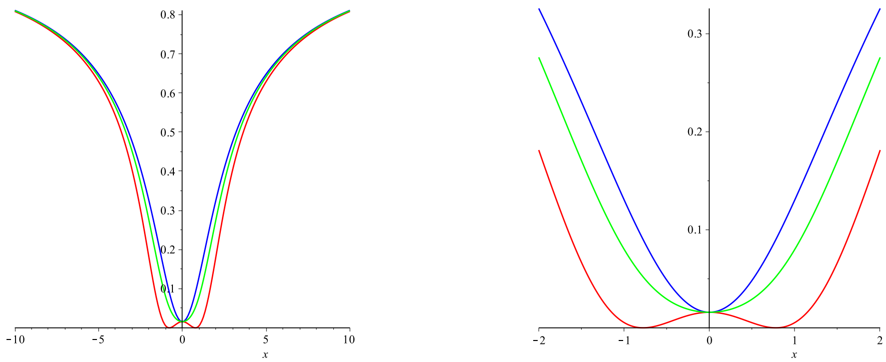

FIGURE 1. $-\frac{1}{2}\operatorname{Tr}\Phi^2$: x_1-axis red; x_2-axis blue; x_3-axis green. $k = 0.8$.

involved in the calculation) but in the range of the second plot this is only[5] $1 \sim 2\%$. Figure 3 compares the energy density: again these are essentially indistinguishable. Closer inspection of the analytic result (red) shows 4 anomalous evaluations: these are close to points of bitangency mentioned above (and described more fully below); these could be removed by using l'Hopital's rule, but we have included them here to illustrate their presence.

Figures 4-7 give a number of views of the energy density as a function of k ($k = 0$ being coincident, and $k = 1$ infinitely separated). In Appendix F (by David E. Braden, Peter Braden and H.W. Braden) we describe and give links to both the scripts that generate the monopole numerics and tools to enable their visualisation. Three tools are given: two are interactive, and the third graphical. The first visualiser encodes energy density as opacity while the second defines a energy density threshold above which to consider as solid (the mesh can be visualised with many mesh viewers, or even 3D printed). Screenshots of these are given in Figure 8. The third method of visualizing the data is a 'Tomogram' that takes slices through the volume. We can plot the contours on these images, or use colour to represent the density at that slice (see Figure 9). The second last column of these figures correspond to the k value of Figure 8.

1.3. **A more detailed outline of the paper.** This paper is long and a more detailed outline of the paper may be helpful beyond the overview and principal results just given. While the index gives a detailed breakdown of what is covered we will give here a synopsis of the strategy of our calculation and the reason behind the various sections. Although the main body of the paper will contain the essential results the Appendices are an integral part of this work serving two purposes. First, we defer many proofs to the appendices: some of these may be straightforward, or, once having proven an illustrative case, proving a number of related results; other proofs are less straightforward but nonetheless a distraction from the progression of the main calculation. Second, this work is built upon nearly 40 years of research and to connect this to our own it is helpful to relate some of the many conventions and results in a unified fashion.

[5]The numerical technique here is, given the Nahm data, to solve Nahm's equations (ODE's) on the requisite interval $[0, 2]$ using a shooting algorithm and then numerically integrate. The ODE (see $\Delta^\dagger$ in the text) is linear in the spatial coordinate x and so this approach necessitates smaller step size the larger the x-values being considered. Inherently this approach has greater errors for larger x. The numerical works do not give error bounds; results are usually given for as large an x-domain as possible where quantities appear stable under step-size changes. Our analytic results allow a significant test of these numerical results.

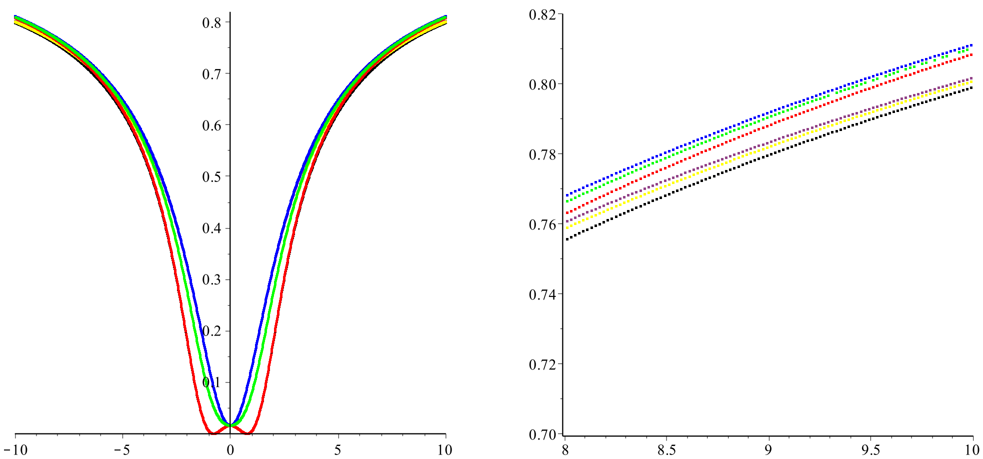

FIGURE 2. $-\frac{1}{2}\operatorname{Tr}\Phi^2$ Analytic vs Numerical: x_1-axis red vs black; x_2-axis blue vs violet; x_3-axis green vs yellow. The second plot focusses on a smaller interval. $k = 0.8$.

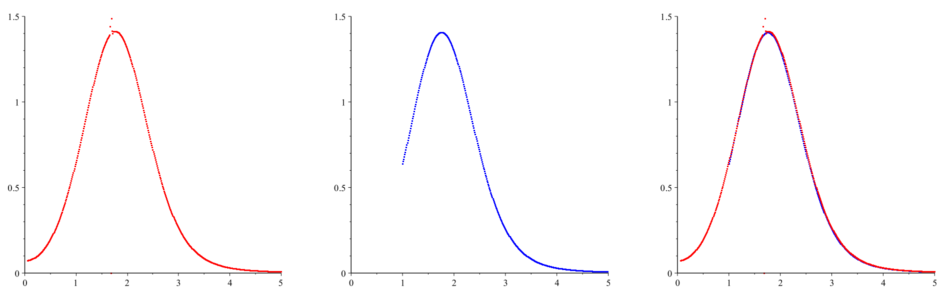

FIGURE 3. Energy Density on the x_1-axis: analytic (red), numerical (blue) and comparison for $k = 0.99$.

Section 2 of the paper gives an overview of our approach which is based upon the ADHM construction. Here we summarise a number of results that underly or implement this construction, including some more recent ones of our own: these are the points (i)-(vii) noted above. The ADHM construction expresses the gauge and Higgs fields in terms of integrals to (normalisable) solutions to on ODE $\Delta^\dagger \boldsymbol{v} = 0$; Δ is built out of 'Nahm data'. Let V denote a fundamental matrix of solutions to this equation. The integrals of the ADHM

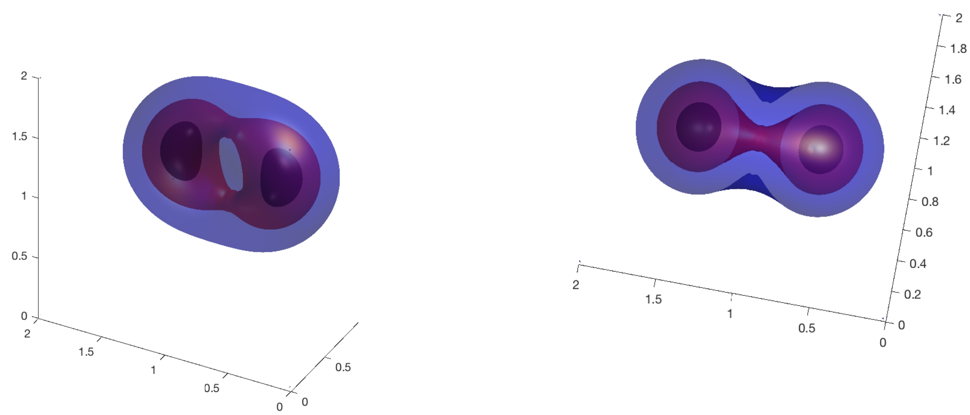

FIGURE 4. Two views of the Energy density $\mathcal{E}(x)$ for $k = 0.8$. Blue corresponds to the isocontour $\mathcal{E}(x) = 0.2$, red to $\mathcal{E}(x) = 0.42$, and dark red to $\mathcal{E}(x) = 0.7$.

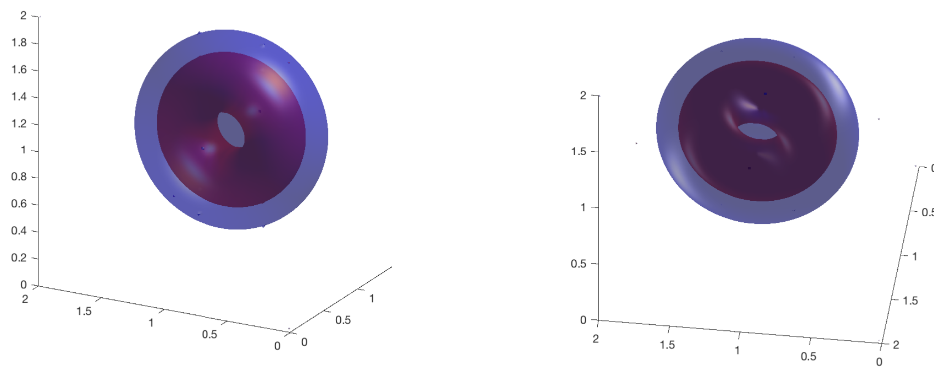

FIGURE 5. Two views of the Energy density $\mathcal{E}(x)$ for $k = 0.05$. Blue corresponds to the isocontour $\mathcal{E}(x) = 0.2$, red to $\mathcal{E}(x) = 0.42$. The energy density $\mathcal{E}(x) = 0.7$ is not achieved.

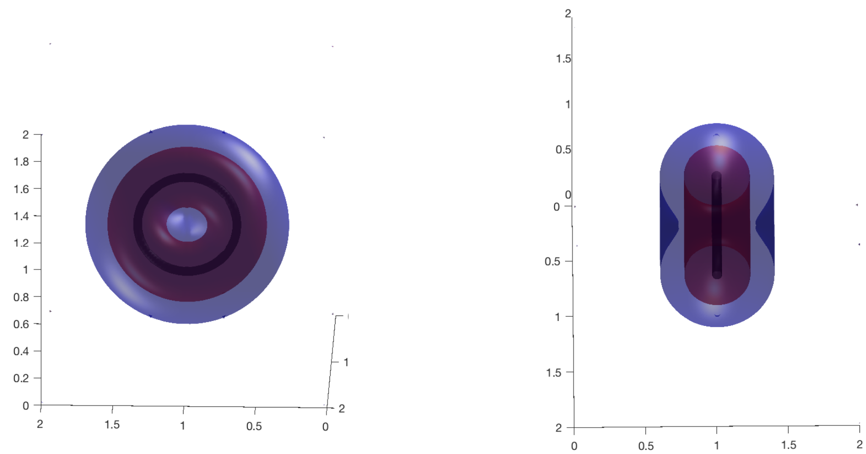

FIGURE 6. Two views of the Energy density $\mathcal{E}(x)$ for $k = 0.25$. Blue corresponds to the isocontour $\mathcal{E}(x) = 0.2$, red to $\mathcal{E}(x) = 0.42$, dark red to $\mathcal{E}(x) = 0.65$.

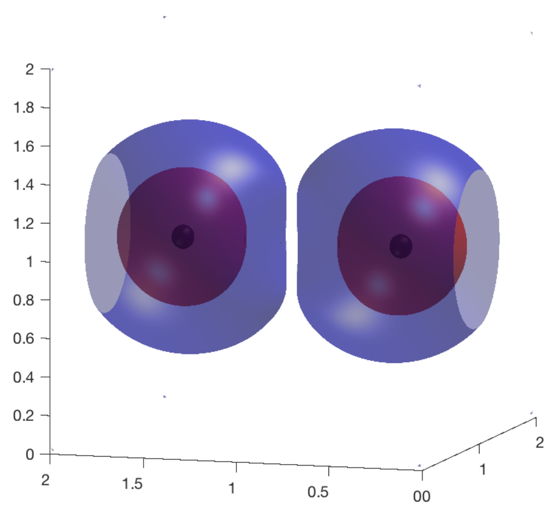

FIGURE 7. The Energy density $\mathcal{E}(x)$ for $k = 0.99$. Blue corresponds to the isocontour $\mathcal{E}(x) = 0.09$, red to $\mathcal{E}(x) = 0.42$, dark red to $\mathcal{E}(x) = 1.35$.

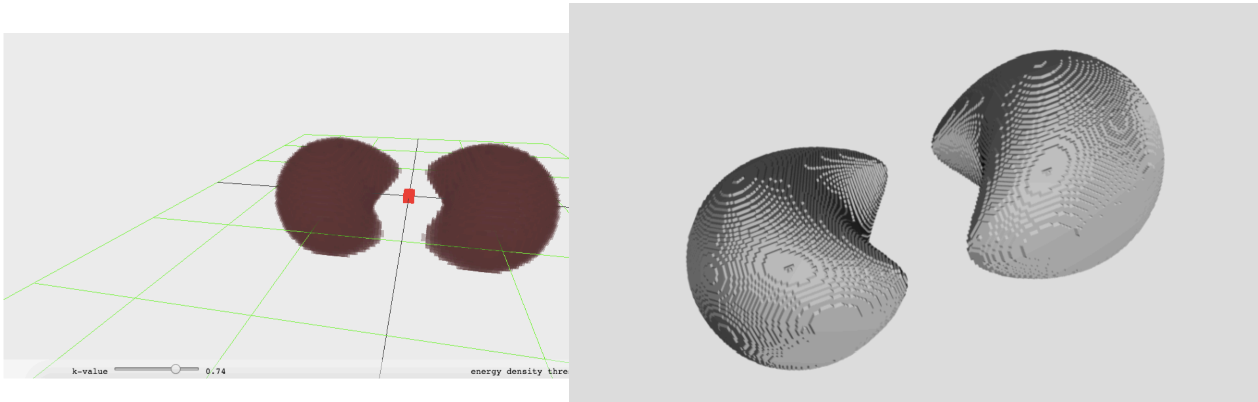

Figure 8. Two interactive visualisers: the first represents energy density by opacity while the second uses energy density to give a threshold producing a solid above the threshold. Here $k = 0.74$ and the energy density threshold is 0.5.

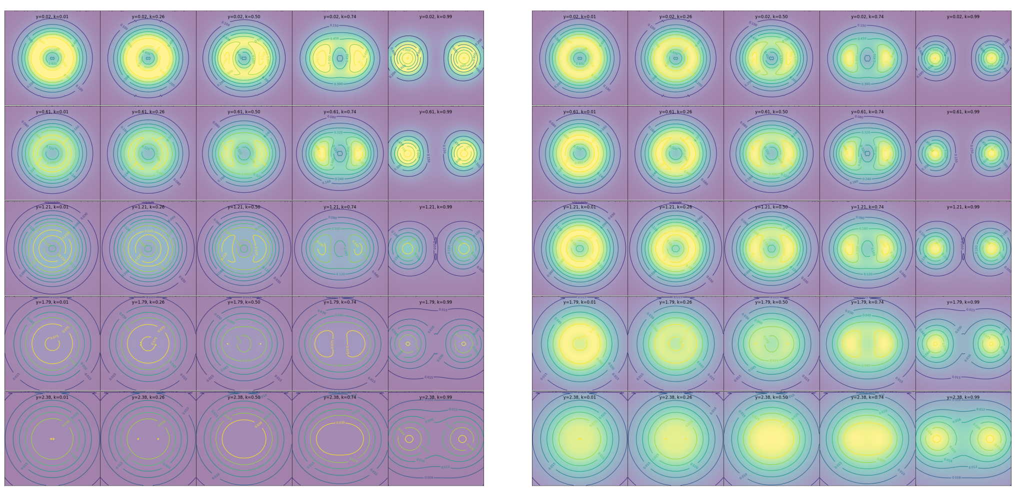

Figure 9. A Monopole Tomogram with (a) uniform colouring and (b) nonuniform colouring.

theorem may in fact be performed and knowing V (which also yields the projector to the normalisable solutions) is sufficient to reconstruct the gauge and Higgs fields. The aim is to construct V. Thus far integrable systems are not in the fore. Hitchin related the construction of Nahm data to an integrable system and linear flow on the Jacobian of an associated spectral curve $\mathcal{C}$. There is a twistorial basis to Hitchin's construction and the Atiyah-Ward equation will relate spatial coordinates to coordinates on the spectral curve. This integrable system allows us to construct solutions to the adjoint equation $\Delta \boldsymbol{w} = 0$; taking $W = (\boldsymbol{w})$ then $V = (W^\dagger)^{-1}$ and we may obtain the desired V. The columns of W are determined by the solutions of the Atiyah-Ward equation. Together with our recent result expressing W in terms of the Baker-Akhiezer function for Hitchin's flow on $\mathcal{C}$ (questions of gauge choice arise here) we have a means of reconstructing the gauge and Higgs fields and then relevant physical quantities: we wish to determine W and thence V.

To follow our route we need to construct the Baker-Akhiezer function and flow of the associated integrable system. This necessitates a good understanding and parameterisation of the curve and Section 3 treats this. Although we use the Ercolani-Sinha paramaterisation of the curve and homology throughout we will need to compare with results expressed in different parameterisations and Appendix A will do this giving the explicit transformations with a number of authors and providing further calculational details for the section. Various needed expansions at the end-points of the flow and these are gathered here. At this stage we have the basic building blocks for the construction.

Although the Baker-Akhiezer function (and so W) only depends on a single point on the curve $V = (W^\dagger)^{-1}$ depends on all of the solutions to the Atiyah-Ward equation and their Abel-Jacobi images. Section 4 deals with with the addition formulae that will appear and how they interact with conjugation and other symmetries. Some of these addition formulae are general and Appendix B first lists Weierstrass's Trisecant θ-formulae and then applies them in our setting. There are however a number of addition formulae arising because of Atiyah-Ward equation and these are also dealt with. (Appendix C will prove many of these.)

Thus far all our expressions have been for general $x \in \mathbb{R}^3$ and Section 5 describes those loci where the general analysis simplifies. These include each of the coordinate axes and here we examine these and give alternate parameterizations that will facilitate comparison with results in the literature. For general $x \in \mathbb{R}^3$ the roots of the Atiyah-Ward equation are distinct: the locus for which we have multiple roots is described in Section 6. These are the points of bitangency referred to above and whose significance will be described in the text.

We are now at the stage where we can calculate W and from this $V = (W^\dagger)^{-1}$. Section 7 constructs W and examines its pole structure and expansion (results used later). In this section we express W in the form $(1_2 \otimes \mathcal{O}\, C(z))\, \Psi\, \mathcal{D}$ (the notation is defined in the sequel) and calculate the determinant $|\mathcal{D}|$ and adjugate matrix of Ψ in preparation for calculating V. Here the addition formulae previously established become critical. While we illustrate these in the calculation of $|\mathcal{D}|$ we defer the proof of the adjugate matrix of Ψ (Theorem 7.1) to Appendix D.

The all important matrix V may now be determined, though its easier to treat its complex conjugate $\overline{V} = (W^T)^{-1}$ rather than conjugate all expressions. We do this in Section 8 with the important Theorem 8.1 giving its expansion at one end of the spectral flow; the common pole structure required by the theory is exposed. Further, we show the expansion at the other end may be described by a (constant) monodromy matrix which simplifies the problem to expansions at one end of the interval. The appearance of this matrix and the consequent simplifications appears new. A convenient normalisation is introduced. We find the expansion of the matrix $\overline{V}$ can determined from our earlier expansion W; this serves as a check in the present work but may prove a useful observation for the general monopole setting.

The common pole structure of $\overline{V}$ together with the monodromy matrix allows the projector to the normalisable states to be constructed in Section 9. We are now at the stage where we have the ingredients of the ADHM theorem and can perform the integrations to recover the gauge fields. As a warm-up we use our result to reconstruct the Higgs field on the x_2-axis in Section 10. Here we recover previous results obtained via Lamé's Equation. In Appendix E we do this calculation for each of the coordinate axes and perform the analysis needed to compare with the parameterisation of the body of the paper.

Section 11 now calculates the Higgs field and energy density for a generic $x \in \mathbb{R}^3$. To integrate the normalisable solutions we already have all of the needed quantities bar one and this is determined. Here our recent Theorem 2.3 proves useful. We show how to combine our results to give the gauge invariant $-\frac{1}{2}\operatorname{Tr}\Phi^2$ and, via a formula of Ward, the energy density $\mathcal{E}(\boldsymbol{x}) = -\frac{1}{2}\nabla^2\operatorname{Tr}\Phi^2$. While the paper determines all of the partial derivatives needed to evaluate the energy density we do not write this out here; we do however determine the new result of the energy density at the origin described in the Overview. Appendix F indicates where code implementing the formulae of this section may be found.

The final two sections specialise our results. In Section 12 we again focus on (now all of the) coordinate axes. Again we make contact with Lamé's Equation and Appendix E and show these approaches coincide. Further we discuss the zeros of the Higgs field identifying a transcendental equation that determines these. The final Section 13 looks at the $k \to 0$ limit which reproduces the charge 2 axially symmetric monopole showing how our results reproduce those of Ward.

2. Background

To make this paper more self-contained we will elaborate a little on the points noted in the construction: the ADHM construction and Panagopoulos formulae; the spectral curve and Hitchin's constraints; the Ercolani-Sinha Baker-Akhiezer function for the curve; and Nahm's lesser known ansatz. Here we will simply cite the critical formulae.

The field equations for the three dimensional Yang-Mills-Higgs Lagrangian with gauge group $SU(2)$

$$L = -\frac{1}{2}\operatorname{Tr} F_{ij}F^{ij} + \operatorname{Tr} D_i\Phi\, D^i\Phi,$$

are

$$D_i\Phi = \frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}F_{jk}, \quad i = 1, 2, 3. \tag{2.1}$$

Here Φ is the Higgs field, $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ is the curvature of the (spatial) connection of the gauge field $A_i(\boldsymbol{x})$ and D_i the covariant derivative $D_i\Phi = \partial_i\Phi + [A_i, \Phi]$, $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. These equations may be viewed as a reduction of the self-dual Yang Mills equations to three dimensions under the assumption that all fields are independent of time. Configurations minimizing the energy of the system are given by the *Bogomolny equation* (2.1). A solution with the boundary conditions

$$\left.\sqrt{-\frac{1}{2}\operatorname{Tr}\Phi(r)^2}\right|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

is called a *monopole* of charge n. The aim is to construct the Higgs and gauge field satisfying the Bogomolny equation and this boundary condition.

2.1. The ADHM construction and Panagopoulos formulae.

Nahm, in modifying the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of instanton solutions to the (Euclidean) self-dual Yang-Mills equations, introduced the operator

$$\Delta = \imath\frac{d}{dz} + x_4 - \imath T_4 + \sum_{j=1}^{3}\sigma_j \otimes (T_j + \imath x_j 1_n), \tag{2.2}$$

where the $T_j(z)$ are $n \times n$ matrices and σ_j the Pauli matrices. Here n is the charge of the $su(2)$ monopole. Following the instanton construction the operator $\Delta^\dagger\Delta$ must commute with quaternions which happens if and only if ${T_i}^\dagger = -T_i$, $T_4^\dagger = -T_4$ and

$$\dot{T}_i = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j(z), T_k(z)]. \tag{2.3}$$

Equations (2.3) are known as Nahm's equations; one often encounters them in the more familiar gauge with $T_4 = 0$. When $\Delta^\dagger\Delta$ commutes[6] with quaternions it is a positive operator; in particular this means that $\left(\Delta^\dagger\Delta\right)(z)$ is an invertible operator and consequently Δ has no zero modes. The ADHM construction further requires Δ to be quaternionic linear, which means that $T_i(z) = -\overline{T}_i(-z)$, $T_4(z) = -\overline{T}_4(-z)$. To describe monopoles the matrices $T_j(z)$ are further required to be regular for $z \in (-1, 1)$ and have simple poles at $z = \pm 1$, the residues of which define an irreducible n-dimensional representation of the $su(2)$ algebra. Hitchin's analysis [25][§2] of the equation $\Delta^\dagger \boldsymbol{v} = 0$ tells us this has two normalizable solutions and it is in terms of these that the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction gives the gauge and Higgs field solutions.

Theorem 2.1 (**ADHMN**)**.** *The charge n monopole solution of the Bogomolny equation (2.1) is given by*

$$\Phi_{ab}(\boldsymbol{x}) = \imath \int_{-1}^{1} \mathrm{d}z\, z \boldsymbol{v}_a^\dagger(z, \boldsymbol{x}) \boldsymbol{v}_b(z, \boldsymbol{x}), \quad a, b = 1, 2, \tag{2.4}$$

$$A_{i\,ab}(\boldsymbol{x}) = \int_{-1}^{1} \mathrm{d}z\, \boldsymbol{v}_a^\dagger(z, \boldsymbol{x}) \frac{\partial}{\partial x_i} \boldsymbol{v}_b(z, \boldsymbol{x}), \quad i = 1, 2, 3, \quad a, b = 1, 2. \tag{2.5}$$

Here the two ($a = 1, 2$) $2n$-column vectors $\boldsymbol{v}_a(z, \boldsymbol{x}) = (v_1^{(a)}(z, \boldsymbol{x}), \ldots, v_{2n}^{(a)}(z, \boldsymbol{x}))^T$ form an orthonormal basis on the interval $z \in [-1, 1]$

$$\int_{-1}^{1} \mathrm{d}z\, \boldsymbol{v}_a^\dagger(z, \boldsymbol{x}) \boldsymbol{v}_b(z, \boldsymbol{x}) = \delta_{ab}, \tag{2.6}$$

for the normalizable solutions to the Weyl equation

$$\Delta^\dagger \boldsymbol{v} = 0, \tag{2.7}$$

where

$$\Delta^\dagger = \imath \frac{d}{dz} + x_4 - \imath T_4 - \sum_{j=1}^{3} \sigma_j \otimes (T_j + \imath x_j 1_n). \tag{2.8}$$

The normalizable solutions form a two-dimensional subspace of the full $2n$-dimensional solution space to the formal adjoint equation (2.7). The $n\times n$-matrices $T_j(z)$, $T_4(z)$, called Nahm data, satisfy Nahm's equation (2.3) and the $T_j(z)$ are required to be regular for $z \in (-1, 1)$ and have simple poles at $z = \pm 1$, the residues of which define an irreducible n-dimensional representation of the $su(2)$ algebra; further

$$T_i(z) = -T_i^\dagger(z), \quad T_4(z) = -T_4^\dagger(z), \quad T_i(z) = T_i^T(-z), \quad T_4(z) = T_4^T(-z). \tag{2.9}$$

[6] Throughout the superscript † means conjugated and transposed. We will at times emphasise the vectorial nature of an object by printing this in bold, e.g for vector $\boldsymbol{a}^\dagger = \overline{\boldsymbol{a}}^T$.

Although the integrations in (2.4, 2.5) look intractable work of Panagopoulos enables their evaluation. Define the Hermitian matrices

$$\mathcal{H} = -\sum_{j=1}^{3} x_j \sigma_j \otimes 1_n, \qquad \mathcal{F} = \imath \sum_{j=1}^{3} \sigma_j \otimes T_j, \qquad \mathcal{Q} = \frac{1}{r^2}\mathcal{H}\mathcal{F}\mathcal{H} - \mathcal{F}. \tag{2.10}$$

Then

Proposition 2.2 (Panagopoulos [43, 8])**.**

$$\int \mathrm{d}z\, \boldsymbol{v}_a^\dagger \boldsymbol{v}_b = \boldsymbol{v}_a^\dagger \mathcal{Q}^{-1} \boldsymbol{v}_b. \tag{2.11}$$

$$\int \mathrm{d}z\, z \boldsymbol{v}_a^\dagger \boldsymbol{v}_b = \boldsymbol{v}_a^\dagger \mathcal{Q}^{-1} \left(z + \mathcal{H}\, \frac{x_i}{r^2} \frac{\partial}{\partial x_i} \right) \boldsymbol{v}_b. \tag{2.12}$$

$$\int \boldsymbol{v}_a^\dagger \frac{\partial}{\partial x_i} \boldsymbol{v}_b \mathrm{d}z = \boldsymbol{v}_a^\dagger \mathcal{Q}^{-1} \left[\frac{\partial}{\partial x_i} + \mathcal{H} \frac{z}{r^2}\, x_i + \mathcal{H} \frac{\imath}{r^2} \left(\boldsymbol{x} \times \boldsymbol{\nabla}\right)_i \right] \boldsymbol{v}_b. \tag{2.13}$$

At this stage we see that to reconstruct the gauge and Higgs fields we need knowledge of the normalizable solutions to $\Delta^\dagger \boldsymbol{v} = 0$ at the endpoints $z = \pm 1$. We will construct a fundamental FV $= (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{2n})$ of solutions to this equation and then extract the normalizable solutions using a ($2n \times 2$ matrix) projector μ

$$V\mu = (\boldsymbol{v}_1, \boldsymbol{v}_2).$$

The work of [8] shows that μ is z-independent and so may be removed from the integrals; thus for example the matrix

$$\left(\int \mathrm{d}z\, \boldsymbol{v}_a^\dagger \boldsymbol{v}_b \right) = \int dz\, \mu^\dagger V^\dagger V \mu = \mu^\dagger \left(\int dz\, V^\dagger V \right) \mu = \mu^\dagger V^\dagger \mathcal{Q}^{-1} V \mu.$$

We also note a further result of [8] that will prove useful:

Theorem 2.3. *With the notation above, and for $W = (V^\dagger)^{-1}$*

$$\left(V^\dagger \mathcal{Q}^{-1} \mathcal{H} V\right)(z) = \textit{constant}, \qquad \left(W^\dagger \mathcal{Q} \mathcal{H} W\right)(z) = \textit{constant}. \tag{2.14}$$

Towards constructing the fundamental matrix V we next turn to the spectral curve.

2.2. **The Spectral Curve and Hitchin's constraints.** One may readily associate an integrable system and spectral curve to Nahm's equations. Hitchin's seminal work [25] provided a geometric setting for this, the global geometry yielding necessary and sufficient conditions for such to be a monopole spectral curve. Here we will recall the salient features.

Upon setting (with ${T_i}^\dagger = -T_i$, $T_4^\dagger = -T_4$)

$$\alpha = T_4 + \imath T_3, \quad \beta = T_1 + iT_2, \quad L = L(\zeta) := \beta - (\alpha + \alpha^\dagger)\zeta - \beta^\dagger \zeta^2, \quad M = M(\zeta) := -\alpha - \beta^\dagger \zeta,$$

one finds

$$\begin{aligned} \dot{T}_i = [T_4, T_i] + \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} [T_j(z), T_k(z)] &\Longleftrightarrow \dot{L} = [L, M] \\ &\Longleftrightarrow \quad \left\{ \begin{aligned} &\left[\frac{d}{dz} - \alpha, \beta \right] = 0, \\ &\frac{d(\alpha + \alpha^\dagger)}{dz} = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]. \end{aligned} \right. \end{aligned} \tag{2.15}$$

Focussing on the first equivalence, Nahm's equations may be expressed as a Lax pair, to which we may associate the spectral curve $\mathcal{C}$ given by

$$P(\zeta,\eta) := \det(\eta - L(\zeta)) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta) = 0, \quad \deg a_k(\zeta) \le 2k. \tag{2.16}$$

The genus of $\mathcal{C}$ is $g = (n-1)^2$. Hitchin's construction shows that the spectral curve naturally lies in mini-twistor space[7] $T\mathbb{P}^1$, the space of lines in $\mathbb{R}^3$. The spectral curve is an algebraic curve $\mathcal{C} \subset T\mathbb{P}^1$. If ζ is the inhomogeneous coordinate on the Riemann sphere then (ζ,η) are the standard local coordinates on $T\mathbb{P}^1$ defined by $(\zeta,\eta) \to \eta\frac{d}{d\zeta}$. The mini-twistor correspondence relates $(x_1,x_2,x_3) \in \mathbb{R}^3$ with (ζ,η) by

$$\eta = (x_2 - \imath\, x_1) - 2x_3\,\zeta - (x_2 + \imath\, x_1)\,\zeta^2. \tag{2.17}$$

The anti-holomorphic involution

$$\mathfrak{J} : (\zeta,\eta) \to (-\frac{1}{\overline{\zeta}}, -\frac{\overline{\eta}}{\overline{\zeta}^2}), \tag{2.18}$$

which takes a point on $\mathbb{P}^1$ to its antipodal point reversing the orientation of a line, endows $T\mathbb{P}^1$ with its standard real structure. The hermiticity properties of the Nahm matrices mean that $\mathcal{C}$ is invariant under $\mathfrak{J}$.

If the homogeneous coordinates of $\mathbb{P}^1$ are $[\zeta_0,\zeta_1]$ we consider the standard covering of this by the open sets $U_0 = \{[\zeta_0,\zeta_1]\,|\,\zeta_0 \neq 0\}$ and $U_1 = \{[\zeta_0,\zeta_1]\,|\,\zeta_1 \neq 0\}$, with $\zeta = \zeta_1/\zeta_0$ the usual coordinate on U_0. We denote by $\hat{U}_{0,1}$ the pre-images of these sets under the projection map $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$. Let L^λ denote the holomorphic line bundle on $T\mathbb{P}^1$ defined by the transition function $g_{01} = \exp(-\lambda\eta/\zeta)$ on $\hat{U}_0 \cap \hat{U}_1$, and let $L^\lambda(m) \equiv L^\lambda \otimes \pi^*\mathcal{O}(m)$ be similarly defined in terms of the transition function $g_{01} = \zeta^m \exp(-\lambda\eta/\zeta)$. A holomorphic section of such line bundles is given in terms of holomorphic functions f_α on $\hat{U}_\alpha$ satisfying $f_\alpha = g_{\alpha\beta} f_\beta$. We denote line bundles on $\mathcal{C}$ in the same way, where now we have holomorphic functions f_α defined on $\mathcal{C} \cap \hat{U}_\alpha$. Hitchin's conditions for a monopole spectral curve are:

H1 Reality conditions: $\mathcal{C}$ is invariant under $\mathfrak{J}$, $a_k(\zeta) = (-1)^k \zeta^{2k}\overline{a_k(-1/\overline{\zeta}\,)}$.
H2 L^2 is trivial on $\mathcal{C}$ and $L^1(n-1)$ is real.
H3 $H^0(\mathcal{C}, L^\lambda(n-2)) = 0$ for $\lambda \in (0,2)$.

2.3. **The mini-twistor correspondence and the Abel-Jacobi map.** Given the mini-twistor correspondence (2.17) and the spectral curve (2.16), a point $\boldsymbol{x} \in \mathbb{R}^3$ yields an equation of degree $2n$ in ζ and gives us $2n$ points $P_i = (\zeta_i, \eta_i)$ (perhaps with multiplicity) on the curve $\mathcal{C}$. We will refer to this degree $2n$ equation in ζ and $\boldsymbol{x}$ as the Atiyah-Ward equation (it having appeared in their work). As both the curve and the correspondence satisfy the antiholomorphic involution $\mathfrak{J}$, so to do the solutions and we may choose an ordering such that

$$P_{i+n} = \mathfrak{J}(P_i). \tag{2.19}$$

Recall that the fundamental matrices V and W were $2n \times 2n$; the points P_i will be used to label the columns of W.

The points $\{P_i\}$ satisfy a number of relations. Let $\phi(P) = \int_{P_0}^{P} \boldsymbol{v}$ denote the Abel-Jacobi map for our curve $\mathcal{C}$ and for a choice of suitably normalized holomorphic differentials $\boldsymbol{v}$,

[7] If we set $\boldsymbol{y} = \left(\frac{1+\zeta^2}{2\,\imath}, \frac{1-\zeta^2}{2}, -\zeta\right) \in \mathbb{C}^3$, then $\boldsymbol{y}\cdot\boldsymbol{y} = 0$ and $\boldsymbol{y}\cdot\overline{\boldsymbol{y}} = (1+|\zeta|^2)^2/2$. Thus with $\boldsymbol{T} = (T_1,T_2,T_3)$, $\boldsymbol{x} = (x_1,x_2,x_3)$, then

$$L(\zeta) := 2\,\imath\,\boldsymbol{y}\cdot\boldsymbol{T} = (T_1 + \imath\,T_2) - 2\,\imath\,T_3\,\zeta + (T_1 - \imath\,T_2)\,\zeta^2.$$

We have $\eta = 2\,\boldsymbol{y}\cdot\boldsymbol{x}$.

Abel's theorem says that $\sum_{\mathfrak{p}\in\mathrm{Div}(w)}\phi(\mathfrak{p})$ lies in the period lattice Λ for any function w. Consider first the function $w(P)=-\eta+(x_2-\imath x_1)-2\zeta x_3-(x_2+\imath x_1)\zeta^2$ on $\mathcal{C}$ which has divisor

$$\mathrm{Div}(w)=P_1+\ldots+P_{2n}-2(\infty_1+\ldots+\infty_n).$$

Then

$$\sum_{i=1}^{2n}\int_{P_0}^{P_i}\boldsymbol{v}-2\sum_{i=1}^{n}\int_{P_0}^{\infty_i}\boldsymbol{v}\in\Lambda. \tag{2.20}$$

Further identities arise by taking $f(P):=\int_{P_0}^{P}\gamma$ for some (possibly meromorphic) differential γ and appropriate functions $w(P)$; a dissection of $\mathcal{C}$ along a canonical homology basis $\{\mathfrak{a}_i,\mathfrak{b}_i\}_{i=1}^{g}$ (suitably avoiding poles) yields

$$0=\frac{1}{2i\pi}\int_{\mathcal{C}}d\,f(P)\wedge d\,\ln w(P)=\frac{1}{2i\pi}\int_{\partial\mathcal{C}}f(P)\,d\,\ln w(P)=\sum_{P\in\mathrm{Div}(w)}\mathrm{Res}\,(f(P)\,d\,\ln w(P))$$

$$=\sum_{j=1}^{g}\frac{1}{2i\pi}\left[\oint_{\mathfrak{a}_i}d\,f(P)\oint_{\mathfrak{b}_i}d\,\ln w(P)-\oint_{\mathfrak{b}_i}d\,f(P)\oint_{\mathfrak{a}_i}d\,\ln w(P)\right] \tag{2.21}$$

Generically the points P_i are distinct with non-generic points corresponding to points of bitangency of the spectral curve. There are typically a number of components to these loci and Hurtubise's study of the asymptotic behaviour of the Higgs field [30] discussed one of these .

2.4. **The Ercolani-Sinha Construction.** Ercolani and Sinha [16] sought to use integrable systems techniques to solve Nahm's equations by solving

$$(L-\eta)U=0,$$
$$\left[\frac{d}{dz}+M\right]U=0. \tag{2.22}$$

To understand the Ercolani-Sinha results its useful to focus on the second of the equivalences of (2.15) which expresses Nahm's equations in the form of a complex and a real equation (respectively) [15]. The complex Nahm equation is readily solved,

$$\beta g=g\nu,\quad \left(\frac{d}{dz}-\alpha\right)g=0\Longleftrightarrow\beta=g\nu g^{-1},\quad \alpha=\dot g g^{-1}, \tag{2.23}$$

where ν is constant and generically diagonal, $\nu=\mathrm{Diag}(\nu_1,\ldots,\nu_n)$; by conjugating[8] by the constant matrix $g(0)$ we may assume $\beta(0)=\nu$ and $g(0)=1_n$. The meaning of ν follows from the equation of the curve (2.16). For large ζ we see that $\det(\eta/\zeta^2-L/\zeta^2)\sim\prod_{i=1}^{n}(\eta/\zeta^2+\nu_i^\dagger)$ and so $\eta/\zeta\sim-\nu_i^\dagger\zeta$; we shall denote by $\{\infty_i\}_{i=1}^{n}$ the preimages of $\zeta=\infty$ with this behaviour. The real equation is more difficult; Donaldson proved the existence of this equation in the monopole context in [15]. Upon setting

$$h=g^\dagger g \tag{2.24}$$

then

$$\dot h h^{-1}=g^\dagger(\alpha+\alpha^\dagger)g^{\dagger\,-1},\qquad h(0)=1_n, \tag{2.25}$$

[8] $\tilde\beta=g(0)^{-1}\beta g(0)$, $\tilde g(z)=g(0)^{-1}g(z)$, $\tilde\alpha=g(0)^{-1}\alpha g(0)$.

and the real equation yields the (possibly) nonabelian Toda equation

$$\frac{d}{dz}\left(\dot{h}h^{-1}\right) = \left[h\nu h^{-1}, \nu^\dagger\right]. \tag{2.26}$$

Now (2.22) is not a standard scattering equation, but upon setting $U = g^{\dagger\,-1}\Phi$ we may use the complex equation to transform (2.22) into a standard scattering equation for Φ,

$$\left[\frac{d}{dz} - g^\dagger(\alpha+\alpha^\dagger)g^{\dagger\,-1}\right]\Phi = \zeta\nu^\dagger\Phi. \tag{2.27}$$

"Standard" here simply means that the matrix $\zeta\nu^\dagger$ on the right-hand side is z-independent. In terms of h we have (2.25) and

$$g^\dagger L g^{\dagger\,-1} = h\nu h^{-1} - \dot{h}h^{-1}\zeta - \nu^\dagger\zeta^2. \tag{2.28}$$

The point to note is that we can solve the standard scattering equation (2.27) explicitly in terms of the function theory of $\mathcal{C}$ by what is known as a Baker-Akhiezer function [31]. If $\Phi := (\Phi_i)$ (with i labelling the rows) the Baker-Akhiezer function is uniquely specified by requiring the behaviour

$$\lim_{P=P(\zeta,\eta)\to\infty_j} \Phi_i(z,P)e^{z\frac{\eta}{\zeta}(P)} = \lim_{P=P(\zeta,\eta)\to\infty_j} \Phi_i(z,P)e^{-z\zeta\nu_j^\dagger} = \delta_{ij} \tag{2.29}$$

and that $\Phi(z,P)$ is meromorphic for $P\in\mathcal{C}\setminus\{\infty_1,\ldots,\infty_n\}$ with poles at a suitably generic degree $g+n-1$ divisor δ. If $\widehat{\Phi}(z,\zeta)$ is the $n\times n$ matrix whose columns[9] are the Baker-Akhiezer functions for the preimages of ζ, then $g^\dagger L g^{\dagger\,-1} = \widehat{\Phi}\,\mathrm{Diag}(\eta_1\ldots,\eta_n)\widehat{\Phi}^{-1}$. Thus the Baker-Akhiezer function enables us to solve for (the gauge transform) $g^\dagger L g^{\dagger\,-1}$ and so too $\dot{h}h^{-1}$; Ercolani and Sinha [16] gave an expression for $\dot{h}h^{-1}$.

The Baker-Akhiezer function may be explicitly constructed: the asymptotics of the essential singularity of (2.27) is encoded by seeking an abelian differential γ_∞ on the curve such that near ∞_j $(j=1,\ldots,n)$

$$z\nu_j^\dagger\zeta \sim -z\eta/\zeta \sim z\left[\int_{P_0}^{P}\gamma_\infty(P) - \tilde{\nu}_j\right].$$

This behaviour defines a differential γ_∞ of the second kind on $\mathcal{C}$ which is unique if we further require the $\mathfrak{a}$-normalization $\oint_{\mathfrak{a}_k}\gamma_\infty(P) = 0$, $(k=1,\ldots,g)$. The constant $\tilde{\nu}_j$ here is defined by $\tilde{\nu}_j = \lim_{P\to\infty_j}\left[\int_{P_0}^{P}\gamma_\infty(P) + \frac{\eta}{\zeta}\right]$.

In the Baker-Akiezer description the flow of line bundles given by Hitchin's exponential transition functions corresponds to a flow on the Jacobian of $\mathcal{C}$ in the direction of the winding vector $\boldsymbol{U}$ of $\mathfrak{b}$-periods of the differential $\gamma_\infty(P)$,

$$\boldsymbol{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty,\ldots,\oint_{\mathfrak{b}_g}\gamma_\infty\right)^T. \tag{2.30}$$

This connection with Hitchin's monopole constraints comes from

Lemma 2.4 (Ercolani-Sinha Constraints)**.** *The following are equivalent:*

(i) L^2 *is trivial on* $\mathcal{C}$.

[9] $\widehat{\Phi}(z,\zeta) := (\boldsymbol{\Phi}_1(z,P_1),\ldots,\boldsymbol{\Phi}_n(z,P_n))$, where $P_i = (\zeta,\eta_i)$.

(ii) There exists a 1-cycle $\mathfrak{es} = \boldsymbol{n}\cdot\mathfrak{a} + \boldsymbol{m}\cdot\mathfrak{b}$ *such that for every holomorphic differential* $\Omega = \left(\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)\right) d\zeta/(\partial\mathcal{P}/\partial\eta)$,

$$\oint_{\mathfrak{es}} \Omega = -2\beta_0, \tag{2.31}$$

(iii) $2\boldsymbol{U} \in \Lambda \Longleftrightarrow \boldsymbol{U} = \dfrac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \dfrac{1}{2}\boldsymbol{n} + \dfrac{1}{2}\tau\boldsymbol{m}.$

Thus for a monopole spectral curve we require that $\boldsymbol{U}$ is a half-period. Further, from **H3** the vector $\boldsymbol{U}$ should be *primitive*, i.e. $\lambda\boldsymbol{U}$ belongs to the period lattice Λ if and only if $\lambda = 0$ or $\lambda = 2$. Although a general Baker-Akhiezer function depends on a generic divisor δ the real structure demanded by Hitchin's **H2** imposes constraints on this. This reduces [16] to

$$c_{ij} = -c_{ji}, \qquad \text{where} \quad c_{ij} := \lim_{P=P(\zeta,\eta)\to\infty_j} \zeta\,\Phi_i(0,P). \tag{2.32}$$

Braden and Fedorov [6] show that these constraints may always be solved for.

It is worth clarifying what the Ercolani-Sinha construction does and does not yield. Given the Baker-Akhiezer function the construction yields the gauge transformed $T_i' := g^\dagger T_i g^{\dagger\,-1}$ which satisfy

$$\dot{T}_i' = [\frac{1}{2}\dot{h}h^{-1}, T_i'] + [T_j', T_k']$$

(and cyclic). Here $T_3' = \frac{1}{2}\dot{h}h^{-1}$ and the $i = 3$ equation becomes (2.26). Although Ercolani and Sinha only solved for $\dot{h}h^{-1}$ the recent work of [8] shows how one may obtain h. Thus the Ercolani-Sinha construction *will* yield solutions of the Nahm equations, but not in the standard gauge with $T_4 = 0$. To obtain a solution of the Nahm equations with $T_4 = 0$ requires

$$\alpha = \alpha^\dagger \Longleftrightarrow h^{-1}\dot{h} = 2g^{-1}\dot{g} \Longleftrightarrow \dot{h} = 2g^\dagger\dot{g} = 2\dot{g}^\dagger g, \tag{2.33}$$

viewed as a differential equation for g with h specified; the solution for g is only defined up to left multiplication by a constant unitary matrix. Although a solution exists we cannot as yet explicitly write one down; such however is not needed to solve for the gauge and Higgs fields.

To make connection with the notation of Ercolani and Sinha [16] that we will at times employ, we record

$$g^{\dagger\,-1} := C, \qquad \nu^\dagger = -\operatorname{Diag}(\rho_1, \ldots, \rho_n). \tag{2.34}$$

2.5. **A lesser known ansatz of Nahm and the construction of V.** It remains to give the fundamental matrix V of solutions to $\Delta^\dagger\boldsymbol{v} = 0$. The solution we follow is based on another ansatz of Nahm: we construct a fundamental solution W to the equation $\Delta\boldsymbol{w} = 0$ and then take $V = \left(W^\dagger\right)^{-1}$.

Theorem 2.5 (Nahm [40]; the modification of [8])**.** *Let* $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ *and let* $\widehat{\boldsymbol{u}}(\boldsymbol{x})$ *be a unit vector independent of* z*. Let* $|s>$ *be an arbitrarily normalized spinor not in* $\ker(1_2 + \widehat{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{\sigma})$*. Then*

$$\boldsymbol{w} := \boldsymbol{w}(\zeta) = (1_2 + \widehat{\boldsymbol{u}}(\boldsymbol{x})\cdot\boldsymbol{\sigma})\, e^{-\imath z[(x_1 - \imath x_2)\zeta - \imath x_3 - x_4]}|s> \otimes U(z) \tag{2.35}$$

satisfies $\Delta\boldsymbol{w} = 0$ *if and only if*

$$0 = (L(\zeta) - \eta)\, U(z), \tag{2.36}$$

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}z} + M(\zeta)\right) U(z), \tag{2.37}$$

where

$$\eta = (x_2 - \imath\, x_1) - 2x_3\zeta - (x_2 + \imath\, x_1)\zeta^2, \tag{2.38}$$

and $L(\zeta)$ *and* $M(\zeta)$*, as above, satisfy the Lax equation* $\dot{L} = [L, M]$.

Although early workers sought to explicitly perform these integrations we may use the connection with integrable systems to solve

$$U(z) = g^{\dagger\,-1}\,\Phi \tag{2.39}$$

in terms of the earlier (and unknown) gauge transformation $C := g^{\dagger\,-1}$ and the Baker-Akhiezer function Φ. We may write the k-th column to the fundamental matrix $W = \left(\boldsymbol{w}^{(k)}(x.z)\right)$ of $\Delta\boldsymbol{w} = 0$ as

$$\begin{aligned}
\boldsymbol{w}^{(k)}(z,x) &= (1_2 + \widehat{\boldsymbol{u}}(\zeta)\cdot\boldsymbol{\sigma})\, e^{-iz[(x_1 - ix_2)\zeta - ix_3 - x_4]} |s> \otimes C(z)\boldsymbol{\Phi}(z, P_k) \\
&= (1_2 \otimes C(z)) \Big((1_2 + \widehat{\boldsymbol{u}}(\zeta)\cdot\boldsymbol{\sigma})\, e^{-iz[(x_1 - ix_2)\zeta - ix_3 - x_4]} |s> \otimes \boldsymbol{\Phi}(z, P_k) \Big) \\
&:= (1_2 \otimes C(z))\; \widehat{\boldsymbol{w}}^{(k)}(z,x)
\end{aligned} \tag{2.40}$$

and $P_k = (\zeta_k, \eta_k)$ are the $2n$ solutions to the mini-twistor correspondence described earlier. These $2n$ points come in n pairs of points related by the antiholomorphic involution $\mathfrak{J}$. To each point we have the associated values $\widehat{\boldsymbol{u}}(\zeta_j)$ and for each of these we solve for $U(z)$ yielding a $2n\times 1$ matrix $\boldsymbol{w}(P_j)$. Taking each of the $2n$ solutions we obtain a $2n\times 2n$ matrix of solutions W. As noted earlier, there may be non-generic points for which $\zeta_i = \zeta_j$ correspond to points of bitangency of the spectral curve; at such points we modify this discussion by taking a derivative $\boldsymbol{w}'(P_j)$.

The equation for $\Delta\boldsymbol{w} = 0$ may be rewritten as

$$0 = \left[\frac{d}{dz} - \mathcal{H} - \mathcal{F}' - \frac{1}{2}\dot{h}h^{-1}\, 1_2\right] \widehat{\boldsymbol{w}} \tag{2.41}$$

where we have used (2.33) and set

$$\mathcal{F}' = \imath \sum_{j=1}^{3} \sigma_j \otimes g^\dagger T_j g^{\dagger\,-1} = \begin{pmatrix} \frac{1}{2}\dot{h}h^{-1} & -\imath\nu^\dagger \\ \imath h\nu h^{-1} & -\frac{1}{2}\dot{h}h^{-1} \end{pmatrix}.$$

Both h and the solution $\widehat{\boldsymbol{w}}$ of (2.41) are determined entirely in terms of the Baker-Akhiezer function and the only unknown in the account at this stage is the gauge transform g. But as shown in [8], this unknown gauge transform combines in all of the integrals in Proposition 2.2 into the known $h(z)$:

$$\mu^\dagger V^\dagger \mathcal{Q}^{-1}\mathcal{O}V\mu = \mu^\dagger \widehat{V}^\dagger \left[\left(1_2\otimes g^\dagger\right)\mathcal{Q}^{-1}\left(1_2\otimes g\right)\right]\mathcal{O}\widehat{V}\mu = \mu^\dagger\widehat{V}^\dagger\mathcal{Q}'^{-1}\left(1_2\otimes h\right)\mathcal{O}\widehat{V}\mu.$$

Here $\mathcal{O}$ is one of the operators appearing on the right-hand side of Proposition 2.2 and (using the definitions (2.10))

$$\mathcal{Q}' = \left(1_2\otimes g^\dagger\right)\mathcal{Q}\left(1_2\otimes g^{\dagger\,-1}\right) := \frac{1}{r^2}\mathcal{H}\mathcal{F}'\mathcal{H} - \mathcal{F}', \qquad \mathcal{F}' = \imath\sum_{j=1}^{3}\sigma_j\otimes g^\dagger T_j g^{\dagger\,-1}.$$

The conclusion is that we may reconstruct the gauge and Higgs fields from just a knowledge of the Baker-Akhiezer function.

2.6. **Remarks.** At this stage we have presented the ingredients needed to reconstruct the gauge and Higgs fields apart from the general construction of the Baker-Akhiezer function.

Although we only need an expansion of the solutions of V at the end points $z = \pm 1$ to reconstruct the gauge theory data and we have in fact described the solution $V(z)$ for all z. The asymptotic behaviour of the Nahm matrices given by the ADMHN theorem tells us that $T_j(z)$ expanded in the vicinity of the end point $z = 1 - \xi$ behaves as

$$T_j(1-\xi) = -\imath\, \frac{l_j}{\xi} + O(1), \quad j = 1, 2, 3,$$

where (the Hermitian) l_j define the irreducible n-dimensional representation of the $su(2)$ Lie algebra, $[l_j, l_k] = \imath\, \epsilon_{jkl}\, l_l$. Then (2.7) behaves in the vicinity of the pole as

$$\left[\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{1}{\xi} \left(\sum_{j=1}^{3} \sigma_j \otimes l_j \right) + \left(\sum_{j=1}^{3} \sigma_j \otimes x_j 1_2 \right) + \mathcal{O}(1) \right] \boldsymbol{v}(1-\xi, \boldsymbol{x}) = 0. \tag{2.42}$$

One can show (see for example [51]) that $\sum_{j=1}^{3} \sigma_j \otimes l_j$ has only two distinct eigenvalues, $\lambda_a = (n-1)/2$ with multiplicity $n+1$ and $\lambda_b = -(n+1)/2$ with multiplicity $n-1$. If $\boldsymbol{a}_i$ are eigenvectors associated with λ_a ($i = 1, \ldots, n+1$), and $\boldsymbol{b}_j$ eigenvectors associated with λ_b ($j = 1, \ldots, n-1$), then (2.42) has solutions $\boldsymbol{v} = \xi^{\lambda_a} \boldsymbol{a}_i + \ldots$ and $\boldsymbol{v} = \xi^{\lambda_b} \boldsymbol{b}_j + \ldots$. Therefore normalizable solutions must lie in the subspace with positive $\lambda_a = (n-1)/2$ and so we require that $\boldsymbol{v}(1, \boldsymbol{x})$ is orthogonal to the subspace with eigenvalue $-(n+1)/2$, i.e.

$$\lim_{z \to 1^-} \boldsymbol{v}(z, \boldsymbol{x})^T \cdot \boldsymbol{b}_j = 0, \quad j = 1, \ldots, n-1.$$

These $n-1$ conditions coming from the behaviour at $z = 1$ thus yield a $n+1$ dimensional space of solutions to $\Delta^\dagger \boldsymbol{v} = 0$. A similar analysis at $z = -1$ again yields a further $n-1$ constraints resulting in two normalisable solutions on the interval. Now although local analysis at each of the end points lets us construct normalizable solutions, the difficulty is in relating normalizable solutions at both ends: $V(z)$ does this for us while numerically this has been done via shooting methods (see [27] for an algebraic implementation of these).

3. Basic Properties of the Spectral Curve

3.1. **The Curve.** The spectral curve $\mathcal{C}$ for $n = 2$ was constructed by Hurtubise [29] and we shall employ the Ercolani-Sinha [16] choice of homology basis (see Fig. 10) and form of the curve,

$$0 = \eta^2 + \frac{K^2}{4} \left(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1 \right), \tag{3.1}$$

where $k'^2 = 1 - k^2$ and $K = K(k)$ is a complete elliptic integral[10]. Here η is related to the spatial coordinates by (2.17). With our conventions the monopoles are on the x_1 axis (for $k > 0$) and at $k = 0$ the monopoles are axially symmetric about the x_2 axis. These properties, together with a comparison with other curve conventions in the literature, are given for convenience in Appendix A.

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$$K(k) = \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 u}}.$$

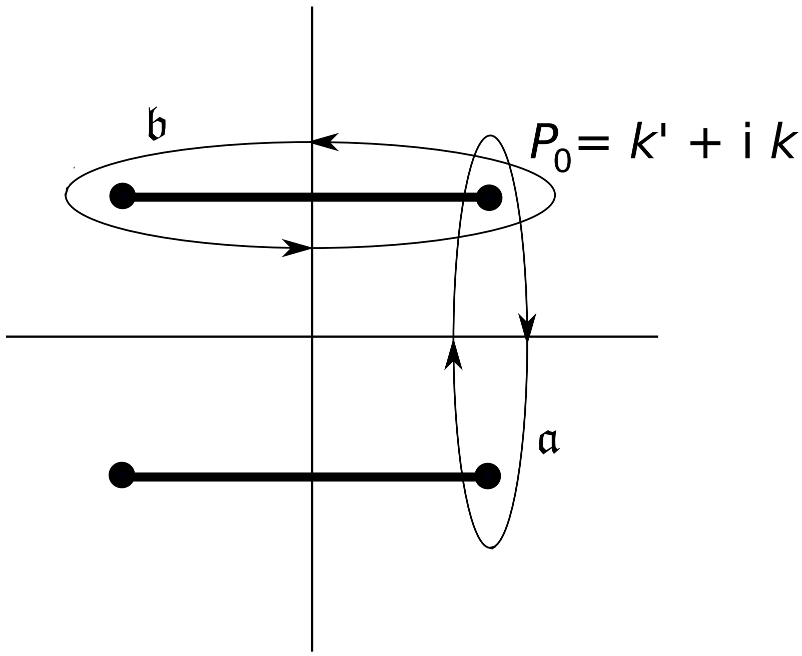

FIGURE 10. The homology basis for the curve with dark lines representing the cuts.

3.2. **Homology, differentials and the Ercolani-Sinha vector.** The roots of the quartic $\zeta^4+2(k^2-k'^2)\zeta^2+1$ are $\pm k'\pm\imath\, k$; these give us the branch points. With $k'=\cos\alpha$, $k=\sin\alpha$ they be written as $\pm e^{\pm i\alpha}$ and these lie on the unit circle. We may take $0\le\alpha\le\pi/4$. We choose cuts between $-k'+ik=-e^{-i\alpha}$ and $k'+ik=e^{i\alpha}$ as well as $-k'-ik$ and $k'-ik$. Let $\mathfrak{b}$ encircle $-k'+ik$ and $k'+ik$ with $\mathfrak{a}$ encircling $k'+ik$ and $-k'+ik$ on the two sheets as shown in Figure 10. We take as our assignment of sheets ($j=1,\,2$, with analytic continuation from $\zeta=0$ avoiding the cuts) to be

$$\eta_j=(-1)^j\, i\frac{K}{2}\sqrt{\zeta^4+2(k^2-k'^2)\zeta^2+1}.$$

With our choice of homology basis the normalized holomorphic differential is then[11]

$$\boldsymbol{v}=\frac{d\zeta}{4\eta} \tag{3.2}$$

and the period matrix for $\mathcal{C}$ is $\tau=i\mathbf{K}'/\mathbf{K}$. Comparison with (2.31) shows that the Ercolani-Sinha constraint is satisfied for for the normalization of our curve (3.1) and $\mathfrak{es}=-\mathfrak{a}$. Thus we have

$$\boldsymbol{U}=-1/2. \tag{3.3}$$

Also from the (2.34) and with our assignment of sheets for $\mathcal{C}$ we have that

$$\rho_1=-\frac{i}{2}\,\mathbf{K},\qquad \rho_2=\frac{i}{2}\,\mathbf{K}.$$

[11] See Appendix A.3.

The normalized second kind differential γ_∞ written in the curve coordinates is

$$\gamma_\infty(P) = \frac{K^2}{4\eta}\left(\zeta^2 - \frac{2E-K}{K}\right)\mathrm{d}\,\zeta. \tag{3.4}$$

3.3. **Abel Maps and Notation.** In [7] we used the Abel map $\phi(P) = \int_{P_0}^{P} \boldsymbol{v}$ with respect to a fixed base point $P_0 = (k' + ik, 0_1)$ of the $\mathfrak{a}$-normalized differential $\boldsymbol{v}$, and using symmetry established that

$$\boldsymbol{\phi}(\infty_1) = \frac{1+\tau}{4} = -\boldsymbol{\phi}(\infty_2), \qquad \boldsymbol{\phi}(0_1) = \frac{1-\tau}{4} = -\boldsymbol{\phi}(0_2).$$

For a degree zero divisor the choice of P_0 doesn't matter. Here we shall often use ∞_1 as a limit of our integrals and to distinguish this we introduce

$$\alpha(P) = \int_{\infty_1}^{P} \boldsymbol{v} = \phi(P) - \phi(\infty_1); \quad \phi(P) = \alpha(P) - \alpha(P_0).$$

It will be convenient to introduce the shorthand notation

$$\theta_i[D] := \theta_i\left(\sum_{P\in D}\alpha(P)\right).$$

Upon using the identification of $\theta(z)$ with the Jacobi theta function $\theta_3(z) := \theta_3(z|\tau) = \sum_{n\in\mathbb{Z}} \exp(\imath\,\pi[n^2\tau + 2nz])$ and the periodicities of the Jacobi theta functions $\theta_*(z)$ we note that

$$\theta(-[z+1]/2 - 1 - \tau) = -e^{-i\pi(z+\tau)}\,\theta_4(z/2),$$
$$\theta(-[z+1]/2 - (1+\tau)/2) = e^{-i\pi(z/2+\tau/4)}\,\theta_2(z/2).$$

We record for later use:

$$\phi(\infty_1) - \phi(0_1) = \frac{\tau}{2}, \qquad \phi(\infty_1) - \phi(\infty_2) = \frac{1+\tau}{2},$$
$$\phi(\infty_1) - \phi(0_2) = \frac{1}{2}, \qquad \phi(\infty_1) = -\alpha(P_0) = \frac{1+\tau}{4}. \tag{3.5}$$

3.4. **Parameterization of the Curve.** We establish in Appendix A.4 that

Lemma 3.1. *With $\theta_i := \theta_i(0)$ the curve (3.1) is parameterized by*

$$\zeta = -i\,\frac{\theta_2[P]\theta_4[P]}{\theta_1[P]\theta_3[P]}, \quad \eta = \frac{i\pi\,\theta_3\theta_2^2\theta_4^2}{4}\,\frac{\theta_3[2P]}{\theta_1[P]^2\theta_3[P]^2}. \tag{3.6}$$

The following θ-quotients are also expressible in terms of coordinates and parameters of the curve

$$\begin{aligned}
\frac{\theta_1[P]^2}{\theta_4[P]^2} &= \frac{K(\zeta^2-1) - 2\imath\eta}{2Kk\zeta^2},\\
\frac{\theta_2[P]^2}{\theta_4[P]^2} &= \frac{K(\zeta^2(k^2-k'^2)+1) + 2\imath\eta}{2Kkk'\zeta^2},\\
\frac{\theta_3[P]^2}{\theta_4[P]^2} &= \frac{K(\zeta^2+1) + 2\imath\eta}{2Kk'\zeta^2}.
\end{aligned} \tag{3.7}$$

3.5. **The Baker-Akhiezer function.** We shall now gather together a number of functions on $\mathcal{C}$ including the Baker-Akhiezer function. (The construction of these from first principles is described in [7].)

The unique meromorphic functions $g_{1,2}(P)$ on the curve such that

$$g_j(\infty_l) = \delta_{jl}$$

and with poles for P such that $\alpha(P) = \pm 1/4$ are

$$\begin{aligned} g_1(P) &= \frac{\theta_2[P]\theta_3[P]}{\theta_2[P]\theta_3[P] - \theta_1[P]\theta_4[P]} = \frac{1+\zeta^2+2i\eta/\boldsymbol{K}}{1+\zeta^2+2i\eta/\boldsymbol{K}+2ik'\zeta}, \\ g_2(P) &= \frac{\theta_1[P]\theta_4[P]}{\theta_1[P]\theta_4[P] - \theta_2[P]\theta_3[P]} = \frac{2ik'\zeta}{1+\zeta^2+2i\eta/\boldsymbol{K}+2ik'\zeta}. \end{aligned}$$

The pole behaviour of these may be seen from

$$\begin{aligned} \theta_1(\alpha(P)-1/4)\theta_4(\alpha(P)+1/4)\,\theta_2\theta_3 &= \theta_1[P]\theta_4[P]\,\theta_2(1/4)\theta_3(1/4) - \theta_1(1/4)\theta_4(1/4)\,\theta_2[P]\theta_3[P] \\ &= -\theta_1(1/4)\theta_4(1/4)\,(\theta_2[P]\theta_3[P] - \theta_1[P]\theta_4[P]) \end{aligned}$$

which holds as a consequence of

$$\theta_1(x+y)\theta_4(x-y)\,\theta_2\theta_3 = \theta_1(x)\theta_4(x)\,\theta_2(y)\theta_3(y) + \theta_1(y)\theta_4(y)\,\theta_2(x)\theta_3(x)$$

and

$$\theta_1(1/4) = \theta_2(1/4), \qquad \theta_3(1/4) = \theta_4(1/4).$$

There is a unique $\mathfrak{a}$-normalized differential γ_∞ on $\mathcal{C}$ with second order poles at $\infty_{1,2}$ such that in the vicinity of $P = \infty_{1,2}$ we have $\int_{P_0}^{P}\gamma_\infty \sim -\eta/\zeta$. We set

$$\tilde{\nu}_i := \tilde{\nu}_i(P_0) = \lim_{P\to\infty_i}\left(\int_{P_0}^{P}\gamma_\infty(P') + \frac{\eta}{\zeta}(P)\right).$$

The following lemma (proved in Appendix A.5) will be useful.

Lemma 3.2.

$$\int_{P_0}^{P}\gamma_\infty(P') = \frac{1}{4}\left\{\frac{\theta_1'[P]}{\theta_1[P]} + \frac{\theta_1'[P-\infty_2]}{\theta_1[P-\infty_2]}\right\} = \frac{1}{4}\left\{\frac{\theta_1'[P]}{\theta_1[P]} + \frac{\theta_3'[P]}{\theta_3[P]} - \imath\pi\right\}, \tag{3.8}$$

$$\oint_{\mathfrak{b}}\gamma_\infty = 2\pi i\boldsymbol{U} = -i\pi, \tag{3.9}$$

$$\tilde{\nu}_1 = -\frac{i\pi}{4}, \quad \tilde{\nu}_2 = \frac{i\pi}{4}, \quad \tilde{\nu}_2 - \tilde{\nu}_1 = \frac{i\pi}{2}, \quad \tilde{\nu}_1 + \tilde{\nu}_2 = 0. \tag{3.10}$$

It is important to note that this lemma relates the choice of contours on each side of the identity by the vanishing of each side at $P = P_0$; adding a $\mathfrak{b}$-cycle then to one side is compensated by adding a $\mathfrak{b}$-cycle to the other and so on.

If we define $\beta_i(P) = \int_{P_0}^{P}\gamma_\infty - \tilde{\nu}_i$, then $\beta_1(P) = \beta_2(P) + \frac{\imath\pi}{2}$ and we are able to work with just the one function, which we will choose to be

$$\beta_1(P) = \frac{1}{4}\left\{\frac{\theta_1'[P]}{\theta_1[P]} + \frac{\theta_3'[P]}{\theta_3[P]}\right\} = \int_{P_0}^{P}\gamma_\infty(P') + \frac{\imath\pi}{4}. \tag{3.11}$$

Combining these expressions yields the Baker-Akhiezer function $\Phi(z, P)$ for our problem; its chief properties are given by:

Lemma 3.3. *$\Phi(z,P)$ defined by*

$$\Phi(z,P)=\chi(P)\begin{pmatrix}-\theta_3(\alpha(P))\theta_2(\alpha(P)-z/2)\\ \theta_1(\alpha(P))\theta_4(\alpha(P)-z/2)\end{pmatrix}\frac{e^{\beta_1(P)z}}{\theta_2(z/2)} \tag{3.12}$$

where

$$\chi(P)=\frac{\theta_2(1/4)\theta_3(1/4)}{\theta_3(0)\theta_1(\alpha(P)-1/4)\theta_4(\alpha(P)+1/4)} \tag{3.13}$$

satisfies

(i) $\Phi(z,P)$ *is meromorphic for* $P\in\mathcal{C}\setminus\{\infty_1,\infty_2\}$ *and with poles at* $\alpha(P)=\pm 1/4$.
(ii) $\Phi(z,P)$ *has simple poles at* $z=\pm 1$ *and is regular for* $z\in(-1,1)$.
(iii) $$\lim_{P=P(\zeta,\eta)\to\infty_i}\Phi(z,P)\,e^{-z\nu_i^\dagger\zeta(P)}=\lim_{P=P(\zeta,\eta)\to\infty_i}\Phi(z,P)\,e^{z\frac{\eta}{\zeta}(P)}=\begin{pmatrix}\delta_{i1}\\ \delta_{i2}\end{pmatrix}.$$
(iv) $\Phi(0,P)=\begin{pmatrix}g_1(P)\\ g_2(P)\end{pmatrix}$.
(v) $c_{12}=\lim_{P=P(\zeta,\eta)\to\infty_2}\zeta\,g_1(P)=-\imath\,k'=-c_{21}=-\lim_{P=P(\zeta,\eta)\to\infty_1}\zeta\,g_2(P)$.

Hence $\Phi(z,P)$ *is a Baker-Akhiezer function for the charge* 2 *spectral curve.*

We remark that the reality conditions (2.32) determine the pole structure of Φ, here $\alpha(P)=\pm 1/4$, only up to a discrete number of choices (see [7]). We will work throughout with the above.

3.6. **Nahm Data and expansions.** The Nahm data for the $n=2$ spectral curve has long been known. With $T_j(z)=\frac{\sigma_j}{2\imath}\,f_j(z)$ Nahm's equation reduce to the equations of the spinning top $\dot f_1=f_2\,f_3$ (and cyclic) with solutions

$$\begin{aligned} f_1(z)&=K\,\frac{\mathrm{dn}\,Kz}{\mathrm{cn}\,Kz}=\frac{\pi\theta_2\theta_3}{2}\,\frac{\theta_3(z/2)}{\theta_2(z/2)},\quad f_2(z)=Kk'\,\frac{\mathrm{sn}\,Kz}{\mathrm{cn}\,Kz}=\frac{\pi\theta_3\theta_4}{2}\,\frac{\theta_1(z/2)}{\theta_2(z/2)},\\ f_3(z)&=Kk'\,\frac{1}{\mathrm{cn}\,Kz}=\frac{\pi\theta_2\theta_4}{2}\,\frac{\theta_4(z/2)}{\theta_2(z/2)}.\end{aligned} \tag{3.14}$$

(This choice of solution yields the spectral curve (3.1)[12].) These solutions were derived from first principles for the $n=2$ curve in the work of [16] and (with corrections in) [7]. We shall rederive this solution using the recent general approach of [8]; this enables us to introduce a number of functions and their expansions that will be used throughout.

As noted in our general description, $h(z)$ may be constructed directly for a monopole spectral curve [8]. The $n=2$ example of that reference gives

$$h(z)=\frac{1}{K}\begin{pmatrix}f_1&-f_2\\ -f_2&f_1\end{pmatrix},\qquad h^{-1}(z)=\frac{1}{K}\begin{pmatrix}f_1&f_2\\ f_2&f_1\end{pmatrix},\qquad \dot h h^{-1}=-f_3\begin{pmatrix}0&1\\ 1&0\end{pmatrix}. \tag{3.15}$$

To put the Nahm data into standard gauge one solves the differential equation $\alpha=\alpha^\dagger$ (with $h^{-1}=C^\dagger C$)

$$C^{-1}\dot C=\dot C C^{-1}=\frac{f_3}{2}\begin{pmatrix}0&1\\ 1&0\end{pmatrix}.$$

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$$|\eta 1_2-L(\zeta)|=\eta^2+\frac14(f_1^2-f_2^2)\zeta^4+\frac12(f_1^2+f_2^2-2\,f_3^2)\zeta^2+\frac14(f_1^2-f_2^2)=\eta^2+\frac14\boldsymbol{K}^2\left(\zeta^4+2(k^2-k'^2)\zeta^2+1\right)$$

Upon writing

$$C(z) = \begin{pmatrix} F(z) & G(z) \\ G(z) & F(z) \end{pmatrix}, \tag{3.16}$$

we find $\dot{F} = f_3 G/2$, $\dot{G} = f_3 F/2$ with solution

$$F = \cosh\left(\int_0^z f_3(s)ds/2\right) = \left[p(z) + 1/p(z)\right]/2, \quad G = \sinh\left(\int_0^z f_3(s)ds/2\right) = \left[p(z) - 1/p(z)\right]/2,$$

where[13]

$$p(z) = \exp\left(\int_0^z f_3(s)ds/2\right) = \exp\left(k'K\int_0^z \frac{ds}{\mathrm{cn}\,Kz}\right) = \left[\frac{\mathrm{dn}\,Kz + k'\mathrm{sn}\,Kz}{\mathrm{cn}\,Kz}\right]^{1/2}.$$

Now

$$G^2(z) = \frac{1}{2}\left(\frac{\mathrm{dn}(Kz;k)}{\mathrm{cn}(Kz;k)} - 1\right) = \frac{1}{2}\left(\frac{f_1}{K} - 1\right), \qquad 2F(z)G(z) = k'\frac{\mathrm{sn}(Kz;k)}{\mathrm{cn}(Kz;k)} = \frac{f_2}{K}, \tag{3.17}$$

$$F^2 - G^2 = 1, \qquad F^2 + G^2 = \frac{f_1}{K}.$$

The Nahm data now follows from

$$\tilde{T}_3 = \frac{\imath}{2}C(\dot{h}h^{-1})C^{-1} = \frac{\imath}{2}f_3\sigma_1, \qquad \beta = \tilde{T}_1 + \imath\tilde{T}_2 = g\nu g^{-1} = -\frac{\imath}{2}\begin{pmatrix} f_1 & f_2 \\ -f_2 & -f_1 \end{pmatrix},$$

which leads to

$$\tilde{T}_1(z) = \frac{1}{2\imath}f_1(z)\sigma_3, \quad \tilde{T}_2(z) = \frac{1}{2\imath}f_2(z)\sigma_2, \quad \tilde{T}_3(z) = -\frac{1}{2\imath}f_3(z)\sigma_1.$$

Now g and C are only defined up to left multiplication by a constant unitary matrix. Let $\mathcal{O}$ be the orthogonal matrix

$$\mathcal{O} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{3.18}$$

for which

$$\mathcal{O}^{-1}\sigma_1\mathcal{O} = \sigma_3, \qquad \mathcal{O}^{-1}\sigma_2\mathcal{O} = \sigma_2, \qquad \mathcal{O}^{-1}\sigma_3\mathcal{O} = -\sigma_1,$$

With this we obtain

$$T_j(z) = \mathcal{O}\,\tilde{T}_j(z)\mathcal{O}^{-1} = \frac{\sigma_j}{2\imath}\,f_j(z), \quad j = 1, 2, 3. \tag{3.19}$$

3.6.1. *Expansions.* For later use we record

$$\begin{aligned} f_1(1-\xi) &= \frac{1}{\xi} + \frac{1}{6}\,K^2\left(k^2+1\right)\xi + \mathcal{O}(\xi^3), \\ f_2(1-\xi) &= \frac{1}{\xi} - \frac{1}{6}\,K^2\left(-k^2+2\right)\xi + \mathcal{O}(\xi^3), \\ f_3(1-\xi) &= \frac{1}{\xi} + \frac{1}{6}\,K^2\left(-2k^2+1\right)\xi + \mathcal{O}(\xi^3), \end{aligned} \tag{3.20}$$

and note that

$$f_1(\xi - 1) = f_1(1-\xi) + \mathcal{O}(\xi^2),$$

[13] Here we have made use of

$$\int \frac{du}{\mathrm{cn}\,u} = \frac{1}{k'}\ln\frac{\mathrm{dn}u + k'\mathrm{sn}u}{\mathrm{cn}u}.$$

$$f_2(\xi-1) = -f_2(1-\xi) + \mathcal{O}(\xi^2),\\ f_3(\xi-1) = f_3(1-\xi) + \mathcal{O}(\xi^2). \tag{3.21}$$

Then

$$T_j(1-\xi) \sim -\frac{\imath}{2}\frac{\sigma_j}{\xi} + \mathcal{O}(\xi), \quad j=1,2,3, \tag{3.22}$$

$$T_j(-1+\xi) \sim \begin{cases} -\frac{\imath}{2}\frac{\sigma_j}{\xi} + \mathcal{O}(\xi), & j=1,3,\\ \frac{\imath}{2}\frac{\sigma_j}{\xi} + \mathcal{O}(\xi), & j=2. \end{cases} \tag{3.23}$$

The expansion of the entries of $F(z)$ and $G(z)$ in the vicinity of points $z=\pm 1$ may be obtained as follows. Taking into account the expressions for F^2 and G^2 we find that

$$F(\pm 1 \mp \xi) = \pm\left(\frac{1}{\sqrt{\pi}\theta_3(0)\sqrt{\xi}} + \frac{1}{4}\sqrt{\pi}\theta_3(0)\sqrt{\xi} + O(\xi^{3/2})\right)$$
$$G(\pm 1 \mp \xi) = \pm\left(\frac{1}{\sqrt{\pi}\theta_3(0)\sqrt{\xi}} - \frac{1}{4}\sqrt{\pi}\theta_3(0)\sqrt{\xi} + O(\xi^{3/2}\right)$$

The final choice of sign follows from the relation $2F(z)G(x) = k'\mathrm{tn}(Kz;k)$ from which it follows that the coefficients of $\xi^{-1/2}$ in F and G should be of the same sign at $z=1-\xi$ and of opposite sign at $z=-1+\xi$. Therefore we will fix the signs as follows

$$\begin{aligned} F(1-\xi) &= \frac{1}{\sqrt{\pi}\theta_3(0)\sqrt{\xi}} + \frac{1}{4}\sqrt{\pi}\theta_3(0)\sqrt{\xi} + \mathcal{O}(\xi^{3/2}),\\ G(1-\xi) &= \frac{1}{\sqrt{\pi}\theta_3(0)\sqrt{\xi}} - \frac{1}{4}\sqrt{\pi}\theta_3(0)\sqrt{\xi} + \mathcal{O}(\xi^{3/2}),\\ F(-1+\xi) &= -\frac{1}{\sqrt{\pi}\theta_3(0)\sqrt{\xi}} - \frac{1}{4}\sqrt{\pi}\theta_3(0)\sqrt{\xi} + O(\xi^{3/2}),\\ G(-1+\xi) &= \frac{1}{\sqrt{\pi}\theta_3(0)\sqrt{\xi}} - \frac{1}{4}\sqrt{\pi}\theta_3(0)\sqrt{\xi} + \mathcal{O}(\xi^{3/2}). \end{aligned} \tag{3.24}$$

These then yield

$$C(1-\xi) = \frac{1}{\sqrt{\xi}}\frac{1}{\sqrt{2K}}\begin{pmatrix} 1+\xi\,K/2 & 1-\xi\,K/2\\ 1-\xi\,K/2 & 1+\xi\,K/2\end{pmatrix} + \mathcal{O}(\xi^{3/2}), \tag{3.25}$$

$$C(\xi-1) = \frac{1}{\sqrt{\xi}}\frac{1}{\sqrt{2K}}\begin{pmatrix} -1-\xi\,K/2 & 1-\xi\,K/2\\ 1-\xi\,K/2 & -1-\xi\,K/2\end{pmatrix} + \mathcal{O}(\xi^{3/2}). \tag{3.26}$$

3.7. **Asymptotic Expansions.** We consider the expansion of the Weyl equation $\Delta^\dagger v = 0$,

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\left(\sum_{j=1}^{3}\sigma_j\otimes\sigma_j f_j(z)\right) - \left(\sum_{j=1}^{3}\sigma_j\otimes x_j 1_2\right)\right]\boldsymbol{v}(z,\boldsymbol{x}) = 0,$$

in the vicinity of the pole $z=\pm 1$. First, with $z = 1-\xi$ we find from (3.22) and (3.20) the leading behaviour

$$\left[\frac{\mathrm{d}}{\mathrm{d}\xi} - \frac{1}{2\xi}\left(\sum_{j=1}^{3}\sigma_j\otimes\sigma_j\right) + \left(\sum_{j=1}^{3}\sigma_j\otimes x_j 1_2\right) + \mathcal{O}(\xi)\right]\boldsymbol{v}_1(\xi,\boldsymbol{x}) = 0.. \tag{3.27}$$

Now $\frac{1}{2}\sum_{j=1}^{3}\sigma_j\otimes\sigma_j$ has an eigenvector $(0,1,-1,0)^T$ with eigenvalue $-3/2$ and eigenvectors $(0,0,0,1)^T$, $(0,1,1,0)^T$, $(1,0,0,0)^T$ each with eigenvalue $1/2$. The singular solution at $z=1$ then behaves as

$$\boldsymbol{v}_1(\xi,\boldsymbol{x})=\frac{1}{\xi^{3/2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}+\frac{1}{\xi^{1/2}}\begin{pmatrix}\imath\, x_2-x_1\\x_3\\x_3\\\imath\, x_2+x_1\end{pmatrix}+\xi^{1/2}\begin{pmatrix}a\\b-r^2/2\\b+r^2/2\\c\end{pmatrix}+\mathcal{O}(\xi^{3/2}). \tag{3.28}$$

The undetermined coefficients a,b,c here reflect that we can get contributions from the regular solutions that begin at this order.

A similar analysis at $z=-1$ now using (3.23) leads to consideration of the matrix $\frac{1}{2}\left(\sum_{j=1,3}\sigma_j\otimes\sigma_j-\sigma_2\otimes\sigma_2\right)$ which has the eigenvector $(1,0,0,1)^T$ with eigenvalue $3/2$ and eigenvectors $(0,1,0,0)^T$, $(0,0,1,0)^T$, $(1,0,0,-1)^T$ each with eigenvalue $-1/2$. We then obtain the expansion of the singular solution at $z=-1$ to be

$$\boldsymbol{v}_{-1}(\xi,\boldsymbol{x})=\frac{1}{\xi^{3/2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}+\frac{1}{\xi^{1/2}}\begin{pmatrix}-x_3\\\imath\, x_2-x_1\\-\imath\, x_2-x_1\\x_3\end{pmatrix}+\xi^{1/2}\begin{pmatrix}a'-r^2/2\\b'\\c'\\-a'-r^2/2\end{pmatrix}+\mathcal{O}(\xi^{3/2}), \tag{3.29}$$

where again the coefficients a',b',c' reflect that we can get contributions from the regular solutions at this order.

4. Identities

This section will gather a number of useful identities to be used in the sequel. Recall we have defined the Atiyah-Ward equation to be the quartic equation (in general, the degree $2n$ equation) obtained by substituting the mini-twistor relation (2.17) into the equation for the curve (3.1). Throughout we let $\{P_k=(\zeta_k,\eta_k)\}_{k=1}^4$ be the corresponding four solutions to this and denote their Abelian images by

$$\alpha_k=\frac{1}{4}\int_{\infty_1}^{P_k}\frac{\mathrm{d}\zeta}{\eta},\qquad k=1,\dots,4. \tag{4.1}$$

Throughout all our calculations have been checked numerically and some comment now may be helpful. The general Abel image α_k is only defined up to a shift by a lattice point reflecting the ambiguity in choice of contour integral. Abel's theorem tells us that the degree zero divisor $\sum_i(\mathfrak{p}_i-\mathfrak{q}_i)$ corresponds to that of a function if and only if its Abel image is a lattice point. When constructing this function we typically specify sheets by choosing $\sum_i(\mathfrak{p}_i-\mathfrak{q}_i)=0$ (see Mumford [36]). The real structure of our curve will enable us to further specify our choice of contours and we shall see that we may take

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=N\tau,\quad N\in\mathbb{Z}, \tag{4.2}$$

where N reflects a remaining choice of contours in the Abel-map. Assume henceforth that such a choice has been made and N is then fixed for a given set of solutions to the Atiyah-Ward equation. The points may generically be ordered by

$$P_{i+2}=\mathfrak{I}(P_i).$$

We also introduce the important functions

$$\mu_k=\beta_1(P_k)+\imath\,\pi N-\imath\left[(x_1-ix_2)\zeta_k-ix_3-x_4\right] \tag{4.3}$$

and derive a number of relations for them. This function combines the exponential term in Nahm's ansatz (2.35) and that coming from the Baker-Akhiezer function (3.12); there is an additional phase proportional to N that comes from the choice of contours.

The identities we describe are grouped as follows: those that follow from the mini-twistor correspondence and Abel's theorem; those arising from the real structure of the curve; and those related to single points on the curve. Because of (4.2) we may express theta functions with arguments depending on three points to those involving a single point. Next we derive a number of identities for the functions μ defined by (4.3). Finally we derive a number of identities involving theta functions whose arguments have more than one Abel image: some of these hold true for arbitrary arguments and are based on the Weierstrass trisecant identities which (together with other properties of the theta functions) are gathered together in Appendix B; others depend on properties peculiar to our curve. All proofs unless given are deferred to Appendix C.

Before turning to the identities we note that any three solutions to the mini-twistor correspondence may be solved to give the monopole coordinates (x_1, x_2, x_3), or equivalently $x_\pm := x_1 \pm x_2$, x_3. Let $i, j, k, l \in \{1, 2, 3, 4\}$ be distinct solutions of

$$\eta_j = -\imath x_- \zeta_j^2 - 2\zeta_j x_3 - \imath x_+, \quad j = 1, \ldots, 4. \tag{4.4}$$

Then given $\{i, j, k\}$

$$x_- = \frac{\imath \eta_i}{(\zeta_i - \zeta_j)(\zeta_i - \zeta_k)} + \text{cyclic permutations of } i, j, k, \tag{4.5}$$

$$x_+ = \frac{\imath \zeta_j \zeta_k \eta_i}{(\zeta_i - \zeta_j)(\zeta_i - \zeta_k)} + \text{cyclic permutations of } i, j, k, \tag{4.6}$$

$$x_3 = \frac{1}{2} \frac{(\zeta_j + \zeta_k)\eta_i}{(\zeta_i - \zeta_j)(\zeta_i - \zeta_k)} + \text{cyclic permutations of } i, j, k. \tag{4.7}$$

The compatibility condition of all 4 equations (4.4) shows

$$\eta_l = \frac{\eta_i (\zeta_j - \zeta_l)(\zeta_k - \zeta_l)}{(\zeta_i - \zeta_j)(\zeta_i - \zeta_k)} + \text{cyclic permutations of } i, j, k. \tag{4.8}$$

One can check that the permutations of (i, j, k, l) in equation (4.8) leads to a solvable homogeneous system with respect to $\eta_1, \ldots, \eta_4$. By considering the two Atiyah-Ward equations with $\zeta = \zeta_i$ and ζ_j and eliminating the variable k^2 from the two equations by computing resultant one also finds that

$$x_3 = \frac{\imath}{16} \frac{(\zeta_i + \zeta_j)[\zeta_i^2 \zeta_j^2 (K^2 - 4x_-^2) - K^2 + 4x_+^2]}{\zeta_i \zeta_j (x_- \zeta_i \zeta_j - x_+)}. \tag{4.9}$$

We have that

$$\frac{x_- \zeta_i \zeta_j - x_+}{\zeta_i + \zeta_j} + \frac{x_- \zeta_k \zeta_l - x_+}{\zeta_k + \zeta_l} = 0, \quad i \neq j \neq k \neq l \in \{1, 2, 3, 4\} \tag{4.10}$$

or equivalently,

$$x_+(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4) = x_-(\zeta_1\zeta_2\zeta_3 + \zeta_1\zeta_2\zeta_4 + \zeta_1\zeta_3\zeta_4 + \zeta_2\zeta_3\zeta_4)$$

This relation follows from the Atiyah-Ward equation.

4.1. **Derivatives.** In later calculations we will need various derivatives with respect to the spatial coordinates (x_1, x_2, x_3). Implicit differentiation of the Atiyah-Ward equation gives us $\partial_i\zeta := \partial\zeta/\partial x_i$; the differential of (2.17) together with $\partial_i\zeta$ then yields $\partial_i\eta$. From (3.4) we have

$$\partial_i\beta_1(P) = \frac{K^2}{4\eta}\left(\zeta^2 - \frac{2E-K}{K}\right)\partial_i\zeta,$$

from which we obtain $\partial_i\mu(P)$.

4.2. **Reflection.** We see that if $P = (\zeta, \eta)$ is a point on the spectral curve corresponding to $\boldsymbol{x}$ then $P' = (\zeta, -\eta)$ corresponds to $-\boldsymbol{x}$. Now using $\gamma_\infty(P') = -\gamma_\infty(P)$ and that P_0 is a branch point we have

$$\begin{aligned}\beta_1(P') - \frac{\imath\pi}{4} &= \int_{P_0}^{P'} \gamma_\infty(P) = -\int_{P_0}^{P'} \gamma_\infty(P') = -\int_{P_0'}^{P} \gamma_\infty(P) \\ &= -\left[\beta_1(P) - \frac{\imath\pi}{4}\right] + \int_{P_0}^{P_0'} \gamma_\infty(P) = -\beta_1(P) + \frac{\imath\pi}{4}.\end{aligned}$$

Then

$$\mu(P) \equiv -\mu(P') \pmod{\frac{i\pi}{2}}. \tag{4.11}$$

Although we have used the reflected path to define $\beta_1(P)$ the path need not be given this way and there is an ambiguity of half $\mathfrak{b}$-periods.

4.3. **Abel-Jacobi Constraints.** As noted in section 2.3 the solutions to the Atiyah-Ward equation satisfy a number of relations.

Proposition 4.1. *Let the four points $P_i = (\zeta_i, \eta_i)$ $i = 1, \dots, 4$ solve (2.17) for the curve (3.1). Then the following hold*

$$\int_{\infty_1}^{P_1} \boldsymbol{v} + \int_{\infty_1}^{P_2} \boldsymbol{v} + \int_{\infty_1}^{P_3} \boldsymbol{v} + \int_{\infty_1}^{P_4} \boldsymbol{v} = N\tau + M, \quad N, M \in \mathbb{Z} \tag{4.12}$$

and for the choice of paths given by Lemma 3.2

$$\int_{P_0}^{P_1} \gamma_\infty + \int_{P_0}^{P_2} \gamma_\infty + \int_{P_0}^{P_3} \gamma_\infty + \int_{P_0}^{P_4} \gamma_\infty = \frac{4K^2 x_3}{K^2 - 4x_-^2} - \imath\pi(N+1), \tag{4.13}$$

and

$$\sum_k \mu_k = 3\imath\pi N. \tag{4.14}$$

Here $\boldsymbol{v}$ in the normalized holomorphic, γ_∞ the $\mathfrak{a}$-normalized differential of the second kind, and base point $P_0 = (k' + \imath k, 0_1)$.

The proposition is proven in Appendix C.1. If we do not relate the paths $\int_{\infty_1}^{P_k} \boldsymbol{v}$ and $\int_{P_0}^{P_k} \gamma_\infty$ via Lemma 3.2 then (4.13, 4.14) are only defined mod $\imath\pi$.

Using (4.12) and the periodicities of the theta functions (see Appendix B) we have the further relations:

Corollary 4.2. *Set $E_i := e^{i\pi[-N^2\tau + 2N\int_{\infty_1}^{P_i}\boldsymbol{v} + Nz]}$. Then for distinct $i, j, k, l \in \{1, 2, 3, 4\}$ if (4.12) holds we have*

$$\begin{aligned}\theta_1(P_j + P_k + P_l - z/2) &= (-1)^{N+M+1}\, E_i\, \theta_1(P_i + z/2), \\ \theta_2(P_j + P_k + P_l - z/2) &= \qquad (-1)^{M}\, E_i\, \theta_2(P_i + z/2),\end{aligned}$$

$$\theta_3(P_j+P_k+P_l-z/2) = \qquad E_i\,\theta_3(P_i+z/2),$$
$$\theta_4(P_j+P_k+P_l-z/2) = \qquad (-1)^N\,E_i\,\theta_4(P_i+z/2),$$

and similarly

$$\theta_r(P_i+P_j)\theta_r(P_i+P_k)e^{2i\pi N\int_{\infty_1}^{P_i}\boldsymbol{v}} = \theta_r(P_l+P_j)\theta_r(P_l+P_k)e^{2i\pi N\int_{\infty_1}^{P_l}\boldsymbol{v}}.$$

4.4. **Conjugation.** We will need the complex conjugates of the Baker-Akhiezer functions to implement our strategy and we investigate this here. At the outset we note that choices of contours are implicit in the results stated; the proofs make these clear, but they are the natural ones: given a contour λ between two points P and Q, then we integrate between $\mathfrak{I}(P)$ and $\mathfrak{I}(P)$ along $\mathfrak{I}_*(\lambda)$ and so forth.

Proposition 4.3. *Let P_j, $j=1,\ldots,4$ be the solutions of the Atiyah-Ward equation constraint. Then*

$$\overline{\int_{\infty_1}^{P_1} v} = -\int_{\infty_1}^{P_3} v - \frac{\tau}{2},\qquad \overline{\int_{\infty_1}^{P_2} v} = -\int_{\infty_1}^{P_4} v - \frac{\tau}{2}. \tag{4.15}$$

Proof. For the first of relations (4.15) we have

$$\frac{1}{4}\overline{\int_{\infty_1}^{P_1}\frac{\mathrm{d}\zeta}{\eta}} = \frac{1}{4}\int_{\infty_1}^{P_1}\overline{\frac{\mathrm{d}\zeta}{\eta}} = \frac{1}{4}\int_{\infty_1}^{P_1}\frac{\mathrm{d}\overline{\zeta}}{\overline{\eta}} = -\frac{1}{4}\int_{\infty_1}^{P_1}\frac{\mathrm{d}\mathfrak{I}(\zeta)}{\mathfrak{I}(\eta)} = -\frac{1}{4}\int_{\infty_1}^{P_1}\mathfrak{I}\left(\frac{\mathrm{d}\zeta}{\eta}\right)$$
$$= -\frac{1}{4}\int_{\mathfrak{I}(\infty_1)}^{\mathfrak{I}(P_1)}\frac{\mathrm{d}\zeta}{\eta} = -\frac{1}{4}\int_{0_1}^{P_3}\frac{\mathrm{d}\zeta}{\eta} = -\frac{1}{4}\int_{\infty_1}^{P_3}\frac{\mathrm{d}\zeta}{\eta} - \frac{1}{4}\int_{0_1}^{\infty_1}\frac{\mathrm{d}\zeta}{\eta} = -\int_{\infty_1}^{P_3} v - \frac{\tau}{2}.$$

The second relation is proven analogously. ☐

Using the fact that τ is pure imaginary for our curve we have that

$$\sum_{k=1}^{4}\int_{\infty_1}^{P_k} v = -\sum_{k=1}^{4}\overline{\int_{\infty_1}^{P_k} v}$$

whence

Corollary 4.4. *In Proposition 4.1 we have $M=0$.*

The conjugation rule induces the following conjugation rule of theta functions. Again the purely imaginary period matrix for (3.1) yields that

$$\overline{\theta_k(z)} = \theta_k(\overline{z}),\quad k=1,\ldots,4.$$

The following relations are valid

$$\begin{aligned}
\overline{\vartheta_1\left(\int_{\infty_1}^{P_{1,2}} v\right)} &= -\imath\vartheta_4\left(\int_{\infty_1}^{P_{3,4}} v\right)\exp\left\{-\imath\pi\int_{\infty_1}^{P_{3,4}} v - \frac{\imath\pi\tau}{4}\right\},\\
\overline{\vartheta_4\left(\int_{\infty_1}^{P_{1,2}} v\right)} &= \imath\vartheta_1\left(\int_{\infty_1}^{P_{3,4}} v\right)\exp\left\{-\imath\pi\int_{\infty_1}^{P_{3,4}} v - \frac{\imath\pi\tau}{4}\right\},\\
\overline{\vartheta_{2,3}\left(\int_{\infty_1}^{P_{1,2}} v\right)} &= \vartheta_{3,2}\left(\int_{\infty_1}^{P_{3,4}} v\right)\exp\left\{-\imath\pi\int_{\infty_1}^{P_{3,4}} v - \frac{\imath\pi\tau}{4}\right\}.
\end{aligned} \tag{4.16}$$

We will need also:

$$\overline{\vartheta_2\left(\int_{\infty_1}^{P_{1,2}} v - \frac{z}{2}\right)} = \vartheta_3\left(\int_{\infty_1}^{P_{3,4}} v + \frac{z}{2}\right)\exp\left\{-\imath\pi\int_{\infty_1}^{P_{3,4}} v - \frac{\imath\pi z}{2} - \frac{\imath\pi\tau}{4}\right\},$$
$$\overline{\vartheta_4\left(\int_{\infty_1}^{P_{1,2}} v - \frac{z}{2}\right)} = \imath\vartheta_1\left(\int_{\infty_1}^{P_{3,4}} v + \frac{z}{2}\right)\exp\left\{-\imath\pi\int_{\infty_1}^{P_{3,4}} v - \frac{\imath\pi z}{2} - \frac{\imath\pi\tau}{4}\right\}. \tag{4.17}$$

4.5. **The curve.** We shall now prove various properties of theta functions depending on one and (via Corollary 4.2) three distinct solutions to the Atiyah-Ward equation.

Lemma 4.5. *Let $P = (\zeta, \eta) \in \mathcal{C}$ for the curve (3.1). Then*

$$\frac{\theta_1'[P]}{\theta_1[P]} - \frac{\theta_3'[P]}{\theta_3[P]} = 2\imath K\zeta, \tag{4.18}$$

$$\frac{\theta_1''[P]}{\theta_1[P]} - \frac{\theta_3''[P]}{\theta_3[P]} = 8\imath K\left(\zeta\,\beta_1(P) + \eta\right), \tag{4.19}$$

$$\frac{d}{d\alpha(P)}\zeta = 4\eta, \tag{4.20}$$

$$\frac{d}{d\alpha(P)}\eta = -2K^2\left(\zeta^3 + (k^2 - k'^2)\zeta\right), \tag{4.21}$$

where $\beta_1(P)$ was defined in (3.11).

Corollary 4.6. *With $\alpha = \int_{\infty_1}^{P} v$, where $P = (\zeta, \eta) \in \mathcal{C}$ for the curve (3.1), we have*

$$\frac{\theta_1'(\alpha)}{\theta_1(\alpha)} = 2\beta_1(P) + \imath K\zeta, \tag{4.22}$$

$$\frac{\theta_3'(\alpha)}{\theta_3(\alpha)} = 2\beta_1(P) - \imath K\zeta, \tag{4.23}$$

$$\frac{\theta_1''(\alpha)}{\theta_1(\alpha)} = K^2\zeta^2 - 4EK + 2K^2 + 4\beta_1(P)^2 + 4\imath K\zeta\beta_1(P) + 4\imath K\eta, \tag{4.24}$$

$$\frac{\theta_3''(\alpha)}{\theta_3(\alpha)} = K^2\zeta^2 - 4EK + 2K^2 + 4\beta_1(P)^2 - 4\imath K\zeta\beta_1(P) - 4\imath K\eta, \tag{4.25}$$

$$\begin{aligned}\frac{\theta_1'''(\alpha)}{\theta_1(\alpha)} &= 4K[K\zeta + 6\imath\beta_1(P)]\eta + 6K^2\beta_1(P)\zeta^2 - 24KE\beta_1(P) + 12K^2\beta_1(P)\\ &\quad + 8\beta_1(P)^3 - 3\imath K^3\zeta^3 - 2\imath K[-8k'^2K^2 + 6KE + K^2 - 6\beta_1(P)^2]\zeta,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\frac{\theta_3'''(\alpha)}{\theta_3(\alpha)} &= 4K[K\zeta - 6\imath\beta_1(P)]\eta + 6K^2\beta_1(P)\zeta^2 - 24KE\beta_1(P) + 12K^2\beta_1(P)\\ &\quad + 8\beta_1(P)^3 + 3\imath K^3\zeta^3 + 2\imath K[-8k'^2K^2 + 6KE + K^2 - 6\beta_1(P)^2]\zeta.\end{aligned} \tag{4.27}$$

Corollary 4.7. *Let $i, j, k, l \in \{1, 2, 3, 4\}$ be distinct with α_i the Abel image of P_i subject to $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = N\tau$. The following relations are valid:*

$$\frac{\theta_1'(\alpha_i + \alpha_j + \alpha_k)}{\theta_1(\alpha_i + \alpha_j + \alpha_k)} = -2\beta_1(\alpha_l) - 2\imath\pi N - \imath K\zeta_l, \tag{4.28}$$

$$\frac{\theta_3'(\alpha_i + \alpha_j + \alpha_k)}{\theta_3(\alpha_i + \alpha_j + \alpha_k)} = -2\beta_1(\alpha_l) - 2\imath\pi N + \imath K\zeta_l, \tag{4.29}$$

$$\frac{\theta_1''(\alpha_i+\alpha_j+\alpha_k)}{\theta_1(\alpha_i+\alpha_j+\alpha_k)} = K^2\zeta_l^2 - 4\pi^2N^2 + 8\imath\beta_1(\alpha_l)\pi N - 2(2EK-K) + 4\beta_l^2 - 4K(\pi N - \imath\beta_1(\alpha_l))\zeta_l + 4\imath K\eta_l, \tag{4.30}$$

$$\frac{\theta_3''(\alpha_i+\alpha_j+\alpha_k)}{\theta_3(\alpha_i+\alpha_j+\alpha_k)} = K^2\zeta_l^2 - 4\pi^2N^2 + 8\imath\beta_1(\alpha_l)\pi N - 2(2EK-K) + 4\beta_l^2 + 4K(\pi N - \imath\beta_1(\alpha_l))\zeta_l - 4\imath K\eta_l. \tag{4.31}$$

4.6. **Conjugation and properties of μ_k.** We may now use the results of the previous subsections to place useful constraints on the μ_k's.

Proposition 4.8. *With μ_k defined by (4.3) the ordering $\mathcal{J}(P_i) = P_{i+2}$ and the conjugate contours then the following relations for μ_k are valid for all (x_1, x_2, x_3)*

$$\mu_1 + \overline{\mu}_3 = -\frac{\imath\pi}{2}, \quad \mu_2 + \overline{\mu}_4 \equiv -\frac{\imath\pi}{2}. \tag{4.32}$$

Proof. Upon noting that $\mathcal{J}(P_1) = P_3$ we have

$$\begin{aligned}\mu_1 + \overline{\mu}_3 &= [\beta_1(P_1) - x_3 - (x_2 + \imath x_1)\zeta_1] + \overline{[\beta_1(P_3) - x_3 - (x_2 + \imath x_1)\zeta_3]} \\ &= \beta_1(P_1) + \overline{\beta_1(P_3)} - (x_2 + \imath x_1)\zeta_1 - 2x_3 + (x_2 - \imath x_1)\frac{1}{\zeta_1} \\ &= \beta_1(P_1) + \overline{\beta_1(P_3)} + \frac{\eta_1}{\zeta_1}.\end{aligned}$$

Taking into account (3.8) and (4.15) we find

$$\mu_1 + \overline{\mu}_3 = \frac{1}{4}\left[\frac{\theta_1'(P_1)}{\theta_1(P)} + \frac{\theta_3'(P_1)}{\theta_3(P)} - \frac{\theta_2'(P_1)}{\theta_2(P)} - \frac{\theta_4'(P_1)}{\theta_4(P)}\right] - \frac{\imath\pi}{2} + \frac{\eta_1}{\zeta_1}.$$

Upon using the representation,

$$\zeta(P) = -\imath\frac{\theta_2(P)\theta_4(P)}{\theta_1(P)\theta_3(P)}$$

this may be transformed into the form

$$\mu_1 + \overline{\mu}_3 = -\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\alpha_1}\ln\zeta(\alpha_1) + \frac{\eta_1}{\zeta_1} - \frac{\imath\pi}{2}.$$

Finally, the first two terms cancel because of the relation (4.20). Thus the first of the stated relation follows; the second is proven in analogous way. □

Here we have chosen contours in a specified way; if we had chosen arbitrary contours then the relations are only defined mod $\imath\pi$. We also prove in Appendix C.5 that

Proposition 4.9. Suppose that $\zeta_1 + \zeta_* = 0$, $\zeta_2 + \zeta_{*'} = 0$. Then for the (x_1, x_2) plane, $x_3 = 0$,

$$\mu_1 + \mu_* \equiv 0 \pmod{\imath\pi}, \quad \mu_2 + \mu_{*'} \equiv 0 \pmod{\imath\pi}. \tag{4.33}$$

4.7. **Combined results.** We next present a number of nontrivial identities that arise from the Weierstrass trisecant identities or combining our expressions for the curve together with the Weierstrass trisecant identities.

Lemma 4.10. *For arbitrary $\alpha_i, \alpha_j, \alpha_k$ the following relations are valid*

$$\begin{aligned}
&[\theta_1(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k)+\theta_3(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)] \\
&\qquad\qquad \times\theta_1(\alpha_i-\alpha_j)\theta_1(\alpha_j-\alpha_k)\theta_1(\alpha_i-\alpha_k) \\
&= \sum_{\text{cyclic permutations } i,j,k} \theta_1(\alpha_i)\theta_3(\alpha_i)\theta_1(\alpha_j)\theta_3(\alpha_j)\theta_1(\alpha_i-\alpha_j)\theta_3(\alpha_i+\alpha_j)\theta_2(2\alpha_k),
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
&\theta_1(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k)-\theta_3(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k) \\
&\qquad = -\theta_3(0)\theta_1(\alpha_i+\alpha_j)\theta(\alpha_j+\alpha_k)\theta_1(\alpha_i+\alpha_k).
\end{aligned} \tag{4.35}$$

Lemma 4.11. *Let $i,j,k,l \in \{1,2,3,4\}$ be distinct and $P_{i,j,k,l}$ be points on the curve corresponding to solutions of the Atiyah-Ward equation. Let α_i be the Abel image of P_i subject to $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = N\tau$. Set $\beta_i := \beta_1(P_i)$, $x_\pm = x_1 \pm \imath x_2$ and $\mu_i = \beta_i + \imath\pi N - (x_2+\imath x_1)\zeta_i - x_3$. The following relations are valid*

$$\pi\theta_3(0)\frac{\theta_1(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k)}{\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k)} = 2x_- - K, \tag{4.36}$$

$$\pi\theta_3(0)\frac{\theta_3(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)}{\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k)} = 2x_- + K, \tag{4.37}$$

$$\begin{aligned}
&\theta_1(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k)+\theta_3(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k) \\
&\qquad = 4\frac{x_1-\imath x_2}{\pi\theta_3(0)}\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_i+\alpha_k),
\end{aligned} \tag{4.38}$$

$$\frac{\theta_1'(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k)}{\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k)} = -(2\beta_1(\alpha_l)+2\imath\pi N+\imath K\zeta_l)\frac{2x_- - K}{\pi\theta_3(0)}, \tag{4.39}$$

$$\frac{\theta_3'(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)}{\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k)} = -(2\beta_1(\alpha_l)+2\imath\pi N-\imath K\zeta_l)\frac{2x_- + K}{\pi\theta_3(0)}, \tag{4.40}$$

$$\begin{aligned}
&\theta_1'(\alpha_j+\alpha_k+\alpha_l)\theta_1(\alpha_j)\theta_1(\alpha_k)\theta_1(\alpha_l)-\theta_3'(\alpha_j+\alpha_k+\alpha_l)\theta_3(\alpha_j)\theta_3(\alpha_k)\theta_3(\alpha_l) \\
&\quad = 2(\mu_i+x_3)\theta_3(0)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_k+\alpha_l)\theta_3(\alpha_j+\alpha_l),
\end{aligned} \tag{4.41}$$

(4.42)
$$\frac{\theta_1''(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k)}{\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k)} = [-4K(\pi N - \imath\beta_1(\alpha_l)) + 4\imath K\eta_l + K^2\zeta_l^2 - 4\pi^2N^2 + 8\imath\pi\beta_1(\alpha_l)N - 2K(2E-K) + 4\beta_1(\alpha_l)^2] \, \frac{2x_- - K}{\pi\theta_3(0)},$$

(4.43)
$$\frac{\theta_3''(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)}{\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k)} = [4K(\pi N - \imath\beta_1(\alpha_l)) - 4\imath K\eta_l + K^2\zeta_l^2 - 4\pi^2N^2 + 8\imath\pi\beta_1(\alpha_l)N - 2K(2E-K) + 4\beta_1(\alpha_l)^2] \, \frac{2x_- + K}{\pi\theta_3(0)},$$

(4.44)
$$\frac{\theta_3''(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k) - \theta_1''(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k)}{\theta_3(0)\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_k+\alpha_k)} = K^2\zeta_l^2 + 8(\pi N - \imath\beta_1(\alpha_l))x_- - 4(\pi N - \imath\beta_1(\alpha_l))^2 - 8\imath\eta_l x_- - 2K(2E-K),$$

(4.45)
$$\imath\pi\zeta_l\theta_3(0)^2[\theta_1(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k) + \theta_3(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)] = -2[\theta_1'(\alpha_i+\alpha_j+\alpha_k)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k) - \theta_3'(\alpha_i+\alpha_j+\alpha_k)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)] + 4\theta_3(0)(\imath\pi N + \beta_l)\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_i+\alpha_k).$$

We note that from (4.36, 4.37) we may also express

$$x_- = -\frac{\pi\theta_3^2(0)}{4}\frac{\theta_1(\alpha_l)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k) + \theta_3(\alpha_l)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)}{\theta_1(\alpha_l)\theta_1(\alpha_i)\theta_1(\alpha_j)\theta_1(\alpha_k) - \theta_3(\alpha_l)\theta_3(\alpha_i)\theta_3(\alpha_j)\theta_3(\alpha_k)}. \tag{4.46}$$

Finally we consider expressions of the form

$$\frac{\theta_3'(\alpha_i+\alpha_j)}{\theta_3(\alpha_i+\alpha_j)}, \quad i,j,\in\{1,2,3,4\}, \tag{4.47}$$

involving the addition of two points. Using that $\alpha_k+\alpha_l = N\tau - \alpha_i - \alpha_j$ we first note that

$$\frac{\theta_3'(\alpha_i+\alpha_j)}{\theta_3(\alpha_i+\alpha_j)} + \frac{\theta_3'(\alpha_k+\alpha_l)}{\theta_3(\alpha_k+\alpha_l)} = -2\imath\pi N. \tag{4.48}$$

Proposition 4.12. *Let $i,j,k,l \in \{1,2,3,4\}$ be distinct and $P_{i,j,k,l}$ be points on the curve corresponding to solutions of the Atiyah-Ward equation. Let α_i be the Abel image of P_i subject to $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = N\tau$. Then*

$$\frac{\theta_3'(\alpha_i+\alpha_j)}{\theta_3(\alpha_i+\alpha_j)} = 2(\mu_i+\mu_j) \mod(2\imath\pi), \tag{4.49}$$

$$\frac{\theta_1'(\alpha_i+\alpha_j)}{\theta_1(\alpha_i+\alpha_j)} = 2(\mu_i+\mu_j) + \frac{4\imath(x_-\zeta_i\zeta_j - x_+)}{\zeta_i+\zeta_j} \mod(2\imath\pi), \tag{4.50}$$

$$\frac{\theta_3''(\alpha_i+\alpha_j)}{\theta_3(\alpha_i+\alpha_j)} = 4(2\imath x_3 - x_-(\zeta_k+\zeta_l))^2 + 4(\imath\pi N - \mu_k - \mu_l)^2 - K^2(\zeta_k+\zeta_l)^2 - 4K(E - Kk'^2) \mod(2\imath\pi), \tag{4.51}$$

$$\frac{\theta_1''(\alpha_i+\alpha_j)}{\theta_1(\alpha_i+\alpha_j)} = \frac{\theta_3''(\alpha_i+\alpha_j)}{\theta_3(\alpha_i+\alpha_j)} + 4K^2\zeta_i^2\zeta_j^2 + 16\imath(\beta_i+\beta_j)\frac{x_-\zeta_i\zeta_j - x_+}{\zeta_i+\zeta_j} \mod(2\imath\pi). \tag{4.52}$$

Corollary 4.13. *Let $i,j,k,l \in \{1,2,3,4\}$ be distinct and $P_{i,j,k,l}$ be points on the curve corresponding to solutions of the Atiyah-Ward equation. Let α_i be the Abel image of P_i subject to $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = N\tau$. Then with the paths specified in Lemma (3.2) the following relation is valid*

$$\frac{\theta_3'(\alpha_i+\alpha_j)}{\theta_3(\alpha_i+\alpha_j)} = -2(\mu_k+\mu_j) + 2\imath\pi N. \tag{4.53}$$

Observe the consistency between (4.14, 4.48) and (4.53).

5. Special Loci

In this section we shall describe those loci in $\mathbb{R}^3$ where the general analysis simplifies, in particular describing the corresponding transcendent μ. These loci include each of the coordinate axes and we shall give an alternate parameterization for these involving Jacobi elliptic functions that will facilitate comparison with results in the literature.

For charge 2 the Atiyah-Ward equation is a general quartic and while this is solvable, there are several simplifying loci for which it becomes a biquadratic. This occurs for the x_3-axis and the $x_3 = 0$ plane, the latter including the $x_{1,2}$ axes.

***x_3*-axis.** Here $\eta = -2x_3\zeta$. For $x_3 < Kk'/2$ we may take

$$\zeta_1 = \frac{i\sqrt{K^2k^2+4\,{x_3}^2} + \sqrt{K^2{k'}^2 - 4\,{x_3}^2}}{K}.$$

Then $|\zeta_1| = 1$ and so with $P_1 = (\zeta_1, \eta_1)$ our ordering yields $P_3 = (-1/\overline{\zeta_1}, -\overline{\eta_1}/\overline{\zeta_1}^2) = -(\zeta_1,\eta_1)$. We may then take $P_2 = (\overline{\zeta_1}, \overline{\eta_1})$ and $P_4 = -(\overline{\zeta_1}, \overline{\eta_1})$. For this range $|\zeta_i| = 1$.

In general we may set $\beta = k^2 - {k'}^2 + 8x_3^2/K^2$ and

$$\zeta_1 = i\left[\sqrt{\frac{\beta+1}{2}} - \sqrt{\frac{\beta-1}{2}}\right]$$

which reduces to the previous. For $x_3 > Kk'/2$ now ζ_1 is purely imaginary and

$$\zeta_3 = -i\left[\sqrt{\frac{\beta+1}{2}} + \sqrt{\frac{\beta-1}{2}}\right],\ \zeta_2 = -i\left[\sqrt{\frac{\beta+1}{2}} - \sqrt{\frac{\beta-1}{2}}\right] = -\zeta_1 = \overline{\zeta}_1,\ \zeta_4 = -\zeta_3.$$

***$x_3 = 0$*-plane.** We give the solutions of the general plane $x_3 = 0$ before specialising to the simpler cases of the x_1 and x_2 axes. Upon setting $x_\pm = x_1 \pm \imath x_2$ and $\eta = -\imath x_-\zeta^2 - \imath x_+$ the Atiyah-Ward equation becomes

$$\left(\frac14 K^2 - x_-^2\right)\zeta^4 + \left(\frac12 K^2 - K^2{k'}^2 - 2x_+x_-\right)\zeta^2 + \left(\frac14 K^2 - x_+^2\right) = 0.$$

This has solutions $\tilde\zeta_i(\boldsymbol{x})$, $i \in \{1,\ldots,4\}$

$$\begin{aligned}
\tilde\zeta_1(x_1,x_2) &= S_3^{-1}\sqrt{2K^2{k'}^2 + 2KS_1 - S_4^2}, &\quad \tilde\zeta_2(x_1,x_2) &= S_3^{-1}\sqrt{2K^2{k'}^2 - 2KS_1 - S_4^2},\\
\tilde\zeta_3(x_1,x_2) &= -\tilde\zeta_1(x_1,x_2), & \tilde\zeta_4(x_1,x_2) &= -\,\tilde\zeta_2(x_1,x_2),
\end{aligned}$$

where we have set

$$S_1 = S_1(x_1, x_2) = \sqrt{-k'^2(K^2k^2 - 4x_+x_-) + (x_+ - x_-)^2}, \quad S_2 = S_2(x_1, x_2) = \sqrt{K^2 - 4x_+^2},$$
$$S_3 = S_3(x_1, x_2) = \sqrt{K^2 - 4x_-^2}, \qquad S_4 = S_4(x_1, x_2) = \sqrt{K^2 - 4x_+x_-}.$$

We need to order the roots. We are free to choose $\zeta_1 = \tilde{\zeta}_1$ once and for all, but the choice of $\zeta_3 := -1/\overline{\zeta}_1$ depends on (x_1, x_2) according to whether $\pm S_1^2 > 0$. Noting that $\tilde{\zeta}_1\tilde{\zeta}_2 = S_2/S_3$ then

$$\overline{\zeta}_1 = \begin{cases} \tilde{\zeta}_2 \dfrac{S_3}{S_2} = \dfrac{1}{\tilde{\zeta}_1} = -\dfrac{1}{\tilde{\zeta}_3} & \text{if } S_1^2 < 0, \\ \tilde{\zeta}_1 \dfrac{S_3}{S_2} = \dfrac{1}{\tilde{\zeta}_2} = -\dfrac{1}{\tilde{\zeta}_4} & \text{if } S_1^2 > 0. \end{cases}$$

Thus

$$\zeta_3 = \tilde{\zeta}_3 \text{ if } S_1^2 < 0, \qquad \zeta_3 = \tilde{\zeta}_4 \text{ if } S_1^2 > 0.$$

We observe that if $S_1^2 < 0$ we have four roots ζ_i each of modulus 1. We are similarly free to choose $\zeta_2 = \tilde{\zeta}_2$, giving a double root $\zeta_1 = \zeta_2$ when $S_1^2 = 0$. As we cross $S_1^2 = 0$ the ordering of P_3 and P_4 interchanges:

$$\begin{aligned} S_1^2 < 0: &\qquad \zeta_1 = \tilde{\zeta}_1,\ \zeta_2 = \tilde{\zeta}_2,\ \zeta_3 = \tilde{\zeta}_3,\ \zeta_4 = \tilde{\zeta}_4, \\ S_1^2 > 0: &\qquad \zeta_1 = \tilde{\zeta}_1,\ \zeta_2 = \tilde{\zeta}_2 = 1/\overline{\zeta}_1,\ \zeta_3 = \tilde{\zeta}_4,\ \zeta_4 = \tilde{\zeta}_3. \end{aligned}$$

The solutions are ordered to satisfy

$$\begin{aligned} \zeta_1(\mathbf{0}) &= k' + \imath k, & \zeta_2(\mathbf{0}) &= k' - \imath k, \\ \zeta_3(\mathbf{0}) &= -k' - \imath k, & \zeta_4(\mathbf{0}) &= -k' + \imath k. \end{aligned} \tag{5.1}$$

When we restrict to the coordinate axes we can say more.

***x*₂-axis.** On the x_2-axis $S_1^2 < 0$ and so each $|\zeta_i| = 1$. Further, as $\eta = x_2(1 - \zeta^2)$ is invariant under complex conjugation we have the four points

(5.2)

$$P_1 = (\zeta, \eta), \quad P_3 = (-1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2) = (-\zeta, \eta), \quad P_2 = (\overline{\zeta}, \overline{\eta}), \quad P_4 = (-1/\zeta, -\eta/\zeta^2) = (-\overline{\zeta}, \overline{\eta}).$$

Explicitly we take

$$\zeta_1 = \frac{\imath Kk + \sqrt{K^2k'^2 + 4x_2^2}}{\sqrt{K^2 + 4x_2^2}}, \tag{5.3}$$

***x*₁-axis.** Finally consider the x_1-axis. Here $\eta = -\imath x_1(1 + \zeta^2)$ is invariant under $(\zeta, \eta) \to (\pm\overline{\zeta}, -\overline{\eta})$. We have ordered the points $P_1 = (\zeta, \eta)$, $P_3 = (-1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2)$, and with $P_4 = \mathcal{J}(P_2)$ we are left to determine $P_2 = (\zeta_2, \eta_2)$. We determine these depending on the sign of S_1^2.

$$\begin{array}{llll} (5.4) \quad & x_1\text{-axis:} & P_1 = (\zeta, \eta), & P_3 = (-1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2) = (-\zeta, \eta), \\ & S_1^2 < 0 & P_2 = (\overline{\zeta}, -\overline{\eta}), & P_4 = (-1/\zeta, \eta/\zeta^2) = (-\overline{\zeta}, \overline{\eta}). \end{array}$$

We take here

$$\zeta_1 = \zeta_1(x_1) = \frac{Kk' + \imath\sqrt{K^2k^2 - 4x_1^2}}{\sqrt{K^2 - 4x_1^2}}. \tag{5.5}$$

For $S_1^2 > 0$ we have chosen $\overline{\zeta_1} = 1/\zeta_2$.

$$\text{(5.6)} \qquad x_1\text{-axis:} \quad P_1 = (\zeta, \eta), \quad P_3 = (-1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2),$$
$$S_1^2 > 0 \quad P_2 = (1/\overline{\zeta}, -\overline{\eta}/\overline{\zeta}^2), \quad P_4 = (-\zeta, \eta).$$

We remark that the solutions $\zeta_i(\boldsymbol{x}) = \zeta_i(x_1, x_2)$ are singular on the x_1-axis when

$$\text{(5.7)} \qquad \zeta_i(K/2, 0) = \infty, \quad i = 1, \ldots, 4.$$

At this point the degree of the AWC is no longer a quartic. With $0 \le k \le 1$ we see that ζ_1 has a different analytic behaviour for each of the following domains of the x_1-axis

$$\begin{aligned} \mathbf{I}: &\qquad x_1 \in [-Kk/2,\ kK/2]\,, \\ \mathbf{II}: &\qquad x_1 \in [-K/2,\ -kK/2] \cup [Kk/2,\ K/2]\,, \\ \mathbf{III}: &\qquad x_1 \in [-\infty,\ -K/2] \cup [K/2,\ \infty]\,. \end{aligned}$$

The interval **I** corresponds to $S_1^2 < 0$ and we have $\overline{\zeta}_1 = \zeta_2$. On interval **II** we have that ζ_1 is real while on interval **III** it is purely imaginary. It follows from $\eta = -\imath x_1(1 + \zeta^2)$ that η is purely imaginary on intervals **II** and **III**.

5.1. $\boldsymbol{\mu_i}$**.** We now calculate the transcendents μ for these loci.

$\boldsymbol{x_2}$**-axis.** The invariance under complex conjugation gives

$$\overline{\alpha(P_1)} = \overline{\int_{\infty_1}^{P_1} v} = \int_{\overline{\infty_1}}^{\overline{P_1}} v = \int_{\overline{\infty_1}}^{P_2} v = \int_{\infty_1}^{P_2} v + \int_{\overline{\infty_1}}^{\infty_1} v = \alpha(P_2) + \int_{\infty_2}^{\infty_1} v = \alpha(P_2) + \frac{1+\tau}{2},$$

where we have used our definition of sheets to give $\overline{\infty_1} = \infty_2$. Therefore, from (3.11), we have that

$$\overline{\beta(P_1)} = \beta(P_2) - \frac{\imath\pi}{2}$$

and consequently from (4.3) and that $\overline{\zeta}_1 = \zeta_2$ we see that

$$\overline{\mu}_1 = \mu_2 - \frac{\imath\pi}{2}.$$

Combining this with Proposition (4.9) we obtain

Proposition 5.1. *The transcendents μ_i on the x_2-axis are given* (mod $\imath\pi$) *by:*

$$\text{(5.8)} \qquad \mu_1 = \lambda_2 + \frac{\imath\pi}{4},\ \mu_2 = \lambda_2 + \frac{\imath\pi}{4},\ \mu_3 = -\lambda_2 - \frac{\imath\pi}{4},\ \mu_4 = -\lambda_2 - \frac{\imath\pi}{4},\ \lambda_2 \in \mathbf{R}.$$

Thus on the x_2-axis there is only one transcendental function $\lambda_2 = \lambda_2(x_2)$ to evaluate; we shall identify this function shortly. Now

$$\text{(5.9)} \qquad \mu_1(\mathbf{0}) = \mu_2(\mathbf{0}) = \frac{\imath\pi}{4}, \quad \mu_3(\mathbf{0}) = \mu_4(\mathbf{0}) = \frac{3\imath\pi}{4}.$$

The first equality is evident as $\zeta_1(\mathbf{0}) = k' + \imath k = a$. To prove $\mu_2(\mathbf{0}) = \imath\pi/4$ we note

$$\frac{\imath K}{2} \int_{k'+\imath k}^{k'-\imath k} \frac{(z^2 - c)\mathrm{d}z}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = 0$$

because of the normalization condition $\int_{\mathfrak{a}} \gamma_\infty = 0$. The remaining equalities follow from the propositions. Thus we have that $\lambda_2(0) = 0$.

$\boldsymbol{x_1}$**-axis.** The behaviour of μ depends on which interval x_1 belongs. A similar proof to the above (given in Appendix C.9) shows that:

Proposition 5.2. *The transcendents $\mu_{1,2}$ on the x_1-axis behave* $(\mathrm{mod}\ \imath\pi)$ *as*

$$\begin{aligned}
&\mathbf{I}: \mu_1(x_1,0)=\lambda_1+\frac{\imath\pi}{4}, \quad \mu_2(x_1,0)=-\lambda_1+\frac{\imath\pi}{4}, \mu_3=-\lambda_1-\frac{\imath\pi}{4},\ \mu_4=\lambda_1-\frac{\imath\pi}{4},\\
&\mathbf{II}: \mu_1=\imath\lambda_1',\ \mu_2=-\imath\lambda_1'-\frac{\imath\pi}{2},\ \mu_3=\imath\lambda_1'+\frac{\imath\pi}{2},\ \mu_4=-\imath\lambda_1',\\
&\mathbf{III}: \mu_1=\lambda_1'',\ \mu_2=\lambda_1''-\frac{\imath\pi}{2},\ \mu_3=-\lambda_1''+\frac{\imath\pi}{2},\ \mu_4=-\lambda_1'',
\end{aligned} \tag{5.10}$$

where $\lambda_1=\lambda_1(x_1)$, $\lambda_1'=\lambda_1'(x_1)$, $\lambda_1''=\lambda''(x_1)\in\mathbb{R}$ are such that

$$0=\lambda_1(0)=\lambda_1(\pm Kk/2)=\lambda_1''(\pm K/2), \qquad \lambda_1'(\pm Kk/2)=\frac{\pi}{4},$$

and $\lambda_1'(\pm K/2)=0\ (\mathrm{mod}\ \pi)$.

$x_3=0$-plane. More generally the same symmetry arguments together with (5.2), (5.4) and (5.6) show that

Corollary 5.3. *For the plane $x_3=0$ and the choices above, we have that* $(\mathrm{mod}\ \imath\pi)$

$$\begin{aligned}
&S_1^2<0: \quad \mu_1=\lambda+\frac{\imath\pi}{4},\ \mu_3=-\lambda-\frac{\imath\pi}{4},\ \mu_2=\lambda'+\frac{\imath\pi}{4},\ \mu_4=-\lambda'-\frac{\imath\pi}{4},\\
&S_1^2>0: \quad \mu_1=\lambda''+\imath\alpha,\ \mu_3=-\lambda''+\imath\alpha+\frac{\imath\pi}{2},\ \mu_2=\lambda''-\imath\alpha-\frac{\imath\pi}{2},\ \mu_4=-\lambda''-\imath\alpha,
\end{aligned}$$

where $\lambda,\lambda',\lambda'',\alpha\in\mathbb{R}$.

We observe that Proposition 4.8 tells us that in the general case we have two complex functions to consider; Proposition 4.9 and Corollary 5.3 reduces this to two real functions. Although the transcendents $\mu_1(\boldsymbol{x})$ and $\mu_2(\boldsymbol{x})$ are analytically independent they are related when we reduce to either of the axes $(x_1,0)$, $(0,x_2)$.

We record:

Proposition 5.4. *The derivatives of $\mu_{1,2}$ are*

$$\begin{aligned}
\frac{\partial}{\partial x_1}\mu_1(x_1,x_2) &= -\frac{2\imath(\imath(E-K)x_2-Ex_1)(-\imath S_1x_2+Kk'^2x_1)K\zeta_1}{S_1(x_-\zeta_1^2+x_+)x_+S_3^2}\\
&\quad -\frac{\imath(2EK-S_4^2)(Kk'^2+S_1)x_1}{S_1\zeta_1x_+S_3^2},\\
\frac{\partial}{\partial x_1}\mu_2(x_1,x_2) &= \frac{2\imath(\imath(E-K)x_2-Ex_1)(\imath S_1x_2+Kk'^2x_1)K\zeta_2}{S_1(x_-\zeta_2^2+x_+)x_+S_3^2}\\
&\quad +\frac{\imath(2EK-S_4^2)(Kk'^2-S_1)x_1}{S_1\zeta_2x_+S_3^2},\\
\frac{\partial}{\partial x_2}\mu_1(x_1,x_2) &= -\frac{2\imath(\imath(E-K)x_2-Ex_1)(\imath S_1x_1-Kk^2x_2)K\zeta_1}{S_1(x_-\zeta_1^2+x_+)x_+S_3^2}\\
&\quad -\frac{\imath(2EK-S_4^2)(-Kk^2+S_1)x_2}{S_1\zeta_1x_+S_3^2},\\
\frac{\partial}{\partial x_2}\mu_2(x_1,x_2) &= -\frac{2\imath(\imath(E-K)x_2-Ex_1)(\imath S_1x_1+Kk^2x_2)K\zeta_2}{S_1(x_-\zeta_2^2+x_+)x_+S_3^2}\\
&\quad -\frac{\imath(2EK-S_4^2)(Kk^2+S_1)x_2}{S_1\zeta_2x_+S_3^2}.
\end{aligned} \tag{5.11}$$

x_3-axis. We show in Appendix C.10 that

Proposition 5.5. *The transcendents μ_i on the x_3-axis are given* (mod $\imath\pi$) *by:*

$$x_3 < \frac{Kk'}{2}: \quad \mu_1 = \imath\lambda \qquad \mu_2 = -\imath\lambda - \frac{\imath\pi}{2}, \quad \mu_3 = \imath\lambda + \frac{\imath\pi}{2}, \quad \mu_4 = -\imath\lambda, \qquad \lambda \in \mathbf{R},$$
$$x_3 > \frac{Kk'}{2}: \quad \mu_1 = \lambda_3' - \frac{\imath\pi}{2}, \quad \mu_2 = \lambda_3', \qquad \mu_3 = -\lambda_3', \qquad \mu_4 = -\lambda_3' + \frac{\imath\pi}{2}, \quad \lambda_3' \in \mathbf{R}.$$

Again there is only one transcendental function to evaluate.

5.2. **Parameterizing of the axes in terms of Jacobi's Elliptic functions.** To compare with existing results shall need in the sequel to parameterize the axes in terms of Jacobi's Elliptic functions. For reasons that will later be clearer we take

$$x_1: \quad \mathrm{sn}^2(t) = \frac{4x_1^2}{k^2K^2}, \qquad \zeta^2 = \frac{k'^2 - k^2\mathrm{cn}^2(t) \pm 2\imath kk'\mathrm{cn}(t)}{\mathrm{dn}^2(t)}, \qquad \zeta = \pm\frac{k' \pm \imath k\,\mathrm{cn}(t)}{\mathrm{dn}(t)},$$
$$x_2: \quad \mathrm{dn}^2(t) = -\frac{4x_2^2}{K^2} \qquad \zeta^2 = 1 + \frac{-2 \pm 2\,\mathrm{cn}(t)}{\mathrm{sn}^2(t)}, \qquad \zeta = \pm\imath\frac{1 \pm \mathrm{cn}(t)}{\mathrm{sn}(t)},$$
$$x_3: \quad \mathrm{cn}^2(t) = -\frac{4x_3^2}{k^2K^2}, \quad \zeta^2 = 2\,\mathrm{dn}^2(t) - 1 \pm 2\imath k\,\mathrm{sn}(t)\mathrm{dn}(t), \quad \zeta = \pm(\mathrm{dn}(t) \pm \imath k\,\mathrm{sn}(t)).$$

There are 4 choices of ζ in each of the above corresponding to the 4 signs, each point giving a solution to the Atiyah-Ward equation. Given one solution the other solutions are generated by $t \to t + 2K$ and $t \to t + 2\imath K'$ so only one solution must be found. Further we note that if

$$t = t' + \imath K' = t'' + K + \imath K' \quad \text{then} \quad \imath\frac{1 + \mathrm{cn(t)}}{\mathrm{sn(t)}} = \mathrm{dn(t')} + \imath\mathrm{k}\,\mathrm{sn(t')} = \frac{\mathrm{k'} + \imath\mathrm{k}\,\mathrm{cn(t'')}}{\mathrm{dn(t'')}}. \tag{5.12}$$

Previously we have parameterised $\zeta = -i\,\theta_2[P]\theta_4[P]/\theta_1[P]\theta_3[P]$, and now we have, for example on the x_2-axis

$$\zeta = \pm\imath\frac{1 \pm \mathrm{cn}(t)}{\mathrm{sn}(t)} = \pm\imath\,\frac{\theta_2\theta_4(z) \pm \theta_4\theta_2(z)}{\theta_3\theta_1(z)}$$

where $z = t/(2K)$. Can we relate the Abel images of P to z? We show in Appendix C.11

Proposition 5.6. *The Jacobi parameterizations of the axes, given above, follow upon taking $z = -2\,\alpha(P) - 1/2 - \tau/2$ for the x_1-axis, $z = -2\,\alpha(P)$ for the x_2-axis and $z = -2\,\alpha(P) - \tau/2$ for the x_3-axis.*

5.3. **Expressions for μ in terms of the Jacobi Zeta function.** We have as yet to identify the functions $\lambda_{1,2,3}$ beyond their definition: we do this now. Set

$$\lambda_1 := \frac{1}{2}KZ(\mathrm{sn}^{-1}\left(\frac{2x_1}{kK}, k\right), k), \tag{5.13}$$

$$\lambda_2' := \frac{1}{2}KZ(\mathrm{dn}^{-1}\left(\frac{2\imath x_2}{K}, k\right), k), \tag{5.14}$$

$$\lambda_3 := \frac{1}{2}KZ(\mathrm{cn}^{-1}\left(\frac{2\imath x_3}{Kk}, k\right), k). \tag{5.15}$$

where $Z(v)$ is the Jacobi Zeta function. Then

Lemma 5.7.

$$\frac{\mathrm{d}\mu_1(x_1, 0, 0)}{\mathrm{d}x_1} = \frac{\mathrm{d}\lambda_1}{\mathrm{d}x_1} = -\frac{EK - K^2 + 4x_1^2}{\sqrt{K^2 - 4x_1^2}\sqrt{K^2k^2 - 4x_1^2}},$$

$$\frac{\mathrm{d}\mu_1(0,x_2,0)}{\mathrm{d}x_2} = \frac{\mathrm{d}\lambda_2'}{\mathrm{d}x_2} = -\frac{EK + 4x_2^2}{\sqrt{K^2+4x_2^2}\sqrt{K^2k'^2+4x_2^2}},$$
$$\frac{\mathrm{d}\mu_1(0,0,x_3)}{\mathrm{d}x_3} = \frac{\mathrm{d}\lambda_3}{\mathrm{d}x_3} = \frac{K^2k^2 + EK - K^2 + 4x_3^2}{\sqrt{K^2k^2+4x_3^2}\sqrt{-K^2k'^2+4x_3^2}}.$$

This may be proven directly or via (5.11). For example,

$$\frac{\mathrm{d}\mu_1}{\mathrm{d}x_2} = \frac{\mathrm{d}\beta_1}{\mathrm{d}\alpha_1}\frac{\mathrm{d}\alpha_1}{\mathrm{d}\zeta_1}\frac{\mathrm{d}\zeta_1}{\mathrm{d}x_2} - \zeta_1 - x_2\frac{\mathrm{d}\zeta_1}{\mathrm{d}x_2},$$

and $\mathrm{d}\beta_1/\mathrm{d}\alpha_1$ was determined in (C.5) while from (4.20) we have $\mathrm{d}\alpha_1/\mathrm{d}\zeta_1 = 1/(4\eta_1)$. Finally the $\mathrm{d}\zeta_1/\mathrm{d}x_2$ term comes by implicit differentiation of the Atiyah-Ward equation. After simplification we obtain the derivative given above.

It follows then that $\mu_1(0,x_2,0) = \lambda_2'$+constant and similarly for the other axes. The constant may be determined by comparison at the origin. (There is a choice of square root here that may be appropriately chosen.) Now $\mathrm{dn}(K \pm iK') = 0$ and $KZ(K \pm iK')/2 = \mp i\pi/4$ and we may identify $\mu_1(0,x_2,0) = \mp\lambda_2'(x_2)$. The choice of sign ultimately makes no difference (it will correspond to the symmetry of the x_2 Higgs field about the origin) and we choose $\mu_1(0,x_2,0) = \lambda_2'(x_2)$. For the identification on the x_3 axis we recall that $\mathrm{cn}(K) = 0$ and $KZ(K)/2 = 0$. Noting (5.9) we find[14]

$$\begin{aligned} \mu_1(x_1,0,0) &= \lambda_1 + \frac{\imath\pi}{4}, && (x_1 \in \mathbf{I}),\\ \mu_1(0,x_2,0) &= \lambda_2 + \frac{\imath\pi}{4} = \lambda_2'(x_2), \\ \mu_1(0,0,x_3) &= \lambda_3 + \frac{\imath\pi}{4} = \imath\lambda, && (x_3 < Kk'/2). \end{aligned} \tag{5.16}$$

We plot these in Figures 11, 12, 13 respectively and note that the points of discontinuity on the x_1 and x_3 axes correspond to the vanishing of the Atiyah-Ward discriminant corresponding to points of bitangency; these will be described in the next section.

6. The points of bitangency

Substituting the Atiyah-Ward constraint

$$\eta = (x_2 - \imath x_1) - 2\zeta x_3 - (x_2 + \imath x_1)\zeta^2$$

into the equation of the spectral curve $P(\zeta,\eta) = 0$ gives an equation of (in general) degree $2n$ in ζ and generically this has $2n$ solutions. There are however loci of points $\boldsymbol{x} \in \mathbb{R}^3$ for which we have multiple roots, the points of bitangency we have referred to; calculating the discriminant of our equation describes the locus. This discriminant is of degree $2n$ in the x_i's but because of the reality conditions it may be expressed as a polynomial of degree n in the x_i^2's with no odd power of x_i appearing. Focussing on the $n = 2$ case the discriminant Q is a quartic in $X = x_1^2$, $Y = x_2^2$, $Z = x_3^2$. This discriminant is not very enlightening: we find real loci

$$0 = K^2{k'}^2 - 4\,{k'}^2{x_1}^2 - 4\,{x_3}^2 \tag{6.1}$$

[14] We remark that Maple's inbuilt function `InverseJacobiDN` has the unwanted behaviour

`Re(InverseJacobiDN(JacobiDN(x, k), k)) = |x|`

and must be used cautiously.

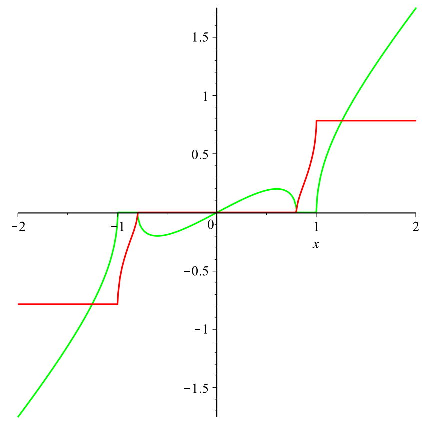

FIGURE 11. The real (green) and imaginary (red) parts of λ_1 restricted to x_1-axis $k = 0.8$. The points of discontinuity correspond to points of bitangency.

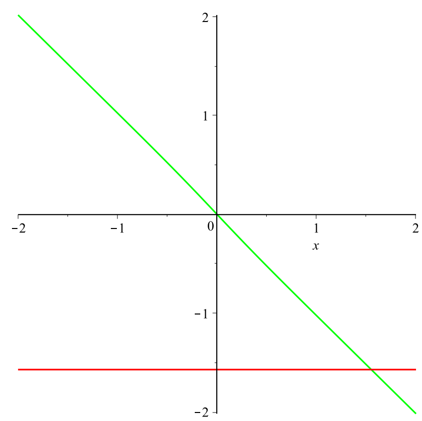

FIGURE 12. The real (green) and imaginary (red) parts of λ_2 restricted to x_2-axis $k = 0.8$

$$x_1{}^2 = \frac{1}{4}\,\frac{\left(K^2 k'^2 + 4\,x_2{}^2\right) k^2}{k'^2}. \tag{6.2}$$

Thus on the x_1 axis there are the 4 solutions $\pm K/2$, $\pm Kk/2$, no solutions on the x_2 axis and $\pm Kk'/2$ on the x_3 axis. These are the points we have previously noted. We remark that in the case of (6.1) we obtain double roots with modulus less than one and greater than one while for (6.2) we find the roots of the Atiyah-Ward constraint have modulus one. Hurtubise [30] in his study of the asymptotics of the Higgs field gave the first of these loci. In Appendix A.2 we will relate Hurtubise's curve and our own.

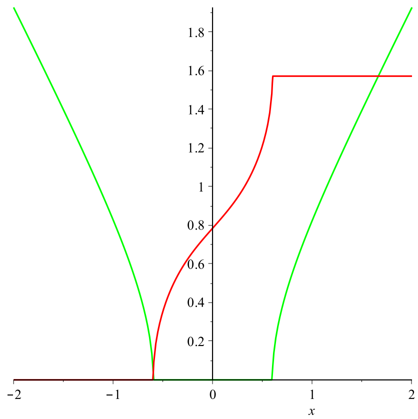

FIGURE 13. The real (green) and imaginary (red) parts of μ_1 restricted to x_3-axis $k = 0.8$. The points of discontinuity correspond to points of bitangency.

7. THE MATRIX W

In this section we will determine the matrix W (2.40) and its inverse. We shall calculate W^{-1} via cofactors and this will involve a number of the identities established in the previous section. We will also look at the behaviour of W at the $z = \pm 1$.

The 4×4 matrix

$$W = \left(\boldsymbol{w}^{(k)}(z,x) \right)$$

is constructed from

$$\boldsymbol{w}^{(k)}(z,x) = (1_2 \otimes C(z)) \left((1_2 + \widehat{\boldsymbol{u}}(\zeta) \cdot \boldsymbol{\sigma}) \, e^{-iz[(x_1 - ix_2)\zeta - ix_3 - x_4]} |s> \otimes \boldsymbol{\Phi}(z, P_k) \right)$$

where $P_k = (\zeta_k, \eta_k)$ are solutions to the mini-twistor constraint. From (3.12, 3.13) the Baker-Akhiezer function takes the form

$$\boldsymbol{\Phi}(z, P_k) = \begin{pmatrix} a_k \\ b_k \end{pmatrix} \mathcal{D}'_k$$

where (again with $\alpha_k := \alpha(P_k)$) we have

$$a_k = -\theta_3(\alpha_k)\theta_2(\alpha_k - z/2), \quad b_k = \theta_1(\alpha_k)\theta_4(\alpha_k - z/2), \tag{7.1}$$

and

$$\mathcal{D}'_k = \frac{\theta_2(1/4)\theta_3(1/4)}{\theta_3(0)\theta_1(\alpha_k - 1/4)\theta_4(\alpha_k + 1/4)} \frac{e^{\beta_1(P_k)z}}{\theta_2(z/2)}.$$

Now the kernel of

$$1_2 + \widehat{\boldsymbol{u}}(\zeta) \cdot \boldsymbol{\sigma} = \frac{2}{1 + |\zeta|^2} \begin{pmatrix} 1 & 0 \\ 0 & \imath\,\zeta \end{pmatrix} \begin{pmatrix} 1 & -\imath\,\bar{\zeta} \\ 1 & -\imath\,\bar{\zeta} \end{pmatrix}$$

has basis $\begin{pmatrix} \imath\,\bar{\zeta} \\ 1 \end{pmatrix}$ for finite ζ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the infinite case. Thus we may take for our construction

$$|s> = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

for all directions apart from $\zeta = \infty$ which gives us

$$(1_2 + \widehat{\boldsymbol{u}}(\zeta)\cdot\boldsymbol{\sigma})\,[|s>= \frac{1}{1+|\zeta|^2}\begin{pmatrix}1\\ i\zeta\end{pmatrix}.$$

We may therefore write

$$W = (1_2 \otimes C(z))\,\Psi\,\mathcal{D}, \qquad \mathcal{D} = \mathrm{Diag}(\mathcal{D}_k), \tag{7.2}$$

where

$$\mathcal{D}_k = \frac{1}{1+|\zeta_k|^2}\,\frac{\theta_2(1/4)\theta_3(1/4)}{\theta_3(0)\theta_1(\alpha_k-1/4)\theta_4(\alpha_k+1/4)}\,\frac{e^{\beta_1(P_k)z-iz[(x_1-ix_2)\zeta_k-ix_3-x_4]}}{\theta_2(z/2)} \tag{7.3}$$

and

$$\Psi = \left(\begin{pmatrix}1\\ i\zeta_k\end{pmatrix}\otimes\begin{pmatrix}a_k\\ b_k\end{pmatrix}\right) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\\ i\zeta_1 a_1 & i\zeta_2 a_2 & i\zeta_3 a_3 & i\zeta_4 a_4\\ i\zeta_1 b_1 & i\zeta_2 b_2 & i\zeta_3 b_3 & i\zeta_4 b_4\end{pmatrix}. \tag{7.4}$$

We remark that if we work with with the standard Nahm matrices (3.19) then the corresponding conjugation by $\mathcal{O}$ replaces $\widehat{\boldsymbol{w}}(z) = C(z)\boldsymbol{\Psi}(z,\zeta,\eta)$ by $\mathcal{O}\,C(z)\boldsymbol{\Psi}(z,\zeta,\eta)$ and consequently we would have

$$W = (1_2 \otimes \mathcal{O}\,C(z))\,\Psi\,\mathcal{D}.$$

We shall calculate W^{-1} via cofactors. Upon noting that

$$|W| = |C(z)|\,|\Psi|\,|\mathcal{D}| = |\Psi|\,|\mathcal{D}|$$

in the following subsections we shall calculate $|\mathcal{D}|$, $|\Psi|$ and finally the adjoint $\mathrm{Adj}\,\Psi$ but before turning to this a helpful check is to look at the pole structure of W which provides a nontrivial check of the solution.

7.1. **The pole structure of $\boldsymbol{W}$.** We have seen in Lemma 3.3 that for the case at hand the Baker-Akhiezer function only has simple poles at $z=\pm 1$. (For general n one has that $\dot{h}h^{-1}$ only has simple poles [16, 7] though the the Baker-Akhiezer function has higher order poles [7].) Using

$$\theta_2\left((1-\xi)/2\right) = \theta_1\left(\xi/2\right) = \frac{\xi}{2}\,\theta_1'(0) + \frac{\xi^3}{48}\,\theta_1'''(0) + \mathcal{O}(\xi^5), \qquad \theta_1'(0) = \pi\,\theta_2\theta_3\theta_4, \tag{7.5}$$

we obtain (in the following c, c' etc are constants)

$$\begin{aligned}\boldsymbol{\Phi}(1-\xi,P_1) &= \frac{2c}{\xi}\begin{pmatrix}-\theta_3\left(P_1\right)\theta_2\left(P_1-1/2+\xi/2\right)\mathrm{e}^{\beta(P_1)(1-\xi)}\\ \theta_1\left(P_1\right)\theta_4\left(P_1-1/2+\xi/2\right)\mathrm{e}^{\beta(P_1)(1-\xi)}\end{pmatrix} + \mathcal{O}(\xi)\\ &= \frac{c}{\xi}\begin{pmatrix}-2\,\theta_3\left(P_1\right)\theta_1\left(P_1\right) + \theta_3\left(P_1\right)\xi\,\left(2\,\beta\left(P_1\right)\theta_1\left(P_1\right) - \theta_1'\left(P_1\right)\right)\\ 2\,\theta_3\left(P_1\right)\theta_1\left(P_1\right) - \theta_1\left(P_1\right)\xi\,\left(2\,\beta\left(P_1\right)\theta_3\left(P_1\right) - \theta_3'\left(P_1\right)\right)\end{pmatrix}\mathrm{e}^{\beta(P_1)} + \mathcal{O}(\xi)\end{aligned}$$

and we see this simple pole behaviour. Now from (3.25) and (3.18) we have

$$\mathcal{O}\,C(1-\xi) = \frac{1}{\sqrt{\xi}}\frac{1}{\sqrt{K}}\begin{pmatrix}\xi\,K/2 & -\xi\,K/2\\ 1 & 1\end{pmatrix} + \mathcal{O}(\xi^{3/2}),$$

and consequently that

$$\mathcal{O}\,C(1-\xi)\boldsymbol{\Psi}(1-\xi,P_1) = \frac{c'}{\sqrt{\xi}}\begin{pmatrix}-2\,\theta_3\left(P_1\right)\theta_1\left(P_1\right)K\\ \theta_1\left(P_1\right)\theta_3'\left(P_1\right) - \theta_3\left(P_1\right)\theta_1'\left(P_1\right)\end{pmatrix} + \mathcal{O}(\xi^{1/2}).$$

Upon making use of (4.18) this takes the form

$$= \frac{c_1}{\sqrt{\xi}} \begin{pmatrix} 1 \\ \imath\,\zeta_1 \end{pmatrix} + \mathcal{O}(\xi^{1/2}).$$

Therefore the pole structure of the first column (and similarly the remaining columns) of W takes the form

$$W_1(1-\xi) = \frac{c_1}{\sqrt{\xi}} \begin{pmatrix} 1 \\ \imath\,\zeta_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \imath\,\zeta_1 \end{pmatrix} + \mathcal{O}(\xi^{1/2}) = \frac{c_1}{\sqrt{\xi}} \begin{pmatrix} 1 \\ \imath\,\zeta_1 \\ \imath\,\zeta_1 \\ -\zeta_1^2 \end{pmatrix} + \mathcal{O}(\xi^{1/2}).$$

Linear combinations of the columns of W therefore give us three solutions at $z=1$ with singular behaviour $1/\xi^{1/2}$ proportional to the vectors $(1,0,0,0)^T$, $(0,0,0,1)^T$ and $(0,1,1,0)^T$. This is what the discussion of section 3.7 requires, with the orthogonal direction, spanned by $(0,1,-1,0)^T$, corresponding to the solution with $\xi^{3/2}$ behaviour. We note that to get this behaviour requires the use of the identity (4.18). We shall see significantly more complicated identities are required when we examine the pole behaviour of V. Using the θ-constant relations

$$\frac{\theta_i''(0)}{\theta_i(0)} - \frac{\theta_j''(0)}{\theta_j(0)} = \pi^2\theta_k^4(0) \tag{7.6}$$

for $(ijk) \in \{(4,3,2),(4,2,3),(3,2,4)\}$, we find the analogous expansion near $z=-1$,

$$\begin{aligned} W_1(-1+\xi) &= \frac{c_1}{\sqrt{\xi}} \begin{pmatrix} \imath\zeta_1 \\ -1 \\ -\zeta_1^2 \\ -\imath\zeta_1 \end{pmatrix} e^{-2(\beta_1(P_1)-i[(x_1-ix_2)\zeta_1-ix_3-x_4])} + \mathcal{O}(\xi^{1/2}) \\ &= \frac{c_1}{\sqrt{\xi}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \imath\,\zeta_1 \\ \imath\,\zeta_1 \\ -\zeta_1^2 \end{pmatrix} e^{-2(\beta_1(P_1)-i[(x_1-ix_2)\zeta_1-ix_3-x_4])} + \mathcal{O}(\xi^{1/2}). \end{aligned}$$

The reason for our writing the expansion in the second form will be made clearer in due course.

7.1.1. *Higher expansion terms of W.* The identities of the last section yield further terms in the expansion of the matrix $W(\mp 1 \pm \xi)$, where $W(z,\boldsymbol{x}) = (\boldsymbol{W}_1(z,\boldsymbol{x}),\ldots,\boldsymbol{W}_4(z,\boldsymbol{x}))$. Introduce the 2-vectors

$$\boldsymbol{w}_{0,k} = \begin{pmatrix} 1 \\ \imath\zeta_k \end{pmatrix}, \qquad \boldsymbol{w}_{1,k} = \begin{pmatrix} \imath x_-\zeta_k + x_3 \\ \\ x_+ - \imath\zeta_k x_3 \end{pmatrix}, \tag{7.7}$$

$$\boldsymbol{w}_{2,k} = \begin{pmatrix} \frac{1}{8}\left(K^2-4x_-^2\right)\zeta_k^2 + \imath x_- x_3\zeta_k - \frac{1}{24}(2k'^2-1)K^2 + \frac{1}{2}x_3^2 \\ \\ \left(-\frac{\imath}{24}(2k'^2-1)K^2 + \frac{1}{2}\imath x_3^2\right)\zeta_k - x_+x_3 + \frac{\imath}{8\zeta_k}\left(K^2-4x_+^2\right) \end{pmatrix}, \tag{7.8}$$

and the 4-vectors

$$\boldsymbol{W}_{0,k} = \boldsymbol{w}_{0,k}\otimes\boldsymbol{w}_{0,k}, \quad \boldsymbol{W}_{1,k} = \boldsymbol{w}_{0,k}\otimes\boldsymbol{w}_{1,k}, \quad \boldsymbol{W}_{2,k} = \boldsymbol{w}_{0,k}\otimes\boldsymbol{w}_{2,k}. \tag{7.9}$$

Then we find

$$(7.10)\quad \begin{aligned} \boldsymbol{W}_k(1-\xi) &= c_k \left\{ \frac{1}{\sqrt{\xi}} \boldsymbol{W}_{0,k} + \sqrt{\xi}\boldsymbol{W}_{1,k} + \xi^{3/2}\boldsymbol{W}_{2,k} + \mathcal{O}(\xi^{5/2}) \right\}, \\ \boldsymbol{W}_k(-1+\xi) &= U c_k \left\{ \frac{1}{\sqrt{\xi}} \boldsymbol{W}_{0,k} - \sqrt{\xi}\boldsymbol{W}_{1,k} + \xi^{3/2}\boldsymbol{W}_{2,k} + \mathcal{O}(\xi^{5/2}) \right\} \mathrm{e}^{-2\mu_k}, \end{aligned}$$

where $U = 1_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the matrix already encountered and c_k are constants that we need not specify.

7.2. **The determinant $|\mathcal{D}|$.** From (7.3) we obtain

$$|\mathcal{D}| = \prod_{k=1}^{4} \mathcal{D}_k = \frac{1}{\theta_2^4(z/2)} \frac{\theta_2^4(1/4)\theta_3^4(1/4)}{\theta_3^4(0)} \prod_{k=1}^{4} \frac{1}{1+|\zeta_k|^2} \frac{e^{\beta_1(P_k)z - iz[(x_1-ix_2)\zeta_k - ix_3 - x_4]}}{\theta_1(\alpha_k - 1/4)\theta_4(\alpha_k+1/4)}.$$

Recalling that we have ordered the roots so that

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (\zeta_1, \zeta_2, -1/\bar{\zeta}_1, -1/\bar{\zeta}_2)$$

we consequently find that

$$\begin{aligned} \prod_{k=1}^{4} \frac{1}{1+|\zeta_k|^2} &= \frac{\zeta_1\zeta_2\zeta_3\zeta_4}{(\zeta_1-\zeta_3)^2(\zeta_2-\zeta_4)^2} \\ &= \frac{1}{\theta_2^2(0)\theta_4^2(0)} \frac{\prod_{k=1}^4 \theta_1(\alpha_k)\theta_2(\alpha_k)\theta_3(\alpha_k)\theta_4(\alpha_k)}{\theta_1^2(\alpha_1-\alpha_3)\theta_3^2(\alpha_1+\alpha_3)\theta_1^2(\alpha_2-\alpha_4)\theta_3^2(\alpha_2+\alpha_4)}. \end{aligned}$$

As the identities needed to evaluate $|\boldsymbol{\Psi}|$ are illustrative of the more complicated identities that are employed in calculating $\operatorname{Adj}\Psi$ we shall describe these in the text, and leave the latter to Appendix D.1. We begin by observing that

$$\begin{aligned} |\Psi| &= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \zeta_1 a_1 & \zeta_2 a_2 & \zeta_3 a_3 & \zeta_4 a_4 \\ \zeta_1 b_1 & \zeta_2 b_2 & \zeta_3 b_3 & \zeta_4 b_4 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} (\zeta_1\zeta_2 + \zeta_3\zeta_4) - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} (\zeta_1\zeta_3 + \zeta_2\zeta_4) \\ &\quad + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} (\zeta_1\zeta_4 + \zeta_2\zeta_3). \end{aligned}$$

Noting (7.1) we may may show

$$\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} = \theta_3(0)\theta_2(z/2)\theta_1(\alpha_i - \alpha_j)\theta_4(\alpha_i + \alpha_j - z/2).$$

Now it is clear that the determinant vanishes when $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4$ and so

$$0 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

whence

$$|\Psi| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} [(\zeta_1\zeta_2 + \zeta_3\zeta_4) - (\zeta_1\zeta_4 + \zeta_2\zeta_3)]$$

$$
\begin{aligned}
&-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}[(\zeta_1\zeta_3+\zeta_2\zeta_4)-(\zeta_1\zeta_4+\zeta_2\zeta_3)]\\
&=\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}(\zeta_1-\zeta_3)(\zeta_2-\zeta_4)-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}(\zeta_1-\zeta_2)(\zeta_3-\zeta_4).
\end{aligned}
$$

Now

$$
\begin{aligned}
i\,[\zeta_j-\zeta_k] &= \frac{\theta_2(\alpha_j)\theta_4(\alpha_j)}{\theta_1(\alpha_j)\theta_3(\alpha_j)}-\frac{\theta_2(\alpha_k)\theta_4(\alpha_k)}{\theta_1(\alpha_k)\theta_3(\alpha_k)}\\
&= \frac{\theta_2(\alpha_j)\theta_4(\alpha_j)\theta_1(\alpha_k)\theta_3(\alpha_k)-\theta_1(\alpha_j)\theta_3(\alpha_j)\theta_2(\alpha_k)\theta_4(\alpha_k)}{\theta_1(\alpha_j)\theta_3(\alpha_j)\theta_1(\alpha_k)\theta_3(\alpha_k)}
\end{aligned}
$$

and upon using (W3, Appendix B) with $\boldsymbol{\alpha}=(\alpha_j,\alpha_k,\alpha_k,\alpha_j)$ we obtain

$$
\zeta_j-\zeta_k = i\,\theta_2(0)\theta_4(0)\,\frac{\theta_1(\alpha_j-\alpha_k)\theta_3(\alpha_j+\alpha_k)}{\theta_1(\alpha_j)\theta_3(\alpha_j)\theta_1(\alpha_k)\theta_3(\alpha_k)}. \tag{7.11}
$$

Thus

$$
\begin{aligned}
|\Psi| = &-\frac{\theta_2^2(z/2)\theta_2^2(0)\theta_3^2(0)\theta_4^2(0)}{\prod_{j=1}^4\theta_1(\alpha_j)\theta_3(\alpha_j)}\,\theta_1(\alpha_1-\alpha_2)\theta_1(\alpha_3-\alpha_4)\theta_1(\alpha_1-\alpha_3)\theta_1(\alpha_2-\alpha_4)\\
&\times[\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_2+\alpha_4)\theta_4(\alpha_1+\alpha_2-z/2)\theta_4(\alpha_3+\alpha_4-z/2)\\
&-\theta_3(\alpha_1+\alpha_2)\theta_3(\alpha_3+\alpha_4)\theta_4(\alpha_1+\alpha_3-z/2)\theta_4(\alpha_2+\alpha_4-z/2)]\,.
\end{aligned}
$$

Next, using (W4, Appendix B) with $\boldsymbol{\alpha}=(\alpha_1+\alpha_2-z/2,\alpha_3+\alpha_4-z/2,\alpha_1+\alpha_3,\alpha_2+\alpha_4)$ we find

$$
\begin{aligned}
&\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_2+\alpha_4)\theta_4(\alpha_1+\alpha_2-z/2)\theta_4(\alpha_3+\alpha_4-z/2)\\
&\quad-\theta_3(\alpha_1+\alpha_2)\theta_3(\alpha_3+\alpha_4)\theta_4(\alpha_1+\alpha_3-z/2)\theta_4(\alpha_2+\alpha_4-z/2)\\
&=\theta_2(z/2)\theta_2\left(\sum_{k=1}^4\alpha_k-z/2\right)\theta_1(\alpha_1-\alpha_4)\theta_1(\alpha_2-\alpha_3)
\end{aligned}
$$

where the sign is determined for example by considering $\alpha_1=-\alpha_3$, $\alpha_2=-\alpha_4$. Then

$$
|\Psi| = -\theta_2^3(z/2)\theta_2\left(\sum_{k=1}^4\alpha_k-z/2\right)\theta_2^2(0)\theta_3^2(0)\theta_4^2(0)\,\frac{\prod_{i<j}\theta_1(\alpha_i-\alpha_j)}{\prod_{j=1}^4\theta_1(\alpha_j)\theta_3(\alpha_j)}.
$$

Upon using (4.12)

$$
\theta_2\left(\sum_{k=1}^4\alpha_k-z/2\right)=\theta_2\left(N\tau+M-z/2\right)=\theta_2\left(z/2\right)\,e^{-i\pi[M+N^2\tau-Nz]}
$$

giving finally that

$$
|\Psi| = -\theta_2^4(z/2)\,\theta_2^2(0)\theta_3^2(0)\theta_4^2(0)\,\frac{\prod_{i<j}\theta_1(\alpha_i-\alpha_j)}{\prod_{j=1}^4\theta_1(\alpha_j)\theta_3(\alpha_j)}\,e^{-i\pi[M+N^2\tau-Nz]}. \tag{7.12}
$$

7.4. **The Adjoint of Ψ.** Developing the method just employed we show in Appendix D.1 that

Theorem 7.1. *The i-th column of* $\mathrm{Adj}(\Psi)^T$ *takes the form (for* i, j, k, l *distinct)*

$$\begin{pmatrix} i\zeta_j \frac{\theta_2(z/2)\,\theta_2(\sum_{s\neq i}\alpha_s - z/2)}{\prod_{s\neq i}\theta_3(\alpha_s)} - \frac{\theta_2(0)\theta_4(0)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_j+\alpha_l)\theta_4(\alpha_j-z/2)\theta_4(\alpha_k+\alpha_l-z/2)}{\theta_3(\alpha_j)\prod_{s\neq i}\theta_1(\alpha_s)\theta_3(\alpha_s)} \\ i\zeta_j \frac{\theta_2(z/2)\,\theta_4(\sum_{s\neq i}\alpha_s - z/2)}{\prod_{s\neq i}\theta_1(\alpha_s)} - \frac{\theta_2(0)\theta_4(0)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_j+\alpha_l)\theta_2(\alpha_j-z/2)\theta_4(\alpha_k+\alpha_l-z/2)}{\theta_1(\alpha_j)\prod_{s\neq i}\theta_1(\alpha_s)\theta_3(\alpha_s)} \\ -\frac{\theta_2(z/2)\,\theta_2(\sum_{s\neq i}\alpha_s - z/2)}{\prod_{s\neq i}\theta_3(\alpha_s)} \\ -\frac{\theta_2(z/2)\,\theta_4(\sum_{s\neq i}\alpha_s - z/2)}{\prod_{s\neq i}\theta_1(\alpha_s)} \end{pmatrix} d_i \tag{7.13}$$

where

$$d_i = -\epsilon_{ijkl}\,\theta_2(0)\theta_3(0)\theta_4(0)\theta_1(\alpha_j-\alpha_k)\theta_1(\alpha_j-\alpha_l)\theta_1(\alpha_k-\alpha_l)\,\theta_2(z/2). \tag{7.14}$$

8. The Matrix $\overline{V}$

Recall that

$$V = W^{\dagger\,-1} = \left(\mathcal{D}^{-1}\,\Psi^{-1}\,\left(1_2\otimes C^{-1}(z)\right)\right)^{\dagger} = \left(1_2\otimes C^{-1}(z)\right)\,\Psi^{\dagger\,-1}\,\bar{\mathcal{D}}^{-1}$$

and so

$$\overline{V} = \left(1_2\otimes C^{-1}(z)\right)\,\mathrm{Adj}(\Psi)^T\,\mathcal{D}^{-1}|\Psi|^{-1},$$

where we have made use of the fact that $C(z)$ is real and symmetric. Here $\overline{V}$ is the complex conjugate of the matrix V and it will be convenient to deal with this for the time being. If we work with the standard Nahm basis (3.19) we have instead

$$\overline{V} = \left(1_2\otimes \mathcal{O}\,C^{-1}(z)\right)\,\mathrm{Adj}(\Psi)^T\,\mathcal{D}^{-1}|\Psi|^{-1}$$

as $\mathcal{O}$ is orthogonal. Collecting the z dependent factors together and utilising (4.12) this may be rewritten as

$$\overline{V} = \left(1_2\otimes \mathcal{O}\,C^{-1}(z)\right)\,\frac{1}{\theta_2^2(z/2)}\overline{\Lambda}\,\tilde{\mathcal{D}}$$

where $\tilde{\mathcal{D}}$ is a z-independent diagonal matrix that we shall not need and

$$\overline{\Lambda}_i = \begin{pmatrix} -i\zeta_j \frac{\theta_2(z/2)\,\theta_2(\sum_{r\neq i}\alpha_r - z/2)}{\prod_{s\neq i}\theta_3(\alpha_s)} + \frac{\theta_2(0)\theta_4(0)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_j+\alpha_l)\theta_4(\alpha_j-z/2)\theta_4(\alpha_k+\alpha_l-z/2)}{\theta_3(\alpha_j)\prod_{s\neq i}\theta_1(\alpha_s)\theta_3(\alpha_s)} \\ -i\zeta_j \frac{\theta_2(z/2)\,\theta_4(\sum_{r\neq i}\alpha_r - z/2)}{\prod_{s\neq i}\theta_1(\alpha_s)} + \frac{\theta_2(0)\theta_4(0)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_j+\alpha_l)\theta_2(\alpha_j-z/2)\theta_4(\alpha_k+\alpha_l-z/2)}{\theta_1(\alpha_j)\prod_{s\neq i}\theta_1(\alpha_s)\theta_3(\alpha_s)} \\ \frac{\theta_2(z/2)\,\theta_2(\sum_{r\neq i}\alpha_r - z/2)}{\prod_{s\neq i}\theta_3(\alpha_s)} \\ \frac{\theta_2(z/2)\,\theta_4(\sum_{r\neq i}\alpha_r - z/2)}{\prod_{s\neq i}\theta_1(\alpha_s)} \end{pmatrix} e^{-z\mu_i} \tag{8.1}$$

where we may choose (i, j, k, l) as a cyclic permutation of $(1, 2, 3, 4)$. Here we encounter μ_k defined in (4.3).

8.1. **The pole structure of $\overline{V}$.** We need the expansion (8.1) to determine the projector and for calculating the Higgs field via the Panagopoulos formulae. Set

$$\overline{\boldsymbol{v}}_i(z) = \left(1_2 \otimes \mathcal{O}\, C^{-1}(z)\right) \frac{1}{\theta_2^2(z/2)}\, \overline{\Lambda}_i = \sum_{s\geq 0} \overline{\boldsymbol{v}}_{i,s}\, \xi^{s-5/2}, \qquad \text{where } z = 1-\xi, \tag{8.2}$$

and

$$\overline{\Lambda}_i = \begin{pmatrix} -\imath\,\zeta_j \\ 1 \end{pmatrix} \otimes \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{8.3}$$

Here

$$\begin{aligned}
A &= \theta_2(z/2)\, \frac{\theta_2(\sum_{r\neq i} \alpha_r - z/2)}{\prod_{s\neq i} \theta_3(\alpha_s)}\, e^{-z\mu_i},\\
B &= \theta_2(z/2)\, \frac{\theta_4(\sum_{r\neq i} \alpha_r - z/2)}{\prod_{s\neq i} \theta_1(\alpha_s)}\, e^{-z\mu_i},\\
\alpha &= \frac{\theta_2(0)\theta_4(0)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_j+\alpha_l)\theta_4(\alpha_j - z/2)\theta_4(\alpha_k+\alpha_l - z/2)}{\theta_3(\alpha_j)\, \prod_{s\neq i}\theta_1(\alpha_s)\theta_3(\alpha_s)}\, e^{-z\mu_i},\\
\beta &= \frac{\theta_2(0)\theta_4(0)\theta_3(\alpha_j+\alpha_k)\theta_3(\alpha_j+\alpha_l)\theta_2(\alpha_j - z/2)\theta_4(\alpha_k+\alpha_l - z/2)}{\theta_1(\alpha_j)\, \prod_{s\neq i}\theta_1(\alpha_s)\theta_3(\alpha_s)}\, e^{-z\mu_i}.
\end{aligned}$$

A lengthy calculation given in Appendix D.2 shows that

Theorem 8.1. *Each column of $\overline{\boldsymbol{v}}_i$ has expansion at $z = 1-\xi$*

$$\overline{\boldsymbol{v}}_i = N_i \left(\frac{1}{\xi^{3/2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{\xi^{1/2}} \begin{pmatrix} -x_1 - \imath\, x_2 \\ x_3 \\ x_3 \\ x_1 - \imath\, x_2 \end{pmatrix} + \frac{\overline{\boldsymbol{v}}_{i,3}}{N_i}\, \xi^{1/2} + \mathcal{O}(\xi^{3/2}) \right) \tag{8.4}$$

where

$$N_i := c\sqrt{K}\, \theta_2\theta_4\, \frac{\prod_{j<k\ \ j,k\neq i} \theta_3(P_j + P_k)}{\prod_{r\neq i} \theta_1(P_r)\theta_3(P_r)}\, \mathrm{e}^{-\mu_i}$$

and the finite term $\overline{\boldsymbol{v}}_{i,3}/N_i$ has the equivalent expansions (for $j\neq i$)

$$\frac{\overline{\boldsymbol{v}}_{i,3}}{N_i} = \begin{pmatrix} \frac{\imath}{4}\,(\zeta_i^2 + \zeta_i\zeta_j + \zeta_j^2)\zeta_j\, X - x_+ x_3 - 2\imath\, \left(r^2 - 3x_3^2 + 2\lambda\right)\zeta_j - 4\, x_- x_3\, \zeta_j(\zeta_i + \zeta_j) \\[2ex] -\frac{1}{8} X\, \zeta_i^2 - 2\imath\, x_- x_3\, \zeta_i + \lambda - \frac{3}{2} x_3^2 \\[2ex] -\frac{1}{8} X\, \zeta_i^2 - 2\imath\, x_- x_3\, \zeta_i + \lambda + x_+ x_- - \frac{1}{2} x_3^2 \\[2ex] \frac{\imath}{4}\, X\, \zeta_i - x_- x_3 \end{pmatrix}, \tag{8.5}$$

$$(8.6)\qquad \frac{\overline{\boldsymbol{v}}_{i,3}}{N_i}=\begin{pmatrix} -\frac{\imath}{4}\,X\,\zeta_i^3+4\,x_-x_3\,\zeta_i^2+2\imath\,\left(r^2-3x_3^2+2\lambda\right)\zeta_i+3\,x_+x_3\\ -\frac{1}{8}X\,\zeta_i^2-2\imath\,x_-x_3\,\zeta_i+\lambda-\frac{3}{2}x_3^2\\ -\frac{1}{8}X\,\zeta_i^2-2\imath\,x_-x_3\,\zeta_i+\lambda+x_+x_--\frac{1}{2}x_3^2\\ \frac{\imath}{4}\,X\,\zeta_i-x_-x_3 \end{pmatrix},$$

$$(8.7)\qquad \frac{\overline{\boldsymbol{v}}_{i,3}}{N_i}=\begin{pmatrix} \frac{\imath}{4}\frac{(K^2-4x_+^2)}{\zeta_i}-x_+x_3\\ -\frac{1}{8}X\,\zeta_i^2-2\imath\,x_-x_3\,\zeta_i+\lambda-\frac{3}{2}x_3^2\\ -\frac{1}{8}X\,\zeta_i^2-2\imath\,x_-x_3\,\zeta_i+\lambda+x_+x_--\frac{1}{2}x_3^2\\ \frac{\imath}{4}\,X\,\zeta_i-x_-x_3 \end{pmatrix},$$

with

$$x_\pm=x_1\pm\imath x_2,\qquad \lambda=\frac{1}{8}\,K^2(1-2k^2),\qquad X=K^2-4x_-^2.$$

Several observations are in order. First, up to normalisation, this takes the form of (the complex conjugate of) (3.28) with $\overline{\boldsymbol{v}}'_{i,3}=(a,b-r^2/2,b+r^2/2,c)^T$. The common pole structure means that we may determine a projector (see later) onto the normalisable solution. Next, using the second representation we find that

$$|\overline{\boldsymbol{v}}_{1,3},\,\overline{\boldsymbol{v}}_{2,3},\,\overline{\boldsymbol{v}}_{3,3},\,\overline{\boldsymbol{v}}_{4,3}|=\frac{r^2}{128}\,N_1N_2N_3N_4(K-2x_-)^3(K+2x_-)^3\prod_{i<j}(\zeta_i-\zeta_j).$$

We have already noted that points of bitangency of the spectral curve yield solutions with multiplicity to the mini-twistor constraint and at these nongeneric points we need to take further terms in our expansions to get a basis for solutions to $\Delta W=0$ and $\Delta^\dagger V=0$; this occurs when $\prod_{i<j}(\zeta_i-\zeta_j)=0$. It naively appears that the solutions are also linearly dependent whenever $K=\pm 2x_-$: what is happening here? The reality of K means this only occurs for the lines $x_2=0$, $x_1=\pm K/2$ and x_3 arbitrary; this means one of the roots of the mini-twistor constraint is $\zeta=0$. Indeed if we call this vanishing point P_1 we have

- $x_1=K/2$, $x_2=0$, x_3 arbitrary: $P_1=0_1$, $P_3=\infty_1$; P_2,P_4 determined by x_3.
- $x_1=-K/2$, $x_2=0$, x_3 arbitrary: $P_1=0_2$, $P_3=\infty_2$; P_2,P_4 determined by x_3.

Upon noting that the normalizations N_i ($i=2,3,4$) have a denominator $\theta_1(P_1)$ there is a precise cancellation between numerator and denominators and the determinant does not vanish along these lines. Thus we only need to consider further terms in the expansion at (nongeneric) points of bitangency.

8.2. **The behaviour at $z=-1$ and Monodromy.** Although we must take the expansions of the solutions at both $z=\pm1$ these are related. We establish in Appendix D.3

Theorem 8.2. *Let* $\overline{\boldsymbol{v}}_i(1-\xi)=\sum_{s\geq 0}\overline{\boldsymbol{v}}_{i,s}\,\xi^{s-5/2}$. *Then* $\overline{\boldsymbol{v}}_i(-1+\xi)=\pm\sum_{s\geq 0}\overline{\boldsymbol{v}}'_{i,s}\,\xi^{s-5/2}$ *where*

$$\overline{\boldsymbol{v}}'_{i,s}=(-1)^s\left(1_2\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}e^{2\mu_i}\right)\overline{\boldsymbol{v}}_{i,s}.$$

This explains the origin of the matrix $U=1_2\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ we encountered earlier relating solutions at the endpoints. In particular the expansion at $z=-1+\xi$ then takes the form

$$\overline{\boldsymbol{v}}_i=N_i\,e^{2\mu_i}\left(\frac{1}{\xi^{3/2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}+\frac{1}{\xi^{1/2}}\begin{pmatrix}-x_3\\-\imath\,x_2-x_1\\\imath\,x_2-x_1\\x_3\end{pmatrix}+\mathcal{O}(\xi^{1/2})\right). \tag{8.8}$$

8.3. **A convenient normalization.** We have just established that up to the normalisation factors N_i the pole terms of $\bar{V}$ have a common form; these normalisations may be removed by left multiplication by a constant matrix and is convenient to define

$$\bar{\mathcal{V}}:=\bar{V}\,\mathrm{Diag}(1/N_1,1/N_2,1/N_3,1/N_4). \tag{8.9}$$

Then

$$\bar{\mathcal{V}}_i(1-\xi)=\frac{1}{\xi^{3/2}}\bar{\mathfrak{v}}_0+\frac{1}{\xi^{1/2}}\bar{\mathfrak{v}}_1+\xi^{1/2}\bar{\mathfrak{v}}_{i,2}+\mathcal{O}(\xi^{3/2}) \tag{8.10}$$

where

$$\bar{\mathfrak{v}}_0:=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix},\quad \bar{\mathfrak{v}}_1:=\begin{pmatrix}-x_1-\imath\,x_2\\x_3\\x_3\\x_1-\imath\,x_2\end{pmatrix},\quad \bar{\mathfrak{v}}_{i,2}:=\overline{\boldsymbol{v}}_{i,3}/N_i. \tag{8.11}$$

Similarly we have the non-conjugated quantities

$$\mathcal{V}_i(1-\xi)=\frac{1}{\xi^{3/2}}\mathfrak{v}_0+\frac{1}{\xi^{1/2}}\mathfrak{v}_1+\xi^{1/2}\mathfrak{v}_{i,2}+\mathcal{O}(\xi^{3/2}),\quad\text{where}\quad \mathfrak{v}_1:=\begin{pmatrix}-x_1+\imath\,x_2\\x_3\\x_3\\x_1+\imath\,x_2\end{pmatrix}$$

and so forth. Now we have shown that

$$\bar{\mathcal{V}}_i(-1+\xi)=U\left(\frac{1}{\xi^{3/2}}\bar{\mathfrak{v}}_0-\frac{1}{\xi^{1/2}}\bar{\mathfrak{v}}_1+\xi^{1/2}\bar{\mathfrak{v}}_{i,2}+\mathcal{O}(\xi^{3/2})\right)\exp(2\mu_i)$$

where

$$U:=1_2\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix} \tag{8.12}$$

Thus

$$\bar{\mathcal{V}}(-1+\xi)=U\left[\bar{\mathcal{V}}_+(1-\xi)-\bar{\mathcal{V}}_-(1-\xi)\right]\mathcal{M}$$

where $\bar{\mathcal{V}}_\pm$ are the even (odd) terms $\bar{\mathfrak{v}}_l$ and

$$\mathcal{M}:=\mathrm{Diag}(e^{2\mu_1},e^{2\mu_2},e^{2\mu_3},e^{2\mu_4}). \tag{8.13}$$

With this normalisation $\mathcal{W} := \mathcal{V}^{\dagger\,-1}$ has columns

$$\mathcal{W}_k := \frac{1}{\theta_2(z/2)} \left(1_2 \otimes \begin{pmatrix} 1/p(z) & 0 \\ 0 & p(z) \end{pmatrix} \mathcal{O} \right) \left(\begin{pmatrix} 1 \\ i\zeta_k \end{pmatrix} \otimes \begin{pmatrix} a_k \\ b_k \end{pmatrix} \right) e^{z[\mu_k - \imath\,\pi N]}\, d_k \tag{8.14}$$

where

$$d_k := -c\sqrt{\frac{\pi}{2}}\,\theta_1[P_k]\theta_3[P_k]\,e^{-\mu_k} \left(\frac{\theta_2(0)}{\theta_2[\sum_{j=1}^4 P_j]} \right) \frac{\prod_{\substack{r<s \\ r,s\neq k}} \theta_3[P_r + P_s]}{\prod_{l\neq k} \theta_1[P_l - P_k]}. \tag{8.15}$$

From our expansion (7.10) we find that

$$\begin{aligned} \mathcal{W}_k(1-\xi) &= \mathfrak{d}_k \left\{ \frac{1}{\sqrt{\xi}} \boldsymbol{W}_{0,k} + \sqrt{\xi}\boldsymbol{W}_{1,k} + \xi^{3/2}\boldsymbol{W}_{2,k} + \mathcal{O}(\xi^{5/2}) \right\} \\ &:= \frac{1}{\sqrt{\xi}}\mathfrak{w}_{0,k} + \sqrt{\xi}\mathfrak{w}_{1,k} + \xi^{3/2}\mathfrak{w}_{2,k} + \mathcal{O}(\xi^{5/2}) \end{aligned}$$

where

$$\mathfrak{d}_k = -d_k \sqrt{\frac{2}{\pi}} \frac{\theta_1(\alpha_k)\theta_3(\alpha_k)}{\theta_2(0)\theta_4(0)} \mathrm{e}^{\mu_k - \imath\pi N}.$$

We remark that from $\mathcal{W}^T(z)\overline{\mathcal{V}}(z) = 1_4$ the expansion at $z = 1$ yields the consistency relations

$$\begin{aligned} 0 &= \mathfrak{w}_0^T.\bar{\mathfrak{v}}_0, \\ 0 &= \mathfrak{w}_1^T.\bar{\mathfrak{v}}_0 + \mathfrak{w}_0^T.\bar{\mathfrak{v}}_1, \\ 1_4 &= \mathfrak{w}_0^T.\bar{\mathfrak{v}}_2 + \mathfrak{w}_1^T.\bar{\mathfrak{v}}_1 + \mathfrak{w}_2^T\bar{\mathfrak{v}}_0. \end{aligned} \tag{8.16}$$

The first two of these are easily seen to be true while, upon noting $\mathfrak{w}_1^T.\bar{\mathfrak{v}}_1 = 0$, the third simplifies to

$$1_4 = \mathfrak{w}_0^T.\bar{\mathfrak{v}}_2 + \mathfrak{w}_2^T\bar{\mathfrak{v}}_0. \tag{8.17}$$

This then follows from the relation (for each $j \in \{1, \ldots, 4\}$)

$$\mathfrak{D}_j = 1/\mathfrak{d}_j, \tag{8.18}$$

where

$$\mathfrak{D}_j := -4x_-x_3\zeta_j^2 - \imath((2k'^2 - 1)K^2 + 4r^2 - 12x_3^2)\zeta_j - 12x_+x_3 + \frac{\imath S_+^2}{\zeta_j}, \tag{8.19}$$

which holds for ζ_j, η_j satisfying the Atiyah-Ward equation. Theorem 8.2 then means that the expansion at $z = -1$ also holds true.

8.4. **Derivation of $\mathfrak{v}_2$ from $\mathcal{W}$-data and vice versa.** In the preceding sections we have derived $\mathfrak{v}_2$ by calculating the general inverse of W and taking its expansion. In this subsection we show that the same result can be obtained from the expansion of W itself; this is a useful check. We shall show that using (8.16) we can in fact determine the $\mathfrak{v}_2$ term of the expansion of $\mathcal{V}$ from that of $\mathcal{W}$ and vice versa.

Recall from (3.28) that we have an expansion for $\overline{\mathcal{V}}$ of the form

$$\bar{\mathfrak{v}}_{0,j} = \frac{1}{\xi^{3/2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \bar{\mathfrak{v}}_{1,j} = \frac{1}{\xi^{1/2}} \begin{pmatrix} -x_+ \\ x_3 \\ x_3 \\ x_- \end{pmatrix}, \quad \tilde{\bar{\mathfrak{v}}}_{2,j} = \xi^{1/2} \begin{pmatrix} a_j \\ b_j - r^2/2 \\ b_j + r^2/2 \\ c_j \end{pmatrix}, \tag{8.20}$$

while that of $\mathcal{W}$ is (up to the constants $\mathfrak{d}_j$) in terms of $\boldsymbol{W}_{0,j} = \boldsymbol{w}_{0,j} \otimes \boldsymbol{w}_{0,j}$, $\boldsymbol{W}_{1,j} = \boldsymbol{w}_{0,j} \otimes \boldsymbol{w}_{1,j}$ and $\boldsymbol{W}_{2,j} = \boldsymbol{w}_{0,j} \otimes \boldsymbol{w}_{2,j}$. We choose to rewrite

$$\boldsymbol{w}_{2,j} = \begin{pmatrix} \frac{1}{8}S_-^2\zeta_j^2 + \imath x_- x_3\zeta_j - \frac{1}{24}(2k'^2-1)K^2 + \frac{1}{2}x_3^3 \\ \\ \imath\left(-\frac{1}{24}(2k'^2-1)K^2 + \frac{1}{2}x_3^2\right)\zeta_j - x_- x_3 + \frac{1}{8\zeta_j}S_+^2 \end{pmatrix}$$

with $S_\pm = \sqrt{K^2 - 4x_\pm^2}$, $x_\pm = x_1 \pm \imath x_2$. The first two identities of (8.16) hold and we find

Proposition 8.3. *The matrix $\mathfrak{D}$ in the following relation*

$$\boldsymbol{W}_2^T.\bar{\mathfrak{v}}_0 + \boldsymbol{W}_1^T.\bar{\mathfrak{v}}_1 + \boldsymbol{W}_0^T.\widetilde{\bar{\mathfrak{v}}}_2 = \mathfrak{D} \tag{8.21}$$

is diagonal if and only if the quatities a_j, b_j, c_j are given by the formulae

$$\begin{aligned} a_j &= \frac{\imath}{4}\frac{S_+^2}{\zeta_j} - x_+x_3, \\ b_j &= -\frac{1}{8}S_-^2\zeta_j^2 - 2\imath x_- x_3\zeta_j + \frac{1}{8}K^2(1-2k^2) + \frac{1}{2}x_+x_- - x_3^2, \\ c_j &= \frac{\imath}{4}S_-^2\zeta_j - x_- x_3, \end{aligned} \tag{8.22}$$

where ζ_j, $j = 1, \ldots, 4$ are solutions of the Atiyah-Ward equation and the j-th diagonal element $\mathfrak{D}_j$ is given by (8.19).

Analogously, given $\mathfrak{v}_2$ one can construct $\boldsymbol{W}_2$.

9. The Projector

The common pole structure at $z = \pm 1$ means that the construction of the projection matrix becomes algebraic. It is clear from (8.11) that

$$\bar{\mathcal{V}} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{9.1}$$

gives three vectors vanishing at $z = 1$ and with behaviour at $z = -1$ going as

$$\left(e^{2\mu_1} - e^{2\mu_r}\right)\left(\frac{1}{\xi^{3/2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \frac{1}{\xi^{1/2}}\begin{pmatrix} -x_3 \\ -\imath x_2 - x_1 \\ \imath x_2 - x_1 \\ x_3 \end{pmatrix} + \mathcal{O}(\xi^{1/2})\right) \tag{9.2}$$

for $r = 2, 3, 4$. While it is possible for $e^{2\mu_1} = e^{2\mu_2}$ for certain $\boldsymbol{x}$ (see the x_2-axis below) a consequence of proposition (4.8) is that

$$e^{2\mu_1} - e^{2\mu_3} = e^{2\mu_1} + e^{-2\bar{\mu}_1} \neq 0.$$

Therefore the rank of

$$\begin{pmatrix} e^{2\mu_1} - e^{2\mu_3} & 0 \\ e^{2\mu_2} - e^{2\mu_1} & e^{2\mu_1} - e^{2\mu_4} \\ 0 & e^{2\mu_3} - e^{2\mu_1} \end{pmatrix}.$$

is always 2. Thus we may construct from these the two required normalisable solutions by taking

$$\bar{\mathcal{V}}\begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}e^{2\mu_1}-e^{2\mu_3} & 0\\ e^{2\mu_2}-e^{2\mu_1} & e^{2\mu_1}-e^{2\mu_4}\\ 0 & e^{2\mu_3}-e^{2\mu_1}\end{pmatrix}.$$

We observe that the projector is z independent, as required.[15] Thus we have the projector

$$\overline{\mu} := \begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}e^{2\mu_1}-e^{2\mu_3} & 0\\ e^{2\mu_2}-e^{2\mu_1} & e^{2\mu_1}-e^{2\mu_4}\\ 0 & e^{2\mu_3}-e^{2\mu_1}\end{pmatrix}. \tag{9.3}$$

which is such that

$$(1,1,1,1)\overline{\mu} = (0,0), \qquad (1,1,1,1)\mathcal{M}\overline{\mu} = (0,0),$$

the latter showing this holds for $z=-1$ as well.

10. An Example: the x_2 axis

Before turning to the general formulae its helpful to see an example of our formalism to reconstruct the Higgs field on the x_2-axis. We shall first construct the two normalisable solutions, then use the Panagopoulos formalism to calculate their normalisation and then finally calculate the Higgs field. Aleady at this stage we obtain a new analytic result for the depth of the well.

10.1. The normalizable solutions.

Utilising Theorem 8.1 and restricting to the x_2 direction we find that at $z = 1-\xi$:

Proposition 10.1. *Let $x_1 = x_3 = 0$ and the points P_i be given by (5.3). Then the first column of the expansion of the fundamental solution takes the form*

$$\overline{\boldsymbol{v}}_1(1-\xi)|_{\xi\sim 0} = N_1\left\{\xi^{-3/2}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + \xi^{-1/2}\begin{pmatrix}\imath x_2\\0\\0\\\imath x_2\end{pmatrix}\right.$$
$$\left. + \,\xi^{1/2}\,\frac{Kk}{4}\begin{pmatrix}\sqrt{K^2+4x_2^2}\left(1+\imath\dfrac{1}{Kk}\sqrt{K^2k'^2+4x_2^2}\right)\\ -\imath\sqrt{K^2k'^2+4x_2^2}-\dfrac{2x_2^2}{Kk}\\ -\imath\sqrt{K^2k'^2+4x_2^2}+\dfrac{2x_2^2}{Kk}\\ -\sqrt{K^2-4x_2^2}\left(1+\imath\dfrac{1}{Kk}\sqrt{K^2k'^2+4x_2^2}\right)\end{pmatrix}\right\}.$$

[15] If we had used the projector

$$\begin{pmatrix}e^{2\mu_1}-e^{2\mu_3} & e^{2\mu_1}-e^{2\mu_4}\\ e^{2\mu_2}-e^{2\mu_1} & 0\\ 0 & e^{2\mu_2}-e^{2\mu_1}\end{pmatrix}$$

instead, then on the x_2 axis the first vector $\bar{\mathcal{V}}_1-\bar{\mathcal{V}}_2$ is already normalizable at both ends and we construct the remaining vector as an appropriate linear combination of the final two columns.

It is convenient to set

$$p=\sqrt{K^2+4x_2^2},\quad q=\sqrt{K^2k'^2+4x_2^2} \tag{10.1}$$

in terms of which the whole $\xi^{1/2}$ entry to the expansion of the fundamental solution (for the ordering of roots (5.2))then reads

$$\frac{Kk}{4}\begin{pmatrix} p+\imath pq/Kk & -p+\imath pq/Kk & -p-\imath pq/Kk & p-\imath pq/Kk \\ -\imath q-\frac{2x_2^2}{Kk} & \imath q-\frac{2x_2^2}{Kk} & -\imath q-\frac{2x_2^2}{Kk} & \imath q-\frac{2x_2^2}{Kk} \\ -\imath q+\frac{2x_2^2}{Kk} & \imath q+\frac{2x_2^2}{Kk} & -\imath q+\frac{2x_2^2}{Kk} & \imath q+\frac{2x_2^2}{Kk} \\ -p+\imath pq/Kk & p+\imath pq/Kk & p-\imath pq/Kk & -p-\imath pq/Kk \end{pmatrix}\mathrm{Diag}(N_1,\ldots,N_4).$$

The expansion of the vector $\boldsymbol{v}_i(z)$ near the point $z=-1+\xi$ is then given by Theorem 8.2,

$$\overline{\boldsymbol{v}}_i(1-\xi)=N_i\begin{pmatrix} a_i \\ b_i-x_2^2/2 \\ b_i+x_2^2/2 \\ -\bar{a}_i\end{pmatrix}\Longrightarrow \overline{\boldsymbol{v}}_i(-1+\xi)=N_i\mathrm{e}^{2\mu_i}\begin{pmatrix} b_i-x_2^2/2 \\ -a_i \\ \bar{a}_i \\ b_i+x_2^2/2\end{pmatrix}.$$

Now acting by the projector (9.1) yields three normalizable vectors at $z=1$

$$\xi^{1/2}\,\frac{Kk}{2}\begin{pmatrix} p & p+\imath pq/Kk & \imath pq/Kk \\ -\imath q & 0 & -\imath q \\ -\imath q & 0 & -\imath q \\ -p & -p+\imath pq/Kk & \imath pq/Kk \end{pmatrix}.$$

From (9.2) and (5.8) the first column here is also finite at $z=-1$, and because

$$e^{2\mu_3}=e^{2\mu_4}=e^{-2\mu_1}=e^{-2\mu_2}$$

as a result of (5.8) the remaining two vectors have the same poles and consequently their difference is then finite at $z=-1$. We have then

Proposition 10.2. *The Weyl equation $\Delta^\dagger\boldsymbol{v}=0$ admits precisely two normalizable solutions $\boldsymbol{v}_1(z;\boldsymbol{x})$, $\boldsymbol{v}_2(z;\boldsymbol{x})$, for $z\in[-1,1]$ which for $\boldsymbol{x}=(0,x_2,0)$ vanish at the end points as*

$$\boldsymbol{v}_1(1-\xi;\boldsymbol{x})=\frac{Kk}{2}\begin{pmatrix} p\\ \imath q\\ \imath q\\ -p\end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}),\quad \boldsymbol{v}_1(-1+\xi;\boldsymbol{x})=\frac{Kk}{2}\mathrm{e}^{2\lambda_2}\begin{pmatrix} q\\ \imath p\\ \imath p\\ -q\end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}),$$

$$\boldsymbol{v}_2(1-\xi;\boldsymbol{x})=\frac{Kk}{2}\begin{pmatrix} p\\ -\imath q\\ -\imath q\\ -p\end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}),\quad \boldsymbol{v}_2(-1+\xi;\boldsymbol{x})=\frac{Kk}{2}\mathrm{e}^{-2\lambda_2}\begin{pmatrix} q\\ -\imath p\\ -\imath p\\ -q\end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}),$$

where $p=\sqrt{K^2+4x_2^2}$ and $q=\sqrt{K^2k'^2+4x_2^2}$.

Proof. Recall that we have been working with the matrix $\overline{V}$ and so the complex conjugate of the required solutions of the Weyl equation. The normalisable solutions we have constructed vanish at the end points as

$$\overline{\boldsymbol{v}}_1(1-\xi;\boldsymbol{x}) = \frac{Kk}{2}\begin{pmatrix} p \\ -\imath q \\ -\imath q \\ -p \end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}), \quad \overline{\boldsymbol{v}}_1(-1+\xi;\boldsymbol{x}) = -\frac{Kk}{2}\mathrm{e}^{2\mu_1}\begin{pmatrix} \imath q \\ p \\ p \\ -\imath q \end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}),$$

$$\overline{\boldsymbol{v}}_2(1-\xi;\boldsymbol{x}) = \frac{Kk}{2}\begin{pmatrix} p \\ \imath q \\ \imath q \\ -p \end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}), \quad \overline{\boldsymbol{v}}_2(-1+\xi;\boldsymbol{x}) = \frac{Kk}{2}\mathrm{e}^{-2\mu_1}\begin{pmatrix} \imath q \\ -p \\ -p \\ -\imath q \end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}).$$

Both p, q are real in our setting and so the behaviour at $z=1$ follows. For $z=-1$ we use (5.8) and so

$$\overline{\boldsymbol{v}}_1(-1+\xi;\boldsymbol{x}) = \frac{Kk}{2}\mathrm{e}^{2\lambda_2}\begin{pmatrix} q \\ -\imath p \\ -\imath p \\ -q \end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}), \ \overline{\boldsymbol{v}}_2(-1+\xi;\boldsymbol{x}) = \frac{Kk}{2}\mathrm{e}^{-2\lambda_2}\begin{pmatrix} q \\ \imath p \\ \imath p \\ -q \end{pmatrix}\sqrt{\xi}+O(\xi^{3/2}),$$

with the proposition following. □

10.2. **Orthogonalization.** Given our two normalisable solutions we must now normalise them. To do this we shall calculate the inner products using Panagopoulos's formulae,

$$\begin{aligned}\int_{-1}^{1}\boldsymbol{v}_i^{\dagger}(z,\boldsymbol{x})\boldsymbol{v}_j(z,\boldsymbol{x})\mathrm{d}z &= \mathrm{Lim}_{\xi\to 0}\frac{1}{\xi}\boldsymbol{v}_i^{\dagger}(1-\xi,\boldsymbol{x})\ \mathrm{Res}_{\xi=0}\mathcal{Q}(1-\xi)^{-1}\boldsymbol{v}_j(1-\xi,\boldsymbol{x})\\ &\quad - \mathrm{Lim}_{\xi\to 0}\frac{1}{\xi}\boldsymbol{v}_i^{\dagger}(-1+\xi,\boldsymbol{x})\ \mathrm{Res}_{\xi=0}\mathcal{Q}(-1+\xi)^{-1}\boldsymbol{v}_j(-1+\xi,\boldsymbol{x}).\end{aligned} \tag{10.2}$$

Direct calculation shows that for $\boldsymbol{x}=(0,x_2,0)$

$$\mathrm{Res}_{\xi=0}\mathcal{Q}(1-\xi)^{-1} = \mathrm{Res}_{\xi=0}\mathcal{Q}(-1+\xi)^{-1} = \frac{1}{K^2k^2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

and consequently we find

Proposition 10.3. *The Gram matrix built from the vectors $\boldsymbol{v}_i(z,\boldsymbol{x})$, $i=1,2$ is diagonal,*

$$\left(\int_{-1}^{1}\boldsymbol{v}_i^{\dagger}(z,\boldsymbol{x})\boldsymbol{v}_i(z,\boldsymbol{x})\mathrm{d}z\right)_{i,j=1,2} = \begin{pmatrix} K^2k^2(1+\mathrm{e}^{4\lambda_2}) & 0 \\ 0 & K^2k^2(1+\mathrm{e}^{-4\lambda_2}) \end{pmatrix} \tag{10.3}$$

Therefore the vectors $\boldsymbol{v}_i(z,\boldsymbol{x})$, $i=1,2$ are orthogonal with norms

$$\mathcal{N}_1 = ||\boldsymbol{v}_1(z,\boldsymbol{x})|| = Kk\sqrt{1+\mathrm{e}^{4\lambda_2}}, \qquad \mathcal{N}_2 = ||\boldsymbol{v}_2(z,\boldsymbol{x})|| = Kk\sqrt{1+\mathrm{e}^{-4\lambda_2}}. \tag{10.4}$$

In what follows we shall denote the orthonormal vectors used in the ADHM construction by

$$\boldsymbol{V}_1(z,\boldsymbol{x}) = \frac{1}{\mathcal{N}_1}\boldsymbol{v}_1(z,\boldsymbol{x}), \quad \boldsymbol{V}_2(z,\boldsymbol{x}) = \frac{1}{\mathcal{N}_2}\boldsymbol{v}_2(z,\boldsymbol{x}).$$

10.3. **The Higgs field.** We now compute the Higgs field using (2.12) to evaluate the integrals. For the x_2-axis this takes the form

$$\begin{aligned}
-\imath\,\Phi_{ij} &= \int_{-1}^{1} z\boldsymbol{V}_i^{\dagger}(z,\boldsymbol{x})\boldsymbol{V}_j(z,\boldsymbol{x})\mathrm{d}z\\
&= \mathrm{Lim}_{\xi\to 0}\frac{1}{\xi}\boldsymbol{V}_i^{\dagger}(1-\xi,\boldsymbol{x})\ \mathrm{Res}_{\xi=0}\mathcal{Q}(1-\xi)^{-1}\boldsymbol{V}_j(1-\xi,\boldsymbol{x})\\
&\quad+\mathrm{Lim}_{\xi\to 0}\frac{1}{\xi}\boldsymbol{V}_i^{\dagger}(-1+\xi,\boldsymbol{x})\ \mathrm{Res}_{\xi=0}\mathcal{Q}(-1+\xi)^{-1}\boldsymbol{V}_j(-1+\xi,\boldsymbol{x})\\
&\quad+\mathrm{Lim}_{\xi\to 0}\frac{1}{\xi}\boldsymbol{V}_i^{\dagger}(1-\xi,\boldsymbol{x})\ \mathrm{Res}_{\xi=0}\mathcal{Q}(1-\xi)^{-1}H_0\frac{\mathrm{d}}{\mathrm{d}x_2}\boldsymbol{V}_j(1-\xi,\boldsymbol{x})\\
&\quad-\mathrm{Lim}_{\xi\to 0}\frac{1}{\xi}\boldsymbol{V}_i^{\dagger}(-1+\xi,\boldsymbol{x})\ \mathrm{Res}_{\xi=0}\mathcal{Q}(-1+\xi)^{-1}H_0\frac{\mathrm{d}}{\mathrm{d}x_2}\boldsymbol{V}_j(-1+\xi,\boldsymbol{x})
\end{aligned}\tag{10.5}$$

where

$$H_0=\imath x_2\begin{pmatrix}0 & 1_2\\ -1_2 & 0\end{pmatrix}.$$

We remark in passing that while the Panagopoulos formula used to establish the norms was insensitive to the interchange of V_i and $\overline{V}_i$, this is no longer the case for the above formula as it has been derived assuming $\Delta^{\dagger}V=0$.

Now with $i=j=1$, the first two lines of (10.5) give in this case

$$\frac{1-\mathrm{e}^{4\lambda_2}}{1+\mathrm{e}^{4\lambda_2}}=\tanh(2\lambda_2)$$

whilst the next two lines reduce after simplifications to

$$-\frac{4x_2}{\sqrt{K^2+4x_2^2}\sqrt{K^2k'^2+4x_2^2}}.$$

Therefore,

$$\Phi_{1,1}=\imath\left(\tanh(2\lambda_2)-\frac{4x_2}{\sqrt{K^2+4x_2^2}\sqrt{K^2k'^2+4x_2^2}}\right).\tag{10.6}$$

In analogous way we compute

$$\Phi_{2,2}=\imath\left(-\tanh(2\lambda_2)+\frac{4x_2}{\sqrt{K^2+4x_2^2}\sqrt{K^2k'^2+4x_2^2}}\right)=-\Phi_{1,1}.\tag{10.7}$$

Similar calculations leads to

$$\Phi_{1,2}=\Phi_{2,1}=-\imath\,\frac{Kk^2+2E-2K}{\cosh(2\lambda_2)Kk^2}.\tag{10.8}$$

Recalling (11.16) then $H=\sqrt{-\frac{1}{2}\Phi_{1,1}^2-\frac{1}{2}\Phi_{2,2}^2-\Phi_{1,2}\Phi_{2,1}}$ and so

$$H^2(0,x_2,0)=\left(\tanh 2\lambda_2+\frac{4x_2}{W_2}\right)^2+\frac{(Kk'^2-2E+K)^2}{K^2k^4\cosh^2 2\lambda_2}\tag{10.9}$$

with

$$\lambda_2=\mu_1-\frac{\imath\pi}{4},\quad W_2=\sqrt{(K^2+4x_2^2)(K^2k'^2+4x_2^2)}.$$

We find that as $x_2 \to \infty$ that H approaches to 1 as

$$H = 1 - \frac{1}{|x_2|} + \frac{1}{8}\frac{K^2(1+{k'}^2)}{|x_2|^3} + O\left(\frac{1}{|x_2|^4}\right). \tag{10.10}$$

The depth of the well ($x_2 = 0, \lambda_2 = 0$) is found to be

$$H(0) = \frac{K(1+{k'}^2) - 2E}{Kk^2}, \tag{10.11}$$

reproducing (1.1) found by Brown *et. al.* [11, see their equation 7.2; Appendix E compares notation]; the work of [42, (7.1)] presented this in terms of an infinite series together with an undetermined integral.

11. Formulae for the Higgs Field and Energy Density

The aim of this section is to evaluate the formulae (2.11, 2.12) determining the Higgs field (at a generic point $\boldsymbol{x}$). We have all of the necessary components with the exception of the quantity $\mathcal{Q}^{-1}$ which we will evaluate in the first subsection. We have already noted that the combinations $\left(V^\dagger \mathcal{Q}^{-1}\mathcal{H}V\right)(z)$ and $\left(W^\dagger \mathcal{Q}\mathcal{H}W\right)(z)$ are constant. These structured matrices help us to simplify our results and we next evaluate these. The final subsection then combines preceding results. Again some proofs are relegated to an Appendix.

11.1. $\boldsymbol{\mathcal{Q}^{-1}}$ and related quantities.

We need $\mathcal{Q}^{-1}$ at the endpoints. Here the Hermitian $\mathcal{Q}$, defined in (2.10), takes the form

$$\mathcal{Q} = \begin{pmatrix} -\frac{\left(r^2+{x_1}^2+{x_2}^2-{x_3}^2\right)f_3}{2r^2} & -\frac{x_3(if_2x_2-f_1x_1)}{r^2} & -\frac{x_3f_3(-x_1+ix_2)}{r^2} & * \\ \frac{x_3(if_2x_2+f_1x_1)}{r^2} & \frac{\left(r^2+{x_1}^2+{x_2}^2-{x_3}^2\right)f_3}{2r^2} & * & \frac{x_3f_3(-x_1+ix_2)}{r^2} \\ \frac{(x_1+ix_2)f_3x_3}{r^2} & * & \frac{\left(r^2+{x_1}^2+{x_2}^2-{x_3}^2\right)f_3}{2r^2} & \frac{x_3(if_2x_2-f_1x_1)}{r^2} \\ * & -\frac{(x_1+ix_2)f_3x_3}{r^2} & -\frac{x_3(if_2x_2+f_1x_1)}{r^2} & -\frac{\left(r^2+{x_1}^2+{x_2}^2-{x_3}^2\right)f_3}{2r^2} \end{pmatrix}$$

where

$$\mathcal{Q}_{14} = -\frac{f_1{x_2}^2+f_2{x_2}^2+2\,if_1x_1x_2+2\,if_2x_1x_2+f_1r^2-f_2r^2-f_1{x_1}^2+{x_3}^2f_1-f_2{x_1}^2-{x_3}^2f_2}{2r^2} = \bar{\mathcal{Q}}_{41}$$
$$\mathcal{Q}_{23} = \frac{-f_1{x_2}^2+f_2{x_2}^2+2\,if_1x_1x_2-2\,if_2x_1x_2-f_1r^2-f_2r^2+f_1{x_1}^2-{x_3}^2f_1-f_2{x_1}^2-{x_3}^2f_2}{2r^2} = \bar{\mathcal{Q}}_{32}.$$

Thus $\mathcal{Q}$ has first order poles at $z = \pm 1$. One finds upon use of elliptic function identities that the determinant of $\mathcal{Q}$ is constant, $|\mathcal{Q}| = D \times K^2/r^4$ with D given below. Indeed one finds that the entries of $\operatorname{Adj}\mathcal{Q}$ are again linear in the f_i's and consequently $\mathcal{Q}^{-1}$ has only first order poles at $z = \pm 1$ (the possible third order poles cancelling) and vanishing constant term. One finds that

$$\operatorname{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x}) = \begin{pmatrix} A & B & B & C \\ \overline{B} & -A & -A & -B \\ \overline{B} & -A & -A & -B \\ \overline{C} & -\overline{B} & -\overline{B} & A \end{pmatrix},$$

$$\operatorname{Res}_{z=-1}\mathcal{Q}^{-1}(z,\boldsymbol{x}) = \begin{pmatrix} A & \overline{B} & B & -A \\ B & -A & C & -B \\ \overline{B} & \overline{C} & -A & -\overline{B} \\ -A & -\overline{B} & B & A \end{pmatrix} = U\left(\operatorname{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x})\right)U, \tag{11.1}$$

where

$A = (-{k'}^2 x_+^2 x_-^2 + x_2^2\, x_+ x_- - x_3^2(x_1^2 - x_2^2))/D,$

$$B = x_3\left(x_+({k'}^2x_-^2+x_3^2)+\imath(x_1+x_+)x_-x_2\right)/D,$$
$$C = \left(-(x_+x_-+2\,x_3^2)x_-^2{k'}^2-2x_3^4-(x_++2\imath x_2)x_-x_3^2+x_-^2x_2^2\right)/D,$$
$$D = -K^2\left(2\,ikx_2x_3-k^2{x_1}^2-k^2{x_2}^2+{x_1}^2+{x_3}^2\right)\left(k^2{x_1}^2+k^2{x_2}^2+2\,ikx_2x_3-{x_1}^2-{x_3}^2\right).$$

Towards evaluating $V^\dagger\mathcal{Q}^{-1}$ and $V^\dagger\mathcal{Q}^{-1}\mathcal{H}$ at the end points we record that (recall $\mathcal{H}^{-1}=\mathcal{H}/r^2$)

$$\mathcal{H}\mathfrak{v}_1=-r^2\,\mathfrak{v}_0,\quad \mathcal{H}\mathfrak{v}_0=-\mathfrak{v}_1. \tag{11.2}$$

These, together with $\mathcal{Q}^{-1}\mathcal{H}=-\mathcal{H}\mathcal{Q}^{-1}$, yield that

$$\mathfrak{v}_0^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathfrak{v}_0=-\frac{1}{r^2}\mathfrak{v}_1^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathfrak{v}_1,\quad \mathfrak{v}_0^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathfrak{v}_1+\mathfrak{v}_1^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathfrak{v}_0=0. \tag{11.3}$$

Further, calculations show that

$$\begin{aligned}\mathfrak{v}_0^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathfrak{v}_0&=4\,\imath\,r^2x_1x_2x_3(k^2f_1-k^2f_2-f_1+f_3)/D,\\ \mathfrak{v}_0^\dagger\,\mathrm{Res}_{z=1}\mathcal{Q}^{-1}&=\mathfrak{v}_1^\dagger\,\mathrm{Res}_{z=1}\mathcal{Q}^{-1}=(0,0,0,0).\end{aligned} \tag{11.4}$$

11.2. **The matrices $V^\dagger\mathcal{Q}^{-1}\mathcal{H}V$, $W^\dagger\mathcal{Q}\mathcal{H}W$ and $\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{V}$.** The constancy of the matrix $\left(\mathcal{V}^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathcal{V}\right)(z)$ (and similarly for W) means that the possible poles at the end points must occur in vanishing combinations. Thus, for example, the leading pole $(\mathfrak{v}_0^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathfrak{v}_0)/\xi^3$ (for $z=1-\xi$) together with (11.4) and

$$(k^2f_1-k^2f_2-f_1+f_3)(1-\xi)=\frac{1}{8}k^2k'^2K^4\xi^3+\mathcal{O}(\xi^5)$$

in fact gives a finite contribution. The results of the previous section show that

$$\begin{aligned}\left(\mathcal{V}^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathcal{V}\right)_{ij}=&\frac{\imath\,r^2x_1x_2x_3\,k^2k'^2K^4}{2D}+\mathfrak{v}_{i,2}^\dagger\left(\mathrm{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x})\right)\mathcal{H}\,\mathfrak{v}_{j,2}\\ &+\mathfrak{v}_{i,2}^\dagger\left(\mathrm{Res}_{z=1}\frac{\mathcal{Q}^{-1}(z,\boldsymbol{x})}{(1-z)^2}\right)\mathcal{H}\,\mathfrak{v}_{j,0}+\mathfrak{v}_{i,0}^\dagger\left(\mathrm{Res}_{z=1}\frac{\mathcal{Q}^{-1}(z,\boldsymbol{x})}{(1-z)^2}\right)\mathcal{H}\,\mathfrak{v}_{j,2}.\end{aligned}$$

We can in fact say more about the structure of this matrix. Its constancy means that

$$\begin{aligned}\lim_{z\to1}\left(\mathcal{V}^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathcal{V}\right)_{jk}&=\lim_{z\to-1}\left(\mathcal{V}^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathcal{V}\right)_{jk}\\ &=\mathcal{M}_{jj}\left(\lim_{z\to1}\mathcal{V}^\dagger U^\dagger U\,\mathcal{Q}^{-1}\mathcal{H}\,UU\mathcal{V}\right)_{jk}\overline{\mathcal{M}}_{kk}\\ &=-\exp(2\mu_j+2\overline{\mu}_k)\lim_{z\to1}\left(\mathcal{V}^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,\mathcal{V}\right)_{jk}\end{aligned}$$

upon using (11.1) and that $U^TU=1_4$, $U^2=-1_4$, $U\mathcal{H}=\mathcal{H}U$. Thus the (j,k)-element is non-vanishing only if $\exp(2\mu_j+2\overline{\mu}_k)=-1$. We have seen that this is always the case for the $(1,3)$, $(3,1)$, $(2,4)$ and $(4,2)$ elements. We prove in Appendix D.5:

Theorem 11.1. *With our ordering $\mathcal{J}(P_1)=P_3$, $\mathcal{J}(P_2)=P_4$, then*

$$\mathcal{W}^\dagger\,\mathcal{Q}\mathcal{H}\,\mathcal{W}=\begin{pmatrix}0&0&\mathfrak{f}_3&0\\0&0&0&\mathfrak{f}_4\\\mathfrak{f}_1 0&0&0&\\0&\mathfrak{f}_2&0&0\end{pmatrix} \tag{11.5}$$

and the constant matrix

$$\mathfrak{C} := \mathcal{V}^\dagger\, \mathcal{Q}^{-1}\mathcal{H}\, \mathcal{V} = -r^2 \begin{pmatrix} 0 & 0 & 1/\mathfrak{f}_1 & 0 \\ 0 & 0 & 0 & 1/\mathfrak{f}_2 \\ 1/\mathfrak{f}_3 & 0 & 0 & 0 \\ 0 & 1/\mathfrak{f}_4 & 0 & 0 \end{pmatrix} \tag{11.6}$$

satisfies $\mathcal{M}\mathfrak{C} = -\mathfrak{C}\overline{\mathcal{M}}^{-1}$. *Here*

$$\mathfrak{f}_k := c^2\, \frac{\pi^2}{2} \prod_{s=1}^{4} \theta_s[P_k] \left(\frac{\theta_2(0)}{\theta_2[\sum_{j=1}^4 P_j]} \right)^2 \frac{\prod_{\substack{r<s\\ r,s\neq k}} \theta_3[P_r+P_s]^2}{\prod_{l\neq k}\theta_1[P_l-P_k]^2} \tag{11.7}$$
$$\times \left(\imath\, \theta_2[2P_k]\theta_2(0)^3\, x_2 + \theta_4[2P_k]\theta_4(0)^3\, x_1 - \frac{\theta_2[2P_k]\theta_4[2P_k]}{\theta_1[2P_k]}\theta_3(0)^3\, x_3 \right)$$
$$= \frac{K^2{\zeta_k}^2 \left(R_-\,{\zeta_k}^3 + 2\,R x_3 {\zeta_k}^2 - R_+\,\zeta_k - x_3 \left({S_-}^2 + {S_+}^2 \right)\right)}{S_-^2 \left(4\,i x_3 x_-\,{\zeta_k}^3 - R{\zeta_k}^2 + 12\,i x_+\,x_3\zeta_k + {S_+}^2 \right)^2} \tag{11.8}$$

satisfies $\overline{\mathfrak{f}_k} = -\mathfrak{f}_{\mathcal{J}(k)}$, *where*

$$R_\mp = \imath S_-^2 \,(x_\mp(2{k'}^2-1) + x_\pm) \mp 16\imath x_\mp x_3^2, \quad R = 2K^2{k'}^2 - S^2 - 8x_3^2,$$
$$S_\pm = \sqrt{K^2-4x_\pm^2}, \quad S = \sqrt{K^2-4x_+x_-}.$$

We may now use the matrix $\mathfrak{C}$ in evaluating $\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{V}$. By inserting $1_4 = \mathcal{W}^\dagger\mathcal{V}$, we have

$$\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{V} = \frac{1}{r^2}\,\mathfrak{C}\left(\mathcal{W}^\dagger\mathcal{H}\mathcal{V}\right) = -\frac{1}{r^2}\left(\mathcal{V}^\dagger\mathcal{H}\mathcal{W}(z)\right)\mathfrak{C}$$

and so $\mathfrak{C}\left(\mathcal{W}^\dagger\mathcal{H}\mathcal{V}(z)\right)$ is Hermitian for general z. Although $\mathcal{W}^\dagger\mathcal{H}\mathcal{V}(z)$ diverges as $z\to\pm$ the projector removes these. Thus

$$\lim_{z\to 1}\mu^\dagger\mathfrak{C}\left(\mathcal{W}^\dagger\mathcal{H}\mathcal{V}(z)\right)\mu = \mu^\dagger\mathfrak{C}\mathfrak{w}_0^\dagger\mathcal{H}\mathfrak{v}_2\mu$$

is Hermitian, whence

$$\mu^\dagger\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{V}\right)\mu\Big|_{z=1} = \frac{1}{r^2}\,\mu^\dagger\mathfrak{C}\left(\mathcal{W}^\dagger\mathcal{H}\mathcal{V}\right)\mu\Bigg|_{z=1} = \frac{1}{r^2}\,\mu^\dagger\mathfrak{C}\left(\mathfrak{w}_0^\dagger\mathcal{H}\mathfrak{v}_2\right)\mu.$$

Therefore

$$\mu_{aj}^\dagger\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{V}\big|_{z=-1}^{z=1}\right)_{jk}\mu_{kb} = \frac{1}{r^2}\,\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathcal{W}^\dagger\mathcal{H}\mathcal{V}\big|_{z=-1}^{z=1}\right)_{lk}\mu_{kb},$$
$$= \frac{1}{r^2}\,\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}, \tag{11.9}$$
$$\mu_{aj}^\dagger\left(z\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{V}\big|_{z=-1}^{z=1}\right)_{jk}\mu_{kb} = \frac{1}{r^2}\,\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1+\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}. \tag{11.10}$$

11.3. **Evaluating $\int_{-1}^{1}\mathrm{d}z\, \boldsymbol{v}_a^\dagger\boldsymbol{v}_b$.** We have

$$\int \mathrm{d}z\,\boldsymbol{v}_a^\dagger\boldsymbol{v}_b = \mu_{aj}^\dagger\left(\int \mathrm{d}z\,\mathcal{V}^\dagger\mathcal{V}\right)_{jk}\mu_{kb} = \mu_{aj}^\dagger\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{V}\right)_{jk}\mu_{kb}.$$

Thus

$$\int_{-1}^{1}\mathrm{d}z\,\boldsymbol{v}_a^\dagger\boldsymbol{v}_b = \mu_{aj}^\dagger\left(\mathfrak{v}_{j,2}^\dagger\mathrm{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x})\mathfrak{v}_{k,2} - \mathcal{M}_{jj}\mathfrak{v}_{j,2}^\dagger U^T\mathrm{Res}_{z=-1}\mathcal{Q}^{-1}(z,\boldsymbol{x})U\mathfrak{v}_{k,2}\overline{\mathcal{M}}_{kk}\right)\mu_{kb}$$
$$= \mu_{aj}^\dagger\left(\mathfrak{v}_{j,2}^\dagger\mathrm{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x})\mathfrak{v}_{k,2}\left[1+\exp(2\mu_j+2\overline{\mu}_k)\right]\right)\mu_{kb}$$

where we have used (11.1) together with $U^TU = 1_4$ and $U^2 = -1_4$. Combining this with (11.9) and Theorem 11.1 yields:

Theorem 11.2. *We have the formulae for the orthogonalisation*

$$
\begin{aligned}
\int_{-1}^{1} \mathrm{d}z\, \boldsymbol{v}_a^\dagger \boldsymbol{v}_b &= \mu_{aj}^\dagger \left(\mathfrak{v}_{j,2}^\dagger \mathrm{Res}_{z=1} \mathcal{Q}^{-1}(z,\boldsymbol{x}) \mathfrak{v}_{k,2} \left[1+\exp(2\mu_j + 2\overline{\mu}_k)\right]\right) \mu_{kb}, \\
&= \frac{1}{r^2}\, \mu_{aj}^\dagger \mathfrak{C}_{jl} \left(\mathfrak{w}_0^\dagger \mathcal{H} \mathfrak{v}_2\right)_{lk} \left[1-\exp(-2\overline{\mu}_l + 2\overline{\mu}_k)\right] \mu_{kb}, \\
&= \mu_{aj}^\dagger \mathfrak{F}_{jl} \left(\boldsymbol{W}_0^\dagger \mathcal{H} \mathfrak{v}_2\right)_{lk} \left[1-\exp(-2\overline{\mu}_l + 2\overline{\mu}_k)\right] \mu_{kb},
\end{aligned}
\tag{11.11}
$$

where

$$
\mathfrak{F} = \frac{1}{r^2}\, \mathfrak{C}\mathfrak{d}^\dagger = \begin{pmatrix} 0 & 0 & \overline{\mathfrak{d}_3/\mathfrak{f}_3} & 0 \\ 0 & 0 & 0 & \overline{\mathfrak{d}_4/\mathfrak{f}_4} \\ \overline{\mathfrak{d}_1/\mathfrak{f}_1} & 0 & 0 & 0 \\ 0 & \overline{\mathfrak{d}_2/\mathfrak{f}_4} & 0 & 0 \end{pmatrix} = \mathfrak{F}^\dagger
$$

and

$$
\mathfrak{f}_j/\mathfrak{d}_j = \mathfrak{D}_j \mathfrak{f}_j = \imath \frac{\zeta_j K^2 (R_- \zeta_j^3 + 2Rx_3\zeta_j^2 - R_+\zeta_j - x_3(S_+^2 + S_-^2))}{(4\imath x_3 x_- \zeta_j^3 - R\zeta_j^2 + 12\imath x_+ x_3 \zeta_j + S_+^2)\, S_-^2}.
$$

Here we have defined

$$
\begin{aligned}
&R_\mp = \imath S_-^2 (x_\mp(2{k'}^2 - 1) + x_\pm) \mp 16\imath x_\mp x_3^2, \quad R = 2K^2{k'}^2 - S^2 - 8x_3^2, \\
&S_\pm = \sqrt{K^2 - 4x_\pm^2}, \quad S = \sqrt{K^2 - 4x_+x_-}.
\end{aligned}
$$

Conjugation acts at these quantities by

$$
\overline{S} = S, \quad \overline{S_\pm} = S_\mp, \quad \overline{R} = R, \quad \overline{R_\pm} = -\imath S_+^2 (x_\mp(2{k'}^2 - 1) + x_\pm) \mp 16\imath x_\mp x_3^2 \tag{11.12}
$$

and our convention is that

$$
\overline{(\zeta_1, \zeta_2, \zeta_3, \zeta_4)} = (-1/\zeta_3, -1/\zeta_4, -1/\zeta_1, -1/\zeta_2). \tag{11.13}
$$

11.4. **Evaluating $\int_{-1}^{1} \mathbf{d}z\, z \boldsymbol{v}_a^\dagger \boldsymbol{v}_b$.** Now we must simplify

$$
\int \mathrm{d}z\, z \boldsymbol{v}_a^\dagger \boldsymbol{v}_b = \mu_{aj}^\dagger \left(\int \mathrm{d}z\, z \mathcal{V}^\dagger \mathcal{V} \right)_{jk} \mu_{kb} = \mu_{aj}^\dagger \left(\mathcal{V}^\dagger \mathcal{Q}^{-1} \left[z + \mathcal{H} \frac{x^i}{r^2} \frac{\partial}{\partial x^i} \right] \mathcal{V} \right)_{jk} \mu_{kb}. \tag{11.14}
$$

The first term here is treated as before and we have

$$
\begin{aligned}
\int_{-1}^{1} \mathrm{d}z\, z \boldsymbol{v}_a^\dagger \boldsymbol{v}_b &= \mu_{aj}^\dagger \left(\mathfrak{v}_{j,2}^\dagger \mathrm{Res}_{z=1} \mathcal{Q}^{-1}(z,\boldsymbol{x}) \mathfrak{v}_{k,2} \left[1-\exp(2\mu_j + 2\overline{\mu}_k)\right]\right) \mu_{kb} \\
&\quad + \mu_{aj}^\dagger \left(\mathcal{V}^\dagger \mathcal{Q}^{-1}\, \mathcal{H} \frac{x^i}{r^2} \frac{\partial}{\partial x^i} \mathcal{V} \Big|_{z=-1}^{z=1} \right)_{jk} \mu_{kb}.
\end{aligned}
$$

The constancy of $\mathfrak{C} = \mathcal{V}^\dagger \mathcal{Q}^{-1} \mathcal{H} \mathcal{V}$ means that the derivative acts only on the $\mathcal{V}$. Let us further consider the final term. Writing

$$
\mathcal{V}^\dagger \mathcal{Q}^{-1} \mathcal{H} \frac{x^i}{r^2} \frac{\partial}{\partial x^i} \mathcal{V} = \mathfrak{C} \left(\mathcal{W}^\dagger \frac{x^i}{r^2} \frac{\partial}{\partial x^i} \mathcal{V} \right)
$$

then

$$
\mu^\dagger \left(\mathcal{V}^\dagger \mathcal{Q}^{-1}\, \mathcal{H} \frac{x^i}{r^2} \frac{\partial}{\partial x^i} \mathcal{V} \Big|_{z=-1}^{z=1} \right) \mu = \mu^\dagger \mathfrak{C} \left(\mathcal{W}^\dagger\, \frac{x^i}{r^2} \frac{\partial}{\partial x^i} \mathcal{V} \Big|_{z=-1}^{z=1} \right) \mu
$$

with the only z dependence in $\mathcal{W}^\dagger\mathcal{V}'$, where $'$ abbreviates $\frac{x^i}{r^2}\frac{\partial}{\partial x^i}$. Noting that $x^i\partial_i\mathfrak{v}_{k,0}=0$ and that $x^i\partial_i\mathfrak{v}_{k,1}=\mathfrak{v}_{k,1}$ is annihilated by the projector then

$$\left(\mathcal{W}^\dagger\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}\Big|_{z=1}\right)\mu=\mathfrak{w}_0^\dagger\left(\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right)\mu$$

while

$$\begin{aligned}\left(\mathcal{W}^\dagger\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}\Big|_{z=-1}\right)\mu&=\lim_{\xi\to0}\overline{\mathcal{M}}^{-1}\left(\frac{1}{\sqrt{\xi}}\mathfrak{w}_\mathbf{0}-\sqrt{\xi}\mathfrak{w}_\mathbf{1}+\xi^{3/2}\mathfrak{w}_\mathbf{2}\right)^\dagger U^\dagger U\\&\quad\times\left(\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\left[\left(\frac{1}{\xi^{3/2}}\mathfrak{v}_0-\frac{1}{\xi^{1/2}}\mathfrak{v}_1+\xi^{1/2}\mathfrak{v}_2\right)\overline{\mathcal{M}}\right]\right)\mu\\&=\overline{\mathcal{M}}^{-1}\mathfrak{w}_0^\dagger\left(\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right)\overline{\mathcal{M}}\mu+\left(\overline{\mathcal{M}}^{-1}\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mathcal{M}}\right)\mu\end{aligned}$$

where we have used (8.16) and (8.17). Thus

$$\begin{aligned}\mu_{aj}^\dagger\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\,\mathcal{H}\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}\Big|_{z=-1}^{z=1}\right)_{jk}\mu_{kb}&=\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\left[\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right]\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad-\mu_{aj}^\dagger\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}\end{aligned}$$

and consequently

$$\begin{aligned}\int_{-1}^{1}\mathrm{d}z\,z\boldsymbol{v}_a^\dagger\boldsymbol{v}_b&=\mu_{aj}^\dagger\left(\mathfrak{v}_{j,2}^\dagger\mathrm{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x})\mathfrak{v}_{k,2}\left[1-\exp(2\mu_j+2\overline{\mu}_k)\right]\right)\mu_{kb}\\&\quad+\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\left[\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right]\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad-\mu_{aj}^\dagger\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}.\end{aligned}\tag{11.15}$$

The derivatives $\mathfrak{w}_0^\dagger\left(\partial_i\mathfrak{v}_2\right)\mu$ appearing here may be combined in various ways. From (8.17) and that $\mathfrak{v}_{k,0}$ is constant we have that

$$\mathfrak{w}_0^\dagger\mathfrak{v}_2\mu=\mu,\qquad\mathfrak{w}_0^\dagger\left(\partial_i\mathfrak{v}_2\right)\mu=-\left(\partial_i\mathfrak{w}_0^\dagger\right)\mathfrak{v}_2\mu.$$

Also

$$\begin{aligned}\partial_i\mathfrak{w}_{k,0}=\partial_i\left(\mathfrak{d}_k\boldsymbol{W}_{k,0}\right)&=\left(\partial_i\mathfrak{d}_k\right)\boldsymbol{W}_{k,0}+\mathfrak{d}_k\partial_i\begin{pmatrix}1\\ \imath\zeta_k\\ \imath\zeta_k\\ -\zeta_k^2\end{pmatrix},\\&=\left[\frac{\partial_i\mathfrak{d}_k}{\mathfrak{d}_k}+\frac{\partial_i\zeta_k}{\zeta_k}\right]\mathfrak{w}_{k,0}-\begin{pmatrix}1\\0\\0\\ \zeta_k^2\end{pmatrix}\mathfrak{d}_k\frac{\partial_i\zeta_k}{\zeta_k},\end{aligned}$$

giving

$$\mathfrak{w}_0^\dagger\left(\partial_i\mathfrak{v}_2\right)\mu=-\,\mathrm{Diag}\left[\frac{\partial_i\mathfrak{d}_k}{\mathfrak{d}_k}+\frac{\partial_i\zeta_k}{\zeta_k}\right]^\dagger\mu+\mathrm{Diag}\left[\mathfrak{d}_k\frac{\partial_i\zeta_k}{\zeta_k}\right]^\dagger\begin{pmatrix}1&\ldots&1\\0&&0\\0&&0\\\zeta_1^2&\ldots&\zeta_4^2\end{pmatrix}^\dagger\mathfrak{v}_2\mu.$$

Similarly

$$\mathfrak{w}_0^\dagger\mathfrak{v}_2\overline{\mathcal{M}}\mu=\overline{\mathcal{M}}\mu,\qquad\mathfrak{w}_0^\dagger\left(\partial_i\mathfrak{v}_2\right)\overline{\mathcal{M}}\mu=-\left(\partial_i\mathfrak{w}_0^\dagger\right)\mathfrak{v}_2\overline{\mathcal{M}}\mu,$$

and we can express the derivative similarly at $z=-1$.

Bringing these results together gives

Theorem 11.3.

$$\begin{aligned}\int_{-1}^{1}\mathrm{d}z\,z\boldsymbol{v}_a^\dagger\boldsymbol{v}_b&=\mu_{aj}^\dagger\left(\mathfrak{v}_{j,2}^\dagger\mathrm{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x})\mathfrak{v}_{k,2}\left[1-\exp(2\mu_j+2\overline{\mu}_k)\right]\right)\mu_{kb}\\&\quad+\ \mu_{aj}^\dagger\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\,\mathcal{H}\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}\Big|_{z=-1}^{z=1}\right)_{jk}\mu_{kb}\\&=\mu_{aj}^\dagger\left(\mathfrak{v}_{j,2}^\dagger\mathrm{Res}_{z=1}\mathcal{Q}^{-1}(z,\boldsymbol{x})\mathfrak{v}_{k,2}\left[1-\exp(2\mu_j+2\overline{\mu}_k)\right]\right)\mu_{kb}\\&\quad+\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\left[\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right]\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad-\mu_{aj}^\dagger\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}\\&=\frac{1}{r^2}\,\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1+\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad+\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\left[\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right]\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad-\mu_{aj}^\dagger\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}\\&=\mu_{aj}^\dagger\mathfrak{F}_{jl}\left(\boldsymbol{W}_0^\dagger\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1+\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad+\mu_{aj}^\dagger\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\left[\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right]\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad-\mu_{aj}^\dagger\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}\\&=\mu_{aj}^\dagger\mathfrak{F}_{jl}\left(\boldsymbol{W}_0^\dagger\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1+\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad+\mu_{aj}^\dagger\mathfrak{F}_{jl}\left(\boldsymbol{W_0}^\dagger\left[x^i\frac{\partial}{\partial x^i}\mathfrak{v}_2\right]\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\\&\quad-\mu_{aj}^\dagger\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}\\&=\mu_{aj}^\dagger\mathfrak{F}_{jl}\left(\boldsymbol{W}_0^\dagger\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1+\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}\end{aligned}$$

$$+\,\mu^\dagger_{aj}\mathfrak{F}_{jl}\,\mathrm{Diag}\left[\frac{x^i\,\partial_i\zeta_k}{\zeta_k}\right]^\dagger_{ll}\left(\begin{pmatrix}1 & \dots & 1\\ 0 & & 0\\ 0 & & 0\\ \zeta_1^2 & \dots & \zeta_4^2\end{pmatrix}^\dagger \mathfrak{v}_2\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}$$
$$-\,\mu^\dagger_{aj}\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}.$$

We conclude with some comments on the Hermiticity of these expressions. From

$$\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{H}\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}\right)^\dagger=\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\left(\mathcal{V}^\dagger\mathcal{H}\mathcal{Q}^{-1}\mathcal{V}\right)-\mathcal{V}^\dagger\left(\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\left[\mathcal{H}\mathcal{Q}^{-1}\right]\right)\mathcal{V}-\mathcal{V}^\dagger\mathcal{H}\mathcal{Q}^{-1}\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}$$

together with the homogeneity $x^i\partial_i\left(\mathcal{Q}^{-1}\mathcal{H}\right)=\mathcal{Q}^{-1}\mathcal{H}$ (which easily follows from the simpler $x^i\partial_i\left(\mathcal{Q}\mathcal{H}\right)=\mathcal{Q}\mathcal{H}$) this becomes

$$=\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\left(\mathcal{V}^\dagger\mathcal{H}\mathcal{Q}^{-1}\mathcal{V}\right)+\frac{1}{r^2}\,\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{H}\mathcal{V}+\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{H}\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}.$$

Then the constancy of $\mathcal{V}^\dagger\mathcal{Q}^{-1}\mathcal{H}\mathcal{V}$ then shows that

$$\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\,\mathcal{H}\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}\Big|_{z=-1}^{z=1}\right)^\dagger=\left(\mathcal{V}^\dagger\mathcal{Q}^{-1}\,\mathcal{H}\,\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathcal{V}\Big|_{z=-1}^{z=1}\right)$$

is Hermitian. If we consider the hermiticity of (11.15) the first term on the right-hand side is manifestly Hermitian while the property $\overline{\mathfrak{f}_k}=-\mathfrak{f}_{\mathcal{J}(k)}$ together with (4.32) show that the final term is Hermitian. Indeed the matrix $\mathfrak{C}\Lambda$ is Hermitian for any diagonal matrix $\Lambda=\mathrm{Diag}(\lambda_1,\dots,\lambda_4)$ provided $\overline{\lambda}_j=-\lambda_j$. By rewriting the middle term

$$\mu^\dagger_{aj}\mathfrak{C}_{jl}\left(\mathfrak{w}_0^\dagger\left[\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_2\right]\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb}$$
$$=\mu^\dagger_{aj}\left(\mathfrak{v}^\dagger_{j,2}\mathrm{Res}_{z=1}\mathcal{Q}^{-1}\mathcal{H}\,\left[\frac{x^i}{r^2}\frac{\partial}{\partial x^i}\mathfrak{v}_{k,2}\right]\left[1+\exp(2\mu_j+2\overline{\mu}_k)\right]\right)\mu_{kb}$$

we see that it also is Hermitian.

11.5. **Calculating the Higgs Field.** We now describe how to calculate the Higgs field Φ and the more important gauge invariant quantity

$$H(\boldsymbol{x}):=\sqrt{-\frac{1}{2}\,\mathrm{Tr}\;\Phi^2}=1-\frac{1}{r}+O(r^{-2}).\tag{11.16}$$

Define

$$\mathcal{G}:=\mathrm{Gram}=\left(\int_{-1}^1 dz\,\boldsymbol{v}_a^\dagger.\boldsymbol{v}_b\right)_{a,b=1,2},\qquad \mathcal{H}:=\mathrm{Higgs}'=\left(\int_{-1}^1 dz\,z\boldsymbol{v}_a^\dagger.\boldsymbol{v}_b\right)_{a,b=1,2}.\tag{11.17}$$

The (Hermitian and positive definite) Gram matrix $\mathcal{G}$ may be diagonalized and written as

$$\mathcal{G}=U^\dagger\,\mathrm{Diag}\,U=N^\dagger N,\qquad N:=\sqrt{\mathrm{Diag}}\;U.$$

The Higgs field is then (in terms of the unnormalized Higgs′ expressions)

$$\Phi=N^{\dagger-1}\mathcal{H}N^{-1}.$$

When we calculate the Higgs field we will need to calculate this factorization of $\mathcal{G}$ but this is not necessary to calculate the gauge invariant quantity

$$H^2 = -\frac{1}{2}\,\mathrm{Tr}\ \Phi^2 = -\frac{1}{2}\,\mathrm{Tr}\ \left(N^{\dagger -1}\mathcal{H}N^{-1}\right)\left(N^{\dagger -1}\mathcal{H}N^{-1}\right) = -\frac{1}{2}\,\mathrm{Tr}\ \mathcal{H}\mathcal{G}^{-1}\mathcal{H}\mathcal{G}^{-1}.$$

At this stage we have all the needed formulae to evaluate $H^2(\boldsymbol{x})$ (for generic space time points) and, upon solving the diagonalization, the Higgs field Φ.

Our strategy is then to calculate

$$\begin{aligned}
\mathrm{Gram} &= \mu^{\dagger}_{aj}\mathfrak{F}_{jl}\left(\boldsymbol{W}_0^{\dagger}\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb},\\
\mathrm{Higgs}'_1 &= i\mu^{\dagger}_{aj}\mathfrak{F}_{jl}\left(\boldsymbol{W}_0^{\dagger}\mathcal{H}\mathfrak{v}_2\right)_{lk}\left[1+\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb},\\
\mathrm{Higgs}'_2 &= i\mu^{\dagger}_{aj}\mathfrak{F}_{jl}\,\mathrm{Diag}\left[\frac{x^i\,\partial_i\zeta_k}{\zeta_k}\right]^{\dagger}_{ll}\left(\begin{pmatrix}1&\dots&1\\0&&0\\0&&0\\\zeta_1^2&\dots&\zeta_4^2\end{pmatrix}^{\dagger}\mathfrak{v}_2\right)_{lk}\left[1-\exp(-2\overline{\mu}_l+2\overline{\mu}_k)\right]\mu_{kb},\\
\mathrm{Higgs}'_3 &= -i\mu^{\dagger}_{aj}\mathfrak{C}_{jk}\left(\frac{2x^i}{r^2}\frac{\partial}{\partial x^i}\overline{\mu}_k\right)\mu_{kb}
\end{aligned}$$

where $\mathrm{Higgs}'_{1,2,3}$ simply correspond to the three terms arising in the evaluation of the integral in Theorem 11.3 and Gram the terms of Theorem 11.2.

11.6. **Calculating the Energy Density.** Although one could calculate the gauge fields via (2.13) and from these the energy density $\mathcal{E}$, the easiest way to calculate the energy density is using a formula of Ward [48]

$$\mathcal{E}(\boldsymbol{x}) = -\frac{1}{2}\nabla^2\,\mathrm{Tr}\,\Phi^2, \tag{11.18}$$

which is normalized[16] such that $E=\int_{\mathbb{R}^3}\mathcal{E}(\boldsymbol{x})\,d^3x = 4\pi n$ for the charge n monopole. Then

Lemma 11.4.

$$\begin{aligned}
-\mathcal{E}(\boldsymbol{x}) = {}& \mathrm{Trace}\left(\left[\frac{\partial\mathcal{H}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{H}.\mathcal{G}_{1,i}\right]^2\right)\\
&+\mathrm{Trace}\left(\left\{\frac{\partial^2\mathcal{H}}{\partial x_i^2}.\mathcal{G}^{-1}-2\frac{\partial\mathcal{H}}{\partial x_i}.\mathcal{G}_{1,i}+\mathcal{H}.\left[\mathcal{G}_{1,i}.\frac{\partial\mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{G}_{2,i}+\mathcal{G}^{-1}.\frac{\partial\mathcal{G}}{\partial x_i}.\mathcal{G}_{1,i}\right]\right\}.\mathcal{H}.\mathcal{G}^{-1}\right)
\end{aligned} \tag{11.19}$$

where

$$\mathcal{G}_{1,i} = \mathcal{G}^{-1}.\frac{\partial\mathcal{G}}{\partial x_i}.\mathcal{G}^{-1},\qquad \mathcal{G}_{2,i} = \mathcal{G}^{-1}.\frac{\partial^2\mathcal{G}}{\partial x_i^2}.\mathcal{G}^{-1}.$$

Using (11.2,11.3) all of the derivatives here involve expressions such as $\partial_i\zeta$ and $\partial_i\mu$ and we have described earlier how these are to evaluated. Thus the energy density may be calculated analytically; the large number of terms mean this is best done with computer algebra. As an example we establish in Appendix D.7

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[16] This normalization varies: [34] chooses $E=\int_{\mathbb{R}^3}\mathcal{E}(\boldsymbol{x})\,d^3x = 2\pi n$ while [46] has $E=\int_{\mathbb{R}^3}\mathcal{E}(\boldsymbol{x})\,d^3x = 4\pi n$. For the charge $n=1$ monopole we have $H(\boldsymbol{x}) = \coth(2r)-\frac{1}{2r}$ and $\mathcal{E}(0)=8/3$.

Proposition 11.5. *The Energy density at the origin is given by*

$$\mathcal{E}_{\boldsymbol{x}=0}(k) = \frac{32}{k^8 k'^2 K^4} \left[k^2(K^2 k'^2 + E^2 - 4EK + 2K^2 + k^2) - 2(E-K)^2\right]^2 \tag{11.20}$$

The limiting values of $\mathcal{E}_{\boldsymbol{x}=0}(k)$ at $k=0$ and $k=1$ are

$$\mathcal{E}_{\boldsymbol{x}=0}(0) = \frac{8}{\pi^4}(\pi^2-8)^2, \qquad \mathcal{E}_{\boldsymbol{x}=0}(1) = 0. \tag{11.21}$$

We plot $\mathcal{E}_{\boldsymbol{x}=0}(k)$ in Figure 14.

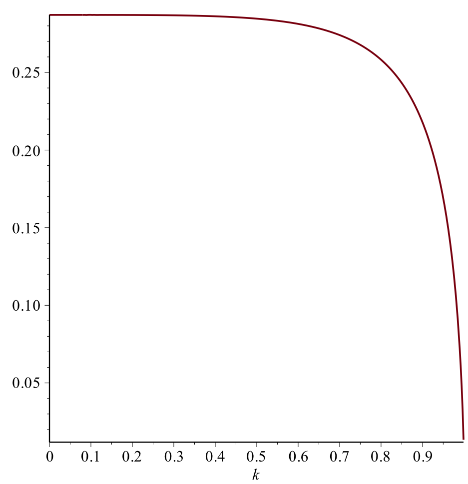

FIGURE 14. $\mathcal{E}_{\boldsymbol{x}=0}(k)$

12. The Higgs Field on the coordinate axes

In this section we shall calculate the Higgs Field on each of the coordinate axes. We have already done this for the x_2 axis using Panagopoulos's formulae for evaluating the Higgs field and determined $H(\boldsymbol{x}) = \sqrt{-\frac{1}{2}\operatorname{Tr}\Phi^2}$. Here we shall employ Theorem 11.3. First though we recall an old approach of Brown, Panagopoulos and Prasad [11] sufficient to make comparison with the results here; the full details are given in Appendix E.

Brown, Panagopoulos and Prasad observed that (a constant) conjugation of the operator $\Delta^\dagger$ took the form

$$\frac{\mathrm{d}}{\mathrm{d}z}1_4 + \begin{pmatrix} (f_3+f_1-f_2)/2 & -x_3 & -x_1 & ix_2 \\ -x_3 & (f_3-f_1+f_2)/2 & ix_2 & -x_1 \\ -x_1 & -ix_2 & (f_1+f_2-f_3)/2 & x_3 \\ -ix_2 & -x_1 & x_3 & -(f_1+f_2+f_3)/2 \end{pmatrix}$$

and so on any coordinate axis this reduced to two 2×2 matrix equations. They focussed on the axis joining the two monopoles where they showed that each of the 2×2 matrix equations reduced to Lamé's equation. For that case they determined the two normalizable solutions to $\Delta^\dagger v = 0$ and then showed that two of the three Higgs field components vanished; denoting the remaining non-vanishing component by ϕ then [11, 6.13] expresses this as

$$H = -i\phi = -k'K + \frac{2k'}{k^2\,\mathrm{sn}^2\,t - S^2}\left(S - \frac{\mathrm{sn}\,t}{\mathrm{cn}\,t\,\mathrm{dn}\,t}\frac{dS}{dt}\right), \tag{12.1}$$

where [11, 6.11]

$$S(t) = -\frac{\mathrm{sn}\,t\,\mathrm{dn}\,t}{\mathrm{cn}\,t}\tanh\left(KZ(t)\right)$$

and t is defined through the relation [11, 6.8]

$$4k'^2x_{1,BPP}^2 - (1+k^2) = -1 - k^2\,\mathrm{cn}^2\,t \Longleftrightarrow \mathrm{sn}^2\,t = \frac{4x_{1,BE}^2}{k^2K^2}.$$

We will relate this to our expressions making use of the results of §5.2 and §5.3. Appendix E performs the analysis for each of the coordinate axes and relates the conventions and scalings[17] of Brown, Panagopoulos and Prasad to those here; the analysis of the Appendix clarifies some of the arguments of [11].

For each coordinate axis we record the calculations of Gram and Higgs'_i $(i = 1, 2, 3)$ of Theorem 11.3 and then evaluate $H(\boldsymbol{x})$. It is convenient to define

$$\mathcal{D}(\lambda) = \mathrm{Diag}(\mathrm{e}^{-2\lambda},\, \mathrm{e}^{2\lambda}), \qquad \mathcal{M}(\lambda) = \begin{pmatrix} \mathrm{e}^{-2\,\lambda}\sinh(2\,\lambda) & -1 \\ -1 & -\ \mathrm{e}^{2\,\lambda}\sinh(2\,\lambda)\end{pmatrix}. \tag{12.2}$$

Then $\mathrm{Tr}\left[\mathcal{D}(\lambda)\mathcal{M}(\lambda)\right]^2/2 = \cosh^2(2\,\lambda)$. We begin first with the simpler case of the x_2 axis, recovering our earlier result, and then turn to the other axes which contain points of bitangency.

12.1. **The x_2 axis.** With $\mu_1 = \lambda_2 + i\pi/4$ and

$$\zeta_1 = \frac{iKk + \sqrt{K^2{k'}^2 + 4\,{x_2}^2}}{\sqrt{K^2 + 4\,{x_2}^2}}$$

we obtain

$$\begin{aligned}
\mathrm{Gram} &= 8\,\cosh^3(2\,\lambda_2)\,K^2k^2\mathcal{D}(-\lambda_2)\\
\mathrm{Higgs}'_1 &= 8\,i\cosh^2(2\,\lambda_2)\left[K^2k^2\mathcal{M}(-\lambda_2) + 2(K^2+4x_2^2)\sigma_1\right]\\
\mathrm{Higgs}'_2 &= -\frac{32\,i\cosh^3(2\,\lambda_2)\,x_2K^2k^2}{\sqrt{K^2+4\,{x_2}^2}\sqrt{K^2{k'}^2+4\,{x_2}^2}}\mathcal{D}(-\lambda_2)\\
\mathrm{Higgs}'_3 &= -16\,i\cosh^2(2\lambda_2)(EK + 4x_2^2)\,\sigma_1
\end{aligned}$$

Assembling these we again obtain (10.9) (for $-\infty < x_2 < \infty$)

$$H^2(0, x_2, 0) = \left(\tanh 2\lambda_2 + \frac{4x_2}{W_2}\right)^2 + \frac{(K{k'}^2 - 2E + K)^2}{K^2k^4\cosh^2 2\lambda_2}$$

with $W_2 = \sqrt{(K^2+4x_2^2)(K^2{k'}^2+4x_2^2)}$. These are plotted in Figure 15 for different scales using $\lambda_2 = \lambda'_2 - i\pi/4$ where λ'_2 is given by (5.14).

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[17] $x_{BPP} = x_{BE}/Kk'$, $\phi_{BE} = \phi_{BPP}/(k'K)$.

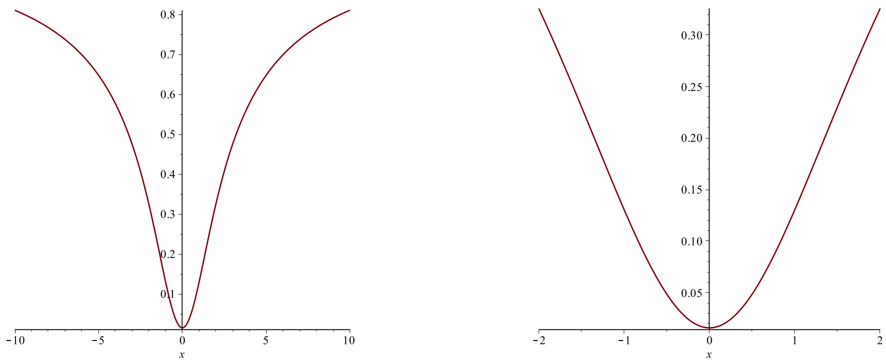

FIGURE 15. $H^2(\boldsymbol{x}) := -\frac{1}{2}\,\mathrm{Tr}\ \Phi^2$ restricted to the x_2-axis $k = 0.8$

12.2. **The x_1 axis.** We have seen that the points $\pm Kk/2$, $\pm K/2$ of the x_1 axis correspond to nongeneric points of bitangency to the spectral curve. Without including higher order terms in the expansion of the eigenfunctions we consider separate cases and we show here that these piece together to one expression for H^2. Recall that we identified in (5.5)

$$\zeta_1 = \zeta_1(x_1) = \frac{Kk' + \imath\sqrt{K^2k^2 - 4x_1^2}}{\sqrt{K^2 - 4x_1^2}}$$

12.2.1. *Calculations for $|x_1| < Kk/2$.* Here $\mu_1 = \lambda_1 + i\pi/4$ with λ_1 given by (5.13).

$$\begin{aligned}
\mathrm{Gram} &= 8\cosh(2\lambda_1)\left(-K^2\,{k'}^2\left(\cosh^2(2\lambda_1)\right) + K^2 - 4\,{x_1}^2\right)\mathcal{D}(\lambda_1)\\
\mathrm{Higgs}'_1 &= 8i\left(-K^2\,{k'}^2\left(\cosh^2(2\lambda_1)\right) + K^2 - 4\,{x_1}^2\right)\mathcal{M}(\lambda_1)\\
\mathrm{Higgs}'_2 &= -\frac{16\imath K^2{k'}^2\,x_1\sinh(4\lambda_1)}{\sqrt{K^2 - 4x_1^2}\sqrt{K^2k^2 - 4x_1^2}}\,\mathcal{M}(\lambda_1)\\
\mathrm{Higgs}'_3 &= 16i\left(K(E-K) + 4\,{x_1}^2\right)\mathcal{M}(\lambda_1)
\end{aligned}$$

12.2.2. *Calculations for $Kk/2 < |x_1| < K/2$.*

$$\begin{aligned}
\mathrm{Gram} &= -8\left(K^2{k'}^2\sinh^2(2\mu_1) + K^2 - 4x_1^2\right)1_2\\
\mathrm{Higgs}'_1 &= -8\imath\left(K^2{k'}^2\sinh^2(2\mu_1) + K^2 - 4x_1^2\right)\sigma_1\\
\mathrm{Higgs}'_2 &= \frac{16\imath K^2{k'}^2\sinh(4\,\mu_1)\,x_1}{\sqrt{K^2 - 4\,{x_1}^2}\sqrt{K^2k^2 - 4\,{x_1}^2}}\sigma_1\\
\mathrm{Higgs}'_3 &= -16i\left(K(E-K) + 4\,{x_1}^2\right)\sigma_1
\end{aligned}$$

12.2.3. *Calculations for $K/2 < |x_1|$.* With $\mu_1 = \lambda''_1$.

$$\begin{aligned}
\mathrm{Gram} &= 8\cosh(2\lambda''_1)({k'}^2K^2\sinh^2(2\lambda''_1) + K^2 - 4x_1^2)\,\mathrm{Diag}(\mathrm{e}^{-2\,\lambda''_1}, \mathrm{e}^{2\,\lambda''_1})\\
\mathrm{Higgs}'_1 &= 8\imath(K^2{k'}^2\sinh^2(2\lambda''_1) + K^2 - 4x_1^2)\mathcal{M}(-\lambda''_1)\\
\mathrm{Higgs}'_2 &= -\frac{16\imath K^2{k'}^2\sinh(4\,\lambda''_1)\,x_1}{\sqrt{K^2 - 4\,{x_1}^2}\sqrt{K^2k^2 - 4\,{x_1}^2}}\,\mathcal{M}(-\lambda''_1)\\
\mathrm{Higgs}'_3 &= 16\,i\left(K(E-K) + 4\,{x_1}^2\right)\mathcal{M}(-\lambda''_1)
\end{aligned}$$

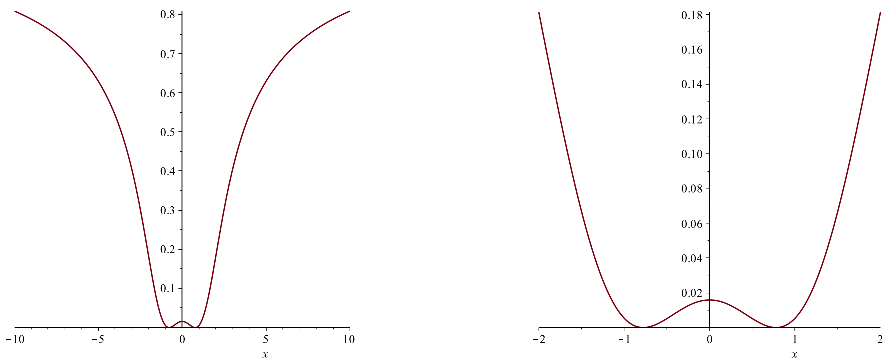

FIGURE 16. $H^2(\boldsymbol{x}) := -\frac{1}{2}\,\mathrm{Tr}\,\Phi^2$ restricted to the x_1-axis $k = 0.8$

Assembling these pieces yields

$$H(x_1,0,0) = \begin{cases} 1 + \dfrac{2K(-Kk'^2\cosh^2 2\lambda_1 + E)}{K^2k'^2\cosh^2 2\lambda_1 - K^2 + 4x_1^2} - \dfrac{1}{W_1}\dfrac{2K^2k'^2 x_1\sinh 4\lambda_1}{K^2k'^2\cosh^2 2\lambda_1 - K^2 + 4x_1^2} & |x| < \frac{Kk}{2} \\ 1 - \dfrac{2K(Kk'^2\sinh^2 2\mu_1 + E)}{K^2k'^2\cosh^2 2\mu_1 + K^2k^2 - 4x_1^2} + \dfrac{1}{W_1}\dfrac{2K^2k'^2 x_1\sinh 4\mu_1}{K^2k'^2\cosh^2 2\mu_1 + K^2k^2 - 4x_1^2} & \frac{Kk}{2} < |x| < \frac{K}{2} \\ 1 - \dfrac{2K(Kk'^2\sinh^2 2\lambda_1'' + E)}{K^2k'^2\cosh^2 2\lambda_1'' + k^2K^2 - 4x_1^2} + \dfrac{1}{W_1}\dfrac{2K^2k'^2 x_1\sinh 4\lambda_1''}{K^2k'^2\cosh^2 2\lambda_1'' + K^2k^2 - 4x_1^2} & \frac{K}{2} < |x| \end{cases}$$

where

$$W_1 = \sqrt{(K^2k^2 - 4x_1^2)(K^2 - 4x_1^2)}.$$

The sign of the square root W_1 in regions **II**, **III** requires a little more care and is best done by analytic continuation. We also note that in all of these formulae there are apparent singularities at $x_1 = \pm Kk/2, \pm K/2$ and the zeros of the denominator

$$K^2k'^2\cosh^2 2\mu_1 = 4x_1^2 - K^2k^2.$$

We may in fact solve this transcendental equation: our previous results $\mu_1(\pm Kk/2) = i\pi/4$ and $\mu_1(\pm K/2) = 0$ show the roots to again be $x_1 = \pm Kk/2, \pm K/2$. The signs may be verified as such that give finite values. Analytic continuation away from the axis around these values shows that the signs of region **II**, **III** coincide with the single expression for the whole axis

$$H(x_1,0,0) = 1 + \frac{2K(-Kk'^2\cosh^2 2\lambda_1 + E)}{K^2k'^2\cosh^2 2\lambda_1 - K^2 + 4x_1^2} - \frac{1}{W_1}\frac{2K^2k'^2 x_1\sinh 4\lambda_1}{K^2k'^2\cosh^2 2\lambda_1 - K^2 + 4x_1^2} \tag{12.3}$$

where λ_1 is given by (5.13). Note that $\lambda(x_1)$ is odd and overall $H^2(x_1,0,0,0)$ is even. From the expression when $|x| > K/2$ we see it has the desired asymptotics. We again observe (1.1), a necessary test of consistency. This is plotted in Figure 16.

12.3. **The x_3 axis.** Again we have the points of bitangency $\pm Kk'/2$ and our analysis proceeds for two intervals which again may be expressed in terms of a single expression.

12.3.1. *Calculations for* $|x_3| < Kk'/2$. With $\mu_1 = i\lambda = \lambda_3 + i\pi/4$ and

$$\zeta_1 = \frac{i\sqrt{K^2k^2 + 4\,x_3{}^2} + \sqrt{K^2k'^2 - 4\,x_3{}^2}}{K}.$$

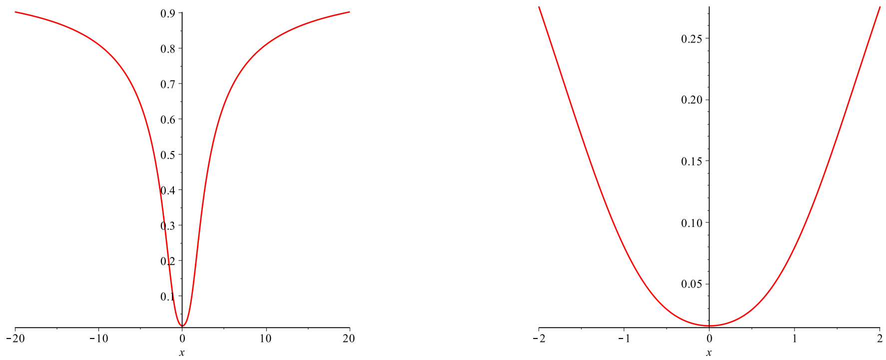

FIGURE 17. $H^2(\boldsymbol{x}) := -\frac{1}{2}\,\mathrm{Tr}\ \Phi^2$ restricted to the x_3-axis $k = 0.8$

$$\begin{aligned}
\mathrm{Gram} &= 8(K^2 \sin^2(2\,\lambda) - K^2 k'^2 + 4\,{x_3}^2) 1_2 \\
\mathrm{Higgs}'_1 &= 8\,i\left(K^2 \sin^2(2\,\lambda) - K^2 k'^2 + 4\,{x_3}^2\right)\sigma_1 \\
\mathrm{Higgs}'_2 &= \frac{-16\,\imath\,\sin(4\lambda) x_3 K^2}{\sqrt{K^2k'^2 - 4x_3^2}\sqrt{K^2k^2+4x_3^2}}\,\sigma_1 \\
\mathrm{Higgs}'_3 &= -16\,i\left(-K^2k'^2 + KE + 4\,{x_3}^2\right)\sigma_1
\end{aligned}$$

12.3.2. *Calculations for $Kk'/2 < |x_3|$.*

$$\begin{aligned}
\mathrm{Gram} &= 8\,\cosh(2\,\mu_1)\left(K^2\cosh^2(2\,\mu_1) - K^2k^2 - 4x_3^2\right)\mathcal{D}(-\mu_1) \\
\mathrm{Higgs}'_1 &= 8\imath\left(K^2\cosh^2(2\,\mu_1) - K^2k^2 - 4\,x_3^2\right)\mathcal{M}(-\mu_1) \\
\mathrm{Higgs}'_2 &= \frac{16\sinh(4\mu_1)x_3K^2}{\sqrt{K^2k'^2 - 4x_3^2}\sqrt{K^2k^2+4x_3^2}}\,\mathcal{M}(-\mu_1) \\
\mathrm{Higgs}'_3 &= 16\imath(-K^2k'^2 + EK + 4x_3^2)\,\mathcal{M}(-\mu_1)
\end{aligned}$$

Recall that we have shown that for $|x_3| < Kk'/2$ then $\mu_1 = \lambda_3 + i\pi/4$ with λ_3 given in (5.15). These combine to yield the single expression for the x_3-axis we have

$$H(0,0,x_3) = 1 - \frac{2K(K\cosh^2 2\mu_1 + E - K)}{K^2\cosh^2 2\mu_1 - K^2k^2 - 4x_3^2} + \frac{1}{W_3}\frac{2i\,K^2x_3\sinh 4\mu_1}{K^2\cosh^2 2\mu_1 - K^2k^2 - 4x_3^2} \tag{12.4}$$

where now

$$W_3 = \sqrt{(K^2k'^2 - 4x_3^2)(K^2k^2+4x_3^2)}. \tag{12.5}$$

Here the expressions for μ_1 combine with the signs of the square roots W_3. The expression given in fact holds for $x_3 > -Kk'/2$ with $\mu_1 = \lambda_3 + i\pi/4$ and λ_3 given by (5.15). We note that both W_3 and the dominators $K^2\cosh^2 2\mu_1 - K^2k^2 - 4x_3^2$ vanish at $x_3 = Kk'/2$ and again careful analysis of the function $H^2(0,0,x_3)$ shows these points it to be regular. The shape of the field $H^2(0,0,x_3)$ is shown in Figure 17 . Again we have the consistency check (1.1) and the correct asymptotics.

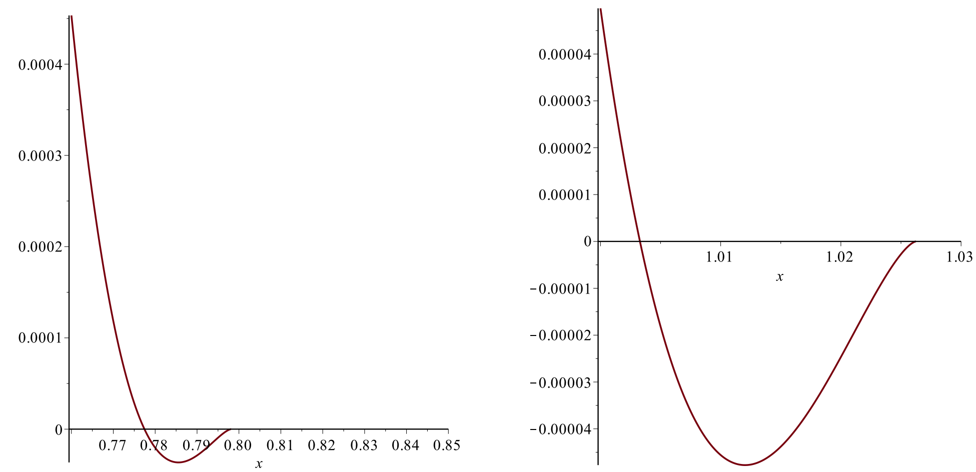

FIGURE 18. Zeros of equation (12.6) for (left) $k = 0.8$; $kK(k)/2 = 0.798$ is the upper zero and (right) $k = 0.9$; $kK(k)/2 = 1.026$.

12.4. **Zeros of the Higgs Field.** The position of the zeros of the Higgs field was an early subject of discussion. Based on the numerical evaluation of their ansatz Forgács, Horváth and Palla [22, (21)] gave these to be (in our units) $\pm kK(k)/2$. Two analytic works then followed. In [42, §6, §9] an analytic expression that needed differentiation was obtained; this 'very complicated' expression was evaluated numerically where the zero was found to be 'very close' and 'barely distinguishable' from $\pm kK(k)/2$. In [11] expansions for the zeros of the Higgs field were given for k near 0 and 1 (the latter being situated near $\pm K(k)/2$). In [23, §6] Forgács, Horváth and Palla expressed that their earlier result was to be viewed as a very good approximation of the zeros.

The position of the zeros of the Higgs field, which lie on the x_1 axis, may be found from (12.3). We have already recorded that at the points of bitangency $\pm kK(k)/2$ and $\pm K(k)/2$ the numerator and denominators of (12.3) vanish, but by l'Hopital's rule for example one sees regular behaviour here. We may express the zero of the Higgs field then as the vanishing of the numerator of (12.3), but discounting $\pm kK(k)/2$ and $\pm K(k)/2$. With

$$Y = \exp(4\lambda_1(x)), \qquad W = \sqrt{(K^2k^2 - 4x^2)(K^2 - 4x^2)},$$

(and λ_1 again defined in 5.13) we obtain the transcendental equation

$$-Y^2\,(W + 4\,x) - 2\,\frac{W\left(K^2k^2 + 4\,EK - 3\,K^2 + 8\,x^2\right)Y}{K^2\,(k^2-1)} - W + 4\,x = 0. \tag{12.6}$$

The vanishing of W at the points of bitangency makes checking the vanishing of this equation there straightforward. One finds for $k \in (0,1)$ a further point of vanishing in $(0, kK(k)/2)$. Figure 18 illustrates this zero for which we have found no analytic expression.

13. THE $k = 0$ LIMIT

Early analytic studies of monopoles followed work of Manton [33] assuming axial symmetry. One of the surprises discovered was that an axially symmetric monopole corresponded to coincident charges [28]. Ward [48] developed the Atiyah-Ward ansatz in the monopole setting and gave an ansatz that produced a charge 2 axially symmetric monopole. Our aim in this section is to reproduce Ward's results as the $k \to 0$ limit of our own. We will first recall Ward's results, then obtain those as a limit and then conclude with a new result (Proposition 13.2).

13.1. **Ward's Results.** Ward [48] expresses the Higgs field (with our enumeration of axes) as

$$\Phi = \begin{pmatrix} U & V\mathrm{e}^{-2\imath\psi} \\ W\mathrm{e}^{2\imath\psi} & -U \end{pmatrix} \tag{13.1}$$

where $x_1 + ix_3 = |x_1 + ix_3|e^{i\psi}$. Then on the x_2-axis

$$\begin{aligned} U &= \frac{x_2}{x_2^2 + c^2} - \tanh(2z) \\ V &= W = 0 \end{aligned} \tag{13.2}$$

while on the (x_1, x_3)-plane

$$\begin{aligned} U &= 0 \\ V = W &= \frac{c^2 \cosh(2a)[\sinh(2a) - 2a\cosh(2a)]}{a(a^2 - c^2 \sinh^2(2a))} - 1 \end{aligned} \tag{13.3}$$

where $a = \sqrt{r^2 - c^2}$. Ward found that only $c = \pi/4$ gave nonsingular solutions.

13.2. **The $k = 0$ limit of the Atiyah-Ward constraint and relevant quantities.** The $k = 0$ limit of the Atiyah-Ward constraint yields the equation

$$\left[(x_2 + \imath x_1)\zeta^2 + 2x_3\zeta - x_2 + \imath x_1\right]^2 + \frac{\pi^2}{16}(\zeta^2 - 1)^2 = 0.$$

With $r^2 = x_1^2 + x_2^2 + x_3^2$ and $x_\pm = x_1 \pm \imath x_2$ the solutions are

$$\zeta_{1,2} = \frac{4\imath x_3 \pm \sqrt{\pi^2 - 16r^2 + 8\imath\pi x_2}}{4x_- - \pi}, \quad \zeta_{3,4} = \frac{4\imath x_3 \pm \sqrt{\pi^2 - 16r^2 - 8\imath\pi x_2}}{4x_- + \pi}, \tag{13.4}$$

where we again order the roots according to the conjugation conditions

$$\zeta_3 = -\frac{1}{\overline{\zeta}_1}, \quad \zeta_4 = -\frac{1}{\overline{\zeta}_2}.$$

Noting that $K(0) = E(0) = \pi/2$ one has that the corresponding μ_i are

$$\begin{aligned} \mu_{1,2} &= \left(\frac{\imath\pi}{4} - x_2 - \imath x_1\right)\zeta_{1,2} - x_3 = \mp\frac{\imath}{4}\sqrt{\pi^2 - 16r^2 + 8\imath\pi x_2}, \\ \mu_{3,4} &= \left(-\frac{\imath\pi}{4} - x_2 - \imath x_1\right)\zeta_{3,4} - x_3 + \frac{\imath\pi}{2} = \mp\frac{\imath}{4}\sqrt{\pi^2 - 16r^2 - 8\imath\pi x_2} + \frac{\imath\pi}{2}. \end{aligned} \tag{13.5}$$

One sees that

$$\mu_1 + \overline{\mu}_3 = -\frac{\imath\pi}{2}, \quad \mu_2 + \overline{\mu}_4 = -\frac{\imath\pi}{2}.$$

Upon introducing the notation

$$R_+ = \sqrt{\pi^2 - 16r^2 + 8\imath\pi x_2}, \quad R_- = \overline{R}_+ = \sqrt{\pi^2 - 16r^2 - 8\imath\pi x_2} \tag{13.6}$$

it is convenient to rewrite the above formulae as

$$\begin{aligned} \zeta_{1,2} &= \frac{4\imath x_3 \pm R_+}{4x_- - \pi}, \quad \zeta_{3,4} = \frac{4\imath x_3 \pm R_-}{4x_- + \pi}, \\ \mu_{1,2} &= \mp\frac{\imath}{4}R_+, \quad \mu_{3,4} = \mp\frac{\imath}{4}R_- + \frac{\imath\pi}{2}. \end{aligned} \tag{13.7}$$

13.3. **The x_2-axis.** We have previously obtained (10.9)

$$H(0,x_2,0)^2 = \left(\tanh 2\lambda + \frac{4x_2}{W_2}\right)^2 + \frac{(Kk'^2 - 2E + K)^2}{K^2k^4\cosh^2 2\lambda}$$

with

$$\lambda = \mu_1 - \frac{\imath\pi}{4}, \quad W_2 = \sqrt{(K^2+4x_2^2)(K^2k'^2+4x_2^2)}.$$

Now on the x_2 axis with $k=0$ then $\zeta_{1,2,3,4} = \pm 1$ and $\mu_{1,2} = \frac{\imath\pi}{4} - x_2$. In the limit $k \to 0$ the second term in $H(0,x_2,0)^2$ vanishes and we obtain

$$H(0,x_2,0)^2_{k=0} = \left(-\tanh(2x_2) + \frac{16x_2}{16x_2^2+\pi^2}\right)^2.$$

This coincides with Ward's result upon his use of $c = \pi/4$.

13.4. **The (x_1,x_3)-plane.** Next we obtain the Higgs field and on the $x_2 = 0$-plane. We remark that our Higgs field is a gauge transform of Ward's by the gauge transformation $\mathrm{Diag}(e^{-i\psi}, e^{i\psi})$. In Appendix D.8 we establish

Proposition 13.1. *The Higgs field at $k=0$ and $x_2=0$ is function of $r = \sqrt{x_1^2+x_3^2}$ in the whole (x_1,x_3)-plane given by*

(13.8)
$$\begin{aligned} &H(x_1,0,x_3) = H(r) \\ &= -1 - \frac{2\pi^2\cos(\frac12\sqrt{\pi^2-16r^2})(2\sin(\frac12\sqrt{\pi^2-16r^2}) - \sqrt{\pi^2-16r^2}\cos(\frac12\sqrt{\pi^2-16r^2}))}{\sqrt{\pi^2-16r^2}(\pi^2\cos^2(\frac12\sqrt{\pi^2-16r^2}) - 16r^2)} \end{aligned}$$

One can see that $H(0)=0$ accords with earlier formulae for depth of the well.

13.5. **The (x_1,x_2)-plane.** Next we obtain the Higgs field and on the $x_3 = 0$-plane. In Appendix D.9 we establish

Proposition 13.2. *Higgs field on to the plane $x_3 = 0$ is given by*

$$H(x_1,x_2,0)^2 = \sum_{j=1}^{6} \mathcal{H}_j(x_1,x_2) \tag{13.9}$$

with

(13.10)

$$\mathcal{H}_1 = -\frac{S_+^2 S_-^2}{R_+^2 R_-^2 G^2}(\pi^2 - 16x_2^2)\left(-R_+^4 R_-^4 + 2048\pi^4 x_2^2 + 16\pi^2(R_+^2 + R_-^2)\right)$$

$$\mathcal{H}_2 = \frac{8\pi S_+ S_-}{G^2}\left(C_- S_+(\pi + 4\imath x_2)R_- + C_+ S_-(\pi - 4\imath x_2)R_+\right)$$

$$\mathcal{H}_3 = -\frac{8\pi S_+}{R_+ R_- G^2}(C_+ C_- + 1)(\pi + 4\imath x_2)(R_- C_-(\pi^2 + 16r^2) - 4(\pi - 4\imath x_2)\pi S_-)$$

$$\mathcal{H}_4 = -\frac{\pi S_-}{R_- G^2}(\pi - 4\imath x_2)\left\{-16\left[8(C_+C_- + 1)(2C_- - 3C_+) - 16S_+S_-(C_- - C_+)\right]r^2 + 8\pi^2 C_+(C_-C_+ + 1) + 64\imath\pi x_2(C_- + C_+)(C_-C_+ - S_-S_+ + 1)\right\}$$

$$\mathcal{H}_5 = \frac{1}{2}\frac{S_-S_+R_-R_+(C_-C_+ + 1)\pi^2}{E_-^2 + E_+^2}\left\{64\left[2S_-S_+(E_-^2 - E_+^2) - (3C_-C_+ + 1)E_-^2 + (2E_-^4 - 2E_-^2E_+^2 - 1)E_+^2 + C_-C_+E_+^2\right]r^2 - 32\imath(E_-^2E_+^2 + 1)(C_- + C_+)\pi x_2 - 4(E_-^2 + E_+^2)(C_-C_+ + 1)\pi^2\right\}$$

$$\mathcal{H}_6 = \frac{1}{(E_-^2 + E_+^2)G}(C_-C_+ + 1)(\pi^2 + 16r^2)\left\{-16\left[4C_- - 4C_+ + (E_-^2E_+^2 - 1)E_-^2 + 2(C_- - 2C_+)E_-^2E_+^2 - C_-C_+(E_-^2 + E_+2)\right]r^2 + 16\imath(E_-^2E_+^2 + 1)(C_- + C_+)\pi x_2 + (E_-^2 + E_+^2)(C_-C_+ + 1)\pi^2\right\}$$

Here

(13.11) $$S_\pm = \sin\left(\frac{1}{2}R_\pm\right), \quad C_\pm = \cos\left(\frac{1}{2}R_\pm\right) \quad E_\pm = \exp\left(\frac{1}{4}\imath R_\pm\right)$$

and $R_\pm = \sqrt{\pi^2 - 16r^2 \pm \imath\pi x_2}$.

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Appendix A. The Curve

A.1. **Properties of the curve.** We may see that the monopoles are on the x_1 axis (for $k > 0$ and at $k = 0$ the monopoles are axially symmetric about the x_2 axis in several ways.

First our spectral curve takes the form

$$0 = \left(\eta + i\frac{K(k)}{2}\left[\zeta^2+1\right]\right)\left(\eta - i\frac{K(k)}{2}\left[\zeta^2+1\right]\right) - K(k)^2 k'^2\zeta^2$$
$$= \left(\eta + i\frac{K(k)}{2}\left[\zeta^2-1\right]\right)\left(\eta - i\frac{K(k)}{2}\left[\zeta^2-1\right]\right) + K(k)^2 k^2\zeta^2.$$

Now upon noting that $K(k) \sim \ln(4/k')$ as $k \to 1$ then in this limit this behaves as

$$0 \sim \left(\eta + i\frac{K(k)}{2}\left[\zeta^2+1\right]\right)\left(\eta - i\frac{K(k)}{2}\left[\zeta^2+1\right]\right)$$

and so upon comparison with the 1-monopole curve we have two widely separated monopoles at $\pm(\frac{K(k)}{2}, 0, 0)$, on the x_1 axis. Alternately, set $\tilde\eta = \eta - iK(k)\left[\zeta^2+1\right]/2$, which corresponds to a shift by $K(k)/2$ along the x_1 axis, then the curve may be written as

$$0 = \frac{\tilde\eta^2}{K(k)} + i(1+\zeta^2)\tilde\eta - K(k)k'^2\zeta^2.$$

Now again letting $k \to 1$ we find $(1+\zeta^2)\tilde\eta = 0$. If $\tilde\eta = 0$ we see the second monopole is at the origin, and so both lie on the x_1 axis; if $\zeta = \pm i$ then $\tilde\eta = \eta = 2\mathbf{y}\cdot\boldsymbol{x} = 2(x_2 \mp i x_3)$ corresponds to a line parallel to the x_1 axis through the point $\tilde\eta = \eta$. Finally, we can read off the axis of symmetry from the curve as follows. If $k = 0$ we have

$$0 = \left(\eta + i\frac{K(0)}{2}\left[\zeta^2-1\right]\right)\left(\eta - i\frac{K(0)}{2}\left[\zeta^2-1\right]\right)$$

where $K(0) = \pi/2$, and this corresponds to the (complex) points $(0, \pm iK(0)/2, 0)$. A rotation around the x_2 axis leaves this invariant.

A.2. **Comparison of Notation.** The charge 2 spectral curve has appeared with many differing conventions. We give here transformations between these to enable comparison with existing results.

$\mathcal{C}$	$0 = \eta^2 + \frac{K^2}{4}(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1) = \lvert\eta - L(\zeta)\rvert$	**Here, Ercolani-Sinha**
$\mathcal{C}_{FHP}$	$0 = y^2 + A\left(x^2 + \frac{1}{x^2}\right) + B$	**Forgács, Horváth &Palla**
$\mathcal{C}_{ORS}$	$0 = \gamma^2 + 1 - \frac{\varepsilon^2}{4}\left(\zeta_B - \zeta_B^{-1}\right)^2$	**O'Raifeartaigh et. al., Brown**
$\mathcal{C}_{H83}$	$w^2 = r_1 z^3 - r_2 z^2 - r_1 z, \qquad r_{1,2} \in \mathbb{R},\ r_1 \geq 0$	**Hurtubise 83**
$\mathcal{C}_{H85}$	$w^2 = \kappa(z^2 - s^2)(s^2 z^2 - 1), \qquad \kappa > 0,\ s \in [0,1)$	**Hurtubise 85**
$\mathcal{C}_{AH}$	$\eta_{AH}^2 = K^2\,\zeta_{AH}\left(kk'[\zeta_{AH}^2 - 1] + (k^2 - k'^2)\zeta_{AH}\right)$	**Atiyah, Hitchin**

The reality properties of the spectral curve are preserved by the transformations of $T\mathbb{P}^1$

$$R := \begin{pmatrix} p & q \\ -\bar q & \bar p \end{pmatrix} \in PSU(2), \qquad \zeta \to \zeta_R := \frac{\bar p\,\zeta - \bar q}{q\,\zeta + p}, \qquad \eta \to \eta_R := \frac{\eta}{(q\,\zeta + p)^2}, \tag{A.1}$$

which correspond to a spatial rotation. In particular with $p = e^{-i\theta/2}$, $q = 0$, we may rotate $(\zeta, \eta) \to e^{i\theta}(\zeta, \eta)$, and so the relative signs between η^2 and the highest powers (ζ^4 or ζ^3) may be chosen so that the leading coefficients are positive. This, for example, enabled Hurtubise [29] to choose his coefficient $r_1 \geq 0$.

We will describe our procedure for establishing the needed birational correspondence between our curve and the other curves with the curve of Forgács, Horváth and Palla as the example.

A.2.1. *Transforming between $\mathcal{C}$ and $\mathcal{C}_{FHP}$.* First we record that the parameters of the Forgács, Horváth and Palla curve are related by.

$$\sqrt{B} = \frac{1}{\sqrt{1+\beta}} K\left(\sqrt{\frac{2\beta}{1+\beta}}\right), \quad A = \frac{1}{2}\beta B, \quad \beta \in [-1, 0].$$

To see that a transformation (A.1) between the curves is possible we compute the Klein absolute invariants of both curves, $j_{\mathcal{C}}$, $j_{\mathcal{C}_{ORS}}$ and find the allowed relations between the parameters k and β given their prescribed domains. Here,

$$j_{\mathcal{C}} = \frac{256(k^4 - k^2 + 1)^3}{k^4(k^2-1)^2}, \quad j_{\mathcal{C}_{FHP}} = \frac{64(3\beta^2+1)^3}{\beta^2(\beta^2-1)^2}. \tag{A.2}$$

The equation $j_{\mathcal{C}} = j_{\mathcal{C}_{FHP}}$ admits a number of solutions, and among them exists one suitable, namely,

$$\beta = -\frac{k^2}{1+{k'}^2}, \qquad \beta \in [-1, 0] \leftrightarrow [0, 1] \ni k^2, \tag{A.3}$$

and equivalently,

$$k^2 = \frac{2\beta}{\beta - 1}. \tag{A.4}$$

The relation (A.3) enables us to connect the parameters of the curves,

$$B = \frac{1+{k'}^2}{2} K(k)^2, \qquad K(k) = \sqrt{\frac{2B}{1+{k'}^2}}. \tag{A.5}$$

To find the explicit transformation (A.1) one can equate the fractional linear transformations of branch points of $\mathcal{C}$ with the branch points of $\mathcal{C}_{ORS}$. There are 24 variants of such homogeneous equations with respect to the parameters of the fractional linear transformations but only four of them admit non-zero solution. An appropriate transformation is given by

$$\begin{aligned} \zeta &= \frac{\imath x - 1}{\imath x + 1}, \quad \eta = \frac{2\imath x y}{(\imath - x)^2}, \\ x &= \imath \frac{\zeta + 1}{\zeta - 1}, \quad y = \frac{2\eta}{\zeta^2 - 1}. \end{aligned} \tag{A.6}$$

Note, that transformation (A.6) maps the 4 complex branch points $\pm k' \pm \imath k$ of the curve $\mathcal{C}$ to four real branch points of the curve $\mathcal{C}_{FHP}$ as follows

$$\begin{aligned} \pm(k' + \imath k) &\longleftrightarrow \pm\sqrt{\frac{-1+\sqrt{1-\beta^2}}{\beta}} \equiv \pm\frac{1+k'}{k}, \\ \pm(-k' + \imath k) &\longleftrightarrow \pm\imath\sqrt{\frac{1+\sqrt{1-\beta^2}}{\beta}} \equiv \pm\frac{1-k'}{k}. \end{aligned} \tag{A.7}$$

A.2.2. *Transforming between $\mathcal{C}$ and $\mathcal{C}_{ORS}$.* We note that Brown's curve [10] is the same as O'Raifeartaigh, Rouhani and Singh's [42] with $d \leftrightarrow \varepsilon$. Now

$$j_{\mathcal{C}} = \frac{256(k'^4 - k'^2 + 1)^3}{k'^4(1-k'^2)^2} = \frac{256(\varepsilon^4 + \varepsilon^2 + 1)^3}{\varepsilon^4(\varepsilon^2+1)^2} = j_{\mathcal{C}_{ORS}}$$

has solutions $\varepsilon^2 = -k'^2,\ -k^2,\ -\frac{1}{k'^2},\ -\frac{1}{k^2},\ \frac{k^2}{k'^2},\ \frac{k'^2}{k^2}$. Take

$$\varepsilon = -\frac{\imath}{k}. \tag{A.8}$$

Following the method outlined above we find the transformations

$$\begin{aligned} \zeta &= \frac{1-z}{1+z}, \quad \eta = \frac{2wzK(k)}{(1+z)^2}, \\ z &= \frac{1-\zeta}{1+\zeta}, \quad w = \frac{2\eta}{K(k)(\zeta^2-1)}, \end{aligned} \tag{A.9}$$

(where the parameters k and ϵ are related according (A.8)) take the branch points $\pm k' \pm \imath k$ of $\mathcal{C}$ to the branch points of $\mathcal{C}_{ORS}$, $(\pm 1 \pm \sqrt{\varepsilon^2+1})/\varepsilon$.

A.2.3. *Transforming between $\mathcal{C}$ and $\mathcal{C}_{H85}$.* Consider the transformation

$$\frac{1}{\sqrt{2}}\begin{pmatrix} u & \bar{u} \\ -u & \bar{u}\end{pmatrix}, \qquad u = e^{i\pi/4}; \qquad \zeta = -\frac{iz+1}{iz-1}, \quad \eta = \frac{2iw}{(iz-1)^2}.$$

This transforms our curve (3.1) into

$$w^2 = \frac{K^2}{4}\left(kz^2 + 2z + k\right)\left(kz^2 - 2z + k\right).$$

The substitution

$$k = \frac{2s}{1+s^2}, \qquad k' = \frac{1-s^2}{1+s^2}, \qquad s = \frac{k}{1+k'},$$

then yields

$$w^2 = \frac{K^2}{(1+s^2)^2}\,(z^2 - s^2)(s^2z^2 - 1)$$

which is Hurtubise's curve upon the identification $\kappa = K^2/(1+s^2)^2$. When we substitute this transformation into our Atiyah-Ward constraint $\eta = (x_2 - \imath x_1) - 2\zeta x_3 - (x_2 + \imath x_1)\zeta^2$ we obtain

$$w = -(x_1 - ix_3) + 2x_2 z + (x_1 + ix_3)z^2$$

which corresponds to an interchange $x_2 \leftrightarrow x_3$ with Hurtubise's conventions. With the above identifications we find that our curve of bitangency (6.1) is that of Hurtubise [30] whose curve of bitangency is

$$\frac{\kappa(s^4-1)^2}{4} = (s^2-1)^2x_1^2 + (s^2+1)^2x_2^2, \qquad x_3 = 0.$$

A.2.4. *Transforming between $\mathcal{C}$ and $\mathcal{C}_{AH}$.* Then with the rotation $R = \frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix} -a & 1 \\ -1 & -\bar{a}\end{pmatrix}$ we have

$$\sqrt{\frac{\bar{a}\bar{b}}{ab}}(\zeta-a)(\zeta+\frac{1}{\bar{a}})(\zeta-b)(\zeta+\frac{1}{\bar{b}})\times\frac{1}{(\zeta-a)^4}\to\frac{(b-a)(1+a\bar{b})}{|a||b|(1+|a|^2)^2}\,\zeta_R\left(\zeta_R+\frac{1+\bar{a}b}{b-a}\right)\left(\zeta_R+\frac{\bar{a}-\bar{b}}{1+a\bar{b}}\right).$$

We have previously ordered the four roots of our curve as $a = e^{i\alpha} = k'+ik$, $-1/\bar{a} = -k'-ik$, $b = e^{-i\alpha} = k'-ik$, $-1/\bar{b} = -k'+ik$, where $2\alpha \geq 0$ is the angle between the lines. Then $k = \sin\alpha \geq 0$ and so

$$|a-b| = 2k, \quad |1+\bar{a}b| = 2k', \quad \frac{1+\bar{a}b}{b-a} = i\,\frac{e^{-i\alpha}}{\tan\alpha}, \quad \frac{\bar{a}-\bar{b}}{1+a\bar{b}} = -i\,e^{-i\alpha}\tan\alpha,$$

giving

$$-ie^{i\alpha}\zeta_R\,kk'\left(\zeta_R^2 - ie^{-i\alpha}(\tan\alpha-\cot\alpha)\zeta_R + e^{-2i\alpha}\right).$$

Thus the further rotation $(\zeta_R,\eta_R)\to(\zeta',\eta') := ie^{i\alpha}(\zeta_R,\eta_R)$ gives

$$0 = \eta^2 + \frac{K^2}{4}(\zeta^4 + 2(k^2-k'^2)\zeta^2+1) \to 0 = \eta'^2 - \frac{kk'K^2}{4}\zeta'(\zeta'-k'/k)(\zeta'+k/k'),$$

where $\cot\alpha = k'/k$, $\tan\alpha = k/k'$. Thus we have the Atiyah-Hitchin curve upon the identifications $(\zeta_{AH},\eta_{AH}) = (\zeta', 2\eta')$.

We note that the sign of the term $(\tan\alpha-\cot\alpha)\zeta' = (k^2-k'^2)/kk'\,\zeta'$ depends on whether $0\leq\alpha<\pi/4$ or $\pi/2<\alpha\leq\pi/2$. When $\alpha>\pi/4$ the angle between the lines is obtuse.

A.3. **Calculation of Periods and the Abel Map.** We determine here the periods of the holomorphic differential $\boldsymbol{v}$, the meromorphic differential γ_∞ and express the Abel map in terms of incomplete elliptic integrals.

A.3.1. *The periods of $\boldsymbol{v}$.* Upon using the substitutions $\zeta = e^{i\theta}$ and $k\sin u = \sin\theta$ on sheet 1,

$$\frac{d\zeta}{\eta} = i\frac{2}{K}\frac{d\zeta}{\sqrt{(\zeta^2-e^{2i\alpha})(\zeta^2-e^{-2i\alpha})}} = \frac{-1}{kK}\frac{d\theta}{\sqrt{1-\frac{1}{k^2}\sin^2\theta}} = \frac{-1}{K}\frac{du}{\sqrt{1-k^2\sin^2 u}}.$$

Thus

$$\oint_{\mathfrak{a}}\frac{d\zeta}{\eta} = \frac{-2}{K}\int_{\alpha}^{-\alpha}\frac{d\theta}{\sqrt{k^2-\sin^2\theta}} = \frac{4}{K}\int_0^{\pi/2}\frac{du}{\sqrt{1-k^2\sin^2 u}} = 4.$$

Similarly (with $\zeta = \exp i(w+\pi/2)$, $\sin w = k'\sin u$)

$$\frac{d\zeta}{\eta} = i\frac{2}{K}\frac{d\zeta}{\zeta}\frac{1}{2\sqrt{\sin^2 w - k'^2}} = \frac{-1}{K}\frac{d\zeta}{\zeta}\frac{1}{\sqrt{k'^2-\sin^2 w}} = \frac{-i}{K}\frac{dw}{\sqrt{k'^2-\sin^2 w}} = \frac{-i}{K}\frac{du}{\sqrt{1-k'^2\sin^2 u}}.$$

One determines the sign of the square root in the second equality, for when $\zeta = i$, $w=0$ and $\eta = -iKk'$ on sheet 1 (and crossing no cuts). We then have

$$\oint_{\mathfrak{b}}\frac{d\zeta}{\eta} = -\frac{2i}{K}\int_{\pi/2-\alpha}^{\alpha-\pi/2}\frac{dw}{\sqrt{k'^2-\sin^2 w}} = \frac{4i}{K}\int_0^{\pi/2-\alpha}\frac{dw}{\sqrt{k'^2-\sin^2 w}}$$
$$= \frac{4i}{K}\int_0^{\pi/2}\frac{du}{\sqrt{1-k'^2\sin^2 u}} = \frac{4\,\imath\,\mathbf{K}'(k)}{\mathbf{K}(k)} = 4\tau,$$

where $\tau = \imath\,\mathbf{K}'(k)/\mathbf{K}(k)$ is the period matrix.

A.3.2. *The periods of γ_∞.* Consider

$$\gamma_\infty(P)=\frac{K^2}{4\eta}\left(\zeta^2-\frac{2E-K}{K}\right)\mathrm{d}\,\zeta.$$

Then with the earlier substitutions and again on sheet 1,

$$\frac{K^2}{4}\frac{\zeta^2\mathrm{d}\,\zeta}{\eta}=i\frac{K}{2}\frac{\zeta^2 d\zeta}{\sqrt{(\zeta^2-e^{2i\alpha})(\zeta^2-e^{-2i\alpha})}}=\frac{-K}{4k}\frac{e^{2i\theta}\,d\theta}{\sqrt{1-\frac{1}{k^2}\sin^2\theta}}$$
$$=\frac{-K}{4}\frac{(1-2k^2\sin^2 u+2ik\sin u\sqrt{1-k^2\sin^2 u})du}{\sqrt{1-k^2\sin^2 u}}.$$

Then

$$\frac{K^2}{4}\oint_{\mathfrak{a}}\frac{\zeta^2\mathrm{d}\,\zeta}{\eta}=K\int_0^{\pi/2}\frac{(2\left[1-k^2\sin^2 u\right]-1)du}{\sqrt{1-k^2\sin^2 u}}=K\,(2E-K)$$

and so $\oint_{\mathfrak{a}}\gamma_\infty=0$. Now

$$\frac{K^2}{4}\oint_{\mathfrak{b}}\left(\zeta^2-\frac{2E-K}{K}\right)\frac{d\zeta}{\eta}=-i\frac{K}{2}\int_{\pi/2-\alpha}^{\alpha-\pi/2}\frac{\left[-e^{2iw}-(2E-K)/K\right]dw}{\sqrt{k'^2-\sin^2 w}}$$
$$=-iK\int_0^{\pi/2-\alpha}\frac{\left[\cos 2w+(2E-K)/K\right]dw}{\sqrt{k'^2-\sin^2 w}}$$
$$=-iK\int_0^{\pi/2}\frac{(2\left[1-k'^2\sin^2 u\right]-1+(2E-K)/K)du}{\sqrt{1-k'^2\sin^2 u}}$$
$$=-i2\,(KE'+E'K-KK')=-i\pi=2i\pi\boldsymbol{U},$$

where we have use Legendre's relation.

A.3.3. *The Abel map.* We may express the Abel maps ϕ and α in terms of incomplete elliptic integrals. Denote

$$a=k'+\imath k,\quad b=k'-\imath k,\quad c=\frac{2E-K}{K}\in\mathbb{R}.$$

Here $a=P_0$ is the base point of the Abel map $\phi(\zeta)=\frac{1}{4}\int_a^\zeta\frac{\mathrm{d}\zeta}{\eta}$. One can represent $\phi(\zeta)$ in terms of Jacobian incomplete integrals

$$\phi(\zeta)=\frac{\imath}{2Kb}\left(F\left(\frac{\zeta}{a},\frac{a}{b}\right)-F\left(1,\frac{a}{b}\right)\right),\tag{A.10}$$

and this representation accords with the relations of [7],

$$\phi(\infty_1)=\frac{1+\tau}{4}=-\phi(\infty_2),\quad \phi(0_1)=\frac{1-\tau}{4}=-\phi(0_2).$$

We note the relations

$$\frac{a}{b}=\frac{1+ik/k'}{1-ik/k'}=\frac{1-\dot{k}'}{1+\dot{k}'},\quad \dot{k}'=-\frac{ik}{k'},\quad \dot{k}=\frac{1}{k'},\quad K\left(\frac{a}{b}\right)=\frac{1+\dot{k}'}{2}K(\dot{k})=\frac{b}{2}\left(K'(k)+iK(k)\right).$$

Using these our normalized Abel map now reads

$$\alpha(\zeta(\boldsymbol{x}))=\phi(\zeta(\boldsymbol{x}))-\phi(\infty_1)=\frac{\imath}{2K}\frac{1}{b}F\left(\frac{\zeta(\boldsymbol{x})}{a},\frac{a}{b}\right)-\frac{\tau}{2}.\tag{A.11}$$

A.3.4. *Numerical Computation.* We have shown that

$$\gamma_\infty(P)=\frac{K^2}{4\eta}\left(\zeta^2-\frac{2E-K}{K}\right)\mathrm{d}\,\zeta.$$

Then

(A.12)
$$\int_{P_0}^{P}\gamma_\infty(P')=\frac{\imath K}{2}\int_a^{\zeta(\boldsymbol{x})}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}}=\frac14\left(\frac{\theta_1'\left(\alpha(\zeta(\boldsymbol{x}))\right)}{\theta_1\left(\alpha(\zeta(\boldsymbol{x}))\right)}+\frac{\theta_3'\left(\alpha(\zeta(\boldsymbol{x}))\right)}{\theta_3\left(\alpha(\zeta(\boldsymbol{x}))\right)}\right)-\frac{\imath\pi}{4}$$

with $\alpha(\zeta(\boldsymbol{x}))$ being the normalized Abel map (A.11). Now

(A.13)
$$\begin{aligned}&\int_a^{\zeta(\boldsymbol{x})}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}}\\&=bE\left(\frac{a}{b}\right)-\frac{b^2-c}{b}K\left(\frac{a}{b}\right)-bE\left(\frac{\zeta(\boldsymbol{x})}{a},\frac{a}{b}\right)+\frac{b^2-c}{b}F\left(\frac{\zeta(\boldsymbol{x})}{a},\frac{a}{b}\right)\end{aligned}$$

where $K(\kappa),E(\kappa)$ and $F(z,\kappa),E(z,\kappa)$ with $\kappa=a/b$ are standard complete and incomplete elliptic integrals of the first and second kind respectively. We remark that some care is needed in keeping track of the sheets when using this representation.

A.4. **Proof of Lemma 3.1.** To show that (3.1) is parameterized by

$$\zeta=-i\,\frac{\theta_2[P]\theta_4[P]}{\theta_1[P]\theta_3[P]},\qquad \eta=\frac{i\pi\,\theta_3\theta_2^2\theta_4^2}{4}\,\frac{\theta_3[2P]}{\theta_1[P]^2\theta_3[P]^2}.$$

we use (with $\theta_i:=\theta_i(0)$)

$$\begin{aligned}&k=\frac{\theta_2^2}{\theta_3^2},\quad k'=\frac{\theta_4^2}{\theta_3^2},\quad \theta_2^4+\theta_4^4=\theta_3^4,\quad K=\frac{\pi}{2}\,\theta_3^2\\&\theta_1[P]^2\theta_3[P]^2+\theta_2[P]^2\theta_4[P]^2=\theta_2^2\theta_4\,\theta_4[2P],\\&2\,\theta_1[P]\theta_2[P]\theta_3[P]\theta_4[P]=\theta_2\theta_3\theta_4\,\theta_1[2P],\\&\theta_3(x+y)\theta_3(x-y)\,\theta_4^2=\theta_4^2(x)\theta_3^2(y)-\theta_1^2(x)\theta_2^2(y)=\theta_3^2(x)\theta_4^2(y)-\theta_2^2(x)\theta_1^2(y);\end{aligned}$$

the latter with $x=0$ and $y=2\alpha(P)$. Then

$$\begin{aligned}\zeta^4+2(k^2-k'^2)\zeta^2+1&=\frac{\theta_3^4\left(\theta_1[P]^2\theta_3[P]^2+\theta_2[P]^2\theta_4[P]^2\right)^2+4(\theta_4^4-\theta_3^4)\,\theta_1[P]^2\theta_2[P]^2\theta_3[P]^2\theta_4[P]^2}{\theta_3^4\,\theta_1[P]^4\theta_3[P]^4}\\&=\frac{\theta_3^4\theta_2^4\theta_4^2\,\theta_4[2P]^2-\theta_2^6\theta_3^2\theta_4^2\,\theta_1[2P]^2}{\theta_3^4\,\theta_1[P]^4\theta_3[P]^4}\\&=\frac{\theta_2^4\theta_3^2\theta_4^2\left(\theta_3^2\,\theta_4[2P]^2-\theta_2^2\,\theta_1[2P]^2\right)}{\theta_3^4\,\theta_1[P]^4\theta_3[P]^4}\\&=\frac{\theta_2^4\theta_4^4}{\theta_3^2}\,\frac{\theta_3[2P]^2}{\theta_1[P]^4\theta_3[P]^4}\end{aligned}$$

so establishing the lemma.

A.5. **Proof of Lemma 3.2.** We will show that both sides have the same periodicity, zeros, poles and residues. Both sides of (3.8) are constant under shifts of $\mathfrak{a}$-periods. A shift in the theta functions under a $\mathfrak{b}$-period is immediate giving $-\imath\,\pi=2\pi i\boldsymbol{U}$ (using $\boldsymbol{U}=-1/2$). That is the right-hand side shifts by ($2\pi\,\imath$ times) the Ercolani-Sinha vector. But

$$\oint_{\mathfrak{b}}\gamma_\infty=2\pi\,\imath\,\boldsymbol{U}$$

is fundamental to its definition and follows from a bilinear relation [7]. Now observe that

$$d\ln\theta_1\left(\int_{P_*}^{P}\boldsymbol{v}\right)=\boldsymbol{v}(P)\,\frac{\theta_1'\left(\int_{P_*}^{P}\boldsymbol{v}\right)}{\theta_1\left(\int_{P_*}^{P}\boldsymbol{v}\right)}$$

and that for a local parameter t at P_*,

$$\boldsymbol{v}=\frac{d\zeta}{4\eta}=[\mu(P_*)+O(t)]dt,\qquad \int_{P_*}^{P}\boldsymbol{v}=\mu(P_*)t+O(t^2).$$

Thus

$$d\ln\theta_1\left(\int_{P_*}^{P}\boldsymbol{v}\right)=dt\,[\mu(P_*)+O(t)]\left(\frac{\theta_1'(0)}{\theta_1(\mu(P_*)t)}+O(t^2)\right)=\frac{dt}{t}\,[1+O(t)]$$

has a simple pole at P_*. Thus expanding the right-hand side of (3.8) at ∞_1, for example, gives

$$\frac{1}{4}\left\{\frac{1}{-t/(4\rho_1)}+\frac{\theta_1'[\infty_1-\infty_2]}{\theta_1[\infty_1-\infty_2]}+\ldots\right\}=\tilde\nu_1-\frac{\rho_1}{t}+\ldots$$

where

$$\tilde\nu_1=\frac{1}{4}\frac{\theta_1'[\infty_1-\infty_2]}{\theta_1[\infty_1-\infty_2]}=-\frac{\imath\,\pi}{4}.$$

We know that in the vicinity of ∞_i the left-hand side looks like,

$$\int_{P_0}^{P}\gamma_\infty(P')=\tilde\nu_i-\frac{\eta}{\zeta}.$$

Thus both sides have the same pole and residue and similarly at ∞_2. Finally at P_0 both sides vanish so establishing the lemma. As remarked after the lemma, this identifying of the vanishing relates the choice of contours on each side of the identity.

Appendix B. Theta function Identitities

B.1. **Weierstrass Trisecant θ-formulae.** In this appendix we describe the Weierstrass Trisecant θ-formulae that we implemented in the course of calculation. Following [50][p47] we introduce 3 vectors $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $\boldsymbol{\alpha}'=(\alpha_1',\alpha_2',\alpha_3',\alpha_4')$, $\boldsymbol{\alpha}''=(\alpha_1'',\alpha_2'',\alpha_3'',\alpha_4'')$ that transformed one to another by the rule:

$$T:\boldsymbol{\alpha}^T\to\boldsymbol{\alpha}'^T=\frac{1}{2}\begin{pmatrix}\alpha_1+\alpha_2+\alpha_3+\alpha_4\\ \alpha_1+\alpha_2-\alpha_3-\alpha_4\\ \alpha_1-\alpha_2+\alpha_3-\alpha_4\\ -\alpha_1+\alpha_2+\alpha_3-\alpha_4\end{pmatrix}$$

which leads to the relations: $T(\boldsymbol{\alpha})=\boldsymbol{\alpha}'$, $T(\boldsymbol{\alpha}')=\boldsymbol{\alpha}''$, $T(\boldsymbol{\alpha}'')=\boldsymbol{\alpha}$.

The following 6 Weierstrass Trisecant θ-relations are valid

$$\begin{aligned}\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_3)\theta_1(\alpha_4)+\theta_1(\alpha_1')\theta_1(\alpha_2')\theta_1(\alpha_3')\theta_1(\alpha_4')\\ +\theta_1(\alpha_1'')\theta_1(\alpha_2'')\theta_1(\alpha_3'')\theta_1(\alpha_4'')=0,\end{aligned}\tag{W1}$$

$$\begin{aligned}\theta_i(\alpha_1)\theta_i(\alpha_2)\theta_1(\alpha_3)\theta_1(\alpha_4)+\theta_i(\alpha_1')\theta_i(\alpha_2')\theta_1(\alpha_3')\theta_1(\alpha_4')\\ +\theta_i(\alpha_1'')\theta_i(\alpha_2'')\theta_1(\alpha_3'')\theta_1(\alpha_4'')=0,\end{aligned}\tag{W2}$$

$$\begin{aligned}\theta_i(\alpha_1)\theta_j(\alpha_2)\theta_k(\alpha_3)\theta_1(\alpha_4)+\theta_i(\alpha_1')\theta_j(\alpha_2')\theta_k(\alpha_3')\theta_1(\alpha_4')\\ +\theta_i(\alpha_1'')\theta_j(\alpha_2'')\theta_k(\alpha_3'')\theta_1(\alpha_4'')=0,\end{aligned} \qquad (W3)$$

$$\begin{aligned}\theta_i(\alpha_1)\theta_i(\alpha_2)\theta_j(\alpha_3)\theta_j(\alpha_4)-\theta_i(\alpha_1')\theta_i(\alpha_2')\theta_j(\alpha_3')\theta_j(\alpha_4')\\ \pm\theta_k(\alpha_1'')\theta_k(\alpha_2'')\theta_1(\alpha_3'')\theta_1(\alpha_4'')=0,\end{aligned} \qquad (W4)$$

$$\begin{aligned}\theta_2(\alpha_1)\theta_2(\alpha_2)\theta_2(\alpha_3)\theta_2(\alpha_4)-\theta_3(\alpha_1')\theta_3(\alpha_2')\theta_3(\alpha_3')\theta_3(\alpha_4')\\ +\theta_4(\alpha_1'')\theta_4(\alpha_2'')\theta_4(\alpha_3'')\theta_4(\alpha_4'')=0,\end{aligned} \qquad (W5)$$

$$\begin{aligned}\theta_i(\alpha_1)\theta_i(\alpha_2)\theta_i(\alpha_3)\theta_i(\alpha_4)-\theta_i(\alpha_1')\theta_i(\alpha_2')\theta_i(\alpha_3')\theta_i(\alpha_4')\\ \pm\theta_1(\alpha_1'')\theta_1(\alpha_2'')\theta_1(\alpha_3'')\theta_1(\alpha_4'')=0.\end{aligned} \qquad (W6)$$

We present here particular cases of these relations that used in our development. From (W2) it follows that:

Proposition B.1. *Let $\alpha_i, \alpha_j, \alpha_k$, $i, j, k \in \{1, 2, 3, 4\}$ be three arbitrary complex numbers. Then for $n = 2, 3, 4$ and $z \in \mathbb{C}$ the following trisecant addition formula is valid*

$$\begin{aligned}&\theta_1(\alpha_i)\theta_1(\alpha_j-\alpha_k)\theta_n\left(\alpha_i\pm\frac{z}{2}\right)\theta_n\left(\alpha_j+\alpha_k\pm\frac{z}{2}\right)\\ &\quad+\theta_1(\alpha_k)\theta_1(\alpha_i-\alpha_j)\theta_n\left(\alpha_k\pm\frac{z}{2}\right)\theta_n\left(\alpha_i+\alpha_j\pm\frac{z}{2}\right)\\ &\qquad+\theta_1(\alpha_j)\theta_1(\alpha_k-\alpha_i)\theta_n\left(\alpha_j\pm\frac{z}{2}\right)\theta_n\left(\alpha_i+\alpha_k\pm\frac{z}{2}\right)=0.\end{aligned} \tag{B.1}$$

From (W3) it follows that:

Proposition B.2. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be four arbitrary complex numbers. Then for any $i, j = 1, \ldots 4$ and arbitrary $z \in \mathbb{C}$ the following trisecant addition formula is valid*

$$\begin{aligned}&\theta_1(\alpha_i)\theta_2\left(\alpha_j\pm\frac{z}{2}\right)\theta_3(\alpha_j)\theta_4\left(\alpha_i\pm\frac{z}{2}\right)-\theta_1(\alpha_j)\theta_2\left(\alpha_i\pm\frac{z}{2}\right)\theta_3(\alpha_i)\theta_4\left(\alpha_j\pm\frac{z}{2}\right)\\ &\quad=\theta_1(\alpha_j-\alpha_i)\theta_2\left(\frac{z}{2}\right)\theta_3(0)\theta_4\left(\alpha_i+\alpha_j\pm\frac{z}{2}\right).\end{aligned} \tag{B.2}$$

From (W4) it follows that:

Proposition B.3. *Let $\alpha_i, \alpha_j, \alpha_k$, $i, j, k \in \{1, 2, 3, 4\}$ be three arbitrary complex numbers. Then for $p = 3, q = 4$ or $p = 4, q = 3$ and $z \in \mathbb{C}$ the following trisecant addition formula is valid*

$$\begin{aligned}&\theta_p(\alpha_i)\theta_p(\alpha_j+\alpha_k)\theta_q\left(\alpha_k\pm\frac{z}{2}\right)\theta_q\left(\alpha_i+\alpha_j-\frac{z}{2}\right)\\ &\quad-\theta_p(\alpha_k)\theta_p(\alpha_i+\alpha_j)\theta_q\left(\alpha_i\pm\frac{z}{2}\right)\theta_q\left(\alpha_j+\alpha_k\pm\frac{z}{2}\right)\\ &\qquad=\theta_2\left(\frac{z}{2}\right)\theta_2\left(\alpha_i+\alpha_j+\alpha_k\pm\frac{z}{2}\right)\theta_1(\alpha_i-\alpha_k)\theta_1(\alpha_j).\end{aligned} \tag{B.3}$$

Also for arbitrary four complex numbers $\alpha_1, \ldots, \alpha_4$, $i, j, k \in \{1, 2, 3, 4\}$ the following trisecant addition formula is valid

$$\begin{aligned}
&\theta_p(\alpha_4+\alpha_2)\theta_p(\alpha_1+\alpha_3)\theta_q\left(\alpha_2+\alpha_1\pm\frac{z}{2}\right)\theta_q\left(\alpha_4+\alpha_3\pm\frac{z}{2}\right)\\
&\quad-\theta_p(\alpha_4+\alpha_3)\theta_p(\alpha_2+\alpha_1)\theta_q\left(\alpha_4+\alpha_2\pm\frac{z}{2}\right)\theta_q\left(\alpha_3+\alpha_1\pm\frac{z}{2}\right)\\
&\quad-\theta_2\left(\frac{z}{2}\right)\theta_2\left(\alpha_1+\alpha_2+\alpha_3+\alpha_4\pm\frac{z}{2}\right)\theta_1(\alpha_2-\alpha_3)\theta_1(\alpha_1-\alpha_4)=0.
\end{aligned}\tag{B.4}$$

Suppose we now have that

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=N\tau,\quad N\in\mathbb{Z}\tag{B.5}$$

the last trisecant relation turns to the following:

$$\begin{aligned}
&\theta_p(\alpha_4+\alpha_2)\theta_p(\alpha_1+\alpha_3)\theta_q\left(\alpha_2+\alpha_1\pm\frac{z}{2}\right)\theta_q\left(\alpha_4+\alpha_3\pm\frac{z}{2}\right)\\
&\quad-\theta_p(\alpha_4+\alpha_3)\theta_p(\alpha_2+\alpha_1)\theta_q\left(\alpha_4+\alpha_2\pm\frac{z}{2}\right)\theta_q\left(\alpha_3+\alpha_1\pm\frac{z}{2}\right)\\
&\quad=\theta_2\left(\frac{z}{2}\right)^2\theta_1(\alpha_2-\alpha_3)\theta_1(\alpha_1-\alpha_4)\exp\left\{-\imath\pi(N^2\tau\pm Nz)\right\}.
\end{aligned}\tag{B.6}$$

The combination of relations (W3) written in the form

$$\begin{aligned}
&\theta_1(\alpha_i)\theta_4\left(\alpha_i-\frac{z}{2}\right)\theta_4\left(\alpha_j\right)\theta_1\left(\alpha_j+\frac{z}{2}\right)\\
&\quad-\theta_3\left(\alpha_i\right)\theta_2\left(\alpha_i-\frac{z}{2}\right)\theta_2\left(\alpha_j\right)\theta_3\left(\alpha_j+\frac{z}{2}\right)\\
&\quad+\theta_2\left(\frac{z}{2}\right)\theta_2\left(\alpha_i+\alpha_j\right)\theta_3\left(0\right)\theta_3\left(\alpha_i-\alpha_j-\frac{z}{2}\right)=0
\end{aligned}\tag{B.7}$$

and

$$\begin{aligned}
&\theta_1\left(\alpha_i+\frac{z}{2}\right)\theta_2\left(\alpha_j-\frac{z}{2}\right)\theta_3\left(\alpha_j\right)\theta_4\left(\alpha_i\right)\\
&\quad-\theta_1\left(\alpha_i\right)\theta_2\left(\alpha_j\right)\theta_3\left(\alpha_j+\frac{z}{2}\right)\theta_4\left(\alpha_i-\frac{z}{2}\right)\\
&\quad+\theta_1\left(\alpha_i-\alpha_j-\frac{z}{2}\right)\theta_2\left(\frac{z}{2}\right)\theta_3\left(0\right)\theta_4\left(\alpha_i+\alpha_j\right)=0
\end{aligned}\tag{B.8}$$

together with (W4) leads to the addition formula that we implemented to calculate the Gram matrix,

$$\begin{aligned}
&\theta_1(\alpha_i)\theta_4\left(\alpha_i-\frac{z}{2}\right)\begin{vmatrix}\theta_3\left(\frac{z}{2}\right)\theta_2(0) & \theta_1\left(\frac{z}{2}\right)\theta_4(0)\\ \theta_1\left(\alpha_j+\frac{z}{2}\right)\theta_4(\alpha_i) & \theta_3\left(\alpha_j+\frac{z}{2}\right)\theta_2(\alpha_j)\end{vmatrix}\\
&+\theta_3(\alpha_i)\theta_2\left(\alpha_i-\frac{z}{2}\right)\begin{vmatrix}\theta_3\left(\frac{z}{2}\right)\theta_2(0) & \theta_1\left(\frac{z}{2}\right)\theta_4(0)\\ \theta_3\left(\alpha_j+\frac{z}{2}\right)\theta_2(\alpha_i) & \theta_1\left(\alpha_j+\frac{z}{2}\right)\theta_4(\alpha_j)\end{vmatrix}\\
&=\theta_2\left(\frac{z}{2}\right)^2\theta_3^2(0)\theta_1(\alpha_i+\alpha_j)\theta_4\left(\alpha_i-\alpha_j-\frac{z}{2}\right).
\end{aligned}\tag{B.9}$$

Finally we note:

Proposition B.4. *Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ be four complex numbers satisfying condition*

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=N\tau,\quad N\in\mathbb{Z}$$

and $z \in \mathbb{C}$. *Then*

$$(-1)^N \prod_{m=1}^{4} \theta_4(\alpha_m) + (-1)^M \prod_{m=1}^{4} \theta_2(\alpha_m) = \theta_3(0)\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k) \times \exp\left\{-2\imath\pi N\alpha_l + \imath\pi N^2\tau\right\}, \tag{B.10}$$

and

$$(-1)^{(N+M)} \prod_{m=1}^{4} \theta_1(\alpha_m) + \prod_{m=1}^{4} \theta_3(\alpha_m) = \theta_3(0)\theta_3(\alpha_i+\alpha_j)\theta_3(\alpha_i+\alpha_k)\theta_3(\alpha_j+\alpha_k) \times \exp\left\{-2\imath\pi N\alpha_l + \imath\pi N^2\tau\right\}, \tag{B.11}$$

with $i \neq j \neq k \neq l \in \{1,2,3,4\}$

Proof. If we use $\boldsymbol{\alpha} = (\alpha_3, \alpha_2, \alpha_4, -\alpha_2-\alpha_3-\alpha_4)$, $\boldsymbol{\alpha'} = (0, \alpha_2+\alpha_3, \alpha_3+\alpha_4, \alpha_2+\alpha_4)$ and $\boldsymbol{\alpha''} = (\alpha_2+\alpha_3+\alpha_4, -\alpha_4, -\alpha_2, \alpha_3)$ in (W6) for $i=3$, together with the Abel relation we obtain the second identity while the first similarly follows from (W5). □

B.2. **Periodicities.**

$$\begin{aligned}
\theta_1(z+M+N\tau) &= (-1)^{N+M+1}\, e^{-i\pi[N^2\tau+2Nz]}\,\theta_1(z),\\
\theta_2(z+M+N\tau) &= (-1)^{M}\, e^{-i\pi[N^2\tau+2Nz]}\,\theta_2(z),\\
\theta_3(z+M+N\tau) &= e^{-i\pi[N^2\tau+2Nz]}\,\theta_3(z),\\
\theta_4(z+M+N\tau) &= (-1)^{N}\, e^{-i\pi[N^2\tau+2Nz]}\,\theta_4(z).
\end{aligned}$$

We also note

$$\theta_2(z \pm 1/2) = \mp\theta_1(z), \tag{B.12}$$

$$\theta_4(z \pm 1/2) = \theta_3(z). \tag{B.13}$$

Appendix C. Identities

C.1. **Proof of Proposition 4.1.** For the curve (3.1) we may write

$$w(P) = -\eta + (x_2 - \imath x_1) - 2\zeta x_3 - (x_2 + \imath x_1)\zeta^2 = c\,\frac{\prod_{i=1}^4 \theta_1[P-P_i]}{\theta_1[P]^2\theta_1[P-\infty_2]^2}, \tag{C.1}$$

some constant c. Here we encounter a subtlety referred to earlier when discussing Abel images. In writing the function in the specified form with the given theta functions, periodicity requires choosing the Abel images so that

$$\alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4 = \sum_k \alpha(P_k) = 2\alpha(\infty_1 + \infty_2).$$

The first part of the proposition is then proven upon establishing that $\phi(2[\infty_1 - \infty_2]) \in \Lambda$, which follows from either (3.5) or observing that this is the divisor of the function

$$\eta + \imath\,\frac{K}{2}\left(\zeta^2 + (k^2 - k'^2)\right).$$

Although our numerical calculation of Abel images was such that $\sum_k \alpha_k = N\tau$, the choice of of sheets in defining (C.1) has $\sum_k \alpha'_k = -(1+\tau)$; agreement can be achieved simply by shifting the argument of one of the θ_1's in (C.1) by the appropriate lattice point, for example

$$\alpha_1 = \alpha'_1, \quad \alpha_2 = \alpha'_2, \quad \alpha_3 = \alpha'_3, \quad \alpha_4 = \alpha'_4 + (N+1)\tau + 1.$$

If we expand $w(P)$ at ∞_1 on sheet 1 we see

$$w(P) = \frac{i}{2}(K-2x_-)\zeta^2 - 2x_3\zeta + \ldots \qquad c_1 = \frac{i}{2}(K-2x_-)\frac{\theta_1^2[\infty_1-\infty_2]\theta_1'^{\,2}}{\prod_{i=1}^4 \theta_1[\infty_1 - P_i]},$$

while on sheet 2

$$w(P) = -\frac{i}{2}(K+2x_-)\zeta^2 - 2x_3\zeta + \ldots \qquad c_2 = -\frac{i}{2}(K+2x_-)\frac{\theta_1^2[\infty_2-\infty_1]\theta_1'^{\,2}}{\prod_{i=1}^4 \theta_1[\infty_2 - P_i]}.$$

Consistency requires that $c_1 = c_2$ or that

$$-\frac{K-2x_-}{K+2x_-} = \frac{\prod_{i=1}^4 \theta_1[\infty_1 - P_i]}{\prod_{i=1}^4 \theta_1[\infty_2 - P_i]} = \exp(i\pi[\sum_k \alpha_k' + \tau])\,\frac{\prod_{i=1}^4 \theta_1(\alpha_i')}{\prod_{i=1}^4 \theta_3(\alpha_i)} = -\frac{\prod_{i=1}^4 \theta_1(\alpha_i')}{\prod_{i=1}^4 \theta_3(\alpha_i')}$$

upon using $\theta_1[\infty_2 - P_k] = -\theta_1(\alpha_k' + (1+\tau)/2) = -\exp(-i\pi[\alpha_k' + \tau/4)\,\theta_3(\alpha_k')$. Now from (4.36, 4.37)

$$\frac{K-2x_-}{K+2x_-} = -\frac{\theta_1(N\tau-\alpha_4)\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_3)}{\theta_3(N\tau-\alpha_4)\theta_3(\alpha_1)\theta_3(\alpha_2)\theta_3(\alpha_3)} = (-1)^{N+1}\frac{\prod_{i=1}^4 \theta_1(\alpha_i)}{\prod_{i=1}^4 \theta_3(\alpha_i)}$$

and the shifts given above establish the needed consistency.

To establish the second identity we use (2.21). With $\zeta = 1/t$ a local parameter we have at ∞_1 on sheet 1

$$w(P) = \frac{i}{2}(K-2x_-)\frac{1}{t^2}\left[1 - \frac{2x_3}{\frac{i}{2}(K-2x_-)}t + \ldots\right] \qquad d\ln w(P) = -2\frac{dt}{t} + \frac{4ix_3}{K-2x_-}\,dt + \ldots$$

while on sheet 2

$$w(P) = -\frac{i}{2}(K+2x_-)\frac{1}{t^2}\left[1 + \frac{2x_3}{\frac{i}{2}(K-2x_-)}t + \ldots\right] \qquad d\ln w(P) = -2\frac{dt}{t} - \frac{4ix_3}{K+2x_-}\,dt + \ldots$$

while with $f(P) = \int_{P_0}^P \gamma_\infty$,

$$f(P) \sim_{P\sim\infty_j} \tilde\nu_j - \frac{\rho_j}{t}.$$

Thus

$$0 = \sum_{\text{Residues}} f(P)d\ln w(P) = \sum_k f(P_k) - 2(\tilde\nu_1 + \tilde\nu_2) - \rho_1\frac{4ix_3}{K-2x_-} + \rho_2\frac{4ix_3}{K+2x_-}$$

giving

$$\sum_k \int_{P_0}^{P_k} \gamma_\infty = \sum_k \frac{1}{4}\left\{\frac{\theta_1'(\alpha_k')}{\theta_1(\alpha_k')} + \frac{\theta_3'[(\alpha_k')}{\theta_3(\alpha_k')} - \imath\pi\right\} = \frac{4K^2x_3}{K^2-4x_-^2}$$

establishing (4.13).

A consequence of this result is that

$$\sum_k \frac{1}{4}\left\{\frac{\theta_1'(\alpha_k)}{\theta_1(\alpha_k)} + \frac{\theta_3'[(\alpha_k)}{\theta_3(\alpha_k)}\right\} + \imath\pi N = \sum_k \beta_1(P_k) + \imath\pi N = \frac{4K^2x_3}{K^2-4x_-^2}$$

and so

$$\sum_k \mu(P_k) = 3\imath\pi N - \sum_k (x_3 + ix_-\zeta_k) + \frac{4K^2x_3}{K^2-4x_-^2} = 3\imath\pi N - 4x_3 - ix_-\left(\sum_k \zeta_k\right) + \frac{4K^2x_3}{K^2-4x_-^2}.$$

Using $\sum_k \zeta_k = -16\imath x_3 x_-/(K^2 - 4x_-^2)$ we obtain

$$\sum_k \mu_k = 3\imath\pi N \tag{C.2}$$

establishing the proposition.

C.2. **Proof of Lemma 4.5.** We use

$$\begin{aligned}
\frac{\theta_1'[P]}{\theta_1[P]} - \frac{\theta_3'[P]}{\theta_3[P]} &= \frac{\theta_3[P]}{\theta_1[P]} d\left(\frac{\theta_1[P]}{\theta_3[P]}\right) = \frac{\theta_3[P]}{\theta_1[P]}\pi\theta_3^2 \frac{\theta_2[P]\theta_4[P]}{\theta_3[P]^2} = 2i\boldsymbol{K}\zeta,\\
\frac{\theta_1''[P]}{\theta_1[P]} - \frac{\theta_3''[P]}{\theta_3[P]} &= \frac{d}{d\alpha(P)}\left(\frac{\theta_1'[P]}{\theta_1[P]} - \frac{\theta_3'[P]}{\theta_3[P]}\right) + \left(\frac{\theta_1'[P]}{\theta_1[P]}\right)^2 - \left(\frac{\theta_3'[P]}{\theta_3[P]}\right)^2\\
&= 2\boldsymbol{K}\frac{d}{d\alpha(P)}\left(\frac{\theta_2[P]\theta_4[P]}{\theta_1[P]\theta_3[P]}\right) + \left(\frac{\theta_1'[P]}{\theta_1[P]} - \frac{\theta_3'[P]}{\theta_3[P]}\right)\left(\frac{\theta_1'[P]}{\theta_1[P]} + \frac{\theta_3'[P]}{\theta_3[P]}\right)\\
&= 2\boldsymbol{K}\left[\frac{\theta_4[P]}{\theta_1[P]}\frac{d}{d\alpha(P)}\left(\frac{\theta_2[P]}{\theta_3[P]}\right) + \frac{\theta_2[P]}{\theta_3[P]}\frac{d}{d\alpha(P)}\left(\frac{\theta_4[P]}{\theta_1[P]}\right)\right] + 2i\boldsymbol{K}\zeta\left(\frac{\theta_1'[P]}{\theta_1[P]} + \frac{\theta_3'[P]}{\theta_3[P]}\right)\\
&= 8i\boldsymbol{K}\eta + 2i\boldsymbol{K}\zeta\left(\frac{\theta_1'[P]}{\theta_1[P]} + \frac{\theta_3'[P]}{\theta_3[P]}\right)\\
&= 8i\boldsymbol{K}\eta + 8i\boldsymbol{K}\zeta\,\beta_1(P).
\end{aligned}$$

In going from the third last to the penultimate line here we are using standard theta function identities such as $(\theta_2[P]/\theta_3[P])' = -\pi\theta_4^2\theta_1[P]\theta_4[P]/\theta_3[P]^2$. Examination of the latter proof shows we have in fact also established (4.20). The final identity follows upon differentiating both sides of

$$\eta^2 = -\frac{K^2}{4}\left(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1\right)$$

and using (4.20).

C.3. **Proof of Corollary 4.6.** The first set two relations follow upon combining (4.18) with (3.11),

$$\beta_1(P) = \int_{P_0}^{P} \gamma_\infty(P') - \nu_1 = \frac{1}{4}\frac{\theta_1'(\alpha)}{\theta_1(\alpha)} + \frac{1}{4}\frac{\theta_3'(\alpha)}{\theta_3(\alpha)}. \tag{C.3}$$

Next, upon differentiating (C.3), using (4.22) and upon noting that the normalized second kind differential γ_∞ written in the curve coordinates is

$$\gamma_\infty(P) = \frac{K^2}{4\eta}\left(\zeta^2 - \frac{2E-K}{K}\right)\mathrm{d}\,\zeta \tag{C.4}$$

while $\boldsymbol{v} = d\zeta/(4\eta)$ we get

$$\beta_1'(P) := \frac{d\beta_1(P)}{d\alpha(P)} = K^2\left(\zeta^2 - \frac{2E-K}{K}\right) \tag{C.5}$$

and consequently

$$\frac{\theta_1''(\alpha)}{\theta_1(\alpha)} + \frac{\theta_3''(\alpha)}{\theta_3(\alpha)} = 2K^2\zeta^2 - 4(2E-K)K + 8\beta^2(P). \tag{C.6}$$

Combining this with (4.19) then yields (4.24) and (4.25). Equally, upon defining

$$T := \frac{\theta_1'(\alpha)}{\theta_1(\alpha)} = 2\beta_1(P) + \imath K\zeta$$

then

$$\frac{\theta_1''(\alpha)}{\theta_1(\alpha)} = 2\beta_1'(P) + \imath K\zeta' + T^2 = 2K^2\left(\zeta^2 - \frac{2E-K}{K}\right) + 4\imath\eta + T^2.$$

The final results follow upon further differentiation and using the earlier results.

C.4. **Proof of Corollary 4.7.** Using the constraint $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = N\tau$ the periodicity of the θ-functions (Corollary 4.2) yields

(C.7)
$$\theta_1(\alpha_i+\alpha_j+\alpha_k) = (-1)^{N+1}\theta_1(\alpha_l)\mathrm{e}^{-\imath\pi N^2\tau+2\imath\pi N\alpha_l},$$
(C.8)
$$\theta_3(\alpha_i+\alpha_j+\alpha_k) = \theta_3(\alpha_l)\mathrm{e}^{-\imath\pi N^2\tau+2\imath\pi N\alpha_l},$$
(C.9)
$$\theta_1'(\alpha_i+\alpha_j+\alpha_k) = (-1)^N\left[\theta_1'(\alpha_l) + 2\imath\pi N\theta_1(\alpha_l)\right]\mathrm{e}^{-\imath\pi N^2\tau+2\imath\pi N\alpha_l},$$
(C.10)
$$\theta_3'(\alpha_i+\alpha_j+\alpha_k) = -\left[\theta_3'(\alpha_l) + 2\imath\pi N\theta_3(\alpha_l)\right]\mathrm{e}^{-\imath\pi N^2\tau+2\imath\pi N\alpha_l},$$
(C.11)
$$\theta_1''(\alpha_i+\alpha_j+\alpha_k) = (-1)^{N+1}\mathrm{e}^{-\imath\pi N^2\tau+2\imath\pi N\alpha_l}\times\left(-4\pi^2N^2\theta_1(\alpha_l) + 4\imath\pi N\theta_1'(\alpha_l) + \theta_1''(\alpha_l)\right),$$
(C.12)
$$\theta_3''(\alpha_i+\alpha_j+\alpha_k) = \mathrm{e}^{-\imath\pi N^2\tau+2\imath\pi N\alpha_l}\times\left(-4\pi^2N^2\theta_3(\alpha_l) + 4\imath\pi N\theta_3'(\alpha_l) + \theta_3''(\alpha_l)\right).$$

The Corollary now follows upon employing Corollary 4.6. We note that

(C.13)
$$\frac{\theta_1'(\alpha_i+\alpha_j+\alpha_k)}{\theta_1(\alpha_i+\alpha_j+\alpha_k)} + \frac{\theta_3'(\alpha_i+\alpha_j+\alpha_k)}{\theta_3(\alpha_i+\alpha_j+\alpha_k)} = -4\beta_1(\alpha_l) - 4\imath\pi N,$$
$$\frac{\theta_1'(\alpha_i+\alpha_j+\alpha_k)}{\theta_1(\alpha_i+\alpha_j+\alpha_k)} - \frac{\theta_3'(\alpha_i+\alpha_j+\alpha_k)}{\theta_3(\alpha_i+\alpha_j+\alpha_k)} = -2\imath K\zeta_l,$$

and

(C.14)
$$\frac{\theta_1''(\alpha_i+\alpha_j+\alpha_k)}{\theta_1(\alpha_i+\alpha_j+\alpha_k)} + \frac{\theta_3''(\alpha_i+\alpha_j+\alpha_k)}{\theta_3(\alpha_i+\alpha_j+\alpha_k)}, = 2K^2\zeta_l^2 - 8\pi^2N^2 + 16\imath\beta_1(\alpha_l)\pi N - 4K(2E-K) + 8\beta_1(\alpha_l)^2$$
$$\frac{\theta_1''(\alpha_i+\alpha_j+\alpha_k)}{\theta_1(\alpha_i+\alpha_j+\alpha_k)} - \frac{\theta_3''(\alpha_i+\alpha_j+\alpha_k)}{\theta_3(\alpha_i+\alpha_j+\alpha_k)} = -8K(\pi N - \imath\beta_1(\alpha_l))\zeta_l + 8\imath K\eta_l.$$

C.5. **Proof of Proposition 4.9.** To prove the first of these relations we compute

$$
\begin{aligned}
&\mu_1(\boldsymbol{x}) + \mu_*(\boldsymbol{x}) \\
&= \frac{\imath K}{2}\left(\int_a^{\zeta_1(\boldsymbol{x})} + \int_a^{\zeta_*(\boldsymbol{x})}\right)\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} - (x_2+\imath x_1)(\zeta_1(\boldsymbol{x})+\zeta_*(\boldsymbol{x})) + \frac{\imath\pi}{2} \\
&= \frac{\imath K}{2}\left(\int_a^{\zeta_1(\boldsymbol{x})} + \int_a^{-\zeta_1(\boldsymbol{x})}\right)\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} + \frac{\imath\pi}{2} \\
&= \frac{\imath K}{2}\left(\int_a^{\zeta_1(\boldsymbol{x})} - \int_{-a}^{\zeta_1(\boldsymbol{x})}\right)\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} + \frac{\imath\pi}{2} \\
&= \frac{\imath K}{2}\int_{k'+\imath k}^{-k'-\imath k}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} + \frac{\imath\pi}{2}.
\end{aligned}
\tag{C.15}
$$

Taking into account that

$$
\begin{aligned}
0 = \frac{\imath K}{2}\oint_{\mathfrak{a}}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} &= \imath K\int_{k'+\imath k}^{k'-\imath k}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} \\
= -\imath K\int_{-k'-\imath k}^{-k'+\imath k}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}}
\end{aligned}
$$

one can transform the last integral in (C.15) into

$$
\begin{aligned}
&\frac{\imath K}{2}\int_{k'+\imath k}^{-k'-\imath k}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} + \frac{\imath K}{2}\int_{-k'-\imath k}^{-k'+\imath k}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} \\
&= \frac{\imath K}{4}\oint_{\mathfrak{b}}\frac{(z^2-c)\mathrm{d}z}{\sqrt{(z^2-a^2)(z^2-b^2)}} = \frac{\imath\pi}{2}
\end{aligned}
$$

which completes the proof. The second relation in (4.32) can be proved similarly.

C.6. **Proof of Lemma 4.10.**

(4.34): Let us fix for definiteness $i=1, j=2, k=3$. To prove (4.34) group and factorize the first term from the left-hand side of (4.34) with the first term of the right-hand side and then do the same with next pair to get

$$
\begin{aligned}
&-\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_1-\alpha_2)[\theta_3(\alpha_1)\theta_3(\alpha_2)\theta_3(\alpha_1+\alpha_2)\theta_3(2\alpha_3) \\
&\qquad - \theta_1(\alpha_1-\alpha_3)\theta_1(\alpha_2-\alpha_3)\theta_1(\alpha_1+\alpha_2+\alpha_3)\theta_1(\alpha_3))],
\end{aligned}
\tag{C.16}
$$

and

$$
\begin{aligned}
&-\theta_3(\alpha_2)\theta_3(\alpha_3)\theta_1(\alpha_2-\alpha_3)[\theta_1(\alpha_2)\theta_1(\alpha_3)\theta_3(\alpha_2+\alpha_3)\theta_3(2\alpha_1) \\
&\qquad -\theta_1(\alpha_1-\alpha_3)\theta_1(\alpha_1-\alpha_2)\theta_3(\alpha_1+\alpha_2+\alpha_3)\theta_3(\alpha_1))].
\end{aligned}
\tag{C.17}
$$

Using the Weierstrass trisecants (W6)

$$
\begin{aligned}
&\theta_1(\alpha_1+\alpha_2+\alpha_3)\theta_1(\alpha_1-\alpha_3)\theta_1(\alpha_2-\alpha_3)\theta_1(\alpha_3) \\
&\qquad - \theta_3(\alpha_2)\theta_3(\alpha_1)\theta_3(\alpha_1+\alpha_2)\theta_3(2\alpha_3) \\
&\qquad\qquad + \theta_3(\alpha_2+\alpha_3-\alpha_1)\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_2+\alpha_3)\theta_3(\alpha_3) = 0,
\end{aligned}
\tag{C.18}
$$

and (W2),

$$
\begin{aligned}
&\theta_1(\alpha_1-\alpha_2)\theta_1(\alpha_1-\alpha_3)\theta_3(\alpha_1+\alpha_2+\alpha_3)\theta_3(\alpha_1)\\
&\qquad -\theta_1(\alpha_2)\theta_1(\alpha_3)\theta_3(\alpha_3+\alpha_2)\theta_3(2\alpha_1)\\
&\qquad\qquad +\theta_1(\alpha_2+\alpha_3-\alpha_1)\theta_1(\alpha_1)\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_1+\alpha_2)=0.
\end{aligned} \tag{C.19}
$$

Correspondingly we factorise expressions (C.16) and (C.17) to the form

$$-\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_1-\alpha_2)\theta_3(\alpha_3)\theta_3(\alpha_1+\alpha_2-\alpha_3))\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_2+\alpha_3),$$

and

$$-\theta_3(\alpha_3)\theta_3(\alpha_2)\theta_1(\alpha_2-\alpha_3)\theta_3(\alpha_2+\alpha_3-\alpha_1)\theta_1(\alpha_1)\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_1+\alpha_2).$$

Adding to these two expressions with the remaining term from (4.34) we observe the vanishing of the overall sum because of the trisecant (W2),

$$
\begin{aligned}
&\theta_1(\alpha_3)\theta_1(\alpha_2-\alpha_3)\theta_3(2\alpha_1)\theta_3(\alpha_2)\\
&\qquad +\theta_1(\alpha_1)\theta_1(\alpha_1-\alpha_2)\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_1+\alpha_2-\alpha_3)\\
&\qquad\qquad -\theta_1(\alpha_1-\alpha_3)\theta_1(\alpha_1+\alpha_3-\alpha_2)\theta_3(\alpha_1)\theta_3(\alpha_1+\alpha_2)=0.
\end{aligned}
$$

(4.35): This follows from the Weierstrass trisecant (W6).

C.7. **Proof of Lemma 4.11.**

(4.36): Let us fix $i=1$, $j=2$, $k=3$. Then from (4.5),

$$x_-=\frac{\imath\eta_1}{(\zeta_1-\zeta_2)(\zeta_1-\zeta_3)}+\frac{\imath\eta_2}{(\zeta_2-\zeta_1)(\zeta_2-\zeta_3)}+\frac{\imath\eta_3}{(\zeta_3-\zeta_1)(\zeta_3-\zeta_2)}.$$

Substituting the θ-functional expressions (3.6) into this and using (7.11) we may rewrite x_- in the form

$$x_-=\frac{\pi\theta_3(0)}{4}\frac{\theta_3(2\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_3)\theta_3(\alpha_2)\theta_3(\alpha_3)}{\theta_1(\alpha_1-\alpha_2)\theta_1(\alpha_1-\alpha_3)\theta_3(\alpha_1+\alpha_2)\theta_3(\alpha_1+\alpha_3)}+\text{cyclic}. \tag{C.20}$$

Substituting (C.20) into (4.36) we get

$$
\begin{aligned}
&2\theta_1(\alpha_1+\alpha_2+\alpha_3)\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_3)\\
&\qquad =-\theta_3(0)\theta_3(\alpha_1+\alpha_2)\theta_3(\alpha_1+\alpha_3)\theta_3(\alpha_2+\alpha_3)\\
&\qquad +\frac{\theta_1(2\alpha_1)\theta_3(\alpha_2+\alpha_3)\theta_1(\alpha_2)\theta_1(\alpha_3)\theta_3(\alpha_2)\theta_3(\alpha_3)}{\theta_1(\alpha_1-\alpha_2)\theta_1(\alpha_1-\alpha_3)}+\text{cyclic}\ .
\end{aligned}
$$

Now using Weierstrass trisecant (W6) written in the form

$$
\begin{aligned}
&-\theta_3(0)\theta_1(\alpha_1+\alpha_2)\theta(\alpha_2+\alpha_3)\theta_1(\alpha_1+\alpha_3)\\
&\qquad =\theta_1(\alpha_1+\alpha_2+\alpha_3)\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_3)\\
&\qquad\qquad -\theta_3(\alpha_1+\alpha_2+\alpha_3)\theta_3(\alpha_1)\theta_3(\alpha_2)\theta_3(\alpha_3)
\end{aligned} \tag{C.21}
$$

in the first term of the right hand side we get

$$
\begin{aligned}
&\theta_1(\alpha_1+\alpha_2+\alpha_3)\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_3)\\
&\qquad +\theta_3(\alpha_1+\alpha_2+\alpha_3)\theta_3(\alpha_1)\theta_3(\alpha_2)\theta_3(\alpha_3)\\
&=\frac{\theta_1(2\alpha_1)\theta_3(\alpha_2+\alpha_3)\theta_1(\alpha_2)\theta_1(\alpha_3)\theta_3(\alpha_2)\theta_3(\alpha_3)}{\theta_1(\alpha_1-\alpha_2)\theta_1(\alpha_1-\alpha_3)}+\text{cyclic}.
\end{aligned}
$$

After multiplication of both sides by $\theta_1(\alpha_1-\alpha_2)\theta_1(\alpha_1-\alpha_3)\theta_1(\alpha_2-\alpha_3)$ the last relation becomes the already proven relation (4.34).

(4.37): Is proven in the same way as (4.36).

(4.38): This follows immediately from (4.36) and (4.37).

(4.39, 4.40, 4.41): This group of relations represent composition of (4.36) and (4.22) and (4.37) and (4.23) with the final identity given by subtracting them and using the definition of μ_l.

(4.42, 4.43, 4.44): This group of relations represent composition of (4.36) and (4.26) and (4.37) and (4.27) with the final identity given by subtracting them.

(4.45), (4.46): These follow directly from those just obtained.

C.8. **Proof of Proposition 4.12.** Let us prove (4.49). Fix values $i = 1$, $j = 2$. Taking the partial derivatives with respect to α_1 and α_2 of both sides of the equality

$$\theta_3(\alpha_1+\alpha_2)\theta_3(\alpha_1-\alpha_2) = \theta_4^{-1}(0)(\theta_4^2(\alpha_1)\theta_3^2(\alpha_2) - \theta_1^2(\alpha_1)\theta_2^2(\alpha_2)) \tag{C.22}$$

and the adding the results we get

$$\begin{aligned}\frac{\theta_3'(\alpha_1+\alpha_2)}{\theta_3(\alpha_1+\alpha_2)} &= \frac{1}{\theta_4^2(\alpha_1)\theta_3^2(\alpha_2) - \theta_1^2(\alpha_1)\theta_2^2(\alpha_2)}[\theta_4(\alpha_1)\theta_3^2(\alpha_2)\theta_4'(\alpha_1) \\ &\quad - \theta_1(\alpha_1)\theta_2^2(\alpha_2)\theta_1'(\alpha_1) + \theta_3(\alpha_2)\theta_4^2(\alpha_1)\theta_3'(\alpha_2) - \theta_2(\alpha_2)\theta_1^2(\alpha_1)\theta_2'(\alpha_2)].\end{aligned} \tag{C.23}$$

Now we find from taking logarithmic derivatives of (3.7) and using Corollary 4.6 that

$$\frac{\theta_2'(\alpha)}{\theta_2(\alpha)} = 2\beta_1(\alpha) + \frac{\imath K + 2\eta}{\zeta}, \quad \frac{\theta_4'(\alpha)}{\theta_4(\alpha)} = 2\beta_1(\alpha) + \frac{-\imath K + 2\eta}{\zeta}. \tag{C.24}$$

Substituting these expressions together with (4.22), (4.23) and the expressions for θ-squares (3.7) into (C.23) we obtain after simplification

$$\frac{\theta_3'(\alpha_1+\alpha_2)}{\theta_3(\alpha_1+\alpha_2)} = 2(\beta_1(\alpha_1)+\beta_1(\alpha_2)) - \imath K(\zeta_1+\zeta_2)\frac{1-X}{1+X} \tag{C.25}$$

with

$$X = \frac{K(\zeta_1^2-1) - 2\imath\eta_1}{K(\zeta_1^2+1)+2\imath\eta_1} \cdot \frac{K(\zeta_2^2-1)-2\imath\eta_2}{K(\zeta_2^2+1)+2\imath\eta_2}\left(\frac{k'}{k}\right)^2 = \frac{\theta_1^2(\alpha_1)\theta_1^2(\alpha_2)}{\theta_3^2(\alpha_1)\theta_3^2(\alpha_2)}. \tag{C.26}$$

Taking into account expression for μ_k, $\mu_k = \beta_1(\alpha_k) + \imath\pi N - \imath(x_-\zeta_k - \imath x_3)$, we conclude that the proof of (4.49) will follow upon establishing that

$$X \equiv \frac{(\zeta_1+\zeta_2)(K-2x_-)+4\imath x_3}{(\zeta_1+\zeta_2)(K+2x_-)-4\imath x_3} \tag{C.27}$$

where in the expression for X the variables η_i should be expressed in terms of ζ_i via the mini-twistor correspondence.

To proceed, one can find k^2 from the relation

$$P(\zeta_1, x_2 - \imath x_1 - 2\zeta_1 x_3 - (x_2+\imath x_1)\zeta_1^2) - P(\zeta_2, x_2 - \imath x_1 - 2\zeta_2 x_3 - (x_2+\imath x_1)\zeta_2^2) = 0$$

giving

$$\begin{aligned}k^2 &= -\frac{1}{4}(\zeta_1^2+\zeta_2^2) + \frac{1}{2} + \frac{x_-^2}{K^2}(\zeta_1^2+\zeta_2^2) + \frac{2x_+x_-}{K^2} - \frac{4x_3^2}{K^2} \\ &\quad - \frac{4\imath x_3}{K^2(\zeta_1+\zeta_2)}[x_-(\zeta_1^2+\zeta_1\zeta_2+\zeta_2^2)+x_+].\end{aligned} \tag{C.28}$$

Expression (C.27) factorises after using (C.28) with one of the factors vanishing because of relation (4.9).

The proof of (4.50) parallels that of (4.49). We find an expression for $\theta_1'(\alpha_1+\alpha_2)/\theta(\alpha_1+\alpha_2)$ similarly to (C.23). Next computing

$$\frac{\theta_1'(\alpha_1+\alpha_2)}{\theta_1(\alpha_1+\alpha_2)}-\frac{\theta_3'(\alpha_1+\alpha_2)}{\theta_3(\alpha_1+\alpha_2)} \tag{C.29}$$

and making all of the above substitutions the result follows.

To prove (4.51) compute the α_1 derivative of both sides of (C.23) and use the expressions $\theta_i'(\alpha)/\theta_i(\alpha)$, $\theta_i''(\alpha)/\theta_i(\alpha)$ from the list of formulae (4.22)-(4.25) together with the formulae

$$\begin{aligned}\frac{\theta_2''(\alpha)}{\theta_2(\alpha)}&=4\beta_1^2(\alpha)+\frac{8\eta_1\beta_1(\alpha)}{\zeta(\alpha)}+4K^2k'^2-4EK-K^2\zeta^2(\alpha)+\frac{4\imath K\beta_1(\alpha)}{\zeta(\alpha)},\\ \frac{\theta_4''(\alpha)}{\theta_4(\alpha)}&=4\beta_1^2(\alpha)+\frac{8\eta_1\beta_1(\alpha)}{\zeta(\alpha)}+4K^2k'^2-4EK-K^2\zeta^2(\alpha)-\frac{4\imath K\beta_1(\alpha)}{\zeta(\alpha)},\end{aligned} \tag{C.30}$$

and those for θ-squares to get algebraic expression of $\eta_{1,2},\zeta_{1,2},\beta_1(\alpha_{1,2})$ and $(x_\pm,x_3)$, K,k For the right hand side of (4.51) one can transform from the group of variables labeled by indices 3 and 4 to variables labeled by 1 and 2 using the formulae,

$$\zeta_3+\zeta_4=-\frac{16\imath x_3x_-}{K^2-4x_-^2}-\zeta_1-\zeta_2, \tag{C.31}$$

$$\mu_3+\mu_4=-\mu_1-\mu_2 \pmod{i\pi}. \tag{C.32}$$

Now upon subtracting theses expressions for the left and right hand sides of (4.51) and using the expressions for μ_j and those for η_j following from the mini-twistor correspondence one obtains a rather cumbersome expression that again factorises as in the proof of (4.49). Here we find a vanishing factor

$$(\zeta_1+\zeta_2)[\zeta_1^2\zeta_2^2(K^2-4x_-^2)-(K^2-4x_+^2)]+16\imath\zeta_1\zeta_2(x_+\zeta_1\zeta_2-x_+)x_3 \tag{C.33}$$

so proving (4.51).

The final expression (4.52) is proved analogously.

C.9. **Proof of Proposition 5.2.** For $S_1^2<0$ and the invariance of the curve under conjugation we have that

$$\overline{\alpha(P_1)}=\overline{\int_{\infty_1}^{P_1}v}=\int_{\overline{\infty_1}}^{(\overline{\zeta},\overline{\eta})}v=\int_{\infty_2}^{(\overline{\zeta},\overline{\eta})}v=-\int_{\infty_1}^{(\overline{\zeta},-\overline{\eta})}v=-\alpha(P_2). \tag{C.34}$$

This together with (4.3) and the even/oddness properties of the theta functions shows that on interval **I**

$$\overline{\beta_1(P_1)}=-\beta_1(P_2).$$

Accordingly

$$\overline{\mu}_1(x_1,0)=-\mu_2(x_1,0),\qquad \overline{\mu}_2(x_1,0)=-\mu_1(x_1,0).$$

Taken together with Proposition (4.9) we obtain upon noting $\zeta_1+\zeta_3=0$ that

$$\mu_1=\lambda_1(x_1)+\frac{\imath\pi}{4},\ \mu_2=-\lambda_1(x_1)+\frac{\imath\pi}{4},\ \mu_3=-\lambda_1(x_1)-\frac{\imath\pi}{4},\ \mu_4=\lambda_1(x_1)-\frac{\imath\pi}{4}.$$

where $\lambda_1(x_1)$ is a real function. The initial conditions of μ_i give $\lambda_1(0)=0$.

Given the reality properties noted earlier, we have that on the remaining intervals

II $\overline{\zeta}=\zeta,\quad \overline{\eta}=-\eta,\quad \overline{\alpha(P_1)}=-\alpha(P_1),\quad \overline{\beta(P_1)}=-\beta(P_1),\quad \overline{\gamma_\infty}=-\gamma_\infty,\quad \overline{\imath x_1\zeta}=-\imath x_1\zeta,$

III $\overline{\zeta}=-\zeta,\quad \overline{\eta}=-\eta,\quad \overline{\alpha(P_1)}=\alpha(P_1),\quad \overline{\beta(P_1)}=\beta(P_1),\quad \overline{\gamma_\infty}=\gamma_\infty,\quad \overline{\imath x_1\zeta}=\imath x_1\zeta.$

From these it follows that

(C.35) $$\mathbf{II}: \overline{\mu}_1(x_1,0) = -\mu_1(x_1,0), \qquad \mathbf{III}: \overline{\mu}_1(x_1,0) = \mu_1(x_1,0).$$

Again taken together with Proposition (4.9)

$$\mu_1 = \lambda'' + \imath\alpha,\ \mu_2 = \lambda'' - \imath\alpha - \frac{\imath\pi}{2},\ \mu_3 = -\lambda'' + \imath\alpha + \frac{\imath\pi}{2},\ \mu_4 = -\lambda'' - \imath\alpha,$$

the result follows. The remaining boundary conditions now follow.

C.10. **Proof of Proposition 5.5.** For $x_3 < Kk'/2$ the proof follows that of the the x_2-axis. For $x_3 > Kk'/2$ then $P_2 = -P_1$ and

$$\mu(-\zeta,-\eta) = \frac{\imath K}{2}\int_a^{-\zeta}\frac{z^2-c}{-\eta}dz - x_3 = \frac{\imath K}{2}\int_{-a}^{\zeta}\frac{z^2-c}{\eta}dz - x_3. = \mu(\zeta,\eta) + \int_{-a}^{a}\gamma_\infty.$$

Then

$$-\int_{-a}^{a}\gamma_\infty = \int_{k'+\imath k}^{k'-\imath k}\gamma_\infty + \int_{k'-\imath k}^{-k'-\imath k}\gamma_\infty = \frac{1}{2}\oint_{\mathfrak{a}}\gamma_\infty + \frac{1}{2}\oint_{\mathfrak{b}}\gamma_\infty = -\imath\frac{\pi}{2}$$

and the result follows.

C.11. **Proof of Proposition 5.6.** We note that in all cases the possible signs are generated by $z \to z+1$ and $z \to z+\tau$ and because of (5.12) we need only solve this for one axis to determine the answer for each axis. We begin by focussing on the x_2 axis and a choice of signs such that

(C.36) $$\theta_2[P]\theta_4[P]\theta_3\theta_1(z) = \theta_1[P]\theta_3[P]\theta_2\theta_4(z) + \theta_1[P]\theta_3[P]\theta_4\theta_2(z).$$

Now we have (for any distinct $i,j,k \in \{2,3,4\}$) the trisecant identity (W3)

$$\theta_i(\alpha_1)\theta_j(\alpha_2)\theta_k(\alpha_3)\theta_1(\alpha_4) + \theta_i(\alpha_1')\theta_j(\alpha_2')\theta_k(\alpha_3')\theta_1(\alpha_4') + \theta_i(\alpha_1'')\theta_j(\alpha_2'')\theta_k(\alpha_3'')\theta_1(\alpha_4'') = 0$$

where $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $\boldsymbol{\alpha'}$ and $\boldsymbol{\alpha''}$ are described in Appendix B. Then if we take $\boldsymbol{\alpha} = (0,P,-P,2P)$ and $(i,j,k) = (3,4,2)$ we recover (C.36) with $z = -2\,\alpha(P)$.

For the x_1 axis

$$\zeta = \pm\frac{k' \pm \imath k\,\mathrm{cn}(t)}{\mathrm{dn}(t)} = \pm\frac{\theta_4\theta_4(z) \pm \imath\,\theta_2\theta_2(z)}{\theta_3\theta_3(z)}$$

we now wish to solve

(C.37) $$\theta_2[P]\theta_4[P]\theta_3\theta_3(z) = \pm\left[\theta_2\theta_2(z) \pm \imath\,\theta_4\theta_4(z)\right]\theta_1[P]\theta_3[P].$$

This may be rewritten as

$$\theta_2[P]\theta_4[P]\theta_3\theta_2(z+\tau/2) = \pm\left[\theta_2\theta_3(z+\tau/2) \pm \theta_4\theta_1(z+\tau/2)\right]\theta_1[P]\theta_3[P]$$

or

$$\begin{aligned}\theta_3\theta_3(\alpha(P)+1/2)\theta_2(z+\tau/2)\theta_2[P] = {}&\pm\theta_2\theta_2(\alpha(P)+1/2)\theta_3(z+\tau/2)\theta_3[P]\\ &\pm\theta_4\theta_4(\alpha(P)+1/2)\theta_1(z+\tau/2)\theta_1[P].\end{aligned}$$

This is in the form if the trisecant identity

$$\theta_2(\alpha_1)\theta_2(\alpha_2)\theta_3(\alpha_3)\theta_3(\alpha_4) - \theta_2(\alpha_1')\theta_2(\alpha_2')\theta_3(\alpha_3')\theta_3(\alpha_4') \pm \theta_4(\alpha_1'')\theta_4(\alpha_2'')\theta_1(\alpha_3'')\vartheta_1(\alpha_4'') = 0.$$

We find a solution $\boldsymbol{\alpha} = (-2P-1/2, P, 0, P+1/2)$, that is $z = -2\,\alpha(P) - 1/2 - \tau/2$.

Finally, for the x_3 axis we have

$$\zeta = \pm(\mathrm{dn}(t) \pm \imath k\,\mathrm{sn}(t)) = \pm\frac{\theta_4\theta_3(z) \pm \imath\,\theta_2\theta_1(z)}{\theta_3\theta_4(z)}$$

and we are led to

$$\theta_2[P]\theta_4[P]\theta_3\theta_4(z) = \pm\left[\theta_2\theta_1(z) \pm \imath\,\theta_4\theta_3(z)\right]\theta_1[P]\theta_3[P]. \tag{C.38}$$

For an appropriate set of signs we may rewrite this as

$$\theta_1(z+\tau/2)\theta_2[P]\theta_3\theta_4[P] = \theta_1[P]\theta_2\theta_3[P]\theta_4(z+\tau/2) + \theta_1[P]\theta_2(z+\tau/2)\theta_3[P]\theta_4$$

which we solved earlier, $z = -2\,\alpha(P) - \tau/2$.

Appendix D. The matrices W and V

D.1. **Proof of Theorem 7.1.** The form of (7.4) shows that its principal cofactors are either linear or bilinear in the ζ's. Now $\Psi\,\mathrm{Adj}(\Psi) = |\Psi| 1_4$ and the first two columns are bilinear in the ζ's while the third and fourth are linear. Let us consider first the linear case. We find for example that

$$\begin{aligned}
\mathrm{Adj}(\Psi)_{13} &= i\theta_1(\alpha_2)\,\theta_4(\alpha_2 - z/2)\,\theta_1(\alpha_3-\alpha_4)\,\theta_4(\alpha_4 - z/2 + \alpha_3)\,\theta_3(0)\theta_2(z/2)\zeta_2 \\
&\quad - i\theta_1(\alpha_3)\,\theta_4(\alpha_3 - z/2)\,\theta_1(\alpha_2-\alpha_4)\,\theta_4(\alpha_4 - z/2 + \alpha_2)\,\theta_3(0)\theta_2(z/2)\zeta_3 \\
&\quad + i\theta_1(\alpha_4)\,\theta_4(\alpha_4 - z/2)\,\theta_1(\alpha_2-\alpha_3)\,\theta_4(\alpha_3 - z/2 + \alpha_2)\,\theta_3(0)\theta_2(z/2)\zeta_4 \\
&:= a\zeta_2 + b\zeta_3 + c\zeta_4
\end{aligned}$$

where $a+b+c=0$

$$= a(\zeta_2-\zeta_4) + b(\zeta_3-\zeta_4)$$

and upon using (7.11) we find

$$= -\frac{\theta_2(\sum_{k\neq 1}\alpha_k - z/2)}{\prod_{k\neq 1}\theta_3(\alpha_k)}\theta_2^2(z/2)\theta_2(0)\theta_3(0)\theta_4(0)\prod_{\substack{k<l\\ k,l\neq 1}}\theta_1(\alpha_k-\alpha_l).$$

Here we have used the trisecant identity to simplify each of the coefficients of ζ_i in obtaining the first line, and we remark that whichever of a, b or c we eliminate in the third step we arrive at the same final expression. We have then for the third and fourth columns of $\mathrm{Adj}(\Psi)$ that

$$\mathrm{Adj}(\Psi)_{i3} = \frac{\theta_2(\sum_{k\neq i}\alpha_k - z/2)}{\prod_{k\neq i}\theta_3(\alpha_k)}\left[(-1)^i\theta_2^2(z/2)\theta_2(0)\theta_3(0)\theta_4(0)\prod_{\substack{k<l\\ k,l\neq 1}}\theta_1(\alpha_k-\alpha_l)\prod_{\substack{k<l\\ k,l\neq 1}}\theta_1(\alpha_k-\alpha_l)\right],$$

$$\mathrm{Adj}(\Psi)_{i4} = \frac{\theta_4(\sum_{k\neq i}\alpha_k - z/2)}{\prod_{k\neq i}\theta_1(\alpha_k)}\left[(-1)^i\theta_2^2(z/2)\theta_2(0)\theta_3(0)\theta_4(0)\prod_{\substack{k<l\\ k,l\neq 1}}\theta_1(\alpha_k-\alpha_l)\right],$$

and we note that the terms $\theta_{2,4}(\sum_{k\neq i}\alpha_k - z/2)$ may be rewritten using Proposition 4.2.

Let us now consider the quadratic terms. Taking for example $\mathrm{Adj}(\Psi)_{11}$ with the same a, b, c appearing as in $\mathrm{Adj}(\Psi)_{13}$ we have

$$\begin{aligned}
\mathrm{Adj}(\Psi)_{11} &= ic\,\zeta_2\zeta_3 + ib\,\zeta_2\zeta_4 + ia\,\zeta_3\zeta_4 \\
&= ia\,\zeta_3(\zeta_4-\zeta_2) + ib\,\zeta_2(\zeta_4-\zeta_3) \\
&= -i\left[a(\zeta_2-\zeta_4) + b(\zeta_3-\zeta_4)\right]\zeta_3 + ib\,(\zeta_3-\zeta_4)(\zeta_3-\zeta_2) \\
&= -i\zeta_3\,\mathrm{Adj}(\Psi)_{13} + (\zeta_3-\zeta_4)(\zeta_3-\zeta_2)\times \\
&\qquad \theta_1(\alpha_3)\theta_1(\alpha_2-\alpha_4)\theta_4(\alpha_3-z/2)\theta_4(\alpha_2+\alpha_4-z/2)\theta_3(0)\theta_2(z/2)
\end{aligned}$$

or grouping factors differently

$$\begin{aligned}
&= -i\zeta_2 \operatorname{Adj}(\Psi)_{13} + (\zeta_2-\zeta_3)(\zeta_2-\zeta_4)\times \\
&\qquad \theta_1(\alpha_2)\theta_1(\alpha_3-\alpha_4)\theta_4(\alpha_2-z/2)\theta_4(\alpha_3+\alpha_4-z/2)\theta_3(0)\theta_2(z/2) \\
&= -i\zeta_4 \operatorname{Adj}(\Psi)_{13} + (\zeta_4-\zeta_2)(\zeta_4-\zeta_2)\times \\
&\qquad \theta_1(\alpha_4)\theta_1(\alpha_2-\alpha_3)\theta_4(\alpha_4-z/2)\theta_4(\alpha_2+\alpha_3-z/2)\theta_3(0)\theta_2(z/2).
\end{aligned}$$

In general, for $j,k,l \neq i$ and $k<l$ we have

$$\begin{aligned}
\operatorname{Adj}(\Psi)_{i1} &= -i\zeta_j \operatorname{Adj}(\Psi)_{i3} + (\zeta_j-\zeta_k)(\zeta_j-\zeta_l)\times \operatorname{Coeff}(\operatorname{Adj}(\Psi)_{i1},\zeta_k\zeta_l),\\
\operatorname{Adj}(\Psi)_{i2} &= -i\zeta_j \operatorname{Adj}(\Psi)_{i4} + (\zeta_j-\zeta_k)(\zeta_j-\zeta_l)\times \operatorname{Coeff}(\operatorname{Adj}(\Psi)_{i2},\zeta_k\zeta_l).
\end{aligned}$$

which may be written as

$$\begin{aligned}
\operatorname{Adj}(\Psi)_{i1} &= -i\zeta_j \operatorname{Adj}(\Psi)_{i3} - \epsilon_{ijkl}(\zeta_j-\zeta_k)(\zeta_j-\zeta_l)\times \\
&\qquad \theta_1(\alpha_j)\theta_1(\alpha_k-\alpha_l)\theta_4(\alpha_j-z/2)\theta_4(\alpha_k+\alpha_l-z/2)\theta_3(0)\theta_2(z/2),\\
\operatorname{Adj}(\Psi)_{i2} &= -i\zeta_j \operatorname{Adj}(\Psi)_{i4} - \epsilon_{ijkl}(\zeta_j-\zeta_k)(\zeta_j-\zeta_l)\times \\
&\qquad \theta_3(\alpha_j)\theta_1(\alpha_k-\alpha_l)\theta_2(\alpha_j-z/2)\theta_4(\alpha_k+\alpha_l-z/2)\theta_3(0)\theta_2(z/2).
\end{aligned}$$

We find upon using (7.11) and taking the transpose

D.2. **Proof of Theorem 8.1.** Towards expanding (8.2) we first note that

$$\mathcal{O}\, C^{-1}(z) = \operatorname{Diag}(F+G, F-G)\,\mathcal{O} = \operatorname{Diag}(p(z), 1/p(z))\,\mathcal{O}.$$

Then, from the integral representation of $p(z)$ and that $f_3(z)$ is even, we have $p(z)p(-z)=1$ and so

(D.1)

$$\overline{\boldsymbol{v}}_i(z) = \frac{1}{\theta_2^2(z/2)}\frac{1}{\sqrt{2}}\left(1_2\otimes\begin{pmatrix} p(z) & 0\\ 0 & p(-z)\end{pmatrix}\right)\left[\begin{pmatrix}-\imath\,\zeta_j\\ 1\end{pmatrix}\otimes\begin{pmatrix}A-B\\ A+B\end{pmatrix}+\begin{pmatrix}1\\ 0\end{pmatrix}\otimes\begin{pmatrix}\alpha-\beta\\ \alpha+\beta\end{pmatrix}\right].$$

We shall now define

$$\begin{aligned}
\hat{A}(\xi) &:= A(1-\xi) = \sum_{s\geq 1} A_s\,\xi^s, \quad \hat{B}(\xi) := B(1-\xi) = \sum_{s\geq 1} B_s\,\xi^s,\\
\hat{\alpha}(\xi) &:= \alpha(1-\xi) = \sum_{s\geq 0} \alpha_s\,\xi^s, \quad \hat{\beta}(\xi) := \beta(1-\xi) = \sum_{s\geq 0} \beta_s\,\xi^s,\\
\frac{1}{\theta_2^2((1-\xi)/2)} &= \frac{c}{\xi^2}\sum_{s\geq 0} c_{2s}\,\xi^{2s}, \qquad c = \frac{4}{(\theta_1'(0))^2},\ c_0 = 1,\ c_2 = -\frac{1}{12}\frac{\theta_1'''(0)}{\theta_1'(0)},\\
p(1-\xi) &= \frac{1}{\sqrt{\xi}}\frac{\sqrt{2}}{\sqrt{K}}\sum_{s\geq 0} p_{2s}\,\xi^{2s}, \qquad p_0 = 1,\ p_2 = \frac{1}{24}(2k^2-1)K^2,\\
p(-1+\xi) &= \sqrt{\xi}\frac{\sqrt{2}}{\sqrt{K}}\sum_{s\geq 0} q_{2s}\,\xi^{2s}, \qquad q_0 = \frac{K}{2},\ q_2 = -p_2 q_0,
\end{aligned}$$

where here the expansion of A and B begin with ξ because of

$$\theta_2\left((1-\xi)/2\right) = \theta_1\left(\xi/2\right) = \frac{\xi}{2}\,\theta_1'(0) + \frac{\xi^3}{48}\,\theta_1'''(0) + \mathcal{O}(\xi^5) = \xi\,\frac{\pi\,\theta_2\theta_3\theta_4}{2}\left(1-\frac{1}{2}c_2\xi^2+\mathcal{O}(\xi^4)\right).$$

We then have that
(D.2)
$$\overline{\boldsymbol{v}}_{i,s}=\frac{c}{\sqrt{K}}\sum_{2l+2m+n=s}c_{2l}\left[\begin{pmatrix}-\imath\,\zeta_j\\1\end{pmatrix}\otimes\begin{pmatrix}p_{2m}(A_n-B_n)\\q_{2m}(A_{n-1}+B_{n-1})\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}p_{2m}(\alpha_n-\beta_n)\\q_{2m}(\alpha_{n-1}+\beta_{n-1})\end{pmatrix}\right].$$

In particular we have

$$\overline{\boldsymbol{v}}_{i,0}=\frac{c}{\sqrt{K}}\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}\alpha_0-\beta_0\\0\end{pmatrix}=0, \tag{D.3}$$

where we have used that $\alpha_0=\beta_0$ which follows from (B.12, B.12). Making use of this then yields

$$\overline{\boldsymbol{v}}_{i,1}=\frac{c}{\sqrt{K}}\begin{pmatrix}\alpha_1-\beta_1-\imath\,\zeta_2(A_1-B_1)\\ \alpha_0K\\ A_1-B_1\\ 0\end{pmatrix}, \tag{D.4}$$

$$\overline{\boldsymbol{v}}_{i,2}=\frac{c}{\sqrt{K}}\begin{pmatrix}\alpha_2-\beta_2-\imath\,\zeta_2(A_2-B_2)\\ (\alpha_1+\beta_1-\imath\,\zeta_2(A_1+B_1))\,K/2\\ A_2-B_2\\ (A_1+B_1)K/2\end{pmatrix}, \tag{D.5}$$

$$\overline{\boldsymbol{v}}_{i,3}=\frac{c}{\sqrt{K}}\begin{pmatrix}\alpha_3-\beta_3-\imath\,\zeta_2(A_3-B_3)\\ (\alpha_2+\beta_2-\imath\,\zeta_2(A_2+B_2))\,K/2\\ A_3-B_3\\ (A_2+B_2)K/2\end{pmatrix}-2Kp_2\alpha_0\,\frac{c}{\sqrt{K}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+(c_2+p_2)\overline{\boldsymbol{v}}_{i,1}. \tag{D.6}$$

$$A_1=\frac{1}{2}\,\frac{\pi\,\theta_2\theta_3\theta_4\,\theta_1\,(P_4+P_3+P_2)}{\theta_3\,(P_3)\,\theta_3\,(P_2)\,\theta_3\,(P_4)}\,\mathrm{e}^{-\mu_1},\qquad A_2=\left(\frac{1}{2}\frac{\theta_1'\,(P_4+P_3+P_2)}{\theta_1\,(P_4+P_3+P_2)}+\mu_1\right)A_1,$$

$$B_1=\frac{1}{2}\,\frac{\pi\,\theta_2\theta_3\theta_4\,\theta_3\,(P_4+P_3+P_2)}{\theta_1\,(P_3)\,\theta_1\,(P_2)\,\theta_1\,(P_4)}\,\mathrm{e}^{-\mu_1},\qquad B_2=\left(\frac{1}{2}\frac{\theta_3'\,(P_4+P_3+P_2)}{\theta_3\,(P_4+P_3+P_2)}+\mu_1\right)B_1,$$

$$\alpha_0=\frac{\theta_2\theta_4\theta_3\,(P_4+P_3)\,\theta_3\,(P_3+P_2)\,\theta_3\,(P_4+P_2)}{\theta_1\,(P_2)\,\theta_3\,(P_2)\,\theta_1\,(P_3)\,\theta_3\,(P_3)\,\theta_1\,(P_4)\,\theta_3\,(P_4)}\,\mathrm{e}^{-\mu_1},\qquad \alpha_1=\left(\frac{1}{2}\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+\frac{1}{2}\frac{\theta_3'\,(P_2)}{\theta_3\,(P_2)}+\mu_1\right)\alpha_0,$$

$$\beta_0=\alpha_0,\qquad \beta_1=\left(\frac{1}{2}\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+\frac{1}{2}\frac{\theta_1'\,(P_2)}{\theta_1\,(P_2)}+\mu_1\right)\alpha_0,$$

$$A_3=\left(\frac{1}{8}\frac{\theta_1''\,(P_4+P_3+P_2)}{\theta_1\,(P_4+P_3+P_2)}+\frac{\mu_1}{2}\frac{\theta_1'\,(P_4+P_3+P_2)}{\theta_1\,(P_4+P_3+P_2)}+\frac{\mu_1^2}{2}-\frac{1}{2}c_2\right)A_1,$$

$$B_3=\left(\frac{1}{8}\frac{\theta_3''\,(P_4+P_3+P_2)}{\theta_3\,(P_4+P_3+P_2)}+\frac{\mu_1}{2}\frac{\theta_3'\,(P_4+P_3+P_2)}{\theta_3\,(P_4+P_3+P_2)}+\frac{\mu_1^2}{2}-\frac{1}{2}c_2\right)B_1,$$

$$\alpha_2=\left(\frac{1}{8}\frac{\theta_3''\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+\frac{1}{8}\frac{\theta_3''\,(P_2)}{\theta_3\,(P_2)}+\frac{1}{4}\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}\frac{\theta_3'\,(P_2)}{\theta_3\,(P_2)}+\frac{1}{2}\left(\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+\frac{\theta_3'\,(P_2)}{\theta_3\,(P_2)}\right)\mu_1+\frac{\mu_1^2}{2}\right)\alpha_0,$$

$$\beta_2=\left(\frac{1}{8}\frac{\theta_3''\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+\frac{1}{8}\frac{\theta_1''\,(P_2)}{\theta_1\,(P_2)}+\frac{1}{4}\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}\frac{\theta_1'\,(P_2)}{\theta_1\,(P_2)}+\frac{1}{2}\left(\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+\frac{\theta_1'\,(P_2)}{\theta_1\,(P_2)}\right)\mu_1+\frac{\mu_1^2}{2}\right)\alpha_0,$$

$$\alpha_3=\left[\frac{1}{8}\left(\frac{\theta_3'''\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+3\,\frac{\theta_3''\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}\frac{\theta_3'\,(P_2)}{\theta_3\,(P_2)}+3\,\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}\frac{\theta_3''\,(P_2)}{\theta_3\,(P_2)}+\frac{\theta_3'''\,(P_2)}{\theta_3\,(P_2)}\right)\right.$$
$$\left.+\frac{3\mu_1}{4}\left(\frac{\theta_3''\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}+2\,\frac{\theta_3'\,(P_4+P_3)}{\theta_3\,(P_4+P_3)}\frac{\theta_3'\,(P_2)}{\theta_3\,(P_2)}+\frac{\theta_3''\,(P_2)}{\theta_3\,(P_2)}\right)\right.$$

$$+\frac{3\mu_1^2}{2}\left(\frac{\theta_3'(P_4+P_3)}{\theta_3(P_4+P_3)}+\frac{\theta_3'(P_2)}{\theta_3(P_2)}\right)+\mu_1^3\Bigg]\frac{\alpha_0}{6},$$

$$\beta_3=\Bigg[\frac{1}{8}\left(\frac{\theta_3'''(P_4+P_3)}{\theta_3(P_4+P_3)}+3\frac{\theta_3''(P_4+P_3)}{\theta_3(P_4+P_3)}\frac{\theta_1'(P_2)}{\theta_1(P_2)}+3\frac{\theta_3'(P_4+P_3)}{\theta_3(P_4+P_3)}\frac{\theta_1''(P_2)}{\theta_1(P_2)}+\frac{\theta_1'''(P_2)}{\theta_1(P_2)}\right)$$
$$+\frac{3\mu_1}{4}\left(\frac{\theta_3''(P_4+P_3)}{\theta_3(P_4+P_3)}+2\frac{\theta_3'(P_4+P_3)}{\theta_3(P_4+P_3)}\frac{\theta_1'(P_2)}{\theta_1(P_2)}+\frac{\theta_1''(P_2)}{\theta_1(P_2)}\right)$$
$$+\frac{3\mu_1^2}{2}\left(\frac{\theta_3'(P_4+P_3)}{\theta_3(P_4+P_3)}+\frac{\theta_1'(P_2)}{\theta_1(P_2)}\right)+\mu_1^3\Bigg]\frac{\alpha_0}{6},$$

Now

$$A_1-B_1=\frac{1}{2}\,\frac{\pi\,\theta_2\theta_3\theta_4\left(\theta_1(P_4+P_3+P_2)\,\theta_1(P_3)\,\theta_1(P_2)\,\theta_1(P_4)-\theta_3(P_4+P_3+P_2)\,\theta_3(P_3)\,\theta_3(P_2)\,\theta_3(P_4)\right)}{\theta_1(P_2)\,\theta_3(P_2)\,\theta_1(P_3)\,\theta_3(P_3)\,\theta_1(P_4)\,\theta_3(P_4)}\,\mathrm{e}^{-\mu_1},$$
$$=-K\,\alpha_0,$$

where we have employed the Weierstrass trisecant identity (W6) which says for any $\alpha_{1,2,3}$ that

$$\theta_1(\alpha_1+\alpha_2+\alpha_3)\theta_1(\alpha_1)\theta_1(\alpha_2)\theta_1(\alpha_3)-\theta_3(\alpha_1+\alpha_2+\alpha_3)\theta_3(\alpha_1)\theta_3(\alpha_2)\theta_3(\alpha_3)$$
$$=-\theta_3(0)\theta_3(\alpha_1+\alpha_2)\theta_3(\alpha_2+\alpha_3)\theta_3(\alpha_1+\alpha_3)$$

and $K=\pi\theta_3^2/2$. Further

$$\alpha_1-\beta_1-\imath\,\zeta_2(A_1-B_1)=\left(\frac{1}{2}\frac{\theta_3'(P_2)}{\theta_3(P_2)}-\frac{1}{2}\frac{\theta_1'(P_2)}{\theta_1(P_2)}+\imath\,\zeta_2K\right)\alpha_0=0$$

upon making use of (4.18) and we then have the leading order pole term of $\overline{\boldsymbol{v}}$ at $z=1-\xi$ behaving as

$$\frac{1}{\xi^{3/2}}c\,\alpha_0\sqrt{K}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.$$

Now from (4.38) we see that

$$\frac{A_1+B_1}{2\alpha_0}=x_1-\imath\,x_2.$$

Writing

$$\frac{\alpha_1+\beta_1-\imath\,\zeta_2(A_1+B_1)}{2\alpha_0}=\frac{1}{2}\left(\frac{\theta_3'(P_4+P_3)}{\theta_3(P_4+P_3)}+\frac{1}{2}\frac{\theta_3'(P_2)}{\theta_3(P_2)}+\frac{1}{2}\frac{\theta_1'(P_2)}{\theta_1(P_2)}-\frac{1}{2}\frac{\theta_3'(P_2)}{\theta_3(P_2)}+2\mu_1\right)$$
$$-\imath\,\zeta_2[x_1-\imath\,x_2]$$

and making use of (4.22, 4.23, 4.53) yields

$$=\frac{1}{2}\left([-\imath\,K+2(x_2+\imath\,x_1)]\,\zeta_2-2\mu_1+2x_3+\imath\,K\zeta_2+2\mu_1\right)-\imath\,\zeta_2[x_1-\imath\,x_2]$$
$$=x_3.$$

Similarly making use of (4.36, 4.37, 4.39, 4.40) we find

$$A_2-B_2=\alpha_0Kx_3,$$

$$
\begin{aligned}
\frac{\alpha_2-\beta_2-\imath\,\zeta_2(A_2-B_2)}{\alpha_0} &= \frac{\alpha_2-\beta_2}{\alpha_0} - \imath\,\zeta_2 K x_3 \\
&= \left(\frac{1}{8}\frac{\theta_3''(P_2)}{\theta_3(P_2)} - \frac{1}{8}\frac{\theta_1''(P_2)}{\theta_1(P_2)} + \frac{1}{4}\left[\frac{\theta_3'(P_4+P_3)}{\theta_3(P_4+P_3)} + 2\mu_1\right]\left[\frac{\theta_3'(P_2)}{\theta_3(P_2)} - \frac{\theta_1'(P_2)}{\theta_1(P_2)}\right]\right) \\
&\quad - \imath\,\zeta_2 K x_3
\end{aligned}
$$

and using (4.22, 4.23, 4.24, 4.25,4.53) we obtain

$$
\begin{aligned}
&= -\imath\,K\left(\beta(P_2)\zeta_2+\eta_2\right) - \frac{\imath\,K}{2}\zeta_2\left(2(x_2+\imath\,x_1)\zeta_2 + 2x_3 - 2\beta(P_2)\right) - \imath\,\zeta_2 K x_3 \\
&= -\imath\,K\left(2x_3\zeta_2 + [x_2+\imath\,x_1]\zeta_2^2 + \eta_2\right)
\end{aligned}
$$

and upon making use of the mini-twistor constraint simplifies to

$$
\begin{aligned}
&= -\imath\,K(x_2-\imath\,x_1) \\
&= K(-x_1-\imath\,x_2).
\end{aligned}
$$

Thus we obtain the subleading pole

$$
\frac{1}{\xi^{1/2}} c\,\alpha_0\sqrt{K}\begin{pmatrix} -x_1-\imath\,x_2 \\ x_3 \\ x_3 \\ x_1-\imath\,x_2 \end{pmatrix}
$$

which agrees with (the complex conjugate of) (3.28). At this stage we have shown that the first column has an expansion

$$
c\sqrt{K}\,\frac{\theta_2\theta_4\theta_3(P_3+P_2)\,\theta_3(P_4+P_2)\,\theta_3(P_4+P_3)}{\theta_1(P_4)\,\theta_3(P_4)\,\theta_1(P_2)\,\theta_3(P_2)\,\theta_1(P_3)\,\theta_3(P_3)}\,\mathrm{e}^{-\mu_1}\left(\frac{1}{\xi^{3/2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + \frac{1}{\xi^{1/2}}\begin{pmatrix} -x_1-\imath\,x_2 \\ x_3 \\ x_3 \\ x_1-\imath\,x_2 \end{pmatrix} + \mathcal{O}(\xi^{1/2})\right)
$$

and analogously each column has expansion at $z=1-\xi$

$$
\overline{\boldsymbol{v}}_i = N_i\left(\frac{1}{\xi^{3/2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + \frac{1}{\xi^{1/2}}\begin{pmatrix} -x_1-\imath\,x_2 \\ x_3 \\ x_3 \\ x_1-\imath\,x_2 \end{pmatrix} + \mathcal{O}(\xi^{1/2})\right) \tag{D.7}
$$

where

$$
N_i := c\sqrt{K}\,\theta_2\theta_4\,\frac{\prod_{j<k\ j,k\neq i}\theta_3(P_j+P_k)}{\prod_{r\neq i}\theta_1(P_r)\theta_3(P_r)}\,\mathrm{e}^{-\mu_i}.
$$

The remaining terms of the theorem follow from (D.6) and the relations of Lemma 4.11 and Lemma 4.12. Thus for example the fourth entry of $\overline{\boldsymbol{v}}_i$ uses (4.38) and the second entry (4.45).

D.3. **Proof of Theorem 8.2.** We have seen that

$$
\overline{\boldsymbol{v}}_i(1-\xi) = \frac{1}{\theta_1^2(\xi/2)}\frac{1}{\sqrt{2}}\left(1_2\otimes\begin{pmatrix}p(1-\xi) & 0\\ 0 & p(-1+\xi)\end{pmatrix}\right)\lambda_i(\xi)
$$

where we now define

$$
\lambda_i(\xi) = \begin{pmatrix}-\imath\,\zeta_j\\ 1\end{pmatrix}\otimes\begin{pmatrix}\hat{A}(\xi)-\hat{B}(\xi)\\ \hat{A}(\xi)+\hat{B}(\xi)\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}\hat{\alpha}(\xi)-\hat{\beta}(\xi)\\ \hat{\alpha}(\xi)+\hat{\beta}(\xi)\end{pmatrix}.
$$

Next we easily obtain

Lemma D.1.

$$A(-1+\xi) = \hat{A}(-\xi)e^{2\mu_i}, \quad B(-1+\xi) = -\hat{B}(-\xi)e^{2\mu_i},$$
$$\alpha(-1+\xi) = \hat{\alpha}(-\xi)e^{2\mu_i}, \quad \beta(-1+\xi) = -\hat{\beta}(-\xi)e^{2\mu_i},$$

Then from (D.1) and the previous lemma,

$$\begin{aligned}\overline{\boldsymbol{v}}_i(-1+\xi) &= \frac{1}{\theta_1^2(\xi/2)}\frac{1}{\sqrt{2}}\left(1_2\otimes\begin{pmatrix}p(-1+\xi) & 0\\ 0 & p(1-\xi)\end{pmatrix}\right)\left(1_2\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)\lambda_i(-\xi)e^{2\mu_i}\\ &= \left(1_2\otimes\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}e^{2\mu_i}\right)\frac{1}{\theta_1^2(\xi/2)}\frac{1}{\sqrt{2}}\left(1_2\otimes\begin{pmatrix}p(1-\xi) & 0\\ 0 & p(-1+\xi)\end{pmatrix}\right)\lambda_i(-\xi)\\ &= \left(1_2\otimes\begin{pmatrix}0 & \frac{p(1+\xi)}{p(1-\xi)}\\ \frac{p(1-\xi)}{p(1+\xi)} & 0\end{pmatrix}e^{2\mu_i}\right)\frac{1}{\theta_1^2(\xi/2)}\frac{1}{\sqrt{2}}\left(1_2\otimes\begin{pmatrix}p(1+\xi) & 0\\ 0 & p(-1-\xi)\end{pmatrix}\right)\lambda_i(-\xi)\end{aligned}$$

where we have used $p(z)p(-z)=1$. Upon comparing this with

$$\overline{\boldsymbol{v}}_i(1-\xi) = \frac{1}{\theta_1^2(\xi/2)}\frac{1}{\sqrt{2}}\left(1_2\otimes\begin{pmatrix}p(1-\xi) & 0\\ 0 & p(-1+\xi)\end{pmatrix}\right)\lambda_i(\xi) = \sum_{s\geq 0}\overline{\boldsymbol{v}}_{i,s}\,\xi^{s-5/2}$$

we obtain

$$\overline{\boldsymbol{v}}_i(-1+\xi) = -\imath\left(1_2\otimes\begin{pmatrix}0 & \frac{p(1+\xi)}{p(1-\xi)}\\ \frac{p(1-\xi)}{p(1+\xi)} & 0\end{pmatrix}e^{2\mu_i}\right)\sum_{s\geq 0}(-1)^s\,\overline{\boldsymbol{v}}_{i,s}\,\xi^{s-5/2}.$$

Here we use the definition of $p(z)$ and the periodicity of the theta functions to see that $p^2(1-\xi) = -p^2(1+\xi)$ to give that $p(1+\xi) = \pm\,\imath\, p(1-\xi)$ yielding

$$\overline{\boldsymbol{v}}_i(-1+\xi) = \pm\left(1_2\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}e^{2\mu_i}\right)\sum_{s\geq 0}(-1)^s\,\overline{\boldsymbol{v}}_{i,s}\,\xi^{s-5/2}. \tag{D.8}$$

Finally we may use continuity and the explicit formula (D.1) to determine the overall sign, which is found to be 1, so establishing the theorem.

D.4. **Proof of Proposition 8.3.** Write the (2,1), (3,1) and (4,1) matrix elements of the matrix equation (8.21) which are linear equations with respect to a_1, b_1, c_1. Solving these via Kramer's rule we find quantities a_1, b_1, c_1 in the form of symmetric functions of $\zeta_{2,3,4}$. Thus, for example,

$$a_1 = -\frac{\imath S_+^2}{8}\frac{\zeta_2\zeta_3+\zeta_2\zeta_4+\zeta_3\zeta_4}{\zeta_2\zeta_3\zeta_4} + \frac{\imath}{8}\zeta_2\zeta_3\zeta_4 S_-^2 + x_+x_3. \tag{D.9}$$

Now from the mini-twistor constraint,

$$\begin{aligned}&\zeta_1\zeta_2\zeta_3\zeta_4 = \frac{K^2-4x_+^2}{K^2-4x_-^2},\\ &\zeta_1+\zeta_2+\zeta_3+\zeta_4 = -\frac{16\imath x_3x_-}{K^2-4x_-^2},\\ &\zeta_1(\zeta_2+\zeta_3+\zeta_4)+\zeta_2\zeta_3+\zeta_2\zeta_4+\zeta_3\zeta_4 = \frac{-8x_+x_- + 16x_3^2 + 2K^2(1-2k'^2)}{K^2-4x_-^2}.\end{aligned} \tag{D.10}$$

Using these we may solve to give a_1 as given by (8.22); we similarly obtain b_1 and c_1. Now the (1,1) diagonal entry is

$$\text{(D.11)}\qquad \frac{\imath}{8}S_-^2\zeta_j^3-(c_1-x_-x_3)\zeta_1^2+2\imath b_1\zeta_j-x_+x_3+a_1+\frac{\imath}{8}\frac{S_+^2}{\zeta_j}=\mathfrak{D}_{1,1}\delta_{1,j}.$$

Substituting the expressions for a_1, b_1 and c_1 leads to (8.19). The same arguments work for the remaining entries to $\widetilde{\widetilde{\mathfrak{v}}}_2$. The mini-twistor constraint is used at all stages of the derivation.

D.5. **Proof of Theorem 11.1.** The matrix is skew-hermitian and the block structure is preserved by left and right multiplication by diagonal matrices, so it suffices to show that

$$(V^\dagger\,\mathcal{Q}^{-1}\mathcal{H}\,V)^{-1}=W^\dagger\,\mathcal{H}^{-1}\mathcal{Q}\,W=-\frac{1}{r^2}\,W^\dagger\,\mathcal{Q}\mathcal{H}\,W$$

has the desired structure. The constancy of the matrix enables us to choose any convenient z to evaluate this; we will choose $z=0$ where $C(0)=1_2$. Then (3.6, 3.7, 7.2) give

$$W_k=\begin{pmatrix}1\\ i\zeta_k\end{pmatrix}\otimes\mathcal{O}\begin{pmatrix}-\imath\,\dfrac{K(\zeta_k^2+1)+2\,\imath\,\eta_k}{2Kk'\zeta_k}\\ 1\end{pmatrix}d_k,$$

$$\overline{W_k}=\begin{pmatrix}\zeta_{\mathcal{J}(k)}\\ i\end{pmatrix}\otimes\mathcal{O}\begin{pmatrix}-\imath\,\dfrac{K(\zeta_{\mathcal{J}(k)}^2+1)+2\,\imath\,\eta_{\mathcal{J}(k)}}{2Kk'\zeta_{\mathcal{J}(k)}}\\ 1\end{pmatrix}d'_{\mathcal{J}(k)},$$

for appropriate nonzero d_k, d'_k. Now

$$(\mathcal{Q}\mathcal{H})(0)=\begin{pmatrix}0 & iKx_2 & -Kk'\,(-x_1+ix_2) & -x_3K\\ iKx_2 & 0 & -x_3K & Kk'\,(-x_1+ix_2)\\ -Kk'\,(ix_2+x_1) & x_3K & 0 & -iKx_2\\ x_3K & Kk'\,(ix_2+x_1) & -iKx_2 & 0\end{pmatrix}.$$

Substitution of these into $W^\dagger\,\mathcal{H}^{-1}\mathcal{Q}\,W$ and using (2.17) yields (i,j)-matrix entries which, for $j\neq\mathcal{J}(i)$ have the form $\mathrm{poly}(\zeta_i,\zeta_j)/\zeta_i\zeta_j$ and this polynomial is in the ideal generated by each of the quartics that $\zeta_{i,j}$ individually satisfy.[18]

The nonzero elements of the matrix (11.6) $\mathfrak{f}_j$ $j=1,\ldots,4$ may be represented in the θ function form (11.7) by using duplication θ-formulae, formulae (7.8) representing θ-quotients in term of coordinates of the curve together with the relations (for all permutations of α_j)

(D.12)

$$\theta_1^2(\alpha_1)\theta_3^2(\alpha_1)\frac{\theta_3(\alpha_2+\alpha_3)\theta_3(\alpha_2+\alpha_4)\theta_3(\alpha_3+\alpha_4)}{\theta_1(\alpha_2-\alpha_1)\theta_1(\alpha_3-\alpha_1)\theta_1(\alpha_4-\alpha_1)}=\frac{4\zeta_1(kk')^{3/2}K^4\mathrm{e}^{-\imath\pi N^2\tau}}{\pi^2(4x_-x_3\zeta_1^3+\imath R\zeta_1^2+12x_+x_3\zeta_1-\imath S_+^2)}.$$

D.6. **Proof of Lemma 11.4.** We want to compute

$$-4\mathcal{E}(\boldsymbol{x})=\left\{\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right\}\mathrm{Tr}\left(\mathcal{H}.\mathcal{G}^{-1}.\mathcal{H}.\mathcal{G}^{-1}\right).$$

Recall we have defined

$$\mathcal{G}_{1,i}=\mathcal{G}^{-1}.\frac{\partial\mathcal{G}}{\partial x_i}.\mathcal{G}^{-1},\quad \mathcal{G}_{2,i}=\mathcal{G}^{-1}.\frac{\partial^2\mathcal{G}}{\partial x_i^2}.\mathcal{G}^{-1}.$$

---

[18] An elementary way to verify this is as follows. Let q_i be the quartic that ζ_i satisfies. Then the resultant of $\mathrm{poly}(\zeta_i,\zeta_j)$ and q_j with respect to ζ_j is a polynomial in ζ_i with a factor (amongst others) of q_i^3.

First observe that

$$\frac{\partial}{\partial x_i}\left(\mathcal{H}.\mathcal{G}^{-1}\right)=\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}^{-1}+\mathcal{H}.\frac{\partial \mathcal{G}^{-1}}{\partial x_i}=\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{H}.\mathcal{G}^{-1}.\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}=\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{H}.\mathcal{G}_{1,i}.$$

Further,

$$\frac{\partial^2}{\partial x_i^2}\left(\mathcal{H}.\mathcal{G}^{-1}\right)=\frac{\partial^2 \mathcal{H}}{\partial x_i^2}.\mathcal{G}^{-1}-2\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}_{1,i}-\mathcal{H}.\frac{\partial \mathcal{G}_{1,i}}{\partial x_i}.$$

Because

$$\begin{aligned}\frac{\partial \mathcal{G}_{1,i}}{\partial x_i}&=\frac{\partial \mathcal{G}^{-1}}{\partial x_i}\cdot\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}+\mathcal{G}^{-1}\cdot\frac{\partial^2 \mathcal{G}}{\partial x_i^2}.\mathcal{G}^{-1}+\mathcal{G}^{-1}\cdot\frac{\partial \mathcal{G}}{\partial x_i}\cdot\frac{\partial \mathcal{G}^{-1}}{\partial x_i}\\&=-\mathcal{G}_{1,i}.\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}+\mathcal{G}_{2,i}-\mathcal{G}^{-1}\cdot\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}_{1,i}\end{aligned}$$

we get

$$\frac{\partial^2}{\partial x_i^2}\left(\mathcal{H}.\mathcal{G}^{-1}\right)=\frac{\partial^2 \mathcal{H}}{\partial x_i^2}.\mathcal{G}^{-1}-2\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}_{1,i}+\mathcal{H}.\left[\mathcal{G}_{1,i}.\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{G}_{2,i}+\mathcal{G}^{-1}\cdot\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}_{1,i}\right].$$

We conclude that

$$\begin{aligned}&\frac{\partial^2}{\partial x_i^2}\left(\mathcal{H}.\mathcal{G}^{-1}.\mathcal{H}.\mathcal{G}^{-1}\right)\\&=\left\{\frac{\partial^2 \mathcal{H}}{\partial x_i^2}.\mathcal{G}^{-1}-2\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}_{1,i}+\mathcal{H}.\left[\mathcal{G}_{1,i}.\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{G}_{2,i}+\mathcal{G}^{-1}\cdot\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}_{1,i}\right]\right\}.\mathcal{H}.\mathcal{G}^{-1}\\&\quad+\mathcal{H}.\mathcal{G}^{-1}.\left\{\frac{\partial^2 \mathcal{H}}{\partial x_i^2}.\mathcal{G}^{-1}-2\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}_{1,i}+\mathcal{H}.\left[\mathcal{G}_{1,i}.\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{G}_{2,i}+\mathcal{G}^{-1}\cdot\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}_{1,i}\right]\right\}\\&\quad+2\left[\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{H}.\mathcal{G}_{1,i}\right]^2.\end{aligned}$$

Upon noting

$$\begin{aligned}&\frac{\partial^2}{\partial x_i^2}\operatorname{Tr}\left(\mathcal{H}.\mathcal{G}^{-1}.\mathcal{H}.\mathcal{G}^{-1}\right)\\&=2\,\mathrm{Trace}\left(\left\{\frac{\partial^2 \mathcal{H}}{\partial x_i^2}.\mathcal{G}^{-1}-2\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}_{1,i}+\mathcal{H}.\left[\mathcal{G}_{1,i}.\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{G}_{2,i}+\mathcal{G}^{-1}\cdot\frac{\partial \mathcal{G}}{\partial x_i}.\mathcal{G}_{1,i}\right]\right\}.\mathcal{H}.\mathcal{G}^{-1}\right)\\&\quad+2\,\mathrm{Trace}\left(\left[\frac{\partial \mathcal{H}}{\partial x_i}.\mathcal{G}^{-1}-\mathcal{H}.\mathcal{G}_{1,i}\right]^2\right)\end{aligned}$$

the Lemma follows.

D.7. **Proof of Proposition 11.5.** Let matrices $\mathcal{G}$ and $\mathcal{H}$ be given by (11.17) and use (11.19) . Each of the terms depend on the solutions ζ_j of the Atiyah-Ward equation and their derivatives. We note that at $\boldsymbol{x}=0$ we have $\zeta_j=\pm k'\pm \imath k$. Then,

$$\begin{aligned}\mathcal{G}(\mathbf{0})&=8k^2K^2\begin{pmatrix}1&0\\0&1\end{pmatrix},\\\frac{\partial}{\partial x_1}\mathcal{G}(\boldsymbol{x})\bigg|_{\boldsymbol{x}=0}&=16Kk(E-K)\begin{pmatrix}-1&0\\0&1\end{pmatrix},\\\frac{\partial}{\partial x_2}\mathcal{G}(\boldsymbol{x})\bigg|_{\boldsymbol{x}=0}&=16EK\frac{k^2}{k'^2}\begin{pmatrix}-1&0\\0&1\end{pmatrix},\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial x_3}\mathcal{G}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= 0,\\
\frac{\partial^2}{\partial x_1^2}\mathcal{G}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= 16EK\frac{64}{k^2}\left((E-K)^2(2k^2-1)-k^2\right)\begin{pmatrix}1&0\\0&1\end{pmatrix},\\
\frac{\partial^2}{\partial x_2^2}\mathcal{G}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= \frac{128E^2k^2}{k'^2}\begin{pmatrix}1&0\\0&1\end{pmatrix},\\
\frac{\partial^2}{\partial x_3^2}\mathcal{G}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= -\frac{64}{k^2k'^2}\left[(E-k'^2K)^2-k^2k'^2\right]\begin{pmatrix}1&0\\0&1\end{pmatrix}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\mathcal{H}(\mathbf{0}) &= 8\imath K[K(1+k'^2)-2E]\begin{pmatrix}0&1\\1&0\end{pmatrix},\\
\frac{\partial}{\partial x_1}\mathcal{H}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= 16\imath(E-K)[2E-K(1+k'^2)]\begin{pmatrix}-1&0\\0&1\end{pmatrix},\\
\frac{\partial}{\partial x_2}\mathcal{H}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= \frac{16\imath k^2}{k'}(EK-2)\begin{pmatrix}1&0\\0&-1\end{pmatrix},\\
\frac{\partial}{\partial x_3}\mathcal{H}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= 0,\\
\frac{\partial^2}{\partial x_1^2}\mathcal{H}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= \frac{64\imath}{Kk^2}[K(1+k'^2)-2E]\begin{pmatrix}1&0\\0&1\end{pmatrix}\\
&\quad+\frac{64\imath}{Kk^2}\left[(K(E-K)-2)(E-K)k'^2-Kk^2\right]\begin{pmatrix}0&1\\1&0\end{pmatrix},\\
\frac{\partial^2}{\partial x_2^2}\mathcal{H}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= \frac{64\imath k^2E}{Kk'^2}(2-EK)\begin{pmatrix}1&0\\0&1\end{pmatrix}\\
&\quad+\frac{64\imath E^2}{Kk'^2}[K(1+k'^2)-2E]\begin{pmatrix}0&1\\1&0\end{pmatrix},\\
\frac{\partial^2}{\partial x_3^2}\mathcal{H}(\boldsymbol{x})\Big|_{\boldsymbol{x}=0} &= -\frac{64\imath}{Kk'^2k^2}[K(E-Kk'^2)^2+Kk^2k'^2+2(Kk'^2-E)]\begin{pmatrix}0&1\\1&0\end{pmatrix}.
\end{aligned}
$$

Bringing all these together in (11.19) yields (11.20). To find (11.21) we use for $k=0,1$ the corresponding expansions of $K(k)$ and $E(k)$,

(D.13)
$$
\begin{aligned}
\mathcal{E}_{\boldsymbol{x}=0,k\sim 0} &= \frac{8}{\pi^4}(\pi^2-8)^2+\frac{(\pi^2-8)(\pi^2-16)}{4\pi^4}k^4+O(k^6),\\
\mathcal{E}_{\boldsymbol{x}=0,k'\sim 0} &= 32k'^2+O(k'^4).
\end{aligned}
$$

D.8. **Proof of Proposition 13.1.** In the case $x_2=0$ we have from (13.6) that $R_+=R_-=R=\sqrt{\pi^2-16r^2}$, $x_+=x_-=x_1$. We order the solutions of Atiyah-Ward equation in such the way that in the limit $x_2=x_3=0$ they coincide with our case **II**, $|x_1|\le\frac{\pi}{4}$ of the x_1-axis, namely,

(D.14)
$$
\zeta_1=\frac{4\imath x_3-R}{4x_1-\pi},\ \zeta_2=\frac{4\imath x_3+R}{4x_1+\pi},\ \zeta_3=\frac{4\imath x_3-R}{4x_1+\pi},\ \zeta_4=\frac{4\imath x_3+R}{4x_1-\pi}.
$$

The corresponding exponentials μ_i are

(D.15) $$\mu_1 = \lambda, \ \ \mu_2 = -\lambda + \frac{1}{2}\imath\pi, \ \ \mu_3 = \lambda + \frac{1}{2}\imath\pi, \ \ \mu_4 = -\lambda,$$

with

(D.16) $$\lambda = \frac{\imath}{4}R = \frac{\imath}{4}\sqrt{\pi^2 - 16r^2}, \quad r = \sqrt{x_1^2 + x_3^3}.$$

We then obtain

(D.17) $$\begin{aligned} \mathrm{Gram} &= 32r^2 - 2\pi^2\cosh^2(2\lambda)1_2 \\ \mathrm{Higgs}'_1 &= \imath(32r^2 - 2\pi^2\cosh^2(2\lambda))\sigma_1 \\ \mathrm{Higgs}'_2 &= \frac{4\pi^2}{R}\sinh(4\lambda)\sigma_1 \\ \mathrm{Higgs}'_3 &= -64\imath r^2\sigma_1 \end{aligned}$$

and the corresponding normalized Higgs field is

(D.18) $$\Phi_{\mathrm{norm}} = \imath\sigma_1\left[\frac{\cosh(2\lambda)^2\pi^2R + 2\imath\pi^2\sinh(4\lambda) + 16r^2R}{R(\pi^2\cosh^2(2\lambda) - 16r^2)}\right]$$

and formula (13.8) follows.

Although this formula was derived under the assumption $16r^2 < \pi^2$ the discussion of §11 shows that (13.8) admits a continuation to the area $16r^2 > \pi^2$. Indeed the points $r = \pm\pi/4$ are ordinary points of the function $H(r)$, namely

$$H(\pm\pi/4) = -\frac{12 - \pi^2}{3(\pi^2 - 4)} \sim 0.121.$$

D.9. **Proof of Proposition 13.2.** In the case $x_3 = 0$ we have from (13.6)

(D.19) $$R_+ = \sqrt{\pi^2 - 16r^2 + 8\imath\pi x_2}, \quad R_- = \sqrt{\pi^2 - 16r^2 - 8\imath\pi x_2}, \quad r^2 = x_1^2 + x_2^2.$$

Solutions of Atiyah-Ward equation are ordered as follows

(D.20) $$\zeta_1 = \frac{R_+}{\pi - 4x_-}, \ \zeta_2 = \frac{R_-}{\pi + 4x_-}, \ \zeta_3 = -\frac{R_-}{\pi + 4x_-}, \ \zeta_4 = -\frac{R_+}{\pi - 4x_-}.$$

The associated μ_i are then

(D.21) $$\mu_1 = \frac{1}{4}\imath R_+, \ \mu_2 = -\frac{\imath}{4}R_- + \frac{1}{2}\imath\pi, \ \mu_3 = \frac{\imath}{4}R_+ + \frac{1}{2}\imath\pi, \ \mu_4 = \frac{1}{4}\imath R_-.$$

The usual calculation leads to the following expression for Gram matrix,

(D.22) $$\mathrm{Gram} = \frac{1}{2}G(x_1, x_2)\mathrm{Diag}\left(\exp\left(\frac{\imath}{2}(R_+ - R_-)\right), \exp\left(-\frac{\imath}{2}(R_+ - R_-)\right)\right)$$

where

(D.23) $$G(x_1, x_2) = (\pi^2 + 16r^2)\sin\frac{R_+}{2}\sin\frac{R_-}{2} - R_+R_-\left(\cos\frac{R_+}{2}\cos\frac{R_-}{2} + 1\right).$$

It is easy to see that $G(0, x_2) \equiv 0$ and we have the series expansion in $\xi \sim 0$,

(D.24) $$G(\xi, x_2) = -\frac{32\mathrm{cosh}^2(2x_2)\pi^2}{\pi^2 + 16x_2^2}\xi^2 + O(\xi^4).$$

Introducing the shorthand,

(D.25) $$S_\pm = \sin\left(\frac{1}{2}R_\pm\right), \quad C_\pm = \cos\left(\frac{1}{2}R_\pm\right) \quad E_\pm = \exp\left(\frac{1}{4}\imath R_\pm\right),$$

we arrive at the formula (11.3) upon substitution of (D.20), (D.21) into the Higgs field Φ.

To check the result obtained, consider the reduction of (13.9) to the x_1 and x_2 axes. To reduce expression (13.9) x_1-axis set $x_2 = 0$ and

$$R_+ = R_- = \sqrt{\pi^2 - 16x_1^2} = R. \tag{D.26}$$

Substituting these into the general formula together with $k = 0$, $K = E = \pi/2$ one obtains Ward's expression (13.3) with $r = x_1$,

$$H(x_1, 0, 0) = -1 - \frac{2\pi^2 C(2S - RC)}{R(\pi^2 C^2 - 16x_1^2)}, \tag{D.27}$$

where $S = \sin(R/2), C = \cos(R/2)$.

To reduce expression (13.9) to the x_2-axis set $x_1 = 0$ and

$$R_+ = \pi + 4\imath x_2, \quad R_- = \pi - 4\imath x_2. \tag{D.28}$$

Now in this case $G(0, x_2) = 0$ for all $x_2 \in \mathbb{R}$ while also the numerator of the expression vanishes and therefore the value of $H(0, x_2, 0)$ results from the limit as $x_1 \to 0$. To do that expand the quantities $R_\pm$ with $x_1 = \xi$ up to order 4,

$$R_\pm = \pi \pm 4\imath x_2 - \frac{8\xi^2}{\pi \pm 4\imath x_2} - \frac{32\xi^4}{(\pi \pm 4\imath x_2)^3} + O(\xi^6). \tag{D.29}$$

Substituting these into the numerator and denominator one finds that both vanish to order ξ^4 and the quotient reads

$$H(0, x_2, 0)^2 = \left(-\tanh(2x_2) + \frac{16x_2}{16x_2^2 + \pi^2}\right)^2 \tag{D.30}$$

which again coincides with Ward's answer in this case.

Appendix E. Lamé's Equation

Here we adopt the approach of Brown, Panagopoulos and Prasad [11] to conjugate the equation $\Delta^\dagger v = 0$ into a convenient form. The aim of this appendix is to show that for each coordinate axis we may reduce the matrix differential equation $\Delta^\dagger v = 0$ to solving

$$\frac{d^2u}{dz^2}(z) + \mathcal{U}(z)\, u(z) = \lambda_1\, u(z),$$

or the same equation for a shifted argument giving $w(z)$. Here

$$\lambda_1 = x_1^2 - \frac{1}{4}(1 + k^2)K^2, \qquad \lambda_2 = x_2^2 - \frac{1}{4}k^2K^2, \qquad \lambda_3 = x_3^2 - \frac{1}{4}K^2.$$

With $Kz = 3K + 2iK' + 2s$ we may put $\mathcal{U}(z)$ into the standard Lamé form

$$\left(\frac{d^2}{ds^2} - 2k^2\mathrm{sn}^2(s)\right)F(s) = -\lambda\, F(s)$$

with $\lambda = 1 + k^2\mathrm{cn}^2(t) = -4\lambda_j/K^2$ and the parameterization of §5.2:

$$x_1: \quad \mathrm{sn}^2(t) = \frac{4x_1^2}{k^2K^2}, \qquad x_2: \quad \mathrm{dn}^2(t) = -\frac{4x_2^2}{K^2}, \qquad x_3: \quad \mathrm{cn}^2(t) = -\frac{4x_3^2}{k^2K^2}.$$

We caution at the outset that both the order of our tensor products and our Nahm data differ from those of [11] and we shall relate our conventions shortly. Set

$$\mathcal{P} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{Q} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{R} := \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ -i & 0 & 0 & -i \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Then we have $\mathcal{P}\left(\sum_{j=1}^{3} \sigma_j \otimes x_j 1_2\right)\mathcal{P} = \sum_{j=1}^{3} x_j 1_2 \otimes \sigma_j$. Then with

$$\begin{aligned} \Delta^\dagger v &= \left[\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}\left(\sum_{j=1}^{3} \sigma_j \otimes \sigma_j f_j(z) \right) - \left(\sum_{j=1}^{3} \sigma_j \otimes x_j 1_2 \right) \right] \boldsymbol{v}(z, \boldsymbol{x}) \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}z} 1_4 + \begin{pmatrix} f_3/2 - x_3 & 0 & -x_1 + i x_2 & f_1/2 - f_2/2 \\ 0 & -f_3/2 - x_3 & f_1/2 + f_2/2 & -x_1 + i x_2 \\ -x_1 - i x_2 & f_1/2 + f_2/2 & -f_3/2 + x_3 & 0 \\ f_1/2 - f_2/2 & -x_1 - i x_2 & 0 & f_3/2 + x_3 \end{pmatrix} \right] \boldsymbol{v}(z, \boldsymbol{x}) \end{aligned}$$

the conjugation

$$\begin{aligned} \tilde{\Delta}^\dagger &:= \mathcal{Q}^{-1}\mathcal{P}\,\Delta^\dagger\,\mathcal{P}\mathcal{Q} \\ &= \frac{\mathrm{d}}{\mathrm{d}z} 1_4 + \begin{pmatrix} (f_3 + f_1 - f_2)/2 & -x_3 & -x_1 & i x_2 \\ -x_3 & (f_3 - f_1 + f_2)/2 & i x_2 & -x_1 \\ -x_1 & -i x_2 & (f_1 + f_2 - f_3)/2 & x_3 \\ -i x_2 & -x_1 & x_3 & -(f_1 + f_2 + f_3)/2 \end{pmatrix} \end{aligned} \tag{E.1}$$

brings this to the *form* (6.2) of [11]. This is the form of the equation to be studied. The key observation of [11] is that for a coordinate axis (there the x_2-axis) the 4×4 problem (E.1) reduces to two 2×2 uncoupled equations. These in turn may be reduced to an $n = 1$ Lamé equation. Before turning to each of these reductions we first comment on the relation of the solutions to our earlier general solution and relate the conventions of [11] with those here.

In general the solutions $\mathcal{V}_{BPP}$ to $\tilde{\Delta}^\dagger \mathcal{V}_{BPP} = 0$ are related to our earlier solutions by

$$\mathcal{V}_{\mathcal{BPP}} = \mathcal{Q}^{-1}\mathcal{P}\,V = \mathcal{Q}^{-1}\mathcal{P}\,\left(1_2 \otimes \mathcal{O}\,C^{-1}(z)\right)\,\frac{1}{\theta_2^2(z/2)}\,\Lambda_i$$

where (the conjugate of) Λ_i was given in (8.1). Thus

$$\begin{aligned} \overline{\mathcal{V}}_{BPP} &= \mathcal{Q}^{-1}\mathcal{P}\,\left(1_2 \otimes \mathrm{Diag}(p(z), p^{-1}(z))\,\mathcal{O}\right)\,\frac{1}{\theta_2^2(z/2)}\,\overline{\Lambda_i} \\ &= \frac{1}{\theta_2^2(z/2)}\,\frac{1}{2} \begin{pmatrix} p\,(z) & -p\,(z) & p\,(-z) & p\,(-z) \\ p\,(z) & -p\,(z) & -p\,(-z) & -p\,(-z) \\ p\,(-z) & p\,(-z) & p\,(z) & -p\,(z) \\ -p\,(-z) & -p\,(-z) & p\,(z) & -p\,(z) \end{pmatrix} \overline{\Lambda_i} \\ &= \frac{1}{\theta_2^2(z/2)} \begin{pmatrix} (1/2\,iB - 1/2\,iA)\,p\,(z)\,\zeta_i + (1/2\,\alpha - 1/2\,\beta)\,p\,(z) + (1/2\,B + 1/2\,A)\,p\,(-z) \\ (1/2\,iB - 1/2\,iA)\,p\,(z)\,\zeta_i + (1/2\,\alpha - 1/2\,\beta)\,p\,(z) + (-1/2\,B - 1/2\,A)\,p\,(-z) \\ (-1/2\,iB - 1/2\,iA)\,p\,(-z)\,\zeta_i + (-1/2\,B + 1/2\,A)\,p\,(z) + (1/2\,\alpha + 1/2\,\beta)\,p\,(-z) \\ (1/2\,iB + 1/2\,iA)\,p\,(-z)\,\zeta_i + (-1/2\,B + 1/2\,A)\,p\,(z) + (-1/2\,\alpha - 1/2\,\beta)\,p\,(-z) \end{pmatrix} \end{aligned}$$

With our expansion
(E.2)

$$V(1-\xi)\sim\begin{pmatrix}\dfrac{-x_1+ix_2}{\sqrt{\xi}}+\sqrt{\xi}\,a\\ \xi^{-3/2}+\dfrac{x_3}{\sqrt{\xi}}+\sqrt{\xi}\left(b-1/2\,r^2\right)\\ -\xi^{-3/2}+\dfrac{x_3}{\sqrt{\xi}}+\sqrt{\xi}\left(b+1/2\,r^2\right)\\ \dfrac{x_1+ix_2}{\sqrt{\xi}}+\sqrt{\xi}\,c\end{pmatrix}\quad\text{then}\quad\mathcal{V}_{BPP}(1-\xi)\sim\begin{pmatrix}\dfrac{(a+c)}{\sqrt{2}}\sqrt{\xi}+\dfrac{i\sqrt{2}x_2}{\sqrt{\xi}}\\ \dfrac{(a-c)}{\sqrt{2}}\sqrt{\xi}-\dfrac{\sqrt{2}x_1}{\sqrt{\xi}}\\ \sqrt{2}\,b\,\sqrt{\xi}+\dfrac{\sqrt{2}x_3}{\sqrt{\xi}}\\ \dfrac{r^2}{\sqrt{2}}\sqrt{\xi}-\dfrac{\sqrt{2}}{\xi^{3/2}}\end{pmatrix}$$

and we find $\tilde{\Delta}^\dagger\mathcal{V}_{BPP}=O(\xi^{1/2})$. Similarly
(E.3)

$$V(\xi-1)\sim\begin{pmatrix}\xi^{-3/2}-\dfrac{x_3}{\sqrt{\xi}}+\sqrt{\xi}\left(b-1/2\,r^2\right)\\ \dfrac{-x_1+ix_2}{\sqrt{\xi}}-\sqrt{\xi}a\\ -\dfrac{x_1+ix_2}{\sqrt{\xi}}+\sqrt{\xi}c\\ \xi^{-3/2}+\dfrac{x_3}{\sqrt{\xi}}-\sqrt{\xi}\left(b+1/2\,r^2\right)\end{pmatrix}\quad\text{then}\quad\mathcal{V}_{BPP}(\xi-1)\sim\begin{pmatrix}-\dfrac{r^2}{\sqrt{2}}\sqrt{\xi}+\dfrac{\sqrt{2}}{\xi^{3/2}}\\ \sqrt{2}\,b\sqrt{\xi}-\dfrac{\sqrt{2}x_3}{\sqrt{\xi}}\\ -\dfrac{(a-c)}{\sqrt{2}}\sqrt{\xi}-\dfrac{\sqrt{2}x_1}{\sqrt{\xi}}\\ \dfrac{(a+c)}{\sqrt{2}}\sqrt{\xi}-\dfrac{i\sqrt{2}x_2}{\sqrt{\xi}}\end{pmatrix}.$$

E.1. **Comparison of Notation.** To compare with [11] we note their choice of the functions $f_j(z)$ are cyclically shifted from ours and their spatial coordinate are the opposite of ours. Denoting the [11] choices by $\tilde{f}_j$, $\tilde{x}_j$ and $\tilde{z}$ then

$$\tilde{f}_1=f_3/Kk',\quad \tilde{f}_2=f_1/Kk',\quad \tilde{f}_3=f_2/Kk'.$$

By comparing

$$\mathcal{R}\,\Delta^\dagger\,\mathcal{R}^{-1}=\frac{\mathrm{d}}{\mathrm{d}z}1_4$$
$$+\begin{pmatrix}\dfrac{Kk'}{2}(\tilde{f}_3+\tilde{f}_1-\tilde{f}_2) & -x_2 & -x_3 & ix_1\\ -x_2 & \dfrac{Kk'}{2}(\tilde{f}_3-\tilde{f}_1+\tilde{f}_2) & ix_1 & -x_2\\ -x_3 & -ix_1 & \dfrac{Kk'}{2}(\tilde{f}_1+\tilde{f}_2-\tilde{f}_3) & x_2\\ -ix_1 & -x_3 & x_2 & -\dfrac{Kk'}{2}(\tilde{f}_1+\tilde{f}_2+\tilde{f}_3)\end{pmatrix}$$

with [11] we see that

$$\tilde{x}_1=-x_3/Kk',\quad \tilde{x}_2=-x_1/Kk',\quad \tilde{x}_3=-x_2/Kk',\qquad \tilde{z}=Kk'z.$$

E.2. **The x_2 axis.** We shall now reduce (E.1) for each of the coordinate axes in turn to a $n=1$ Lamé's equation; having done that we will solve for this, and so solve (E.1) for the three axes. First the x_2 axis. With $x_1=0=x_3$ the equations $\tilde{\Delta}^\dagger\mathcal{V}=0$ decouple into

$$\begin{pmatrix}\dfrac{d}{dz}v_1(z)+(1/2\,f_3(z)+1/2\,f_1(z)-1/2\,f_2(z))\,v_1(z)+ix_2v_2(z)\\ \dfrac{d}{dz}v_2(z)-ix_2v_1(z)+(-1/2\,f_1(z)-1/2\,f_2(z)-1/2\,f_3(z))\,v_2(z)\end{pmatrix}=0$$

and

$$\begin{pmatrix} \frac{d}{dz}w_1(z) + (1/2\, f_3(z) - 1/2\, f_1(z) + 1/2\, f_2(z))\, w_1(z) + i x_2 w_2(z) \\ \frac{d}{dz}w_2(z) - i x_2 w_1(z) + (1/2\, f_1(z) + 1/2\, f_2(z) - 1/2\, f_3(z))\, w_2(z) \end{pmatrix} = 0.$$

Consider the first of these. Solving the first entry for v_2 yields

$$v_2(z) = \frac{i}{2x_2}\left(v_1(z)\, f_3(z) + v_1(z)\, f_1(z) - v_1(z)\, f_2(z) + 2\,\frac{d}{dz}v_1(z) \right)$$

which after substituting in the second component gives a second order equation of the form

$$v_1''(z) - f_2(z)\, v_1'(z) + \tilde{A}(z) v_1(z) = 0.$$

Now

$$f_1 = \frac{d}{dz}\ln(f_2+f_3), \quad f_2 = \frac{d}{dz}\ln(f_3+f_1), \quad f_3 = \frac{d}{dz}\ln(f_1+f_2), \tag{E.4}$$

and so upon introducing the integrating factor

$$v_1(z) = u(z)\, \exp(\frac{1}{2}\int^z f_2(s)ds) = u(z)\,\sqrt{f_1(z)+f_3(z)}$$

we obtain the equation

$$\frac{d^2u}{dz^2}(z) + A(z)\, u(z) = 0$$

where

$$A(z) = -1/4\,(f_3(z))^2 - 1/2\, f_1(z)\, f_3(z) - 1/4\,(f_1(z))^2 + 1/2\, f_1(z)\, f_2(z) - {x_2}^2 + 1/2\, f_3(z)\, f_2(z)\,.$$

We shall show that this is a Lamé equation. In terms of $u(z)$ we have

$$v_2(z) = \frac{i}{2x_2}\left[2u'(z) + (f_3(z)+f_1(z))\, u(z)\right]\,\sqrt{f_1(z)+f_3(z)}. \tag{E.5}$$

When performing the same eliminations for the second set of equations we find that the integrating factor is the inverse of that found. We have

$$w_1(z) = \frac{w(z)}{\sqrt{f_1(z)+f_3(z)}}, \quad w_2(z) = \frac{i}{2x_2}\frac{2w'(z) + (f_3(z)-f_1(z))\, w(z)}{\sqrt{f_1(z)+f_3(z)}},$$

where now

$$\frac{d^2w}{dz^2}(z) + B(z)\, w(z) = 0, \quad B(z) = A(z) + f_3(z)\left[f_1(z) - f_2(z)\right].$$

At this stage we have a solution of the form

$$\mathcal{V} = c_1\mathcal{V}_1 + c_2\mathcal{V}_2$$

where

$$\mathcal{V}_1 = \begin{pmatrix} u(z)\,\sqrt{f_1(z)+f_3(z)} \\ 0 \\ 0 \\ \frac{i}{2x_2}\left[2u'(z) + (f_3(z)+f_1(z))\, u(z)\right]\,\sqrt{f_1(z)+f_3(z)} \end{pmatrix}, \tag{E.6}$$

$$\mathcal{V}_2 = \begin{pmatrix} 0 \\ \dfrac{w(z)}{\sqrt{f_1(z)+f_3(z)}} \\ \dfrac{i}{2x_2}\dfrac{2w'(z)+(f_3(z)-f_1(z))\,w(z)}{\sqrt{f_1(z)+f_3(z)}} \\ 0 \end{pmatrix} \tag{E.7}$$

and this has precisely the asymptotics of (E.2, E.3) for regular solutions u and w.

To proceed we now construct the relevant Lamé equations. Although our choices for the f_j's differ from those of [11] we will obtain the same equation though for shifted arguments. First, noting that $f_3^2(z) = -k^2K^2 + f_1^2(z)$ we obtain the equation

$$\frac{d^2u}{dz^2}(z) + \mathcal{U}(z)\,u(z) = \lambda_2\,u(z), \qquad \lambda_2 = x_2^2 - \frac{1}{4}k^2K^2, \tag{E.8}$$

with

$$\mathcal{U}(z) = \frac{1}{2}\left[f_2(z)\,f_3(z) - f_1(z)\,f_3(z) + f_1(z)\,f_2(z) - f_1^2(z)\right],$$

and also

$$\frac{d^2w}{dz^2}(z) + \mathcal{W}(z)\,w(z) = \lambda_2\,w(z), \qquad \lambda_2 = x_2^2 - \frac{1}{4}k^2K^2,$$

with

$$\mathcal{W}(z) = \frac{1}{2}\left[-f_2(z)\,f_3(z) + f_1(z)\,f_3(z) + f_1(z)\,f_2(z) - f_1^2(z)\right].$$

Let us record the translation properties of our functions $f_j(z)$ (3.14):

$$\begin{aligned} &f_1(\pm z) = f_1(z), && f_1(2\pm z) = -f_1(z), && f_1(2\tau\pm z) = f_1(z),\\ &f_2(\pm z) = \pm f_2(z), && f_2(2\pm z) = \pm f_2(z), && f_2(2\tau\pm z) = \mp f_2(z),\\ &f_3(\pm z) = f_3(z), && f_3(2\pm z) = -f_3(z), && f_3(2\tau\pm z) = -f_3(z). \end{aligned}$$

Thus

$$\mathcal{W}(z) = \mathcal{U}(2\tau - z), \qquad w(z) = u(2\tau - z),$$

and our analysis reduces to the study of $\mathcal{U}(z)$.

Observe that from our asymptotics (3.20, 3.21) of $f_j(z)$ that

$$\mathcal{U}(1-\xi) = O(\xi), \qquad \mathcal{U}(\xi - 1) \sim -\frac{2}{\xi^2} - \frac{1+k^2}{6}K^2,$$

which means that from (E.8)

$$u(1-\xi) = \text{constant} + O(\xi), \tag{E.9}$$

$$\begin{aligned} u(\xi-1) &= \frac{1}{\xi} - \left[\frac{1}{12}(1+k^2)K^2 + \frac{1}{2}\lambda_2\right]\xi + O(\xi^2)\\ &= \frac{1}{\xi} - \left[\frac{x_2^2}{2} + \frac{1}{24}(2-k^2)K^2\right]\xi + O(\xi^2). \end{aligned} \tag{E.10}$$

When substituted into (E.6) this yields (E.2). We also see that both $w(1-\xi)$ and $w(\xi-1)$ are regular.

Finally, let

$$Kz = 3K + 2iK' + 2s.$$

Then

$$\mathcal{U}(z) = -\frac{1}{2}k^2K^2\mathrm{sn}^2(s)$$

and we arrive at the first Lamé equation

$$\frac{d^2F}{ds^2}(s) - 2k^2\mathrm{sn}^2(s)\,F(s) = \left(\frac{4x_2^2}{K^2} - k^2\right)\,F(s) := -\lambda\,F(s), \qquad F(s) = u(z).$$

If we set (this is the parameterization of §5.2)

$$\lambda = 1 + k^2\mathrm{cn}^2(t) = k^2 - \frac{4x_2^2}{K^2} = -\frac{4\lambda_2}{K^2}, \quad \text{or} \quad \mathrm{dn}^2(t) = -\frac{4x_2^2}{K^2},$$

and noting that if $F(s)$ is a solution so too is $F(-s)$ (because $\mathrm{sn}^2(-s) = \mathrm{sn}^2(s)$), then in section §E.5 we show that Hermite's eigenfunctions in this case are

$$\frac{H(s+t)}{\Theta(s)}\exp\{-sZ(t)\}\,, \qquad \frac{H(-s+t)}{\Theta(s)}\exp\{sZ(t)\}\,.$$

We note that

$$x_2 = 0 \Longleftrightarrow t = K + iK'.$$

E.3. **The x_1 axis.** With $x_2 = 0 = x_3$ the equations $\tilde{\Delta}^{\dagger}\mathcal{V} = 0$ decouple into

$$\begin{pmatrix} \frac{d}{dz}v_1(z) + (1/2\,f_3(z) + 1/2\,f_1(z) - 1/2\,f_2(z))\,v_1(z) - x_1v_2(z) \\ \frac{d}{dz}v_2(z) - x_1v_1(z) + (1/2\,f_1(z) + 1/2\,f_2(z) - 1/2\,f_3(z))\,v_2(z) \end{pmatrix} = 0$$

and

$$\begin{pmatrix} \frac{d}{dz}w_1(z) + (1/2\,f_3(z) - 1/2\,f_1(z) + 1/2\,f_2(z))\,w_1(z) - x_1w_2(z) \\ \frac{d}{dz}w_2(z) - x_1w_1(z) + (-1/2\,f_1(z) - 1/2\,f_2(z) - 1/2\,f_3(z))\,w_2(z) \end{pmatrix} = 0.$$

Following the same steps as before we now have a solution of the form

$$\mathcal{V} = c_1\mathcal{V}_1 + c_2\mathcal{V}_2$$

where

$$\mathcal{V}_1 = \begin{pmatrix} \frac{u(z)}{\sqrt{f_2(z) + f_3(z)}} \\ 0 \\ \frac{1}{2x_1}\frac{2u'(z) + (f_3(z) - f_2(z))\,u(z)}{\sqrt{f_2(z) + f_3(z)}} \\ 0 \end{pmatrix},$$

$$\mathcal{V}_2 = \begin{pmatrix} 0 \\ w(z)\,\sqrt{f_2(z) + f_3(z)} \\ 0 \\ \frac{1}{2x_1}\left[2w'(z) + (f_3(z) + f_2(z))\,w(z)\right]\sqrt{f_2(z) + f_3(z)} \end{pmatrix}.$$

Here

$$\frac{d^2u}{dz^2}(z) + \mathcal{U}(z)\,u(z) = \lambda_1\,u(z), \qquad \lambda_1 = x_1^2 - \frac{1}{4}(1 + k^2)K^2,$$

where again

$$\mathcal{U}(z) = \frac{1}{2}\left[f_2(z)\,f_3(z) - f_1(z)\,f_3(z) + f_1(z)\,f_2(z) - f_1^2(z)\right],$$

and also

$$\frac{d^2w}{dz^2}(z) + \mathcal{W}(z)\,w(z) = \lambda_1\,w(z), \qquad \lambda_1 = x_1^2 - \frac{1}{4}(1+k^2)K^2,$$

with

$$\mathcal{W}(z) = \frac{1}{2}\left[-f_2(z)\,f_3(z) + f_1(z)\,f_3(z) + f_1(z)\,f_2(z) - f_1^{\,2}(z)\right].$$

Now we have the parameterisation (this is the parameterization of §5.2)

$$\lambda = 1 + k^2\mathrm{cn}^2(t) = \frac{4}{K^2}\left(\frac{1}{4}(1+k^2)K^2 - x_1^2\right) \quad \text{or} \quad \mathrm{sn}^2(t) = \frac{4x_1^2}{k^2K^2}.$$

We note that depending on whether $x_1 < Kk/2$ or not, the value of t changes from real to a general complex number. This effects the nature of the functions $u(z)$ and the argument of Brown *et al.* for a real solution breaks down.

E.4. **The x_3 axis.** There are a few differences in this case. With $x_1 = 0 = x_2$ we have the decoupled equations

$$\begin{pmatrix} \frac{d}{dz}v_1(z) + (1/2\,f_3(z) + 1/2\,f_1(z) - 1/2\,f_2(z))\,v_1(z) - x_3v_2(z) \\ \frac{d}{dz}v_2(z) - x_3v_1(z) + (1/2\,f_3(z) - 1/2\,f_1(z) + 1/2\,f_2(z))\,v_2(z) \end{pmatrix} = 0$$

and

$$\begin{pmatrix} \frac{d}{dz}w_1(z) + (1/2\,f_1(z) + 1/2\,f_2(z) - 1/2\,f_3(z))\,w_1(z) + x_3w_2(z) \\ \frac{d}{dz}w_2(z) + x_3w_1(z) + (-1/2\,f_1(z) - 1/2\,f_2(z) - 1/2\,f_3(z))\,w_2(z) \end{pmatrix} = 0.$$

Again solutions have the form

$$\mathcal{V} = c_1\mathcal{V}_1 + c_2\mathcal{V}_2$$

where now

$$\mathcal{V}_1 = \begin{pmatrix} \dfrac{u(z)}{\sqrt{f_1(z)+f_2(z)}} \\ \dfrac{1}{2x_3}\dfrac{2u'(z) + (f_1(z) - f_2(z))\,u(z)}{\sqrt{f_1(z)+f_2(z)}} \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V}_2 = \begin{pmatrix} 0 \\ 0 \\ w(z)\sqrt{f_1(z)+f_2(z)} \\ -\dfrac{1}{2x_3}\left[2w'(z) + (f_1(z)+f_2(z))\,w(z)\right]\sqrt{f_1(z)+f_2(z)} \end{pmatrix}.$$

Here

$$\frac{d^2u}{dz^2}(z) + \mathcal{U}(z)\,u(z) = \lambda_3\,u(z), \qquad \lambda_3 = x_3^2 - \frac{1}{4}K^2,$$

where again

$$\mathcal{U}(z) = \frac{1}{2}\left[f_2(z)\,f_3(z) - f_1(z)\,f_3(z) + f_1(z)\,f_2(z) - f_1^{\,2}(z)\right],$$

but now

$$\frac{d^2w}{dz^2}(z) + \tilde{\mathcal{W}}(z)\, w(z) = \lambda_3\, w(z), \qquad \lambda_3 = x_3^2 - \frac{1}{4}K^2,$$

with

$$\tilde{\mathcal{W}}(z) = \mathcal{W}(2K+z) = \frac{1}{2}\left[f_2\,(z)\, f_3\,(z) + f_1\,(z)\, f_3\,(z) - f_1\,(z)\, f_2\,(z) - f_1^2\,(z)\right].$$

Thus

$$\tilde{\mathcal{W}}(z) = \mathcal{U}(2K + 2iK' - z), \qquad w(z) = u(2K + 2iK' - z),$$

and our analysis again reduces to the study of $\mathcal{U}(z)$.

E.5. $n = 1$ **Lamé Equation.** We have shown that our matrix differential equation may be reduced to the same Lamé equation for each of the coordinate axes and here we recall the solutions to this. We have that

$$\Phi(u,a) = \frac{\sigma(u+a)}{\sigma(u)\sigma(a)} \exp\{-u\,\zeta(a)\}$$

solves

$$\left[\frac{d^2}{du^2} - 2\wp(u)\right]\Phi(u,a) = \wp(a)\,\Phi(u,a).$$

The Jacobi functions are

$$H(u) = \theta_1(u/\theta_3^2(0)), \quad \Theta(u) = \theta_4(u/\theta_3^2(0)), \quad Z(u) = \frac{\Theta'(u)}{\Theta(u)}.$$

Set $\omega_1 = \frac{1}{2}\,\theta_3^2(0) = K$. Then

$$\sigma(u) = \frac{2\omega_1}{\theta_1'(0)}\,\theta_1(u/(2\omega_1))\,\exp(\eta_1 u^2/(2\omega_1)),$$

$$\zeta(u) = \frac{\eta_1 u}{\omega_1} + \frac{1}{2\omega_1}\frac{\theta_1'(u/(2\omega_1))}{\theta_1(u/(2\omega_1))}$$

and

$$\Phi(u,a) = c\,\frac{\theta_1([u+a]/(2\omega_1))}{\theta_1(u/(2\omega_1))}\,\exp\left\{-\frac{u}{2\omega_1}\,\frac{\theta_1'(a/(2\omega_1))}{\theta_1(a/(2\omega_1))}\right\}.$$

Thus

$$\begin{aligned}\Phi(u-\omega_1\tau, a+\omega_1\tau) &= c\,\frac{H(u+a)}{\theta_1(u/(2\omega_1)-\tau/2)}\,\exp\left\{-\left(\frac{u}{2\omega_1}-\frac{\tau}{2}\right)\frac{\theta_1'(a/(2\omega_1)+\tau/2)}{\theta_1(a/(2\omega_1+\tau/2)}\right\}\\ &= c'\,\frac{H(u+a)}{\theta_4(u/(2\omega_1))\exp(i\pi u/(2\omega_1))}\,\exp\left\{-\frac{u}{2\omega_1}\left[\frac{\theta_4'(a/(2\omega_1))}{\theta_4(a/(2\omega_1))} - i\pi\right]\right\}\\ &= c'\,\frac{H(u+a)}{\Theta(u)}\,\exp\{-uZ(a)\}\end{aligned}$$

and using $\tau = \omega_3/\omega_1$,

$$\wp(u+\omega_3) = -\frac{1}{3}(1+k^2) + k^2\mathrm{sn}^2(u)$$

we obtain Hermite's solution.

We may write our solutions to (E.8) as

$$u(z) = \frac{\theta_4\left(\frac{z+1}{4} + \frac{t}{2K}\right)}{\theta_1\left(\frac{z+1}{4}\right)}\,\exp\left\{-\frac{z+1}{4}\,\frac{\theta_4'\left(\frac{t}{2K}\right)}{\theta_4\left(\frac{t}{2K}\right)}\right\} \tag{E.11}$$

Now

$$\frac{\theta_4\left(v+\alpha\right)\theta_1'(0)}{\theta_1\left(v\right)\theta_4\left(\alpha\right)}\exp\left\{-v\frac{\theta_4'\left(\alpha\right)}{\theta_4\left(\alpha\right)}\right\}=\frac{1}{v}+\frac{1}{2}\left[\frac{\theta_4''\left(\alpha\right)}{\theta_4\left(\alpha\right)}-\left(\frac{\theta_4'\left(\alpha\right)}{\theta_4\left(\alpha\right)}\right)^2-\frac{1}{3}\frac{\theta_1'''(0)}{\theta_1'\left(0\right)}\right]v+O(v^2)$$
$$=\frac{1}{v}-2K^2\,\wp(2K\alpha+\omega_3)\,v+O(v^2)$$
$$=\frac{1}{v}-16\left[\frac{1}{12}(1+k^2)K^2+\frac{1}{2}\lambda_2\right]v+O(v^2)$$

which gives (E.10).

Using our observation that if $F(s)$ is a solution of Lamé's equation then so is $F(-s)$ we may construct a solution vanishing at $z=-1$ by taking

$$u(z)=\frac{\theta_4\left(\frac{z+1}{4}+\frac{t}{2K}\right)}{\theta_1\left(\frac{z+1}{4}\right)}\exp\left\{-\frac{z+1}{4}\frac{\theta_4'\left(\frac{t}{2K}\right)}{\theta_4\left(\frac{t}{2K}\right)}\right\}-\frac{\theta_4\left(-\frac{z+1}{4}+\frac{t}{2K}\right)}{\theta_1\left(\frac{z+1}{4}\right)}\exp\left\{\frac{z+1}{4}\frac{\theta_4'\left(\frac{t}{2K}\right)}{\theta_4\left(\frac{t}{2K}\right)}\right\}.$$

Appendix F. Monopole Numerics and Visualisation (by David E. Braden, Peter Braden and H.W. Braden)

The numerical evaluation and visualisation of the charge 2 monopole is described.

We describe here the numerical evaluation and visualisation of the charge 2 monopole. The code and numerical evaluation of the energy density are available via GitHub. The numerical evaluation of the Higgs field and energy density implements the functions of the main text in python: the main procedures are described below. The key procedures are those that calculate $-\frac{1}{2}\operatorname{Tr}\Phi^2$ and the energy density $\mathcal{E}$ for a given k and point in space (x_1,x_2,x_3). These are then utilised to calculate the same quantities on planes or axes as desired. We have given the energy density output in the directory *python_smoothed* for k from $k=0.01$ to $k=0.99$ in steps of 0.01.[19]

We have provided three tools to visualize the five dimensional datasets $(\mathcal{E},k,x_1,x_2,x_3)$: two are interactive, and the third graphical. These may be also used for more complicated monopole configurations. We consider the data as three dimensional volumetric data. Each of the interactive viewers allow data to be dragged around and resized. The first https://www.maths.ed.ac.uk/ hwb/browse.html (see the first of Figure 8) uses the energy density as opacity, and hides all volumes below a specified value in order to look inside the volume. One may vary k and the threshold energy density value. The opacity here is on a $0-255$ scale and the double precision energy density is converted to byte format. (The code also includes options for *ab initio* creating energy density in byte form.)

The second visualizer defines a threshold above which to consider as solid, and uses the Marching Cubes algorithm [32] to construct a mesh of that threshold's contour. These meshes can be visualised with many mesh viewers, or even 3D printed (see the second of Figure 8). The procedure *generatemesh.py* will take the value $k=0.6$ and threshold 0.55 to produce a standard .obj file via '`python generatemesh.py 0.6 0.55 > test.obj`'. The resolution of the cubes is that coming from the numerical evaluation (in the Figure these are cubes of size 0.05^3).

The third method of visualizing the data is a 'Tomogram' that takes slices through the volume. We can plot the contours on these images, or use colour to represent the density at

[19] Each of the subdirectories, for example *python_smoothed/k* = 0.01, contains 60 files, each with the results of an xy-plane with a specified z-value from $z=0.025$ to $z=2.975$ in steps of 0.05. The xy-plane themselves are 60×60 arrays of double precision output for x,y from 0.025 to 2.975 in steps of 0.05. The procedures allow arbitrary grids to be specified.

that slice; Figure 9 shows Tomograms with uniform and nonuniform colourings. The second last column of these figures correspond to the k value of Figure 8.

Numerical and Visualisation Scripts

The key scripts will now be described, breaking these into the numerical determination of the relevant quantities and then their visualisation.

Numerical Scripts. The requirements for running these are python 2.7 and the following pip packages: numpy, scipy, mpmath.

higgs_squared.py: For input (k, x_1, x_2, x_3) calculates $-\frac{1}{2}\operatorname{Tr}\Phi^2$ at the point (x_1, x_2, x_3) of space and a parameter k (between 0 and 1). This file also calculates $-\frac{1}{2}\operatorname{Tr}\Phi^2$ for various planes. It may be run by

```
python -c "from higgs_squared import higgs_squared; print(higgs_squared(0.8,
0.1, 0.2, 0.3))"
```

energy_density.py: For input (k, x_1, x_2, x_3) calculates the trace of the Higgs field squared for a point (x_1, x_2, x_3) of space and a parameter k (between 0 and 1). This file also calculates $-\frac{1}{2}\operatorname{Tr}\Phi^2$ for various planes. The file tests if the spatial point corresponds to a multiple root or branch point. It may be run by

```
python -c "from energy_density import energy_density; print(energy_density(0.8,
0.1, 0.2, 0.3))"
```

Scripts for basic Functions: These functions are in the previous scripts. Given k, determining the curve, and a point in space (k, x_1, x_2, x_3) the elementary operations are

- *quartic_roots*(k, x_1, x_2, x_3) gives (unordered) solutions ζ_i to the Atiyah-Ward constraint.
- *order_roots(roots)* orders the roots using the real structure (5.1).
- *calc_zeta*(k, x_1, x_2, x_3) = *order_roots(quartic_roots*(k, x_1, x_2, x_3))
- *calc_eta*(k, x_1, x_2, x_3) gives the corresponding μ_i's.
- *calc_abel(k, zeta, eta)* calculates the Abel image of a point $P = (\zeta, \eta)$. To get the correct choice of contour we compare the η-value given by the theta function (3.6) given by *calc_eta_by_theta*(k, z) and use *abel_select*: if they agrees the Abel image is accepted and if not it is shifted by the half period to the correct sheet.
- *calc_mu*$(k, x1, x2, x3, \zeta, abel)$ calculates the one transcendental function μ.
- *is_awc_multiple_root*(k, x_1, x_2, x_3) tests if there are multiple roots.
- *is_awc_branch_point*(k, x_1, x_2, x_3) tests if we get a branch point as a roots; these are numerically unstable.

python_expressions, modified_expressions: These directories contains the code for such expressions as the Gram matrix, the Higgs, and the various first and second derivatives of ζ_i's, μ_i's. When the the expressions are very long, the appropriate matrix element of the matrices is given.

python_smoothed: As described above, this directory gives the double precision output for the energy density for k from $k = 0.01$ to $k = 0.99$ in steps of 0.01.

Visualisation Scripts. The dependencies for the visualisation tools may be installed via the Makefile. The README.md contains instructions for using the tools.We have

contours-image.py: Generates the Tomogram image.

generatemesh.py: Generates a 3d `.obj` file from the data.

generate-image-data.py: Generates the image data for the interactive web visualiser.

visualise: This directory contains the interactive visualiser.

File Handling and Smoothing. A number of files deal with file handling.

array_tools.py, array_tools_float.py: General file handling and reflection of first quadrant data.
data.py: Load and manipulate the data.
files.py: read and write floating point files.
file_converter.py: converts floating point to bytes.
file_smoother.py: smooths the data on the exceptional loci arising from bitangency.
simplify_script.py: modifies a number of python expressions in order to evaluate them faster; the resulting files are in the modified_expressions directory.
smoothing_tools.py: for smoothing arrays.

References

[1] J. H. Merlin, D. E. L. Pottinger, Sir M. F. Atiyah, N. J. Hitchen and M. W. Ricketts. Monopoles in motion: A study of the low-energy scattering of magnetic monopoles. IBM UK Scientific Centre video and accompanying Pamphlet 207, 1989.
[2] M. F. Atiyah and R. S. Ward. Instantons and algebraic geometry. *Comm. Math. Phys.*, 55(2):117–124, 1977.
[3] H. W. Braden, Antonella D'Avanzo, and V. Z. Enolski. On charge-3 cyclic monopoles. *Nonlinearity*, 24(3):643–675, 2011.
[4] H. W. Braden and V. Z. Enolski. A Platonic monopole, a related Riemann surface, and an identity of Ramanujan. *Ukraïn. Fīz. Zh.*, 50(8):866–874, 2005.
[5] H. W. Braden and V. Z. Enolski. On the tetrahedrally symmetric monopole. *Comm. Math. Phys.*, 299(1):255–282, 2010.
[6] H. W. Braden and Yu. N. Fedorov. An extended Abel-Jacobi map. *J. Geom. Phys.*, 58(10):1346–1354, 2008.
[7] H. W. Braden and V. Z. Enolski. Some remarks on the Ercolani-Sinha construction of monopoles. *Theoretical and Mathematical Physics*, 165(3):5671597, 2010.
[8] H. W. Braden and V. Z. Enolski. The Construction of Monopoles. *Comm. Math. Phys.*, 362(2):547–570, 2018.
[9] H. W. Braden. Spectral Curves are Transcendental. `arXiv:1812.11093.`
[10] S. A. Brown, Atiyah-Ward two-monopole solution, *Phys. Rev. D (3)*, 27(12):2968–2973, 1983.
[11] S. A. Brown, H. Panagopoulos and M. K. Prasad, Two separated SU(2) Yang-Mills-Higgs monopoles in the Atiyah-Drinfel′d-Hitchin-Manin-Nahm construction. *Phys. Rev. D* 26(3): 854–863, 1982.
[12] S. A. Brown, M. K. Prasad, and P. Rossi. Multimonopole solutions. *Phys. Rev. D (3)*, 24(8):2217–2224, 1981.
[13] E. F. Corrigan, D. B. Fairlie, R. G. Yates, and P. Goddard. The construction of self-dual solutions to SU(2) gauge theory. *Comm. Math. Phys.*, 58(3):223–240, 1978.
[14] E. Corrigan and P. Goddard. An n monopole solution with $4n-1$ degrees of freedom. *Comm. Math. Phys.*, 80(4):575–587, 1981.
[15] S. K. Donaldson. Nahm's equations and the classification of monopoles. *Comm. Math. Phys.*, 96(3):387–407, 1984.
[16] N. Ercolani and A. Sinha. Monopoles and Baker functions. *Comm. Math. Phys.*, 125(3):385–416, 1989.
[17] Peter Forgács, Zalán Horváth, and László Palla. Generating the Bogomolny-Prasad-Sommerfield one-monopole solution by a Bäcklund transformation. *Phys. Rev. Lett.*, 45(7):505–508, 1980.
[18] Peter Forgács, Zalán Horváth, and László Palla. Exact multimonopole solutions in the Bogomolny-Prasad-Sommerfield limit. *Phys. Lett. B*, 99(3):232–236, 1981.
[19] Peter Forgács, Zalán Horváth, and László Palla. Generating monopoles of arbitrary charge by Bäcklund transformations. *Phys. Lett. B*, 102(2-3):131–135, 1981.
[20] Peter Forgács, Zalán Horváth, and László Palla. Nonlinear superposition of monopoles. *Nuclear Phys. B*, 192(1):141–158, 1981.
[21] Peter Forgács, Zalán Horváth, and László Palla. Towards complete integrability of the self-duality equations. *Phys. Rev. D (3)*, 23(8):1876–1879, 1981.
[22] Peter Forgács, Zalán Horváth, and László Palla. Finitely separated multimonopoles generated as solitons. *Phys. Lett. B*, 109(3):200–204, 1982.

[23] Peter Forgács, Zalán Horváth, and László Palla. Solution-generating technique for self-dual monopoles, *Nuclear Phys. B*, 229(1): 77–104, 1983.
[24] N. J. Hitchin. Monopoles and geodesics. *Comm. Math. Phys.*, 83(4):579–602, 1982.
[25] N. J. Hitchin. On the construction of monopoles. *Comm. Math. Phys.*, 89(2):145–190, 1983.
[26] C. J. Houghton, N. S. Manton and N. M. Romão. On the constraints defining BPS monopoles. *Comm. Math. Phys.*, 212(1):219–243, 2000.
[27] C. J. Houghton and P. M. Sutcliffe. Tetrahedral and cubic monopoles, *Comm. Math. Phys.*, 180(2):343–361, 1996.
[28] P. Houston and L. O'Raifeartaigh. On the charge distribution of static axial and mirror symmetric monopole systems. *Phys. Lett. B*, 94(2):153–156, 1980.
[29] Jacques Hurtubise. SU(2) monopoles of charge 2. *Comm. Math. Phys.*, 92(2):195–202, 1983.
[30] Jacques Hurtubise. The asymptotic Higgs field of a monopole. *Comm. Math. Phys.*, 97(3):381–389, 1985.
[31] I. M. Kričever. Integration of nonlinear equations by the methods of algebraic geometry. *Funkcional. Anal. i Priložen.*, 11(1):15–31, 96, 1977.
[32] W. E. Lorensen and Harvey E. Cline. Marching cubes: A high resolution 3d surface construction algorithm. *ACM Computer Graphics* 21(4): 163–169, 1987. doi:10.1145/37402.37422
[33] N. S. Manton. Complex Structure of Monopoles, *Nucl. Phys.* B135:319-332, 1978.
[34] Nicholas Manton and Paul Sutcliffe. *Topological solitons*, Cambridge University Press, Cambridge, 2004.
[35] J. H. Merlin and M. W. Ricketts. Transputers bring monopoles into sight. *New Scientist*, page 49, 17 Sept 1987.
[36] David Mumford. *Tata lectures on theta. I*, Birkhäuser Boston, Inc., Boston, MA, 2007.
[37] W. Nahm. A Simple Formalism for the BPS Monopole. *Phys. Lett. B*, 90:413-414, 1980.
[38] W. Nahm. On abelian self-dual multimonopoles. *Phys. Lett. B*, 93(1-2):42–46, 1980.
[39] W. Nahm. Multi-monopoles in the adhm construction. *ICTP preprint IC/81/238*, pages 1–8, 1981.
[40] Werner Nahm. The algebraic geometry of multi-monopoles. In *Group Theoretical Methods in Physics. Proceedings, 11th International Colloquium, Istanbul, Turkey, August 23-28, 1982*, 1982.
[41] L. O'Raifeartaigh, S. Rouhani, and L. P. Singh. Explicit solution of the Corrigan-Goddard conditions for n monopoles for small values of the parameters. *Phys. Lett. B*, 112(4-5):369–372, 1982.
[42] L. O'Raifeartaigh, S. Rouhani. and L. P. Singh. On the finitely separated two-monopole solution, *Nuclear Phys. B*, 206(1): 137–151, 1982.
[43] H. Panagopoulos. Multimonopoles in arbitrary gauge groups and the complete SU(2) two-monopole system. *Phys. Rev. D (3)*, 28(2):380–384, 1983.
[44] M. K. Prasad and P. Rossi. Construction of exact Yang-Mills-Higgs multimonopoles of arbitrary charge. *Phys. Rev. Lett.* 46(13):806–809, 1981.
[45] Martin William Ricketts, Michael T. Vaughn, Anthony J. G. Hey, John H. Merlin and David C. Williams. Topological solutions in gauge theory and their computer graphic representation. *Science*, 240:1163–1168, 1988.
[46] Paul M. Sutcliffe. BPS monopoles, *Internat. J. Modern Phys. A*, 12(26):4663–4705, 1997.
[47] R. S. Ward. On self-dual gauge fields. *Phys. Lett. A*, 61(2):81–82, 1977.
[48] R. S. Ward. A Yang-Mills-Higgs monopole of charge 2 *Comm. Math. Phys.*, 79(3):317–325, 1981.
[49] R. S. Ward. Ansätze for self-dual Yang-Mills fields. *Comm. Math. Phys.*, 80(4):563–574, 1981.
[50] K. Weierstrass. Formeln und Lehrsätze der elliptiechen Functionen bearbeited und herausgegeben von H.A.Schwarz Göttingen, 1885
[51] Erick J. Weinberg and Piljin Yi. Magnetic monopole dynamics, supersymmetry, and duality. *Phys. Rep.*, 438(2-4):65–236, 2007.
[52] Chen Ning Yang. Condition of self-duality for SU(2) gauge fields on Euclidean four-dimensional space. *Phys. Rev. Lett.*, 38(24):1377–1379, 1977.

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