A MODEL FOR PRIVATE STORAGE BEHAVIOR UNDER COMPETITION AND MONOPOLY WITH AN APPLICATION TO KOREAN RICE STORAGE

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Abstract

This paper analyzes storage behavior and price response when industry-wide profit maximization occurs (the case of a monopoly or a cartel) and when the price rise just equals the average cost of storage (a characterization of competitive equilibrium for a storage industry). The model is based on a linear demand curve with inventory costs per metric ton stored which are assumed to be the sum of interest costs plus a fixed storage rate per month.

An empirical example using the parameters for Korean rice storage and marketing is used to illustrate the workings of the model and compare its results under the monopoly and competitive assumptions. The model serves to illustrate the extent to which a monopoly can insulate itself from the external forces of the market.

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Introduction

Inventory policies are important aspects of the economics underlying the markets for all commodities. They are particularly important in agriculture because of the temporally discrete production process and the continuous consumption processes.

The time frame for the development of such storage policies may be from month to month or from year to year or possibly the focus is on the long run equilibrium storage levels. This study concentrates upon the month to month (short term) storage decisions, with the capability to adjust the yearly storage levels in the face of changing demand prospects. Gustafson [6] focused his attention on the year to year carryover with some analysis of the long run storage levels in equilibrium.

The primary sector involved in the storage policy formulation may be a government agency [6, 8] or a private firm. The approach to the derivation of the storage policies may be to simulate the consequences of a number of alternatives and then select one [5, 8] or it may involve specifying an optimality criterion and then analytically deriving the optimal policy on that basis [6]. The analytical solutions to the optimal storage policy might be derived for discrete points in time by a dynamic programming algorithm [6] or as a continuous function of time using the methods of the calculus of variations, as in the monopoly model developed in this study.

This paper, then, is addressing inventory policy by examining the behavior of private, profit motivated individuals, with an analytical means to determine their behavior, utilizing the theory of differential equations in both the competitive and monopolistic forms of the model, with the primary relation in the monopolistic case derived using the methods of the calculus of variations.

This paper has four subsequent sections, dealing with the structure of the model, interpreting the assumptions implied by the structure of the model, mathematically analyzing the behavior of the model under competitive and monopolistic assumptions, and using the model to analyze the intra-year pattern of rice storage and marketing in Korea. Following these are brief statements which present a summary of the conclusions of the paper and offer suggestions for further related research.

There are three appendices to this paper. The first two present the detailed algebraic analysis of the storage model under the competitive assumption and under the monopolistic assumption. The third appendix reviews the basic principles of the calculus of variations, including both the necessary and sufficient conditions for minimizing a functional.

Structure of the Model

Storage Level

$$S(t) = S(0) + \int_0^t (H(y) + G(y) - D(y))\, dy.$$

Inventory Cost

$$IC(t) = r\, P(t)\, S(t) + k\, S(t).$$

Demand Equation

$$P(t) = a - b\ (D(t) + G(t)).$$

Storage Profit Level

$$\pi(t) = \pi(0) + \int_0^t [P(y)\ (-S'(y)) - IC(y)]dy.$$

Farm Harvest Rate[1/]

$$H(t) = H\ \frac{256}{6}\ t^3\ e^{-4t}.$$

Government Sales Rate[1/]

$$G(t) = C(-1/3 + t/18).$$

Gross Farm Income

$$FY(t) = FY(0) + \int_0^t P(y)\ H(y)\ dy.$$

Consumer Expenditure

$$X(t) = X(0) + \int_0^t P(y)\ D(y)\ dy.$$

Revenue from Government Sales

$$TX(t) = TX(0) + \int_0^t P(y)\ G(y)\ dy.$$

Behavior Assumptions

In the monopolistic case, the storage level $S(t)$, and hence sales to consumers $D(t)$[2/], is determined in such a way as to maximize year-end profits from private storage activity, $\pi(12)$.

Table 1: Variables and Parameters Within the Model

Endogenous Variables

P(t)	= Price	W/MT*
D(t)	= Sales rate to consumers	MT/Month
S(t)	= Private storage level	MT
IC(t)	= Inventory cost rate	W/Month
$\pi(t)$	= Accumulated profit level	W
FY(t)	= Accumulated farm income	W
X(t)	= Accumulated consumer expenditure on the commodity	W
TX(t)	= Accumulated revenue from government sales, ignoring government storage costs	W

Predetermined Variables

H(t)	= Harvest rate	MT/Month
G(t)	= Government sales rate	MT/Month

Parameters

r	= Interest rate	%/Month
k	= Warehousing costs	(W/MT)/Month
a	= Price intercept	W
b	= Price response to total demand	W/(MT/Month)
C	= Government inventory capacity	MT
H	= Level of farm harvest	MT/Year

* Won per metric ton. At the current time $1.00 U.S. = 400 Won Korean, approximately.

In the competitive case, storage levels and sales to consumers are such that the month-to-month rise in prices is just sufficient to equal the average cost of holding inventories.

Interpretation of the Model

The storage equation reflects the identity which states that if additions to the private supply (harvest plus government sales) exceed the withdrawals for private consumption (or sales by the private sector to the public) the inventory of the commodity will increase.

The inventory cost equation states that there are two components of the (variable) costs of storage. The interest charge is based upon the instantaneous value of the inventory. By so doing the annual aggregate (integral over the year) of these charges is equal to the annual interest rate multiplied by the average value of the inventory over the year. The second component of inventory cost indicates that there are storage costs that vary directly with the physical size of the inventory irrespective of its value. Such costs might include the costs of operating heating, cooling of drying equipment, the costs from quality deterioration and theft, and the losses due to rodents and bin leakage. Costs which depend more upon the changes in the inventory, such as loading and unloading, and buying and selling costs, could affect the form or the parameters of the solution under the two cases, but are not analyzed in this paper[3].

The demand equation, which determines the price in this model, states that price responds to the rates of private plus government demand in the same degree. If the government is purchasing (G(t) is negative), the price will be higher than it would be if the government is selling, for a given level of private consumption. This formulation assumes that any income effects which may alter the level or form of the demand relationship occur at the outset of the year, but are unchanging through the year. It further assumes that there is no seasonal price functions which cannot be explained by quantity variations.

The profit equation assumes that the same price is paid or received for the commodity whether it is being purchased or sold. Moreover, it assumes that the government pays the same prices for both buying and selling as consumers or private storage firms do. No buying or selling costs are assumed in this model. The profits in the model derive from buying the commodity when it is plentiful and the prices are low and selling it later when the prices are higher, taking account of the costs of holding the inventory. The time interval [0, 12] is assumed to be such that it begins at the outset of harvest and ends just prior to the next harvest.

The interpretation of H(t) as a harvest rate implies that the farmers participate in the storage activities.4/ The sales from farmers to commercial warehouses at times other than harvest are ignored under this interpretation because the profit from the farmer's storage to that point would be a cost to the commercial warehouse and the storage profit equation represents the sum of the farm plus commercial storage profits. The model assumes that the farmer's production activity is reimbursed at the price which prevails at the moment of harvest and that farm consumption is included with urban consumption in D(t).

The particular form of the harvest equation assumes that harvest is distributed according to a fourth order gamma probability function with mean of one month and standard deviation of ½ month. Table 1 shows that the assumed harvest is more than 95% completed by the end of the second month of the harvest season. In this model, the crop year and the time indexing begins at the time the harvest begins.

Table 1: Assumed Harvest Rate

Months since the Start of Harvest	Percentage of Crop yet to be Harvested
.000	100.0%
.918	50.0%
1.670	10.0%
1.940	5.0%
2.511	1.0%
3.266	0.1%

The government purchase equation reflects the assumption that the government agency buys the grain at a linearly decreasing rate for six months, at which time all of its storage facilities will be full, and then begin to sell from its stocks at the same linearly increasing rate for the next six months[5/]. The government behavior equation analyzed in this paper assumes that there would be no net government imports and no quantity losses from storage.

Analysis of the Model

In this section of the paper we will analyze the storage model under an assumption of competitive behavior and under an assumption of monopolistic behavior. The competitive assumption is that prices will rise only as much as the per unit cost of storing the commodity [2, p. 211]. If the prices were expected to rise by more than this amount, larger inventories will be held, resulting in higher current period prices and lower future prices and hence a smaller price change. Likewise, should prices be expected to rise by less than this amount, smaller inventories would be held increasing future prices at the expense of current prices, and the price change would be larger[6].

The monopolistic assumption is that the industry can manage its inventory levels and selling behavior in such a way as to maximize the profits from storage for the entire industry. Implicit in this situation is the assumption that the industry does have all the relevant information in terms of parameters and relationships available to it when it is deriving its storage strategy.

Before beginning the analysis, two important algebraic substitutions will be made: from the definition of storage level, we obtain that $S'(t) = H(t) + G(t) - D(t)$ or $D(t) = H(t) + G(t) - S'(t)$. This can be substituted into the demand equation or obtain $P(t) = a + bS'(t) - bH(t) - 2bG(t)$. This relates the price level to the storage function which is going to be our primary control instrument.

Storage Behavior under Competitive Assumptions. Our assumption regarding the competitive market was that prices would change the same amount as the storage costs per metric ton of the commodity. Algebraically, this is equivalent to $P'(t) = IC(t)/S(t) = rP(t) + k$. This differential equation could be solved as it is, or as is done here, solved to determine the storage levels by substituting the demand equation for P(t). This gives rise to the following differential equation involving storage

$$S''(t) - rS'(t) = \frac{ra+k}{b} - rf(t) + f'(t),$$

where $f(t) = H(t) + 2G(t)$ is the portion of the price equation which depends explicitly upon time. With our particular storage and government sales equations the equation becomes

$$S''(t) - rS'(t) = \frac{ra+k}{b} + 2C(\frac{1}{18} + \frac{r}{3}) - rCt/9 + (3t^2-(4+r)t^3)H \frac{4^4}{3!} e^{-4t}.$$

The general solution to this nonhomogeneous second order differential equation is

$$S(t) = S_0 + S_1 e^{rt} - (\frac{ra+k}{rb} + \frac{2C}{3})t + \frac{Ct^2}{18} - He^{-4t}\ (1+4t+8t^2+\frac{32t^3}{3}).$$

The numerical constants S_0 and S_1 are determined by the boundary conditions imposed upon storage behavior in the particular situation. If we impose the conditions that there is no carry-in and no carry-out (i.e. $S(0) = S(12) = 0$) we can derive the values of the S_0 and S_1 parameters[7].

$$S_0 = (He^{12r} - 12(\frac{ra+k}{rb}))/(e^{12r}-1) \qquad \text{and}$$

$$S_1 = (12\ (\frac{ra+k}{rb}) - H)/(e^{12r}-1)\ .$$

If the carry-in equaled the carry-out, but equaled a minimum level of working stocks, rather than zero, S_0 would be increased by the amount of these working stocks and S_1 would not be changed.

From this solution we can determine the rate at which the storage level changes through time to be

$$S'(t) = r\ S_1 e^{rt} - (\frac{ra+k}{rb}) + 2G(t) + H(t).$$

This implies that the price level is determined to be

$$P(t) = b\ r\ S_1 e^{rt} - k/r.$$

From these equations it is possible to derive explicit algebraic formulas for the remaining endogenous variables in the model (gross farm income, consumer expenditure, gross margin on government sales). Indeed, these have been derived and are included in an algebraic appendix to this paper. For the most part, the equations provide few insights into the operation of the model.

One exception is the derivation of profit in this competitive case. The profit is found to be:

$$\pi(12) = -br\ (12\ (\frac{ra+k}{rb}) - H)\ S(0).$$

This means that if the carry-in at the beginning of the year is zero, there will be, on balance, no net profits realized by the storage sector. Thus the expectation of zero profit in competitive equilibrium is fulfilled.

The next subsection of this paper will present the analysis of the storage model under the assumption of a monopoly organization of the storage industry.

Storage Behavior under Monopolistic Assumptions. If the storage industry was a monopoly, the single firm would be able to manage its inventory levels in such a way that the industry profit from storage would be maximized. Thus our analysis of storage under monopolistic conditions is derived from the assumption that year-end storage profits, $\pi(12)$, are maximized by the choice of an appropriate storage policy.

$$\max_{\substack{S(t)\\ 0\leq t\leq 12}} \left\{ \pi(12) = \pi(0) + \int_0^{12} \left[P(y)\,(-S'(y)) - IC(y) \right] dy \right\}$$

Because both the price and the inventory cost are functions of the storage level, its rate of change and time, we can put our objective function into the form

$\int_0^{12} F(S,S't)dt$, the maximization (or minimization) of which is the first general problem in the Calculus of Variations.

A necessary condition for a function S(t) to be an extremal solution to the problem is that it satisfy the

<u>Euler Equation:</u>

$$\frac{\partial F(S,S',t)}{\partial S} - \frac{d}{dt}\frac{\partial F(S,S't)}{\partial S'} = 0.$$

For our particular problem[8] we have

$$\frac{\partial F}{\partial S} = r\ P(t) + k$$

$$\frac{\partial F}{\partial S'} = P(t) + (S'(t) + rS(t))\frac{\partial P}{\partial S'}\ , \text{ and}$$

$$\frac{d}{dt}\frac{\partial F}{\partial S'} = b\ S' - bf' + b(S'' + rS'),$$

where $f(t) = H(t) + 2G(t)$ is the portion of the price equation which depends explicitly upon time. Thus, the Euler equation is

$$S'' = \frac{ra+k}{2b} - \frac{r}{2}f + \frac{1}{2}f'.$$

With our particular assumptions regarding the harvest rates and government behavior made explicit, the Euler Equation becomes

$$2S'' = \frac{ra+k}{rb} + 2C\left(\frac{r}{3} + \frac{1}{18}\right) - \frac{Crt}{9} + 128\ Ht^2e^{-4t} - (4+r)\ \frac{128}{3}\ Ht^3e^{-4t}.$$

The solution to this differential equation, with exception of two parameters (S_0 and S_1) to be determined by the boundary conditions, is

$$S(t) = S_0 + S_1t\ \left(\frac{1}{4}\ \left(\frac{ra+k}{b}\right) + \frac{bC}{36}\ (1+6r)\right)t^2 - \frac{Cr}{108}\ t^3$$

$$- He^{-4t}\ \left(\left(\frac{1+r}{2}\right) + (2+1.5r)t + (4+2r)t^2 + \left(\frac{16+4r}{3}\right)t^3\right).$$

If, as in the competitive case, we determine these two parameters based on the assumption of zero carry-in and carry-out ($S(0) = S(12) = 0$), these are the specific parameters[9/] which result:

$$S_0 = \left(\frac{1+r}{2}\right)H$$

$$S_1 = -\left(\frac{1+r}{24}\right)H - 3\left(\frac{ra+k}{b}\right) - \frac{C}{3}\ (1+2r).$$

Again, if a minimum level of working stocks were desired, S_0 would be increased accordingly, but S_1 would not be changed.

The rate of change of storage is determined from this to be

$$S'(t) = S_1 + \left(\frac{ra+k}{2b} + \frac{C}{18}(1+6r)\right)t - \frac{rC}{36}t^2 + rHe^{-4t}\left(0.5+2t+4t^2+\frac{16}{3}t^3\right) + \frac{1}{2}H(t).$$

And this gives rise to the price equation, which is

$$P(t) = a + \frac{2bC}{3} + bS_1 + \left(\frac{ra+k}{2} + \frac{bC}{18}(6r-1)\right)t - \frac{rbC}{36}t^2 - \frac{1}{2}bH(t) + rbHe^{-4t}\left(0.5+2t+4t^2+\frac{16}{3}t^3\right).$$

All of these equations together enable one to obtain explicit algebraic representations for all of the other endogenous variables under this monopolistic form of the model. In an algebraic appendix to this paper the expressions for gross farm income, consumer expenditure and gross margin from government sales are presented. The expression for private sector profit in this monopolistic case was not calculated because of the complexity of the expression. In most empirical applications one would use a computer program to trace out the consequences for a particular set of data and the numerical evaluation would be most expediant.

In the next section of this paper we will discuss the results of an empirical application of this model to the case of rice storage and marketing in Korea. In this discussion the competitive and the monopolistic versions of the model are contrasted and some observations are made regarding the empirical generalizations of the model for policy purposes.

Korean Rice Storage and Marketing - An Empiricial Application

This discussion will present an empirical application of this model of storage behavior to describe the intra-year response of rice marketings and storage in Korea. This will consist of a presentation of the basic parameter values used in the model, a description of the response of the storage model under the competitive assumption, a description of the response of the model under the monopolistic assumption, and some concluding remarks on the two forms of the model and its implications for policy makers.

Basic Data Used in the Model: The empirical numbers which are used in this application of the model are taken to represent the situation pertaining to the marketing environment for domestic Korean rice in quite broad terms. While careful thought preceded the choice of the parameter values, they are not being represented as the results of any exhaustive econometric study.

At the heart of this model is the demand equation. This is assumed to be linear, with no intra-year income effects, and in price-dependent form. This is assumed to have an own-price elasticity of -0.4 at the price of 125 won/kilogram and the consumption rate of one-third of a million metric tons per month[10]. Thus, price (in won per metric ton) is given by this relation:

$$P = 437500 - 0.9375\ Q.$$

The harvest level is assumed to be four million metric tons per year, which is virtually the same as the 1971 production of 3.998 million metric tons[11]. The implicit assumptions regarding the timing of harvest are discussed in the section "Interpretation of the Model."

The assumed interest rate is 1.5 percent per month, which is the same rate charged by the Medium Industry Bank for operation loans[12]. Some interest rates on loans are higher and some lower. If anything, this rate may be lower than the opportunity cost of capital for many storage enterprises, since the so-called "curb-market" rates are much higher than the official bank rates.

The noninterest cost of storage is assumed to be 250 won per metric ton per month. The current year MAF specification of storage charges for polished rice in Seoul is 9.55 won per metric ton per day, with an insurance fee of 1.32 won/ton/day, in addition to that[13].

The government storage capacity is assumed to be 700,000 metric tons[14/]. This is assumed to be filled by buying for six months at a linearly decreasing rate, and then emptied by selling at that same rate.

Response of the Competitive Model: The competitive model assumed that prices rose only to the extent that the average cost per metric ton of storage was covered by that price rise. As was mentioned in the theoretical analysis of the model, this assumption gives rise to a specific differential equation which determined storage levels, and hence, consumption rates and prices.

The competitive model determined that it was necessary at the beginning of the crop year to have working stocks of rice in the private storage sector at least in the amount of 135,200 metric tons to enable all demands to be met without resulting in negative storage levels. The maximum level of privately stored rice calculated by the model is slightly more than 2.5 million metric tons. This is somewhat more than twice the 1971 USAID estimate of total storage capacity [13], but the two estimates may not be comparable since the estimate of this model includes farm storage for farm consumption and for off-farm sale at later times of the year, as well as commercial storage.

The rate of total rice consumption through the year decreases in an almost linear fashion from the rate of 580 thousand tons per month as the harvest begins to 86 thousand tons per month at the end of the crop year. Perfectly uniform consumption would be at the rate of 333 thousand tons per month throughout the year.

Corresponding to this consumption pattern, the price level steadily increases from 112.6 won per kilogram at the beginning of the year to 138.1 won per kilogram at the end of the year. This increase follows an exponential growth path (similar to compound interest).

The value of the rice production in terms of these prices is 458.4 billion won. This is about 20 percent higher than the 1971 value of rice production of 373.4 billion won, but there are no marketing and transportation costs assumed in this model.

The total consumer expenditure on rice in the example is 487.4 billion won.

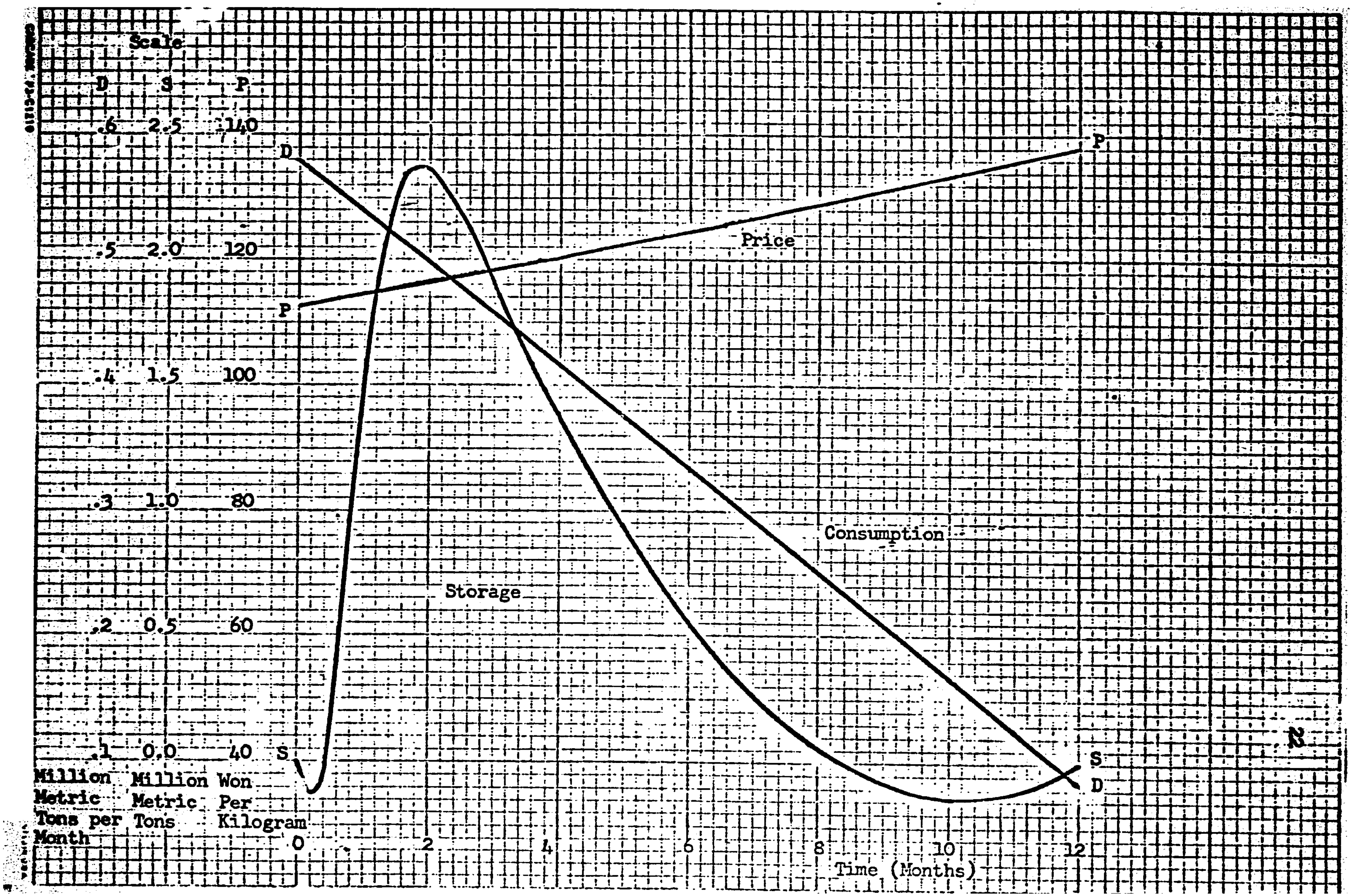
The accumulated gross margin at the end of the year from the governmental storage activity is 11.9 billion won, with the largest deficit which is incurred during the year equal to 81.6 billion won.

The costs and returns of the private sector storage operation which are indicated by the model are as follows. The accumulated interest plus noninterest costs of storage at the end of the year amount to 20.6 billion won. The model calculates a year-end loss from storage activities of 3.468 billion won, which is equal to the annual cost of holding a carryover of 135,200 tons plus a numerical discrepancy[15/] of 21 million won, about 10 won per year per person who might be involved in storage activity. The largest deficit incurred during the year by the private storage sector (indicating the magnitude of the working credit requirements of the sector) is 277.3 billion won.

Figure 1 illustrates the time pattern of consumption rates, price level, and storage level as calculated by the model under the competitive assumption.

As a crude test of the applicability of the model in describing the pattern of rice marketing in Korea both the output of this storage model and Korean corresponding data

Figure 1
Response of the Competitive Storage Model



were indexed so that the maximum intra-year value was 1000, and the variance of the errors of the indexed data around the indexed output was compared with the variance of the indexed data. This ratio will be 1 - $\bar{R}^2$, the adjusted correlation coefficient, for the proportional relationship between the model output and the actual data.

The relation between the storage level of the model and per capita farm storage for September through September rice years from 1964 to 1971 indicated an $\bar{R}^2$ of .830 for this competitive model.

The calculated consumption rates compare more with farm marketing than with consumption *per se*. The equivalent $\bar{R}^2$ was .324 for the comparison with rice taken into Seoul for November through October years from 1963 to 1968[16/]. The comparisons of the model's calculated consumption with either farm or nonfarm household consumption resulted in negative $\bar{R}^2$ equivalents.

The comparison of the indexed output to the indexed prices of rice received by farmers in October through October rice years from 1964 to 1971 resulted in an $\bar{R}^2$ equivalent of .580.

Since there seems to be a reasonable correspondence between the real world data for Korea and the results indicated by this competitive model, there would appear to be a degree of empirical validity in the comparison of this model with the monopolistic model of the next section.

Response of the Monopolistic Model: The storage model is being analyzed under monopolistic assumptions in the Korean setting primarily to provide a contrasting case for the competitive model. However, there is a dominating firm (Korean Express Company) in the private storage industry which controls storage capacity for 244 thousand metric tons of bagged grain [13]. This was almost as much capacity as all other private storage firms in Korea by the USAID estimate. If the response of the monopolistic version of the model provided a better explanation of the data for the Korean rice marketing system than the competitive model, we could then conclude that this dominant firm is able to exercise significant control over the market.

The response of the monopolistic model will be described as the solution to the storage profit maximizing equation and, because of the nature of that response, the response of the model when the storage levels and corresponding derivative is twice that indicated by the profit maximizing equations. In the modified version of the monopolistic model, the same forcing function acts on the differential equation for storage as is felt in the competitive model.

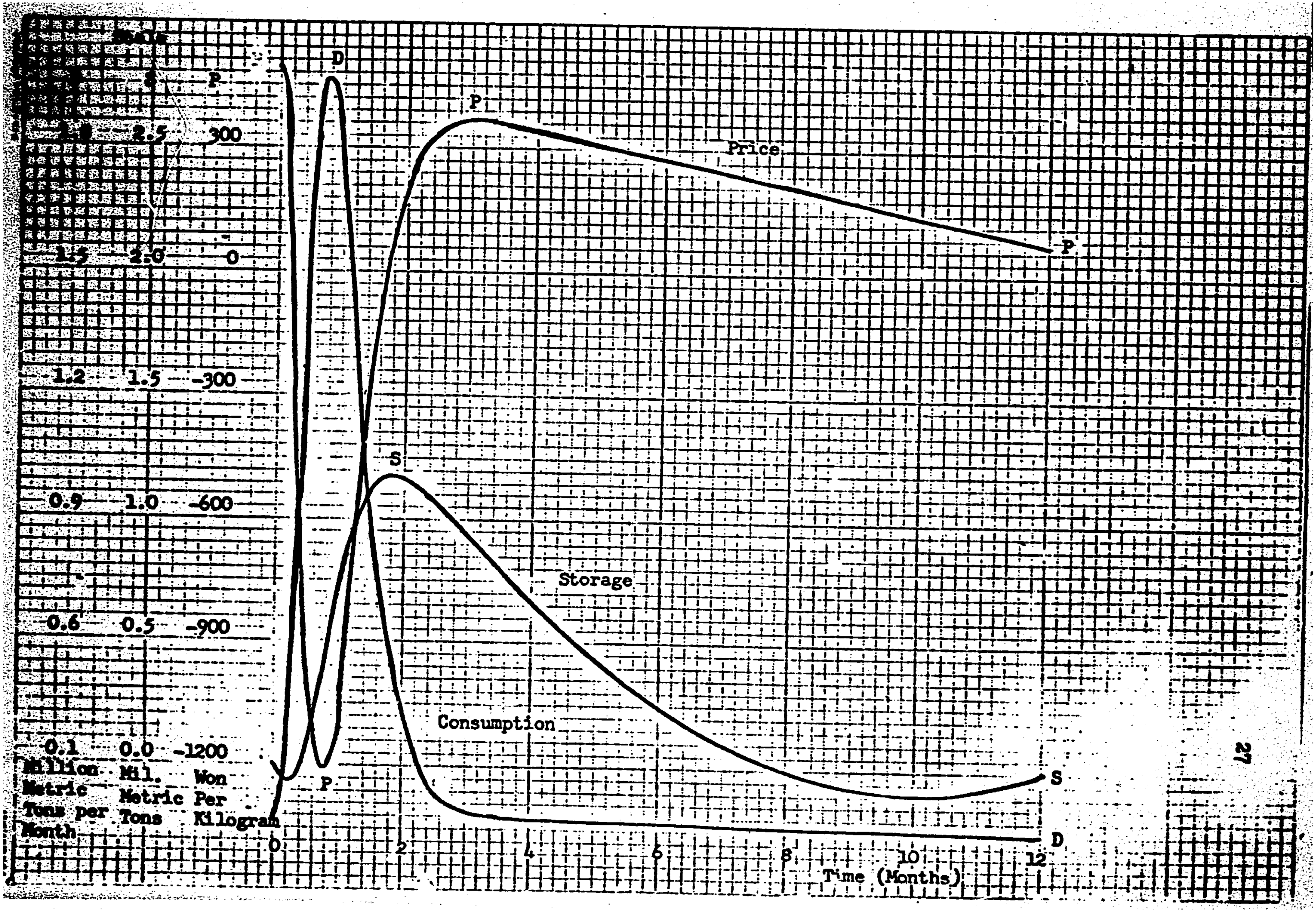
The Euler equation for this maximizing model states that the storage levels of a storage monopoly are changed so that at each time point the rate of change of the marginal revenue from storage is equal to the marginal cost (which in this case equals the average cost) of storage. This compares with the competitive assumption that the rate of change in prices equals the average cost (which is equal to the marginal cost) of storage.

In the profit maximizing model the working stocks necessary at the beginning of the year to meet all demands while maintaining positive storage levels in 79110 metric tons. Under this version of the model the maximum storage levels are 1.26 million tons, or about 100 thousand tons more than the 1971 USAID estimate [13] of total capacity to store bagged grain in Korea.

In this profit maximizing version of the model, the rate of consumption rises from 168 thousand tons per month at the beginning of the year to a peak rate of 1.955 million tons per month at the time of peak harvest (about 25 days after the start of harvest) and then drops off to a rate of 154 thousand tons per month at the end of the year. As before, the uniform rate of consumption is 333 thousand tons per month.

The price pattern which corresponds to this consumption rate starts at 500 won per kilogram at the beginning of the year, drops to the negative price of 1206 won per kilogram at the point of peak consumption, rises to a local maximum of 366

Figure 2
Response of the Monopolistic Storage Model



won per kilogram three months after harvest, and then drops to 74 won per kilogram at the end of the crop year.

Because of the negative prices at the time of peak production and peak consumption, this version of the model computes a negative value of production (-2869 billion won) and a negative value of consumption (-1140 billion won).

The year end cumulative gross margin from government storage activity is 173.6 billion won, or nearly 15 times the level in the competitive model, reflecting the impact of the negative prices during its acquisition period.

The private sector storage profits in this profit maximizing version of the model are 1498 billion won. In this version of the model the storage sector is generating revenue on both the purchase and sale of inventory stocks, i.e. stocks are being built up while the price is negative and are being drawn down when the price is positive (the ultimate in a buy-low, sell-high strategy).

Figure 2 presents the response of the storage model under the monopolistic assumption. The same scale is used for storage levels as for the competitive model, however, the scale for consumption is three times that in the competitive case and for price is fifteen times the scale in the competitive case.

The storage pattern conformed reasonably well to the farm storage pattern for rice with an $\bar{R}^2$ equivalent of .811. The difference between the level of storage under this monopolistic model and that under the competitive model has already been noted.

The comparison of the calculated consumption of the monopoly model with the rice shipments to Seoul indicated an equivalent $\bar{R}^2$ of -0.307, which meant that the average monthly shipment described that variation better than the monopoly model output.

The comparison with the prices received by farmers for rice indicated an $\bar{R}^2$ of -69, which indicated the predicted prices were greatly inferior to the average price as an explanation. This would be expected because of the negative predicted values.

These results seem to serve as a quantitative support for the rejection of this monopolistic form of the storage model as a representation of the behavior of the Korean rice markets within a year.

Because the difference between the response of the monopolistic model above and that of the competitive model of the preceding subsection is so great, one might wonder if it is due to the form of the differential equation or due to the difference in the amplitude of the forcing function being applied to the equation. To answer this equation, the forcing function in the differential equation for monopolistic storage was set equal to the forcing function of the competitive model (i.e. it was doubled) and the model's response was observed.

In this version, then, the minimum carryin necessary to satisfy all demands while maintaining positive storage levels is 158220 metric tons. The maximum level of private storage is 2.52 million metric tons. These exceed the corresponding values from the competitive model by 23 thousand and 10 thousand tons, respectively.

The rate of consumption in this version decreases almost monotonically from 569 thousand metric tons per month at the beginning of the year to 75 thousand metric tons per month at the end of the year with an almost imperceptible trend reversal in the first month followed by a more rapid rate of decrease starting in the second month of the crop year. That is, consumption does not decrease as fast in the first two months of the year as it does later.

Corresponding to this rate of consumption, the prices start at 123 won per kilogram and drop to 93 won per kilogram in the second month after the beginning of harvest. After that, the prices continuously rise at a decreasing rate to the year-end price of 148 won per kilogram.

This version of the model values rice production at 428.5 billion won and consumers spend 469.4 billion won on rice. Both of these values are less than those in the competitive case (30 billion won and 18 billion won less, respectively).

The government's accumulated gross margin from its storage activities is 26.4 billion won, with the largest accumulated deficit in this storage account being 74.6 billion won.

The profit and loss accruing to the private storage sector under this variant of the monopolistic model is quite similar to that in the competitive model. The sector as a whole showed a year-end loss of 4.062 billion won of which all but 28 million won is the cost of holding the necessary working stocks. The accumulated private inventory costs are 18.5 billion won at the end of the year. The largest intra-year deficit in the private storage sector is 260 billion won, which is the amount of short term financing which the private storage sector will need.

Figure 3
Response of the Force-Increased Monopoly Storage Model

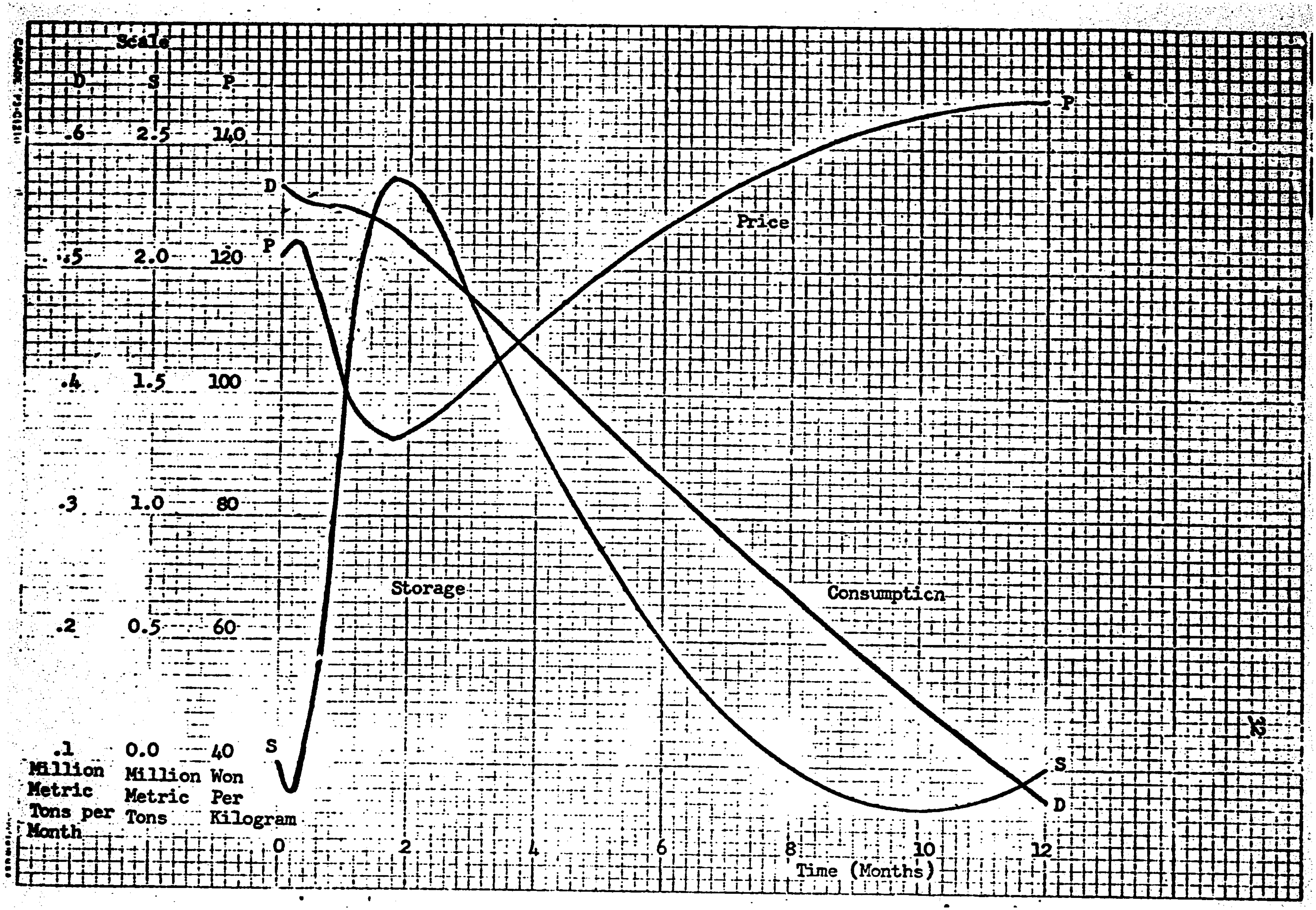


Figure 3 presents the response of prices, consumption and storage under this version of the monopolistic model with increased external forces acting on it. All scales in this figure are identical to those in the competitive model to facilitate direct comparison.

The comparison of this storage response with farm storage levels indicates the same $\bar{R}^2$ as the pure monopoly model (.811) because of the proportionality of the two responses, but the absolute level is more comparable to that of the competitive model.

The comparison of the predicted consumption rates with farm marketings, as indicated by rice shipments to Seoul, resulted in an $\bar{R}^2$ equivalent of .283, which is somewhat less than that for the competitive model.

The predicted price pattern from this version of the model was not as good a predictor of farm rice prices as was the pattern generated by the competitive model. The indicated $\bar{R}^2$ for this case was .106.

Taken in total, thse comparisons seem to indicate that the predictions of the monopoly storage models, in either form, are inferior to those of the competitive model.

Even though the largest private grain storage company (Korean Express Company) has about ten percent of the indicated total private storage requirements (244 thousand metric tons in 1970 [13]), its behavior has not resulted in other than competitive performance of the rice marketing and storage sector, at least as indicated by the published data consulted in this study. However, if the dominant firm in an industry controls only ten percent of that market, there is usually little expectation of monopolistic behavior.

This analysis of the storage model under the monopolistic assumption demonstrated that if the private storage sector were allowed to collude in order to jointly maximize storage sector profits, performance which is not usually desired would be the result. Less of the crop would be stored after harvest, there would be a large intra-year price variation - including negative prices, and producer income would be depressed.

The reason that the pure monopolistic model resulted in such performance is that the force that the market is applying to the storage sector is much weaker under monopoly than under competition. When the same external force is applied to the differential equation for the monopolistic form of the model, the overall response is quite similar to that of the competitive model with the only major exception being a midyear sag in prices.

<u>Concluding Remarks on Korean Rice Storage</u>: Even though this paper is not intended to be a thorough analysis of the policy options which are available to rice price planners, there are some observations on market behavior that show through the model's results.

The first observation is that the operation of government's grain management fund for purposes of intra-year price stabilization can be a profitable operation, particularly if there is no interference with the normal operation of the pricing mechanism. The non-interest costs of storing the government inventory is about 117 million won, or less than one percent of the accumulated gross margin from government storage activity under the competitive model. If the government paid the same interest rate as the private sector the total (interest plus noninterest) costs would be about 20 million won less than the government's accumulated gross margin in the competitive case, and equal to 14.6 billion won giving the government an indicated profit of 11.8 billion won in the case of the augmented monopoly model. Because the grain management fund receives subsidized credit, these figures would understate the government gains from storage.

The second observation is that other rules governing government purchase and sales behavior, while not examined in this paper, can also be analyzed by this model with a minimum of modification. If purchases were proportional to the difference between the actual price and a target price, all that would be necessary is the re-interpretation of the two parameters of the demand equation. In this case, there would not necessarily be the balance between purchase and sales which is built into this formulation, and imports (or exports) would make up the difference. In this case, however, I have not devised a way of incorporating the government storage capacity constraint into this model without substantial revision of the current structure.

It is also possible to use this model to determine the storage pattern which maximizes gross farm income. The storage pattern under that policy would probably be much different from those analyzed in this paper.

One might want to analyze the effect of different levels of government storage capacity on the response of the model, e.g. prices, storage levels, farm income, government storage profit, etc. By so doing, much of the material needed for the benefits of storage could be generated to be used in a benefit-cost analysis. One limit in this analysis would be the response in the case of no government storage.

A third observation regards the intertemporal pattern of prices. This model shows that under any reasonably free market in which production occurs only in certain times of the year when consumption is spread through the year, prices must rise enough to cover the costs of storage, or there will not be a freely motivated transfer of goods from one time period to the next. In that circumstance, there would be substantial pressure to force up the price on the remaining supplies. Thus, a program which expects to achieve total price stability (constant prices throughout the year) while relying on private industry to carry out the storage function, is destined to fail, because the private storage sector requires a minimum price rise to enable it to cover its costs.

The model shows that no excessive profits are being made by the storage industry and, as a matter of fact, the industry is not quite obtaining the opportunity cost of their working capital. This is because of the costs of maintaining the working stocks and the relatively small losses that the sector experiences in addition to the costs of the working stock. The only reasons that actual storage profits might be excessive are that the fixed storage charges contained profit rates which were themselves excessive or that the interest rates used in the analysis exceed the true opportunity cost of capital to storage firms.

Finally, this model seems to present a fairly reasonably representation of storage behavior and inter-temporal price and consumption response. It should then be used as a basis for analyzing the effects of price and storage policy decisions, at least until a better model is available for these purposes.

Summary and Conclusions

In this paper we presented a model of storage behavior in which the average costs of storage consist of the interest on the inventory investment plus a fixed amount per month, and price is set by the demand curve, given a predetermined level of harvest. Particular assumptions were made regarding the timing of the harvest and the form of the purchase response followed by the government storage sector, so as to obtain more concrete results from the analysis. Specifically, it was assumed that the government bought and sold the commodity at the prevailing free market price at the time of sale, and filled its inventory capacity by purchasing at a linearly decreasing rate for six months and then selling at the same linearly increasing rate for the next six months and that the harvest was distributed according to a fourth order Gamma density function with the mean equal to one month and the standard deviation equal to one-half month.

This model was analyzed under two behavioral assumptions. First, the rate of change of prices is equal to the average cost of storage, which defined the response of the model under competitive conditions. The second assumption is that total industry storage profits are maximized during the storage process, which is equivalent to equating the rate of change of the marginal revenue from storage to the marginal cost of storage. The second assumption defined the response of the model under monopolistic conditions.

Based upon the response of the two forms of the model under parameters approximating the situation of rice storage in Korea we were able to compare the performance of the storage industry under competition and monopoly. The most striking conclusion was that the storage level in the monopolistic case rose to only half the level achieved in the competitive case, as a result of the insulation that a monopolistic storage sector has from the external forces of the market. This level of storage was not sufficient to prevent the price of the commodity from plummeting to negative levels at harvest time and rapidly rising during a post-harvest storage before the government starts to sell from their stocks.

Because the monopolistic form of the model appears to follow a policy of buying while the price is negative and selling at positive prices, the competitive form of the model seems to be preferred for most analyses of storage behavior.

The results of the competitive model when applied to the case of Korean rice seem sufficiently credible to warrant its use in analyzing the effects of pricing or storage policies affecting that sector of Korean agriculture.

Suggestions for Further Research

The analysis of this paper deals with the situation of a single commodity being stored and consumed. In later studies it would be particularly useful to generalize the results of this model to the case of multiple commodities, with non-zero cross elasticities of demand and possible competition for the use of both private and government storage facilities.

From a policy-maker's viewpoint, it would be desirable to view the consequences of a number of alternative policies regarding government storage behavior, possibly including tests of different policy response parameters. Some of these that might be considered were mentioned in the "Concluding Remarks on Korean Rice Storage."

A third extension of the model incorporate the effects of constraints on storage capacity (both public and private) into the model in a more explicit fashion than is currently being done.

Citations

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Footnotes

1/ The structure of these two equations is determined at least in part by the empirical content of the Korean situation which will be analyzed later in the paper. Any other functions with continuous first derivatives should work as well in other situations. Justification of these two forms will come later (see pp. 9-10).

2/ The sales to consumers, D(t), are determined from the change in inventory, given the predetermined harvest rate and government market activity.

3/ In the competitive case, this type of specification would give rise to a nonlinear, nonhomogeneous differential equation of second order.

4/ If H(t) is interpreted as a farm sales rate, the storage profit is that of the commercial storage sector, rather than that which accrues to all storage. In this case, D(t) would represent only urban (or nonfarm) consumption.

5/ It is realized that this is only one of many alternative assumptions which could be used to describe this exogeneous input to the private storage model. Another quite plausible, and equally as tractible, assumption is that government sales are proportional to the difference between the realized price and a target price, i.e. if the realized price is "too high" the government would be selling, while it would buy if the actual price were "too low." Under this assumption net imports would be calculated as the residual difference between accumulated purchases and accumulated sales.

6/ Such dynamic disequilibrium phenomena as recognition and decision lags are not being considered in this argument.

7/ The actual parameters, S_0, and S_1, differ from the values computed in the expressions by a term of the order of magnitude approximately 10^{-10}. $19633\ He^{-48}/(e^{12r}-1)$ is subtracted from S_0 and added to S_1.

Footnotes
Cont.

8/ For our problem we can interpret the first term of the Euler equation as the marginal cost of storage and the second term as the rate of change of the marginal revenue from storage. Thus the Euler equation states that a monopolistic storage industry is inequilibrium when the rate of change of the marginal revenue equals the marginal cost of storage.

9/ S_1 actually also contains another term which is of order less than 10^{-11} in magnitude. This is $+ \frac{H}{12} e^{-48}$ (9816.5+2610.5r).

10/ The elasticity is the same as that used in other KASS analyses [8]. The price level is the level at which rice prices are currently set by the government and the quantity level is one-twelfth of the assumed harvest.

11/ [9, p. 136].

12/ [3, p. 337].

13/ This data was provided by the planning section of the Food Bureau of MAF in a September 12, 1973 interview. In January 1971 the corresponding data were 7.70 and 1.09 for warehouse and grain insurance, respectively [10, p. 224].

14/ A USAID study in 1971 [13, p. B-5-13] estimated the MAF and NACF capacity to store bagged grain in December 1970 to be 625000 metric tons.

15/ It can be proved that the annual profit from storage activities under this model is equal to the negative of the cost of holding the carryover when the carryin equals the carryout. Specifically, it is $-br(12(\frac{ra+k}{rb})-H)$ times these working stocks. When there is a zero level of working stocks, there is zero net profit.

16/ The terminal date of October 1968 was chosen because a different government buying policy went into effect at that time which altered the pattern of shipments in favor of early, rather than later, sales. The opening data point for each year (October) was dropped because of the difficulties of the comparison of a discontinuous result with a moderately continuous data series.

APPENDIX

Algebraic Analysis of the Competitive Model

In mathematical terms our assumption governing the competitive case can be stated as

$P'(t) = IC(t)/S(t) = r\,P(t) + k$, that the price rise equals the average cost of storage. Recall that $P(t) = a + bS'(t) - b\,f(t)$, where $f(t) = H(t) + 2G(t)$ is the portion of the demand curve that depends upon time alone.

$bS'' - b\,f'(t) = ra + rbS'(t) - r\,b\,f(t) + k$, or

$$S''(t) - rS'(t) = \frac{ra + k}{b} - r\,f(t) + f'(t).$$

The solution to the homogeneous part is $S(t) = S_0 + S_1 e^{rt}$, where S_0 and S_1 are constants to be determined by the initial conditions of the problem. Substituting the assumed forms of the G and H functions into the equation we obtain the explicit form of the differential equation

$$S'' - rS' = \frac{ra + k}{b} + 2C/18 + 2rC(\frac{1}{3} - t/18) + (3t^2 - 4t^3)H\frac{4^4}{3!}e^{-4t} - rH\frac{4^4}{3!}t^3e^{-4t},$$

$$\text{or } S'' - rS' = \frac{ra + k}{b} + 2C(\frac{1}{18} + \frac{r}{3}) - rCt/9 + (3t^2 - (4+r)t^3)\,H\frac{4^4}{3!}\,e^{-4t}.$$

To obtain the particular solutions to the equation, we will separate the expression to the right of the equality into two distinct portions.

For $S'' - rS' = A - Bt$ we will try $S = k_1t + k_2t^2$

$S' = k_1 + 2k_2t$; $S'' = 2k_2$

$$-r2k_2 = -B \qquad k_2 = \frac{B}{2r} = \frac{rC}{9(2r)} = \frac{C}{18}$$

$$2k_2 - rk_1 = A \qquad k_1 = \frac{2k_2}{r} - \frac{A}{r} = \frac{B}{r^2} - \frac{A}{r} = \frac{C}{9r} - \frac{ra+k}{rb} - 2C(\frac{1}{18r}+\frac{1}{3}).$$

Thus the first particular solution is $S(t) = -(\frac{ra+k}{rb} + \frac{2C}{3})t + \frac{Ct^2}{18}$.

For $S'' - rS' = \frac{4^4}{3!} H(3t^2 - (4+r)t^3)e^{-4t}$, we will try

$S(t) = (k_0 + k_1t + k_2t^2 + k_3t^3)e^{-4t}$ as the particular solution.

$$S'(t) = (k_1 + 2k_2t + 3k_3t^2 - 4k_0 - 4k_1t - 4k_2t^2 - 4k_3t^3)e^{-4t}.$$

$$S''(t) = (2k_2 - 4k_1 + (6k_3 - 8k_2)t - 12k_3t^2 - 4(k_1 - 4k_0)$$
$$- 4(2k_2 - 4k_1)t - 4(3k_3 - 4k_2)t^2 + 16k_3t^3)e^{-4t}.$$

$$16k_3 + 4rk_3 = -\frac{4^4}{3!}H(4 + r) \qquad k_3 = -\frac{4^3}{3!}H$$

$$16k_2 - 12k_3 - 12k_3 - 3rk_3 + 4rk_2 = \frac{4^4}{2!}H \qquad k_2 = -8H$$

$$16k_1 - 8k_2 - 8k_2 + 6k_3 + 4rk_1 - 2rk_2 = 0$$

$$4(4 + r)k_1 = 16k_2 + 2rk_2 - 6k_3 \qquad k_1 = -4H$$

$$16k_0 - 4k_1 - 4k_1 + 2k_2 - rk_1 + 4rk_0 = 0$$

$$4(4 + r)k_0 = (8+r)k_1 - 2k_2 = (8+r)(-H) - 2(-2H) \qquad K_0 = -H$$

So, the second particular solution is $S(t) = -H(1+4t+8t^2+\frac{32}{3}t^3)e^{-4t}$.

Thus, the general equation describing storage behavior in this model is

$$S(t) = S_0 + S_1e^{rt} - (\frac{ra+k}{rb} + \frac{2C}{3})t + Ct^2/18 - He^{-4t}(1+4t+8t^2+\frac{32t^3}{3}),$$

where S_0 and S_1 will be determined by the boundary conditions imposed upon storage behavior.

If we impose the conditions that $S(0)=S(12)=0$ on the solution, we obtain the following values for S_0 and S_1:

$S(0) = 0$ implies $S_0 + S_1 - H = 0$.

$S(12) = 0$ implies $S_0 + e^{12r}S_1 - 12(\frac{ra+k}{rb} + \frac{2C}{3}) + 8C - He^{-48}(19633) = 0$.

Because e^{-48} is of the order of magnitude 10^{-21}, the entire product which involves it will be smaller than 10^{-10} and is negligible. With that convention, we obtain

$S_0 = (e^{12r}H - 12(\frac{ra+k}{rb}) - 10^{-10})/(e^{12r}-1)$ and

$S_1 = (12\ (\frac{ra+k}{rb}) - H + 10^{-10})/(e^{12r}-1)$.

If the carry-in equals the carry-out, but is different from zero, S_0 will be increased by the amount of the carryover.

Now, some of the secondary relations within the model will be derived. Each of these are determined once the storage levels are known. The rate at which storage changes within the year is

$S'(t) = rS_1e^{rt} - (\frac{ra+k}{rb} + \frac{2C}{3}) + Ct/9 + He^{-4t}(\frac{128}{3}\ t^3)$.

The price level within the year is

$P(t) = a + brS_1e^{rt} - (\frac{ra+k}{r} + \frac{2bC}{3}) + bCt/9 + bH\frac{4^4}{3!}\ t^3e^{-4t} - 2bC(t/18 - \frac{1}{3})$

$- bH\ (\frac{4^4}{3!})t^3\ e^{-4t}$. Or

$P(t) = brS_1e^{rt} - k/r$.

The level of gross farm income is $\int_0^{12} P(t)\,H(t)\,dt$.

$$FY(12) = \int_0^{12} brS_1H\,\frac{4^4}{3!}\,t^3e^{(r-4)t}\,dt - Hk/r = \left(\left(\frac{4}{4-r}\right)^4 brS_1 - k/r\right)H.$$

The profit from government sales, ignoring storage costs, is $\int_0^{12} P(t)G(t)dt$.

$$TX(12) = \int_0^{12} brS_1C\,(-1/3 + t/18)e^{rt}\,dt - k/r\int_0^{12} C(-1/3 + t/18)dt.$$

$$= bS_1C\left((1-e^{12r})/3 + \frac{1}{18}\left(12e^{12r} - \frac{1}{r}\,(e^{12r}-1)\right) - 0\left(\frac{kC}{r}\right)\right.$$

$$= bS_1C\left(\frac{2e^{12r}}{3} - (e^{12r}-1)\,(1/3 + 1/18r)\right).$$

$$= \frac{bS_1C}{3}\,(e^{12r}+1) - \frac{bS_1C}{18r}\,(e^{12r}-1)$$

The total consumer expenditure on the commodity is

$$\int_0^{12} P(t)\,D(t)dt = FY(12) + T(12) - \int_0^{12} P(t)\,S'(t)dt.$$

$$\int_0^{12} P(t)\,S'(t) = \int_0^{12} \Big(br^2S_1^2e^{2rt} - S_1(ra+k)e^{rt} - \frac{2brS_1C}{3}e^{rt} + \frac{brS_1C}{9}\,te^{rt}$$

$$+ \frac{128}{3}\,brS_1Ht^3e^{(r-4)t}\Big)dt - (k/r)\,(S(12) - S(0))$$

$$= \frac{brS_1^2}{2}\,(e^{24r}-1) - \left(\frac{ra+k}{r} + \frac{2bC}{3}\right)S_1(e^{12r}-1)$$

$$+ \frac{bS_1C}{9}\,(12e^{12r}-(e^{12r}-1)/r) + \left(\frac{4}{4-r}\right)^4 brS_1H$$

$$= \frac{brS_1^2}{2}\,(e^{24r}-1) + \frac{2bS_1C}{3}\,(e^{12r}+1) - \left(S_1\frac{(ra+k)}{r} + \frac{bS_1C}{9r}\right)(e^{12r}-1)$$

$$+ \left(\frac{4}{4-r}\right)^4 brS_1H.$$

Thus, consumer expenditure is

$$X(12) = (e^{12r}-1)\left(S_1\left(\frac{ra+k}{r}\right) + \frac{bS_1C}{18r}\right) - \frac{bS_1C}{3}\,(e^{12r}+1) - \frac{brS_1^2}{2}\,(e^{24r}-1) - \frac{kH}{r}$$

The derivation of private storage profit in the competitive case consists of the derivation of the year-end cost of inventory and the negative of the accumulated gross margin from private sales and then adding the two to obtain the negative of net private storage profits.

A. Derivation of Inventory Cost

$$\int ICdt = \int (r\ P + k)Sdt = \int (br^2 S_1 e^{rt} S - kS + kS)dt$$

$$= \int br^2 S_1 e^{rt} Sdt$$

$$\int_0^{12} e^{rt} Sdt = \frac{S_0}{r}(e^{12r}-1) + \frac{S_1}{2r}(e^{24r}-1) - \left(\frac{ra+k}{rb} + \frac{2C}{3}\right)\left(\frac{12e^{12r}}{r} - \frac{e^{12r}}{r^2} + \frac{1}{r^2}\right)$$

$$+ \frac{Ce^{12r}}{18}\left(\frac{144}{r} - \frac{24}{r^2} + \frac{2}{r^3}\right) - \frac{2C}{18r^3} - H\left(\frac{1}{4-r}(1-e^{-(4-r)12}) + \frac{4\Gamma(2)}{(4-r)^2}\right)$$

$$- H\left(\frac{8\Gamma(3)}{(4-r)^3} + \frac{32\Gamma(4)}{3(4-r)^4}\right)$$

$$= \frac{S_0}{r}(e^{12r}-1) + \frac{S_1}{2r}(e^{24r}-1) - \left(\frac{ra+k}{rb}\right)\left(\frac{12e^{12r}}{r} - \frac{e^{12r}}{r^2} + \frac{1}{r^2}\right)$$

$$- Ce^{12r}\left(\frac{8}{r} - \frac{2}{3r^2} - \frac{8}{r} + \frac{4}{3r^2} - \frac{1}{9r^3}\right) - \frac{2C}{3r^2} - \frac{C}{9r^3}$$

$$- H\left(\frac{1}{4-r} + \frac{4}{(4-r)^2} + \frac{16}{(4-r)3} + \frac{3.2.32}{3.(4-r)^4}\right)$$

$$= S_0\left(\frac{e^{12r}-1}{r}\right) + S_1\left(\frac{e^{24r}-1}{2r}\right) - \left(\frac{ra+k}{rb}\right)\left(\frac{12e^{12r}}{r} - \frac{e^{12r}}{r^2} + \frac{1}{r^2}\right)$$

$$+ \left(\frac{C}{9r^3}\right)(e^{12r}-1) - \frac{2C}{3r^2}(e^{12r}+1) - \frac{H}{4}\left(\frac{4}{4-r} + \frac{4^2}{(4-r)^2} + \left(\frac{4}{4-r}\right)^3 + \left(\frac{4}{4-r}\right)^4\right)$$

$$= S_0\left(\frac{e^{12r}-1}{r}\right) + S_1\left(\frac{e^{24r}-1}{2r}\right) - \left(\frac{ra+k}{rb}\right)\left(\frac{12e^{12r}}{r} - \frac{1}{r^2}(e^{12r}-1)\right)$$

$$+ \frac{C}{9r^3}(e^{12r}-1) - \frac{2C}{3r^2}(e^{12r}+1) + H\left(\frac{1-\left(\frac{4}{4-r}\right)^4}{r}\right)$$

$$\int_0^{12} e^{rt} S dt = S_0\left(\frac{e^{12r}-1}{r}\right) + S_1\left(\frac{e^{24r}-1}{2r}\right) - \left(\frac{ra+k}{rb}\right)\left(\frac{12e^{12r}}{r} - \left(\frac{e^{12r}-1}{r^2}\right)\right)$$
$$+ \frac{C}{9r^3}\left(e^{12r}-1\right) - \frac{2C}{3r^2}\left(e^{12r}+1\right) + \frac{H}{r}\left(1 - \left(\frac{4}{4-r}\right)^4\right)$$

Thus year-end inventory cost is

$$IC(12) = brS_1S_0\left(e^{12r}-1\right) + \frac{brS_1^2}{2}\left(e^{24r}-1\right) - S_1(ra+k)\left(12e^{12r} - \left(\frac{e^{12r}-1}{r}\right)\right)$$
$$+ \frac{bCS_1}{9r}\left(e^{12r}-1\right) - \frac{2bCS_1}{3}\left(e^{12r}+1\right) + brS_1H\left(1-\left(\frac{4}{4-r}\right)^4\right)$$

B. Derivation of Gross Margin

$$\int_0^{12} P\, S'dt = bS_1\int_0^{12} e^{rt}S'dt - bS_1k(S(12)-S(0)) = bS_1\int_0^{12} e^{rt}S'dt$$

since $S(12) = S(0)$.

$$\int_0^{12} e^{rt}S'dt = rS_1\left(\frac{e^{24r}-1}{2r}\right) - \left(\frac{ra+k}{rb} + \frac{2C}{3}\right)\left(\frac{e^{12r}-1}{r}\right) + \frac{C}{9}\left(\frac{12e^{12r}}{r} - \left(\frac{e^{12r}-1}{r^2}\right)\right)$$
$$+ H\left(\frac{4}{4-r}\right)^4$$

Thus gross margin is

$$\int_0^{12} P\, S'dt = \frac{brS_1^2}{2}\left(e^{24r}-1\right) - S_1(ra+k)\left(\frac{e^{12r}-1}{r}\right) - \frac{2bCS_1}{3}\left(e^{12r}-1\right)$$
$$+ \frac{bS_1^2C}{9}\left(12e^{12r} - \left(\frac{e^{12r}-1}{r}\right)\right) + brS_1H\left(\frac{4}{4-r}\right)^4$$

C. Derivation of profit

$$\int_0^{12} -(\text{Profit}) = IC(12) + \int_0^{12} P\, S'dt$$
$$= brS_1^2\left(e^{24r}-1\right) - S_1(ra+k)\left(\frac{e^{12r}-1}{r}\right) + brS_1S_0\left(e^{12r}-1\right)$$
$$- S_1(ra+k)\left(12e^{12r} - \left(\frac{e^{12r}-1}{r}\right)\right) + brS_1H$$
$$= brS_1\left(S_1\left(e^{24r}-1\right) + S_0\left(e^{12r}-1\right) - 12e^{12r}\left(\frac{ra+k}{rb}\right) + H\right)$$
$$= brS_1\left(S_1\left(e^{24r}-1\right) + He^{12r} - 12\left(\frac{ra+k}{rb}\right) - 12e^{12r}\left(\frac{ra+k}{rb}\right) + H\right)$$
$$= brS_1\left(S_1\left(e^{24r}-1\right) + \left(H-12\left(\frac{ra+k}{rb}\right)\right)\left(e^{12r}+1\right)\right)$$

from the boundary conditions $S(0)=S(12)=0$ we obtain

$$= brS_1\left(S_1\left(e^{24r}-1\right) - S_1\left(e^{12r}-1\right)\left(e^{12r}+1\right)\right)$$
$$= brS_1\left(S_1\left(e^{24r}-1\right) - S_1\left(e^{24r}-1\right)\right)$$
$$= 0$$

If the carry-in were positive, profit would be equal to $-brS_1(e^{12r}-1)$ times the carry-in.

Algebraic Analysis of the Monopolistic Model

In the monopolistic model we maximize year-end private storage profits:

$$\text{Max} \int_0^{12} [P(y)\ (-S/(y)) - IC(y)]dy \quad \text{or} \quad \text{Min} \int_0^{12} [P(y)\ S'(y) + IC(y)]dy$$

Recall that

$S'(t) = H(t) + G(t) - D(t)$, so that $D(t) = H(t) + G(t) - S'(t)$ and $P(t) = a - b(D(t) + G(t))$. If we let $f(t) = H(t) + 2G(t)$, then the demand curve becomes $P(t) = a + bS'(t) - b\ f(t)$. The inventory cost equation is $IC(t) = r\ P(t)\ S(t) + k\ S(t)$.

For this variation problem, $F(S,S',\ t) = p(t)\ S'(t) + IC(t)$. To obtain the components of the Euler Equation:

$$\frac{\partial F}{\partial S} = r\ P(t) + k,$$

$$\frac{\partial F}{\partial S'} = P(t) + S'(t)\frac{\partial P}{\partial S'} + r\ S(t)\frac{\partial P}{\partial S'} = P(t) + b(S'+rS), \text{ and}$$

$$\frac{d}{dt}\frac{\partial F}{\partial S'} = bS'' - bf' + b(S'' + rS').$$

Thus the Euler equation is

$$ra + rbS' - rbf + k - bS'' + bf' - bS'' - rbS' = 0.$$

$$ra + k - rbf + bf' = 2bS''.$$

Or $$S'' = \frac{ra+k}{2b} - \frac{r}{2}f + \frac{1}{2}f'.$$

The solution to the homogeneous part of the equation is $S = S_0 + S_1t$, where S_0 and S_1 are constants to be determined later, using the initial conditions. Substituting back for f in the nonhomogeneous equation, we will proceed to obtain the set of particular solutions to be equation.

$$2S'' = \frac{ra + k}{b} - rH_1t^3e^{-4t} + 2rC/3 - 2Crt/18 + H_1(3t^2 - 4t^3)e^{-4t} + 2C/18.$$

This can be rearranged to be

$$2S'' = \frac{ra + k}{b} + 2C(r/3 + 1/18) - Crt/9 + 3H_1t^2e^{-4t} - (rH_1 + 4H_1)t^3e^{-4t}, \text{ or}$$

$$2S'' = A - Crt/9 + H_1(3t^2 - (4 + r)t^3)e^{-4t}.$$

Consider the portion which is

$$S'' = \frac{A}{2} - C(r/18)t.$$

$S = k_2t^2 + k_3t^3$ would be a particular solution.

$$S' = 2k_2t + 3k_3t^2$$

$$S'' = 2k_2 + 6k_3t$$

From this, we get

$$6k_3 = -rC/18 \qquad k_3 = -rC/108$$

$$2k_2 = A/2 \qquad k_2 = A/4 = 1/4\left(\frac{ra + k}{b} + 2C\left(\frac{r}{3} + \frac{1}{18}\right)\right)$$

Now, consider the portion which is

$$S'' = (H_1/2)(3t^2 - (4 + r)t^3)e^{-4t}.$$

Try $S = (k_0 + k_1t + k_2t^2 + k_3t^3)e^{-4t}$ as a particular solution.

$$S' = e^{-4t}(k_1 + 2k_2t + 3k_3t^2 - 4(k_0 + k_1t + k_2t^2 + k_3t^3)).$$

$$S'' = e^{-4t}(2k_2 - 4k_1 + (6k_3 - 8k_2)t - 12k_3t^2 - 4(k_1 - 4k_0 + (2k_2 - 4k_1)t$$
$$+(3k_3 - 4k_2)t^2 - 4k_3t^3)).$$

From this, we get

$$16k_3 = -(H_1/2)(4 + r) \qquad \text{or} \qquad k_3 = \frac{-H_1}{32}(4 + r).$$

$$-12k_3 + 16k_2 - 12k_3 = \frac{3H_1}{2}$$

$$16k_2 = \frac{3H_1}{2} - \frac{24}{32}H_1(4 + r) = H_1(3/2 - 3/4(4+r)) = -H_1(3/2 + \frac{3r}{4})$$

$$k_2 = -H_1(\frac{3}{32} + \frac{3r}{64})$$

$$16k_1 - 8k_2 - 8k_2 + 6k_3 = 0 = k_1 - k_2 + 3/8k_3$$

$$k_1 = -H_1(\frac{3}{32} + \frac{3r}{64} - (3/8)(\frac{1}{8} + \frac{r}{32})) = -H_1(\frac{3}{64} + \frac{9r}{256})$$

$$16k_0 - 4k_1 - 4k_1 + 2k_2 = 0 \quad k_0 = k_1/2 - k_2/8$$

$$k_0 = -H_1(3/128 + \frac{9r}{512} - \frac{3}{256} - \frac{3r}{512}) = -H_1(\frac{3}{256} + \frac{3r}{256})$$

Combining both the homogeneous and the particular solution to the Euler equation, storage level is determined to be

$$S = S_0 + S_1 t + t^2(\frac{ra + k}{4b} + C(\frac{r}{6} + \frac{1}{36})) - \frac{Crt^3}{108}$$

$$- H\frac{256}{6}(\frac{3}{256} + \frac{3r}{256} + (\frac{3}{64} + \frac{9r}{256})t + (\frac{3}{32} + \frac{3r}{64})t^2 + (\frac{1}{8} + \frac{r}{32})t^3)e^{-4t}, \text{ or}$$

$$S = S_0 + S_1 t + (\frac{9k + bC}{36b} + (\frac{3a + 2bC}{12b})r)t^2 - \frac{Crt^3}{108}$$

$$- H((\frac{1 + r}{2}) + (2 + 1.5r)t + (4 + 2r)t^2 + (\frac{16 + 4r}{3})t^3)e^{-4t}.$$

To determine the S_0 and S_1 parameters, we need to specify two conditions on storage behavior.

If $S(0) = 0$, this implies that

$$S_0 - (\frac{1+r}{2})H = 0 \text{ or that}$$

$$S_0 = (\frac{1+r}{2})H.$$

If $S(12) = 0$, this implies that

$$S_0 + 12(S_1 + r(3a+2bC)/b) + 4(\frac{9k+bC}{b}) - 16rC$$

$$- He^{-48}(\frac{1+r}{2} + 24 + 18r + 576 + 288r + 1728(\frac{16+4r}{3})) = 0, \text{ or}$$

$$S_0 + 12S_1 + 8rC + 4C + 36ar/b + 36k/b - He^{-48}(9816.5 + 2610.5r).$$

Since $\log_{10}(e^{-48}) = -20.846135$, we can neglect the last term.

$$12S_1 = -(\frac{1+r}{2})H - 8rC - 4C - (\frac{36ar+36k}{b}) + (10^{-10}) \quad \text{or}$$

$$S_1 = -(\frac{1+r}{24})H - 2rC/3 - C/3 - 3ra/b - 3k/b + (10^{-11}).$$

If the carry-in at the beginning of the year were equal to the carry-out at the end of the year, the only thing which would differ is S_0, which would then be increased by the amount of the carryover.

At this point some of the secondary relations within the model will be derived. Each of these is determined once the storage levels are known.

The rate of change of storage is described by

$$S'(t) = S_1 + \left(\frac{9k+bC}{18b} + \left(\frac{3a+2bC}{6b}\right)r\right)t - rC\left(\frac{t}{6}\right)^2$$

$$- He^{-4t}\left(\left(\frac{4+3r}{2}\right) + (8+4r)t + (16+4r)t^2 - 2 - 2r - (8+6r)t - (16+8r)t^2\right.$$

$$\left. - \left(\frac{64+16r}{3}\right)t^3\right) .$$

$$S'(t) = S_1 + \left(\frac{k+ra}{2b} + C\left(\frac{1}{18} + \frac{r}{3}\right)\right)t - rC(t/6)^2 - He^{-4t}(-r/2 - 2rt - 4rt^2$$

$$- \left(\frac{64+16r}{3}\right)t^3).$$

$$S'(t) = S_1 + \left(\frac{k+ra}{2b} + C\left(\frac{1}{18} + \frac{r}{3}\right)\right)t - rC(t/6)^2 + rHe^{-4t}(1/2+2t+4t^2+16/3t^3)$$

$$+ \frac{64}{3}\, t^3He^{-4t},$$

The price equation is $P(t) = a - bH(t) - 2bG(t) + bS'(t)$, or

$$P(t) = a - bH\,\frac{256}{6}\,t^3e^{-4t} + 2bC(1/3 - t/18) + bS_1 + \left(\frac{k+ra}{2} + bC\left(\frac{1}{18} + \frac{r}{3}\right)\right)t$$

$$-rbC(t/6)^2 + \frac{64}{3}\,bHt^3e^{-4t} + bHe^{-4t}\,\left(1/2+2t+4t^2+\frac{16}{3}\,t^3\right), \text{ or}$$

$$P(t) = a - bH\,\frac{64}{3}t^3e^{-4t} + 2bC/3 + bS_1 + \left(\frac{k+ra}{2} + bC(r/3 - 1/18)\right)t - rbC(t/6)^2$$

$$+ rbHe^{-4t}\left(1/2+2t+4t^2+\frac{16}{3}t^3\right).$$

In the evaluation of the integrals which represent gross farm income, government net revenue from sales. consumer expenditure, etc., the upper limit (twelve) is sufficiently large that we can approximate the definite integrals involving e^{-4t} by the corresponding Gamma functions with no discernible error.

Gross farm income for the year is $\int_0^{12} P(t)\, H(t)dt$.

$$FY(12) = \int_0^{12} (a+bS_1+2bC/3)H\,\frac{128}{3}\,t^3e^{-4t} + \int_0^{12} \left(\frac{k+ra}{2}+bC\left(\frac{r}{3}-\frac{1}{18}\right)\right)H\,\frac{128}{3}\,t^4e^{-4t}dt$$

$$-\int_0^{12} brC(H/36)\frac{128}{3}\,t^5e^{-4t}dt + rbH^2\int_0^{12}(1/2)\,\frac{128}{3}\,t^3e^{-8t}dt$$

$$+ 2rbH^2\int_0^{12}\frac{128}{3}\,t^4e^{-8t}dt + 4rbH^2\int_0^{12}\frac{128}{3}\,t^5e^{-8t}dt$$

$$+ bH^2\left(\frac{16r}{3}-\frac{64}{3}\right)\int_0^{12}\frac{128}{3}\,t^6e^{-8t}dt.$$

$$FY(12) = H(a+bS_1+2bC/3) + H\left(\frac{k+ra}{2} + bC(r/3-\frac{1}{18})\right) - HrbC\,\frac{5}{144}$$

$$+ rbH^2/32 + rbH^2/16 + 5rbH^2/64 + 5rbH^2/64 - 5bH^2/16.$$

$$FY(12) = H(a+bS_1+2bC/3) + H\left(\frac{k+ra}{2} + bC(r/3-\frac{1}{18})\right) - 5rbCH/144$$

$$+ rbH^2/4 - 5bH^2/16.$$

The profits from government sales, ignoring their holding costs, is $\int_0^{12} P(t)\, G(t)dt$.

$$TX(12) = (a+2bC/3+bS_1)\int_0^{12} C(-1/3+t/18)dt + \left(\frac{k+ra}{2}+bC\left(\frac{r}{3}-\frac{1}{18}\right)\right)\int_0^{12} C\left(\frac{-t}{3}+\frac{t^2}{18}\right)dt$$

$$-\frac{rbC^2}{36}\int_0^{12}\left(\frac{-t^2}{3}+\frac{t^3}{18}\right)dt - rbCH\int_0^{12}\left(\frac{1}{6}+\frac{22}{36}t+\frac{11}{9}t^2+\left(\frac{14}{9}-\frac{64}{9r}\right)t^3\right.$$

$$\left.-\left(\frac{8}{27}-\frac{32}{27r}\right)t^4\right)e^{-4t}dt.$$

$$TX(12) = 8C\left(\frac{k+ra}{2} + bC(r/3 - \frac{1}{18})\right) - 8rbC^2/3 - rbCH\left(\frac{1}{24} + \frac{11}{288} + \frac{11}{288} + \frac{7}{192} - \frac{\text{[illegible]}}{6r}\right)$$

$$-rbCH\left(\frac{1}{144} - \frac{1}{36}r\right)$$

$$TX(12) = 8C\left(\frac{k+ra}{2} - \frac{bC}{18}\right) - rbCH\left(\frac{31}{192}\right) + \frac{7}{36}bCH.$$

Total consumer expenditure on the commodity for the year is

$\int_0^{12} P(t)\, D(t)dt.$ Since $D(t) = H(t) + G(t) - S'(t)$,

$$X(12) = FY(12) + TX(12) - \int_0^{12} P(t)\ S'(t)dt.$$

$$\int_0^{12} P(t)S'(t)dt = \int_0^{12} (a+2bC/3+bS_1)S'(t)dt + \int_0^{12} \left(\frac{k+ra}{2}+bC\left(\frac{r}{3} - \frac{1}{18}\right)\right)S_1 t\ dt$$

$$+\int_0^{12} \left[\left(\left(\frac{k+ra}{2b}\right)^2+\left(\frac{k+ra}{b}\right)C\frac{r}{3}+C^2\left(\frac{r^2}{9} - \left(\frac{1}{18}\right)^2\right) - \frac{rCS_1}{36}\right)bt^2-\left(k+ra+\frac{2rCb}{3}\right)\frac{rC}{36}t^3+br^2C^2(t/6)^4\right]dt$$

$$+\int_0^{12} rbHe^{-4t}\left(S_1/2+\left(2S_1+\frac{k+ra}{2b}+rC/3\right)t+\left(4S_1+2\left(\frac{k+ra}{b}\right)+\frac{4rC}{3} - \frac{rC}{36}\right)t^2\right)dt$$

$$+\int_0^{12} rbHe^{-4t}\left(\left(\frac{16}{3}S_1+4\left(\frac{k+ra}{b}\right)+\frac{8rC}{3} - \frac{rC}{9}\right)t^3+\left(16\left(\frac{k+ra}{3b}\right)+\frac{32rC}{9} - \frac{2rC}{9}\right)t^4 - \frac{32rC}{9}t^5\right)dt$$

$$-\int_0^{12} bH\left(S_1\frac{64}{3}t^3e^{-4t}+C\frac{64}{27}t^4e^{-4t}+\left(\frac{64}{3}\right)^2Ht^6e^{-8t}\right)dt$$

$$= 72\left(\frac{k+ra}{2} +bC(r/3 - 1/18)\right)S_1+576b\left(\left(\frac{k+ra}{2b}\right)^2+\frac{rC(k+ra)}{3b}+C^2\left(\frac{r^2}{9} - \frac{1}{324}\right)- \frac{rCS_1}{36}\right)$$

$$-144rC\left(k+ra+\frac{2rCb}{3}\right)+\frac{192}{5}br^2C^2+rbH\left(\frac{S_1}{8}+\left(2S_1+\frac{k+ra}{2b}+rC/3\right)/16+\left(4S_1+2\left(\frac{k+ra}{b}\right)+\frac{47rC}{36}\right)/32\right)$$

$$+rbH\left(16S_1+12\left(\frac{k+ra}{b}\right)+\frac{23rC}{3}\right)/128+\left(16\left(\frac{k+ra}{b}\right)+10rC\right)/128- \frac{5rC}{48}\Big)- \frac{bHS_1}{2} - \frac{CbH}{18} - \frac{5bH^2}{32}$$

$$\int_0^{12} P(t)S'(t)dt = rbH\left(\frac{S_1}{2}+\frac{5}{16}\left(\frac{k+ra}{b}\right)+\frac{55}{576}rC\right)- \frac{bHS_1}{2} - \frac{bCH}{18} - \frac{5bH^2}{32} + \frac{32}{5}br^2C^2$$

$$- \frac{16bC^2}{9} + 48rC(k+ra) - 144\left(\frac{k+ra}{b}\right)^2+36S_1(k+ra)+bCS_1(8r-4).$$

Total consumer expenditure will be

$$X(12) = H(a+bS_1+\frac{2bC}{3})+H(\frac{k+ra}{2})+bCH(\frac{r}{3} - \frac{1}{18}) - \frac{5rbCH}{144} + \frac{rbH^2}{4}$$

$$- \frac{5bH^2}{16}+4C(k+ra)+8bC^2(\frac{r}{3} - \frac{1}{18}) - \frac{8rbC^2}{3} - \frac{31rbCH}{192} + \frac{7bCH}{36}$$

$$- rbH(\frac{S_1}{2} + \frac{5}{16}(\frac{k+ra}{b})+ \frac{55}{576}rC)+ \frac{bHS_1}{2} + \frac{bCH}{18} + \frac{5bH^2}{32} - \frac{32br^2C^2}{5}$$

$$+ \frac{16bC^2}{9} - 48rC(k+ra) + \frac{144}{b} (k+ra)^2 - 36S_1(k+ra) - bCS_1(8r-4).$$

$$X(12) = H(a+\frac{3bS_1}{2} + 2bC/3) + H(\frac{k+ra}{2}) + \frac{79}{576}rbCH + \frac{rbH^2}{4} - \frac{5bH^2}{32} + \frac{4bC^2}{3}$$

$$+ 4C(k+ra) + \frac{7bCH}{36} - rbH \frac{S_1}{2} - \frac{5}{16}rH(k+ra) - \frac{55}{576} r^2bCH$$

$$- \frac{32br^2C^2}{5} - 48rC(k+ra) - 36S_1(k+ra) + \frac{144}{b} (k+ra)^2 - bCS_1(8r-4).$$

$$X(12) = aH + bHS_1(\frac{3}{2} - r/2) - bCS_1(8r-4) + bHC(\frac{31}{36} + \frac{79r}{576} - \frac{55r^2}{576})$$

$$+ bH^2(r/4 - 5/32) + \frac{4bC^2}{3} + (k+ra) \left[H(\frac{1}{2} - \frac{5r}{16})+C(4-48r)-36S_1\right]$$

$$+ \frac{144}{b} (k+ra)^2 - \frac{32br^2C^2}{5} .$$

Basic Principles from the Calculus of Variations

The method of optimization applied to the profit equation in the analysis of the monopoly model differs from that in much of economic analysis in that it selects a time-varying path of optimal response rather than a single rate - it selects an optimal function rather than an optimal value of a variable. Because the theory underlying this process (the calculus of variations) is likely unfamiliar to many economists,[1] it is given some detail here.[2]

First Order Conditions Necessary for an Extremum: We start the discussion by introducing the concept of a functional. A functional is a rule of correspondence (i.e. function) which assigns a specific (real) number to each function or curve belonging to some class of functions or curves. This parent class of functions will usually be a normed linear space, which, heuristically speaking, is a set of elements and a function (its norm) with which measure the "distance" between the elements of the set. For example, the class $D_1(a,b)$ of continuous functions which have continuous first derivatives on the interval $[a,b]$, together with the norm $\|y\|_1 = \max_{a \leq x \leq b} |y(x)| + \max_{a \leq x \leq b} |y'(x)|$, is a normed linear space.

The concept of continuity of a functional is analogous to the concept with respect to functions, with the norm of the space upon which the functional is defined used as the measure of "distance" between arguments of the functional.

It is often useful to distinguish linear functionals from others. If $J(h)$ is a functional defined on a normed linear space R, it is said to be a continuous linear functional if

1. $J(ah) = aJ(h)$ for any h in R and any real number a;
2. $J(h_1 + h_2) = J(h_1) + J(h_2)$ for any h_1 and h_2 in R; and
3. $J(h)$ is continuous for all h in R.

Suppose that the increment of the functional, which is $\Delta J(h) = J(y+h) - J(y)$, can be written as the sum of a linear functional and an error: $\Delta J(h) = \delta J(h) + \varepsilon //h//$. If the limit of the error (ε) is zero as the norm of h goes to zero, the functional is said to be differentiable and the principal linear part of the increment, $\delta J(h)$, is called the first variation of the functional $J(h)$.

The functional $J(y)$ is said to have a (relative) extremum for the curve $y = \hat{y}(x)$ if $J(y) - J(\hat{y})$ does not change its sign for all curves y in a neighborhood[3] of $y = \hat{y}(x)$. This extremum is called a weak extremum if the norm used in defining the neighborhood is the norm for the class of first order continuously differentiable functions, and is called a strong extremum if the norm is that for the class of continuous functions.[4]

A necessary condition for the differentiable functional $J(y)$ to have an extremum for $y = \hat{y}$ is that its variation vanish for $y = \hat{y}$. That is, for all functions h belonging to the set $\{h: //\hat{y}-y// = //h// < \varepsilon\}$ defined by some fixed, but perhaps small $\varepsilon > 0$, $\delta J(h) = 0$.

The first application of this theorem gives rise to Euler's equation in the calculus of variations. Let $J(y)$ be a functional of the form $\int_a^b F(x,y,y')dx$ defined on the set of functions which have continuous first derivatives in $[a,b]$ and satisfy the boundary conditions[5] $y(a) = A$ and $y(b) = B$. Then a necessary condition for $J(y)$ to have an extremum for a function $y(x)$ is that $y(x)$ satisfy <u>Euler's equation</u> $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$.

<u>Second Order Conditions Sufficient for Minimization</u>: The first order condition for minimizing a functional requires that the first variation of the functional vanish for the extremal curve. This requirement is very similar to the first order condition for the minimization of a function, namely that the first derivative is zero at the extremum. This would lead one to expect that there may be similarities between the second order conditions pertaining to the calculus of variations and those used in real analysis. To illustrate these similarities, the second order conditions will be stated in two ways - the conditions in general, and the conditions which correspond to functionals of

the type analyzed in this paper. However, before the second order conditions can be given, our vocabulary must be extended and some background results stated.[6]

A functional J(y) is said to be <u>twice differentiable</u> if its increment can be written in the form

$$\Delta J(h) = \delta J(h) + \delta^2 J(h) + \varepsilon \|h\|^2$$

where $\delta J(h)$ is a linear functional (its first variation), $\delta^2 J(h)$ is a quadratic functional,[7] and the error ε has the limit of zero as the norm of h approaches zero. The quadratic functional $\delta^2 J(h)$ is called the <u>second variation</u> of the functional J(y).

A necessary condition for the functional J(y) to have a minimum for the extremal curve $y = \hat{y}(x)$ is that $\delta^2 J(y) \geq 0$, for $y = \hat{y}$ and all admissable h. However, this condition alone is not sufficient to assure a minimum, nor is the imposition of the requirement that $\delta^2 J(y) > 0$ sufficient.[8] In order to define a sufficient condition another concept is needed: A quadratic functional $\varphi_2(h)$ defined on a normed linear space is <u>strongly positive</u>[9] if there exists some positive constant k such that $\varphi_2(h) \geq k\|h\|^2$ for all h.

Thus, a sufficient condition for the functional J(y) to have a minimum at $y = \hat{y}$, given that $\delta J(y) = 0$ for $y = \hat{y}$, is that its second variation $\delta^2 J(y)$ be strongly positive for $y = \hat{y}$.

These necessary and sufficient conditions for the existence of a minimum for a functional describe bounds on the behavior of the functional which may not, necessarily, lead to recognisable bounds on the structure of the functional. In particular, it is not immediatly clear what restrictions on $F(x,y,y')$ are implied for our analysis of $J(y) = \int_a^b F(x,y,y')dx$.

Legendre's inequality corresponds to the necessary condition stated above: A necessary condition for the functional $J(y) = \int_a^b F(x,y,y')dx$, $y(a) = A$, $y(b) = B$ to have a minimum at $y = \hat{y}(x)$ is that $\frac{\partial^2 F}{\partial (y')^2}(x,y,y') \geq 0$ for every point of the curve $y = \hat{y}(x)$.

In order to give us the strong positivity condition used in the sufficient condition, we must have a condition which rules out, for example, the eastward path on the great circle as the shortest distance between New York and San Francisco.[10] This condition is the absence of conjugate points.

The second variation of the functional $J(y) = \int_a^b F(x,y,y')dx$ $y(a) = A$, $y(b) = B$ can be written as

$$\delta^2 J(h) = \int_a^b (P\,h'^2 + Q\,h^2)dx \text{ with } h(a) = h(b) = 0$$

where $P = \frac{1}{2}\frac{\partial^2 F}{\partial (y')^2}$ and $Q = \frac{1}{2}\left(\frac{\partial^2 F}{\partial y^2} - \frac{d}{dx}\frac{\partial^2 F}{\partial y'}\right)$.

Notice that the second variation is also a functional, which if positive definite, is nonnegative with a minimum value of zero and otherwise is negative and unbounded. Thus by minimizing this functional we can determine whether the second variation is positive definite. The concept of conjugate points is derived from this optimization process. The point a^c ($\neq a$) is said to be <u>conjugate to the point</u> a with respect to the functional J(y) if the solution, h*(x), to the equation[11/] $Qh - \frac{d}{dx} Ph' = 0$ vanishes for $x = a$(i.e. $h^*(a) = 0$) and for $x = a^c$, but h*(x) is not identically zero. The generalization of this definition to the case of more than one unknown function involves demonstrating the linear dependence of the set of disturbance functions at the point $x = a^c$.

The absence of conjugate points is predicted by <u>Jacobi's necessary condition</u>. If the extremal $y = \hat{y}(x)$ corresponds to a minimum of the functional $\int_a^b F(x,y,y')dx$ and if $\frac{\partial^2 F}{\partial (y')^2} > 0$ (or is positive definite in the multivariate case) along this extremal, then the open interval (a,b) contains no points conjugate to a.

This bring us to the point where we can now state a set of conditions sufficient to establish the minimum of the functional $\int_a^b F(x,y,y')dx$, $y(a) = A$ and $y(b) = B$. If some admissable curve $y = \hat{y}(x)$ simultaneously satisfies the following three conditions, a weak minimum[12] of the stated functional exists at $y=\hat{y}(x)$:

1. The curve $y=\hat{y}(x)$ is extremal, which means it satisfies the Euler equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0.$$

2. Along the curve $y = \hat{y}(x)$, the strengthened Legendre condition is satisfied:

$$P(x) = \frac{1}{2}\frac{\partial^2}{\partial (y')^2} F(x,y,y') > 0 \text{ (or is positive definite).}$$

3. And it fulfills the strengthened Jacobi condition, that the closed interval [a,b] contains no points conjugate to the point a.

The characteristic of these sufficient conditions which makes them sufficient is the fact that all three must hold simultaneously. If any one of the conditions breaks down, we can no longer be assured that a minimum has been attained.

In summary, this section of the appendix has presented the conditions sufficient to establish the minimum of a functional in two cases. First, the minimization of a general functional was discussed. Then, the conditions regarding the minimization of the particular type of functional used in this paper, namely where the functional is defined by the integral of a function whose arguments are an independent variable, a dependent variable which responds to the independent variable and the rate of change of that dependent variable.

A Brief Summary: The necessary and sufficient conditions for the minimization of a functional are quite similar to those for minimizing a function. The first variation of the functional must necessarily vanish for any extremal solution, as must the first derivative of a function. To assure that a minimum of the functional has been obtained, the second variation of the functional must be strongly positive for the extremal solution. This is an obvious parallel to the requirement that the second derivative of a function be positive at its minimum.

For functionals that can be represented as $\int_a^b F(x,y,y')dx$ these necessary and sufficient conditions are represented by the Euler equation, the strengthened Legendre condition, and the strengthened Jacobi condition, all of which have been referred to above. For a much more complete discussion of this topic, the reader is advised to consult the author's reference [4] or some other text on the subject.

Footnotes

1/ Pesek and Saving [11] present an application of the calculus of variations in a different setting, namely an analysis of money demand and saving behavior. R.G.D. Allen [1] devotes one chapter of his <u>Mathematical Analysis for Economists</u> book to a discussion of the calculus of variations. A section of a chapter and one topic in the Mathematical Review section of Lancaster's <u>Mathematical Economics</u> [7] deals with the topic.

2/ This development is patterned after Gelfand and Fomin [4, Chapter 1].

3/ An ε-neighborhood of $\hat{y}(x)$ is defined as $\{ y(x): //y-\hat{y}// < \varepsilon \}$ for some fixed positive real number ε. The norm which is used in this definition is the norm of the function space in which we are operating.

4/ The norm for the set of first order differentiable functions is defined as $//y//_1 = \max_x /y(x)/ + \max_x /y'(x)/$. The norm for the set of continuous functions is defined as $//y//_0 = \max_x /y(x)/$. In each case the maximization is occurring over a domain for which $y(x)$ and $J(y)$ are defined.

5/ Euler's equation is a second order differential equation whose solution contains two arbitrary constants. The two stated conditions, or any pair of independent conditions, are used to determine the specific parameters of the solution equation. In particular, we could just as well use $y(a) = A$ and $y'(a) = B$ as alternative conditions on $y(x)$.

6/ This presentation is patterned after Gelfand and Fomin [4, Chapter 5].

7/ A functional $B(x,y)$ of two elements of a normed linear space is said to be <u>bilinear</u> if it is a linear functional of y for any fixed x and a linear functional of x for any fixed y. If the two arguments of a bilinear functional are the same, the expression is called a <u>quadratic functional</u>, i.e. $B(x,x)$ is a quadratic functional if $B(x,y)$ is a bilinear functional.

Footnotes
Cont.

8/ If we can determine a sequence of functions $\{h_1, h_2, h_3 \ldots\ldots\}$ with $\lim_{n\to\infty} //h_n// = 0$, such that for every $k > 0$ we can find N_k such that $k > -\varepsilon(h_n) > \frac{\delta^2 J(h_n)}{//h_n//^2} > 0$ for all n greater than N_k, then $\Delta J(h) < 0$ for this sequence and hence the functional would not be minimized. This would be the situation if a saddle point occurred.

9/ This concept is to be compared with the concept of positive definiteness. A quadratic functional $A(x)$ is positive definite if $A(x) > 0$ for every nonzero element x. In a finite dimensional space, a strongly positive quadratic form is equivalent to a positive definite quadratic form, but in the general case strong positivity is a more stringent condition than positive definiteness.

10/ This example is due to Lancaster [7, p. 382].

11/ This equation is called the Jacobi equation of the original functional $J(y)$. It is the Euler equation derived in the minimization of $\delta^2 J(h)$.

12/ The derived curve $y = \hat{y}(x)$ is that curve in the class of first order continuously differentiable curves which minimizes the functional. A strong extremum would occur if the derived curve is that element of the set of continuous curves which minimizes the functional.