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A DYNAMIC NONLINEAR PLANNING MODEL FOR KOREA

by

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1. Introduction[1]

During recent years economists have formulated and solved a number of dynamic multisectoral planning models in the form of linear programs, viz., Bruno [5], Eckaus and Parikh [12], and Chakravarty and Lefeber [9]. While most of these studies have used linear production and welfare relations, some have employed piecewise linear segments to make linear approximations to nonlinear functions, e.g., Adelman and Sparrow [1], Barr and Manne [3], and Carter [7].

This paper discusses a set of numerical experiments using a dynamic multisectoral planning model with <u>nonlinear</u> welfare and production relations. In finding optimal solutions to this planning problem, we employed computational techniques developed in recent years by control theorists. Since we are reporting elsewhere [20] on our numerical methods,[2] we will concentrate in this paper on the formulation and interpretation of the solutions to a four sector model of the Korean economy.

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[1] This research has been financed in part by the Agency for International Development and in part by the Harvard Institute for Economic Research under a grant from the National Science Foundation. We are indebted to Rod Dobell, Hollis Chenery, Louis Lefeber, Thomas Vietorisz, J. A. Mirrlees, and Arthur Bryson for comments and suggestions and to Andy Szasz for programming assistance.

[2] See also Bryson and Ho [6], who give a complete survey of solution methods for optimal control problems which have been proposed to date. Other techniques for solving planning models with various types of nonlinearities have been employed by Chenery and Uzawa [10], Frisch [15], Chakravarty [8], Johansen and Lindholt [18], Mirrlees [23], Stoleru [30], and Radner and Friedman [25], [26], [27].

In Sections 2 and 3 we develop the four sector model and provide numerical values for the parameters. In Section 4 we discuss briefly the solution method employed. Section 5 is devoted to an analysis of the results obtained from a number of numerical experiments with the model. Finally, Section 6 includes a discussion of some of the advantages and disadvantages incumbent upon the use of control theory models for development planning.

2. The Model

The basic structure of the model is to maximize a welfare function over a thirty-year period subject to constraints in the form of distribution relations, production functions, absorptive capacity functions, foreign exchange constraints and initial and terminal capital stock and foreign debt constraints. The four sectors are (1) agriculture and mining, (2) heavy industry, (3) light industry, and (4) services.

In a number of linear programming models (e.g., Bruno [5]) the welfare function has been specified in something like the following form:

$$\xi = \sum_{i=1}^{N} (1+z)^{-i} \sum_{j=1}^{4} c_{ji} \tag{2.1}$$

where

z = discount rate

i = time period index

j = sector index

c = consumption

That is, the discounted sum over time of each year's total consumption is the maximand. To make the problem well-posed, each c_{ji} has usually been constrained by a linearized income elasticity formula to bear a certain relationship to $\sum_j c_{ji}$, i.e., total consumption in period i.

We have adopted a superficially different but actually rather similar welfare function,

$$(2.2) \qquad \xi = \sum_{i=1}^{N} (1+z)^{-i} \sum_{j=1}^{4} a_j c_{ji}^{b_j} \qquad 0 \leq b_j \leq 1, \quad a_j \geq 0$$

where the a's and b's are parameters whose interpretation is given in the next section. In the same place, numerical values are derived in a consistent with currently observed income elasticities and consumption shares.

Again in keeping with the linear programming tradition, we built absorptive capacity constraints into the model by assuming capital (or capacity) accumulation equations of the form

$$(2.3) \qquad k_{j,i+1} = k_{ji} + g_j(\delta_{ji}, k_{ji}) \qquad \text{all } i, j$$

where k_{ji} is the capital stock in sector j at time i, δ_{ji} is an "activity" representing total resources devoted to investment in that sector at that time, and $g_j(\delta_{ji}, k_{ji})$ is a function imposing decreasing returns to investment in creating new capacity. To avoid the possibility of unrealistic decumulations in capital stock we impose the constraints

(2.4) $\delta_{ji} \geq 0$, all i, j

The functions $g_j(\delta_{ji}, k_{ji})$, which appear in (2.3) are of the form[1/]

$$g_j(\delta_{ji}, k_{ji}) = \Delta k_{ji} = \mu_j k_{ji} \left\{ 1 - \left[1 + \frac{\varepsilon_j}{\mu_j} \frac{\delta_{ji}}{k_{ji}} \right]^{-\frac{1}{\varepsilon_j}} \right\} \qquad (2.5)$$

$\varepsilon_j \geq -1$

$\mu_j \geq 0$

The assumption behind this specification is simply that as the increase in capacity (Δk) approaches some fraction μ of existing capacity k, then investment (δ) becomes less and less effective in increasing Δk. Thus,

$\frac{d(\Delta k)}{d\delta}$ = a decreasing function of δ

or in a convenient functional form:

$$\frac{d(\Delta k)}{d\delta} = \left(1 - \frac{\Delta k/k}{\mu}\right)^{1+\varepsilon}. \qquad 0 < \mu, \quad -1 \leq \varepsilon \qquad (2.6)$$

[1/]This function was suggested by Robert Dorfman for a different model. We have adopted it for use here. Both Sam Bowles and Louis Lefeber have suggested to us that an absorptive capacity relationship should include educated or highly skilled labor as one of the inputs. While we are in agreement, we have not implemented that suggestion in the present model.

The parameter ε has been introduced to indicate how rapidly the decrease in investment efficiency takes place. The differential equation (2.6) can be solved to give Δk in terms of δ and k:

$$\text{(2.7)} \qquad \frac{\Delta k}{k} = \mu\left\{1 - \left[1 + \frac{\varepsilon}{\mu}\frac{\delta}{k}\right]^{-\frac{1}{\varepsilon}}\right\} .$$

Note that $\varepsilon = -1$ means that $\Delta k = \delta$, so that a linear relationship holds between change in capacity and investment (although there is still an implied upper bound of μ on $\Delta k/k$). For $-1 < \varepsilon < 0$, $\Delta k/k$ is a concave and monotonically increasing function of δ/k until $\Delta k/k = \mu$ at which point the function in (2.4) becomes complex-valued.

For $\varepsilon = 0$, we have $\frac{\Delta k}{k} = \mu[1-e^{\delta/\mu k}]$,

which increases asymptotically to μ. In general when $\varepsilon \geq 0$, this sort of behavior occurs, so there are both diminishing returns and an absolute upper bound μ on $\Delta k/k$ which is approached when $\delta/k \to \infty$.

The distribution relations are of the standard input-output type:

$$\text{(2.8)} \qquad q + Dq + m = Aq + B\delta + e + c$$

q = vector of output levels

δ = vector of investment levels

m = vector of untied imports

e = vector of exports

c = vector of consumption levels

D = diagonal matrix of marginal propensities to import for production

A = input-output matrix

B = capital coefficient matrix[1/]

In line with recent empirical work, we assume that the production functions are CES (constant elasticity of substitution):[2/]

$$(2.9) \qquad q_{ji} = \tau_j (1+\nu_j)^i \left[\beta_j k_{ji}^{-\rho_j} + (1-\beta_j) \ell_{ji}^{-\rho_j} \right]^{-\frac{1}{\rho_j}}$$

where

q = output

τ = efficiency parameter

ν = rate of technical progress

β = distribution parameter

k = capital input

ℓ = labor input

$\rho_j = \frac{1}{\sigma_j} - 1$ where σ_j is the elasticity of substitution for the j^{th} sector.

The labor constraint implied by a neoclassical assumption of full employment is

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[1/] We assume that the top and bottom rows of B consist of zero elements, i.e., that the agriculture and mining sector and the services sector provide negligible amounts of inputs to capital formation. Actually this assumption is also an empirical result. We aggregated an 18-sector Korean input-output "B" matrix to get our matrix. Only 5 out of the 18 sectors actually produce capital goods, and these were all aggregated into our "Heavy" and "Light" industry sectors.

[2/] We use the constant returns to scale form. Diminishing returns specification would add no essential complica-

(2.10) $$\sum_{j=1}^{4} \ell_{ji} = \ell_i$$

where ℓ_i is the exogenously given total labor force in period i.

In the interest of minimizing the number of control variables in the model, we specified sectoral exports exogenously,[1/]

(2.11) e_{ji} given, all i, j .

Also in the interests of simplicity we allowed no untied imports into sectors one (primary production) and four (services):

(2.12) $m_{1j} = m_{4j} = 0,$ all i .

Using the given export paths and all the different kinds of imports, we may write a foreign debt "accumulation equation" of the form

(2.13) $$\gamma_{i+1} = (1+\theta)\gamma_i + \sum_{j=1}^{4} (d_{jj}q_{ji} - e_{ij} + \pi_{jji} + m_{ji})$$

where

γ_i = foreign debt

θ = interest rate on foreign debt

---

tion, but the increasing returns specification would make the problem nonconvex.

[1/]See the next section for details on how we made export projections. We could in principle make exports endogenous to the model, but to have done this, we would have required estimates of price elasticities to use in convex functions relating foreign exchange earnings to volume exports.

d_{jj} = elements of D, i.e., marginal propensities to import for production.

π_j = marginal differential propensity to import for capital formation.

We know initial foreign debt and can constrain terminal debt to be at a given level,

(2.14) γ_1 known; $\bar{\gamma}_{N+1}$ chosen,

but we have no explicit constraints on the level of debt at any intermediate time period.[1]

The system has 5 state variables (4 k_{ji}'s and γ_i) and 14 control variables (4 δ's, 4 c's, 2 m's, and 4 ℓ's). However, these are effectively only nine controls because constraints

(2.8) $q = Dq + m - Aq + B\delta + e + c$

and

(2.10) $\sum_{j=1}^{4} \ell_{ji} = \ell_i$

can be used to eliminate the four c's and one ℓ_j. To carry this out, let

(2.15) $P = I - A + D$

and let P^j and B^j denote the j^{th} row of P and B respectively.

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[1]We are thus solving an "isoperimetric" problem with respect to foreign debt, specifying a given change in debt (from γ_1 to γ_{N+1}) and letting the model optimally allocate this change over time. (An analogous problem in the classical calculus of variations is finding the maximum area which can be enclosed by a given length of rope.) Possible alternative

(2.16) $$c = Pq - B\delta - e + m$$

and from (2.10)

(2.17) $$\ell_{1i} = \ell_i - \ell_{2i} - \ell_{3i} \; \ell_{4i}.$$

Thus in summary the problem is

(2.2) $$\max \xi = \sum_{i=1}^{N} (1+z)^{-i} \left[\sum_{j=1}^{4} a_j c_{ji} \right]^{b_j}$$

subject to

(2.3) $$k_{j,i+1} = k_{ji} + g_j(\delta_{ji}, k_{ji}) \qquad \text{all } i,j$$

(2.13) $$\gamma_{i+1} = (1+\theta)\gamma_i + \sum_{j=1}^{4} (d_{jj} q_{ji} - e_{ji} + \pi_j \delta_{ji} + m_{ji}) \quad \text{all } i$$

(2.18) $$k_{j1} = \bar{k}_{j1} \qquad \text{all } j$$

(2.19) $$\gamma_1 = \bar{\gamma}_1$$

(2.20) $$k_{j,N+1} = \bar{k}_{j,N+1} \qquad \text{all } j$$

(2.21) $$\gamma_{N+1} = \bar{\gamma}_{N+1}$$

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treatments of foreign debt are (a) using penalty functions to hold debt at any time "close" to some predetermined level; (b) putting inequality constraints on the level of debt in each period. The former alternative is computationally feasible, although our debt paths seemed well enough behaved for us not to bother with it. The latter approach, involving state variable inequality constraints, is difficult to handle computationally.

with

$$(2.22) \qquad c_{ji} = P^j q_i - B^j \delta_i - e_{ji} + m_{ji}$$

$$(2.17) \qquad \ell_{1i} = \ell_i - \ell_{2i} - \ell_{3i} - \ell_{4i}$$

$$(2.9) \qquad q_{ji} = \tau_j (1+\nu_j)^i \left[\beta_j k_{ji}^{-\rho_j} + (1-\beta_j) \ell_{ji}^{-\rho_j} \right]^{-\frac{1}{\rho_j}}$$

$$(2.5) \qquad g_{ji} = \mu_j k_{ji} \left\{ 1 - \left[\frac{\varepsilon_j \delta_{ji}}{\mu_j k_{ji}} \right]^{-\frac{1}{\varepsilon_j}} \right\} \qquad \varepsilon_j \geq 1, \quad \mu_j \geq 0$$

3. Data and Sources[1/]

Perusal of the equations of the last section shows that the four sector model requires a considerable amount of data: numerical values for 70 or 80 parameters in addition to yearly export levels for the four sectors. In this section we describe our sources of data, and list the parameters actually used.

Input-Output Coefficients

The basic sources here are "A" and "B" matrices aggregated to 18 sectors by Larry Westphal for use in his integer programming study of scale economies in Korean manufacturing.[2/]

[1/] We are most grateful to David Cole and Larry Westphal for providing us with most of our data, as well as with a number of helpful suggestions on the formulation of the model. Since our primary concern in this research was to determine the feasibility of solving multisectoral nonlinear planning models and to learn a little about the characteristics of such models, we invite the reader's tolerance when we seem a bit cavalier with the data.

[2/] Westphal's sources are 43 sector matrices put together

We aggregated these 18 sectors to four sectors, using Westphal's data on 1965-66 flows and sectoral capacity levels. The aggregation scheme is given in Table 1, while Tables 2 and 3 give the a_{ij} and b_{ij} coefficients respectively. (The latter two tables also list "non-competitive" import requirements per unit of sectoral output and investment.)

Table 1

AGGREGATION TO FOUR SECTORS

Sector 1 - "Primary Production"

Agriculture, Forestry, Fishing, Coal and Other Minerals

Sector 2 - "Heavy Industry"

Fiber spinning, Lumber and plywood, Paper products, Rubber products, Chemicals, Chemical fertilizers, Petroleum products, Cement, Other Ceramic, Clay, Stone, Glass, Iron, Steel (through Ingot), Steel products, Finished metal products, Non-ferrous metals, Machinery, Transportation equipment, Building maintenance, Construction, Electricity, Water, Commercial, Transportation, Storage, Scrap.

Sector 3 - "Light Industry"

Processed food, Beverages, Textiles, Printing, Publishing, Leather, Wood products, Miscellaneous manufactures.

Sector 4 - "Services"

Banking, Insurance, Real estate, Trade margins, Other services.

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by him and based on the work of Marshall Wood and the Bank of Korea for the year 1965-66. For a discussion, see Westphal [31].

Table 2

INPUT-OUTPUT COEFFICIENTS

Sector	1	2	3	4
1	.100	.090	.170	.010
2	.090	.330	.240	.120
3	.040	.020	.120	.050
4	.030	.090	.090	.080
Imports	.0008	.090	.030	.004

Table 3

CAPITAL COEFFICIENTS

Sector	1	2	3	3
2	.6908	1.3109	.1769	.1500
3	.001	.0199	.0022	.0000
Imports	.63	.98	.10	.10

Production Functions

The parameters of the CES production functions are given in Table 4. We have assumed a relatively high elasticity of substitution in the agriculture-mining sector and a relatively low elasticity in the services sector. Also, we have assumed lower rates of technical change in services and light manufacturing than in the other sectors. The efficiency parameters were computed from the production functions by using the base year (1965) labor force, capital stock, and output (See Table 5) along with the assumed values of the other parameters.

Table 4

PRODUCTION FUNCTION PARAMETERS

Sector	Elasticity of Substitution σ	Distribution β	Technical Progress ν	Efficiency τ	Initial Labor Force (million workers)
1	1.20	.35	.03	.41	5.10
2	.90	.30	.035	1.26	.84
3	.90	.25	.025	1.89	.36
4	.60	.20	.025	.47	2.30

Table 5

CAPACITY AND OUTPUT LEVELS (1965)

Sector	Capacity	Gross Output
1	2.02	1.53
2	2.13	1.38
3	1.26	.91
4	1.27	.94

Data is in billions of U.S. dollars.

Export Projections

The export projections shown in Table 6 were made as follows. A base year GNP (1965) of $3.4 billion was used along with assumed growth rates of GNP of 8 percent in the first ten years and 7 percent thereafter. Next total exports were projected by assuming, (1) that they would increase linearly from 8.5 percent of GNP in the base year to 15 percent of GNP in the tenth year, (2) that this percentage would hold constant at 15 percent over the next ten years, and (3) that the percentage would decline linearily from 15 percent to 13 percent over the next ten years. Finally the export path for each sector was computed from the total export projection by assuming the footnoted percentages[1/] at years zero, ten, twenty, and thirty and linear changes of percentages over each ten year interval.

[1/] PERCENTAGE OF TOTAL EXPORTS IN EACH SECTOR

	Sector			
Year	1	2	3	4
0	.20	.10	.30	.40
10	.10	.15	.45	.30
20	.05	.25	.45	.25
30	.05	.40	.35	.20

Table 6

EXPORT LEVELS BY SECTOR

	Sector			
Year	1	2	3	4
1	.06	.04	.11	.13
5	.09	.07	.22	.21
10	.11	.14	.42	.30
15	.12	.31	.69	.42
20	.12	.54	.97	.54
25	.15	.97	1.19	.67
30	.20	1.62	1.42	.81

Data is in billions of U.S. dollars.

Absorptive Capacity Constraints

As indicated in the last section we use a nonlinear function of the form:

$$(3.1) \qquad \frac{\Delta k}{k} = \mu \left\{ 1 - \left[1 + \frac{\varepsilon}{\mu} \frac{\delta}{k} \right]^{-\frac{1}{\varepsilon}} \right\} \qquad \begin{matrix} -1 \leq \varepsilon \\ 0 < \mu \end{matrix}$$

to impose decreasing returns in amounts of capacity (Δk) created by investment expenditures (δ). Recall that μ is an upper bound on the percentage rate of change of the sectoral capital stock.

For our initial exercise, we set ε equal to 0.5, and imposed upper bounds on absorptive capacity in the sectors as follows:

Sector	Capacity Constraint μ
1	.275
2	.35
3	.30
4	.35

Thus, we have assumed that capacity expansion is relatively difficult in agriculture, less difficult in light industry, and relatively easy in heavy industry and services. The actual relationships between δ/k (the sectoral investment rate) and $\Delta k/k$ (increase in capacity) for our parameter choices are shown in Figure 1.

Welfare Function

Again from the last section, our welfare function is,

$$(3.2) \qquad \sum_{i=1}^{N} \frac{1}{(1+z)^i} \sum_{j=1}^{4} a_j c_{ji}^{\,b_j} \qquad 0 \le b_j < 1, \quad a_j > 0$$

where the c_{ji} are consumptions of product j at time i.

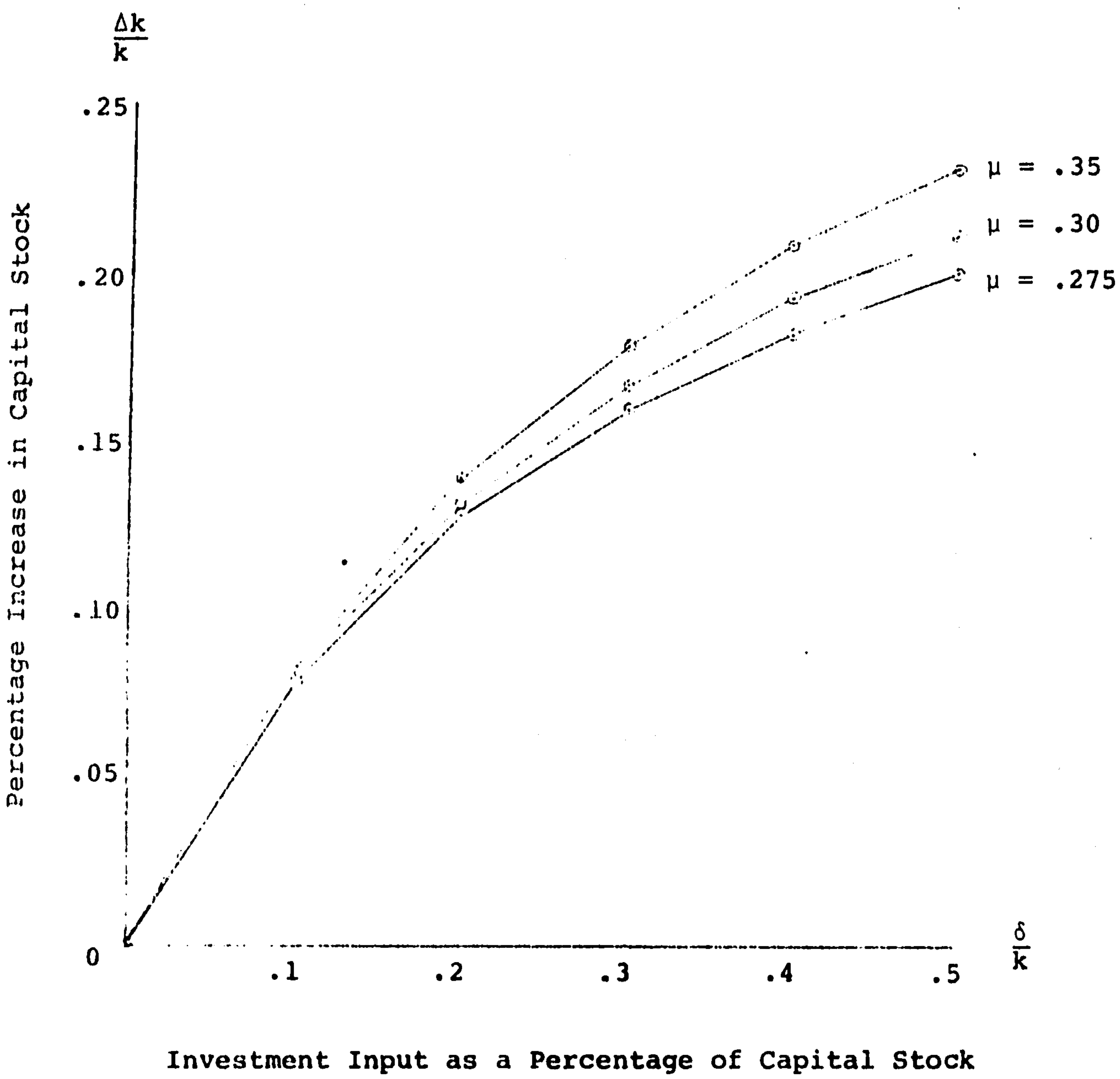
Connoisseurs of consumption theory will recognize that the inner sum is just Houthakker's "direct addilog" utility function [17], which implies a demand function of the form

$$(3.3) \qquad c_j = \left(\frac{a_j b_j y}{p_j \pi}\right)^{\frac{1}{b_j - 1}}$$

where p_j is the price of c_j, y is total consumption expenditure, and the equation-interaction parameter π is determined as the unique solution of the equation

Figure 1

ABSORPTIVE CAPACITY FUNCTION



$$(3.4)\qquad \sum_j \left[\frac{a_j b_j}{\pi}\right]^{\frac{1}{b_j-1}} \left[\frac{y}{p_j}\right]^{\frac{b_j}{b_j-1}} = 1.$$

If we set the p_j nominally to one (as we can if we take our sector outputs as initially given in value terms) the income elasticities from the demand fractions are

$$(3.5)\qquad E_{j/y} = \frac{1}{1-b_j}\left[1 - \frac{\sum \frac{b_j}{1-b_j} c_j}{\sum \frac{1}{1-b_j} c_j}\right].$$

By appropriate choice of our a_j and b_j, then, we can specify initial demand levels and income elasticities. The model will choose optimal consumption levels over time in line with its own generated prices, but still roughly in line with the income elasticities in (3.5).

For our exercise, we chose parameters as in Table 7. Table 8 gives the implied consumption shares and income elasticities assuming no price changes.

Table 7

WELFARE FUNCTION PARAMETERS

Sector	a	b
1	.48	.85
2	.33	.90
3	.345	.91
4	.3925	.87

Miscellaneous Parameters

Additional parameters for the model include the following: Labor force growth rate, r, was set to 2.5 percent per year. The consumption discount rate, z, was chosen to be 3 percent per year. The rate of interest on foreign debt (0) was set to 5 percent.

We chose targets for the four terminal capital stocks by specifying growth rates over the period. The results are:

Sector	Final Stock	Growth Rate
1	14.2	6.5 (%)
2	20.0	7.5
3	10.2	7.0
4	10.3	7.0

Terminal debt was set at $8.0 billion. This is an amount such that the interest and amortization on the debt should be about equal to 20 percent of the export earnings of the country in the terminal year.

Table 8

CONSUMPTION SHARES AND INCOME ELASTICITIES
IMPLIED BY TABLE 7 PARAMETERS

Income	Shares				Elasticities			
50	.482	.099	.219	.200	.816	1.223	1.359	.941
100	.420	.114	.276	.190	.787	1.181	1.312	.908
150	.384	.122	.312	.182	.771	1.157	1.285	.890
200	.359	.127	.338	.176	.760	1.140	1.267	.877
250	.340	.131	.358	.171	.752	1.128	1.253	.867
300	.325	.134	.374	.167	.745	1.118	1.242	.860
350	.312	.136	.388	.163	.740	1.109	1.233	.853
400	.301	.138	.400	.160	.735	1.102	1.225	.848
450	.292	.140	.411	.157	.731	1.097	1.218	.844
500	.284	.141	.420	.155	.728	1.091	1.213	.839

4. Method of Solution[1]

The optimization problem implied by the model is to find the consumption paths[2] for each of the four sectors which maximize the welfare index (2.2) subject to the capital accumulation constraints (2.3), the foreign debt accumulation constraints (2.13), the initial and terminal capital stocks (2.18) and (2.20), and the initial and terminal foreign debt constraints (2.19) and (2.21).

Our method of solution is discussed in detail in [20]. In general terms the technique is as follows. First, we substitute out the consumption variables, one of the labor force variables, the output variables, and the capacity creation variables using (2.22), (2.17), (2.9), and (2.5). This leaves us with a problem of maximizing a function of the five state variables (the four capital stocks and foreign debt) and the nine control variables (three labor force variables, four investment variables, and two import variables) subject to the difference equations (2.3) and (2.13) and the initial and terminal boundary conditions.

Next, additional terms are added to the performance function (2.2) to penalize any separation between the actual

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[1] We are indebted to Raman Mehra for providing us a copy of his conjugate gradient program and to Robert Kierr for excellent programming assistance in modifying the program to meet our requirements.

[2] A "path" consists of the values of a particular variable over the time period of the model.

and the target terminal condition. These penalty functions are quadratic functions and thus give symmetric penalties.

To begin the actual process of finding a solution, arbitrary initial paths for each of the control variables are calculated. These values of the control variables and the initial values of the state variables are used to obtain the paths for the state variables by integrating the difference equations (2.9) and (2.14) forward. The first order conditions for a constrained optimum are then used to see if the arbitrary initial (or nominal) path is optimal. If not, this information is used in making changes in the control variable paths and the process is then repeated.

Also, each time the state variables are integrated forward the terminal values of these variables are checked against the target values, the penalties are computed, and subtracted from the performance index. Thus to achieve an optimum the targets must be met.[1/]

5. Some Numerical Solutions

A limited number of solutions of the model were obtained in order to explore some of its more interesting properties--(1) the turnpike properties (or lack of them), (2) the effects on sectoral labor allocations of changes

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[1/] The interested reader can see the application of these techniques to a simple one sector model in [19].

in the production elasticities of substitution, and (3) the variations in the upper bound parameters in the absorptive capacity functions.

We begin by discussing some of the characteristics of a "base" solution and then turn to an analysis of the properties mentioned above. Since our primary interest was in the computational problem itself rather than in this particular model, the solutions reported are indicative of the kinds of results which one can obtain, rather than being an exhaustive analysis of the properties of the model.

5.1 A Basic Solution

One of the more interesting problems that one would like to attack with a formal planning model is that of the sectoral allocation of investment over time. The optimal phasing of these investment aggregates is bound to have an important influence on decisions made at the project level, in addition to influencing the growth of the other final demand aggregates. Clearly, optimal investment phasing will depend on a number of interdependent factors. These include, among others, (i) the initial sectoral capital stocks; (ii) import and export possibilities; (iii) the relative weights given different consumption bundles in the welfare function; and (iv) the terminal capital stock targets. One of the main purposes of a general equilibrium model such as ours is to take account of many of the interrelationships of these factors and to indicate to the planner which of these are critical.

Figure 2 shows the investment activity levels for the basic solution of the model.[1/] The solution is characterized by relatively high levels of investment in the early years in the service and agriculture-mining sectors, which decline between years 5 and 10 as investment in heavy industry swings up from an initial lower level. Given our "guesstimates" of the sectoral production functions, these results suggest a relative oversupply of capital in the heavy industry sector at time zero, although one should not make too much of this supposition.

In Figure 3, the consumption paths for the basic solution are shown. These reflect the investment pattern, particularly in the heavy industry sector, where consumption falls off as that sector's investment increases between years 5 and 10.[2/] As it turns out, the consumption paths in

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[1/]The rather uneven character of the paths in the figure would fade into smoothness if we decreased the plot interval from five to one or two years.

[2/]A consumption decline, even though it is confined to one of four sectors, may not be a desirable "optimal" policy, although there is no reason why a decline should be unlikely when the welfare function depends only on <u>levels</u> of consumption. Inclusion of rates of change terms in the welfare function--to reward consumption increases and penalize decreases--would make decreases less likely to occur.

Figure 2

BASE SOLUTION
INVESTMENT INPUT BY SECTOR

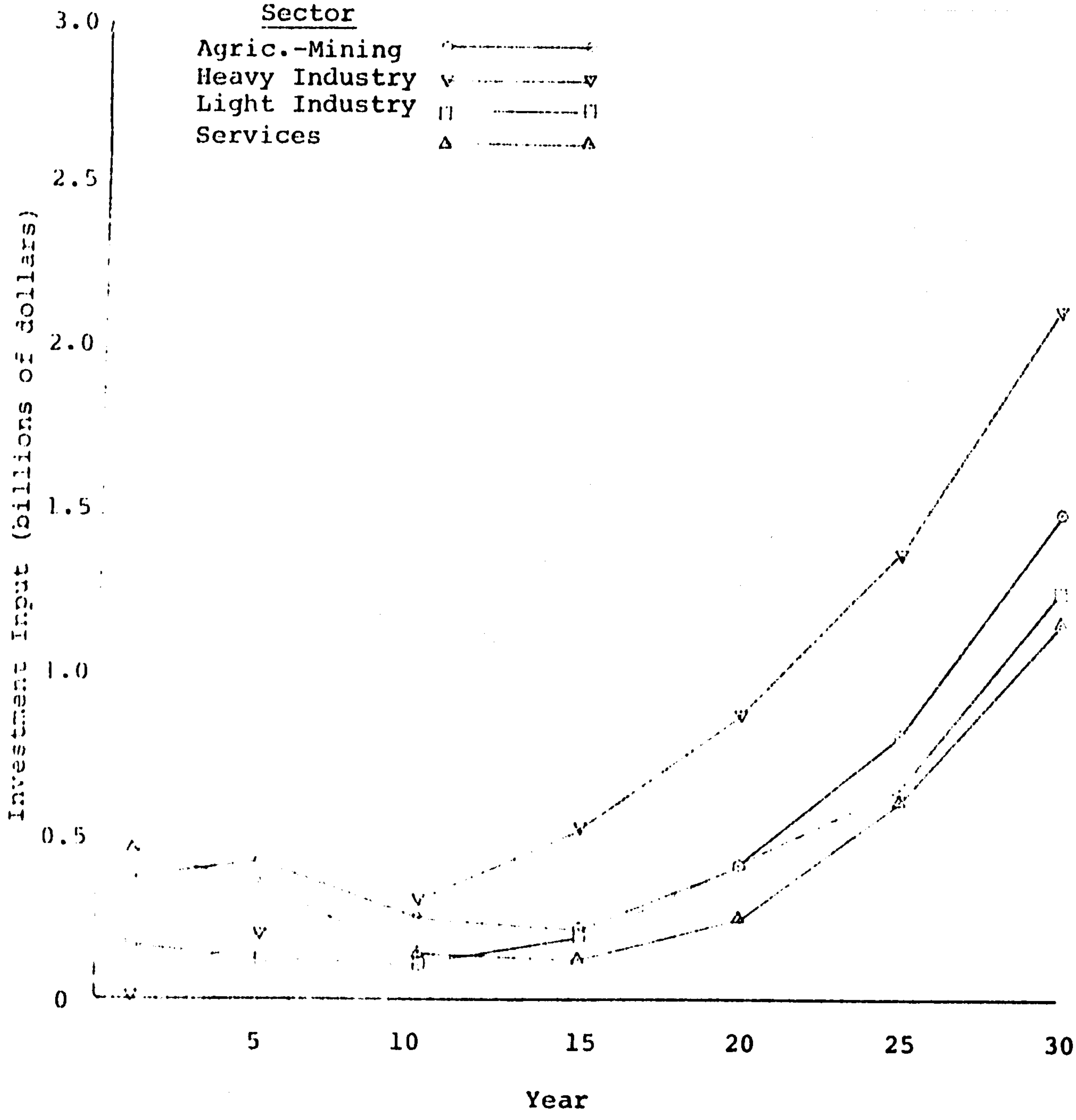
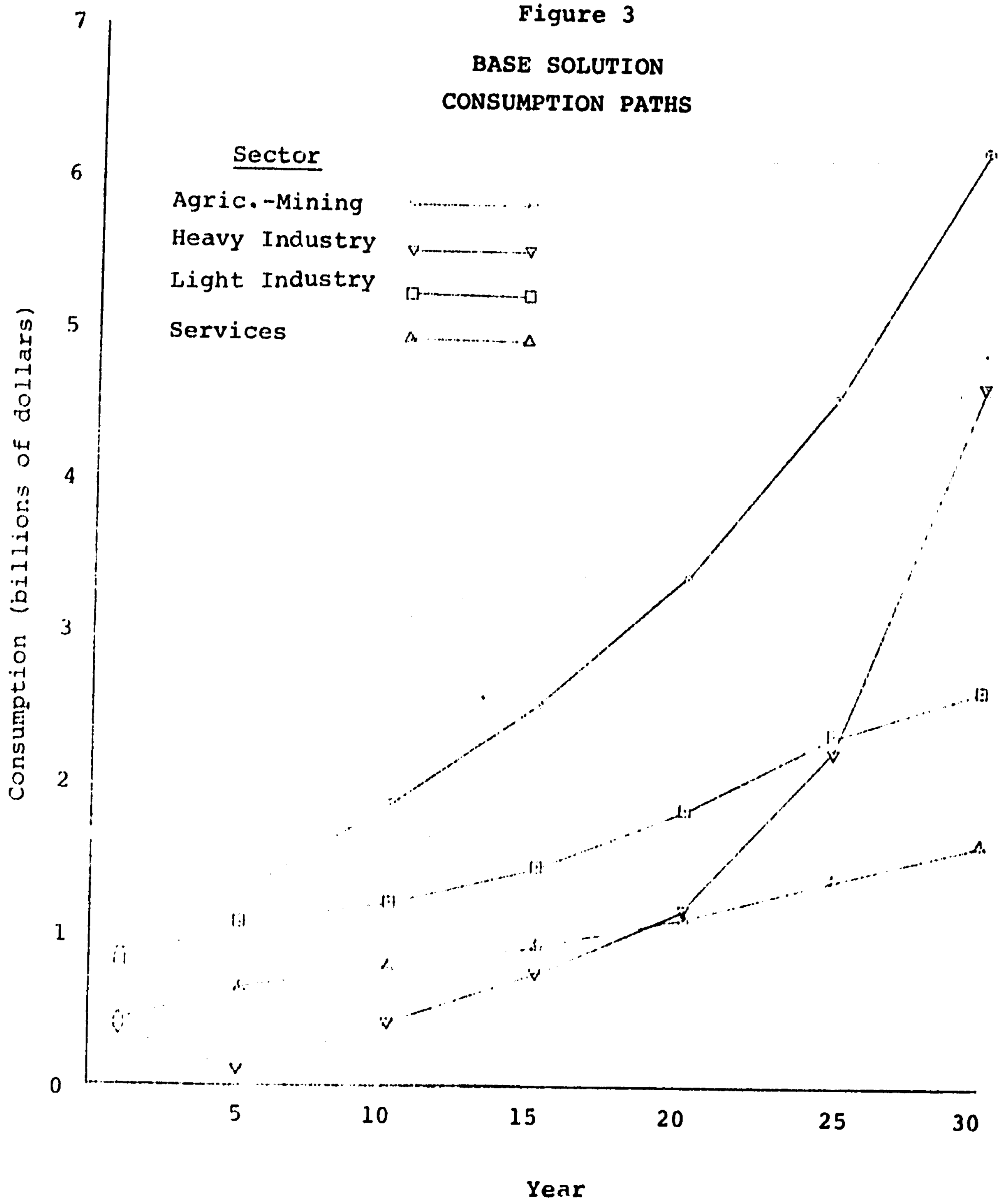


Figure 3

BASE SOLUTION
CONSUMPTION PATHS



sectors one and three are quite sensitive to the terminal capital stock targets. In year 30, consumption of goods from sector one decreases by a factor of over 25% when all the over-the-period capital stock growth rates are reduced by 1.5% (see section 5.2 for details), while terminal consumption from capital-producing sector three increases by about 20%. The other two sectoral consumption paths are little affected by these changes in terminal capital stock targets, except that in the early years of the planning period, sector two consumption in the low-terminal-target solution benefits from an inflow of untied imports which is drastically reduced in the basic solution.

Figure 4 shows two measures of total saving and investment in the basic solution. The dashed line gives the domestic investment rate (total final product from domestic sources devoted to investment as a share of GNP), while the solid line shows the standard GNP savings rate, taking account of foreign trade flows. The savings rate begins at a rather low level and then increases steadily up to year twenty as an initial rise in foreign debt (see the top line of Figure 7) levels off. The initial debt increase finances the high domestic investment rate at the beginning of the planning period, as does another debt increase (or burst of dissaving in GNP terms) toward the end of the 30-year plan.

Finally, Figure 5 shows the labor inputs by sector. As might be expected, the labor input to heavy industry grows more rapidly than that for other sectors. Somewhat more

Figure 4

BASE SOLUTION

SAVINGS RATES

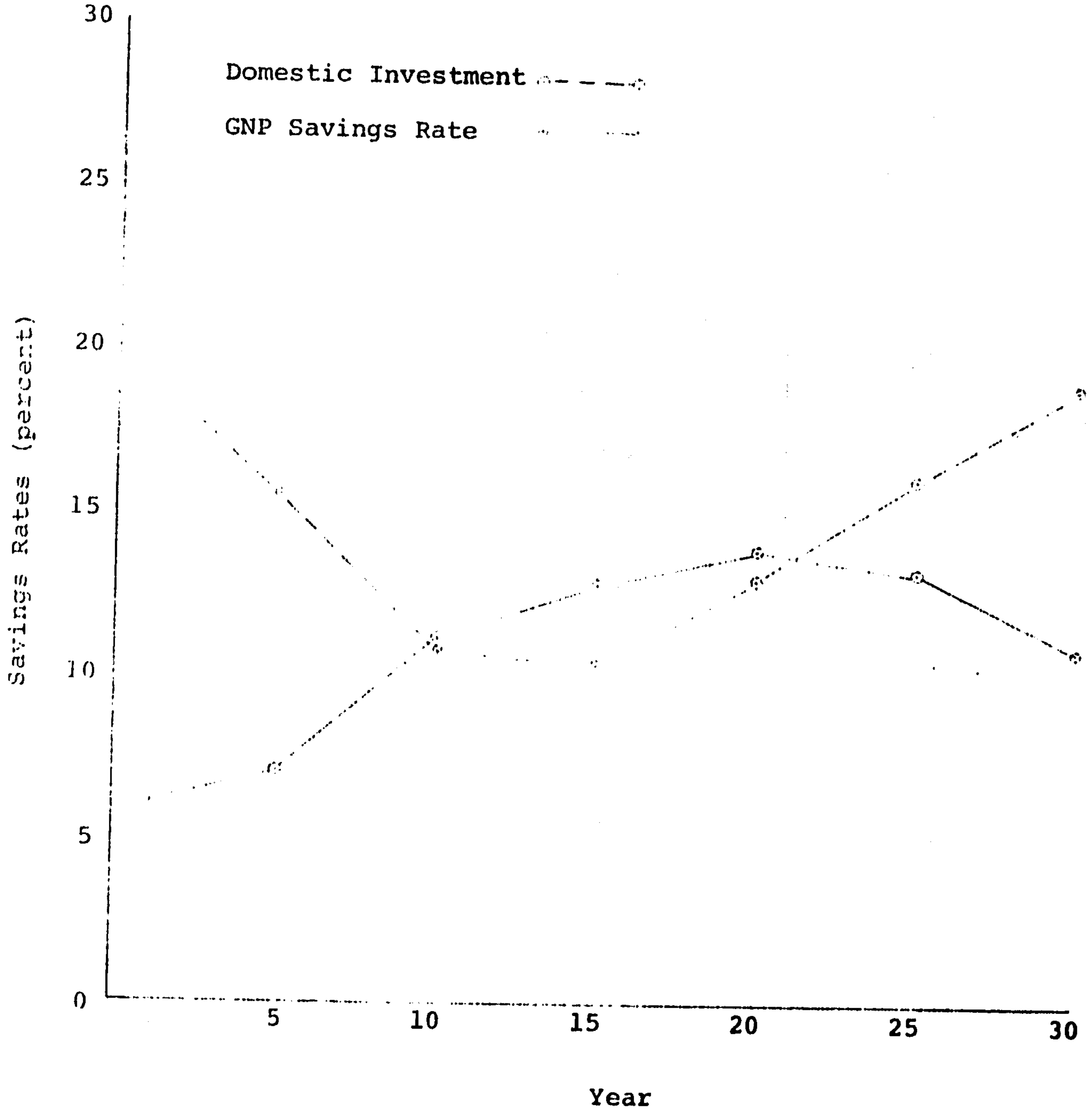
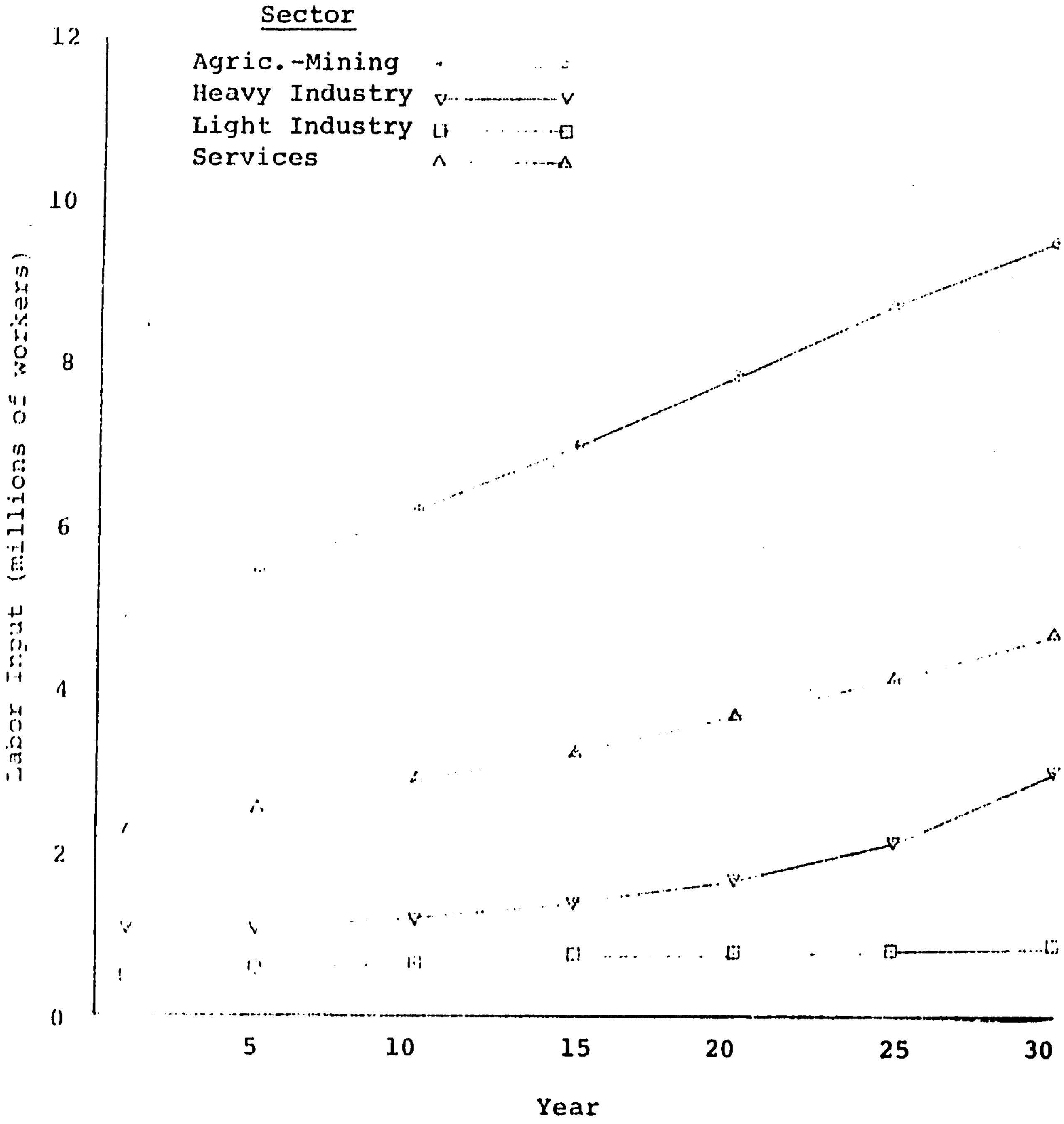


Figure 5

BASE SOLUTION

LABOR INPUT BY SECTOR



surprising are the continued absolute increases in the primary labor force, and the slow growth of labor force in the light industry sector. Unlike the consumption paths in these two sectors, the labor force allocations are not significantly affected by a decrease in the terminal capital stock targets.

5.2 Turnpike Properties

It is well known from economic theory that certain types of closed economy neoclassical models exhibit turnpike behavior in the sense that for most of a sufficiently long planning period, such a model will be in a balanced growth state with resource allocations approximately equal to those prevailing asymptotically in an infinite horizon plan [28]. As a corollary to this theorem, one might expect that the initial stages of a sufficiently long plan would be quite insensitive to terminal conditions. In some previously reported experiments with one-sector closed economy models [19], we found this type of behavior--in a model with a fifty-year planning horizon, the first twenty years of the plan were essentially unaffected by a wide range of terminal conditions.

The more complex model of this paper also displays the hypothesized getting-to-the-turnpike properties, but to a more limited extent. Figures 6 and 7 illustrate.

Figure 6a shows capital stock accumulation paths for sector one. We see that for the two lower terminal stocks (which were calculated using thirty-year capital

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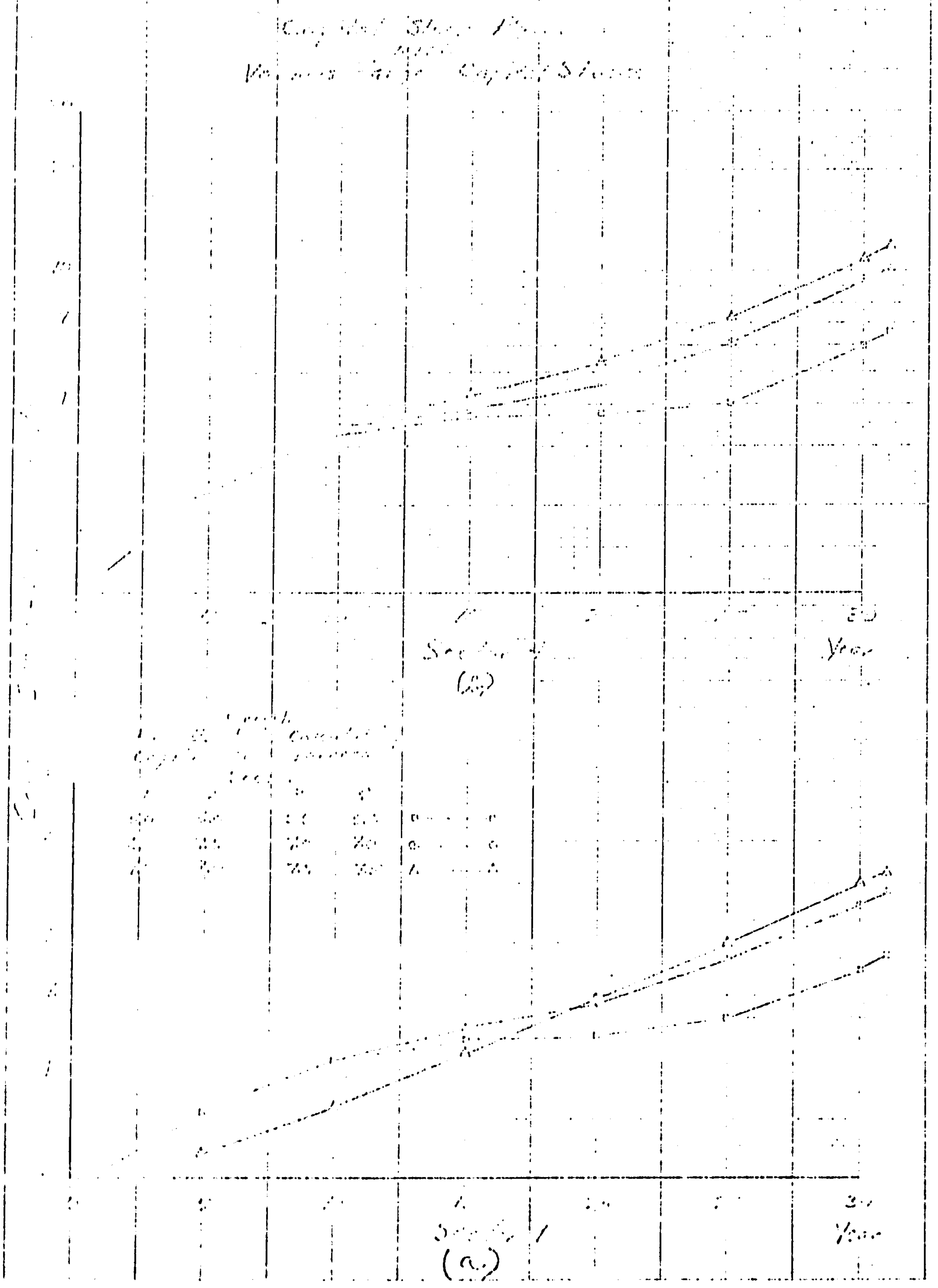

stock growth rates of 5% and the basic solution's 6.5%) the first 10 to 15 years of the plan are largely independent of the terminal conditions, while for the high terminal stock (based on a 7.0% growth rate), the accumulation path differs greatly from the other two. In Figure 6b, by contrast, the first ten years of the plan for sector four are unaffected by terminal conditions for all three terminal stock targets.[1/]

Figure 7, which shows foreign debt paths, helps explain this contrasting behavior. As it turns out, sectors one and four respectively have relatively high and low foreign exchange components in investment, i.e., π_1 from Table 3 takes the value of 0.63 in equation (2.13), while π_4 is only 0.10. It can be seen from Figure 7 that as the terminal stock targets are increased, total foreign debt over the planning period is reduced, even becoming negative during the middle years of the high target plan. The mechanism by which the reduction of debt between the low target and medium target plans takes place involves the untied imports m_{2i} and m_{3i}. (See again equation (2.13).) These are drastically reduced between the two lower target solutions.[2/] In the high

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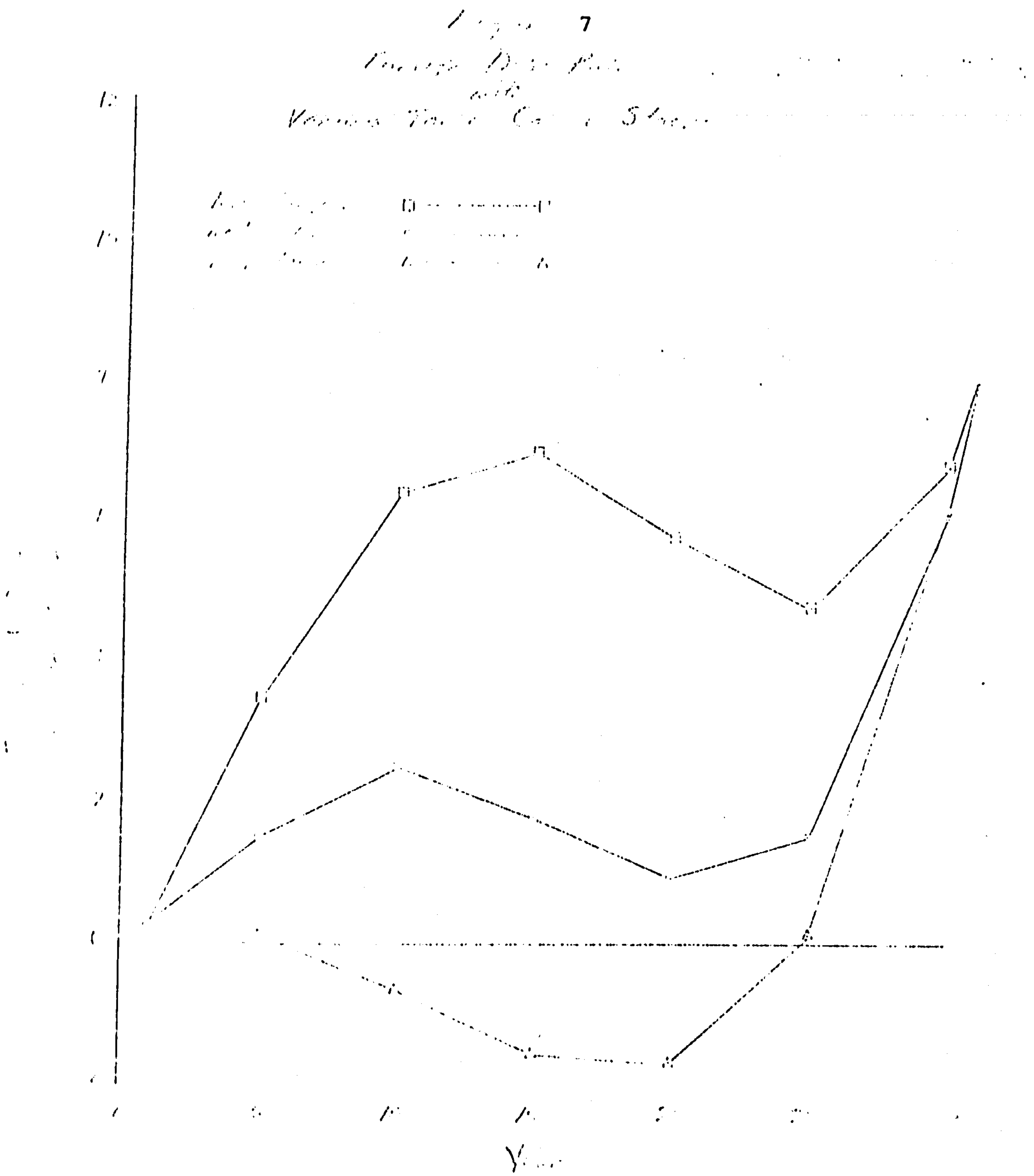
[1/] The growth rates used to calculate the target were 5.5%, 7.0% (basic solution) and 7.5%.

[2/] The decline in untied imports between the low target and basic solutions allows the consumption increases in sectors two and three mentioned in the discussion of Figure 3.



target solution, these "slack" import variables are forced to zero, and other things must be adjusted by the model in order for it to hold terminal debt down to the required level of 8.0 (billion dollars). As it turns out, investment in sectors with a high import component in capital formation is deferred, and the anomaly displayed in Figure 6a results. The initial accumulation pattern in sector four (Figure 6b) is not much affected by the target increase, again because the import coefficient π_4 is relatively small.

One might conjecture that if more slack were built into the debt constraint (e.g., by the inclusion of activities allowing import substitution and/or export promotion), the initial phases of an optimal plan would be independent of a wider range of terminal capital stocks. In any event, the examples given here demonstrate that generalization of the desirable getting-to-the-turnpike property to open economy models is likely not to be a completely straightforward process.

5.3 Varying Elasticities of Substitution

Conjectures vary as to the importance of differential elasticities of substitution in influencing the economic growth process. On the aggregate level, Nelson (as summarized by Nerlove [24]) has shown that when capital and labor are growing at roughly equal rates, changes in the aggregate elasticity of substitution will have little influence on the overall growth rate. In a disaggregated analysis, however, Arrow, et al. [2], point out that differences in elasticities of substitution among industries will have significant effects on sectoral allocations of capital and labor (and ultimately on the aggregate elasticity of substitution). In particular, high elasticities in the primary sector and lower elasticities in the secondary and tertiary sectors are a means of explaining the well known shift of labor from the former sector toward the latter.

Using our four-sector laboratory, we made some partial tests of these hypotheses, especially the latter one, by varying sectoral elasticities of substitution while at the same time recalculating the efficiency parameters (τ_j in equation (2.9)) to bring initial outputs in line with those of Korea in 1965-66. Given this means of normalizing our three-parameter production functions to fit three pieces of data (initial capital stocks, labor forces, and gross production levels), we calculated optimal solutions to the model under the conditions shown in Table 9:

Table 9

VARYING ELASTICITIES OF SUBSTITUTION

Solution	Sector 1	Sector 2	Sector 3	Sector 4
a	1.2	0.9	0.9	0.6
b	1.2	1.3	0.5	0.6
c	0.1	0.9	0.9	3.0
d	3.0	0.9	0.9	0.1

In the first three of these solutions, sectoral labor forces were essentially unchanged, although there were some shifts in the time-phasing of investment. There were major labor force shifts only in solution d where the elasticity of substitution in primary production was raised by a factor of more than two. In this case labor in the primary sector decreased by 0.6 million workers in the terminal year, representing a shift of about three percent of the total labor force of 18 million. In terms of the conventional GNP aggregate, the growth rate in solution a over the thirty-year plan was 6.7%, while it was 6.8% in solution d. Here again the effects of changing substitution elasticities were relatively minor.

How well these preliminary results would stand up under further experimentation is, needless to say, open to question. In particular, shifts of the elasticities of substitution in connection with different welfare functions and/or different normalizations for the efficiency parameter might have more significant effects. It does appear, however, that further experimentation along the lines suggested here would provide a partial answer to the troubling empirical questions regarding the relevance of the elasticity of substitution to actual planning exercises.

5.4 Varying Parameter of Absorptive Capacity Function

Since we know relatively little about the parameters of our absorptive capacity functions--particularly the μ

parameters which represented upper bounds on the percentage rates of change of the capital stocks--we were hopeful that the solution would not be too sensitive to variations in these parameters. Such was the case in the range of variation we tested.

We made two runs as variants on the base solution. In the first we increased each of the upper bound parameters by ten percentage points from the base solution levels, and then in a second run we decreased them by ten percentage points. The change that occurred was primarily one of moving investment earlier in time as the upper bound parameters were raised. This is as might be expected since large investment was made relatively more efficient by increasing the upper bounds.

6. Conclusions

Our purpose on the initiation of this study was to determine whether or not it is now feasible to find numerical solutions for dynamic nonlinear multisectoral planning models. Our conclusion is that with existing algorithms and second generation computers (of the IBM 7094 vintage) it is feasible to solve models with four and more sectors. With third generation computers (of the IBM 360 vintage), a more efficient computer language than we used, and existing algorithms it should be feasible to solve models with ten and more sectors.

Since most development planners have employed linear programming models in the past, a few comments about the comparative advantages of control theory models are in order.

(A) Disaggregation into a larger number of time periods can be done at relatively less cost in control theory models than in linear programming models. This results from the fact that the addition of more time periods only adds to the number of difference equation integration steps in the control theory formulation, while it requires the addition of more constraints in the linear programming formulation.[1/]

(B) Adding state and control variables in a control theory model appears to increase computation time <u>per iteration</u> in a roughly linear fashion, although it is not possible to conduct precise experiments on the matter. Each additional state variable means one additional difference equation to be integrated forward and backwards in time, while an extra control variable entails the evaluation of as many partial derivatives at each time step as there are state variables. The relative computation times of these operations depends greatly on how closely the additional variables are integrated with the rest of the model. This integration factor is also the most important determinant of how many additional iterations in a gradient method the new variables would require.

[1/] The time required to solve a linear program goes up roughly as the cube of the number of constraints.

(C) Inequality constraints on the control variables in control theory models are easier to incorporate into the solution techniques than are inequality constraints on the state variables. However, inequality constraints are in general more troublesome in control theory problems than in linear programming problems. Of course this situation is mitigated by the fact that models specified with nonlinear functions require fewer inequality constraints.

(D) Time lags of greater than one period in the investment process can be incorporated into control theory models by a standard technique for transforming n-th order linear difference equations into systems of the first order. The same procedures could be followed if it was deemed appropriate to include arguments for the rate of change of consumption in the welfare index.[1/]

(E) While it would be worthwhile to compare numerical control theory methods for solving nonlinear planning models with other methods for solving nonlinear programming problems, we have not yet accumulated enough experience to make such comparisons. R. Bove is presently solving a nonlinear model using the Wilson [32] algorithm;[2/] however we have not yet been able to make comparisons of computational efficiency on equivalent models.

[1/]See Broschi and Rossi [4] for a model with rate of change of consumption in the performance index, and also footnote 2, page 24.

[2/]This is a Ph.D. thesis which is currently in preparation in the Economics Department at Harvard University under the supervision of R. Dorfman and H. Chenery.

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