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A new Ai-Zhang type interior point algorithm for sufficient linear complementarity problems

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Abstract In this paper, we propose a new long-step interior point method for solving sufficient linear complementarity problems. The new algorithm combines two important approaches from the literature: the main ideas of the long-step interior point algorithm introduced by Ai and Zhang, and the algebraic equivalent transformation technique proposed by Darvay.
Similarly to the method of Ai and Zhang, our algorithm also works in a wide neighbourhood of the central path and has the best known iteration complexity of short-step variants.
We implemented the new method in Matlab and tested its efficiency on both sufficient and non-sufficient problem instances. In addition to presenting our numerical results, we also make some interesting observations regarding the analysis of Ai-Zhang type methods.

Keywords Mathematical programming · Linear complementarity optimization · Interior point algorithms · Algebraic equivalent transformation technique · sufficient matrices

JEL Classification Number: C61

1 Introduction

In this paper, we introduce a new long-step interior point method for solving linear complementarity problems (LCPs). LCPs have a wide range of applications in numerous different fields, for example solving the Arrow-Debreu market exchange model with Leontief utilities [41], finding equilibrium points in bimatrix games [30], and several engineering applications can be found in the survey [20]. The LCP class also contains the linear programming problem and the quadratic programming problem as special cases. Many of the classical applications and results can be found in the monographs of Cottle et al. [7] and Kojima et al. [28].
In general, the LCP is an NP-complete problem [6], but many efficient algorithms have been introduced assuming that the coefficient matrix has a special property. In this paper, we suppose that the coefficient matrix is a $\mathcal{P}_*(\kappa)$ matrix. In this case, a nonnegative number κ can be assigned to the matrix, which is called its handicap. With this assumption, several authors could introduce interior point algorithms that are polynomial in the size of the problem and the handicap. However, de Klerk and E.-Nagy proved that there are matrices for which the value of the handicap is exponential in the problem size [27].
Based on the used step-length, interior point algorithms (IPAs) can be divided into two main groups, short-step and long-step methods. Even though long-step algorithms perform better in practice, in general, for short-step variants a better theoretical complexity can be proved, i.e. for many years, there was a gap between theory and practice. To resolve this issue, several attempts have been made, e.g. [5,34, 36].
In 2005, Ai and Zhang introduced a long-step IPA for solving monotone LCPs [1]. Their method works in a wide neighbourhood of the central path and has the best known theoretical complexity of short-step

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variants. Based on their approach, several authors proposed long-step methods with the best known theoretical complexity, for different problem classes, e.g., for linear optimization [14,32,40], horizontal linear complementarity problems (HLCPs) [37], symmetric cone Cartesian $\mathcal{P}_*(\kappa)$-HLCPs [4,3], and also for semidefinite optimization [19,31,35].

In 2003, Darvay introduced the algebraic equivalent transformation (AET) technique to determine new search directions for IPAs [8]. His main idea was to apply a continuously differentiable, invertible function φ to the centering equation of the central path problem. Then by applying Newton's method to this transformed system, the new search directions can be determined. A new version of the AET method has been examined in the paper of Darvay and Takács for linear optimization [15], based on a different rearrangement of the centering equation. Using this new type of transformation, recently Darvay et al. introduced a predictor-corrector IPA for sufficient LCPs [12].

By changing the function φ, different methods can be introduced. Most IPAs from the literature can be considered as a special case of the AET technique, with the function $\varphi(t) = t$ (in this case, the central path problem is not transformed). In his first paper, Darvay applied the function $\varphi(t) = \sqrt{t}$. This function has been used in the paper of Darvay and Rigó as well [14], where they introduced an Ai-Zhang type long-step IPA for linear optimization with the best known theoretical complexity, and using the same function, Illés et al. recently proposed a predictor-corrector IPA in [24]. This function has also been applied by Asadi and Mansouri to $\mathcal{P}_*(\kappa)$ HLCPs [2].

The function $\varphi(t) = t - \sqrt{t}$ has been proposed also by Darvay et al. [13], and in the last few years, it has been applied in several different papers by Darvay and his coauthors. They introduced a corrector predictor IPA for solving linear programming problems [9], and proposed another corrector predictor IPA for sufficient LCPs [11], and they also presented a short-step IPA for sufficient LCPs [10].

Moreover, the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ has been introduced by Kheirfam and Haghighi [26], to solve $\mathcal{P}^*(\kappa)$ LCPs with a short-step IPA.

In this paper, we also apply the AET technique, with the function $\varphi(t) = t - \sqrt{t}$ and introduce an Ai-Zhang type long-step interior point method for solving sufficient LCPs. This is the first such algorithm to the best of our knowledge. We prove that our IPA has the best known iteration complexity of short-step variants. This result can be considered as the generalization of the IPA we introduced for linear optimization in [18]. In addition to generalizing the algorithm, we could also improve some of our estimations [18], and for this reason, better parameter settings can be applied here.

Throughout this paper, the following notations will be used. We denote scalars and indices by lowercase Latin letters, and vectors by bold lowercase Latin letters. Matrices are denoted by uppercase Latin letters. We denote sets by capital calligraphic letters. $\mathbb{R}^n_+$ denotes the set of n-dimensional vectors with strictly positive coordinates, and $\mathbb{R}^n_\oplus$ is the set of n-dimensional nonnegative vectors. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be two given vectors. Then $\mathbf{uv}$ is the Hadamard product (namely, the componentwise product) of $\mathbf{u}$ and $\mathbf{v}$. If $v_i \neq 0$ holds for all index i, then the fraction of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u}/\mathbf{v} = [u_1/v_1, \ldots, u_n/v_n]$. If $\alpha \in \mathbb{R}$, let $\mathbf{u}^\alpha = [u_1^\alpha, \ldots, u_n^\alpha]$. Let $\mathcal{I}$ denote the index set $\mathcal{I} = \{1, \ldots, n\}$. We denote the positive and negative part of the vector $\mathbf{u}$ by $\mathbf{u}^+$ and $\mathbf{u}^-$, i.e.,

$$\mathbf{u}^+ = \max\{\mathbf{u}, \mathbf{0}\} \in \mathbb{R}^n \quad \text{and} \quad \mathbf{u}^- = \min\{\mathbf{u}, \mathbf{0}\} \in \mathbb{R}^n,$$

where the maximum and minimum are taken componentwise. We use the standard notation $\|\mathbf{u}\|$ for the Euclidean norm of $\mathbf{u}$, $\|\mathbf{u}\|_1 = \sum_{i=1}^n |u_i|$ denotes the Manhattan-norm of $\mathbf{u}$, and $\|\mathbf{u}\|_\infty = \max_{i=1}^n |u_i|$ is the infinity norm of $\mathbf{u}$. The matrix $\operatorname{diag}(\mathbf{u})$ is the diagonal matrix with the elements of the vector $\mathbf{u}$ in its diagonal. Finally, $\mathbf{e}$ denotes the vector of all ones.

This paper is organized as follows. Section 2 summarizes the most important properties of LCPs and the related matrix classes. In Section 3, we give an overview of the algebraic equivalent transformation technique and the method of Ai and Zhang. In Section 4, we introduce a new, Ai-Zhang type wide neighbourhood and describe our new algorithm. In Section 5, we prove that the method is convergent and has the best known iteration complexity. Section 6 presents our numerical results. In Section 7, we make some interesting observations on the coordinates of the vector $\mathbf{v}$. Section 8 summarizes our results.

2 The linear complementarity problem

Let us consider the linear complementarity problem (LCP) in the following form:

$$\left.\begin{aligned} -M\mathbf{x} + \mathbf{s} &= \mathbf{q} \\ \mathbf{xs} &= \mathbf{0} \\ \mathbf{x}, \mathbf{s} &\geq \mathbf{0}, \end{aligned}\right\}$$

where $M \in \mathbb{R}^{n\times n}$ and $\mathbf{q} \in \mathbb{R}^n$ are given, and our goal is to find a vector pair $(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^n \times \mathbb{R}^n$ that satisfies the system.
Let $\mathcal{F} = \{(\mathbf{x}, \mathbf{s}) : \ -M\mathbf{x} + \mathbf{s} = \mathbf{q}, \mathbf{x} \geq \mathbf{0}, \ \mathbf{s} \geq \mathbf{0}\}$ denote the set of feasible solutions, $\mathcal{F}_+ = \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{x} > \mathbf{0}, \ \mathbf{s} > \mathbf{0}\}$ the set of strictly positive feasible solutions and $\mathcal{F}^* = \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F} : \mathbf{xs} = \mathbf{0}\}$ the set of solutions to the linear complementarity problem.
The class of sufficient matrices has been introduced by Cottle et al. [7]. A matrix $M \in \mathbb{R}^{n\times n}$ is column sufficient if the following implication holds for all $\mathbf{x} \in \mathbb{R}^n$:

$$\text{If} \ \ x_i(M\mathbf{x})_i \leq 0 \ \ \text{for all} \ i \in \mathcal{I} \qquad \text{then} \qquad x_i(M\mathbf{x})_i = 0 \ \ \text{for all} \ i \in \mathcal{I}.$$

M is row sufficient if M^T is column sufficient, and a matrix M is sufficient if it is both row and column sufficient.
In 1991, Kojima et al. introduced the class of $\mathcal{P}_*(\kappa)$ matrices [28]. Let κ be a given nonnegative number. A matrix $M \in \mathbb{R}^{n\times n}$ is a $\mathcal{P}_*(\kappa)$ matrix, if

$$\mathbf{x}^\top M\mathbf{x} + 4\kappa \sum_{i\in\mathcal{I}} (\mathbf{x}(M\mathbf{x}))_i^+ \geq 0$$

holds for all $\mathbf{x} \in \mathbb{R}^n$. This class can be considered as the generalization of positive semidefinite matrices since $\mathcal{P}_*(0)$ is the set of positive semidefinite matrices.
The smallest κ value for which M is a $\mathcal{P}_*(\kappa)$ matrix is called the handicap of M. The matrix class $\mathcal{P}_*$ can be defined in the following way:

$$\mathcal{P}_* := \bigcup_{\kappa\geq 0} \mathcal{P}_*(\kappa).$$

Kojima et al. proved that if a matrix belongs to the set $\mathcal{P}_*$, then it is column sufficient [28]. Later, Guu and Cottle showed that a $\mathcal{P}_*$-matrix is also row sufficient [22], meaning that all $\mathcal{P}_*$-matrices are sufficient. In 1996, Väliaho proved the other inclusion, therefore the class of sufficient matrices is equivalent to the class of $\mathcal{P}_*$-matrices [38].

2.1 The central path problem

The central path problem of LCP can be formulated as follows:

$$\left.\begin{aligned} -M\mathbf{x} + \mathbf{s} &= \mathbf{q} \\ \mathbf{xs} &= \nu\mathbf{e} \\ \mathbf{x}, \ \mathbf{s} &> \mathbf{0}, \end{aligned}\right\} \tag{1}$$

where $\nu > 0$ is a given parameter.
The next theorem highlights the importance of the $\mathcal{P}_*$ matrix class. Illés, Roos and Terlaky gave an elementary proof of these statements in an unpublished manuscript in 1997 [25]. The proof can be found in [33].

Theorem 1 *Let us consider a linear complementarity problem with a $\mathcal{P}_*(\kappa)$ coefficient matrix M. Then the following three statements are equivalent:*

1. $\mathcal{F}^+ \neq \emptyset$.
2. $\forall\ \mathbf{w} \in \mathbb{R}^n_+ \ \exists!\ (\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \mathbf{xs} = \mathbf{w}$.
3. $\forall\ \nu > 0 \ \exists!\ (\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \mathbf{xs} = \nu\mathbf{e}$.

According to the last statement, for $\mathcal{P}_*(\kappa)$ linear complementarity problems, when $\mathcal{F}^+ \neq \emptyset$, the central path exists and it is unique. Moreover, as ν tends to 0, the solutions of the central path problem (1) converge to a solution of the LCP.
From now on, we assume that the coefficient matrix M of the LCP is sufficient, more precisely $\mathcal{P}_*(\kappa)$, furthermore, $\mathcal{F}_+ \neq \emptyset$ and an initial point $(\mathbf{x}_0, \mathbf{s}_0) \in \mathcal{F}_+$ is given.

3 The theoretical background of the algorithm

As it has already been mentioned in the introduction, our method combines two important results from the literature, the algebraic equivalent transformation technique (AET) proposed by Darvay [8] and the main approach of the long-step IPA introduced by Ai and Zhang [1].
According to the AET technique, we apply a continuously differentiable function $\varphi : (\xi, \infty) \to \mathbb{R}$ with $\varphi'(t) > 0$ for all $t \in (\xi, \infty)$, $\xi \in [0, 1)$ to the central path problem (1):

$$\left.\begin{aligned} -M\mathbf{x} + \mathbf{s} &= \mathbf{q} \\ \varphi\left(\frac{\mathbf{xs}}{\nu}\right) &= \varphi(\mathbf{e}) \\ \mathbf{x}, \mathbf{s} &> \mathbf{0}. \end{aligned}\right\} \tag{2}$$

If we apply Newton's method to system (2), we obtain

$$\left.\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} &= \nu\frac{\varphi(\mathbf{e}) - \varphi\left(\frac{\mathbf{xs}}{\nu}\right)}{\varphi'\left(\frac{\mathbf{xs}}{\nu}\right)} =: \mathbf{a}_\varphi. \end{aligned}\right\} \tag{3}$$

As it can be seen from the previous formulation, the right-hand side of system (3) depends on the choice of the function φ, and by modifying φ, we can determine different search directions and introduce new interior point algorithms.
One of the important ideas of Ai and Zhang was to decompose the Newton directions to positive and negative parts and use different step-lengths for the two components. Namely, we consider the following two systems:

$$\left.\begin{aligned} -M\Delta\mathbf{x}_- + \Delta\mathbf{s}_- &= \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x}_- + \mathbf{x}\Delta\mathbf{s}_- &= \mathbf{a}_\varphi^- \end{aligned}\right\} \qquad \left.\begin{aligned} -M\Delta\mathbf{x}_+ + \Delta\mathbf{s}_+ &= \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x}_+ + \mathbf{x}\Delta\mathbf{s}_+ &= \mathbf{a}_\varphi^+ \end{aligned}\right\} \tag{4}$$

where $\mathbf{a}_\varphi^+$ and $\mathbf{a}_\varphi^-$ are the positive and negative part of the vector $\mathbf{a}_\varphi$, respectively.
It is important to notice that the coordinates of $\Delta\mathbf{x}_+$ are not necessarily nonnegative, since this is the solution of the system with the positive part of $\mathbf{a}_\varphi$ on the right-hand side (we have a subscript in the notation, instead of a superscript). The similar can be stated for $\Delta\mathbf{x}_-$, $\Delta\mathbf{s}_+$ and $\Delta\mathbf{s}_-$ as well.
If α_1 and α_2 are given step-lengths, after solving systems (4), we can calculate the new iterates as

$$\mathbf{x}(\alpha) := \mathbf{x} + \alpha_1\Delta\mathbf{x}_- + \alpha_2\Delta\mathbf{x}_+ \quad \text{and} \quad \mathbf{s}(\alpha) := \mathbf{s} + \alpha_1\Delta\mathbf{s}_- + \alpha_2\Delta\mathbf{s}_+.$$

To simplify the analysis of interior point methods, we usually work with a scaled version of the Newton-system. To determine the scaled systems from (4), we introduce the following notations:

$$\mathbf{v} = \sqrt{\frac{\mathbf{xs}}{\nu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{dx}_+ = \frac{\mathbf{v}\Delta\mathbf{x}_+}{\mathbf{x}}, \quad \mathbf{ds}_+ = \frac{\mathbf{v}\Delta\mathbf{s}_+}{\mathbf{s}}, \quad \mathbf{dx}_- = \frac{\mathbf{v}\Delta\mathbf{x}_-}{\mathbf{x}}, \quad \mathbf{ds}_- = \frac{\mathbf{v}\Delta\mathbf{s}_-}{\mathbf{s}}.$$

Let $D = diag(\mathbf{d})$ and $\overline{M} = DMD$, then the scaled systems can be written as

$$\left.\begin{aligned} -\overline{M}\mathbf{dx}_- + \mathbf{ds}_- &= \mathbf{0} \\ \mathbf{dx}_- + \mathbf{ds}_- &= \mathbf{p}_\varphi^- \end{aligned}\right\} \qquad \left.\begin{aligned} -\overline{M}\mathbf{dx}_+ + \mathbf{ds}_+ &= \mathbf{0} \\ \mathbf{dx}_+ + \mathbf{ds}_+ &= \mathbf{p}_\varphi^+ \end{aligned}\right\}$$

where

$$\mathbf{p}_\varphi = \frac{\varphi(\mathbf{e}) - \varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}.$$

In this paper, we focus on the function $\varphi(t) = t - \sqrt{t}$, which has been introduced by Darvay et al. for linear optimization [13]. In this case,

$$\mathbf{p}_\varphi = \frac{2(\mathbf{v} - \mathbf{v}^2)}{2\mathbf{v} - \mathbf{e}}.$$

Since we fixed the function φ, from now on we simply use the notation $\mathbf{p}$ instead of $\mathbf{p}_\varphi$.
Throughout the analysis, we need to ensure that $\mathbf{p}$ is well-defined. Therefore we assume that $v_i > 1/2$ is satisfied for all $i \in \mathcal{I}$.
Because of the decomposition applied in Ai-Zhang type methods, we also introduce the notations for the index sets $\mathcal{I}_+$ and $\mathcal{I}_-$. Let $\mathcal{I}_+ = \{i \in \mathcal{I} : x_i s_i \leq \tau\mu\} = \{i \in \mathcal{I} : v_i \leq 1\}$, and $\mathcal{I}_- = \mathcal{I} \setminus \mathcal{I}_+$. Notice that

under the assumption $v_i > 1/2$ for all index $i \in \mathcal{I}$, the nonnegativity of a coordinate p_i is equivalent to $i \in \mathcal{I}_+$.
The vector $\mathbf{p}$ has been defined as a componentwise transformation of the vector $\mathbf{v}$, therefore let p denote the transforming function, namely for which $p(v_i) = p_i$ holds for all $v_i \in (1/2, \infty)$, i.e.,

$$p : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}, \quad p(t) = \frac{2(t - t^2)}{2t - 1}.$$

In the analysis, we use some estimations on the function p, namely for all $t \in (1/2, \infty)$

$$p(t) \geq 2(1 - t), \tag{5}$$
$$p(t) \geq -t, \tag{6}$$
$$p(t) \geq \frac{1 - t^2}{t}. \tag{7}$$

From now on, we fix the value of ν as $\tau\mu$, where $\tau \in (0, 1)$ is a given update parameter, and $\mu = \frac{\mathbf{x}^T\mathbf{s}}{n}$, i.e. if in the current iteration we are in the point $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+$, then our goal is to take a step towards the $\tau\mu$-center, that is, towards the solution of the central path problem (1) for $\nu = \tau\mu$.

4 The algorithm

The neighbourhood we use is based on the approach of Ai and Zhang [1], however, we slightly modified their definition. To achieve the desired complexity, we limit only the norm of the positive part of the vector $\mathbf{p}$, while the paper of Ai and Zhang uses the norm of the vector $\mathbf{vp}^+$. Furthermore, our definition depends on the handicap of the matrix, and $0 < \beta < 1/2$ is a given real number. Due to the properties of the function $\varphi(t) = t - \sqrt{t}$, we also need to ensure that the technical condition $\mathbf{v} > 1/2\mathbf{e}$ is satisfied throughout the iterations, therefore it is also included in the definition of the neighbourhood:

$$\mathcal{W}(\tau, \beta, \kappa) = \left\{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}_+ : \left\|\mathbf{p}^+\right\| \leq \frac{\beta}{1 + 4\kappa} \text{ and } \mathbf{v} > \frac{1}{2}\mathbf{e}\right\}.$$

The wide neighbourhood $\mathcal{N}_\infty^-(1 - \tau)$ has been introduced by Kojima et al. [29]:

$$\mathcal{N}_\infty^-(1 - \tau) = \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}_+ : \mathbf{xs} \geq \tau\mu\mathbf{e}\}.$$

The following lemma shows that $\mathcal{W}(\tau, \beta, \kappa)$ is indeed a wide neighbourhood.

Lemma 1 *Let $0 < \tau < 1$ and $0 < \beta < 1/2$ be given parameters, and let $\gamma = \tau\left(1 - \frac{\beta}{2(1+4\kappa)}\right)^2$. Then*

$$\mathcal{N}_\infty^-(1 - \tau) \subseteq \mathcal{W}(\tau, \beta, \kappa) \subseteq \mathcal{N}_\infty^-(1 - \gamma)$$

holds.

Proof. If $(\mathbf{x}, \mathbf{s}) \in \mathcal{N}_\infty^-$, then $(\tau\mu - x_i s_i)^+ = 0$ for all $i \in \mathcal{I}$, therefore $\|\mathbf{p}^+\| = 0 < \beta/(1 + 4\kappa)$. The condition $\mathbf{v} > 1/2\mathbf{e}$ is also satisfied, since $v_i^2 = (x_i s_i)/(\tau\mu) \geq 1 > 1/4$ for all $i \in \mathcal{I}$.
For the other inclusion, let $(\mathbf{x}, \mathbf{s}) \in \mathcal{W}(\tau, \beta, \kappa)$ and assume that there exists an index $i \in \mathcal{I}$ for which $x_i s_i < \gamma\mu$ holds. In this case, $v_i^2 = \frac{x_i s_i}{\tau\mu} < \frac{\gamma}{\tau} = \left(1 - \frac{\beta}{2(1+4\kappa)}\right)^2$. Using (5), we get

$$p_i = p(v_i) \geq 2(1 - v_i) > 2\left(1 - \sqrt{\frac{\gamma}{\tau}}\right) = \frac{\beta}{1 + 4\kappa},$$

which is a contradiction. ∎

Remark 1 Let $\tilde{\gamma} = \tau\left(1 - \beta/2\right)^2$. Since $\gamma > \tilde{\gamma}$, $\mathcal{W}(\tau, \beta, \kappa) \subseteq \mathcal{N}_\infty^-(1 - \tilde{\gamma})$ also holds.

In the next corollary, we give lower and upper bounds on the coordinates of $\mathbf{v}$.

Corollary 1 *If $(\mathbf{x}, \mathbf{s}) \in \mathcal{W}(\tau, \beta, \kappa)$ then*

1. $1 - \frac{\beta}{2(1+4\kappa)} \leq v_i \leq 1$ *for all* $i \in \mathcal{I}_+$,
2. $1 < v_i \leq \sqrt{n/\tau}$ *for all* $i \in \mathcal{I}_-$.

Proof. The first statement follows from the second inclusion of Lemma 1. Furthermore, $v_i \leq \sqrt{n/\tau}$ is satisfied for all $i \in \mathcal{I}$, since

$$\sum_{i\in\mathcal{I}} v_i^2 = \sum_{i\in\mathcal{I}} \frac{x_i s_i}{\tau\mu} = \frac{1}{\tau\mu}\mathbf{x}^T\mathbf{s} = \frac{n}{\tau}. \tag{8}$$

■

Since we have already defined all main elements of our method, we are ready to give its pseudocode:

Input: a sufficient matrix $M \in \mathbb{R}^{n\times n}$, and $\mathbf{q} \in \mathbb{R}^n$
an update parameter $0 < \tau < 1$, a neighbourhood parameter $0 < \beta < 1$,
an accuracy parameter $\varepsilon > 0$,
an initial point $(\mathbf{x}_0, \mathbf{s_0}) \in \mathcal{W}(\tau, \beta, \kappa)$ with $\mu_0 = \frac{\mathbf{x}_0^T\mathbf{s}_0}{n}$.
$\mathbf{x} := \mathbf{x}_0$, $\mathbf{s} := \mathbf{s}_0$ and $\mu := \mu_0$
while $\mathbf{x}^T\mathbf{s} > \varepsilon$ **do**
  Determine $\Delta\mathbf{x}_+, \Delta\mathbf{s}_+$ and $\Delta\mathbf{x}_-, \Delta\mathbf{s}_-$ by solving systems (4);
  $\alpha_2 := 1$ and $\alpha_1 := \underset{\alpha_1\in(0,1]}{\operatorname{argmin}} \left\{\mu(\alpha) = (\mathbf{x}(\alpha)^T\mathbf{s}(\alpha))/n : \ (\mathbf{x}(\alpha), \mathbf{s}(\alpha)) \in \mathcal{W}(\tau, \beta, \kappa)\right\}$;
  $\mathbf{x} := \mathbf{x}(\alpha)$;
  $\mathbf{s} := \mathbf{s}(\alpha)$;
  $\mu := \frac{\mathbf{x}^T\mathbf{s}}{n}$;
end

Algorithm 1: Outline of the algorithm

5 Analysis of the algorithm

The next lemma contains some well-known results from the theory of interior point algorithms for the Newton-system of sufficient LCPs. The proof of the first and third statements can be found in [28], and for the second statement, see for example [21, Lemma 2].

Lemma 2 *Let us consider the following system:*

$$\begin{aligned} -M\Delta\mathbf{x} + \Delta\mathbf{s} &= \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} &= \mathbf{a}, \end{aligned}$$

where M is a $\mathcal{P}_*(\kappa)$-matrix, $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$ and $\mathbf{a} \in \mathbb{R}^n$ are given vectors.

1. *Then the system has a unique $(\Delta\mathbf{x}, \Delta\mathbf{s})$ solution.*
2. *The next estimations hold for the solutions of the above system:*

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_r \leq \frac{2^{1/r} + 4\kappa}{4} \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2 \quad \text{for} \ \ r = 1, 2, \infty,$$

   where $1/\infty := 0$.
3. $-\kappa \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2 \leq \Delta\mathbf{x}^T\Delta\mathbf{s} \leq \frac{1}{4} \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2$.

We need to prove that after an iteration the decrease of the duality gap is suitable and that the new iterate will also be in the neighbourhood. Therefore we examine the new iterate after taking a Newton-step with step-length $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \in (0, 1]$ are given.
Let us introduce the following notations:

$$\mathbf{dx}(\alpha) = \alpha_1\mathbf{dx}_- + \alpha_2\mathbf{dx}_+, \quad \mathbf{ds}(\alpha) = \alpha_1\mathbf{ds}_- + \alpha_2\mathbf{ds}_+,$$

$$\mathbf{h}(\alpha) = \tau\mu\mathbf{v}^2 + \alpha_1\tau\mu\mathbf{vp}^- + \alpha_2\tau\mu\mathbf{vp}^+.$$

Using these notations, $\mathbf{x}(\alpha)\mathbf{s}(\alpha) = (\mathbf{x} + \alpha_1\Delta\mathbf{x}_- + \alpha_2\Delta\mathbf{x}_+)(\mathbf{s} + \alpha_1\Delta\mathbf{s}_- + \alpha_2\Delta\mathbf{s}_+)$ can be written as

$$\mathbf{x}(\alpha)\mathbf{s}(\alpha) = \mathbf{h}(\alpha) + \tau\mu\mathbf{dx}(\alpha)\mathbf{ds}(\alpha).$$

Corollary 2 *Let* $(\mathbf{x},\mathbf{s}) \in \mathcal{W}(\tau,\beta,\kappa)$ *and* $\alpha_1,\alpha_2 \in (0,1]$ *be given. Then*

$$\|\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)\|_r \le \frac{2^{1/r}+4\kappa}{4}\left(\alpha_1^2\frac{n}{\tau}+\alpha_2^2\frac{\beta^2}{(1+4\kappa)^2}\right) \quad \text{for } r=1,2,\infty,$$

where $1/\infty := 0$, *and*

$$-\kappa\left(\alpha_1^2\frac{n}{\tau}+\alpha_2^2\frac{\beta^2}{(1+4\kappa)^2}\right) \le \mathbf{dx}(\alpha)^T\mathbf{ds}(\alpha) \le \frac{1}{4}\left(\alpha_1^2\frac{n}{\tau}+\alpha_2^2\frac{\beta^2}{(1+4\kappa)^2}\right).$$

Proof. The vector $\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)$ is the same as $\tau\mu\Delta\mathbf{x}^\alpha\Delta\mathbf{s}^\alpha$, where $\Delta\mathbf{x}^\alpha$ and $\Delta\mathbf{s}^\alpha$ are the solutions of the system

$$\left.\begin{aligned} -M\Delta\mathbf{x}+\Delta\mathbf{s} &= \mathbf{0}\\ \mathbf{s}\Delta\mathbf{x}+\mathbf{x}\Delta\mathbf{s} &= \tau\mu\mathbf{v}(\alpha_1\mathbf{p}^- + \alpha_2\mathbf{p}^+)\end{aligned}\right\}$$

since $\mathcal{I}_-\cap\mathcal{I}_+=\emptyset$.
If we apply Lemma 2 to the above system, we get

$$\|\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)\|_r \le \frac{2^{1/r}+4\kappa}{4}\left\|\alpha_1\mathbf{p}^- + \alpha_2\mathbf{p}^+\right\|^2,$$

since $\frac{\tau\mu\mathbf{v}}{\sqrt{\mathbf{xs}}} = \sqrt{\tau\mu}$.
Furthermore, $\|\alpha_1\mathbf{p}^- + \alpha_2\mathbf{p}^+\|^2 = \alpha_1^2\|\mathbf{p}^-\|^2 + \alpha_2^2\|\mathbf{p}^+\|^2$ by the orthogonality of $\mathbf{p}^-$ and $\mathbf{p}^+$. According to the definition of the neighbourhood $\mathcal{W}(\tau,\beta,\kappa)$, $\|\mathbf{p}^+\|^2 \le \frac{\beta^2}{(1+4\kappa)^2}$. Moreover,

$$\|\mathbf{p}^-\|^2 = \sum_{i\in\mathcal{I}_-} p^2(v_i) \le \sum_{i\in\mathcal{I}_-} v_i^2 \le \frac{n}{\tau}.$$

In the first inequality we applied (6), and the second follows from (8).
From these estimations, the first statement of the corollary follows. The inequalities regarding the scalar product $\mathbf{dx}(\alpha)^T\mathbf{ds}(\alpha)$ can be proved similarly. ∎
We need to give a lower bound on the coordinates of $\mathbf{h}(\alpha)$. The proof of the following two statements remains the same as it was in the LP case [18], therefore we do not present them here. Later on, the analysis becomes more complicated than in [18], since the search directions here are not orthogonal.

Lemma 3 ([18, Lemma 2]) $h_i(\alpha) \ge \tau\mu$ *holds for all* $i \in \mathcal{I}_-$.

In the following lemma, we give a strictly positive lower bound for all coordinates of $\mathbf{h}(\alpha)$.

Lemma 4 ([18, Lemma 3]) *If* $(\mathbf{x},\mathbf{s}) \in \mathcal{W}(\tau,\beta,\kappa)$, *then* $\mathbf{h}(\alpha) \ge \gamma\mu\mathbf{e}$, *and consequently* $\mathbf{h}(\alpha) > \mathbf{0}$.

From now on, we use the step-lengths $\alpha_1 = \frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$ and $\alpha_2 = 1$. From $\alpha_2 = 1$ it follows that

$$\mathbf{h}_i(\alpha) \ge \tau\mu \tag{9}$$

holds for all indices, not just for the ones in $\mathcal{I}_-$. Indeed, by inequality (7), we get $h_i(\alpha) = \tau\mu(v_i^2+\alpha_2 v_i p_i) \ge \tau\mu(1-(1-\alpha_2)v_i p_i)$ for all $i \in \mathcal{I}_+$.

We have to prove that the new iterates are strictly positive, i.e. $\mathbf{x}(\alpha) > \mathbf{0}$ and $\mathbf{s}(\alpha) > \mathbf{0}$ holds.

Lemma 5 *Let* $(\mathbf{x},\mathbf{s}) \in \mathcal{W}(\tau,\beta,\kappa)$, $\alpha_1 = \frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$ *and* $\alpha_2 = 1$. *Then*

$$\mathbf{x}(\alpha)\mathbf{s}(\alpha) \ge \left(1-\frac{\beta+\beta^2}{4(1+4\kappa)}\right)\tau\mu\mathbf{e}.$$

Proof. By applying (9) and then Corollary 2, we get the following:

$$\begin{aligned}\mathbf{x}(\alpha)\mathbf{s}(\alpha) &= \mathbf{h}(\alpha)+\tau\mu\mathbf{dx}(\alpha)\mathbf{ds}(\alpha) \ge \tau\mu\mathbf{e}-\tau\mu\|\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)\|_\infty \ge\\ &\ge \tau\mu\mathbf{e}-\frac{1}{4(1+4\kappa)}(\beta+\beta^2)\tau\mu\mathbf{e} = \left(1-\frac{\beta+\beta^2}{4(1+4\kappa)}\right)\tau\mu\mathbf{e}.\end{aligned}$$

∎
To prove that the new iterates ($\mathbf{x}(\alpha)$ and $\mathbf{s}(\alpha)$) are strictly positive, we can apply Proposition 3.2 by Ai and Zhang [1]. They analyze the case of monotone LCPs, but the properties of the coefficient matrix do not have any role in their proof, therefore it can be used in this more general setting as well.

Proposition 1 ([1, Proposition 3.2]) *Let* $(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+$ *and* $(\Delta\mathbf{x},\Delta\mathbf{s})$ *be the solution of the system*

$$-M\Delta\mathbf{x} + \Delta\mathbf{s} = \mathbf{0}$$
$$\mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} = \mathbf{z},$$

If $\mathbf{z}+\mathbf{xs} > 0$ *and* $(\mathbf{x}+t_0\Delta\mathbf{x})(\mathbf{s}+t_0\Delta\mathbf{s}) > 0$ *holds for some* $t_0 \in (0,1]$*, then* $\mathbf{x}+t\Delta\mathbf{x} > 0$ *and* $\mathbf{s}+t\Delta\mathbf{s} > 0$ *for all* $t \in (0,t_0]$.

To prove the positivity of the vectors $\mathbf{x}(\alpha)$ and $\mathbf{s}(\alpha)$, we apply Proposition 1 with $\mathbf{z} = \tau\mu(\alpha_1\mathbf{vp}^- + \alpha_2\mathbf{vp}^+)$. Since $\mathbf{z}+\mathbf{xs} = \mathbf{h}(\alpha)$¿0 by Lemma 4, it is enough to prove that their Hadamard product is positive. This is satisfied for all $\beta \in [0,1]$, because in this case

$$1 - \frac{\beta+\beta^2}{4(1+4\kappa)} \geq 1 - \frac{\beta+\beta^2}{4} > 0$$

holds.

5.1 Estimation of the change in the duality gap

The next two lemmas examine the change in the duality gap $\mu(\alpha) = \frac{\mathbf{x}(\alpha)^T\mathbf{s}(\alpha)}{n}$ after the Newton-step.

Lemma 6 *Let* $(\mathbf{x},\mathbf{s}) \in \mathcal{W}(\tau,\beta,\kappa)$*,* $\alpha_1 = \frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$ *and* $\alpha_2 = 1$*. Then*

$$\mu(\alpha) \leq \left(1 - \alpha_1\left(\frac{8}{9}(1-\tau) - \sqrt{\beta\tau} - \frac{1+\beta}{4}\right)\right)\mu.$$

Proof. By the definition of $\mu(\alpha)$,

$$\mu(\alpha) = \mu + \frac{\alpha_1\tau\mu}{n}\mathbf{v}^T\mathbf{p}^- + \frac{\alpha_2\tau\mu}{n}\mathbf{v}^T\mathbf{p}^+ + \tau\mu\frac{\mathbf{dx}(\alpha)^T\mathbf{ds}(\alpha)}{n}.$$

First, we give an upper bound on the expression $\mathbf{v}^T\mathbf{p}^+$:

$$\mathbf{v}^T\mathbf{p}^+ = \left\|\mathbf{vp}^+\right\|_1 \leq \sqrt{n}\left\|\mathbf{vp}^+\right\| \leq \sqrt{n}\frac{\beta}{1+4\kappa}. \tag{10}$$

The first inequality holds since $\mathbf{e}^T\mathbf{u} \leq \|\mathbf{u}\|_1$, and the second estimation can be shown using the Cauchy-Schwartz inequality. The last inequality can be verified using the property that $1 \geq v_i$ when $i \in \mathcal{I}_+$ and the definition of the neighbourhood $\mathcal{W}(\tau,\beta,\kappa)$.
To get an estimation on the term $\mathbf{v}^T\mathbf{p}^-$, consider the inequalities $2v_i - 1 > \mathbf{0}$ and $v_i > 1$ for all $i \in \mathcal{I}_-$:

$$\begin{aligned}\mathbf{v}^T\mathbf{p}^- &= \sum_{i\in\mathcal{I}_-} v_i p(v_i) = \sum_{i\in\mathcal{I}_-}\frac{2v_i^2}{(1+v_i)(2v_i-1)}(1-v_i^2) \leq \\ &\leq \sum_{i\in\mathcal{I}_-}\frac{8}{9}(1-v_i^2) \leq \sum_{i\in\mathcal{I}}\frac{8}{9}(1-v_i^2) = \frac{8}{9}n\left(1-\frac{1}{\tau}\right).\end{aligned} \tag{11}$$

Combining (10) and (11), we obtain

$$\begin{aligned}\mu(\alpha) &\leq \mu + \frac{\alpha_1\tau\mu}{n}\frac{8}{9}n\left(1-\frac{1}{\tau}\right) + \frac{\alpha_2\tau\mu}{n}\sqrt{n}\frac{\beta}{1+4\kappa} + \frac{\tau\mu}{4n}\left(\alpha_1^2\frac{n}{\tau} + \alpha_2^2\frac{\beta^2}{(1+4\kappa)^2}\right) \leq \\ &= \mu\left(1 - \alpha_1\frac{8}{9}(1-\tau) + \frac{\alpha_2\tau}{\sqrt{n}}\frac{\beta}{1+4\kappa} + \frac{\alpha_1^2}{4} + \alpha_2^2\frac{\beta^2\tau}{4n(1+4\kappa)^2}\right).\end{aligned} \tag{12}$$

Since $\alpha_2 = 1 = (1+4\kappa)\sqrt{\frac{n}{\beta\tau}}\alpha_1$, we get

$$\mu(\alpha) \leq \mu\left(1 - \alpha_1\frac{8}{9}(1-\tau) + \alpha_1\sqrt{\beta\tau} + \frac{\alpha_1^2}{4} + \alpha_1^2\frac{\beta}{4}\right) \leq \left(1 - \alpha_1\left(\frac{8}{9}(1-\tau) - \sqrt{\beta\tau} - \frac{1+\beta}{4}\right)\right)\mu.$$

■

This result shows that the step-length α_1 is responsible for the decrease of the duality gap in our analysis, i.e. by choosing its value properly, we can prove the convergence and desired complexity of the method. According to our estimation (12), the terms multiplied by α_2 increase the duality gap, but this step-length has an important role in ensuring that the new iterates remain in the neighbourhood $\mathcal{W}(\tau,\beta,\kappa)$, as it will be discussed later.
For the correctness of our algorithm, we need to ensure that the duality gap decreases after every iteration.

Corollary 3 *Let* $(\mathbf{x},\mathbf{s})\in\mathcal{W}(\tau,\beta,\kappa)$, $\alpha_1=\frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$, $\alpha_2=1$ *and* $\beta=\tau=\frac{1}{4}$. *Then*

$$\mu(\alpha)<\mu$$

holds.

Proof. Substituting $\beta=\tau=\frac{1}{4}$,

$$\left(\frac{8}{9}(1-\tau)-\sqrt{\beta\tau}-\frac{1+\beta}{4}\right)\approx 0.7292,$$

so the statement holds by Lemma 6. ∎
To make sure that the iterates stay in the neighbourhood $\mathcal{W}(\tau,\beta,\kappa)$, we need a lower bound on the duality gap after a Newton-step.

Lemma 7 *Let* $(\mathbf{x},\mathbf{s})\in\mathcal{W}(\tau,\beta,\kappa)$, *then*

$$\mu(\alpha)\geq\left(1-\alpha_1-\kappa\tau\left(\frac{\alpha_1^2}{\tau}+\frac{\alpha_2^2\beta^2}{n(1+4\kappa)^2}\right)\right)\mu.$$

Proof.

$$\begin{aligned}\mu(\alpha)&=\mu+\frac{\alpha_1\tau\mu}{n}\mathbf{v}^T\mathbf{p}^-+\frac{\alpha_2\tau\mu}{n}\mathbf{v}^T\mathbf{p}^++\tau\mu\frac{\mathbf{dx}(\alpha)^T\mathbf{ds}(\alpha)}{n}=\\&\geq\mu+\frac{\alpha_1\tau\mu}{n}\mathbf{v}^T\mathbf{p}^--\frac{\kappa\tau\mu}{n}\left(\alpha_1^2\frac{n}{\tau}+\alpha_2^2\frac{\beta^2}{(1+4\kappa)^2}\right)\geq\\&\geq\left(1-\alpha_1-\kappa\tau\left(\frac{\alpha_1^2}{\tau}+\frac{\alpha_2^2\beta^2}{n(1+4\kappa)^2}\right)\right)\mu,\end{aligned}$$

where we used the estimation

$$\frac{\alpha_1\tau\mu}{n}\mathbf{v}^T\mathbf{p}^-=-\frac{\alpha_1\tau\mu}{n}\sum_{i\in\mathcal{I}_-}\frac{2v_i(v_i^2-v_i)}{2v_i-1}\geq-\frac{\alpha_1\tau\mu}{n}\sum_{i\in\mathcal{I}}v_i^2\geq-\alpha_1\mu,$$

that follows from (6) and (8). ∎

5.2 The new iterates stay in the neighbourhood $\mathcal{W}(\tau,\beta,\kappa)$

To guarantee that the new points after taking the Newton-step stay in the neighbourhood $\mathcal{W}(\tau,\beta,\kappa)$, we need to choose the values of the parameters τ and β properly.
First, we need to ensure that all coordinates of the vector $\mathbf{v}(\alpha)=\sqrt{\frac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\tau\mu(\alpha)}}$ are greater than $1/2$.

Lemma 8 *Let* $(\mathbf{x},\mathbf{s})\in\mathcal{W}(\tau,\beta,\kappa)$, $\alpha_1=\frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$ *and* $\alpha_2=1$. *Then*

$$\mathbf{v}(\alpha)>\frac{1}{2}\mathbf{e}.$$

Proof. Using Lemma 5 and Corollary 3, we obtain

$$\mathbf{v}^2(\alpha)=\frac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\tau\mu(\alpha)}\geq 1-\frac{\beta+\beta^2}{4(1+4\kappa)}\geq 1-\frac{\beta+\beta^2}{4}.$$

To prove the statement, it is enough to show that

$$1-\frac{\beta+\beta^2}{4}>\frac{1}{4},$$

which is satisfied for all $\beta\in[0,1]$. ∎
Finally, we need to show that $\|\mathbf{p}(\alpha)^+\|\leq\frac{\beta}{1+4\kappa}$ holds. To be able to prove this, we need the following technical lemma:

Lemma 9 *Let* $\alpha_1 = \frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$ *and* $(\mathbf{x},\mathbf{s}) \in \mathcal{W}(\tau,\beta,\kappa)$. *Then*

$$\|(\tau\mu(\alpha)\mathbf{e} - \mathbf{h}(\alpha))^+\| \leq \frac{\beta}{1+4\kappa}\tau\mu(\alpha)\,(1-\alpha_2)\,.$$

Proof. By Lemma 3, we have $\tau\mu(\alpha) - h_i(\alpha) \leq 0$ for all $i \in \mathcal{I}_-$, therefore we need to consider indices only from $\mathcal{I}_+$.
From (7), it follows that

$$1 - v_i^2 \leq p_i v_i \text{ for all } i \in \mathcal{I}_+. \tag{13}$$

Using Corollary 3 and (13), we obtain

$$\begin{aligned}\tau\mu(\alpha) - h_i(\alpha) = \tau\mu(\alpha) - \tau\mu\left(v_i^2 + \alpha_2 v_i p_i\right) \leq \tau\mu(\alpha)\left(1 - v_i^2 - \alpha_2 v_i p_i\right) \leq \\ \leq \tau\mu(\alpha)p_i v_i(1-\alpha_2) \leq \tau\mu(\alpha)p_i\,(1-\alpha_2)\,.\end{aligned}$$

This, together with the definition of $\mathcal{W}(\tau,\beta,\kappa)$ yields

$$\|(\tau\mu(\alpha)\mathbf{e} - \mathbf{h}(\alpha))^+\| \leq \tau\mu(\alpha)\|\mathbf{p}^+\|\,(1-\alpha_2) \leq \frac{\beta}{1+4\kappa}\tau\mu(\alpha)\,(1-\alpha_2)\,.$$

■

This result shows that if we fix the value of α_2 as 1, then $\tau\mu(\alpha) - h_i(\alpha) \leq 0$ holds for the indices from $\mathcal{I}_+$ as well, i.e., $\|(\tau\mu(\alpha)\mathbf{e} - \mathbf{h}(\alpha))^+\| = 0$ in this case. This also follows from (9).

Lemma 10 *Let* $\beta = \frac{1}{4}$, $\tau = \frac{1}{4}$. *If* $\alpha_1 = \frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$, $\alpha_2 = 1$ *and* $(\mathbf{x},\mathbf{s}) \in \mathcal{W}(\tau,\beta,\kappa)$, *then the new point* $(\mathbf{x}(\alpha),\mathbf{s}(\alpha)) \in \mathcal{W}(\tau,\beta,\kappa)$.

Proof. We need to prove that

$$\left\|\mathbf{p}(\alpha)^+\right\| = \left\|\left(\frac{2\mathbf{v}(\alpha)(\mathbf{e}-\mathbf{v}(\alpha))}{2\mathbf{v}(\alpha)-\mathbf{e}}\right)^+\right\| \leq \frac{\beta}{1+4\kappa}.$$

We can give an upper bound on the norm the following way:

$$\begin{aligned}\left\|\mathbf{p}(\alpha)^+\right\| = \left\|\left(\frac{2\mathbf{v}(\alpha)(\mathbf{e}-\mathbf{v}(\alpha))}{2\mathbf{v}(\alpha)-\mathbf{e}}\right)^+\right\| = \left\|\frac{2\mathbf{v}(\alpha)}{(2\mathbf{v}(\alpha)-\mathbf{e})\,(\mathbf{e}+\mathbf{v}(\alpha))}\left(\mathbf{e}-\mathbf{v}^2(\alpha)\right)^+\right\| \leq \\ \leq \left\|\frac{2\mathbf{v}(\alpha)}{2\mathbf{v}^2(\alpha)+\mathbf{v}(\alpha)-\mathbf{e}}\right\|_\infty \left\|\left(\mathbf{e}-\mathbf{v}^2(\alpha)\right)^+\right\|.\end{aligned} \tag{14}$$

Let $q : \left(\frac{1}{2},\infty\right) \to \mathbb{R}$ and $q(t) = \frac{2t}{2t^2+t-1}$. This function is strictly decreasing on its domain, therefore using the discussed lower bound on $\mathbf{v}(\alpha)$ and substituting the values of β and τ, the first term in (14) can be estimated as

$$\left\|\frac{2\mathbf{v}(\alpha)}{2\mathbf{v}^2(\alpha)+\mathbf{v}(\alpha)-\mathbf{e}}\right\|_\infty \leq q\left(\sqrt{1-\frac{\beta+\beta^2}{4}}\right) < 1.065. \tag{15}$$

For the other term, we use Corollary 2 and Lemma 9:

$$\begin{aligned}&\left\|\left(\mathbf{e}-\mathbf{v}^2(\alpha)\right)^+\right\| = \frac{1}{\tau\mu(\alpha)}\left\|(\tau\mu(\alpha)\mathbf{e} - \mathbf{x}(\alpha)\mathbf{s}(\alpha))^+\right\| \leq \\ &\leq \frac{1}{\tau\mu(\alpha)}\left(\left\|(\tau\mu(\alpha)\mathbf{e} - \mathbf{h}(\alpha))^+\right\| + \tau\mu\left\|(\mathbf{dx}(\alpha)\mathbf{ds}(\alpha))^-\right\|\right) \leq \\ &\leq \frac{1}{\tau\mu(\alpha)}\left[\frac{\beta}{1+4\kappa}\tau\mu(\alpha)\,(1-\alpha_2) + \tau\mu\left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\alpha_1^2\frac{n}{\tau} + \alpha_2^2\frac{\beta^2}{(1+4\kappa)^2}\right)\right] = \\ &= \frac{\beta}{1+4\kappa}\,(1-\alpha_2) + \frac{\mu}{\mu(\alpha)}\left(\frac{1}{\sqrt{8}}+\kappa\right)\left(\alpha_1^2\frac{n}{\tau} + \alpha_2^2\frac{\beta^2}{(1+4\kappa)^2}\right).\end{aligned} \tag{16}$$

From Lemma 7, we have

$$\frac{\mu}{\mu(\alpha)} \leq \frac{1}{1-\alpha_1-\kappa\tau\left(\frac{\alpha_1^2}{\tau} + \frac{\alpha_2^2\beta^2}{n(1+4\kappa)^2}\right)} \leq \frac{1}{1-\frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}} - \frac{2\beta\tau\kappa}{n(1+4\kappa)^2}} \leq \frac{1}{1-\sqrt{\beta\tau}-\frac{1}{8}\beta\tau},$$

since

$$\frac{\kappa}{(1+4\kappa)^2} \leq \frac{1}{16}$$

for all κ values. Using the previous estimation and substituting $\alpha_2 = 1$, from (16) we obtain

$$\left\|\left(\mathbf{e} - \mathbf{v}^2(\alpha)\right)^+\right\| \leq \frac{\beta}{1+4\kappa}\frac{1}{1-\sqrt{\beta\tau}-\frac{1}{8}\beta\tau}\frac{1+4\kappa}{\sqrt{8}}\frac{1}{1+4\kappa}(1+\beta) =$$
$$= \frac{\beta}{1+4\kappa}\frac{1}{1-\sqrt{\beta\tau}-\frac{1}{8}\beta\tau}\frac{1+\beta}{\sqrt{8}}.$$

Substituting $\beta = \frac{1}{4}$ and $\tau = \frac{1}{4}$, we have

$$\frac{1}{1-\sqrt{\beta\tau}-\frac{1}{8}\beta\tau}\frac{1+\beta}{\sqrt{8}} \approx 0.5955 < 0.6. \tag{17}$$

Using (15) and (17), we get

$$\left\|\mathbf{p}(\alpha)^+\right\| < 1.065 \cdot 0.6\frac{\beta}{1+4\kappa} < \frac{\beta}{1+4\kappa},$$

therefore the new point is in the neighbourhood $\mathcal{W}(\beta, \tau, \kappa)$. ■

5.3 The complexity of the new algorithm

Theorem 2 *Let $\beta = \tau = \frac{1}{4}$, $\alpha_1 = \frac{1}{1+4\kappa}\sqrt{\frac{\beta\tau}{n}}$, $\alpha_2 = 1$, and suppose that a starting point $(\mathbf{x}_0, \mathbf{s}_0) \in \mathcal{W}(\tau, \beta, \kappa)$ is given. Then the algorithm provides an ε-optimal solution in*

$$O\left((1+4\kappa)\sqrt{n}\log\frac{\mathbf{x}_0^T\mathbf{s}_0}{\varepsilon}\right)$$

iterations.

Proof. According to Lemma 6, the following holds for the duality gap in the kth iteration:

$$\frac{\mathbf{x}_k^T\mathbf{s}_k}{n} = \mu_k \leq \mu_{k-1}\left(1-\alpha_1\left[\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right]\right) \leq \mu_0\left(1-\alpha_1\left[\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right]\right)^k.$$

By rearranging, we get

$$\mathbf{x}_k^T\mathbf{s}_k \leq \left(1-\alpha_1\left[\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right]\right)^k \mu_0 n.$$

Therefore $\mathbf{x}_k^T\mathbf{s}_k \leq \varepsilon$ holds if

$$\left(1-\alpha_1\left[\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right]\right)^k \mu_0 n \leq \varepsilon$$

is satisfied.
Taking the natural logarithm of both sides yields

$$k\log\left[1-\alpha_1\left(\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right)\right] + \log(\mu_0 n) \leq \log\varepsilon.$$

Using the inequality $-\log(1-\vartheta) \geq \vartheta$, it is enough to prove that

$$-k\alpha_1\left(\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right) + \log(\mu_0 n) \leq \log\varepsilon.$$

The last inequality is satisfied when

$$k \geq (1+4\kappa)\sqrt{\frac{n}{\beta\tau}}\frac{1}{\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}}\log\left(\frac{\mathbf{x}_0^T\mathbf{s}_0}{\varepsilon}\right),$$

and this proves the statement. ■

6 Numerical results

Since usually the handicap of the coefficient matrix is not known in advance (and the only algorithm that has been introduced to determine it has exponential running time [39]), in most cases the theoretical algorithms for solving sufficient linear complementarity problems cannot be implemented directly. In our case, the κ-dependency of the neighbourhood also raises some important questions about the implementation.
Because of the above-mentioned reasons, for the numerical tests, we implemented a greedy variant of our algorithm:

Input: a matrix $M \in \mathbb{R}^{n\times n}$, and $\mathbf{q} \in \mathbb{R}^n$
an update parameter $0 < \tau < 1$, a neighbourhood parameter $0 < \beta < 1$,
an accuracy parameter $\varepsilon > 0$,
an initial point $(\mathbf{x}_0, \mathbf{s_0}) \in \mathcal{W}(\tau, \beta, 0)$ with $\mu_0 = \frac{\mathbf{x}_0^T \mathbf{s}_0}{n}$.
$\mathbf{x} := \mathbf{x}_0$, $\mathbf{s} := \mathbf{s}_0$ and $\mu := \mu_0$
while $\mathbf{x}^T\mathbf{s} > \varepsilon$ **do**
  Determine $\Delta\mathbf{x}_+, \Delta\mathbf{s}_+$ and $\Delta\mathbf{x}_-, \Delta\mathbf{s}_-$ by solving systems (4);
  $\alpha_2 := 1$;
  Let α_1 be the largest value such that $(\mathbf{x}(\alpha), \mathbf{s}(\alpha)) \in \mathcal{W}(\tau, \beta, 0)$ and the duality gap does not increase;
  $\mathbf{x} := \mathbf{x}(\alpha)$;
  $\mathbf{s} := \mathbf{s}(\alpha)$;
  $\mu := \frac{\mathbf{x}^T\mathbf{s}}{n}$;
end

Algorithm 2: Pseudo-code of the greedy variant

As it can be seen from the pseudo-code, in this case, we ignore the value of the handicap and take the largest step so that the new iterates remain in the neighbourhood $\mathcal{W}(\tau, \beta, 0)$. For safety reasons we also check whether the duality gap actually decreases after an iteration (it is known from the theory of sufficient LCPs that the duality gap is not monotonically decreasing in the value of α_1, and we take a larger step than the one that we proved the convergence for). During our numerical tests, this latter condition was never restrictive, i.e., the step-length was always determined by the constraint on the neighbourhood.
We tested our method for both sufficient and non-sufficient LCPs, even though in the second case we have no theoretical proof that interior point methods work (in general it is not necessarily true that the central path exists and it is unique).
The sufficient matrices that we used are the following:

- ENM_SU: 82 matrices were constructed by E.-Nagy, from size 3×3 to size 10×10 [16].
- MGS_SU: 58 matrices were generated by Morapitiye and Illés, from size 10×10 to size 700×700 [23].
- Lower triangular $\mathcal{P}$-matrices (all of their principal minors are positive) introduced by Csizmadia:

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}$$

It was shown by E.-Nagy that the handicap is exponential in the size of the matrix, $\kappa = 2^{2n-8} - 0.25$ [17].
The examined non-sufficient matrices were also collected on webpage [16]:

- ENM_NSU: 80 instances, from size 3×3 to size 10×10.

We calculated the right-hand sides with the following formula:

$$-M\mathbf{e} + \mathbf{e} = \mathbf{q},$$

therefore $\mathbf{x}_0 = \mathbf{e}$ and $\mathbf{s}_0 = \mathbf{e}$ are feasible starting vectors (and are in the neighbourhood $\mathcal{W}(\tau, \beta, 0)$).
For the numerical tests, we used the settings $\beta = \tau = 0.25$ and $\varepsilon = 10^{-5}$. Table 1 shows our numerical results for the different sets of test problems. These are average values, except for Csizmadia-matrices.

Table 1: Numerical results with $\mathbf{q} = -M\mathbf{e} + \mathbf{e}$, $\mathbf{x}_0 = \mathbf{e}$ and $\mathbf{s}_0 = \mathbf{e}$

n	Iteration	Time (s)
ENM_SU matrices		
3	5.80	0.0117
4	5.90	0.0139
5	6.00	0.0160
6	6.10	0.0163
7	6.30	0.0147
8	6.90	0.0125
9	6.73	0.0117
10	7.82	0.0147
MGS_SU matrices		
10	7.20	0.0134
20	7.90	0.0215
50	6.80	0.0350
100	7.30	0.0215
200	8.20	0.0498
500	9.20	0.1945
700	10.00	0.4545

n	Iteration	Time (s)
ENM_NSU matrices		
3	6.20	0.0113
4	6.60	0.0121
5	6.10	0.0109
6	6.10	0.0109
7	6.30	0.0116
8	6.60	0.0127
9	7.10	0.0112
10	7.75	0.0129
Csizmadia-matrices		
10	12	0.0156
20	15	0.0155
30	19	0.0191
40	23	0.0250
50	27	0.0320
100	47	0.0960
150	66	0.1258

The sufficient LCPs determined by using the ENM_SU and MGS_SU matrices could be solved easily, the running time was less than 1 second even for the largest, 700×700 problem instance.

We could only solve problem instances with Csizmadia-matrices up to the size 150×150, and the number of iterations is larger than the average calculated for the MGS_SU instances of similar size. In the case of the 200×200 problem instance, the step-length α_1 at the first iteration is too small and cannot be handled numerically.

To understand this behavior better, we resolved the same problems ($\mathbf{q} = -M\mathbf{e} + \mathbf{e}$) using different starting points. Let $\mathbf{x}_0 = \lambda\mathbf{e}$ ($\lambda \in (0, 1]$), and $\mathbf{s}_0 = \mathbf{q} + M\mathbf{x}_0$, namely $(s_0)_i = 1 + (i-2) \cdot (1-\lambda)$ for all $i \in \{1, \ldots, n\}$. For $\lambda = 1$, we get back the case $\mathbf{x}_0 = \mathbf{s}_0 = \mathbf{e}$.

Table 2: Results for the Csizmadia-instances with modified starting points

	$\lambda = 0.97$		$\lambda = 0.99$		$\lambda = 1$	
n	**Iter.**	**Time (s)**	**Iter.**	**Time (s)**	**Iter.**	**Time (s)**
10	11	0.9467	11	0.029	12	0.0156
20	14	0.0389	15	0.0163	15	0.0155
30	15	0.0238	17	0.0246	19	0.0191
40	17	0.0225	20	0.0434	23	0.0250
50	19	0.0264	23	0.041	27	0.0320
100	27	0.0959	35	0.0617	47	0.0960
150	31	0.1047	45	0.1288	66	0.1258
200	-	-	53	0.1675	-	-
250	Not in $\mathcal{W}(\tau, \beta, 0)$		62	0.1942	-	-

As it can be observed from Table 2, for the smaller problem instances the required number of iterations and the running time decrease as we decrease the value of λ. However, it is not possible to choose arbitrarily small values for λ if we want to have a special starting point in the neighbourhood $\mathcal{W}(\tau, \beta, 0)$. Therefore for larger problems, this approach is impractical but shows that the frequently applied starting point $\mathbf{x}_0 = \mathbf{s}_0 = \mathbf{e}$ may not be the best choice for the problem instances generated using the Csizmadia-matrices.

We also examined another set of LCPs using the Csizmadia-matrices, but in this case, we modified not just the starting points but the right-hand side vector $\mathbf{q}$ as well. Let $\mathbf{x}_0 = \mathbf{e}$ and $\mathbf{s}_0 = \eta\mathbf{e}$ ($\eta \geq 1$), and $\mathbf{q} = -M\mathbf{e} + \eta\mathbf{e}$. For $\eta = 1$ we get back our original case $\mathbf{x}_0 = \mathbf{s}_0 = \mathbf{e}$. Since $\mathbf{q} \geq \mathbf{0}$ holds for the modified LCPs as well, their solution is still $\mathbf{x} = \mathbf{0}$.

Table 3: Results for the LCPs generated using the Csizmadia-matrices, with modified right-hand side vectors

	$\eta = 1$		$\eta = 10$		$\eta = 50$		$\eta = 100$	
n	**Iter.**	**Time (s)**	**Iter.**	**Time (s)**	**Iter.**	**Time (s)**	**Iter.**	**Time (s)**
10	12	0.0156	8	0.9576	8	0.0269	8	0.0266
20	15	0.0155	9	0.0362	9	0.0124	9	0.0136
30	19	0.0191	10	0.0172	9	0.0126	9	0.0152
40	23	0.0250	11	0.0156	9	0.0145	9	0.0192
50	27	0.0320	11	0.0187	9	0.0167	9	0.01716
100	47	0.0960	14	0.0877	11	0.0335	10	0.0273
150	66	0.1258	17	0.0792	12	0.032	11	0.0304
200	-	-	20	0.1319	12	0.0394	12	0.0416
250	-	-	23	0.0759	13	0.0511	12	0.0491
300	-	-	24	0.1364	13	0.0855	12	0.0727
400	-	-	31	0.3055	13	0.1217	13	0.1189
500	-	-	37	0.458	16	0.2221	14	0.1851
600	-	-	43	0.8769	17	0.3631	14	0.3031
700	-	-	50	1.3263	18	0.5291	15	0.4418
1000	-	-	-	-	21	1.3917	17	1.0954
1500	-	-	-	-	30	4.5627	20	3.2659

As it can be seen from Table 3, by increasing η it is possible to solve significantly larger problem instances. The reason for failure is always the too small initial step-length α_1.

Even though our analysis only works for sufficient LCPs, we also tested the algorithm for non-sufficient problem instances. Surprisingly, we could solve almost all problems correctly and the behaviour of the algorithm was quite similar to the sufficient case. There were only two problematic instances out of the 80 (ENM_NSU_10_07 and ENM_NSU_10_08), the LCPs generated using these matrices could not be solved by our method. (Here also, the step-length α_1 at the first iteration was too small and could not be handled numerically). The results are shown in the third part of Table 1.

7 Observations regarding the coordinates of the vector v

To be able to further examine the behaviour of our theoretical algorithm, we calculated the handicap for some of the smaller test matrices. This way we could run the algorithm exactly as it is described in our analysis and make some important observations that raise interesting questions regarding the theoretical analysis.

We used the parameter settings $\beta = \tau = 0.25$ and $\varepsilon = 10^{-5}$. First we used the starting points $\mathbf{x}_0 = \mathbf{e}$, $\mathbf{s}_0 = \mathbf{e}$ (and calculated $\mathbf{q}$ as $\mathbf{q} = -M\mathbf{e}+\mathbf{e}$). The numerical results are summarized in Table 4, and Figure 1 shows the change in the coordinates of the vector $\mathbf{v}$ during the iterations. As expected, the numbers of iterations are significantly larger than in the greedy case, and they depend on the value of the handicap (since the step-length depends on the handicap as well).

In the case of the Csizmadia-matrix, Figure 1 shows that one of the coordinates (the first one) converges remarkably slower than the others, due to the properties of the Newton directions. This is the main reason why we experienced numerical issues with these starting points.

Table 4: Numerical results obtained with the theoretical algorithm ($\mathbf{x}_0 = \mathbf{e}$ and $\mathbf{s}_0 = \mathbf{e}$)

	Handicap	$\mathbf{v}_{\min}$	$\mathbf{v}_{\max}$	**Iterations**	**Time (s)**
ENM_SU_7_01	1.07	1.9993	2.0007	1121	0.0346
ENM_SU_7_02	2.63	1.9989	2.0005	2454	0.0565
ENM_SU_7_03	1.42	1.9993	2.0007	1420	0.0358
ENM_SU_7_04	2.58	1.9993	2.0006	2411	0.0751
ENM_SU_7_05	1.13	1.9963	2.0033	1172	0.1218
Csizmadia_5	3.75	1.9946	2.0038	2809	0.0521
Csizmadia_6	15.75	1.9981	2.0012	12506	0.1309
Csizmadia_7	63.75	1.9993	2.0004	54686	0.4528

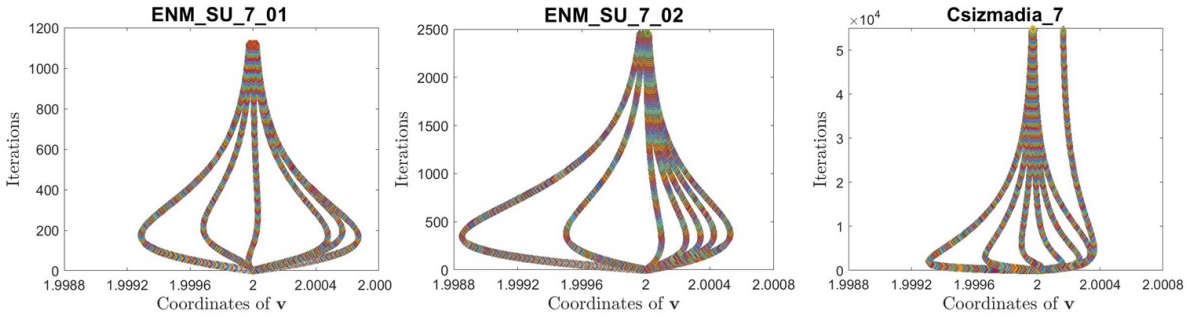

Fig. 1: Coordinates of $\mathbf{v}$ with the theoretical step-length ($\mathbf{x}_0 = \mathbf{e}$ and $\mathbf{s}_0 = \mathbf{e}$)

With these starting points, at the beginning $v_i = \sqrt{\frac{1}{\tau}} = 2$ holds for all coordinates of the vector $\mathbf{v}$. It can be seen from Table 4 and Figure 1 that the coordinates remain in a really narrow interval around 2. However, in our analysis we use the upper bound $v_i \leq \sqrt{\frac{n}{\tau}}$, according to Corollary 1. The value of this upper bound for $n = 7$ and $\tau = 0.25$ is $\sqrt{28} \approx 5.2915$, which is significantly larger than what we experienced in practice, even in this small dimension.

Furthermore, all coordinates of $\mathbf{v}$ are greater than 1, therefore all iterates remain in the narrower neighbourhood $\mathcal{N}_\infty^-$, and never actually get to a point from the set $\mathcal{W}(\beta, \tau, \kappa) \setminus \mathcal{N}_\infty^-$, i.e., in practice the algorithm works in a κ-independent neighbourhood when the starting points are well-centered.

We observed the same phenomenon for linear programming problems while preparing the numerical tests for our recent paper [18], where we had test problems with several thousands of variables. There we applied the self-dual embedding technique, therefore we could use the starting point $\mathbf{x}_0 = \mathbf{e}$, $\mathbf{s}_0 = \mathbf{e}$, similarly to the LCP case examined in this paper, i.e. the starting points were well-centered in both cases. Based on these numerical tests, the size of this interval around $1/\sqrt{\tau}$ seems to be independent of the problem size.

This raises the question whether it would be possible to give constant lower and upper bounds on the coordinates of $\mathbf{v}$, assuming that the starting point is well-centered, i.e. to show that the algorithm is convergent and has the desired complexity using the neighbourhood

$$\mathcal{N}_{\mathbf{v}}(\underline{\nu}, \bar{\nu}) = \{(\mathbf{x}, \mathbf{s}) \in \mathcal{F}^+ : \underline{\nu}\mathbf{e} \leq \mathbf{v} \leq \bar{\nu}\mathbf{e}\},$$

where $1 \leq \underline{\nu} \leq 1/\sqrt{\tau} \leq \bar{\nu}$ are given parameters.

The coordinates of $\mathbf{v}$ for the 5×5 Csizmadia-problems with three different (not well-centered) starting points are shown in Figure 2. We kept the right-hand side as $\mathbf{q} = -M\mathbf{e}+\mathbf{e}$. The starting points were $\mathbf{x}_0 = 0.9\mathbf{e}$ and $\mathbf{s}_0 = [0.9, 1, 1.1, 1.2, 1.3]^T$, $\mathbf{x}_0 = [0.8, 0.6, 0.5, 0.5, 0.6731]^T$ and $\mathbf{s}_0 = [0.8, 0.8, 1.1, 1.6, 2.2731]^T$, $\mathbf{x}_0 = [1.7, 1.7^2, 1.7^3, 1.7^4, 1.7^5]^T$ and $\mathbf{s}_0 = [1.700, 2.1900, 2.3230, 1.8491, 0.3435]^T$, respectively. Here the intervals around 2 become narrower as the algorithm proceeds, and the coordinates are concentrated around this value in the end.

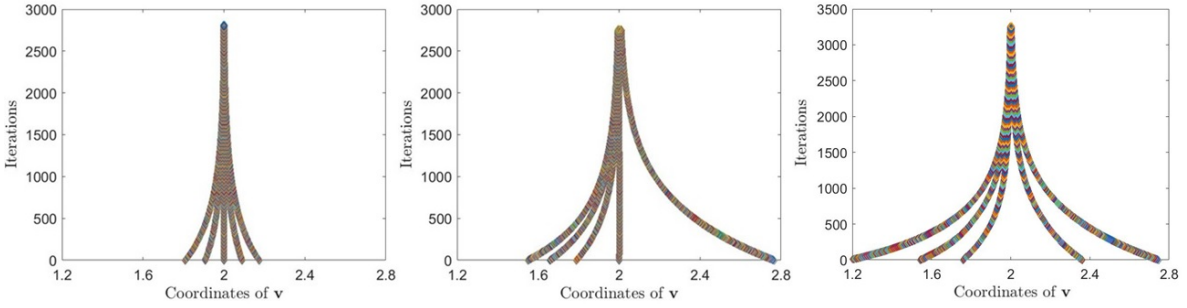

Fig. 2: Coordinates of $\mathbf{v}$ with the theoretical step-length for not well-centered starting points, solving the 5×5 Csizmadia-problem

8 Conclusion

We introduced a new Ai-Zhang type long-step interior point method for $\mathcal{P}_*(\kappa)$ LCPs. The new IPA uses the AET technique with the function $\varphi(t) = t - \sqrt{t}$. We proved that the method is convergent and has the best known iteration complexity.
An interesting question regarding the analysis is whether it would be possible to carry out the analysis in a κ-independent neighbourhood. In his 2014 paper [37], Potra proposed an Ai-Zhang type method for horizontal LCPs. He did not apply the AET method, i.e. in his case φ can be considered as the identity function. In that case, the convergence and best known complexity of the method could be proved using the original, κ-independent neighbourhood of Ai and Zhang. For our method, with the current estimations, the convergence cannot be proved, assuming that the neighbourhood does not depend on the handicap.
Furthermore, we implemented the greedy variant of the IPA in Matlab and tested it on both sufficient and non-sufficient problem instances. The method was very effective on most test problems.
We also run the theoretical variant of the algorithm for some smaller problem instances and investigated the change in the coordinates of the vector $\mathbf{v}$. We found that when the starting points are well-centered, the coordinates remain in a really narrow interval around $\sqrt{1/\tau}$. This phenomenon raises some interesting theoretical questions that we would like to investigate in the future.

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