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**MATRIX-VARIATE, VECTOR-VARIATE AND UNIVARIATE
RISK MEASURES AND RELATED ASPECTS**

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Abstract

Usually, risk measures are functions of a set of real random variables to the real numbers. However, it is often insufficient to consider a single real-valued measure to quantify the risks derived from different economic and financial activities. In the last decade, many extensions of vector-valued risk measures have been investigated. In the last decade, many extensions of vector-valued risk measures have been investigated, see Embrechts and Puccetti [2006], Cousin and Di Bernardino [2013], Torres et al. [2015]. However, as mentioned in Li et al. [2012] some of the univariate transcripts are unrealistic and are based on assumptions that are difficult to elucidate. Probably the most widely used risk measures in economics, insurance, and finance are the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR). The objective of this thesis is to propose new methodologies to quantify VaR and CVaR from an uni-variate, vector-variate and matrix-variate approaches.

In the first chapter of this thesis, a new approach is proposed to model vector-valued risk measures under the Wasserstein barycenter of probability measures. A crucial aspect that underlies here for the new method is that the Wasserstein measure-barycenter remains invariant under the location and scale families, so it is possible to propose exact formulas for the Wasserstein Barycenter VaR and the Wasserstein Barycenter Conditional CVaR. The new method considers a reliable risk measure based on distances among probabilistic models. The underlying suitable probability laws obey, for example, opinions, beliefs, and estimates of data sources, in the context of the financial risk. Explicitly, a concept in probability theory is brought into the financial models by proposing the named Fréchet measures; which are calibrated by certain metrization of the probability measure space. In this case, the well-studied metric of Wasserstein supports the method and provides fundamental connections for the rising concept of barycenter in the sense of Agueh and Carlier in Agueh and Carlier [2011]. Simple and advanced multivariate VaR models are compared with the proposed model. The performance of the model is also checked in the major U.S. stock indices during the COVID-19 pandemic. The introduced model behaves satisfactorily in both common and volatile periods of asset prices, providing a realistic VaR forecast in this era of social distancing.

If we search for a matrix-variate extension for risk measures, the finance literature does not provide us with any approaches. However, from a mathematical point of view, risk measures just requires meaningful percentiles in the context of matrix cumulative density functions. The theory behind the random matrix setting has been deep studied by Muirhead [2005]. In particular, that paper provided a formulation for calculating $P(\mathbf{X} \leq \mathbf{V})$ and $P(\mathbf{X} \geq \mathbf{V})$ when \mathbf{X} follows a Wishart distribution and V is a positive definite matrix. They also demonstrated that its cumulative distribution function can be expressed in terms of a Gaussian hypergeometric function of matrix argument. Based on this theory, in chapter 2 a method is developed to estimate the VaR and the CVaR when the risk factors follow a beta distribution in an univariate and a matrix-variate approach. For this purpose, we connect matrix argument theory of

hypergeometric functions and integration over positive definite matrices. The upper matrix VaR and the lower matrix VaR are defined, which are obtained as the zeros of the Gaussian hypergeometric function. Both extensions are shown to satisfy the properties of monotonicity, positive homogeneity, and translational invariance. Analytical expressions are developed for certain shape parameters, in addition, a numerical solution is presented for any value of said parameters. The proposed risk measures are finally used to quantify the economic loss in credit risk.

Chapter 3, proposes generalized integrals related to the classical Wishart, beta, and F distributions. Then the work defines the termed generalized matrix variate beta and F distributions and the VaR in the matrix setting. As corollaries, a number of published results about cumulative density functions (c.d.f) of Wishart and beta matrices are also revisited and unified. A new c.d.f for a Wishart random matrix and a solution to an open problem proposed by A. C. Constantine in 1963 are also provided. The extreme latent root distributions for Wishart, Beta, and F are obtained by simple derivation. Relations with the Davis' condition number, theory of shape, and VaR are also established; some particular cases are derived and a perspective for future work is set in that novel direction. VaR is provided for gamma, exponential, Erlang, chi-square, beta and uniform distributions for the univariate case and VaR for Wishart, gamma, beta and F distributions for the matrix case. Furthermore, we establish useful results for the upper VaR and the lower matrix VaR and obtain closed expressions when $\mathbf{X} \sim Beta_m(a, \frac{m+1}{2})$ and when $\mathbf{X} \sim W_2(n, \mathbf{I})$.

Keywords: Value-at-Risk; Conditional Value-at-Risk; Wasserstein Barycenter; Location-scatter family; Hypergeometric function of matrix argument; Positive definite matrices; Generalized Wishart; beta and F distribution; Quantil; James' Zonal polynomials.

Résumé

Généralement, les mesures de risque sont considérées comme des mappings d'un ensemble de variables aléatoires réelles vers des nombres réels. Cependant, il est souvent insuffisant de considérer une seule mesure réelle pour quantifier les risques découlant des activités financières. Au cours de la dernière décennie, de nombreuses extensions de la Valeur à risque multivariée ont été étudiées et certains articles proposent des méthodes alternatives de mesure du risque pour les portefeuilles multivariés. Toutefois, comme le mentionne Li et al. [2012], certaines des traductions univariées sont devenues irréalistes et reposent sur des hypothèses inappropriées qui, dans le contexte des mesures de risque, sont difficiles à élucider. Les mesures de risque les plus utilisées en économie, en assurance et en finance sont probablement la valeur à risque (VaR) et la valeur à risque conditionnelle (CVaR). L'objectif de cette thèse est de proposer de nouvelles méthodologies pour quantifier la VaR et la CVaR à partir d'une approche vecteur-variable et matrice-variable.

Dans le premier chapitre de la thèse, une nouvelle approche pour modéliser les mesures de risque vecteur-variable sous le barycentre de Wasserstein des mesures de probabilité est proposée. Un aspect crucial sous-jacent ici pour la nouvelle méthode est que le barycentre de Wasserstein des mesures reste invariant sous les distributions de localisation et d'échelle, il est donc possible de proposer des formules exactes pour le barycentre de Wasserstein de la VaR et de la CVaR. Explicitement, un concept de la théorie des probabilités est incorporé aux modèles financiers en proposant des mesures de Fréchet, qui sont calibrées par une certaine métaréalisation de l'espace des mesures de probabilité. Dans ce cas, la métrique de Wasserstein soutient la méthode et fournit des connexions fondamentales avec le concept émergent de barycentre au sens d'Agueh et Carlier dans Agueh and Carlier [2011]. Le modèle proposé est comparé à d'autres modèles simples et avancés, et ses performances sont vérifiées sur les principaux indices boursiers américains, pendant la pandémie de COVID-19. Le modèle introduit fonctionne de manière satisfaisante dans les périodes de prix d'actifs communs et volatils, fournissant une prévision réaliste de la VaR dans cette ère de distanciation sociale.

Maintenant, lorsque nous cherchons une extension matrice-variable de la VaR, la littérature financière ne fournit aucune approche. Cependant, d'un point de vue mathématique, la VaR ne requiert des percentiles significatifs que dans le contexte des fonctions de densité cumulative matricielle. La théorie des distributions matrice-variable est étudiée en profondeur dans Muirhead [2005]. En particulier, des formules sont fournies pour calculer $P(\mathbf{X} \leq \mathbf{V})$ et $P(\mathbf{X} \geq \mathbf{V})$ lorsque \mathbf{X} suit une distribution de Wishart et \mathbf{V} est une matrice définie positive et il a été démontré que la fonction de distribution cumulative peut être exprimée en termes de fonction hypergéométrique gaussienne. Sur la base de cette théorie, nous développons au chapitre 2 une méthode d'estimation de la valeur à risque et de la valeur à risque conditionnelle lorsque les facteurs de risque suivent une distribution bêta dans un environnement univarié et matriciel-varié. Dans ce but, nous connectons la théorie des fonctions hypergéométriques à argument

matriciel et l'intégration sur les matrices définies positives. Nous définissons la matrice supérieure VaR et la matrice inférieure VaR, qui sont obtenues comme les zéros de la fonction hypergéométrique gaussienne. On montre que les deux extensions satisfont aux propriétés de monotonie, d'homogénéité positive et d'invariance par translation. Des expressions analytiques sont développées pour certains paramètres de forme, et une solution numérique est présentée pour toute valeur de ces paramètres. Les mesures de risque proposées sont finalement utilisées pour quantifier la perte économique dans le risque de crédit.

Le chapitre 3 propose des intégrales généralisées liées aux distributions classiques de Wishart, bêta et F. Ensuite, l'article définit les distributions matrice-variable bêta et F généralisées et la matrice-variable VaR. Comme corollaires, un certain nombre de résultats publiés sur les fonctions de densité cumulative (FDC) des matrices de Wishart et bêta sont également examinés et unifiés. Un nouveau c.d.f. pour une matrice aléatoire de Wishart et la solution à un problème ouvert proposé par A. C. Constantine en 1963. Les distributions extrêmes des racines latentes pour Wishart, beta et F sont obtenues par simple dérivation. Les relations avec le nombre de conditions de Davis, la théorie des formes et la VaR sont également établies ; certains cas particuliers sont dérivés et une perspective pour les travaux futurs dans cette nouvelle direction est établie. Nous fournissons la VaR pour les distributions gamma, exponentielle, Erlang, chi-carré, bêta et uniforme pour le cas univarié et la VaR pour les distributions Wishart, gamma, bêta et F pour le cas matriciel. En outre, nous établissons des résultats utiles pour la VaR supérieure et la VaR inférieure de la matrice et obtenons des expressions fermées lorsque $\mathbf{X} \sim Beta_m(a, \frac{m+1}{2})$ y cuando $\mathbf{X} \sim W_2(n, \mathbf{I})$.

Mots-clé: Valeur à risque, valeur à risque conditionnelle, barycentre de Wasserstein ; famille de localisation et d'échelle ; fonction hypergéométrique ; matrices définies positives ; distributions généralisées de Whishart, bêta et F ; quantile ; polynômes zonaux de James.

Resumen

Usualmente, las medidas de riesgo son funciones de un conjunto de variables aleatorias reales a los números reales. Sin embargo, a menudo es insuficiente considerar una única medida real-variada para cuantificar los riesgos derivados de las diferentes actividades económicas y financieras. En la última década, se han investigado muchas extensiones de las medidas de riesgo vector-variadas ver por ejemplo Embrechts and Puccetti [2006], Cousin and Di Bernardino [2013], Torres et al. [2015]. Sin embargo, como se menciona en Li et al. [2012] algunas de las transcripciones univariadas son poco realistas y se basan en suposiciones que son difíciles de dilucidar. Probablemente, las medidas de riesgo más utilizadas en economía, seguros y finanzas son el Valor en Riesgo (VaR) y el Valor en Riesgo Condicional (CVaR). El objetivo de esta tesis es proponer nuevas metodologías para cuantificar el VaR y el CVaR desde un enfoque univariado, vector-variado y matriz-variado.

En el primer capítulo de la tesis, se propone un nuevo enfoque para modelar medidas de riesgo vector-variadas bajo el baricentro Wasserstein de medidas de probabilidad. Un aspecto crucial que subyace aquí para el nuevo método es que el baricentro de medidas de Wasserstein permanece invariante bajo las distribuciones de localización y escala, entonces es posible proponer fórmulas exactas para el VaR y el CVaR baricentro Wasserstein. Explícitamente, un concepto de la teoría de la probabilidad se incorpora a los modelos financieros al proponer las medidas de Fréchet; que se calibran mediante cierta metrización del espacio de medida de probabilidad. En este caso, la métrica de Wasserstein apoya el método y proporciona conexiones fundamentales para el concepto emergente de baricentro en el sentido de Agueh y Carlier en Agueh and Carlier [2011]. El modelo propuesto es comparado con otros modelos simples y avanzados, además su desempeño se verifica en los principales índices bursátiles de EE. UU, durante la pandemia de COVID-19. El modelo introducido se comporta satisfactoriamente tanto en periodos comunes como volátiles de precios de activos, proporcionando una previsión de VaR realista en esta era de distanciamiento social.

Ahora bien, cuando se busca una extensión matriz variada para la medidas de riesgo en mención, la literatura en finanzas no proporciona ningún enfoque. Sin embargo, desde el punto de vista matemático, dichas medidas de riesgo solo requiere percentiles significativos en el contexto de las funciones de densidad acumulada matricial. La teoría de distribuciones matriz variadas se estudia profundamente en Muirhead [2005]. En particular, se proporcionan fórmulas para calcular $P(\mathbf{X} \leq \mathbf{V})$ y $P(\mathbf{X} \geq \mathbf{V})$ cuando \mathbf{X} sigue una distribución de Wishart y \mathbf{V} es una matriz definida positiva y se demostró que la función de distribución acumulada se puede expresar en términos de una función hipergeométrica de Gauss. Apoyados en dicha teoría, en el capítulo 2 se desarrolla un método para estimar el VaR y el CVaR cuando los factores de riesgo siguen una distribución beta en un entorno univariado y matriz variado. Para este propósito, conectamos la teoría de funciones hipergeométricas de argumento matricial y la integración sobre matrices definidas positivas. Se definen el VaR matricial superior y el VaR matricial inferior, los cuales

se obtienen como los ceros de la función hipergeométrica gaussiana. Se muestra que ambas extensiones satisfacen las propiedades de monotonicidad, homogeneidad positiva e invariancia traslacional. Se desarrollan expresiones analíticas para ciertos parámetros de forma, además se presenta una solución numérica para cualquier valor de dichos parámetros. Las medidas de riesgo propuestas se utilizan finalmente para cuantificar la pérdida económica en riesgo de crédito.

En el capítulo 3 se propone integrales generalizadas relacionadas con las distribuciones clásicas Wishart, beta y F. Luego, el trabajo define las denominadas distribuciones matriz variadas beta y F generalizada y el VaR matricial. Como corolarios, también se revisan y unifican una serie de resultados publicados sobre las funciones de densidad acumulada (c.d.f) de las matrices Wishart y beta. También se proporciona una nueva c.d.f para una matriz aleatoria Wishart y la solución a un problema abierto propuesto por A. C. Constantine en 1963. Las distribuciones de raíces latentes extremas para Wishart, beta y F se obtienen por derivación simple. También se establecen relaciones con el número de condición de Davis, teoría de formas y el VaR; se derivan algunos casos particulares y se establece una perspectiva para el trabajo futuro en esa nueva dirección. Se proporciona el VaR para distribuciones gamma, exponencial, Erlang, chi-cuadrado, beta y uniforme para el caso univariado y el VaR para las distribuciones Wishart, gamma, beta y F para el caso matricial. Además, establecemos resultados útiles para el VaR superior y el VaR inferior matricial y obtenemos expresiones cerradas cuando $\mathbf{X} \sim Beta_m(a, \frac{m+1}{2})$ y cuando $\mathbf{X} \sim W_2(n, \mathbf{I})$.

Palabras clave: Valor en Riesgo, Valor en Riesgo Condicional, Baricentro de Wasserstein; familia de localización y escala; Función hipergeométrica de argumento matricial; matrices definidas positivas; distribuciones Whishart, beta y F generalizadas; cuantil; Polinomios Zonales de James.

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Dedicated to my best friend Juan Guillermo Murillo, one more dead due to Covid-19.

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Chapter 1

Introduction

Matrix variate distribution theory has occupied a central place in the last 70 years around robust applications in several disciplines. However, the sophisticated underlying mathematics and the computation problems of the distributions have restricted the popularization of the results, meanwhile the publications belong to a very small group of authors, compared with other fields of statistics.

The evolution of any theory under the random matrix setting can be traced by the usual matrix Gaussian extension of the real univariate and multivariate cases; the most profuse, elegant and deeply exposition of such enormous advances is given in Muirhead [2005]. Then the results appear in the complex case and sometimes the distributions are generalized for families of distributions of elliptical contours and real normed division algebras; see for example Gupta et al. [2013] and Díaz-García and Gutiérrez-Jáimez [2013], respectively, with the references therein.

The central case has ruled the distributions throughout the decades in terms of suitable polynomials of a positive definite matrix argument studied in a series of works of A.T. James during the 60's; see Muirhead [2005] for details of positive definite James' zonal polynomials. Single James' polynomials in the positive definite case were impossible to compute by 50 years; the Ph.D of Parkust listed the polynomials up the 12th order in the real case Parkhurst and James [1974]; a similar computation for the complex case was given by Caro-Lopera (Caro-Lopera and Nagar [2006], Gupta et al. [2006], Caro-Lopera et al. [2007] and the references therein).

At present, recurrence relations by using the Laplace-Beltrami operator allowed the numerical calculations of single polynomials of a positive definite matrix Koev and Edelman [2006], but infinite series of zonal polynomials still involve strong open problems. A matrix variate distribution approach based on positive semidefinite zonal polynomials is also feasible by extension of the Laplace-Beltrami operator Díaz-García and Caro-Lopera [2006], meanwhile exact formulae for James' polynomials, known as Jack's polynomials in real normed division algebras are only available for second order Caro-Lopera et al. [2007].

The non central distribution theory also appeared under the elliptical and real normed division algebras, but in most situations the distributions are untractable because they are expressed in terms of Davis' invariant polynomials of several matrix arguments (see for example Davis [1980]). The creator of the polynomials conjectured in Davis [1979], that the invariant polynomials could be computed in a similar way to the zonal polynomials; the conjecture was

sustained almost three decades in Davis [2006], but recently, Caro-Lopera [2016] proved the impossibility of constructing those polynomials in a similar recurrence way by using the Laplace-Beltrami operator. This leaves dozens of papers in Davis' polynomials out of any feasible application.

Only few non central cases of certain distributions expressed in terms of zonal polynomials can be really computed. The elliptical statistical shape theory is one of them, an applied approach explored by several transformations (SVD, QR, affine, polar) and real normed division algebras (real, complex, quaternion and octonion); see for example Caro-Lopera et al. [2010], Díaz-García and Caro-Lopera [2017], Díaz-García and Caro-Lopera [2016] and related works of the authors. Working with general families of distributions instead of the usual normal model allows flexible assumptions rather than the Gaussian based studies in shape theory (see for example Goodall and Mardia [1993], Dryden and Mardia [1998] and references therein). When the matrix variate setting is studied in the singular case, the above problems enlarges and the computations and applications are so far to be considered; a very reduced number of publications appear in that line, see for example Díaz-García and Gutiérrez-Jáimez [1997], Diaz-Garcia and Gutierrez-Sanchez [2013], Díaz-García and González-Farías [2008] and references therein.

Some extensions from univariate and multivariate cases into the matrix variate version takes decades and the proportion of the associated publications for the matrix studies were extremely unbalanced. It is the case of the matrix variate extension of the well known univariate Birnbaum-Saunders distribution of the 60's. Birnbaum and Saunders [1969] derived the univariate distribution and promoted a large number of research for more than 50 years. Recently, a detail review of theoretical and applied works was given by Balakrishnan and Kundu [2019]. The review described 281 existence reference of the distribution, but only 1 due to Caro-Lopera et al. [2012] addressed the matrix case. The matrix extension arrived to late, because it required first a connection between the Hadamard and matrix products. This arid branch of research accumulates 4 more works on the matrix case but from the same group of authors (Caro-Lopera and Díaz-García [2016], Sánchez et al. [2015], Díaz-García and Caro-Lopera [2019], Díaz-García and Caro-Lopera [2021]).

Now, when the matrix extension is required from the profuse studied univariate and multivariate risk measures a similar situation appears. Risk measures are usually defined in terms of the *alpha*-th percentile of a certain distribution. In the univariate case, the list of publications covers various theoretical and applied works, see for example, Rockafellar and Uryasev [2002], Wagalath and Zubelli [2018], McNeil et al. [2015], Labopin-Richard et al. [2016], Jorion [2007] and Pflug [2000] and their references. Probably the most commonly used risk measure in finance is the Value-at-Risk (VaR), which is just the α -th percentile of certain distribution; it is widely used measure in economics, insurance, and finance to measure the risk of loss in a specific portfolio of financial assets.

The Value-at-Risk ($VaR_\alpha(X)$) for the random variable X on a probability space $(\Omega, \mathfrak{F}, P)$, at the confidence level $\alpha \in (0, 1)$ is defined in Rockafellar and Uryasev [2000] by

$$VaR_\alpha(X) = \min\{x \in \mathbb{R} \mid P(X \leq x) \geq \alpha\},$$

and equivalently,

$$VaR_\alpha(X) = \sup\{x \in \mathbb{R} \mid P(X \geq x) \geq 1 - \alpha\}.$$

In the univariate case the above problems are equivalent and have received a considerable interest in the literature, (See, for instance, Rockafellar and Uryasev [2000], Pflug [2000], Embrechts and Puccetti [2006], Alexander [2008], Labopin-Richard et al. [2016], McNeil et al. [2015], Wagalath and Zubelli [2018]). If the cumulative distribution function F is strictly increasing, then the Value-at-Risk is the unique threshold $VaR_\alpha(X)$ at which $F_X(VaR_\alpha(X)) = \alpha$, in another words the VaR is a real number such that

$$P(X \leq VaR_\alpha(X)) = \alpha$$

Although the Value-at-Risk by definition is able to calculate risk, it lacks some desirable properties such as subadditivity, which is a mathematical statement of the response of risk concentration, a basic reality in risk management. Among other objections raised against $VaR_\alpha(X)$, we can also mention that it is unable to account for the consequences of the established threshold being surpassed and that, it is generally not continuous on the parameter α Arias-Serna et al. [2016].

A measure of risk, closely related to VaR is the Conditional Value-at-Risk ($CVaR$), defined as the conditional expected value of the $(1 - \alpha)$ -tail. It is defined by Rockafellar and Uryasev [2000] as follows

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_{VaR_\alpha}^{\infty} x f_X(x) dx.$$

Usually, risk measures are thought of as mappings from a set of real-valued random variables to the real numbers. However, it is often insufficient to consider a single real measure to quantify risks created by finance activities. In the last decade, many extensions to Value-at-Risk vector-valued have been investigating and some papers suggest alternative ways of measuring risk for multivariate portfolios. Embrechts and Puccetti [2006] used the notion of quantile curve for defined both the multivariate lower-orthant Value-at-Risk and the multivariate upper-orthant Value-at-Risk at probability level for an increasing function. Cousin and Di Bernardino [2013] proposed two alternative extensions of the Multivariate Value-at-Risk for continuous vectors, based on those level surfaces provided in Embrechts and Puccetti [2006]. They defined the multivariate VaR as the mean of the points belonging to the surface. This means that the measures are real-valued vectors with the same dimension as the considered portfolio of risks. Torres et al. [2015] introduced a directional multivariate VaR, based on the concept of the directional multivariate quantile. They consider the multivariate VaR as a vector-valued point that defines the vertex of an oriented orthant in the direction of analysis.

In the previous literature, the multivariate risk measures are related to a specific partial order, to a specific notion of distance, or to the property of the univariate risk measures that it is desirable to extend. However, as mentioned in Li et al. [2012] some of the univariate translations became unrealistic and are based on inappropriate assumptions that in the context of risk measures which are difficult to elucidate. In chapter 2. of this thesis, we propose a new multivariate risk measure model based on the Wasserstein barycenter of probability measures supported on location-scatter families. The new method considers a reliable risk measure based on distances among probabilistic models. The underlying suitable probability laws obey, for example, opinions, beliefs, and estimates of data sources, in the context of the financial risk. Explicitly, a concept in probability theory is brought into the financial models by proposing the named Fréchet measures; which are calibrated by certain metrization of the probability

measure space. In this case, the well-studied metric of Wasserstein supports the method and provides fundamental connections for the rising concept of barycenter in the sense of Agueh and Carlier in Agueh and Carlier [2011]. Simple and advanced multivariate Value-at-Risk models are compared with the proposed model. The performance of the model is also checked in the major U.S. stock indices during the COVID-19 pandemic.

Now, when we search for a matrix variate extension for VaR, the literature in finance does not provide any approach. However, from the mathematical point of view, the VaR just requires meaningful percentiles of finance in the context of matrix cumulative density functions. This is exactly the domain of the preceding historical notes around matrix variate Gaussian or elliptical non singular or singular distributions in the central or non central cases under real normed division algebras via James' or Davis' polynomials of one of several positive definite or semidefinite matrix arguments.

In this way, the matrix VaR first requires computation of difficult integrals which has been studied separately in the literature since the 60's. The most common involves probabilities of Wishart and beta matrices (results about F distribution can be derived from the beta integrals). Then a second step must consider the solution of a matrix equation for the matrix VaR, a colossal task which can be attained only in some particular cases. The first stage is far to be solved and the classical results on Wishart and beta should be revisited. Integration over cones on Wishart and beta matrices are relatively known, however when we enter into the details we found some curios discrepancies and challenges. For example, computation of probability $P(\mathbf{A} > \mathbf{V})$ for a Wishart matrix \mathbf{A} full derived by Muirhead [2005] and used and cited by several papers, was corrected recently by Caro-Lopera et al. [2016].

Supported by the previous theory, in the the two last chapters of this thesis, we introduce two alternative extensions of the univariate VaR when the underlying risk factors follow matrix-variate beta and Wishart distributions. First, the matrix VaR requires the computation of difficult integrals which have been studied separately in the literature since the 60's. The most commons involves probabilities of Wishart and beta matrices (results about F distribution can be derived from the beta integrals), then these integrals will allow obtaining expressions to directly calculate the probabilities $P(\mathbf{U} > \mathbf{V})$ and $P(\mathbf{U} < \mathbf{V})$ when \mathbf{U} is distributed as a matrix-variate Wishart o beta and \mathbf{V} is a positive definite matrix. Then both extensions are defined, the matrix upper VaR and the matrix lower VaR for beta distributions are obtained as the zeros of the Gaussian hypergeometric function of matrix argument while that the matrix upper VaR and the matrix lower VaR for Wishart distributions are obtained as the zeros of the Confluent hypergeometric function of matrix argument.

In chapter 3, we develop a method for estimating the Value-at-Risk and the Conditional Value-at-Risk when the underlying risk factors follow a beta matrix-variate distribution. The matrix upper VaR and the matrix lower VaR are obtained as the zeros of the Gaussian hypergeometric function of matrix argument. We also developed computational procedures and analytical solutions for estimating the univariate risk measures; under parametric restrictions, some analytical expressions can be found; in other cases, we introduce a numerical algorithm that allows for the computation of these risk measures. For the matricial case, we have used the algorithm proposed by Koev [2021] which calculates hypergeometric functions with matrix arguments. The proposed risk measures are finally utilized for quantifying the potential risk of economic loss in credit risk.

In chapter 4, we propose generalized integrals related with the classical Wishart, beta

and F distributions. Then the work define the termed generalized matrix variate beta and F distributions and the Value-at-Risk in the matrix setting. For completeness a general c.d.f involving Davis' invariant polynomials is derived, then the impossibility of its application is inferred, according to the discussion made before. This centers our goal of the work on expressions that we can really compute by using zonal polynomial theory. In a similar way, for the beta matrix \mathbf{U} , we will provide the solution of the open problem proposed by Constantine in 1963 about the probability $P(\mathbf{U} > \mathbf{V})$, meanwhile the probability $P(\mathbf{U} < \mathbf{V})$ needs to be revisited too (Constantine [1963]).

As corollaries, a number of published results about cumulative density functions of Wishart and beta matrices are also revisited and unified. A new c.d.f for a Wishart random matrix and the solution of an open problem proposed by A. C. Constantine in 1963 are also provided. The extreme latent root distributions for Wishart, beta, and F are obtained by simple derivation. Relations with the VaR are also established. This chapter also proposed a generalized matrix variate beta distribution which involves computable series of zonal polynomials and contains the classical beta matrix as a simple case. Then the generalized matrix variate F distribution can be defined by a matrix transformation of the generalized beta, as we expect.

We also show that both the matrix upper Value-at-Risk and the matrix lower Value-at-Risk satisfy the monotonicity, the positive homogeneity, and the translation invariance properties. As particular cases from the VaR Wishart, we also provide the VaR for exponential, Erlang, Chi-square distributions by the univariate case and the VaR for the beta, F, and the Wishart distributions by the matrix-variate case. Finally, the work also provides a new relationship of the shape theory (cited above) with the condition number and the associated VaR. Some particular cases opens this perspective for future studies.

Chapter 1

Introduction

La théorie des distributions matrice-variable a occupé une place centrale au cours des 70 dernières années autour d'applications robustes dans plusieurs disciplines. Cependant, les mathématiques sophistiquées sous-jacentes et les problèmes de calcul des distributions ont limité la diffusion des résultats, dont les publications appartiennent à un très petit groupe d'auteurs, comparé à d'autres domaines de la statistique. L'évolution de la théorie dans le cadre des matrices aléatoires peut être retracée par l'extension habituelle de la matrice gaussienne aux cas réels univariés et multivariés ; l'exposé le plus élégant et le plus profond de ces énormes avancées se trouve dans Muirhead [2005]. Par la suite, des résultats pour le cas complexe apparaissent et même les distributions sont généralisées pour les familles de distributions de contours elliptiques et les algèbres de division normalisées réelles ; voir par exemple Gupta et al. [2013] et Díaz-García and Gutiérrez-Jáimez [2013], respectivement, avec leurs références. Le cas central a régi les distributions au cours des décennies en termes de polynômes d'argument matriciel étudié dans une série d'articles par A.T. James au cours des années 1960 ; voir Muirhead [2005] pour les détails des polynômes zonaux de James. Les polynômes de James, uniquement dans le cas de matrices définies positives, étaient impossibles à calculer en 50 ans ; le doctorat de Parkurst répertoriait des polynômes d'ordre 12 dans le cas réel Parkhurst and James [1974] ; Caro-Lopera a donné un calcul similaire pour le cas complexe (Caro-Lopera and Nagar [2006], Gupta et al. [2006], Caro-Lopera et al. [2007] et leurs références).

Actuellement, les relations de récurrence en utilisant l'opérateur de Laplace-Beltrami ont permis des calculs numériques de polynômes simples de matrices définies positives Koev and Edelman [2006], mais les séries infinies de polynômes zonaux impliquent encore de forts problèmes ouverts. Une approche matrice-variable de la distribution basée sur des polynômes zonaux semi-définis positifs est également réalisable en étendant l'opérateur Laplace-Beltrami Díaz-García and Caro-Lopera [2006], tandis que les formules exactes pour les polynômes de James, connus sous le nom de polynômes de Jack dans les algèbres de division normées réelles, ne sont disponibles que pour le second ordre Caro-Lopera et al. [2007].

La théorie des distributions non centrales est également apparue dans le cadre des algèbres de division normées réelles et elliptiques, mais dans la plupart des situations les distributions sont intraitables, car elles sont exprimées en termes de polynômes invariants de Davis de divers arguments de matrices (voir par exemple Davis [1980]). L'initiateur des polynômes a conjecturé dans Davis [1979], que les polynômes invariants pouvaient être calculés d'une manière similaire aux polynômes zonaux ; la conjecture a tenu pendant presque trois décennies dans Davis [2006], mais récemment, Caro-Lopera [2016] a prouvé l'impossibilité de construire de tels polynômes d'une manière similaire récurrente en utilisant l'opérateur Laplace-Beltrami. Cela laisse des

dizaines d'articles sur les polynômes de Davis hors de toute application réalisable.

Seuls quelques cas non essentiels de certaines distributions exprimées en termes de polynômes zonaux peuvent effectivement être calculés. La théorie des formes statistiques elliptiques est l'une d'entre elles, une approche appliquée explorée par diverses transformations (SVD, QR, affine, polaire) et des algèbres de division normalisées réelles (réelles, complexes, quaternion et octonion) ; voir par exemple Caro-Lopera et al. [2010], Díaz-García and Caro-Lopera [2017], Díaz-García and Caro-Lopera [2016] et les travaux connexes des auteurs. Le fait de travailler avec des familles générales de distributions au lieu du modèle normal habituel permet des hypothèses flexibles au lieu des études basées sur les gaussiennes dans la théorie des formes (voir, par exemple, Goodall and Mardia [1993], Dryden and Mardia [1998] et leurs références). Lorsque le cas matriciel singulière est étudié, les problèmes ci-dessus augmentent et les calculs et applications doivent être considérés jusqu'à présent ; un très petit nombre de publications apparaissent dans cette ligne, voir, par exemple, Díaz-García and Gutiérrez-Jáimez [1997], Diaz-Garcia and Gutierrez-Sanchez [2013], Díaz-García and González-Farías [2008] et leurs références. Certaines extensions des cas univariés et multivariés à la version matricielle existent depuis des décennies et la proportion de publications associées pour les études matricielles était extrêmement déséquilibrée. C'est le cas de l'extension matrice-variable de la célèbre distribution univariée de Birnbaum-Saunders des années 1960. La distribution univariée a été dérivée par Birnbaum and Saunders [1969] et a suscité de nombreuses recherches pendant plus de 50 ans. Récemment, Balakrishnan and Kundu [2019] a effectué une revue détaillée des travaux théoriques et appliqués. La revue a décrit 281 références d'existence de la distribution, mais seulement 1 parce que Caro-Lopera et al. [2012] a abordé le cas de la matrice.

L'extension matrice-variable arrivée trop tard, car elle nécessitait d'abord une connexion entre les produits Hadamard et la matrice. Cette branche sèche de la recherche accumule 4 autres articles sur le cas de la matrice mais du même groupe d'auteurs (Caro-Lopera and Díaz-García [2016], Díaz-García and Caro-Lopera [2019], Díaz-García and Caro-Lopera [2021]). Lorsque l'extension matricielle des mesures de risque univariées et multivariées, qui ont fait l'objet de nombreuses recherches, est requise, une situation similaire se présente. Les mesures de risque sont généralement définies en termes de percentile d'une distribution donnée. Dans le cas univarié, la liste des publications couvre plusieurs travaux théoriques et appliqués, voir par exemple Rockafellar and Uryasev [2002], Wagalath and Zubelli [2018], McNeil et al. [2015], Labopin-Richard et al. [2016], Jorion [2007], et Pflug [2000] et leurs références.

La Valeur à risque pour la variable aléatoire X dans un espace de probabilité (Ω, \mathcal{F}, P) , au niveau de confiance $\alpha \in (0, 1)$ est définie dans Rockafellar and Uryasev [2000] par. Dans le cas univarié, les problèmes ci-dessus sont équivalents et ont suscité un intérêt considérable dans la littérature, (voir, par exemple, Rockafellar and Uryasev [2000], Pflug [2000], Embrechts and Puccetti [2006], Alexander [2008], Labopin-Richard et al. [2016], McNeil et al. [2015], Wagalath and Zubelli [2018])). Si la fonction de distribution cumulative F est strictement croissante, alors la valeur à risque est le seuil unique $VaR_\alpha(X)$ où $F_X(VaR_\alpha(X)) = \alpha$, en d'autres termes, la VaR est un nombre réel tel que.

Bien que la VaR soit capable de calculer le risque, il lui manque certaines propriétés souhaitables telles que la subadditivité, qui est un énoncé mathématique de la réaction de concentration du risque, une réalité de base dans la gestion du risque. Parmi les autres objections soulevées à l'encontre de $VaR_\alpha(X)$, on peut également mentionner qu'il ne peut rendre compte des conséquences du dépassement du seuil fixé et qu'en général, il n'est pas continu dans le paramètre α Arias-Serna et al. [2016]. Une mesure du risque, étroitement liée à la VaR, est la Valeur à

Risque conditionnelle *CVaR*, définie comme valeur conditionnelle attendue de la queue $(1 - \alpha)$. Rockafellar and Uryasev [2000] le définit comme suit.

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_{VaR_\alpha}^{\infty} x f_X(x) dx$$

Généralement, les mesures de risque sont considérées comme des mappings d'un ensemble de variables aléatoires réelles vers des nombres réels. Cependant, il est souvent insuffisant de considérer une seule mesure réelle pour quantifier les risques découlant des activités financières. Au cours de la dernière décennie, de nombreuses extensions de la Valeur à risque multivariée ont été étudiées et certains articles proposent des méthodes alternatives de mesure du risque pour les portefeuilles multivariés. Dans Embrechts and Puccetti [2006], la notion de courbe quantile a été utilisée pour définir à la fois la valeur à risque multivariée inférieure et la valeur à risque multivariée supérieure. Dans Cousin and Di Bernardino [2013], ils ont proposé deux extensions alternatives de la Value at Risk multivariée pour les vecteurs continus, basées sur les surfaces de niveau fournies dans Embrechts and Puccetti [2006], ils ont défini la VaR multivariée comme la moyenne des points appartenant à la surface, ce qui signifie que les mesures sont des vecteurs de valeur réelle avec la même dimension que le portefeuille de risques considéré. Dans Torres et al. [2015], une VaR directionnelle multivariée a été introduite, basée sur le concept du quantile directionnel multivarié, ils considèrent la VaR multivariée comme un point avec des valeurs vectorielles définissant le sommet d'une ortho orientée dans la direction de l'analyse. Dans la littérature précédente, les mesures de risque multivariées sont liées à un ordre partiel spécifique, à une notion spécifique de distance, ou à la propriété des mesures de risque univariées qu'il est souhaitable d'étendre. Toutefois, comme le mentionne Li et al. [2012], certaines des traductions univariées sont devenues irréalistes et reposent sur des hypothèses inappropriées qui, dans le contexte des mesures de risque, sont difficiles à élucider.

C'est pourquoi dans le chapitre 1. de cette thèse, nous proposons un nouveau modèle de mesure de risque multivarié basé sur le barycentre de Wasserstein des mesures de probabilité supportées par des familles de localisation et d'échelle. La nouvelle méthode considère une mesure de risque fiable basée sur les distances entre les modèles probabilistes. Explicitement, un concept de la théorie des probabilités est incorporé aux modèles financiers en proposant des mesures de Fréchet, qui sont calibrées par une certaine métrisation de l'espace des mesures de probabilité. Dans ce cas, la métrique de Wasserstein soutient la méthode et fournit des connexions fondamentales avec le concept émergent de barycentre au sens d'Agueh et Carlier dans Agueh and Carlier [2011]. Des modèles de valeur à risque multivariés simples et avancés sont comparés au modèle proposé. Les performances du modèle sont également vérifiées sur les principaux indices boursiers américains pendant la pandémie de COVID-19.

Maintenant, lorsque nous cherchons une extension matrice-variable de la VaR, la littérature financière ne fournit aucune approche. Cependant, d'un point de vue mathématique, la VaR ne requiert des percentiles significatifs que dans le contexte des fonctions de densité cumulative matricielle. C'est exactement le domaine des notes historiques précédentes sur les distributions matricielle variables gaussiennes ou elliptiques singulières ou non singulières dans les cas centraux ou non centraux sous des algèbres de division normalisées réelles via des polynômes de James ou de Davis. Ainsi, la VaR matricielle nécessite d'abord le calcul d'intégrales difficiles qui ont été étudiées séparément dans la littérature depuis les années 1960. La plus courante concerne les probabilités de Wishart et les matrices bêta (les résultats sur la distribution F peuvent être dérivés des intégrales bêta). Ensuite, une deuxième étape doit envisager la résolution d'une équation matricielle pour la VaR matricielle, une tâche colossale qui ne peut être réalisée

que dans certains cas particuliers. La première étape est loin d'être résolue et les résultats classiques de Wishart et bêta doivent être révisés. L'intégration sur les cônes dans les matrices de Wishart et bêta est relativement bien connue, cependant, lorsque nous entrons dans les détails, nous trouvons quelques divergences et défis curieux. Par exemple, le calcul de la probabilité $P(\mathbf{A} > \mathbf{V})$ pour une matrice de Wishart \mathbf{A} dérivé dans Muirhead [2005] et utilisé et cité par plusieurs articles, a été récemment corrigé par Caro-Lopera et al. [2016].

En s'appuyant sur la théorie développée dans Muirhead [2005] pour les probabilités des matrices Wishart et beta, dans les deux derniers chapitres de cette thèse, nous présentons deux extensions alternatives de la VaR univariée dans un cadre de matrice variable. Dans ce but, nous connecterons la théorie des fonctions hypergéométriques de l'argument matriciel et l'intégration sur les matrices définies positives. Comme détaillé ci-dessous, la VaR de la matrice supérieure et inférieure pour la distribution bêta est obtenue comme les zéros de la fonction hypergéométrique gaussienne, tandis que la VaR de la matrice supérieure et inférieure des distributions gamma est obtenue comme les zéros de la fonction hypergéométrique confluyente.

Au chapitre 3, nous développons une méthode d'estimation de la Valeur à risque et de la Valeur à risque conditionnelle lorsque les facteurs de risque sous-jacents suivent une distribution bêta matricielle variée. Les valeurs à risque des matrices supérieure et inférieure pour la distribution bêta sont obtenues comme les zéros de la fonction hypergéométrique gaussienne de la matrice-argument. Nous développons des procédures de calcul et des solutions analytiques pour estimer les mesures de risque univariées. Sous des contraintes paramétriques, certaines expressions analytiques peuvent être trouvées ; dans d'autres cas, nous introduisons un algorithme numérique qui permet le calcul de ces mesures de risque. Nous montrons également que les matrices supérieure et inférieure de la valeur à risque satisfont aux propriétés de monotonie, d'homogénéité positive et d'invariance par translation. Les mesures de risque proposées sont enfin utilisées pour quantifier le risque potentiel de pertes économiques dans le risque de crédit.

Dans le chapitre 4, nous proposons des intégrales généralisées liées aux distributions classiques de Wishart, de bêta et de F. Ensuite, le document définit ce qu'on appelle les distributions généralisées de la bêta et F matriciel-varié, ainsi que la matrice de la valeur à risque. De plus, un c.d.f général impliquant des polynômes invariants de Davis est dérivé, puis l'impossibilité de son application est déduite, selon la discussion ci-dessus. Cela concentre notre objectif sur les expressions que nous pouvons effectivement calculer en utilisant la théorie du polynôme zonal. Pour la matrice bêta \mathbf{U} , nous fournirons la solution du problème ouvert proposé par Constantin en 1963 sur la probabilité $P(\mathbf{U} > \mathbf{V})$, de plus, la probabilité $P(\mathbf{U} < \mathbf{V})$ devrait également être révisée. En corollaire, plusieurs résultats publiés sur les Fonction de distribution cumulative des distributions matricielles de Wishart et bêta sont également examinés et unifiés. Nous fournissons également une nouvelle f.c.d. pour une matrice aléatoire de Wishart et la solution d'un problème ouvert proposé par A. C. Constantine en 1963. Les distributions extrêmes des racines latentes pour Wishart, beta et F sont obtenues par simple dérivation. Dans ce chapitre, une distribution bêta matrice-variable généralisée a également été proposée. Elle implique des séries calculables de polynômes zonaux et contient la distribution bêta matricielle classique comme un cas simple. Ensuite, la distribution F généralisée à matrice variable peut être définie par une transformation de la distribution bêta généralisée à matrice variable, comme nous l'attendons. Comme cas particuliers de VaR pour les distributions de Wishart, nous fournissons également VaR pour les distributions gamma, exponentielle, Erlang et chi-carré pour le cas univarié et VaR pour les distributions Gamma, beta et F pour le cas matriciel-varié. Enfin, l'article fournit également une nouvelle relation entre la théorie des formes et le nombre de

conditions et la VaR associée.

Capítulo 1

Introducción

La teoría de distribuciones matriz variadas ha ocupado un lugar central en los últimos 70 años en torno a aplicaciones robustas en varias disciplinas. Sin embargo, las sofisticadas matemáticas subyacentes y los problemas de cálculo de las distribuciones han restringido la divulgación de los resultados, cuyas publicaciones pertenecen a un grupo muy reducido de autores, en comparación con otros campos de la estadística.

La evolución de la teoría en el marco de matrices aleatorias se puede rastrear mediante la extensión habitual de la matriz gaussiana de los casos reales univariados y multivariados; la exposición más elegante y profunda de estos enormes avances se puede consultar en Muirhead [2005]. Posteriormente, aparecen resultados para el caso complejo e inclusive las distribuciones se generalizan para familias de distribuciones de contornos elípticos y álgebras de división normalizadas reales; ver por ejemplo Gupta et al. [2013] y Díaz-García and Gutiérrez-Jáimez [2013], respectivamente, con las referencias allí.

El caso central ha regido las distribuciones a lo largo de las décadas en términos de polinomios de argumento matricial estudiado en una serie de trabajos de A.T. James durante los años 60; ver Muirhead [2005] para detalles de polinomios zonales de James. Los polinomios de James, solo en el caso de matrices definidas positivas, eran imposibles de calcular en 50 años; el doctorado de Parkurst enumeró los polinomios en el orden 12 en el caso real Parkhurst and James [1974]; Caro-Lopera dio un cálculo similar para el caso complejo (Caro-Lopera and Nagar [2006], Gupta et al. [2006], Caro-Lopera et al. [2007] y las referencias allí incluidas).

En la actualidad, las relaciones de recurrencia mediante el uso del operador de Laplace-Beltrami permitieron los cálculos numéricos de polinomios simples de matrices definidas positivas Koev and Edelman [2006], pero las series infinitas de polinomios zonales todavía implican fuertes problemas abiertos. Un enfoque de distribución matriz variada basado en polinomios zonales semidefinidos positivos también es factible mediante la extensión del operador de Laplace-Beltrami Díaz-García and Caro-Lopera [2006], mientras que las fórmulas exactas para los polinomios de James, conocidas como polinomios de Jack en álgebras de división normalizadas reales, solo están disponibles para segundo orden Caro-Lopera et al. [2007].

La teoría de distribuciones no central también apareció bajo las álgebras de división normalizada real y elíptica, pero en la mayoría de las situaciones las distribuciones son intratables porque se expresan en términos de polinomios invariantes de Davis de varios argumentos de matrices (ver por ejemplo Davis [1980]). El creador de los polinomios conjeturó en Davis [1979], que los polinomios invariantes podrían calcularse de manera similar a los polinomios zonales; la

conjetura se mantuvo durante casi tres décadas en Davis [2006], pero recientemente, Caro-Lopera [2016] demostró la imposibilidad de construir esos polinomios de una manera recurrente similar utilizando el operador de Laplace-Beltrami. Esto deja a docenas de artículos en los polinomios de Davis fuera de cualquier aplicación factible.

Sólo se pueden calcular realmente unos pocos casos no centrales de ciertas distribuciones expresadas en términos de polinomios zonales. La teoría de formas estadística elíptica es una de ellas, un enfoque aplicado explorado por varias transformaciones (SVD, QR, afín, polar) y álgebras de división normalizada real (real, compleja, cuaternión y octonión); ver por ejemplo Caro-Lopera et al. [2010], Díaz-García and Caro-Lopera [2017], Díaz-García and Caro-Lopera [2016] y trabajos relacionados de los autores. Trabajar con familias generales de distribuciones en lugar del modelo normal habitual permite suposiciones flexibles en lugar de los estudios basados en Gauss en teoría de formas (ver, por ejemplo, Goodall and Mardia [1993], Dryden and Mardia [1998] y sus referencias). Cuando el caso matricial se estudia en el caso singular, los problemas anteriores aumentan y los cálculos y aplicaciones deben considerarse hasta ahora; un número muy reducido de publicaciones aparece en esa línea, ver por ejemplo Díaz-García and Gutiérrez-Jáimez [1997], Diaz-Garcia and Gutierrez-Sanchez [2013], Díaz-García and González-Farías [2008] y las referencias allí.

Algunas extensiones de casos univariados y multivariados a la versión matricial llevan décadas y la proporción de publicaciones asociadas para los estudios de matriz estaba extremadamente desequilibrada. Es el caso de la extensión matriz variable de la conocida distribución univariada Birnbaum-Saunders de los años 60. Birnbaum and Saunders [1969] derivó la distribución univariante y promovió una gran cantidad de investigaciones durante más de 50 años. Recientemente, Balakrishnan and Kundu [2019] realizó una revisión detallada de trabajos teóricos y aplicados. La revisión describió 281 referencias de existencia de la distribución, pero solo 1 debido a que Caro-Lopera et al. [2012] abordó el caso de la matriz. La extensión de la matriz llegó demasiado tarde, porque primero requería una conexión entre los productos Hadamard y la matriz. Esta árida rama de investigación acumula 4 trabajos más sobre el caso matriz pero del mismo grupo de autores (Caro-Lopera and Díaz-García [2016], Sánchez et al. [2015], Díaz-García and Caro-Lopera [2019], Díaz-García and Caro-Lopera [2021]).

Ahora, cuando se requiere la extensión matricial de las ampliamente investigadas medidas de riesgo univariadas y multivariadas, aparece una situación similar. Las medidas de riesgo generalmente se definen en términos del percentil α -ésimo de una determinada distribución. En el caso univariado, la lista de publicaciones cubre varios trabajos teóricos y aplicados, ver por ejemplo, Rockafellar and Uryasev [2002], Wagalath and Zubelli [2018], McNeil et al. [2015], Labopin-Richard et al. [2016], Jorion [2007] y Pflug [2000] y sus referencias.

El Valor en Riesgo ($VaR_\alpha(X)$) para la variable aleatoria X en un espacio de probabilidad (Ω, \mathcal{F}, P) , en el nivel de confianza $\alpha \in (0, 1)$ se define en Rockafellar and Uryasev [2000] por

$$VaR_\alpha(X) = \min\{x \in \mathbb{R} \mid P(X \leq x) \geq \alpha\},$$

equivalentemente

$$VaR_\alpha(X) = \sup\{x \in \mathbb{R} \mid P(X \geq x) \geq 1 - \alpha\}.$$

En el caso univariado, los problemas anteriores son equivalentes y han recibido un considerable interés en la literatura, (Ver, por ejemplo, Rockafellar and Uryasev [2000], Pflug [2000], Embrechts

and Puccetti [2006], Alexander [2008], Labopin-Richard et al. [2016], McNeil et al. [2015], Wagalath and Zubelli [2018]). Si la función de distribución acumulativa F aumenta estrictamente, entonces el Valor en Riesgo es el umbral único $VaR_\alpha(X)$ en el que $F_X(VaR_\alpha(X)) = \alpha$, en otras palabras, el VaR es un número real tal que

$$P(X \leq VaR_\alpha(X)) = \alpha$$

Aunque el VaR es capaz de calcular el riesgo, carece de algunas propiedades deseables como la subaditividad, que es una declaración matemática de la respuesta de concentración del riesgo, una realidad básica en la gestión del riesgo. Entre otras objeciones planteadas contra $VaR_\alpha(X)$, también podemos mencionar que no puede dar cuenta de las consecuencias de que se supere el umbral establecido y que, en general, no es continuo en el parámetro α Arias-Serna et al. [2016].

Una medida de riesgo, estrechamente relacionada con VaR es el Valor en Riesgo Condicional $CVaR$, definido como el valor condicional esperado de la cola $(1 - \alpha)$. Rockafellar and Uryasev [2000] lo define de la siguiente manera

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_{VaR_\alpha}^{\infty} x f_X(x) dx$$

Por lo general, las medidas de riesgo se consideran como asignaciones de un conjunto de variables aleatorias reales real a los números reales. Sin embargo, a menudo es insuficiente considerar una única medida real para cuantificar los riesgos derivados de las actividades financieras. En la última década, se han investigado muchas extensiones del Valor en Riesgo multivariado y algunos artículos sugieren formas alternativas de medir el riesgo para carteras multivariadas. En Embrechts and Puccetti [2006] se utilizó la noción de curva de cuantiles para definir tanto el Valor en Riesgo multivariado inferior como el Valor en Riesgo multivariado superior. En Cousin and Di Bernardino [2013] se propusieron dos extensiones alternativas del Valor en Riesgo multivariado para vectores continuos, basadas en las superficies de nivel proporcionadas en Embrechts and Puccetti [2006], definieron el VaR multivariado como la media de los puntos pertenecientes a la superficie, lo que significa que las medidas son vectores de valor real con la misma dimensión que la cartera de riesgos considerada. En Torres et al. [2015] se introdujo un VaR direccional multivariado, basado en el concepto del cuantil direccional multivariado, ellos consideran el VaR multivariado como un punto con valores vectoriales que define el vértice de un orto orientado en la dirección de análisis.

En la literatura anterior, las medidas de riesgo multivariante se relacionan con un orden parcial específico, con una noción específica de distancia, o con la propiedad de las medidas de riesgo univariadas que es deseable extender. Sin embargo, como se menciona en Li et al. [2012], algunas de las traducciones univariadas se volvieron poco realistas y se basan en suposiciones inapropiadas que, en el contexto de las medidas de riesgo, son difíciles de dilucidar. Es por ello que en capítulo 1. de esta tesis, proponemos un nuevo modelo de medida de riesgo multivariante basado en el baricentro de Wasserstein de medidas de probabilidad apoyado en familias de localización y escala. El nuevo método considera una medida de riesgo confiable basada en distancias entre modelos probabilísticos. Las leyes de probabilidad adecuadas subyacentes obedecen, por ejemplo, a opiniones, creencias y estimaciones de fuentes de datos, en el contexto del riesgo financiero. Explícitamente, un concepto de la teoría de la probabilidad se incorpora a los modelos financieros al proponer las medidas de Fréchet; que se calibran mediante cierta metrización del espacio de medida de probabilidad. En este caso, la métrica de Wasserstein apoya el método y

proporciona conexiones fundamentales para el concepto emergente de baricentro en el sentido de Agueh y Carlier en Agueh and Carlier [2011]. Modelos de Valor en Riesgo multivariados simples y avanzados se comparan con el modelo propuesto. El desempeño del modelo también se verifica en los principales índices bursátiles de EE. UU. durante la pandemia de COVID-19.

Ahora bien, cuando buscamos una extensión matriz variada para el VaR, la literatura en finanzas no proporciona ningún enfoque. Sin embargo, desde el punto de vista matemático, el VaR solo requiere percentiles significativos en el contexto de las funciones de densidad acumulada matricial. Este es exactamente el dominio de las notas históricas precedentes en torno a distribuciones matriciales variables gaussianas o elípticas no singulares o singulares en los casos centrales o no centrales bajo álgebras de división normalizadas reales a través de los polinomios de James o Davis.

De esta manera, el VaR matricial requiere primero el cálculo de integrales difíciles que se ha estudiado por separado en la literatura desde los años 60. La más común involucra probabilidades de matrices Wishart y beta (los resultados sobre la distribución F pueden derivarse de las integrales beta). Luego, un segundo paso debe considerar la solución de una ecuación matricial para el VaR matricial, una tarea colosal que solo se puede lograr en algunos casos particulares. La primera etapa está lejos de resolverse y los resultados clásicos en Wishart y beta deben revisarse. La integración sobre conos en matrices Wishart y beta es relativamente conocida, sin embargo, cuando entramos en los detalles, encontramos algunas discrepancias y desafíos curiosos. Por ejemplo, el cálculo de la probabilidad $P(\mathbf{A} > \mathbf{V})$ para una matriz Wishart \mathbf{A} derivada en Muirhead [2005] y utilizada y citada por varios artículos, fue corregido recientemente por Carolopera et al. [2016].

Partiendo de la teoría desarrollada in Muirhead [2005] para probabilidades de matrices Wishart y beta, en los dos últimos capítulos de esta tesis, presentamos dos extensiones alternativas del VaR univariado en un entorno matriz variable. Para este propósito, conectaremos la teoría de la función hipergeométrica del argumento matricial y la integración sobre matrices definidas positivas. Como se detalla a continuación, el Valor en Riesgo matricial superior e inferior para la distribución beta se obtienen como los ceros de la función hipergeométrica gaussiana de argumento matricial, mientras que el Valor en Riesgo matricial superior e inferior de distribuciones gamma se obtiene como los ceros de la función hipergeométrica confluyente de argumento matricial.

En el capítulo 3, desarrollamos un método para estimar el Valor en Riesgo y el Valor en Riesgo Condicional cuando los factores de riesgo subyacentes siguen una distribución beta matriz variada. El Valor en Riesgo matricial superior e inferior para la distribución beta se obtienen como los ceros de la función hipergeométrica gaussiana de argumento matricial. Desarrollamos procedimientos computacionales y soluciones analíticas para estimar las medidas de riesgo univariadas. Bajo restricciones paramétricas, se pueden encontrar algunas expresiones analíticas; en otros casos, introducimos un algoritmo numérico que permite el cálculo de estas medidas de riesgo. También mostramos que tanto el Valor en Riesgo matricial superior e inferior satisfacen las propiedades de monotonicidad, homogeneidad positiva e invariancia traslacional. Las medidas de riesgo propuestas se utilizan finalmente para cuantificar el riesgo potencial de pérdidas económicas en riesgo crediticio.

En el capítulo 4, proponemos integrales generalizadas relacionadas con las distribuciones clásicas Wishart, beta y F. Luego, el trabajo define las denominadas distribuciones beta y F matriz variada generalizada y el Valor en Riesgo matricial. También, se deriva un c.d.f general que involucra polinomios invariantes de Davis, luego se infiere la imposibilidad de su aplicación,

de acuerdo con la discusión realizada anteriormente. Esto centra nuestro objetivo del trabajo en expresiones que realmente podemos calcular mediante el uso de la teoría de polinomios zonales. Para la matriz beta \mathbf{U} , proporcionaremos la solución del problema abierto propuesto por Constantine en 1963 sobre la probabilidad $P(\mathbf{U} > \mathbf{V})$, adicionalmente, la probabilidad $P(\mathbf{U} < \mathbf{V})$ también debe revisarse (Constantine [1963]). Como corolario, también se revisan y unifican varios resultados publicados sobre las funciones de densidad acumulada de las distribuciones matriciales Wishart y beta. También se proporciona un nuevo c.d.f para una matriz aleatoria de Wishart y la solución de un problema abierto propuesto por A. C. Constantine en 1963. Las distribuciones de raíces latentes extremas para Wishart, beta y F se obtienen por derivación simple. También se establecen relaciones con el VaR. En este capítulo también se propuso una distribución beta matriz variada generalizada que involucra series computables de polinomios zonales y contiene la distribución matriz variada beta clásica como un caso simple.

Entonces, la distribución matriz variada F generalizada se puede definir mediante una transformación de la distribución matriz variada beta generalizada, como esperamos. Como casos particulares del VaR para distribuciones Wishart, también proporcionamos el VaR para distribuciones gamma, exponencia, Erlang y chi-cuadrado por el caso univariado y el VaR para las distribuciones Gamma, beta y F por el caso matriz variado. Finalmente, el trabajo también proporciona una nueva relación de teoría de formas con el número de condición y el VaR asociado.

Chapter 2

Multi-Variate Risk Measures Under Wasserstein Barycenter

Joint work with Jean-Michel Loubes ¹ and Francisco J. Caro-Lopera ²

Abstract: Randomness in financial markets requires modern and robust multivariate models of risk measures. This paper introduces a new multivariate risk measure based on the notion of Wasserstein barycenter. The particular case of location-scatter families of probability distributions is highlighted and exemplified by the case of Gaussian distributions. The paper introduces the Wasserstein barycenter VaR at the fixed level $\alpha \in (0, 1)$ which is defined as the α -quantile of the Wasserstein barycenter of the aggregated loss distributions μ_1, \dots, μ_N . If μ_1, \dots, μ_N belong to a location-scatter family of distributions, we show that the Wasserstein barycenter VaR belongs to the same location-scatter family, which allows us to obtain closed-form expressions for the Wasserstein barycenter VaR and CVaR. The case where μ_1, \dots, μ_N are Gaussian is provided as an example. Two versions of the Wasserstein barycenter VaR are implemented, assuming either a constant variance and non-constant variance that is estimated using the Exponentially Weighted Moving Average model (EWMA). Also, we compare to classical multivariate VaR alternatives (such as simple summation, var-covar method, various GARCH, and copula models). We use Kupiec's proportion of failures backtesting test. The performance of the model is checked in the major U.S. stock indices log-returns during the COVID-19 pandemic. Results show that the Wasserstein barycenter VaR with non-constant variance provides the best performance for VaR estimation.

Key Words: Wasserstein Barycenter; Transportation Cost; Value-at-Risk; Conditional Value-at-Risk; Location-scatter family.

2.1 Introduction

Regulators issue rules for bankers and insurers to improve their risk management and avoid crises, not always successfully, as recent events illustrate. In the first trimester of 2020, during the COVID-19 pandemic, the stock market crashed. During this trimester, the markets became extremely volatile, with swings of 3% to 7%. Wall Street experienced its biggest percentage drop as on the one on the Black Monday in 1987. The minimum returns of the major U.S. stock market indices were reached on March 16th, 2020. The market crash of March 2020

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demonstrates the need for increasingly sophisticated techniques to adapt to the high volatilities that may occur in unexpected events such as the current pandemic.

The risk measure most used to quantify this type of loss in the market is the Value-at-Risk (VaR). However, when the well-known univariate VaR analysis is generalized into the multivariate setting, many complex theoretical and applied problems appear. This emerging theory extends the univariate case for VaR estimate by using copulas, theory of extreme value, Monte Carlo simulation, historical simulation, variance-covariance, among many others. However, some of the univariate translations became unrealistic and are based on inappropriate assumptions. For example, under the restriction of a perfect dependence, the simple summation approach computes the total risk by summation of the stand-alone risks; a preservative approach with statistical benefit (see, for example, Embrechts et al. [2013] and Li et al. [2015]). Similarly, for a large number of assets, the variance-covariance approach fails, because the estimation of the corresponding matrix is extremely cumbersome due to the high amount of correlations, see McNeil et al. [2015]. Now, in the context of risk management, the multivariate theory of the extreme value (EVT) (see McNeil [1999]) and the technique of multivariate copulas (see Embrechts et al. [2002]) are useful in some scenarios for VaR estimation in portfolios. In particular, the copulas approach attains a robust structure for dependence in financial time series by producing joint distributions with known non-gaussian marginal distributions. Modeling the marginal distributions via copulas allows VaR computations with better performance than the classical approaches, but it involves some intractable assumptions in the context of risk measures which are difficult to elucidate; a similar quotation for the multivariate extreme value theory are also addressed by Jin and Lehnert [2018] and Barone-Adesi et al. [2018].

Inspired by the above discussions and some interesting insights collected by Li et al. [2012] for risk models in the banking industry, this work proposes risk measures based on the Wasserstein barycenter. The new approach considers a reliable risk measure based on distances among probabilistic models. The underlying suitable probability laws obey, for example, opinions, beliefs, and estimates of data sources, in the context of the financial risk. Explicitly, a faraway concept in probability theory is brought into the financial models by proposing the named Fréchet risk measures; which are calibrated by certain metrization of the probability measure space. In this case, the well-studied metric of Wasserstein supports the approach and provides fundamental connections for the rising concept of barycenter in the sense of Agueh and Carlier in Agueh and Carlier [2011]. A seminal work for a number of generalizations and applications, see for example, Bigot et al. [2018], Álvarez Esteban et al. [2018], Le Gouic and Loubes [2017] and the references therein. A crucial aspect underlies here for the new method: the proposed barycenter remains invariant under a class of location-scatter set of (finite or infinite) set of probabilities.

The Wasserstein metric has enriched notably the risk management literature, see for example Kiesel et al. [2016] and Feng and Erik [2018]. In particular, as a canonical metric under well-defined assumptions, the named robust risk management has been studied under the Wasserstein metric. As a consequence, this work represents risk measures via statistical functionals by hybridizing robustness and continuity under the Wasserstein metric. Then, several financial applications of the Wasserstein metric can be obtained in real-time, where the classical approaches provide an excellent scenario for the correctness of the predictions.

The above discussion is organized as follows: preliminaries about risk measures and Wasserstein barycenter are given in Section 2. Section 3 defines the Wasserstein barycenter risk measures

and results for VaR and CVaR calculations are given under a set of location-scatter. Finally, Section 4 applies the results in a portfolio consisting of major U.S. stock market indices. Finally, in section 5 some conclusions are outlined as well as some possible directions for future work.

2.2 Preliminaries

This section provides the necessary background about risk measures and the barycenter in a Wasserstein space.

2.2.1 Risk Measures

Given a random variable X on a probability space $(\Omega, \mathfrak{F}, P)$, we denote its distribution function by F_X , unless otherwise stated, F_X^{-1} is the ordinary inverse of F_X . We will consider X as loss variable, hence $E(X)$ is called the expected loss.

Probably, the most commonly used risk measure in the financial area is the Value-at-Risk, which is defined in Rockafellar and Uryasev [2002] as follows

Value-at-Risk: If the random variable X has a distribution, with mean μ , standard deviation σ and $\hat{X} = \frac{X-\mu}{\sigma}$, then

$$VaR_\alpha(X) = \mu + F_{\hat{X}}^{\leftarrow}(\alpha)\sigma.$$

If F_X is continuous and strictly increasing then,

$$VaR_\alpha(X) = \mu + F_X^{-1}(\alpha)\sigma.$$

Where $F_{\hat{X}}^{\leftarrow}$ is generalized inverse of $F_{\hat{X}}$.

The determination of the $VaR_\alpha(X)$ is completely embedded in the knowledge of their distribution risk factors. This remark entails that the estimation of the Value-at-Risk should be carried determining the parameters of the distribution function of risk factors which, model the changes in the underlying risk factors.

Usually, the approaches described are under the assumption of constant volatility over time. Hence, it is possible to incorporate models describing non-constant volatilities. In practice, there are numerous ARCH and GARCH models that can be chosen from, see Stavroyiannis et al. [2012], Han et al. [2014], Gabrielsen et al. [2015], and references therein.

Although VaR thus defined may reflect risk aversion, and satisfies important properties such as monotonicity, positive homogeneity and translation invariance, it lacks some desirable properties such as subadditivity, which is the mathematical statement of the response of risk concentration, a basic reality in risk management. Among other objections raised on VaR we can also mention that it is unable to account for the consequences of the established threshold being surpassed and that, in general, it is not continuous on the parameter α , Arias-Serna et al. [2017b].

A measure of risk, closely related to Value-at-Risk is the Conditional Value-at-Risk, defined as the conditional expected value of the $(1 - \alpha) - tail$, it is defined in Rockafellar and Uryasev [2000] as follows.

The Conditional Value-at-Risk ($CVaR_\alpha(X)$) of the loss associated at confidence level $\alpha \in (0, 1)$ is the mean of the α -tail loss distribution

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_{VaR_\alpha(X)}^{\infty} x f_X(x) dx. \quad (2.2.1)$$

2.2.2 Barycenters in the Wasserstein space

Start with $\mathcal{P}_2(\mathbf{R}^d)$ as the set of all probability measures defined on \mathbf{R}^d with a finite second order moment. Denote $\mathcal{P}_{2,ac}(\mathbf{R}^d)$ as the subset of absolutely continuous measures, and consider (Ω, σ, P) as a generic probability space. If μ, ν in $\mathcal{P}(\mathbf{R}^d)$, are two measures, then $\mathcal{P}(\mu, \nu)$ will denote the set of all probability measures π in the product set $\mathbf{R}^d \times \mathbf{R}^d$. Here, μ and ν are the corresponding first and second marginals.

Now, for two measures μ, ν in $\mathcal{P}(\mathbf{R}^d)$, **the quadratic transportation cost** between μ and ν (also referred as the transportation cost with a quadratics cost function) is defined as follows

$$\mathcal{T}_2(\mu, \nu) = \inf_{\pi \in \mathcal{P}(\mu, \nu)} \int_{\mathbf{R}^d \times \mathbf{R}^d} d(x, y)^2 d\pi(x, y).$$

The transportation cost with quadratics cost function endows the set $\mathcal{P}_2(\mathbf{R})$ with the metric called **2-Wasserstein distance** or Monge-Kantorovich distance metric, which is given by

$$W_2(\mu, \nu) = \mathcal{T}_2(\mu, \nu)^{\frac{1}{2}}.$$

When $d = 1$, the 2-Wasserstein distance in the real line is just given by the quantile-like expression:

$$W_2^2(\mu, \nu) = \int_0^1 |F_\nu^{-1}(x) - F_\mu^{-1}(x)|^2 dx,$$

here F_ν^{-1} and F_μ^{-1} are the quantile function of ν and μ , respectively.

The Wasserstein metric has enriched notably the risk management literature, see for example Kiesel et al. [2016] and Feng and Erik [2018].

Now, in the Euclidean space, the barycenter of points x_1, \dots, x_N with weights $\lambda_1, \dots, \lambda_N, \lambda_j \geq 0, \sum_{j=1}^N \lambda_j = 1$, is defined as

$$b = \sum_{j=1}^N \lambda_j x_j.$$

In fact, is the unique minimizer

$$E(y) = \sum_{j=1}^N \lambda_j |x_j - y|^2$$

Motivated by the Euclidean version, the Wasserstein barycenter can be defined as follows.

Definition 2.2.1 *We say that the measure $\mu \in \mathcal{P}_2(\mathbf{R}^d)$ is a Wasserstein barycenter for the random probability measures μ_1, \dots, μ_N over \mathbf{R}^d , endowed with positive weights $\lambda_1, \dots, \lambda_N$, where*

$\sum_{j=1}^N \lambda_j = 1$, if μ is a minimizer of

$$E(\mu) = \sum_{j=1}^N \lambda_j W_2^2(\mu; \mu_j). \quad (2.2.2)$$

We will write $\mu_B(\lambda) \in \text{Bar}((\mu_j, \lambda_j)_{1 \leq j \leq N})$.

Empirical consistency of the Wasserstein barycenter has been studied in Agueh and Carlier [2011], Boissard et al. [2015] and Le Gouic and Loubes [2017]. For introducing a fundamental result.

We recall the definition of a location-scatter family.

Definition 2.2.2 *If $M_{d \times d}^+$ denotes the set of $d \times d$ positive definite matrices and X_0 is a random vector with measure $\mu_0 \in \mathcal{P}_{2,ac}(\mathbf{R}^d)$, then the set $\mathcal{F}(\mu_0)$ of probability laws defined by*

$$\mathcal{F}(\mu_0) := \{l(AX_0 + m) : A \in M_{d \times d}^+, m \in \mathbf{R}^d\}$$

is a location-scatter family induced by positive definite affine transformations from μ_0 .

The Wasserstein barycenters of measures on a location-scatter family satisfies the following remarkable property, see Agueh and Carlier [2011], Álvarez Esteban et al. [2016] and Álvarez Esteban et al. [2018].

Proposition 2.2.1 (Theorem 3.10. Álvarez Esteban et al. [2018]) *Let $\mu_0 \in \mathcal{P}_{2,ac}(\mathbf{R}^d)$, and $\mu \in W_2(P_2(\mathbf{R}^d))$, assume that for every $\omega \in \Omega$, the measure $\mu_\omega \in \mathcal{F}(\mu_0)$. Then the unique barycenter, $\bar{\mu}$ of μ also belongs to $\mathcal{F}(\mu_0)$. The mean of $\bar{\mu}$ is $\bar{m} := \int m_\omega P(d\omega)$, and the covariance matrix, $\bar{\Sigma}$, is the only positive definite root of the equation*

$$\bar{\Sigma} = \int (\bar{\Sigma}^{\frac{1}{2}} \Sigma_1 \omega \bar{\Sigma}^{\frac{1}{2}})^{\frac{1}{2}} P(d\omega)$$

This result means that the Wasserstein barycenter is closed respect the location-scatter family.

An interesting case follows for N Gaussian measures on \mathbf{R}^d .

Proposition 2.2.2 (Theorem 2.4. Álvarez Esteban et al. [2016]) *Let μ_1, \dots, μ_N be Gaussian distributions with respective means m_1, \dots, m_N and non singular covariances matrices $\Sigma_1, \dots, \Sigma_N$. The barycenter of μ_1, \dots, μ_N with weights $\lambda_1, \dots, \lambda_N$ is the Gaussian distribution with mean $\bar{m} = \sum_{j=1}^N \lambda_j m_j$ and covariance matrix $\bar{\Sigma}$, defined as the only positive definite satisfying the equation*

$$\bar{\Sigma} = \sum_{i=1}^N \lambda_i (\Sigma^{\frac{1}{2}} \Sigma_i \Sigma^{\frac{1}{2}})^{\frac{1}{2}}.$$

According to Álvarez Esteban et al. [2018], Wasserstein barycenters inherits the strong computational problems of the classical optimal transportation. However, in the real line, some explicit distributions can be obtained.

Proposition 2.2.3 *Let $F_1^{-1}, \dots, F_N^{-1}$ be the quantile functions corresponding to μ_1, \dots, μ_N in the real line. Thus the barycenter of μ_1, \dots, μ_N is the probability with quantile function $\sum_{j=1}^N \lambda_j F_j^{-1}$, where $\lambda_1, \dots, \lambda_N$ are positive weights such that $\sum_{j=1}^N \lambda_j = 1$.*

Finally, using Proposition 2.2.2, with N Gaussian distributions, $N(m_i, \sigma_i^2)$, $i = 1, \dots, N$, on \mathbf{R} , then barycenter is Gaussian $N(\sum_{j=1}^N \lambda_j m_j, (\sum_{j=1}^N \lambda_j \sigma_j)^2)$.

This notable aspect will be used in the context of risk measures.

2.3 Wasserstein Barycenter Risk Measures

This section proposes the Wasserstein Barycenter Risk Measures based on the portfolio loss distributions in finances, these are typically statistical quantities describing the conditional or unconditional loss distribution of the portfolio over some predetermined time horizon.

Now, this research considers risk measures such as VaR and CVaR for a loss random variable defined by $X^+ = \sum_{i=1}^N \omega_i X_i$. Here X_1, \dots, X_N are real random variables attributed to risk types endowed with positive weights $\omega_1, \dots, \omega_N$ (such that $\sum_{j=1}^N \omega_j = 1$) and over a fixed time period T . Now, for computation of these risk measures, a joint law for the random vector $(X_1, \dots, X_N)'$ is required. The Wasserstein barycenter can be regarded as the aggregate model for a certain set of probability measures. It is also suitable for reaching an "average" distribution. The procedure also considers an optimal selection for the positive weights. They are connected with the source credibility for every prior. Moreover, the weights must be chosen equal when all priors remain acceptable. The equality also holds under unknowing performance reliability of the competing laws.

We are in position for definition of the Wasserstein Barycenter Value-at-Risk.

Definition 2.3.1 *Given the aggregate position X^+ , a set of measures $M = (\mu_1, \dots, \mu_N)$, a set of weights $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbf{R}^{N-1}$, a set of quantiles $F = (F_{\mu_1}^{-1}, \dots, F_{\mu_N}^{-1})$ and $\alpha \in (0, 1)$. The Wasserstein Barycenter Value-at-Risk ($VaR_\alpha(X^+, \lambda)$) is defined as:*

$$VaR_\alpha(X^+, \lambda) = F_{\mu_B(\lambda)}^{-1}(\alpha) \quad (2.3.1)$$

Where $F_{\mu_B(\lambda)}^{-1}$ is the quantile function of Wasserstein barycenter $\mu_B(\lambda)$ of μ_1, \dots, μ_N with weights $\lambda_1, \dots, \lambda_N \in \mathbf{R}$, where $\lambda_j \geq 0$, $1 \leq j \leq N$, $\sum_{j=1}^N \lambda_j = 1$.

Next, we use the notable property that the barycenter of distributions of location-scatter families belongs to the same class. This allows deriving closed-form formulas for the Wasserstein Barycenter risk measures for location-scatter distributions.

Theorem 2.3.1 *Let X^+ be an aggregated random variable and μ_1, \dots, μ_N be location-scatter measures, in the real line, with respective means m_1, \dots, m_N and standard deviations $\sigma_1, \dots, \sigma_N$; then the Wasserstein Barycenter Value-at-Risk $VaR_\alpha(X^+, \lambda)$ is given by*

$$VaR_\alpha(X^+, \lambda) = \overline{m_\lambda} + \overline{\sigma_\lambda} G_Z^{-1}(\alpha), \quad (2.3.2)$$

where $Z = \frac{X^+ - \overline{m_\lambda}}{\overline{\sigma_\lambda}}$, G_Z is the cumulative distribution functions of the standard random variable, $\overline{m_\lambda} = \sum_{j=1}^N \lambda_j m_j$, $\overline{\sigma_\lambda} = \sum_{j=1}^N \lambda_j \sigma_j$, and $\lambda_j \geq 0$, $1 \leq j \leq N$, $\sum_{j=1}^N \lambda_j = 1$.

The Wasserstein Barycenter Conditional Value-at-Risk is established next:

Theorem 2.3.2 *Let X^+ be an aggregated random variable and μ_1, \dots, μ_N be location-scatter measures, in the real line, with respective means m_1, \dots, m_N and standard deviations $\sigma_1, \dots, \sigma_N$, then the Wasserstein Barycenter Conditional Value-at-Risk ($CVaR_\alpha(X^+, \lambda)$) is given by*

$$CVaR_\alpha(X^+, \lambda) = \overline{m_\lambda} + \frac{\frac{1}{\overline{\sigma_\lambda}} g_Z(G_Z^{-1}(\alpha))}{1 - \alpha} \overline{\sigma_\lambda}^2 \sigma_Z^2. \quad (2.3.3)$$

where $Z = \frac{X^+ - \overline{m_\lambda}}{\overline{\sigma_\lambda}}$, g_Z and G_Z are the density and cumulative distribution functions of the standard random variable, $\overline{m_\lambda} = \sum_{j=1}^N \lambda_j m_j$, $\overline{\sigma_\lambda} = \sum_{j=1}^N \lambda_j \sigma_j$, and $\lambda_j \geq 0$, $1 \leq j \leq N$, $\sum_{j=1}^N \lambda_j = 1$.

We now illustrate Theorems 2.3.1 and 2.3.2 under the Gaussian distribution.

Corollary 2.3.3 *Let μ_1, \dots, μ_N be Gaussian measures with corresponding means m_1, \dots, m_N and standard deviations $\sigma_1, \dots, \sigma_N$; then the $VaR_\alpha(X^+, \lambda)$ and the $CVaR_\alpha(X^+, \lambda)$ are given by*

$$VaR_\alpha(X^+, \lambda) = \overline{m_\lambda} + \overline{\sigma_\lambda} \Phi^{-1}(\alpha), \quad (2.3.4)$$

$$CVaR_\alpha(X^+, \lambda) = \overline{m_\lambda} + \overline{\sigma_\lambda} \frac{\phi(\Phi^{-1}(\alpha))}{1 - \alpha}. \quad (2.3.5)$$

Here ϕ is the standard Gaussian distribution and Φ^{-1} is the inverse of the standard Gaussian distribution, $\overline{m_\lambda} = \sum_{j=1}^N \lambda_j m_j$, $\overline{\sigma_\lambda} = \sum_{j=1}^N \lambda_j \sigma_j$, and $\lambda_j \geq 0$, $1 \leq j \leq N$, $\sum_{j=1}^N \lambda_j = 1$.

2.4 Empirical Analysis: Portfolio Risk under Gaussian Model

Randomness in financial markets has promoted important research about robust measures of market risk. This problem motivates a profuse study about market risk. An issue involving the risk of loss for investment under multifactor movements in a market. Some dynamical risk factors consider the interest and exchange rates, commodity risks, and capital, among others. Thus, this section focusses on the estimation of the multivariate VaR for a risk portfolio ruled by Nasdaq and S&P500 stock indices. The Nasdaq log-returns and the S&P500 log-returns will be denoted as X_1 and X_2 , respectively. In this case, the portfolio log-return, X^+ , has the form $X^+ = \omega_1 X_1 + \omega_2 X_2$. Here $\omega = (\omega_1, \omega_2)$ and ω_1 and ω_2 are the portfolio weights of the assets 1 and 2, X_1 and X_2 (such that $\sum_{j=1}^2 \omega_j = 1$). Without loss of generality, a portfolio under equal weights, in both indices, is considered. However, it is not a strict restriction, and they can change freely. Finally, for the marginal returns a Gaussian distribution is proposed and a one-day period VaR will be considered.

2.4.1 Results of Wasserstein Barycenter approach

The Wasserstein Barycenter VaR is computed by using 2.3.4. In this case, each stock is ruled by the Gaussian distribution. We follow the 2972 daily closing prices given by Palaro and Hotta [2006]; the database ranged from January 2nd, 1992 to October 1st, 2003. VaR estimate accuracy is measured by using the Kupiec test for backtesting the method in small quantiles

$\alpha = 0.1, 0.05, 0.01, 0.005$. Software R is used for all computations. Table 2.1 shows the descriptive statistics of both series.

Table 2.1 – Descriptive statistics for log-returns series of daily Nasdaq and S&P500 stock indices.

Statistics	Nasdaq	S&P500
Mean	0.00038	0.00030
Mean (annualized)	10.141%	7.857%
Standard Deviation	0.01694	0.01076
Min.	-0.1016800	-0.0711275
Median	0.00122	0.00028
Max.	0.13255	0.05574
Excess of Kurtosis	4.91481	3.78088
Asymmetry	0.01490	-0.10267

According to Table 2.1, the return series distributions of Nasdaq and S&P500 have small asymmetry, but strong kurtosis, in particular the first one. Note also that both series present positive means (annualized).

Thus, the results of Section 2.3 can be used for estimation of the multivariate VaR by using the equation

$$VaR_\alpha(X^+, \lambda) = -\overline{m_\lambda} - \overline{\sigma_\lambda} \Phi^{-1}(\alpha),$$

where Φ^{-1} holds for the inverse of the standard Gaussian distribution, and the mean and the standard deviation are computed via $\overline{m_\lambda} = \sum_{j=1}^N \lambda_j m_j$, and $\overline{\sigma_\lambda} = \sum_{j=1}^N \lambda_j \sigma_j$, respectively.

The approach includes both "unfiltered" and "filtered" models. The filtered model consider the volatility changes of the instrument. Such model will be referred as Wasserstein Barycenter-G*. In the unfiltered model, Wasserstein Barycenter-G, all the σ_j 's, $j = 1, \dots, N$ receive the same value of the sample standard deviation. But in the filtered model the σ_j 's are estimated with a Exponentially Weighted Moving Average model, where $\sigma_t = \sqrt{(1 - \zeta)x_t^2 + \zeta\sigma_{t-1}^2}$.

Kupiec test evaluates the performance by computing the exceptions number in the corresponding test period. In this case, $H_0 : \alpha = p$ represents the null hypothesis. If m is the number of observations for the test period and x denotes the expected frequency of exceptions, then $h = \frac{x}{m}$ is the difference between the observed frequency of losses and VaR. The test statistics. The corresponding test statistics is given by,

$$LR = -2[\ln(p^x(1 - p^{m-x})) - \ln(h^x(1 - h)^{m-x})] \sim \chi^2(1).$$

It rejects the null hypothesis, with a 95% confidence level, for $LR > \chi^2(1)$. In that case, the VaR estimations are not statistical meaningful generated by the particular VaR model, see McNeil [1999].

The dataset under consideration is now divided into sample and test periods, with a selected window of 750 observations; and since there are 2971 observations available, then 2220 VaR tests can be performed at each level can be performed. The corresponding results are presented in Table 2.2.

Table 2.2 – Wasserstein Barycenter VaR, for $t = 751$ to 2971 , number of exceptions where the estimated VaR was exceeded by the portfolio loss with $\alpha = 0.01, 0.05, 0.01, 0.005$. P-values of tests.

Model	0.1 (222)	0.05 (111)	0.01 (22)	0.005 (11)
Wasserstein Barycenter-G	0.0489	0.0442	0.0312	0.0243
Number of exceptions	225	130	46	30
P-Value	0.8323	0.0713	9.1307e-06	2.7020e-06
VaR Model rejected	No	No	Yes	Yes
Wasserstein Barycenter-G*	0.0387	0.0349	0.0247	0.0192
Number of exceptions	207	110	23	16
P-Value	0.2837	0.9223	0.8653	0.1668
VaR Model rejected	No	No	No	No

For all the α levels, the Wasserstein Barycenter-G* model showed the best performance in the VaR estimation. Moreover, for $\alpha = 0.1, 0.05$, the Wasserstein Barycenter-G model also provided a satisfactory performance.. In terms of the Kupiec test, applied to the number of exceptions for the Wasserstein Barycenter-G* model, the null hypothesis was not rejected for all the α levels under consideration. In particular, high p values of 0.2837, 0.9223, 0.8653 and 0.1668 were obtained for Wasserstein Barycenter-G * model, and 0.8323 and 0.07130 for the Wasserstein Barycenter-G model at $\alpha = 0.1, 0.05$ levels.

2.4.2 Comparisons

In the referred context of Li et al. [2012], research by the IFRI and CRO Forum's24 showed that 60% or more of the studied banks consider simple approaches as simple summation and Variance-Covariance method to aggregate risk. And at least a little more advanced approaches (such as the supported approaches by simulation) were used by only 20% or less of the financial institutions in the survey. Next apart will provide a summary of this approaches. Then a performance comparison with the proposed model is given. In the end, the new method is also confronted with the sophisticated hybrid copulas approach given by Palaro and Hotta [2006].

2.4.2.1 Classic Multivariate VaR Approaches

This section computes the multivariate VaR under simple Summation and variance-covariance approaches on different confidence levels. These approaches are briefly described in the next lines.

Simple Summation: The integration of N risks is intuitively reached by aggregating the risks under summation of the particular $VaR_\alpha(X_i)$ of each risk $X_i, i = 1, \dots, N$. Then the total aggregated VaR, $VaR_\alpha(X^+)$, is expressed as:

$$VaR_\alpha(X^+) = - \sum_{i=1}^N VaR_\alpha(X_i), \quad (2.4.1)$$

see Embrechts et al. [2013] and Li et al. [2015].

Now, as usual, the Gaussian model supports several approaches in probability and statistics studies. In particular, the VaR of multivariate Gaussian laws is a common parametric method

for multivariate VaR models. The technique supposes a multivariate Gaussian distribution (with mean μ and covariance Σ) for the returns of the components in the portfolio. The method is characterized as follows:

Variance-Covariance: If $\sigma_+ = \sqrt{\lambda \Sigma \lambda'}$ and $\mu_+ = \lambda \mu$ are respectively the deviation and expected portfolio return, then, the estimation of the VaR for the corresponding multivariate Gaussian distribution returns is given by

$$VaR_\alpha(X^+) = -\mu_+ - \sigma_+ \Phi^{-1}(\alpha), \quad (2.4.2)$$

where Φ^{-1} represents the inverse of the standard Gaussian distribution.

The covariance matrix and the mean vector in the Variance-Covariance (Var-Covar) approach are frequently unknown, then the model requires extra estimates taken from the current observations, see Li et al. [2012] and Li et al. [2015].

Table 2.3 – VaR for $t = 751$ to 2971 , number of exceptions (in brackets) where the estimated VaR was exceeded by the portfolio loss with $\alpha = 0.01, 0.05, 0.01, 0.005$.

Model	0.1 (222)	0.05 (111)	0.01 (22)	0.005 (11)
Simple summation	0.0978 (30)	0.0884 (14)	0.0625 (4)	0.0487(3)
Wasserstein Barycenter-G*	0.0387 (207)	0.0349 (110)	0.0247 (23)	0.0192 (16)
Wasserstein Barycenter-G	0.04891 (225)	0.0442 (130)	0.03123 (46)	0.0243 (30)
Var-Covar	0.0477 (248)	0.0430 (145)	0.0304 (52)	0.0237(38)

A summary of the results are given next:

- According to the Kupiec test, the classic approaches do not predict future losses properly. On one hand, the number of exceptions is small under Simple Summation, this explains an overestimation of future losses. On the other hand, the number of exceptions is large under variance-covariance, providing an underestimation of future losses.
- The Wasserstein Barycenter-G* and Wasserstein Barycenter-G approaches exhibit a remarkable performance for future loss predictions. Moreover, Wasserstein Barycenter's approaches are stronger concerning the other VaR models, because in the same reference time they provide a small exception probability, and then a high-level capital reserve is not required. In the set of the analyzed approaches, the VaR Forecasting at all confidence levels was achieved in high performance by the proposed Wasserstein Barycenter-G* model. In fact, the Wasserstein Barycenter-G exhibited a better VaR forecasting than Var-Covar and Simple Summation approaches in all the confidence levels. Empirical results also demonstrated the known fact that the Simple Approach provides an upper bound for the true VaR. In particular, for a confidence level of 0.1%, a VaR of 0.0387 derived by Wasserstein Barycenter-G* was the third part of the value 0.0978 based on the Simple Summation approach. In such a comparison context, our approach provides several possibilities for a wide banks class. Thus, under a conservative Wasserstein Barycenter VaR compared with the general average, the proposed method, indexed by different types of weights, allow several criteria to improve the results. In contrast, the Var-Covar method is preferably optimistic. Finally, the notable closed property for the barycenter in the class of location-scatter distribution just requires a profuse knowledge of the prior barycenter under the selected distribution. Then, in the Gaussian case, an exact formula for the value at risk

Wasserstein Barycenter can be derived and applied into the complete reference class of distributions. This opens an interesting perspective for manage risk studies of series based on non-Gaussian models because a complete mathematical description of the Wasserstein Barycenter VaR can be found under the selected prior distribution.

2.4.2.2 Others multivariate VaR models

Robust multivariate approaches including Copulas and ARCH models also involve VaR estimation. However, under a big number of assets, such models produce biased parameter estimations and demand a high computational cost. For completeness, we contrast our approaches with those derived by Palaro and Hotta [2006].

Table 2.4 – VaR for $t = 751$ to 2971, number of exceptions (in brackets) where the estimated VaR was exceeded by the portfolio loss with $\alpha = 0.01, 0.05, 0.01, 0.005$.

Model	0.1 (222)	0.05 (111)	0.01 (22)	0.005 (11)
Wasserstein Barycenter-G*	0.0387 (207)	0.0349 (110)	0.0247 (23)	0.0192 (16)
Wasserstein Barycenter-G	0.0489 (225)	0.0442 (130)	0.0312 (46)	0.0243 (30)
SJC Copula + GARCH-E		0.0558 (124)	0.0104 (23)	0.0041 (9)
Bivariate GARCH (BEKK)		0.0819 (182)	0.0338 (75)	0.0248 (55)
Bivariate GARCH (DCC)		0.0432 (96)	0.0140 (31)	0.0113 (25)
EWMA (Bivariate)		0.0387 (86)	0.0144 (32)	0.0104 (23)
GARCH-N (Portfolio)		0.0666 (148)	0.0207 (46)	0.0144 (32)
GARCH-t (Portfolio)		0.0693 (154)	0.0131 (29)	0.0104 (23)
EWMA (Portfolio)		0.0527 (117)	0.0135 (30)	0.0099 (22)
H.S. (Portfolio)		0.1220 (271)	0.0293 (65)	0.0144 (32)

Note that the Wasserstein Barycenter-G* model provides the best performance for $\alpha = 0.1$, $\alpha = 0.05$ and $\alpha = 0.01$. When $\alpha = 0.01$ the estimation equals the result of the hybrid SJC Copula + GARCH-E method; for $\alpha = 0.005$ it is near to the hybrid SJC Copula + GARCH-E model, which was proposed by Palaro and Hotta [2006] as the best approach. In fact, note that the SJC Copula + GARCH-E method succeeds in the VaR forecasting at 99.5% confidence level, but it fails for 90% and 95%. Now, the Kupiec test given in Table 2.2 highlights that our approaches show a high performance in all the confidence levels. Moreover, the inclusion of GARCH models should improve our approaches, but for space reasons, we will leave that study for future work. Given the robustness of GARCH models, the VaR estimation will require a few violations and then a minor capital reserve should be demand.

2.4.3 COVID-19: 2020 Stock Market Crash

We end this section by showing the performance of our method in a crucial modern problem. Explicitly, we will research the impact of COVID-19 in the 2020 Stock Market Crash measured in the Wall Street indexes of NASDAQ Composite, S&P 500, and Dow Jones Industrial Average. These indexes reported historical loss levels, only registered in the financial crisis of 2008. As usual, the complete data set is split into sample and test periods. Sample data ranges from January 4th, 2010 to December 31st, 2018. The 2264 daily return for every stock index also refers to the required historical data for a plausible VaR estimation. The test data, for detecting the performance of the VaR, range from January 2nd, 2019 to December 31st, 2020. The VaR is estimated for each day in the test period (505 days), with the information offered by the

2769 observations ahead of it. Finally, the performance of the VaR model is measured by a comparison of the current loss and the estimated VaR. The division for the sample and test periods are summarized in Table 2.5.

Table 2.5 – Sample and test periods

Period	In-sample period	test period	Total
Date	4/1/2010-31/12/2018	2/1/2019-31/12/2020	
Number of observations	2264	505	2769

We compare the forecasting performance of the Wasserstein Barycenter-G* method with the classic multivariate VaR models. The best challenge for both approaches reside in the financial market behavior of the 2020 year and its high volatility. First, Figure 2.1 shows the forecast of the next trading day VaR for 2019 and 2020 by using both approaches. Under a one day holding period, the models were computed for a 99% confidence level.

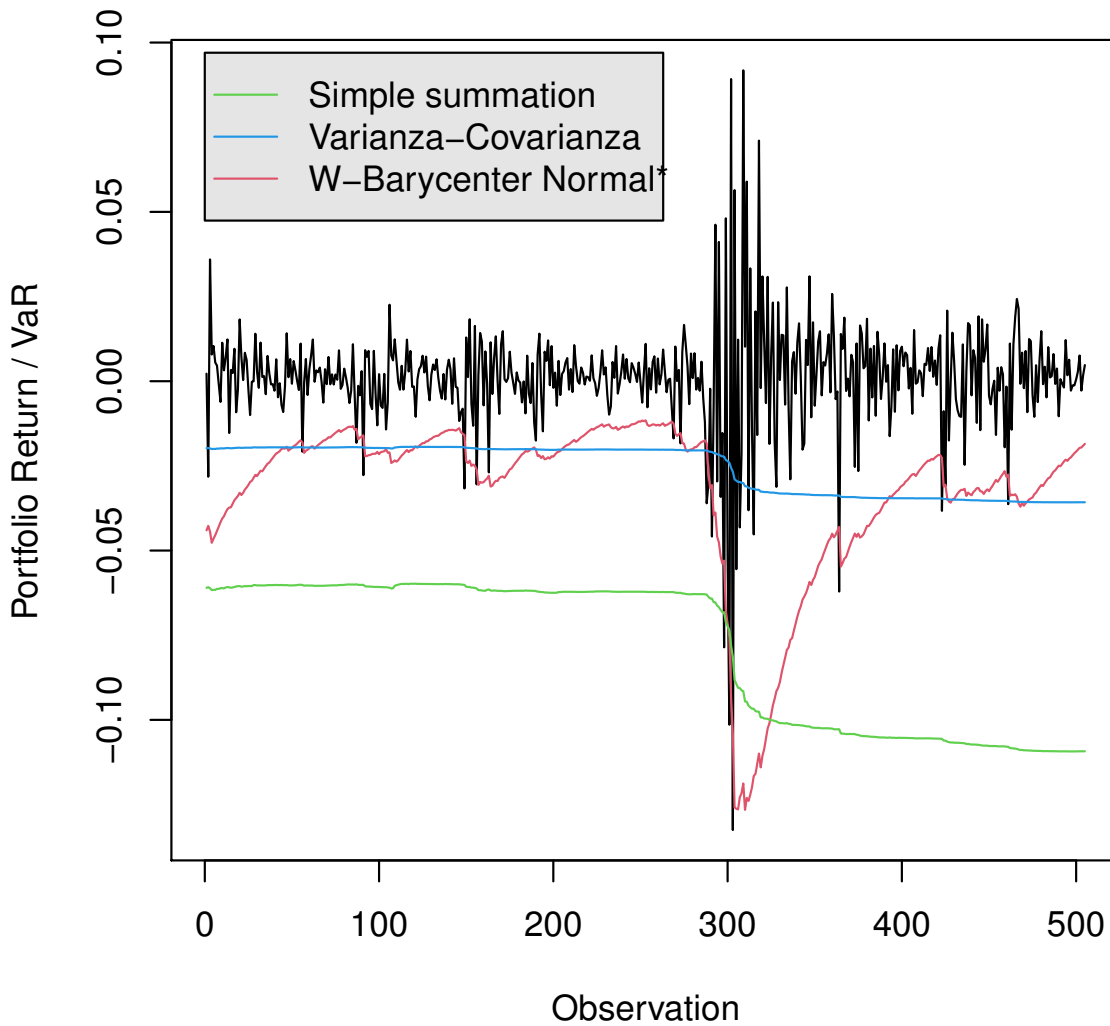


Figure 2.1 – Daily Return series and VaR

As it can be seen the classic multivariate VaR models provide poor estimates for future losses. The VaR estimates rise improperly VaR estimates and unacceptable test period exceptions. However, the VaR estimation given by the Wasserstein Barycenter-G* VaR model is highly accurate. The new approach fits the volatile movements of the returns and predicts future losses notably, in comparison with the classic multivariate VaR approaches. Moreover, according to Figure 2.1, the variance-covariance approach not follows the strong volatility since February 2020; and always presents underestimation. In terms of the Simple Summation model, the test period shows lower exceptions, but a strong conservative characteristic is noted. Finally, our proposed approaches reach satisfactory VaR forecast in normal periods and extremal periods for the high volatility of the periods perform. The results indicate that the proposed approach provides satisfactory forecasts of VaR not only for the "normal periods" but also for the periods of high volatility due to the COVID-19.

2.5 Concluding remarks

This work has proposed a new multivariate risk measure model based on the Wasserstein barycenter of probability distributions under a set of location-scatter. The method was compared with classic and sophisticated models under a portfolio characterized by S&P500 and Nasdaq stock indices. The new model was also compared in United States market indices of high volatility during the current COVID-19. Kupiec test was used for assessing the performance of the existing and new approaches.

The new approach is based on a notable property: Wasserstein barycenter of measures supported on location and scatter family belong to the same class. Then the paper proposed exact formulae for the Wasserstein Barycenter VaR and the Wasserstein Barycenter CVaR for the addressed location and scatter family. This promotes a new setting for building robust risk measures and multivariate of different VaR in multiple financial markets. The closed-form formulae are easily programmed for applications.

The technique could be used for non-Gaussian distributions, opening interesting future research for more robust risk measures.

2.6 Appendix: Proofs

Proof 2.6.1 Theorem 2.3.1.

By definition the $VaR_\alpha(X^+, \lambda)$ is a real number such that

$$P(X^+ \leq VaR_\alpha(X^+, \lambda)) = \alpha, \quad \alpha \in (0, 1),$$

therefore

$$P\left(\frac{X^+ - \overline{m_\lambda}}{\overline{\sigma_\lambda}} \leq \frac{VaR_\alpha(X^+, \lambda) - \overline{m_\lambda}}{\overline{\sigma_\lambda}}\right) = \alpha$$

equivalently

$$P\left(Z \leq \frac{VaR_\alpha(X^+, \lambda) - \overline{m_\lambda}}{\overline{\sigma_\lambda}}\right) = \alpha$$

Thus

$$G_Z\left(\frac{VaR_\alpha(X^+, \lambda) - \overline{m_\lambda}}{\overline{\sigma_\lambda}}\right) = \alpha$$

Finally

$$VaR_\alpha(X^+, \lambda) = \overline{m_\lambda} + \overline{\sigma_\lambda} G_Z^{-1}(\alpha).$$

Proof 2.6.2 Theorem 2.3.2.

Note that

$$CVaR_\alpha(X^+, \lambda) = \frac{1}{1 - \alpha} \int_{VaR_\alpha(X^+, \lambda)}^{\infty} x \cdot \frac{c}{\overline{\sigma_\lambda}} g\left(\frac{1}{2} \left(\frac{x - \overline{m_\lambda}}{\overline{\sigma_\lambda}}\right)^2\right) dx$$

and by letting $Z_q = \frac{VaR_\alpha(X^+, \lambda) - \overline{m_\lambda}}{\overline{\sigma_\lambda}}$, we have

$$CVaR_\alpha(X^+, \lambda) = \frac{1}{1 - \alpha} \int_{Z_q}^{\infty} c(\overline{m_\lambda} + z\overline{\sigma_\lambda}) g\left(\frac{1}{2} z^2\right) dz$$

$$\begin{aligned}
&= \overline{m_\lambda} + \overline{\sigma_\lambda} \frac{1}{1-\alpha} \int_{Z_q}^{\infty} cz.g\left(\frac{1}{2}z^2\right) dz \\
&= \overline{m_\lambda} + \frac{\frac{1}{\overline{\sigma_\lambda}} g_Z(G_Z^{-1}(\alpha))}{1-\alpha} \overline{\sigma_\lambda}^2 \sigma_Z^2.
\end{aligned}$$

Chapter 3

Risk Measures: A Generalization from the Univariate to Matrix-Variate

*The results of this chapter have been recently published in *Journal of Risk*, 2021.

Joint work with Francisco. J. Caro-Lopera ¹ and Jean Michel Loubes ²

Abstract: This paper develops a method for estimating the Value-at-Risk and the Conditional Value-at-Risk when the underlying risk factors follow a beta distribution in a univariate and matrix-variate setting. For this purpose, we will connect the theory of the Gaussian hypergeometric function of matrix argument and integration over positive definite matrixes. For certain choices of the shape parameters, a and b , analytical expressions of the risk measures are developed. More generally, a numerical solution for the risk measures for any parameterization of beta distributed loss variables is presented. The proposed risk measures are finally utilized for quantifying the potential risk of economic loss in credit risk.

Key Words: Beta Distribution; Risk measures; Gaussian Hypergeometric function of matrix argument; Positive definite matrices.

3.1 Introduction

Measuring the risk of a portfolio basically involves determining its distribution function or the functionals describing this distribution function such as its mean, variance or α -th percentile Rockafellar and Uryasev [2002]. Perhaps the most commonly used risk measure in finance is the Value-at-Risk (VaR), which has received the honor of being included in industry regulations (See, for instance, Wagalath and Zubelli [2018], McNeil et al. [2015], Labopin-Richard et al. [2016], Jorion [2007], and Pflug [2000])

Although risk measures based on loss distributions is a subject that has been widely investigated Stavroyiannis et al. [2012], and the inadequacy of Gaussian laws to model the distribution of risk factors, especially in view of applications to risk modeling, is well documented in the literature Marinelli et al. [2012], few studies address the estimation of such risk measures

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for specific probability distributions. These include the standard normal distribution Jorion [2007], Alexander [2008], which does not account for fat tails and is symmetric; the Student's- t distribution Lin and Shen [2006], which is fat-tailed but symmetric and the generalised error distribution (GED), which is more flexible than the Student's- t distribution because it includes both fat and thick tails. Since it is generally very difficult, if not impossible, to obtain analytically tractable expressions for many distribution functions, the risk measures are usually estimated by generating random samples from a distribution and computing the corresponding empirical quantiles.

The aim of this paper is to calculate some risk measures when the underlying risk factors follow a beta distribution in a univariate and matrix-variate setting. As stated Johnson [1997] this is seen as a suitable model in risk analysis since it models a wide class of data with different shapes in closed domains. The beta distribution is of special interest both in the field of finance and in the field of insurance because it allows us to model the loss associated with a loss or the fraction of loss associated with a risk event, see for example Wang [2005]. The distribution can be strongly right-skewed or less skewed as the parameters approach each other, also the distributions would be left-skewed if the parameters' values were switched.

When the univariate Value-at-Risk is considered for matrix-variate distributions a number of new problems arrive. First, we can not define easily an interpretable event associated with the given probability α or $1 - \alpha$ for the risk. Or second, if we reach a well-defined event for the risk, then the problem resides in the computation of the probability. Both problems are also interesting in matrix-variate distribution theory in a different context. First, a restriction over the space of any matrix into the full linear group (invertible matrices) and then into the study of positive definite symmetric matrices, enable us the interpretation of the events. When positive definite matrices are considered, space is reduced to cones which are modeled by the corresponding positive eigenvalues, and then a more manageable space appears. The events can be ordered in such a way that they have a geometrical meaning; i.e., if we have two $m \times m$ random positive definite matrices \mathbf{V} and \mathbf{W} we can ask for the probability that $\mathbf{V} \leq \mathbf{W}$ or $\mathbf{W} \leq \mathbf{V}$. Second, we can compute the probability of an event involving multiple integrals with respect to the Lebesgue measure. The computation of VaR in this context opens an interesting perspective of research. Then, in this paper, we introduce two alternative extensions of the classical univariate VaR for matrix-variate beta distributions, connecting the theory of zonal polynomials and integration over positive definite matrices.

The rest of the paper is organized as follows. Section 2 presents the description of the univariate approach, provides analytical solutions for the values of VaR and the $CVaR$, and also provides numerical solutions for the values of these risk measures. Section 3 presents a description of the matrix-variate approach. Section 4 presents a case study on the credit risk framework. In section 5 some conclusions are outlined as well as some possible directions for future work. Finally, in the appendix the proofs of the theorems and propositions are presented.

3.2 Risk Measures Univariate Under a Family of Beta Distribution

Given a real random variable X on a probability space $(\Omega, \mathfrak{F}, P)$, lets $f_X(x)$ denotes its density function and $F_X(x) = P(X \leq x)$ its associated cumulative distribution function, the Value-at-Risk ($V_\alpha(X)$) for the random variable X , at the confidence level $\alpha \in (0, 1)$ is defined in

Rockafellar and Uryasev [2000] by

$$VaR_\alpha(X) = \min\{x \in \mathbb{R} \mid P(X \leq x) \geq \alpha\},$$

and equivalently,

$$VaR_\alpha(X) = \sup\{x \in \mathbb{R} \mid P(X \geq x) \geq 1 - \alpha\}.$$

In the univariate case the above problems are equivalent and have received a considerable interest in the literature, (See, for instance, Rockafellar and Uryasev [2000], Pflug [2000], Embrechts and Puccetti [2006], Alexander [2008], McNeil et al. [2015], Wagalath and Zubelli [2018]). If F is strictly increasing then the Value-at-Risk is the unique threshold $VaR_\alpha(X)$ at which $F_X(VaR_\alpha(X)) = \alpha$, in another words the VaR is a real number such that

$$P(X \leq VaR_\alpha(X)) = \alpha. \quad (3.2.1)$$

The classical two-parameter probability density function of the beta distribution with shape parameters a and b is given by

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 \leq x \leq 1, \quad a > 0, \quad b > 0 \quad (3.2.2)$$

The beta distribution is of special interest both in the field of finance and in the field of insurance because it allows us to model the fraction of loss associated with a risk event.

In credit risk, for example, consider that Y_1, Y_2 represents the loss associated with the first and second borrower, respectively, and that the amount of each of them follows a gamma distribution $\Gamma(c, a_i)$, $i = 1, 2$. In this case the random variable $X_1 = \frac{Y_1}{Y_1+Y_2}$ represent the fraction of losses associated with the first borrower, while $X_2 = \frac{Y_2}{Y_1+Y_2}$, represents the fraction associated with the second borrower. Then the distribution of X_1 is a beta distribution of parameters a_1, a_2 , while that of X_2 is a beta distribution of parameters a_2, a_1 . Another application is in insurance. For example, consider a policy that covers two types of independent claims Y_1, Y_2 and suppose that their amounts follow a gamma distribution $\Gamma(c, a_i)$, $i = 1, 2$. In this case, $\frac{Y_1}{Y_1+Y_2}$ follow a beta distribution of parameters a_1, a_2 .

Now, since our objective is to find the VaR for a random variable that is distributed as a beta, then we are interested in calculating $P(X \leq VaR_\alpha(X))$. By definition, if $X \sim Beta(a, b)$, then $VaR_\alpha(X)$ is a real such that

$$P(X \leq VaR_\alpha(X)) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^{VaR_\alpha(X)} x^{a-1}(1-x)^{b-1} dx = \alpha.$$

That is, what is relevant to calculate the VaR is the calculation of the integral, which in turn is closely related to the incomplete beta function, which is given by

$$\int_0^x t^{a-1}(1-t)^{b-1} dt = \frac{x^a}{a} {}_2F_1(a, 1-b; a+1; x),$$

where

$${}_2F_1(a, b; c; x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad |x| < 1 \quad (3.2.3)$$

is the Gaussian hypergeometric function, and $(a)_k = a(a+1)\dots(a+k-1) = \Gamma(a+k)/\Gamma(a)$ is Pochhammer's symbol. The hypergeometric function is important in both pure and applied

mathematics, since many elementary functions are special cases of the hypergeometric function, for example, ${}_2F_1(a, b; b; x) = (1 - x)^{-a}$; $-x{}_2F_1(1, 1; 2; x) = \ln(1 - x)$. Some other functions like the incomplete gamma function are also defined in terms of hypergeometric functions, see Erdelyi [1995], Andrews [1998] and references there.

Now, we are ready to propose the main theorem of this section. We shown that finding the VaR of a beta distribution is equivalent to finding the zeros of the Gaussian hypergeometric function. This result will also allow us to find the VaR in the matrix setting.

Theorem 3.2.1 *Let $X \sim \text{Beta}(a, b)$ with $a > 0, b > 0$. The univariate Value-at-Risk ($VaR_\alpha(X)$) of the beta distribution at probability level $\alpha \in (0, 1)$ is the unique solution, in the interval $[0, 1]$, of the hypergeometric equation*

$$\frac{\Gamma(a + b)}{\Gamma(a + 1)\Gamma(b)} VaR_\alpha(X)^a {}_2F_1(a, 1 - b; a + 1; VaR_\alpha(X)) = \alpha. \quad (3.2.4)$$

Under the previous theorem some classical properties of $VaR_\alpha(X)$ are satisfied.

Proposition 3.2.2 *Let $X \sim \text{Beta}(a, b)$ and $Y \sim \text{Beta}(a, b)$, for $\alpha \in (0, 1)$. The univariate Value-at-Risk satisfy the following properties*

1. *Monotonicity: If $X \leq Y$, $VaR_\alpha(X) \leq VaR_\alpha(Y)$.*
2. *Positive homogeneity: For all $\lambda \geq 0$, $VaR_\alpha(\lambda X) = \lambda VaR_\alpha(X)$.*
3. *Translation invariance: For $c \in \mathbb{R}$, $VaR_\alpha(X + c) = VaR_\alpha(X) + c$.*

Although VaR , by definition, is able to calculate risk, it lacks some desirable properties such as subadditivity, which is a mathematical statement of the response of risk concentration, a common reality in risk management. Among other objections raised against $VaR_\alpha(X)$, we can also mention that it is unable to account for the consequences of the established threshold being surpassed and that, it is generally not continuous on the parameter α Arias-Serna et al. [2016]. A measure of risk, closely related to VaR is the $CVaR$, defined as the conditional expected value of the $(1 - \alpha)$ -tail. It is defined by Rockafellar and Uryasev [2000] as follows

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_{VaR_\alpha}^{\infty} x f_X(x) dx \quad (3.2.5)$$

As an immediate consequence of the previous theorem we can find the Conditional Value-at-Risk of the beta distribution.

Corollary 3.2.3 *Let $X \sim \text{Beta}(a, b)$ with $a > 0, b > 0$. The Conditional Value-at-Risk ($CVaR_\alpha(X)$) of the beta distribution at probability level $\alpha \in (0, 1)$ is given by*

$$\begin{aligned} CVaR_\alpha(X) &= \frac{1}{(1 - \alpha)} \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \\ &\times \left[\frac{\Gamma(a + 2)\Gamma(b)}{\Gamma(a + b + 1)} - VaR_\alpha(X)^{a+1} {}_2F_1(a + 1, 1 - b, a + 2, VaR_\alpha(X)) \right]. \end{aligned} \quad (3.2.6)$$

3.2.1 Analytical Expressions of Risk Measures

In this section, we considered some specific values of the parameters a and b that enable us to more precisely analyse the problem. That is, we are seeking values for which it is possible to obtain families of variable changes that provide analytical results on the zeros of the Gauss hypergeometric functions.

An important property of the hypergeometric function is that if $a = -m$ and/or $b = -m$, where $m = 0, 1, 2, \dots$, the series 3.2.3 terminates and reduces to a polynomial of degree $m \in \mathbb{Z}$, the so-called hypergeometric polynomial of grade m (See Erdelyi [1995]). For instance, if $b = -m$, the hypergeometric function is reduced to the next polynomial

$${}_2F_1(a, b; c; x) = \sum_{k=0}^m \frac{(a)_m (-m)_m}{(c)_m m!} x^m = 1 - \frac{am}{c} x - \frac{a(a+1)m(1-m)}{c(c+1)} \frac{x^2}{2!} + \dots + \frac{(a)_k (-m)_m}{(c)_k} \frac{x^m}{m!}.$$

Then, applying this property to Theorem 3.2.1, if $1-b$ is a negative integer, the hypergeometric series 3.2.3 is reduced to a polynomial. Next, we use the fact that $(-m)_k = (-1)^k \frac{m!}{(m-k)!}$, $0 \leq k \leq m$ Driver and Möller [2001], such that by Equation (5.5.2), $VaR_\alpha(X)$ is reduced and we are able to solve the following polynomial equation:

$$\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} VaR_\alpha(X)^a \sum_{k=0}^{b-1} \frac{(-1)^k}{(a+k)\Gamma(b-k)} \frac{VaR_\alpha(X)^k}{k!} - \alpha = 0. \quad (3.2.7)$$

For example, If $b = 2$ and $a = 1$, the equation (3.2.7) is equivalent to

$$-VaR_\alpha(X)^2 + 2VaR_\alpha(X) - \alpha = 0$$

whose zeros are given by

$$VaR_\alpha(X) = 1 \pm \sqrt{1 - \alpha}.$$

As $0 < \alpha < 1$ then $1 - \alpha > 0$, thus $1 - \sqrt{1 - \alpha} \in (0, 1)$.

In particular, note that when $a = b = 1$, the function $Beta(a, b) = U(0, 1)$, and then

$$VaR_\alpha(X) = \alpha.$$

In the following proposition, we will deal with the analytical calculation of the $VaR_\alpha(X)$ seen as real zeros of the hypergeometric polynomial expression of Equation (3.2.7).

Proposition 3.2.4 *The $VaR_\alpha(X)$ for $a, b \in \mathbb{Z}$ such that $a + b \leq 5$ are given by*

Table 3.1 – Analytical Expressions for some parameters a and b

a	b	$VaR_\alpha(X)$
a	1	$\alpha^{1/a}$
1	2	$1 - \sqrt{1 - \alpha}$
2	2	$\frac{1}{2} - \frac{1}{2}(-1 + 2\alpha + 2\sqrt{\alpha^2 - \alpha})^{\frac{1}{3}} - \frac{1}{2}(-1 + 2\alpha + 2\sqrt{\alpha^2 - \alpha})^{-\frac{1}{3}}$
3	2	$\frac{1}{6} \left[2 + \sqrt{4 + \frac{6(\alpha+Q^2)}{R}} - \sqrt{2} \sqrt{4 - \frac{3\alpha}{Q}} \right] + \frac{1}{6} \left[-3Q + \frac{4\sqrt{2}}{\sqrt{2 + \frac{3(\alpha+R^2)}{R}}} \right]$
1	3	$1 + \sqrt[3]{\alpha - 1}$
2	3	$\frac{2}{3} + \frac{1}{2} \sqrt{\frac{8}{9} + \frac{16}{27\sqrt{\frac{4}{9} + \frac{2}{3}SP + \frac{2(1-\alpha)}{3SP}}}} - \frac{2}{3}SP - \frac{2(1-\alpha)}{3SP} - \frac{1}{2} \sqrt{\frac{4}{9} + \frac{2}{3}SP + \frac{2(1-\alpha)}{3SP}}$
1	4	$1 - \sqrt[4]{1 - \alpha}$

Where, $Q = (\alpha - \sqrt{-(-1 + \alpha)\alpha^2})^{\frac{1}{3}}$, $R = (\alpha - \sqrt{-(-1 + \alpha)\alpha})^{\frac{1}{3}}$, $S = (-1 + \sqrt{\alpha})^{\frac{2}{3}}$ and $P = (1 + \sqrt{\alpha})^{\frac{1}{3}}$.

Remark: To find the

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 VaR_\nu(X) d\nu$$

simply calculate the integral of the previous expressions. Continuing with the previous example, if $b = 2$ and $a = 1$, then $VaR_\alpha(X) = 1 - \sqrt{1 - \alpha}$ thus

$$CVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_\alpha^1 (1 - \sqrt{1 - \nu}) d\nu = \frac{3 + 2\sqrt{1 - \alpha}}{3}.$$

3.2.2 Numerical Solutions

According to the Theorem 3.2.1 and Corollary 3.2.3 both $VaR_\alpha(X)$ and $CVaR_\alpha(X)$ for a beta distribution does not always have a closed expression, because by Abel's theorem, there is no formula describing the roots of any general polynomial of degree ≥ 5 . In this section we will deal with the numerical calculation of the risk measures, it will be calculated from the calculation of the zeros of the hypergeometric function. A number of methods for the calculation of zeros of the hypergeometric function have been proposed in the literature. Some of these proposed methods are the Newton method and the method based on the division algorithm Dominici et al. [2013], asymptotic estimates Srivastava et al. [2011] Duren and Guillou [2001], and Matrix methods Ball [1999]. We base our work on the Newton method. This method has the advantage of being easy to understand and has also been completely implemented in the routine packages of R software. In addition, it has internal routines for evaluating the hypergeometric function (See Welbers et al. [2017]).

Our proposed algorithm to compute the VaR_α and the $CVaR_\alpha$ is the following.

The Algorithm

Input: a, b, α, l

$x \leftarrow seq(0, 1, l)$

$poly \leftarrow function(a, b, x, \alpha) \left\{ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{x^a}{a} hypergeo(a, 1 - b, a + 1, x) - \alpha \right\}$

$fun \leftarrow function(x) \{ Re(poly(a, b, x, s)) \}$

$V_\alpha \leftarrow uniroot.all(fun, c(0, 1))$

$CV_\alpha \leftarrow \frac{1}{(1-\alpha)} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left[\frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+1)} - V_\alpha^{a+1} {}_2F_1(a + 1, 1 - b, a + 2, V_\alpha) \right].$

In Table 3.2, we provide the risk measures. We have considered different values for the parameters a and b . The first rows (marked in bold) are contrasted with the results obtained using the analytical expressions obtained in the previous section.

Table 3.2 – Risk Measures beta for parameters a and b

a	b	$VaR_{0.95}(X)$	$CVaR_{0.95}(X)$
1	1	0.9500	0.9750
2	1	0.9747	0.9879
3	1	0.9830	0.9928
4	1	0.9872	0.9960
20	1	0.9974	1.0000
100	1	0.9995	1.0000
1	2	0.7764	0.8510
2	2	0.8647	0.9109
3	2	0.9024	0.9361
1	3	0.6316	0.7238
2	3	0.7514	0.8166
1	4	0.5271	0.6217
6	6	0.7288	0.7756
10	10	0.6799	0.7201
0.6	0.6	0.9874	0.9958
0.8	0.8	0.9700	0.9863
10	2	0.9667	0.9784
40	3	0.9803	0.9857
2	0.9	0.9823	0.9916
4	0.7	0.9968	1.0000
0.9	2	0.7603	0.8398
0.7	4	0.4603	0.5644

3.3 Generalization to Matrix-Variate Approach

In the previous section, we developed the methodology to find the univariate VaR as the zeros of the Gaussian hypergeometric function. In this section, we will show that the philosophy of proposed method can be extended to matrix setting. As a generalization of the univariate beta distribution, in Olkin and Rubin [1964] they derived the beta distribution generalizing the ratio $\frac{X}{Y+X}$, when X and Y both follow a matrix-variate gamma distribution $\Gamma_m(a, \mathbf{I}_m)$ and $\Gamma_m(b, \mathbf{I}_m)$, respectively, getting so the matrix-variate beta distribution with parameters a and b . A random symmetric positive definite matrix $\mathbf{X}_{m \times m}$, is said to have a matrix-variate beta distribution with parameters a and b , and we will write that \mathbf{X} is $Beta_m(a, b)$, if its probability density function (p.d.f) is given by

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{X}|^{a-\frac{m+1}{2}} |\mathbf{I}_m - \mathbf{X}|^{b-\frac{m+1}{2}}, \quad \mathbf{0} < \mathbf{X} < \mathbf{I}_m, \quad a > \frac{m-1}{2}, \quad b > \frac{m-1}{2}.$$

Where $\mathbf{0} < \mathbf{X} < \mathbf{I}_m$, means that $\mathbf{X} > \mathbf{0}$ and $\mathbf{I}_m - \mathbf{X} > \mathbf{0}$ (i.e., \mathbf{X} and $\mathbf{I}_m - \mathbf{X}$ are positive definite matrixes).

As in the univariate case, we are interested in calculating $P(\mathbf{X} \leq VaR_\alpha(\mathbf{X}))$ and $P(\mathbf{X} \geq VaR_\alpha(\mathbf{X}))$ when $\mathbf{X} \sim Beta_m(a, b)$, for this, it is necessary to keep in mind that a unique definition of multivariate VaR does not exist because there are different possible definitions of multivariate quantiles. In the last decade, many extensions to multidimensional settings have

been investigated and recent papers suggest alternative ways of measuring risk for multivariate portfolios. For instance, Embrechts and Puccetti [2006], used the notion of quantile curve for defined both the Multivariate lower-orthant Value at Risk and the Multivariate upper-orthant Value at Risk at probability level α for an increasing function, which is represented by an infinite number of points. In Cousin and Di Bernardino [2013] two alternative extensions of the Multivariate Value at Risk for continuous vectors, based on those level surfaces provided in Embrechts and Puccetti [2006] are proposed. In Torres et al. [2015] introduced a directional multivariate VaR, based on the concept of the directional multivariate quantile. They consider the multivariate VaR as a vector-valued point that defines the vertex of an oriented orthant in the direction of analysis. They presented comparisons in terms of robustness with the alternative multivariate VaR, introduced by Cousin and Di Bernardino [2013].

Now, when we search for a matrix variate extension for VaR, the literature in finance does not provide any approach. However, from the mathematical point of view, the VaR just requires meaningful percentiles in the context of matrix cumulative density functions. The theory under the random matrix setting is deeply studied in Muirhead [2005]. In particular, they provide the formulation for calculating $P(\mathbf{X} \leq \mathbf{V})$ and $P(\mathbf{X} \geq \mathbf{V})$ when \mathbf{X} follows a Wishart distribution and \mathbf{V} is a positive definite matrix and demonstrated that its cumulative distribution function can be expressed in terms of a Gauss hypergeometric function of a matrix argument. Supported by the Theorem 7.2.10 in Muirhead [2005], we provide a couple of theorems that will allow us to find $P(\mathbf{X} \leq \mathbf{V})$ and $P(\mathbf{X} \geq \mathbf{V})$, when $\mathbf{X} \sim \text{Beta}_m(a, b)$ and \mathbf{V} is a positive definite matrix.

Theorem 3.3.1 *If $\mathbf{X} \sim \text{Beta}_m(a, b)$, with $a > \frac{m-1}{2}, b > \frac{m-1}{2}$ and \mathbf{V} is an $m \times m$ positive definite matrix ($\mathbf{V} > \mathbf{0}$) then*

$$P(\mathbf{X} < \mathbf{V}) = \frac{\Gamma_m(a+b)\Gamma_m(\frac{m+1}{2})}{\Gamma_m(b)\Gamma_m(a+\frac{m+1}{2})} |\mathbf{V}|^a {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{V}\right). \quad (3.3.1)$$

Where ${}_2F_1(a, b; c; \mathbf{X}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{(a)_k (b)_k}{(c)_{\kappa}} C_{\kappa}(\mathbf{X})$, is the the Gaussian hypergeometric functions of matrix argument. The series converges for $\|\mathbf{X}\| < 1$, where $\|\mathbf{X}\|$ denote the maximum of the absolute values of the eigenvalues of \mathbf{X} . $(n)_{\kappa} = \pi^{\frac{m(m-1)}{4}} \frac{\prod_{i=1}^m \Gamma[n+\kappa_i - \frac{1}{2}(i-1)]}{\Gamma_m(n)}$ is the generalized hypergeometric coefficient, \sum_{κ} denotes summation over all partitions $\kappa = (\kappa_1, \dots, \kappa_m)$, $\kappa_1 \geq \dots \geq \kappa_m$, and $C_{\kappa}(\mathbf{X})$ denote the Zonal polynomial, see Muirhead [2005] for more details.

We now introduce the $P(\mathbf{X} > \mathbf{V})$, as in Muirhead [2005], we will use the same argument to prove that in the case when $r = a - \frac{m+1}{2}$ is a positive integer, an expression can be obtained in terms of a finite series involving zonal polynomials.

Theorem 3.3.2 *Let $\mathbf{X} \sim \text{Beta}_m(a, b)$, with $a > \frac{m-1}{2}, b > \frac{m-1}{2}$ and \mathbf{V} is an $m \times m$ positive definite matrix ($\mathbf{V} > \mathbf{0}$). If $r = a - \frac{m+1}{2}$ is a positive integer, then*

$$P(\mathbf{X} > \mathbf{V}) = |\mathbf{I}_m - \mathbf{V}|^b \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* (-r)_{\kappa} \frac{\left(\frac{m+1}{2}\right)_{\kappa}}{\left(\frac{m+1}{2} + b\right)_{\kappa}} C_{\kappa}(-(\mathbf{V}^{-1} - \mathbf{I}_m))$$

Where \sum_{κ}^* denotes summation over those partitions $\kappa = (k_1, \dots, k_m)$ of k with $k_1 \leq r$.

Starting from the previous theorems, we introduce here two alternative extensions of the VaR measure. We will denote $\overline{\text{VaR}}_{\alpha}(\mathbf{X})$ our matrix upper Value-at-Risk associate with $P(\mathbf{X} < \text{VaR}_{\alpha}(\mathbf{X}))$ and $\underline{\text{VaR}}_{\alpha}(\mathbf{X})$ the matrix lower Value-at-Risk associated with $P(\mathbf{X} > \text{VaR}_{\alpha}(\mathbf{X}))$.

Definition 3.3.1 Let \mathbf{X} be a random real matrix, if $\mathbf{X} \sim \text{Beta}_m(a, b)$ with $a > \frac{m-1}{2}, b > \frac{m-1}{2}$. The matrix upper Value-at-Risk for the matrix variate beta distribution at probability level $\alpha \in (0, 1)$ is the solution of the hypergeometric equation of matrix argument.

$$\frac{\Gamma_m(a+b)\Gamma_m(\frac{m+1}{2})}{\Gamma_m(b)\Gamma_m(a+\frac{m+1}{2})} |\overline{\mathbf{VaR}}_\alpha(\mathbf{X})|^a {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \overline{\mathbf{VaR}}_\alpha(\mathbf{X})\right) = \alpha, \quad (3.3.2)$$

$$\mathbf{0} < \overline{\mathbf{VaR}}_\alpha(\mathbf{X}) < \mathbf{I}_m.$$

Now, we introduce another possible generalization of the VaR based on $P(\mathbf{X} > \text{VaR}_\alpha(\mathbf{X}))$.

Definition 3.3.2 Let \mathbf{X} be a random real matrix, if $\mathbf{X} \sim \text{Beta}_m(a, b)$ with $a > \frac{m-1}{2}, b > \frac{m-1}{2}$ and $r = a - \frac{m+1}{2}$ be a positive integer. The matrix lower Value-at-Risk for the matrix variate beta distribution at probability level $\alpha \in (0, 1)$ is the solution of the hypergeometric equation of matrix argument.

$$|\mathbf{I}_m - \underline{\mathbf{V}}_\alpha(\mathbf{X})|^b \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* (-r)_\kappa \frac{\left(\frac{m+1}{2}\right)_\kappa}{\left(\frac{m+1}{2} + b\right)_\kappa} C_\kappa(-(\underline{\mathbf{V}}_\alpha(\mathbf{X})^{-1} - \mathbf{I}_m)) = 1 - \alpha. \quad (3.3.3)$$

$$\mathbf{0} < \underline{\mathbf{VaR}}_\alpha(\mathbf{X}) < \mathbf{I}_m.$$

Under the previous definitions the classical properties of VaR are satisfied.

Proposition 3.3.3 Let $\mathbf{X} \sim \text{Beta}(a, b)$ and $\mathbf{Y} \sim \text{Beta}(a, b)$, for $\alpha \in (0, 1)$, the matrix upper Value-at-Risk and matrix lower Value-at-Risk satisfy the following properties

1. *Monotonicity:* If $\mathbf{X} \leq \mathbf{Y}$, then

$$\overline{\mathbf{VaR}}_\alpha(\mathbf{X}) \leq \overline{\mathbf{VaR}}_\alpha(\mathbf{Y}), \quad \underline{\mathbf{VaR}}_\alpha(\mathbf{X}) \leq \underline{\mathbf{VaR}}_\alpha(\mathbf{Y}).$$

2. *Positive homogeneity:* For all symmetric matrix $\Omega \geq 0$,

$$\overline{\mathbf{VaR}}_\alpha(\Omega\mathbf{X}) = \Omega \overline{\mathbf{VaR}}_\alpha(\mathbf{X}), \quad \underline{\mathbf{VaR}}_\alpha(\Omega\mathbf{X}) = \Omega \underline{\mathbf{VaR}}_\alpha(\mathbf{X}).$$

3. *Translation invariance:* For all symmetric matrix $\Omega \geq 0$,

$$\overline{\mathbf{VaR}}_\alpha(\mathbf{X} + \Omega) = \overline{\mathbf{VaR}}_\alpha(\mathbf{X}) + \Omega, \quad \underline{\mathbf{VaR}}_\alpha(\mathbf{X} + \Omega) = \underline{\mathbf{VaR}}_\alpha(\mathbf{X}) + \Omega.$$

As in the univariate case, the properties of monotonicity, positive homogeneity and translation invariance are satisfied since these properties are conserved for matrix-variate beta distributions, see Gupta and Nagar [2000].

To find the upper matrix Value-at-Risk the Algorithm 4.2 in Koev and Edelman [2006] can be used. They have made an algorithm that efficiently approximate the hypergeometric function of a matrix argument through its expansion as a series of zonal polynomials. The implementation of the algorithms in MATLAB (Current version: 1.5, February 12, 2018) is available in Koev [2021]. Basically the algorithm computes the truncated Hypergeometric function ${}_pF_q$ as a series of zonal polynomials, truncated for partitions of size not exceeding M . However, as in the univariate case, an important property of the hypergeometric function is that if $b = \frac{m+1}{2}$ then the previous serie is reduces to 1, and we could obtain a closed expression for the calculation of $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$.

Corollary 3.3.4 *Let \mathbf{X} a random real matrix, if $\mathbf{X} \sim \text{Beta}_m(a, \frac{m+1}{2})$ with $a > \frac{m-1}{2}$. The matrix upper Value-at-Risk at probability level $\alpha \in (0, 1)$ is the solution of equation.*

$$|\overline{\text{VaR}}_\alpha(\mathbf{X})|^a = \alpha, \quad \mathbf{0} < \overline{\text{VaR}}_\alpha(\mathbf{X}) < \mathbf{I}_m. \quad (3.3.4)$$

For example, if $m = 2$ then, $\overline{\text{VaR}}_\alpha(\mathbf{X})$ is a 2×2 matrix positive definite such that $\mathbf{0} < \overline{\text{VaR}}_\alpha(\mathbf{X}) < \mathbf{I}_2$ and $v_{11}v_{22} - v_{12}^2 = \alpha^{1/a}$, whose solution is in the region in R^3 described by the inequalities $0 < v_{11} < 1$ and $1 - v_{11} - v_{22} + v_{11}v_{22} - v_{12}^2 > 0$. Let $v_{11} = \sqrt{\alpha}$, as $v_{12}^2 = \sqrt{\alpha}v_{22} - \alpha^{1/a}$, then $\sqrt{\alpha}v_{22} - \alpha^{1/a} \geq 0$ thus $v_{22} \geq \alpha^{1/a-1/2}$. Let $v_{22} = \alpha^{1/a-1/2} + t$, $t \in \mathbb{R}^+$ then $v_{12} = \sqrt{t\sqrt{\alpha}}$ then

$$\overline{\text{VaR}}_\alpha(\mathbf{X}) = \begin{bmatrix} \sqrt{\alpha} & \sqrt{t\sqrt{\alpha}} \\ \sqrt{t\sqrt{\alpha}} & \alpha^{1/a-1/2} + t \end{bmatrix}.$$

As $1 - v_{11} - v_{22} + v_{11}v_{22} - v_{12}^2 > 0$, then t is such that $t < 1 - \alpha^{1/2} - \alpha^{1/a-1/2} + \alpha^{1/a}$.

Inspired in Theorem 3.3.3 of Muirhead, and focus on $m = 2$, for an intuitive interpretation, the following theorem states that the distribution of a matrix beta can be decomposed into the product of three independent univariate beta distribution. We consider the exact case for $b = 3$, in order to apply the previous corollary as solution of the power determinantal equation. Start with $a = 1$ in the Corollary 3.3.4, then $|\overline{\text{VaR}}_\alpha(\mathbf{X})| = \alpha$, then the Cholesky decomposition, and the previous corollary provides the following relation with the equations underlying the univariate VaR.

Theorem 3.3.5 *Let $\mathbf{X} \sim B_2(a, \frac{3}{2})$, and $\mathbf{X} = \mathbf{T}'\mathbf{T}$, where $\mathbf{T} = (t_{ij})$ is an upper triangular matrix with $t_{ij} > 0$, $i = 1, 2$ then the distribution of a matrix beta can be decomposed into the product of three independent univariate beta distribution.*

A plausible general result using the above argument should holds for any order m . But it involves a cumbersome algebra that will be part of a future work, because we required new representations for nested determinants, according to the definition of positive matrices. The generalization should claim that the distribution of any beta matrix \mathbf{X} can be decomposed into a product of independent univariate beta distributions indexed by the nested determinants $x_{11} > 0$, $|\mathbf{X}_2| > 0$, $|\mathbf{X}_3| > 0$, ..., $|\mathbf{X}| > 0$. Where $|\mathbf{X}_j|$ is the determinant of the $(m - j) \times (m - j)$ left upper corner submatrix of \mathbf{X} , with $j = m - 1, m - 2, \dots, 1, 0$.

Once the general representation is derived, the $\overline{\text{VaR}}_\alpha(\mathbf{X})$, can be interpreted in the same way as the case $m = 2$, but with a more complex Cholesky indexed by the same subdeterminants and univariate VaRs. This opens an interesting future work because the interpretation of the matrix Var not only the beta case, but the F case is plausible in terms of univariate betas or F distributions. Finally note that by replacing $b = 3$, and $\mathbf{X} = \mathbf{I}_m - \mathbf{X}$, the probability beta in $|\mathbf{I}_m - \mathbf{X}| = \beta$ can be easily justify and interpreted in the proof of the Cholesky decomposition of $\overline{\text{VaR}}_\alpha(\mathbf{X})$.

3.4 An Application to Credit Risk

In this section, the proposed risk measures are utilized for quantifying the potential risk of economic loss in credit risk. The Credit risk is the risk that the value of a portfolio changes due to unexpected changes in the credit quality of issuers or trading partners. Only default risk is modeled, not downgrade risk. It is assumed that:

- For a loan, the probability of default in a given period, is the same for any other comparable period.
- For a large number of obligors, the probability of default by any particular obligor is small, and the number of defaults that occur in any given period is independent of the number of defaults that occur in any other period Michel et al. [2000].

In case of a portfolio with N borrowers, the portfolio loss variable X_N is defined as

$$X_N = \sum_{n=1}^N EAD_n \times LGD_n \times PD_n$$

where

X_N is loss, the amount that an institution is contractually owed but does not receive because of the default of the borrower or borrowers;

EAD_n is exposure at default, the total amount of the institution's liability to a borrower;

LGD_n is loss Given Default, the fraction of the exposure that is actually lost given a default of that borrower:

PD_n is the Probability of Default, the binomially random variable that measures whether a borrower has defaulted or not. It takes the value one in the case of default, and zero otherwise. As in the model by Ward et al. [2002], we work with the credit loss rate, which is the total credit loss of the institution divided by the total exposure. Because default is modelled as Bernoulli and does not allow firms to default repeatedly without curing, the sum of a correlated portfolio of loans follows a beta distribution, with the result that $X = \frac{X_n}{X_N} \in [0, 1]$, Ward et al. [2002]. We will consider the consumer portfolio model designed by the Superintendencia Financiera de Colombia (SFC), which is used in Colombia for the evaluation and supervision of internal models presented by financial institutions. In our study, the balance of the debt at the time of calculation was considered as the exposure at default. The Probability of Default is chosen from a rating system (Table 3.3) proposed by the SFC. The Loss Given Default is chosen from the values by the SFC (Table 3.4) Superintendencia [2008].

Table 3.3 – Probability of Default from SFC

Rating	Automobiles	Other	Credit card	CFC Automobiles	CFC Other
AA	0,97%	2,10%	1,58%	1,02%	3,54%
A	3,12%	3,88%	5,35%	2,88%	7,19%
BB	7,48%	12,68%	9,53%	12,34%	15,86%
B	15,76%	14,16%	14,17%	24,27%	31,18%
CC	31,01%	22,57%	17,06%	43,32%	41,01%
Default	100.0%	100.0%	100.0%	100.0%	100.0%

Table 3.4 – Loss Given Default from SFC

Type of Guarantee	LGD	Days	LGD	Days	LGD
Admissible financial collateral	0-12	-	-	-	-
Commercial and residential real estate	40%	360	70%	720	100%
Goods given in real estate leasing	35%	360	70%	720	100%
Goods given in leasing other than real estate	45%	270	70%	540	100%
Receivables	45%	360	80%	720	100%
Other Guarantees suitable	50%	270	70%	540	100%
Non-admissible guarantee	60%	210	70%	420	100%
No guarantee	75%	30	85%	90	100%

3.4.1 The data

The data under consideration is a portfolio of more than 14,000 different loans with an average portfolio balance of \$ 91.454.343.340, in a Colombian financial institution in a period of 12 months (January 2020 to December 2020). In Table 3.5, we show the exposure at default, probability of default, loss given default, credit loss and credit loss rate obligor.

Table 3.5 – Credit loss and Credit loss rate

Obligor	EAD	PD	LGD	Credit loss	Credit loss rate
1	\$ 856.293	22,57%	60%	\$115.959	0,01223 %
2	\$ 22.220.325	2,10%	60%	\$ 279.976	0,02954 %
3	\$ 11.233.923	22,57%	60%	\$ 1.521.298	0,16051%
4	\$ 35.382.090	22,57%	60%	\$ 4.791.443	0,50555 %
5	\$ 9.389.391	22,57%	60%	\$ 1.271.511	0,13416%
6	\$ 39.789.567	22,57%	60%	\$ 5.388.303	0,56852 %
7	\$ 567.710	22,57%	60%	\$ 76.879	0,00811 %
8	\$ 1.079.607	22,57%	60%	\$ 146,200	0,01543 %
9	\$78.921.678	22,57%	60%	\$ 10.687.574	1,12765%
10	\$ 10.987.261	2,10%	75%	\$173.049	0,01826%
.....
14330	\$ 2.876.111	2,10%	75%	\$ 45.299	0,00478 %
TOTAL	\$ 91.454,343.340			\$ 947.773.798	100%

3.4.2 The Implementation

After calculating the loss rate derived from the credit risk (Last column of Table 3.5), the parameters of the beta distribution of each month analyzed were calculated in R. Having estimated the unknown parameters of the model $VaR_\alpha(X)$ and $CVaR_\alpha(X)$ of the beta distribution can be calculated straightforwardly using our algorithm from Section. We will also calculate the Economic Capital ($EC_\alpha(X) = E(X) - VaR_\alpha(X)$), which is a value very used in the context of credit risk and that can be easily calculated with the help of the $VaR_\alpha(X)$. The results obtained were converted into monetary units to have a better idea of the potential losses that the company could incur in each months of the year under consideration. From Table 3.4.2 is possible to infer the following: on average for the year 2020, the total expected credit loss was

\$ 720.495.087, the maximum loss was \$ 2.185.686.058 with up 95% confidence; the economic capital needed as a buffer against unexpected losses was \$1.465.190.972 and the expected loss amount beyond the VaR at probability level 95% was \$4.787.439.041.

Table 3.6 – Credit Risk Measures

Month	$E(X)$	$VaR_\alpha(X)$	$EC\beta_\alpha(X)$	$CVaR_\alpha(X)$
Jan.	\$ 947.773.798	\$ 2.227.701.427	\$ 1.498.566.453	\$ 4.836.908.886
Feb.	\$ 733.684.644	\$ 2.241.519.936	\$ 1.507.835.292	\$ 4.859.629.904
Mar.	\$ 729.848.669	\$ 2.047.888.348	\$ 1.318.039.679	\$ 4.544.330.501
Abr.	\$737.507.092	\$ 2.251.881.449	\$ 1.514.374.358	\$ 4.849.257.893
May.	\$ 732.118.086	\$ 2.234.220.031	\$ 1.502.101.945	\$ 4.793.813.506
Jun.	\$ 734.764.136	\$ 2.244.531.943	\$ 1.509.767.807	\$ 4.918.483.683
Jul.	\$ 727.331.803	\$ 2.222.288.246	\$ 1.494.956.443	\$ 4.863.703.380
Ago.	\$ 729.925.682	\$ 2.229.425.636	\$ 1.499.499.954	\$ 4.926.333.695
Sep.	\$ 737.451.217	\$ 2.252.358.797	\$ 1.514.907.580	\$ 4.967.022.930
Oct.	\$ 748.696.016	\$ 2.285.068.627	\$ 1.536.372.610	\$ 5.082.848.683
Nov.	\$ 654.108.600	\$ 2.001.390.110	\$ 1.347.281.510	\$ 4.417.603.102
Dic.	\$ 647,773,798	\$1.989.958.149	\$ 1.338.588.072	\$ 4.389.332.331

The motivation for using matrix-variate Value-at-Risk measures can be exemplified below.

A financial institution generally offers different lines of credit according to the needs of the clients. Each line of credit is exposed to a random total loss and these losses must be below the given level, simultaneously, with a high probability.

Even consider a single line of credit and we observe its total losses in subsequent periods of time, we would thus have a random vector for each line of credit and therefore we could think of a matrix whose rows are the total losses per period of time and whose columns are the lines of credit. It is reasonable to require that subsequent portfolio losses must be below a given level, with a high probability α . In this case, the matrix-variate VaR provides us with complete information on the maximum loss that can occur in each period of time for each line of credit.

In order to ensure this, we look at the total losses per period of time as components of a random matrix. We find a matrix-variate VaR to know which points are in the $m \times r$ -dimensional space (where m is the number of periods and r is the number lines of credit), which must exceed the matrix of total losses, to guarantee the given level.

Now, suppose further that we have the information on expected losses for two lines of credit in two periods of time and that we have this information for two different banks. Let \mathbf{Y}_1 and \mathbf{Y}_2 , represent the loss associated with the first and second bank, respectively, and that the amount of each of them follows a multivariate gamma distribution. Let us now assume just for the purpose of illustration $\mathbf{Y}_1 \sim \Gamma_2(1, \mathbf{I}_2)$ and $\mathbf{Y}_2 \sim \Gamma_2(1, \mathbf{I}_2)$. In this case the random matrix $\mathbf{X}_1 = \mathbf{Y}_1(\mathbf{Y}_1 + \mathbf{Y}_2)^{-1}$ represents the fraction of loss associated with the first bank, then $\mathbf{X}_1 \sim Beta_2(1, 1)$. By Corollary 3.3.4, the $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$ is a 2×2 matrix such that $|\overline{\mathbf{VaR}}_\alpha(\mathbf{X})| = \alpha$.

If, for instance, $\alpha = 0.95$, then

$$\overline{\mathbf{VaR}}_\alpha(\mathbf{X}_1) = \begin{bmatrix} 0.9747 & 0.0242 \\ 0.0242 & 0.9753 \end{bmatrix}.$$

3.5 Concluding Remarks

In the current literature, multivariate risk measures are related to a specific partial order, or to the property of the univariate risk measures that it is desirable to extend. Under this perspective, in this work, we have considered the Loewner order and we have developed a methodology to calculate univariate Value-at-Risk seen as the zeros of a hypergeometric function, this methodology could be generalized taking advantage of the definition of matrix argument hypergeometric function.

We have developed computational procedures and analytical solutions for estimating the univariate risk measures, under parametric restrictions, some analytical expressions can be found; in other cases, we introduce a numerical algorithm that allows for the computation of these risk measures. For the matricial case we have used the algorithms proposed by Koev [2021] which calculate hypergeometric functions with matrix arguments.

The methodology developed here may be applicable to others distribution functions whose risk measures are defined in terms of hypergeometric functions, such as the Wishard, gamma and F distributions. This is a topic of ongoing research.

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Declarations of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

3.6 Appendix: Proofs

Proof 3.6.1 Theorem 3.2.1. *If $X \sim \text{Beta}(a, b)$, then by (3.2.1), $VaR_\alpha(X)$ is a real such that*

$$P(X \leq VaR_\alpha(X)) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^{VaR_\alpha(X)} x^{a-1}(1-x)^{b-1} dx = \alpha.$$

Expanding $(1-x)^{b-1}$ in the binomial series and integrating term to term, the previous equation is equivalent to,

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(1-b)_k}{k!} \frac{VaR_\alpha(X)^{k+a}}{k+a} = \alpha.$$

Equivalently,

$$\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} VaR_\alpha(X)^a \sum_{k=0}^{\infty} \frac{(a)_k(1-b)_k}{(a+1)_k} \frac{VaR_\alpha(X)^k}{k!} = \alpha.$$

Thus, by the definition of Gaussian hypergeometric function we have the required result. To prove that $VaR_\alpha(X)$ is unique in the interval $(0, 1)$. Let

$$f(VaR_\alpha(X)) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(1-b)_k}{(a+k)k!} VaR_\alpha(X)^{a+k} - \alpha.$$

Thus

$$\begin{aligned} f'(VaR_\alpha(X)) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(1-b)_k}{k!} VaR_\alpha(X)^{a+k-1} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} VaR_\alpha(X)^{a-1} (1 - VaR_\alpha(X))^{b-1}. \end{aligned}$$

Thus, $VaR_\alpha(X) = 0$ and $VaR_\alpha(X) = 1$ are the unique critical points. Given that $f(VaR_\alpha(X))$ is a continuous function over a closed bounded interval $[0, 1]$ and $f(0) = -\alpha < 0$ and $f(1) = 1 - \alpha > 0$, then by continuity, there is at least one root in $(0, 1)$. Suppose that $f(VaR_\alpha(X))$ has two zeros in $(0, 1)$, which we will call x_1 and x_2 , by Rolle's theorem, there exists at least one point c belonging to the interval (x_1, x_2) such that $f'(c) = 0$, ie c is a critical point, which is a contradiction since 0 and 1 are the unique critical points in $(0, 1)$.

Proof 3.6.2 Proposition 3.2.2. The Property 1 is immediate since $0 \leq X \leq 1$ and $0 \leq Y \leq 1$, if $X \leq Y$ then $X^k \leq Y^k$ for $k = 1, 2, \dots$. The properties 2. and 3. are immediates by the positive homogeneity and translation invariance properties of the beta distribution.

Proof 3.6.3 Corollary 3.2.3. If $X \sim \text{Beta}(a, b)$, then by Definition 3.2.5.

$$CVaR_\alpha(X) = \frac{1}{(1-\alpha)} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_{VaR_\alpha(X)}^1 x^{a-1} (1-x)^{b-1}. \quad (3.6.1)$$

Expanding $(1-x)^{b-1}$ in a binomial series and integrating term to term, the integral in 3.6.1 is equivalent to,

$$\frac{1}{(1-\alpha)} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left[\sum_{k=0}^{\infty} \frac{(a+1)_k (1-b)_k}{(a+2)_k k!} 1^k - VaR_\alpha(X)^{a+1} \sum_{k=0}^{\infty} \frac{(a+1)_k (1-b)_k}{(a+2)_k k!} VaR_\alpha(X)^k \right].$$

Thus, by the definition of hypergeometric function we have

$$\frac{1}{(1-\alpha)} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left[{}_2F_1(a+1, 1-b, a+2, 1) - VaR_\alpha(X)^{a+1} {}_2F_1(a+1, 1-b, a+2, VaR_\alpha(X)) \right]$$

$$\text{But, by Ec. (14) in Erdelyi [1995]} \quad {}_2F_1(a+1, 1-b, a+2, 1) = \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+1)},$$

Then, we have

$$CVaR_\alpha(X) = \frac{1}{(1-\alpha)} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left[\frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+1)} - VaR_\alpha(X)^{a+1} {}_2F_1(a+1, 1-b, a+2, VaR_\alpha(X)) \right]$$

Proof 3.6.4 Proposition 3.2.4.

1. If $b = 1$, then the hypergeometric polynomial in 3.2.7 is reduced to 1 and thus to find the VaR consists simply of solving the equation $\frac{\Gamma(a+1)}{\Gamma(a+1)} VaR_\alpha(X)^a - \alpha = 0$, whose solution is given by $VaR_\alpha(X) = \alpha^{1/a}$.

2. If $b = 2$, then the hypergeometric polynomial in (3.2.7) is equivalent to

$$(a + 1)VaR_\alpha(X)^a - aVaR_\alpha(X)^{a+1} - \alpha = 0. \quad (3.6.2)$$

- If $a = 1$, the polynomial in (3.6.2) is reduced to

$$-VaR_\alpha(X)^2 + 2VaR_\alpha(X) - \alpha$$

whose zeros are given by

$$VaR_\alpha(X) = 1 \pm \sqrt{1 - \alpha}.$$

As $0 < \alpha < 1$ then $1 - \alpha > 0$, thus $1 - \sqrt{1 - \alpha} \in (0, 1)$.

- If $a = 2$ the polynomial in (3.6.2) is reduced to

$$-2VaR_\alpha(X)^3 + 3VaR_\alpha(X)^2 - \alpha$$

whose real zero is given by

$$VaR_\alpha(X) = \frac{1}{2} + \frac{1 - \sqrt{-3}}{4(-1 + 2\alpha + 2\sqrt{-a + a^2})^{\frac{1}{3}}} + \frac{1}{4}(1 + \sqrt{-3})(-1 + 2\alpha + 2\sqrt{-a + a^2})^{\frac{1}{3}}$$

- If $a = 3$ the polynomial (3.6.2) is reduced to

$$-3VaR_\alpha(X)^4 + 4VaR_\alpha(X)^3 - \alpha$$

whose real zeros are given by

$$VaR_\alpha(X) = \frac{1}{6} \left[2 + \sqrt{4 + \frac{6(\alpha + Q^2)}{R}} \pm \sqrt{2} \sqrt{4 - \frac{3\alpha}{Q}} \right] + \frac{1}{6} \left[-3Q + \frac{4\sqrt{2}}{\sqrt{2 + \frac{3(\alpha + R^2)}{R}}} \right].$$

Where, $Q = (\alpha - \sqrt{-(-1 + \alpha)\alpha^2})^{\frac{1}{3}}$ and $R = (\alpha - \sqrt{-(-1 + \alpha)\alpha})^{\frac{1}{3}}$, whose real zero in $(0, 1)$ is given by

$$\frac{1}{6} \left[2 + \sqrt{4 + \frac{6(\alpha + Q^2)}{R}} - \sqrt{2} \sqrt{4 - \frac{3\alpha}{Q}} \right] + \frac{1}{6} \left[-3Q + \frac{4\sqrt{2}}{\sqrt{2 + \frac{3(\alpha + R^2)}{R}}} \right].$$

3. If $b = 3$, then the hypergeometric polynomial in (3.2.7) is equivalent to

$$\frac{1}{2}(a + 1)aVaR_\alpha(X)^{a+2} - (a + 2)aVaR_\alpha(X)^{a+1} + \frac{1}{2}(a + 1)(a + 2)VaR_\alpha(X)^a - \alpha = 0. \quad (3.6.3)$$

- If $a = 1$, the polynomial (3.6.3) is reduced to

$$VaR_\alpha(X)^3 - 3VaR_\alpha(X)^2 + 3VaR_\alpha(X) - \alpha$$

whose real zero is given by

$$VaR_\alpha(X) = 1 + \sqrt[3]{\alpha - 1}.$$

- If $a = 2$, the polynomial (3.6.3) is reduced to

$$3VaR_\alpha(X)^4 - 8VaR_\alpha(X)^3 + 6VaR_\alpha(X)^2 - \alpha$$

whose real zeros are given by

$$\frac{2}{3} \pm \frac{1}{2} \sqrt{\left[\frac{8}{9} + \frac{16}{27\sqrt{\frac{4}{9} + \frac{2}{3}SP + \frac{2(1-\alpha)}{3SP}}} - \frac{2}{3}SP - \frac{2(1-\alpha)}{3SP} \right] - \frac{1}{2}\sqrt{\frac{4}{9} + \frac{2}{3}SP + \frac{2(1-\alpha)}{3SP}}},$$

where $S = (-1 + \sqrt{\alpha})^{\frac{2}{3}}$ and $P = (1 + \sqrt{\alpha})^{\frac{1}{3}}$, then the $VaR_\alpha(X) \in (0, 1)$

$$\frac{2}{3} + \frac{1}{2} \sqrt{\left[\frac{8}{9} + \frac{16}{27\sqrt{\frac{4}{9} + \frac{2}{3}SP + \frac{2(1-\alpha)}{3SP}}} - \frac{2}{3}SP - \frac{2(1-\alpha)}{3SP} \right] - \frac{1}{2}\sqrt{\frac{4}{9} + \frac{2}{3}SP + \frac{2(1-\alpha)}{3SP}}}$$

4. If $b = 4$ and $a = 1$, then the hypergeometric polynomial in (3.2.7) is equivalent to

$$-VaR_\alpha(X)^4 + 4VaR_\alpha(X)^3 - 6VaR_\alpha(X)^2 + 4VaR_\alpha(X) - \alpha = 0,$$

whose real zeros are given by $VaR_\alpha(X) = 1 \pm \sqrt[4]{\alpha - 1}$.

As $0 < \alpha < 1$, $1 - \alpha > 0$; thus, $VaR_\alpha(X) = 1 - \sqrt[4]{\alpha - 1} \in (0, 1)$.

Proof 3.6.5 Theorem 3.3.1. Using the Beta $_m(a, b)$ density function for \mathbf{X} , it follows that

$$P(\mathbf{X} < \mathbf{V}) = \frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} \int_{\mathbf{0} < \mathbf{X} < \mathbf{V}} |\mathbf{X}|^{a-\frac{m+1}{2}} |\mathbf{I} - \mathbf{X}|^{b-\frac{m+1}{2}} (d\mathbf{X}) \quad (3.6.4)$$

By Corollary (7.3.5) in Muirhead [2005],

$$\begin{aligned} |\mathbf{I} - \mathbf{X}|^{b-\frac{m+1}{2}} &= {}_1F_0\left(\frac{m+1}{2} - b; \mathbf{X}\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \binom{\frac{m+1}{2} - b}{\kappa} C_{\kappa}(\mathbf{X}) \end{aligned}$$

Then, the right hand of 3.6.4 is equivalent to

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \binom{\frac{m+1}{2} - b}{\kappa} \int_{\mathbf{0} < \mathbf{X} < \mathbf{V}} |\mathbf{X}|^{a-\frac{m+1}{2}} C_{\kappa}(\mathbf{X})(d\mathbf{X}).$$

Put $\mathbf{X} = \mathbf{V}^{\frac{1}{2}} \mathbf{Y} \mathbf{V}^{\frac{1}{2}}$, with Jacobian $(d\mathbf{X}) = |\mathbf{V}|^{\frac{m+1}{2}} (d\mathbf{Y})$ to get

$$P(\mathbf{X} < \mathbf{V}) = \frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^a \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \binom{\frac{m+1}{2} - b}{\kappa} \int_{\mathbf{0} < \mathbf{Y} < \mathbf{I}} |\mathbf{Y}|^{a-\frac{m+1}{2}} C_{\kappa}(\mathbf{V}\mathbf{Y})(d\mathbf{Y})$$

Take $b = \frac{m+1}{2}$ in Theorem (7.2.10) in Muirhead [2005],

$$P(\mathbf{X} < \mathbf{V}) = \frac{\Gamma_m(a+b)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m(b)\Gamma_m\left(a + \frac{m+1}{2}\right)} |\mathbf{V}|^a \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{(a)_{\kappa} \left(\frac{m+1}{2} - b\right)_{\kappa}}{\left(a + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa}(\mathbf{V})$$

Then by the hypergeometric function, we have the required result.

Proof 3.6.6 Theorem 3.3.2. Using the $Beta_m(a, b)$ density function for \mathbf{X} , it follows that

$$P(\mathbf{X} > \mathbf{V}) = \frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} \int_{\mathbf{0} < \mathbf{V} < \mathbf{X} < \mathbf{I}} |\mathbf{X}|^{a-\frac{m+1}{2}} |\mathbf{I} - \mathbf{X}|^{b-\frac{m+1}{2}} (d\mathbf{X}). \quad (3.6.5)$$

Put $\mathbf{X} = \mathbf{V}^{\frac{1}{2}}(\mathbf{I} + \mathbf{Y})\mathbf{V}^{\frac{1}{2}}$, then $\mathbf{V}^{-1} - \mathbf{I} > \mathbf{Y} > \mathbf{0}$, with Jacobian $(d\mathbf{X}) = |\mathbf{V}|^{\frac{m+1}{2}}(d\mathbf{Y})$ then, the right side of the equation 3.6.5 is given by

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^a \int_{\mathbf{0} < \mathbf{Y} < \mathbf{V}^{-1} - \mathbf{I}} |\mathbf{I} + \mathbf{Y}|^{a-\frac{m+1}{2}} |\mathbf{I} - \mathbf{V}(\mathbf{I} + \mathbf{Y})|^{b-\frac{m+1}{2}} (d\mathbf{Y}). \quad (3.6.6)$$

Let, $\mathbf{Y} = (\mathbf{V}^{-1} - \mathbf{I})^{\frac{1}{2}}\mathbf{Z}(\mathbf{V}^{-1} - \mathbf{I})^{\frac{1}{2}}$; with Jacobian $(d\mathbf{Y}) = |\mathbf{V}^{-1} - \mathbf{I}|^{\frac{m+1}{2}}(d\mathbf{Z})$ then the left hand of 4.7.7

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^a \int_{\mathbf{0} < \mathbf{Z} < \mathbf{I}} |\mathbf{I} + (\mathbf{V}^{-1} - \mathbf{I})\mathbf{Z}|^{a-\frac{m+1}{2}} |\mathbf{I} - \mathbf{V} - \mathbf{V}(\mathbf{V}^{-1} - \mathbf{I})\mathbf{Z}|^{b-\frac{m+1}{2}} |\mathbf{V}^{-1} - \mathbf{I}|^{\frac{m+1}{2}} (d\mathbf{Z})$$

This is equivalent to

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^a |\mathbf{I} - \mathbf{V}|^{b-\frac{m+1}{2}} |\mathbf{V}^{-1} - \mathbf{I}|^{\frac{m+1}{2}} \int_{\mathbf{0} < \mathbf{Z} < \mathbf{I}} |\mathbf{I} + (\mathbf{V}^{-1} - \mathbf{I})\mathbf{Z}|^{a-\frac{m+1}{2}} |\mathbf{I} - \mathbf{Z}|^{b-\frac{m+1}{2}} (d\mathbf{Z})$$

By Corollary (7.3.5) in Muirhead [2005], if $r = a - \frac{m+1}{2}$ is a real positive

$$\begin{aligned} |\mathbf{I} + (\mathbf{V}^{-1} - \mathbf{I})\mathbf{Z}|^{a-\frac{m+1}{2}} &= {}_1F_0\left(\frac{m+1}{2} - a; -(\mathbf{V}^{-1} - \mathbf{I})\mathbf{Z}\right) \\ &= \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa^*} (-r)_{\kappa^*} C_{\kappa^*}(-(\mathbf{V}^{-1} - \mathbf{I})\mathbf{Z}) \end{aligned}$$

Then, 5.3.6 is equivalent to

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^a |\mathbf{I} - \mathbf{V}|^{b-\frac{m+1}{2}} |\mathbf{V}^{-1} - \mathbf{I}|^{\frac{m+1}{2}} \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa^*} (-r)_{\kappa^*} \int_{\mathbf{0} < \mathbf{Z} < \mathbf{I}} C_{\kappa^*}(-(\mathbf{V}^{-1} - \mathbf{I})\mathbf{Z}) |\mathbf{I} - \mathbf{Z}|^{b-\frac{m+1}{2}} (d\mathbf{Z}),$$

Take $a = \frac{m+1}{2}$ and $\mathbf{Y} = -(\mathbf{V}^{-1} - \mathbf{I})$ in Theorem (7.2.10) in Muirhead [2005], we have the required result.

Proof 3.6.7 Theorem 3.3.5. Note that for $m = 2$ and $b = 3/2$

$$f(\mathbf{X}) = \frac{\Gamma_2(a+3)}{\Gamma_2(a)\Gamma_2(3)} |\mathbf{X}|^{a-\frac{3}{2}} |\mathbf{I}_2 - \mathbf{X}|^0 (d\mathbf{X})$$

But $|\mathbf{X}| = |\mathbf{T}'\mathbf{T}| = t_{11}^2 t_{22}^2$, then $(d\mathbf{X}) = 2^2 t_{11}^2 t_{22}^2 dt_{11} dt_{12} dt_{22}$ and

$$f(t_{11}, t_{12}, t_{22}) = 2^2 \frac{\Gamma_2(a+3)}{\Gamma_2(a)\Gamma_2(3)} t_{11}^{n_1-1} t_{22}^{n_1-2} dt_{11} dt_{12} dt_{22}$$

$$|\mathbf{I}_2 - \mathbf{T}'\mathbf{T}| = (1 - t_{11}^2)(1 - t_{22}^2) \left[1 - \frac{1}{1-t_{11}^2} t_{12}^2 (1 - t_{22}^2)^{-1}\right].$$

Now let $v = \frac{1}{1-t_{11}^2} t_{12}^2 (1-t_{22}^2)^{-1}$, then $dt_{11} dt_{12} dt_{22} = (1-t_{11}^2)^{1/2} (1-t_{22}^2)^{1/2} dt_{11} dt_{22} dv$, therefore

$$f(t_{11}, t_{12}, v) = 2^2 \frac{\Gamma_2(a+3)}{\Gamma_2(a)\Gamma_2(3)} t_{11}^{2a-1} (1-t_{11}^2)^{\frac{3}{2}-1} t_{22}^{2a-2} (1-t_{22}^2)^{\frac{3}{2}-1} (1-v^2)^0 dt_{11} dt_{22} dv$$

Note, that $dt_{11}^2 = 2t_{11} dt_{11}$, $dt_{22}^2 = 2t_{22} dt_{22}$ and $dv^2 = 2v dv$, then

$$\begin{aligned} f(t_{11}, t_{12}, v) &= \frac{\Gamma_2(a+3)}{2\Gamma_2(a)\Gamma_2(3)} (t_{11}^2)^{a-1} (1-t_{11}^2)^{\frac{3}{2}-1} (t_{22}^2)^{(a-1)-1} (1-t_{22}^2)^{\frac{3}{2}-1} (v^2)^{\frac{-1}{2}} (1-v^2)^0 dt_{11}^2 dt_{22}^2 dv^2 \\ &= \frac{\Gamma_2(a+3)}{2\Gamma_2(a)\Gamma_2(3)} \frac{\Gamma_2(a)\Gamma_2(3/2)}{\Gamma_2(a+3/2)} \left[\frac{\Gamma_2(a+3/2)}{\Gamma_2(a)\Gamma_2(3/2)} (t_{11}^2)^{a-1} (1-t_{11}^2)^{\frac{3}{2}-1} dt_{11}^2 \right] \\ &\quad * \frac{\Gamma_2(a-1)\Gamma_2(3/2)}{\Gamma_2(a+1/2)} \left[\frac{\Gamma_2(a+1/2)}{\Gamma_2(a-1)\Gamma_2(1/2)} (t_{22}^2)^{(a-1)-1} (1-t_{22}^2)^{\frac{3}{2}-1} dt_{22}^2 \right] \\ &\quad * \frac{\Gamma_2(a-1)\Gamma_2(3/2)}{\Gamma_2(a+1/2)} \left[\frac{\Gamma_2(a+1/2)}{\Gamma_2(a-1)\Gamma_2(1/2)} (v^2)^{\frac{-1}{2}} (1-v^2)^0 dv^2 \right] \\ &= \frac{\Gamma_2(a+3)}{2\Gamma_2(a)\Gamma_2(3)} \frac{\Gamma_2(a)\Gamma_2(3/2)}{\Gamma_2(a+3/2)} \frac{\Gamma_2(a-1)\Gamma_2(3/2)}{\Gamma_2(a+1/2)} \frac{\Gamma_2(1/2)\Gamma_2(1)}{\Gamma_2(3/2)} \\ &\quad * B_1(a-1, 3/2, t_{11}) B_1(a, 3/2, t_{11}) B_1(1/2, 1, v^2) \end{aligned}$$

Chapter 4

Matrix-Variate Risk Measures and Related Aspects

Joint work with **Francisco. J. Caro-Lopera** ¹ and **Jean Michel Loubes** ²

Abstract: This paper proposes generalized integrals related to the classical Wishart, Beta, and F distributions. Then the work defines the termed generalized matrix variate Beta and F distributions and the Value-at-Risk (VaR) in the matrix setting. As corollaries, a number of published results about cumulative distribution functions (c.d.f) of Wishart and Beta matrices are also revisited and unified. A new c.d.f for a Wishart random matrix and a solution to an open problem proposed by A. C. Constantine in 1963 are also provided. The extreme latent root distributions for Wishart, Beta, and F are obtained by simple derivation. Relations with the Davis' condition number, theory of shape, and VaR are also established; some particular cases are derived and a perspective for future work is set in that novel direction.

Key Words: Generalized Wishart, Beta and F distribution; Matrix Density Functions; James' Zonal polynomials; Davis' invariant polynomials; Hypergeometric function of matrix argument; Positive definite matrices; Risk measures; Davis' Condition Number; statistical theory of shape.

4.1 Introduction

Matrix-variate distribution theory has occupied a central place in the last 70 years around robust applications in several disciplines. However, the sophisticated underlying mathematics and the computation problems of the distributions have restricted the popularization of the results, meanwhile the publications belong to a very small group of authors, compared with other fields of statistics. The evolution of any theory under the random matrix setting can be traced by the usual matrix Gaussian extension of the real univariate and multivariate cases; the most profuse, elegant and deeply exposition of such enormous advances is given in Muirhead [2005].

Then the results appear in the complex case and sometimes the distributions are generalized for families of distributions of elliptical contours and real normed division algebras; see for example Gupta et al. [2013] and Díaz-García and Gutiérrez-Jáimez [2013], respectively, with the

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references therein.

The central case has ruled the distributions throughout the decades in terms of suitable polynomials of a positive definite matrix argument studied in a series of works of A.T. James during the 60's; see Muirhead [2005] for details of positive definite James' zonal polynomials. Single James' polynomials in the positive definite case were impossible to compute by 50 years; the Ph.D of Parkust listed the polynomials up the 12th order in the real case Parkhurst and James [1974]; a similar computation for the complex case was given by Caro-Lopera (Caro-Lopera and Nagar [2006], Gupta et al. [2006], Caro-Lopera et al. [2007] and the references therein).

At present, recurrence relations by using the Laplace-Beltrami operator allowed the numerical calculations of single polynomials of a positive definite matrix Koev and Edelman [2006], but infinite series of zonal polynomials still involve strong open problems. A matrix variate distribution approach based on positive semidefinite zonal polynomials is also feasible by extension of the Laplace-Beltrami operator Díaz-García and Caro-Lopera [2006], meanwhile exact formulae for James' polynomials, known as Jack's polynomials in real normed division algebras are only available for second order Caro-Lopera et al. [2007].

The non central distribution theory also appeared under the elliptical and real normed division algebras, but in most situations the distributions are untractable because they are expressed in terms of Davis' invariant polynomials of several matrix arguments (see for example Davis [1980]). The creator of the polynomials conjectured in Davis [1979], that the invariant polynomials could be computed in a similar way to the zonal polynomials; the conjecture was sustained almost three decades in Davis [2006], but recently, Caro-Lopera [2016] proved the impossibility of constructing those polynomials in a similar recurrence way by using the Laplace-Beltrami operator. This leaves dozens of papers in Davis' polynomials out of any feasible application.

Only few non central cases of certain distributions expressed in terms of zonal polynomials can be really computed. The elliptical statistical shape theory is one of them, an applied approach explored by several transformations (SVD, QR, affine, polar) and real normed division algebras (real, complex, quaternion and octonion); see for example Caro-Lopera et al. [2010], Díaz-García and Caro-Lopera [2017], Díaz-García and Caro-Lopera [2016] and related works of the authors. Working with general families of distributions instead of the usual normal model allows flexible assumptions rather than the Gaussian based studies in shape theory (see for example Goodall and Mardia [1993], Dryden and Mardia [1998] and references therein).

When the matrix variate setting is studied in the singular case, the above problems enlarges and the computations and applications are so far to be considered; a very reduced number of publications appear in that line, see for example Díaz-García and Gutiérrez-Jáimez [1997], Diaz-Garcia and Gutierrez-Sanchez [2013], Díaz-García and González-Farías [2008] and references therein.

Some extensions from univariate and multivariate cases into the matrix variate version takes decades and the proportion of the associated publications for the matrix studies were extremely unbalanced. It is the case of the matrix variate extension of the well known univariate Birnbaum-Saunders distribution of the 60's. Birnbaum and Saunders [1969] derived the univariate distribution and promoted a large number of research for more than 50 years. Recently, a detail review of theoretical and applied works was given recently by Balakrishnan and Kundu [2019]. The

review described 281 existence reference of the distribution, but only 1 due to Caro-Lopera et al. [2012] addressed the matrix case. The matrix extension arrived to late, because it required first a connection between the Hadamard and matrix products. This arid branch of research accumulates 4 more works on the matrix case but from the same group of authors (Caro-Lopera and Díaz-García [2016], Sánchez et al. [2015], Díaz-García and Caro-Lopera [2019], Díaz-García and Caro-Lopera [2021]).

Now, when the matrix extension is required from the profuse studied univariate and multivariate Value-at-Risk (VaR) a similar situation appears. VaR is just the α -th percentile of certain distribution; in finance, it measures the risk of loss of certain portfolio or financial assets, it also applies for optimization of capital allocation. In the univariate setting the list of publications covers several theoretical and applied works, see for example, Rockafellar and Uryasev [2002], Wagalath and Zubelli [2018], McNeil et al. [2015], Labopin-Richard et al. [2016], Jorion [2007], Pflug [2000] and the references therein. Univariate VaR demands the knowledge of the distribution $F_X(x) = P(X \leq x)$, then it follows as $VaR_\alpha(X) = \mu + F_X^{-1}(\alpha)\sigma$.

Now, when we search for a matrix variate extension for VaR, the literature in finance does not provide any approach. However, from the mathematical point of view, the VaR just requires meaningful percentiles of finance in the context of matrix cumulative distribution functions. This is exactly the domain of the preceding historical notes around matrix variate Gaussian or elliptical non singular or singular distributions in the central or non central cases under real normed division algebras via James' or Davis' polynomials of one of several positive definite or semidefinite matrix arguments.

Once the theoretical domain is elected, the matrix VaR just focuses on computations of probabilities on cones via the well known central (non central), non singular (singular) Wishart, Beta, F distributions based on Gaussian or generalized elliptical or skewelliptical models. Then series of James' and/or Davis polynomials of definite (semidefinite) matrix arguments arrives naturally in the probabilities over positive definite events under general ensembles.

In this way, the matrix VaR first requires computation of difficult integrals which has been studied separately in the literature since the 60's. The most common involves probabilities of Wishart and Beta matrices (results about F distribution can be derived from the Beta integrals). Then a second step must consider the solution of a matrix equation for the matrix VaR, a colossal task which can be attained only in some particular cases. The first stage is far to be solved and the classical results on Wishart and Beta should be revisited. Integration over cones on Wishart and Beta matrices are relatively known, however when we enter into the details we found some curios discrepancies and challenges. For example, computation of probability $P(\mathbf{A} > \mathbf{V})$ for a Wishart matrix \mathbf{A} full derived by Muirhead [2005] and used and cited by several papers, was corrected recently by Caro-Lopera et al. [2016]. In this paper we will see also that $P(\mathbf{A} > \mathbf{V})$ can be computed in another way.

For completeness a general c.d.f involving Davis' invariant polynomials is derived, then the impossibility of its application is inferred, according to the discussion made before. This centers our goal of the work in expressions that we can really compute by using zonal polynomial theory. In a similar way, for the Beta matrix \mathbf{U} , we will provide the solution of the open problem proposed by Constantine in 1963 about the probability $P(\mathbf{U} > \mathbf{V})$, meanwhile the probability $P(\mathbf{U} < \mathbf{V})$ needs to be revisited too Constantine [1963]. For a unified and self-contained exposition, the addressed results for c.d.f of Wishart and Beta matrices will followed as simple corollaries of

two general integrals over positive definite matrices. For completeness, the particular results for latent roots given in literature will follow also as corollaries.

The paper also proposed a generalized matrix variate Beta distribution which involves computable series of zonal polynomials and contains the classical Beta matrix as a simple case. Then the generalized matrix variate F distribution can be defined by a matrix transformation of the generalized Beta, as we expect. Finally, the work also provides a new relationship of the shape theory (cited above) with the condition number and the associated VaR. Some particular cases opens this perspective for future studies.

The paper is structured as follows. In Section 2 some new integrals and probabilistic results involving zonal polynomials are derived. The generalized Wishart cumulative distributions are studied. Then a similar analysis introduces the new generalized Beta distribution and computes the corresponding cumulative distribution functions. This distribution is related with the new generalized F distribution and the cumulative distribution functions follows straightforwardly. Section 3, explores a number of associated extreme latent roots distributions by a simple differentiation, against the cumbersome integration full derived in some isolated works. In Section 4, the matrix-variate Value-at-Risk for Wishart, gamma, beta and F distributions are introduced, then the univariate VaR for χ^2 , gamma, beta, exponential and F distributions are obtained as a particular cases. In Appendix the proofs of the theorems and corollaries are presented. Finally, some conclusions are outlined as well as some possible directions for future work.

4.2 Some Probabilistic Results

It is evident that the estimation of the densities depend strongly on the computation of certain integrals, in this section, we propose some integrals that will allow us to compute probabilities in the matrix case.

Before starting, we want to give sense to the notion of order in the matrix framework. For this we will use the Loewner order, see for example Sharon and Itai [2013]. Let A and B two positive definite matrices, $\mathbf{A} < \mathbf{\Omega}$ means that $\mathbf{\Omega} - \mathbf{A}$ is positive definite. This order will be denoted " $<$ ".

The hypergeometric function of matrix argument occupies a central place in this section, thus we highlight the main properties in the following lines, see Muirhead [2005].

Definition 4.2.1 *The hypergeometric function of matrix argument is given by*

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{A}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_{\kappa} \dots (a_p)_{\kappa} C_{\kappa}(\mathbf{A})}{(b_1)_{\kappa} \dots (b_q)_{\kappa} k!}. \quad (4.2.1)$$

Here $p \geq 0$ and $q \geq 0$ are integers; a_i, b_j are real numbers; $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_m)$ is a partition of k , such that $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 0$ and \sum_{κ} stands for the summation over all the partitions k ; $(a)_{\kappa} = \prod_{i=1}^m (a - i + 1)_{\kappa_i}$, where $(a)_{\kappa} = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$. Note that $(a_i)_{\kappa}, (b_j)_{\kappa}$ are zero if a_i, b_j are negative integers or negative half-integers $\leq \frac{1}{2}(m-1)$. In particular, let $a_1 = -r$ for a positive integer r , then the series vanishes turns into a polynomial of degree mr , given that $(a_1)_k = 0$ for $k \geq mr + 1$. If $p \leq q$, then the series converges for all \mathbf{A} . If $p = q + 1$, the series converges for $\|\mathbf{A}\| < 1$, where $\|\mathbf{A}\|$ is the maximum of the absolutes values of the latent roots of \mathbf{A} . If $p > q + 1$, the series diverges for all $\mathbf{A} \neq 0$, excepting for the addressed polynomial

cases. Finally, for given partition κ , $C_\kappa(\mathbf{A}) = C_\kappa(\lambda_1, \lambda_2, \dots, \lambda_m)$ is the symmetric, homogeneous zonal polynomial of degree $|\kappa|$ in the latent roots $\lambda_1, \lambda_2, \dots, \lambda_m$ of \mathbf{A} . See additional properties of $C_\kappa(\mathbf{A})$ in Muirhead [2005].

The main integrals of this section are given next.

Theorem 4.2.1 *Let \mathbf{Z} be a complex symmetric $m \times m$ matrix, with $\text{Re}(\mathbf{Z}) > \mathbf{0}$, and \mathbf{V}, \mathbf{A} be positive definite $m \times m$ matrices, such that $\mathbf{0} < \mathbf{A} < \mathbf{V} < \mathbf{I}$, then*

$$\begin{aligned} \int_{\mathbf{0} < \mathbf{A} < \mathbf{V}} \text{etr} \left(-\frac{1}{2} \mathbf{Z} \mathbf{A} \right) |\mathbf{A}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1} \mathbf{A}|^{b - \frac{m+1}{2}} (d\mathbf{A}) \\ = \frac{\Gamma_m(a) \Gamma_m(b)}{\Gamma_m(a+b)} |\mathbf{V}|^a {}_1F_1 \left(a; a+b; -\frac{1}{2} \mathbf{Z} \mathbf{V} \right), \end{aligned} \quad (4.2.2)$$

where $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$ and $\text{etr}(\mathbf{A}) = \exp(\text{tr}(\mathbf{A}))$.

Theorem 4.2.2 *Let \mathbf{Z} be a complex symmetric $m \times m$ matrix, with $\text{Re}(\mathbf{Z}) > \mathbf{0}$, and \mathbf{V}, \mathbf{A} be positive definite $m \times m$ matrices, such that $\mathbf{0} < \mathbf{V} < \mathbf{A}$. If $r = a + b - (m+1)$ is a positive integer or a positive half-integer, then*

$$\begin{aligned} \int_{\mathbf{0} < \mathbf{V} < \mathbf{A}} \text{etr} \left(-\frac{1}{2} \mathbf{A} \mathbf{Z} \right) |\mathbf{A}|^{a+b-(m+1)} |\mathbf{V}^{-1} \mathbf{A} - \mathbf{I}|^{-b + \frac{m+1}{2}} (d\mathbf{A}) \\ = \frac{2^{ma} \Gamma_m(a)}{|\mathbf{Z}|^a} |\mathbf{V}|^{b - \frac{m+1}{2}} \text{etr} \left(-\frac{1}{2} \mathbf{V} \mathbf{Z} \right) \\ \times \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* \frac{(-r)_{\kappa}}{\left(-a + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa} \left(\frac{1}{2} \mathbf{V} \mathbf{Z} \right), \end{aligned} \quad (4.2.3)$$

where $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$, and \sum_{κ}^* denotes summation over those partitions $\kappa = (k_1, \dots, k_m)$ of k with $k_1 \leq r$ if r is a positive integer or $k_2 \leq r$ if r is a positive half-integer.

When $b = \frac{m+1}{2}$ the above expression is related with a polemic result back to 60's. Li [1997] tried it by using Muirhead [2005] [Lem. 7.2.12], and then a number of papers used the result. Later the lemma was revised by Díaz-García and Gutiérrez-Jáimez [2010] and proposed a new version in Díaz-García and Gutiérrez-Jáimez [2011] and applications in a series of articles. Finally, Caro-Lopera et al. [2016] gave the right form of the referred lemma and the evaluation of the general integral with consonant corollaries involving the well known particular cases available in the works of A.T. James, A. G. Constantine, C. G. Khatri since the 60's.

In the next section, we will use the two previous theorems to compute $P(\mathbf{A} < \mathbf{V})$ and $P(\mathbf{A} > \mathbf{V})$ when $\mathbf{A} \sim W_m(n, \mathbf{\Sigma})$ and \mathbf{V} is an $m \times m$ positive definite matrix. Finally, we end this section with an integral that provides a generalization of the well known matrix variate beta and F distributions.

Theorem 4.2.3 *Let \mathbf{A} , \mathbf{W} and \mathbf{V} be positive definite $m \times m$ matrices, with $\mathbf{0} < \mathbf{A} < \mathbf{I}$, $\mathbf{0} < \mathbf{W} \mathbf{A} < \mathbf{I}$ and $\mathbf{A} < \mathbf{V}$. Then,*

$$\begin{aligned} \int_{\mathbf{0} < \mathbf{A} < \mathbf{I}} |\mathbf{A}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{W} \mathbf{A}|^{b - \frac{m+1}{2}} (d\mathbf{A}) \\ = \frac{\Gamma_m(a) \Gamma_m \left(\frac{m+1}{2} \right)}{\Gamma_m \left(a + \frac{m+1}{2} \right)} {}_2F_1 \left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W} \right), \end{aligned} \quad (4.2.4)$$

and

$$\begin{aligned} & \int_{\mathbf{0} < \mathbf{A} < \mathbf{V}} |\mathbf{A}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{W}\mathbf{A}|^{b - \frac{m+1}{2}} (d\mathbf{A}) \\ &= \frac{\Gamma_m(a)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(a + \frac{m+1}{2}\right)} |\mathbf{V}|^a {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{V}\mathbf{W}\right); \end{aligned} \quad (4.2.5)$$

where, $Re(a) > \frac{1}{2}(m-1)$, $Re(b) > \frac{1}{2}(m-1)$.

Note that using the integral representation [Muirhead, 2005, Th. 7. 4.2.] of the hypergeometric function, expression (4.2.4) turns into $\Gamma_m(a)\Gamma_m(b)/\Gamma(a+b)$ when $\mathbf{W} = \mathbf{I}_m$; which is the integration constant of a matrix variate beta type distribution, see [Muirhead, 2005, Th. 3.3.1.]. Then (4.2.4) can be used to define a generalized matrix variate beta distribution and a subsequent relation with the associated matrix variate F type distribution can be proposed too.

Before doing so, we note that integrands of (4.2.1) and (4.2.2) generate two new parametric matrix variate distribution which generalizes the Wishart distribution when $b = \frac{m+1}{2}$, however for general b , the constant integration of each case requires the use of A. W. Davis's invariant polynomials of several matrix arguments. This occurs after expanding the exponential and determinant, then products of two zonal polynomials appear, forcing the rising of Davis' polynomials. The polynomials are extensions of zonal polynomials of one matrix argument. Zonal polynomials are eigenfunctions of the Laplace-Beltrami operator, then they can be computed by recurrence relations (James [1968]). For decades, Davis conjectured that the new polynomials can be constructed similarly (Davis [1980] and Davis [2006]), unfortunately, Caro-Lopera [2016] proved that the recurrence construction is not possible, then the computation is only possible for the first small degrees. Literature collects dozens of papers involving Davis' polynomials, but they are so far to be applied. That why we will focus on computable densities in terms of zonal polynomials, which we can handle. Finally, a version of integral (4.2.2) with the generalized beta kernel of (4.2.5), goes also to invariant polynomials. Only if we considered the usual beta distribution, the addressed probability is tractable.

We are now in position to derive a number of results.

4.2.1 Generalized Wishart distribution and cumulative distribution function

Recall that an $m \times m$ matrix $\mathbf{A} = \mathbf{Z}'\mathbf{Z}$, is said to have a matrix-variate Wishart distribution with n degrees of freedom and covariance matrix $\mathbf{\Sigma}$, denoted by $\mathbf{A} \sim W_m(n, \mathbf{\Sigma})$, if its p.d.f is given by

$$\frac{1}{2^{\frac{mn}{2}} \Gamma_m\left(\frac{n}{2}\right) |\mathbf{\Sigma}|^{\frac{n}{2}}} \text{etr}\left(-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{A}\right) |\mathbf{A}|^{\frac{n-m-1}{2}}, \quad \mathbf{A} > \mathbf{0}, \quad n > m-1, \quad (4.2.6)$$

where the $n \times m$ matrix \mathbf{Z} is $N_{n \times m}(\mathbf{0}, \mathbf{I}_n \otimes \mathbf{\Sigma})$, $\Gamma_m(a) = \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^m \Gamma\left[a - \frac{1}{2}(i-1)\right]$, $Re(a) > \frac{1}{2}(m-1)$, $\Gamma(a)$ is the ordinary Gamma function, see Wishart [1928].

Except by a normalization constant, when $b = \frac{m+1}{2}$, the theorem 4.2.1 reduces to a well known cumulative distribution function of the Wishart matrix (see for example [Muirhead, 2005, th. 9.7.1]). However, for a general b , the integral can provide the normalization constant of a new distribution for the positive definite random matrix \mathbf{A} defined in $\mathbf{0} < \mathbf{A} < \mathbf{V}$. If the right hand side of (4.2.2) is denoted by c , then the probability density function of \mathbf{A} is given by:

$$f(\mathbf{A}) = c^{-1} \text{etr}\left(-\frac{1}{2}\mathbf{Z}\mathbf{A}\right) |\mathbf{A}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1}\mathbf{A}|^{b - \frac{m+1}{2}}, \quad (4.2.7)$$

with

$$c = \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} |\mathbf{V}|^a {}_1F_1 \left(a; a+b; -\frac{1}{2}\mathbf{Z}\mathbf{V} \right). \quad (4.2.8)$$

We will say that \mathbf{A} is a *Generalized Wishart Matrix* and this fact will be denoted as $\mathbf{A} \sim GW_m(a, b, \mathbf{Z}, \mathbf{V})$.

Then we can further compute the following cumulative distribution function of \mathbf{A} . Unfortunately, the addressed invariant polynomials of several matrix polynomials appear. We include the computation for completeness and connection with the old results resulting as corollaries.

Theorem 4.2.4 *Assume that $\mathbf{A} \sim GW_m(a, b, \mathbf{Z}, \mathbf{V})$. Let \mathbf{Z} be a complex symmetric $m \times m$ matrix, with $\text{Re}(\mathbf{Z}) > \mathbf{0}$, and \mathbf{V}, \mathbf{G} be positive definite $m \times m$ matrices, such that $\mathbf{0} < \mathbf{G} < \mathbf{V}$, then*

$$\begin{aligned} P(\mathbf{0} < \mathbf{A} < \mathbf{G}) &= \frac{\Gamma_m(a)\Gamma_m(\frac{m+1}{2})}{c\Gamma_m(a + \frac{m+1}{2})} |\mathbf{G}|^a \\ &\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k}{k!} \sum_{\kappa \in P_k} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\lambda \in P_l} \left(\frac{m+1}{2} - b \right)_{\lambda} \\ &\times \sum_{\phi \in \kappa \cdot \lambda} \theta_{\phi}^{\kappa, \lambda} \frac{(a)_{\phi}}{(a + \frac{m+1}{2})_{\phi}} C_{\phi}^{\kappa, \lambda}(\mathbf{Z}\mathbf{G}, \mathbf{V}^{-1}\mathbf{G}), \end{aligned} \quad (4.2.9)$$

where $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$, κ, λ, ϕ are partitions of $k, l, k+l$, respectively; $\phi \in \kappa \cdot \lambda$ represents $2\phi \in 2\kappa \otimes 2\lambda$; $\theta_{\phi}^{\kappa, \lambda}$ comes from $C_{\kappa}(\mathbf{X})C_{\lambda}(\mathbf{Y}) = \sum_{\phi \in \kappa \cdot \lambda} \theta_{\phi}^{\kappa, \lambda} C_{\phi}^{\kappa, \lambda}(\mathbf{X}, \mathbf{Y})$, and $C_{\phi}^{\kappa, \lambda}(\mathbf{X}, \mathbf{Y})$ denote the invariant polynomials of two matrix arguments, extending the zonal polynomials, (see Davis [1980] and the references therein).

Now, we provide a new way of computing an important c.d.f, by using an old integral due to James and Constantine, full derived in [Muirhead, 2005, Th. 7.2.10].

Theorem 4.2.5 *Let $\mathbf{A} \sim W_m(n, \Sigma)$, $n > m-1$ and \mathbf{V} is an $m \times m$ positive definite matrix ($\mathbf{V} > \mathbf{0}$).*

$$P(\mathbf{A} > \mathbf{V}) = \frac{\text{etr}(-\frac{1}{2}\Sigma^{-1}\mathbf{V}) |\frac{1}{2}\Sigma^{-1}\mathbf{V}|^{\frac{n-m-1}{2}}}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m+1}{2})^{-1}} \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* (-r)_{\kappa} \left(\frac{m+1}{2} \right)_{\kappa} C_{\kappa}(-2\Sigma\mathbf{V}^{-1}). \quad (4.2.10)$$

Where $r = \frac{1}{2}(n-m-1)$ is a positive integer or a positive half-integer.

Note that the last expression involves a polynomial, because ${}_2F_0(\cdot, \cdot; \cdot)$ diverges unless the series terminates. The polynomial appears when $r = \frac{1}{2}(n-m-1)$ ranges in the natural domain of the degrees of freedom n and the matrix order m , and the series terminates.

Now, the well known results of upper and lower probabilities in Wishart matrices, full derived in [Muirhead, 2005, Th. 9.7.2, Th. 9.7.3] (and references therein), are consequences of Theorems 4.2.1, 4.2.2 and 4.2.4.

Corollary 4.2.6 *If $\mathbf{A} \sim W_m(n, \Sigma)$, $n > m-1$ and \mathbf{V} is an $m \times m$ positive definite matrix ($\mathbf{V} > \mathbf{0}$) then*

$$P(\mathbf{A} < \mathbf{V}) = \frac{\Gamma_m(\frac{m+1}{2})}{\Gamma_m(\frac{n}{2} + \frac{m+1}{2})} \left| \frac{1}{2}\Sigma^{-1}\mathbf{V} \right|^{\frac{n}{2}} {}_1F_1 \left(\frac{n}{2}; \frac{n+m+1}{2}; -\frac{1}{2}\Sigma^{-1}\mathbf{V} \right). \quad (4.2.11)$$

Recall that $P(\mathbf{A} > \mathbf{V}) \neq 1 - P(\mathbf{A} < \mathbf{V})$ does not hold for $m \geq 2$, then both probabilities must be computed by different methods.

As we quoted before, a correct prove of [Muirhead, 2005, th. 7.2.13] was provided recently, but curiously the integral was unperturbed, because the implicity symmetry of the involved polynomials, see Caro-Lopera et al. [2016] for all the details. The final consequence remains as follows:

Corollary 4.2.7 *Let $\mathbf{A} \sim W_m(n, \mathbf{\Sigma})$, $n > m - 1$ and V is an $m \times m$ positive definite matrix ($\mathbf{V} > \mathbf{0}$).*

$$P(\mathbf{A} > \mathbf{V}) = \text{etr} \left(-\frac{1}{2} \mathbf{\Sigma}^{-1} \mathbf{V} \right) \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* C_{\kappa} \left(\frac{1}{2} \mathbf{\Sigma}^{-1} \mathbf{V} \right). \quad (4.2.12)$$

Where $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer.

Again some interesting hints appear in the history of these particular results. For example, the above corollary was bounded by [Muirhead, 2005, Th. 9.7.3] only for positive integers r , however, it is also true for positive half-integers.

As usual, setting $\mathbf{V} = x\mathbf{I}$, upper and lower probabilities for extreme latent roots distributions can be obtained. We start with a new expression for the c.d.f of a the smallest latent root of a Wishart matrix is given next.

Corollary 4.2.8 *If l_m is the smallest latent root of $\mathbf{A} \sim W_m(n, \mathbf{\Sigma})$, $n > m - 1$, then.*

$$P(l_m > x) = \frac{\text{etr} \left(-\frac{x}{2} \mathbf{\Sigma}^{-1} \right) \left| \frac{x}{2} \mathbf{\Sigma}^{-1} \right|^{\frac{n-m-1}{2}}}{\Gamma_m \left(\frac{n}{2} \right) \Gamma_m \left(\frac{m+1}{2} \right)^{-1}} \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* (-r)_{\kappa} \left(\frac{m+1}{2} \right)_{\kappa} C_{\kappa}(-2x^{-1} \mathbf{\Sigma}). \quad (4.2.13)$$

Where $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer.

As alternatives, the classical results for c.d.f of extreme Wishart latent roots follows from the main integrals derived in Theorems 4.2.2 and 4.2.1.

Corollary 4.2.9 *If l_1 is the largest latent root of $\mathbf{A} \sim W_m(n, \mathbf{\Sigma})$, $n > m - 1$, then its distribution function can be expressed in the form*

$$P(l_1 < x) = \frac{\Gamma_m \left(\frac{m+1}{2} \right)}{\Gamma_m \left(\frac{n}{2} + \frac{m+1}{2} \right)} \left| \frac{1}{2} x \mathbf{\Sigma}^{-1} \right|^{\frac{n}{2}} {}_1F_1 \left(\frac{n}{2}; \frac{n+m+1}{2}; -\frac{1}{2} x \mathbf{\Sigma}^{-1} \right). \quad (4.2.14)$$

Corollary 4.2.10 *If l_m is the smallest latent root of $\mathbf{A} \sim W_m(n, \mathbf{\Sigma})$, $n > m - 1$ and $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer, then*

$$P(l_m > x) = \text{etr} \left(-\frac{1}{2} x \mathbf{\Sigma}^{-1} \right) \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* C_{\kappa} \left(\frac{1}{2} x \mathbf{\Sigma}^{-1} \right). \quad (4.2.15)$$

4.2.2 Generalized Beta distribution and cumulative distribution function

Theorem 4.2.3 provide us the following generalization of the classical matrix variate beta distribution.

Definition 4.2.2 Let \mathbf{U} , \mathbf{W} be positive definite $m \times m$ matrices, with $\mathbf{0} < \mathbf{U} < \mathbf{I}_m$, $\mathbf{0} < \mathbf{W}\mathbf{U} < \mathbf{I}_m$. Then the p.d.f of the termed generalized matrix variate beta is given by

$$f_{\mathbf{U}}(\mathbf{U}) = \frac{\Gamma_m^{-1}(a)\Gamma_m^{-1}\left(\frac{m+1}{2}\right)\Gamma_m\left(a + \frac{m+1}{2}\right)}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} |\mathbf{U}|^{a - \frac{m+1}{2}} |\mathbf{I}_m - \mathbf{W}\mathbf{U}|^{b - \frac{m+1}{2}} \quad (4.2.16)$$

where, $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$.

When $\mathbf{W} = \mathbf{I}_m$, $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, $n_1, n_2 \in \mathcal{N}$, we have the classical beta distribution

$$\frac{\Gamma_m\left(\frac{n_1}{2} + \frac{n_2}{2}\right)}{\Gamma_m\left(\frac{n_1}{2}\right)\Gamma_m\left(\frac{n_2}{2}\right)} |\mathbf{U}|^{\frac{n_1}{2} - \frac{m+1}{2}} |\mathbf{I}_m - \mathbf{U}|^{\frac{n_2}{2} - \frac{m+1}{2}}, \quad \mathbf{0} < \mathbf{U} < \mathbf{I}_m. \quad (4.2.17)$$

which follows from \mathbf{A} and \mathbf{B} independent Wishart distribution, i.e. $\mathbf{A} \sim W_m(n_1, \mathbf{\Sigma})$ and $\mathbf{B} \sim W_m(n_2, \mathbf{\Sigma})$ and $\mathbf{U} = \mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}$. We denote the generalized beta as $\mathbf{U} \sim \text{Beta}_m(a, b, \mathbf{W})$.

Now, Lemma 4.2.2 allows the computation of $P(\mathbf{U} < \mathbf{V})$, for general \mathbf{W} , in terms of zonal polynomials

Corollary 4.2.11 Let $\mathbf{U} \sim \text{Beta}_m(a, b, \mathbf{W})$. If $\mathbf{0} < \mathbf{V} < \mathbf{I}_m$ is an $m \times m$ matrix, then

$$P(\mathbf{U} < \mathbf{V}) = \frac{|\mathbf{V}|^a}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{V}\mathbf{W}\right)} {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{V}\mathbf{W}\right) \quad (4.2.18)$$

Taking $\mathbf{W} = \mathbf{I}_m$, $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, $n_1, n_2 \in \mathcal{N}$, for the classical beta we have

$$P(\mathbf{U} < \mathbf{V}) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right)\Gamma_m\left(\frac{n_1}{2} + \frac{m+1}{2}\right)} |\mathbf{V}|^a {}_2F_1\left(\frac{n_1}{2}, \frac{m+1}{2} - \frac{n_2}{2}; \frac{n_1}{2} + \frac{m+1}{2}; \mathbf{V}\right). \quad (4.2.19)$$

This result was full derived by [Constantine, 1963, Th.7,eq (61)], unfortunately with some errors, as the reader can check. In the same paper Constantine pointed out that the complementary probability $P(\mathbf{U} > \mathbf{V})$ seems difficult to evaluate. For a general beta, the computation goes in terms of Davis' polynomials, then we need to come back to the classical beta distribution. The open problem claimed by Constantine in 1963 is derived next.

Corollary 4.2.12 Let $\mathbf{U} \sim \text{Beta}_m(a, b, \mathbf{I}_m)$ and $\mathbf{0} < \mathbf{V} < \mathbf{I}_m$ is an $m \times m$ matrix. If $r = a - \frac{m+1}{2}$ is a positive integer or a positive half-integer, then:

$$\begin{aligned} P(\mathbf{U} > \mathbf{V}) &= \frac{\Gamma_m(a+b)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m(a)\Gamma_m\left(b + \frac{m+1}{2}\right)} |\mathbf{V}|^{a - \frac{m+1}{2}} |\mathbf{I}_m - \mathbf{V}|^b \\ &\times \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* \frac{(-r)_{\kappa} \left(\frac{m+1}{2}\right)_{\kappa}}{\left(b + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa}(\mathbf{I}_m - \mathbf{V}^{-1}). \end{aligned} \quad (4.2.20)$$

As in the Wishart case, taking $\mathbf{V} = x\mathbf{I}$, we find the c.d.f for the largest and the smallest latent roots of a generalized and regular Beta distribution, respectively.

Corollary 4.2.13 Let u_1 be the largest latent root of $\mathbf{U} \sim \text{Beta}_m(a, b, \mathbf{W})$. The c.d.f of u_1 is given by

$$P(u_1 < x) = \frac{x^{ma}}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; x\mathbf{W}\right)} {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; x\mathbf{W}\right) \quad (4.2.21)$$

The simple case $\mathbf{W} = \mathbf{I}_m$, $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, ($m = p$), $n_1, n_2 \in \mathcal{N}$ turns into the result of Venables [1973].

$$P(u_1 < x) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right)\Gamma_m\left(\frac{n_1}{2} + \frac{m+1}{2}\right)} x^{\frac{mn_1}{2}} {}_2F_1\left(\frac{n_1}{2}, \frac{m+1}{2} - \frac{n_2}{2}; \frac{n_1}{2} + \frac{m+1}{2}; x\mathbf{I}_m\right). \quad (4.2.22)$$

Corollary 4.2.14 *Let u_m be the smallest latent root of $\mathbf{U} \sim \text{Beta}_m(a, b, \mathbf{I})$. The c.d.f of u_m , for a positive integer or a positive half-integer $r = a - \frac{m+1}{2}$, is given by*

$$\begin{aligned} P(u_m > x) &= \frac{\Gamma_m(a+b)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m(a)\Gamma_m\left(b + \frac{m+1}{2}\right)} x^{m(a-\frac{m+1}{2})} (1-x)^{mb} \\ &\quad \times \sum_{k=0}^{mr} \frac{(1-x^{-1})^k}{k!} \sum_{\kappa}^* \frac{(-r)_{\kappa} \left(\frac{m+1}{2}\right)_{\kappa}}{\left(b + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa}(\mathbf{I}_m). \end{aligned} \quad (4.2.23)$$

4.2.3 Generalized F distribution and cumulative distribution function

In this part we connect the generalized matrix variate beta distribution with a new distribution containing the classical F distribution.

Let $\mathbf{U} \sim \text{Beta}_m(a, b, \mathbf{W})$. Taking $\mathbf{U} = \mathbf{F}(\mathbf{I}_m + \mathbf{W}\mathbf{F})^{-1}$ with $\mathbf{F} > \mathbf{0}$, we have $(d\mathbf{U}) = |\mathbf{I} + \mathbf{W}\mathbf{F}|^{-(m+1)}(d\mathbf{F})$ and then by Lemma 4.2.2 we arrive at the following result.

Definition 4.2.3 *Let \mathbf{F} be positive definite $m \times m$ matrix, and $\mathbf{W} > \mathbf{0}$ defined according to (4.2.16). Then the p.d.f of the termed generalized matrix variate F, denoted by $\mathbf{F} \sim F_m(a, b, \mathbf{W})$, is given by*

$$f_{\mathbf{F}}(\mathbf{F}) = \frac{\Gamma_m^{-1}(a)\Gamma_m^{-1}\left(\frac{m+1}{2}\right)\Gamma_m\left(a + \frac{m+1}{2}\right)}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} |\mathbf{F}|^{a-\frac{m+1}{2}} |\mathbf{I} + \mathbf{W}\mathbf{F}|^{-(a+b)} \quad (4.2.24)$$

where, $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$.

When $\mathbf{W} = \mathbf{I}_m$, $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, $n_1, n_2 \in \mathcal{N}$, we have the well known F distribution

$$\frac{\Gamma_m\left(\frac{n_1}{2} + \frac{n_2}{2}\right)}{\Gamma_m\left(\frac{n_1}{2}\right)\Gamma_m\left(\frac{n_2}{2}\right)} |\mathbf{F}|^{\frac{n_1}{2} - \frac{m+1}{2}} |\mathbf{I} + \mathbf{F}|^{-\frac{n_1+n_2}{2}}, \quad \mathbf{0} < \mathbf{F}. \quad (4.2.25)$$

Now, for upper and lower probabilities of \mathbf{F} distribution, we just use the connection with beta matrix and the probabilities found in the previous section. In particular, let $\mathbf{U} \sim \text{Beta}_m(a, b, \mathbf{W})$ and $\mathbf{F} \sim F_m(a, b, \mathbf{W})$, given that $P(\mathbf{F} < \mathbf{V}) = P(\mathbf{U} < \mathbf{V}(\mathbf{I} + \mathbf{W}\mathbf{V})^{-1})$, we have,

Corollary 4.2.15 *Let $\mathbf{F} \sim F_m(a, b, \mathbf{W})$, then*

$$\begin{aligned} P(\mathbf{F} < \mathbf{V}) &= \frac{|\mathbf{V}(\mathbf{I} + \mathbf{W}\mathbf{V})^{-1}|^a}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} \\ &\quad {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\mathbf{V}(\mathbf{I} + \mathbf{W}\mathbf{V})^{-1}\right) \end{aligned} \quad (4.2.26)$$

Trivial cases for meaningful degrees of freedom can be obtained with $\mathbf{W} = \mathbf{I}_m$, $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, $n_1, n_2 \in \mathcal{N}$.

$$P(\mathbf{F} < \mathbf{V}) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right)\Gamma_m\left(\frac{n_1}{2} + \frac{m+1}{2}\right)} |\mathbf{V}(\mathbf{I} + \mathbf{V})^{-1}|^a {}_2F_1\left(\frac{n_1}{2}, \frac{m+1}{2} - \frac{n_2}{2}; \frac{n_1}{2} + \frac{m+1}{2}; \mathbf{V}(\mathbf{I} + \mathbf{V})^{-1}\right) \quad (4.2.27)$$

A computable complementary probability avoiding Davis' invariant polynomials only can be set for $\mathbf{W} = \mathbf{I}_m$, see (4.2.20). Using the fact that $P(\mathbf{F} > \mathbf{V}) = P(\mathbf{U} > \mathbf{V}(\mathbf{I} + \mathbf{V})^{-1})$, we obtain:

Corollary 4.2.16 *Let $\mathbf{F} \sim F_m(a, b, \mathbf{I}_m)$. If $r = a - \frac{m+1}{2}$ is a positive integer, or a positive half-integer.*

$$\begin{aligned} P(\mathbf{F} > \mathbf{V}) &= \frac{\Gamma_m(a+b) \Gamma_m(\frac{m+1}{2})}{\Gamma_m(a) \Gamma_m(b + \frac{m+1}{2})} |\mathbf{V}|^{a - \frac{m+1}{2}} |\mathbf{I} + \mathbf{V}|^{\frac{m+1}{2} - a - b} \\ &\times \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* \frac{(-r)_{\kappa} (\frac{m+1}{2})_{\kappa}}{(b + \frac{m+1}{2})_{\kappa}} C_{\kappa}(-\mathbf{V}^{-1}) \end{aligned} \quad (4.2.28)$$

Cumulative distribution functions of extremal latent roots of general (or regular) F distribution are listed next.

Corollary 4.2.17 *Let f_1 be the largest latent root of $\mathbf{F} \sim F_m(a, b, \mathbf{W})$. The c.d.f of f_1 is given by*

$$\begin{aligned} P(f_1 < x) &= \frac{x^{ma} |\mathbf{I} + x\mathbf{W}|^{-a}}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} \\ &{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; x\mathbf{W}(\mathbf{I} + x\mathbf{W})^{-1}\right) \end{aligned} \quad (4.2.29)$$

The trivial case when $\mathbf{W} = \mathbf{I}_m$, $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, $n_1, n_2 \in \mathcal{N}$, was full derived only for univariate events by [Chikuse, 1977, eq. (3.4)]. Once Euler relation ${}_2F_1(a, b; c; \mathbf{Z}) = |\mathbf{I} - \mathbf{Z}|^{-b} {}_2F_1(c-a, b; c; -\mathbf{Z}(\mathbf{I} - \mathbf{Z})^{-1})$ is applied in the referred c.d.f, $P(f_1 < x)$ simplifies as we expect:

$$\frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right) \Gamma_m\left(\frac{n_1+m+1}{2}\right)} \sum_{k=0}^{\infty} \frac{(1+x)^{-\frac{mn_1}{2}-k}}{x^{-k-\frac{mn_1}{2}} k!} \sum_{\kappa} \frac{\left(\frac{n_1}{2}\right)_{\kappa} \left(\frac{m+1}{2} - \frac{n_2}{2}\right)_{\kappa}}{\left(\frac{n_1}{2} + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa}(\mathbf{I}_m) \quad (4.2.30)$$

Finally, for the smallest latent root of a regular F , we have:

Corollary 4.2.18 *Let f_m be the smallest latent root of $\mathbf{F} \sim F_m(a, b, \mathbf{I}_m)$. For a positive integer or a positive half-integer $r = a - \frac{m+1}{2}$, the c.d.f of f_m is given by.*

$$\begin{aligned} P(f_m > x) &= \frac{\Gamma_m(a+b) \Gamma_m(\frac{m+1}{2})}{\Gamma_m(a) \Gamma_m(b + \frac{m+1}{2})} x^{m(a - \frac{m+1}{2})} (1+x)^{m(\frac{m+1}{2} - a - b)} \\ &\times \sum_{k=0}^{mr} \frac{(-x)^{-k}}{k!} \sum_{\kappa}^* \frac{(-r)_{\kappa} (\frac{m+1}{2})_{\kappa}}{(b + \frac{m+1}{2})_{\kappa}} C_{\kappa}(\mathbf{I}_m) \end{aligned} \quad (4.2.31)$$

Now, applying the Euler relation ${}_2F_1(a, b; c; \mathbf{Z}) = |\mathbf{I} - \mathbf{Z}|^{c-a-b} {}_2F_1(c-a, c-b; c; \mathbf{Z})$, taking $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, $n_1, n_2 \in \mathcal{N}$ and noting that the hypergeometric function becomes a polynomial with the addressed parameters, we have obtained the modified version of [Chikuse, 1977, Lemma 3.2] via [Chikuse, 1977, eq. (3.4)]. In that work the derivation for the smallest latent root was possible because no events involved matrices were considered, and equivalent events in univariate variables are simple. But in the matrix case, as given here, both cumulative distribution functions for more interesting applications in matrix VaR, for example, must be derived separately; then the corollaries for univariate events can be derived easily.

4.3 Extreme Latent Root Distributions

Distributions of the extreme latent roots of positive definite matrices are useful in a number of modern applied areas because they can define bounds for random processes, laws, and principles. They for example provide information about computational problems of inversions of matrices and risks or survival phenomena based on positive definite random matrices or singular value distributions. Curiously, the implicit power of cumulative distribution functions for matrices is not used rarely, and the required univariate distributions for extreme latent roots are obtained by multiple integrations of the joint latent root distribution.

In this section we take advance of the c.d.f before derived, and the extreme latent root distribution follows by simple differentiation. In the sequel we just inform the results which are straightforwardly obtained.

The previous corollaries allow to calculate the probability density function of extreme latent roots of \mathbf{A} .

Corollary 4.3.1 *The pdf of the largest latent roots l_1 of $\mathbf{A} \sim W_m(n, \Sigma)$, $n > m - 1$ is given by*

$$f(l_1) = \frac{\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n}{2} + \frac{m+1}{2}\right)} \left| \frac{1}{2} \Sigma^{-1} \right|^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{\left(k + \frac{mn}{2}\right)}{k!} l_1^{k + \frac{mn}{2} - 1} \sum_{\kappa} \frac{\left(\frac{n}{2}\right)_{\kappa}}{\left(\frac{n+m+1}{2}\right)_{\kappa}} C_{\kappa} \left(-\frac{1}{2} \Sigma^{-1} \right). \quad (4.3.1)$$

Note that this result coincides with the result obtained in the proof of Theorem 1. in Shinozaki et al. [2018], they were based on the results obtained in Caro-Lopera et al. [2016] and then used the results of Sugiyama [1967] to find the marginal of l_1 using the traditional methodology of multiple integration with respect to the other latent roots. We propose we obtain the same result but with an easy derivative.

Corollary 4.3.2 *If $n > m - 1$ and $r = \frac{1}{2}(n - m - 1)$ a positive integer is or a positive half-integer. The pdf of the smallest latent roots l_m of $\mathbf{A} \sim W_m(n, \Sigma)$, given by*

$$f(l_m) = \frac{1}{2} \text{tr}(\Sigma^{-1}) e^{-\frac{1}{2} x \text{tr}(\Sigma^{-1})} \sum_{k=0}^{mr} \frac{\left(\frac{x}{2}\right)^k}{k!} \sum_{\kappa^*} C_{\kappa}(\Sigma^{-1}) - \frac{1}{2} e^{-\frac{1}{2} x \text{tr}(\Sigma^{-1})} \sum_{k=0}^{mr-1} \frac{\left(\frac{x}{2}\right)^k}{k!} \sum_{\kappa^*} C_{\kappa}(\Sigma^{-1}) \quad (4.3.2)$$

In particular we can get Edelman's Theorem 4.2 by taking $n = m + 1$ and $\Sigma = \mathbf{I}$.

Corollary 4.3.3 *(Edelmann's Thesis, th. 4.2) The cdf of the smallest latent roots l_m of $\mathbf{A} \sim W_m(m + 1, \mathbf{I})$ is given by*

$$f(l_m) = \frac{m}{2} e^{-\frac{m}{2} l_m}. \quad (4.3.3)$$

It is important to note that in order to obtain the Theorem 4.2 in Edelman's thesis Edelman [1989] it was necessary to develop a whole chapter in which an expression was first proposed for the pdf of the smallest (similar to what was previously developed by Davis [1972]) and then recursively developed a method considering different for degrees of freedom, however it is surprising that the calculation is reduced to obtain it from the derivative.

Similarly, we can obtain the distributions for the beta and f distributions by deriving the results obtained in the Corollaries 4.2.13 and 4.2.14.

Corollary 4.3.4 The pdf of the largest latent roots u_1 of $\mathbf{A} \sim \text{Beta}_m(a, b, \mathbf{W})$ is given by

$$f(u_1) = \frac{u_1^{ma-1}}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} \sum_{k=0}^{\infty} \frac{(k+ma) u_1^k}{k!} \sum_{\kappa} \frac{(a)_{\kappa} \left(\frac{m+1}{2} - b\right)_{\kappa}}{\left(a + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa}(\mathbf{W}) \quad (4.3.4)$$

In particular if $\mathbf{W} = \mathbf{I}_m$

$$f(u_1) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right) \Gamma_m\left(\frac{n_1}{2} + \frac{m+1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(k + \frac{mn_1}{2}\right)}{k!} u_1^{k + \frac{mn_1}{2} - 1} \quad (4.3.5)$$

$$* \sum_{\kappa} 2^{2\kappa} \kappa! \frac{\left(\frac{m}{2}\right)_{\kappa} \left(\frac{n_1}{2}\right)_{\kappa} \left(\frac{m+1-n_2}{2}\right)_{\kappa}}{\left(\frac{n_1+m+1}{2}\right)_{\kappa}} \frac{\prod_{i<j}^p (2\kappa_i - 2\kappa_j - i + j)}{\prod_{i<j}^p (2\kappa_i + p - i)!}$$

Corollary 4.3.5 If $r = \frac{n_1}{2} - \frac{m+1}{2}$ is a positive integer or a positive half-integer then, the pdf of the smallest latent roots u_m of $\mathbf{A} \sim \text{Beta}_m\left(\frac{n_1}{2}, \frac{n_2}{2}, \mathbf{I}_m\right)$ is given by

$$f(u_m) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_1}{2}\right) \Gamma_m\left(\frac{n_2}{2} + \frac{m+1}{2}\right)} \sum_{k=0}^{mr} \frac{(-1)^k \left(-\frac{1}{2}\right)}{k!} (1 - u_m)^{-1+k+\frac{mn_2}{2}} u_m^{-1-k-\frac{1}{2}m(1+m-n_1)} \quad (4.3.6)$$

$$* (2k - m^2(-1 + u_m) + m(-1 + n_1)(-1 + u_m) + mn_2 u_m)$$

$$* \sum_{\kappa^*} 2^{2\kappa} \kappa! \frac{\left(\frac{m}{2}\right)_{\kappa} (-r)_{\kappa} \left(\frac{m+1}{2}\right)_{\kappa}}{\left(\frac{n_2+m+1}{2}\right)_{\kappa}} \frac{\prod_{i<j}^p (2\kappa_i - 2\kappa_j - i + j)}{\prod_{i<j}^p (2\kappa_i + p - i)!}.$$

Making the corresponding transformations in Corollaries 4.3.4 and 4.3.5 respectively we have the extrem latent roots distribution for the F distribution.

Corollary 4.3.6 The pdf of the largest latent roots f_1 of $\mathbf{A} \sim F_m\left(\frac{n_1}{2}, \frac{n_2}{2}, \mathbf{W}\right)$ is given by

$$f(f_1) = \frac{f_1^{ma-1} (1 + wf_1)^{-ma+1}}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} \sum_{k=0}^{\infty} \frac{(k+ma) f_1^k (1 + wf_1)^{-k}}{k!} \sum_{\kappa} \frac{(a)_{\kappa} \left(\frac{m+1}{2} - b\right)_{\kappa}}{\left(a + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa}(\mathbf{W}) \quad (4.3.7)$$

In particular if $\mathbf{W} = \mathbf{I}_m$

$$f(f_1) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right) \Gamma_m\left(\frac{n_1}{2} + \frac{m+1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(k + \frac{mn_1}{2}\right)}{k!} f_1^{-1-k+\frac{mn_1}{2}} (1 + f_1)^{-1-k-\frac{mn_1}{2}} \quad (4.3.8)$$

$$* \sum_{\kappa} 2^{2\kappa} \kappa! \frac{\left(\frac{m}{2}\right)_{\kappa} \left(\frac{n_1}{2}\right)_{\kappa} \left(\frac{m+1-n_2}{2}\right)_{\kappa}}{\left(\frac{n_1+m+1}{2}\right)_{\kappa}} \frac{\prod_{i<j}^p (2\kappa_i - 2\kappa_j - i + j)}{\prod_{i<j}^p (2\kappa_i + p - i)!}$$

Corollary 4.3.7 If $r = \frac{n_1}{2} - \frac{m+1}{2}$ is a positive integer or a positive half-integer then, the pdf of the smallest latent roots f_m of $\mathbf{A} \sim F_m\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ is given by

$$f(f_m) = \frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n_1}{2}\right) \Gamma_m\left(\frac{n_2}{2} + \frac{m+1}{2}\right)} \sum_{k=0}^{mr} \frac{(-1)^k \left(-\frac{1}{2}\right)}{k!} (1 + f_m)^{-1+\frac{1}{2}m(1+m-n_1-n_2)} f_m^{-1-k-\frac{1}{2}m(1+m-n_1)} \quad (4.3.9)$$

$$* (2k(1 + f_m) + m(1 + m - n_1 + n_2 f_m)) \sum_{\kappa^*} 2^{2\kappa} \kappa! \frac{\left(\frac{m}{2}\right)_{\kappa} (-r)_{\kappa} \left(\frac{m+1}{2}\right)_{\kappa}}{\left(\frac{n_2+m+1}{2}\right)_{\kappa}} \frac{\prod_{i<j}^p (2\kappa_i - 2\kappa_j - i + j)}{\prod_{i<j}^p (2\kappa_i + p - i)!}.$$

4.4 Matrix Value-at-Risk Of Certain Distributions Based On Positive Definite Matrix

In this section, we return to the results of Section 4.2 in order to introduce two alternative extensions of the classical univariate VaR for matrix-variate Wishart, gamma, F and beta distributions.

In the usual univariate setting of Cousin and Di Bernardino [2013], the Value-at-Risk is the minimal amount of the loss which accumulates a probability α to the left tail and $1 - \alpha$ to the right tail. Then, if $F_X(x) = P(X \leq x)$ denotes the cumulative distribution function associated with risk X , and $\overline{F}_X(x) = P(X > x)$ its associated survival function, then the Value-at-Risk ($VaR_\alpha(X)$) at the confidence level $\alpha \in (0, 1)$ is a real number such that

$$VaR_\alpha(X) = \sup\{u \mid \overline{F}(u) \geq 1 - \alpha\}.$$

and equivalently,

$$VaR_\alpha(X) = \min\{u \mid F(u) \geq \alpha\},$$

If F is strictly increasing then the univariate Value-at-Risk means a single real value, in the matrix variate case a set of values satisfy the above definitions. For this set we use the term Matrix-variate Value-at-Risk. Since in the case matrix variate the two previous definitions are not equivalent because $F(X) + \overline{F}_X(X) = 1$ is not true in general, it is necessary to define a Value-at-Risk for each case.

4.4.1 Matrix Value-at-Risk For Wishart distributions

Starting from the Corollaries 4.2.6 and 5.2.5, we introduce here two alternative matrix-variate extensions of the VaR measure. We will denote $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$ our matrix upper VaR associate with $P(\mathbf{X} < VaR_\alpha(\mathbf{X}))$ and $\underline{\mathbf{VaR}}_\alpha(\mathbf{X})$ the matrix lower VaR associate with $P(\mathbf{X} > VaR_\alpha(\mathbf{X}))$.

Definition 4.4.1 Let $\mathbf{X} \sim W_m(n, \Sigma)$, $n > m - 1$. The matrix upper VaR $\overline{\mathbf{VaR}}_\alpha(\mathbf{X}) > \mathbf{0}$ for the matrix variate wishart distribution at probability level $\alpha \in (0, 1)$ is solution of the Confluent hypergeometric equation of matrix argument

$$\frac{\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n}{2} + \frac{m+1}{2}\right)} \left| \frac{1}{2} \Sigma^{-1} \overline{\mathbf{VaR}}_\alpha(\mathbf{X}) \right|^{\frac{n}{2}} {}_1F_1\left(\frac{n}{2}; \frac{n+m+1}{2}; -\frac{1}{2} \Sigma^{-1} \overline{\mathbf{VaR}}_\alpha(\mathbf{X})\right) = \alpha \quad (4.4.1)$$

Now, we introduce $\underline{\mathbf{VaR}}_\alpha(\mathbf{X})$ based on $P(\mathbf{X} > VaR_\alpha(\mathbf{X}))$.

Definition 4.4.2 Let $\mathbf{X} \sim W_m(n, \Sigma)$, $n > m - 1$ and let $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer. The matrix lower VaR $\underline{\mathbf{VaR}}_\alpha(\mathbf{X}) > \mathbf{0}$ for the matrix variate wishart distribution at probability level $\alpha \in (0, 1)$ is obtained solving the following equation

$$\text{etr}\left(-\frac{1}{2} \Sigma^{-1} \underline{\mathbf{VaR}}_\alpha(\mathbf{X})\right) \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* C_{\kappa} \left(\frac{1}{2} \Sigma^{-1} \underline{\mathbf{VaR}}_\alpha(\mathbf{X})\right) = 1 - \alpha. \quad (4.4.2)$$

Proposition 4.4.1 Let $\mathbf{X} \sim W_m(n_1, \Sigma)$, $n_1 > m - 1$ and $\mathbf{Y} \sim W_m(n_2, \Sigma)$, $n_2 > m - 1$, for $\alpha \in (0, 1)$, the matrix upper VaR and matrix lower VaR for the matrix variate wishart distribution satisfy the following properties

1. *Monotonicity: If $\mathbf{X} \leq \mathbf{Y}$, then*

$$\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) \leq \overline{\mathbf{VaR}}_{\alpha}(\mathbf{Y}), \quad \underline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) \leq \underline{\mathbf{VaR}}_{\alpha}(\mathbf{Y}).$$

2. *Positive homogeneity: For all symmetric matrix $\Omega \geq 0$,*

$$\overline{\mathbf{VaR}}_{\alpha}(\Omega\mathbf{X}) = \Omega\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X}), \quad \underline{\mathbf{VaR}}_{\alpha}(\Omega\mathbf{X}) = \Omega\underline{\mathbf{VaR}}_{\alpha}(\mathbf{X}).$$

3. *Translation invariance: For all For all symmetric matrix $\Omega \geq 0$,*

$$\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X} + \Omega) = \overline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) + \Omega, \quad \underline{\mathbf{VaR}}_{\alpha}(\mathbf{X} + \Omega) = \underline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) + \Omega.$$

The properties of monotonicity, positive homogeneity and translation invariance are satisfied since these properties of are conserved for matrix-variate Wishart distributions, see Gupta and Nagar [2000].

If $A \sim W_m(n, \Sigma)$, then $A \sim \Gamma_m(\frac{n}{2}, \frac{1}{2}\Sigma^{-1})$, where Γ_m is the matrix variate gamma distribution. We can get immediatly the matrix upper and lower Value-at-Risk for the matrix variate gamma distribution.

Definition 4.4.3 *Let $\mathbf{X} \sim \Gamma_m(a, \mathbf{C})$, $a > \frac{m-1}{2}$. The matrix upper VaR $\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) > \mathbf{0}$ for the matrix variate gamma distribution at probability level $\alpha \in (0, 1)$ is solution of the Confluent hypergeometric equation of matrix argument*

$$\frac{\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(a + \frac{m+1}{2}\right)} |\mathbf{C}\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X})|^a {}_1F_1\left(a; a + \frac{m+1}{2}; -\mathbf{C}\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X})\right) = \alpha \quad (4.4.3)$$

Definition 4.4.4 *Let $\mathbf{X} \sim \Gamma_m(a, \mathbf{C})$, $a > \frac{m-1}{2}$ and let $r = a - \frac{1}{2}(m+1)$ is a positive integer or a positive half-integer. The matrix lower VaR $\underline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) > \mathbf{0}$ for the matrix variate gamma distribution at probability level $\alpha \in (0, 1)$ is obtained solving the following equation*

$$\text{etr}\left(-\mathbf{C}\underline{\mathbf{VaR}}_{\alpha}(\mathbf{X})\right) \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* C_{\kappa}\left(\mathbf{C}\underline{\mathbf{VaR}}_{\alpha}(\mathbf{X})\right) = 1 - \alpha, \quad (4.4.4)$$

The univariate VaR for Wishart and gamma distribution deserves a detailed study because this corresponds to the Value-at-Risk of the chi-square and gamma distribution. The results can be obtained when $m = 1$ of Definitions 4.4.1, 4.4.2, 4.4.3 and 4.4.4.

Definition 4.4.5 *Let $X \sim \chi_{\nu}^2$ with $\nu > 0$. The univariate VaR of the chi-square distribution at probability level $\alpha \in (0, 1)$ is the unique solution in the interval $[0, \infty)$ of the Confluent hypergeometric equation*

$$\frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2} + 1\right)} (\text{VaR}_{\alpha}(X))^{\frac{\nu}{2}} {}_1F_1\left(\frac{\nu}{2}; \frac{\nu}{2} + 1; \frac{-1}{2}\text{VaR}_{\alpha}(X)\right) = \alpha. \quad (4.4.5)$$

In particular, taking $\nu = 2$ we can obtain a closed analytical expression for the VaR of the exponential distribution.

Definition 4.4.6 *Let $X \sim \text{Exp}(\lambda)$ with $\lambda > 0$ and $x \geq 0$. The univariate VaR, of the exponential distribution at probability level $\alpha \in (0, 1)$ is given by*

$$\text{VaR}_{\alpha}(X) = -\frac{1}{\lambda} \ln(1 - \alpha). \quad (4.4.6)$$

Definition 4.4.7 Let $X \sim \Gamma(a, \lambda)$ with $a > 0$, $\lambda > 0$ and $x \geq 0$. The univariate VaR of the gamma distribution at probability level $\alpha \in (0, 1)$ is the unique solution, in the interval $[0, \infty)$, of the Confluent hypergeometric equation

$$\frac{\lambda^a}{\Gamma(a)} \frac{(VaR_\alpha(X))^a}{a} {}_1F_1(a, ; a+1, -\lambda VaR_\alpha(X)) - \alpha = 0 \quad (4.4.7)$$

Finally, if $a \in \mathcal{Z}^+$, we have the VaR of the Erlang.

Definition 4.4.8 Let X follows a Erlang distribution. The univariate VaR of the Erlang distribution at probability level $\alpha \in (0, 1)$ is the unique solution in the interval $[0, \infty)$ of the Confluent hypergeometric equation

$$\frac{\lambda^a}{a!} (VaR_\alpha(X))^a {}_1F_1(a, ; a+1, -\lambda V(X)) - \alpha = 0 \quad (4.4.8)$$

4.4.2 Matrix Value-at-Risk For Beta distributions

The definitions of matrix Value-at-Risk for the classical matrix variate beta distribution were introduced in Arias-Serna et al. [2021], now we provide the definition for the generalized beta distribution.

Definition 4.4.9 Let $\mathbf{X} \sim Beta_m(a, b, \mathbf{W})$ with $a > \frac{m-1}{2}$, $b > \frac{m-1}{2}$. The matrix upper VaR for the generalized matrix variate beta distribution at probability level $\alpha \in (0, 1)$ is solution of the Gaussian hypergeometric equation of matrix argument.

$$\frac{|\overline{\mathbf{VaR}}_\alpha(\mathbf{X})|^a}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \overline{\mathbf{VaR}}_\alpha(\mathbf{X})\mathbf{W}\right) = \alpha \quad (4.4.9)$$

$$\mathbf{0} < \overline{\mathbf{VaR}}_\alpha(\mathbf{X}) < \mathbf{I}_m.$$

Definition 4.4.10 Let $\mathbf{X} \sim Beta_m(a, b, \mathbf{I}_m)$ with $a > \frac{m-1}{2}$, $b > \frac{m-1}{2}$ and let $r = a - \frac{m+1}{2}$ be a positive integer or a positive half-integer. The matrix lower VaR for the matrix variate beta distribution at probability level $\alpha \in (0, 1)$ is obtained solving the following equation.

$$|\mathbf{I} - \underline{\mathbf{VaR}}_\alpha(\mathbf{X})|^b \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa^*} (-r)_\kappa \frac{\left(\frac{m+1}{2}\right)_\kappa}{\left(\frac{m+1}{2} + b\right)_\kappa} C_\kappa(-(\underline{\mathbf{VaR}}_\alpha(\mathbf{X}))^{-1} - \mathbf{I}) - 1 + \alpha = 0. \quad (4.4.10)$$

$$\mathbf{0} < \underline{\mathbf{VaR}}_\alpha(\mathbf{X}) < \mathbf{I}_m.$$

The univariate VaR for the beta distribution is widely studied in Arias-Serna et al. [2021].

4.4.3 Matrix Value-at-Risk For F distributions

Now we will use the transformation $\mathbf{F} = \mathbf{U}(\mathbf{I}_m - \mathbf{WU})^{-1}$ where $\mathbf{U} \sim Beta_m(a, b, \mathbf{W})$ and introduce the definition of matrix upper and matrix lower VaR for the generalized matrix variate F distribution.

Definition 4.4.11 Let $\mathbf{X} \sim F_m(a, b, \mathbf{W})$ with $a > \frac{m-1}{2}$, $b > \frac{m-1}{2}$. The matrix upper VaR for the generalized matrix variate F distribution at probability level $\alpha \in (0, 1)$ is solution of the Gaussian hypergeometric equation of matrix argument.

$$K * {}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\overline{\mathbf{VaR}}_\alpha(\mathbf{X})(\mathbf{I} + \mathbf{W}\overline{\mathbf{VaR}}_\alpha(\mathbf{X}))^{-1}\right) = \alpha, \quad (4.4.11)$$

$$\text{where } K = \frac{|\overline{\mathbf{VaR}}_\alpha(\mathbf{X})(\mathbf{I} + \mathbf{W}\overline{\mathbf{VaR}}_\alpha(\mathbf{X}))^{-1}|^a}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)}.$$

Definition 4.4.12 Let $\mathbf{X} \sim F_m(a, b, \mathbf{I}_m)$ with $a > \frac{m-1}{2}, b > \frac{m-1}{2}$ and let $r = a - \frac{m+1}{2}$ be a positive integer or a positive half-integer. The matrix lower VaR for the matrix variate F distribution at probability level $\alpha \in (0, 1)$ is obtained solving the following equation.

$$K * \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* \frac{(-r)_{\kappa} \left(\frac{m+1}{2}\right)_{\kappa}}{\left(b + \frac{m+1}{2}\right)_{\kappa}} C_{\kappa}(-\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X})^{-1}) = 1 - \alpha, \quad (4.4.12)$$

where $K = \frac{\Gamma_m(a+b)\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m(a)\Gamma_m\left(b+\frac{m+1}{2}\right)} |\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X})|^{a-\frac{m+1}{2}} |\mathbf{I} + \overline{\mathbf{VaR}}_{\alpha}(\mathbf{X})|^{\frac{m+1}{2}-a-b}$.

Definition 4.4.13 The univariate VaR of the F distribution at probability level $\alpha \in (0, 1)$ is the unique solution in the interval $[0, \infty)$ of the hypergeometric equation

$$\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} (VaR_{\alpha}(X)(1+VaR_{\alpha}(X))^{-1})^a {}_2F_1\left(a, 1-b; a+1; VaR_{\alpha}(X)(1+VaR_{\alpha}(X))^{-1}\right) = \alpha \quad (4.4.13)$$

4.4.4 About the solution for the matrix Value-at-Risk

As stated in Koev and Edelman [2006] the hypergeometric function of a matrix argument has thus acquired a reputation of being notoriously difficult to approximate even in the simplest cases, which makes it difficult to obtain the Value-at-Risk in the matrix variate setting. In what follows, we present some particular cases in which it is possible to have closed expressions to calculate the matrix Value-at-Risk, for the other cases the Algorithm 4.2 of Koev and Edelman [2006] can be used. They have made an algorithm that efficiently approximate the hypergeometric function of a matrix argument through its expansion as a series of zonal polynomials. The implementation of the algorithms in MATLAB (Current version: 1.5, February 12, 2018) is available in Koev [2021]. Basically the algorithm computes the truncated Hypergeometric function ${}_pF_q$ as a series of zonal polynomials, truncated for partitions of size not exceeding M .

The following corollary allows us to obtain a closed expression for the calculation of $\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X})$ taking $b = \frac{m+1}{2}$ in Defintion 4.4.9.

Corollary 4.4.2 Let \mathbf{X} a random real matrix, if $\mathbf{X} \sim Beta_m(a, \frac{m+1}{2})$ with $a > \frac{m-1}{2}$. The matrix upper VaR at probability level $\alpha \in (0, 1)$ is the solution of equation.

$$|\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X})| = \alpha^{1/a}, \quad \mathbf{0} < \overline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) < \mathbf{I}_m. \quad (4.4.14)$$

Only exist a formula for the second order zonal polynomials; the real case was due to James [1968], then complex case was provided by Caro-Lopera et al. [2006], finally, those results were unified by a formula given for Jack polynomials of second order in Caro-Lopera et al. [2007]. This formula allows us to derive in an exact way the VaR of a two order Wishart matrix, as we shall see below.

Corollary 4.4.3 The matrix upper VaR $\overline{\mathbf{VaR}}_{\alpha}(\mathbf{X}) > \mathbf{0}$ of $\mathbf{X} \sim W_2(n, I)$, with significance $\alpha \in (0, 1)$ satisfies the equation

$$\begin{aligned} & \frac{2^{-n-1}\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)} (\lambda_1, \lambda_2)^{\frac{n}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma\left(\frac{n}{2}+\lambda_1\right)\Gamma\left(\frac{n}{2}+\lambda_1-\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{2}+\lambda_1\right)\Gamma\left(\frac{n}{2}+\lambda_1+1\right)} * \\ & \left(\frac{\lambda_1\lambda_1}{4}\right)^{\frac{\kappa}{2}} 2^{2\kappa} (\kappa)! (1)_{\kappa} \frac{\Gamma\left(\frac{1}{2}\right) \prod_{t < j}^p (2\kappa_i - 2\kappa_j - i + j)}{\Gamma\left(\frac{1}{2} + \frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right) \prod_{t < i}^p (2\kappa_i + p - i)!} {}_2F_1\left(\frac{-n}{2}, \frac{n}{2} + \frac{1}{2}; \frac{1}{2}; \frac{(\lambda_1 + \lambda_1)^2}{4\lambda_1\lambda_1}\right) + \\ & \frac{\left(\frac{\lambda_1\lambda_1}{4}\right)^{\frac{\kappa-1}{2}} 2^{2\kappa} (\kappa)! (1)_{\kappa}}{2\left(-\frac{\lambda_1}{2}, -\frac{\lambda_2}{2}\right)^{-1}} \frac{\Gamma\left(\frac{-1}{2}\right) \prod_{t < j}^p (2\kappa_i - 2\kappa_j - i + j)}{\Gamma\left(\frac{-n}{2}\right)\Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \prod_{t < i}^p (2\kappa_i + p - i)!} {}_2F_1\left(\frac{1}{2} + \frac{1+n}{2}, \frac{1}{2} - \frac{n}{2}; \frac{3}{2}; \frac{(\lambda_1 + \lambda_1)^2}{4\lambda_1\lambda_1}\right) = \alpha. \end{aligned} \quad (4.4.15)$$

To finishing this section, we establish a useful result for the bounds for the Matrix Value-at-Risk, which we will apply later.

Corollary 4.4.4 *Let ζ_1 and ζ_m are the largest and smallest latent root of $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$ respectively, then $\zeta_m \mathbf{I} < \overline{\mathbf{VaR}}_\alpha(\mathbf{X}) < \zeta_1 \mathbf{I}$.*

Note that the same result is obtained for $\underline{\mathbf{VaR}}_\alpha(\mathbf{X})$, i.e $\iota_m \mathbf{I} < \underline{\mathbf{VaR}}_\alpha(\mathbf{X}) < \iota_1 \mathbf{I}$, where ι_m and ι_1 are the smallest and larger latent roots of $\underline{\mathbf{VaR}}_\alpha(\mathbf{X})$ respectively.

Corollary 4.4.5 *Let ζ_1 and ζ_m are the largest and smallest latent root of $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$ respectively, then $m\zeta_m < \text{tr}(\overline{\mathbf{VaR}}_\alpha(\mathbf{X})) < m\zeta_1$.*

Corollary 4.4.6 *Let ζ_1 and ζ_m are the largest and smallest latent root of $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$ respectively, then $\zeta_m^m < |(\overline{\mathbf{VaR}}_\alpha(\mathbf{X}))| < \zeta_1^m$.*

4.5 Condition Number, VaR and Shape Theory

In this section we explore some related aspects of certain joint densities with application in VaR, condition number densities and shape theory.

Let l_1, \dots, l_m the latent roots of $\mathbf{A} \sim W_m(n, \mathbf{I}_m)$, $n \geq m$ integer $l_1 > \dots > l_m > 0$. As a simple case of a general elliptical joint density derived in Caro-Lopera et al. [2014], the distribution function of l_1, \dots, l_m is given by

$$\frac{\pi^{m^2}}{2^{\frac{mn}{2}} \Gamma_m(\frac{m}{2}) \Gamma_m(\frac{n}{2})} \exp\left(-\frac{1}{2} \sum_{i=1}^m l_i\right) \prod_{i=1}^m l_i^{\frac{n-m-1}{2}} \prod_{i<j}^m (l_i - l_j) \bigwedge_{i=1}^m dl_i.$$

Let $c_i = \frac{l_i}{\text{tr}(A)}$, $i = 1, \dots, m$. In this paper, the variable c_1 will receive the name of Davis' condition number. The legacy of A.W. Davis about condition number in the 60's and 70's has been forgotten by recent literature. Instead, the quantity c_1^{-1} has received all the honors, but based on original works and procedures of Davis.

Now, taking $l_i = c_i \text{tr}(A)$, then $\bigwedge_{i=1}^m dl_i = \text{tr}^{m-1}(A) d\text{tr}(A) \bigwedge_{i=1}^{m-1} dc_i$, and the joint density of $c_1, \dots, c_{m-1}, \text{tr}(A)$ is factorized as follows

$$\begin{aligned} df(c_1, \dots, c_{m-1}, \text{tr}(A)) &= \\ &= \frac{1}{2^{\frac{mn}{2}} \Gamma(\frac{mn}{2})} \exp\left(-\frac{1}{2} \text{tr}(A)\right) \text{tr}(A)^{\frac{mn}{2}-1} d\text{tr}(A) \\ &\times \frac{\pi^{\frac{m^2}{2}} \Gamma(\frac{mn}{2})}{\Gamma_m(\frac{m}{2}) \Gamma_m(\frac{n}{2})} \prod_{i=1}^m c_i^{\frac{n-m-1}{2}} \prod_{i<j}^m (c_i - c_j) \bigwedge_{i=1}^{m-1} dc_i. \end{aligned} \quad (4.5.1)$$

Then $\text{tr}(A) \sim^2 \chi_{mn}$ and

$$df(c_1, \dots, c_{m-1}) = \frac{\pi^{\frac{m^2}{2}} \Gamma(\frac{mn}{2})}{\Gamma_m(\frac{m}{2}) \Gamma_m(\frac{n}{2})} \prod_{i=1}^m c_i^{\frac{n-m-1}{2}} \prod_{i<j}^m (c_i - c_j) \bigwedge_{i=1}^{m-1} dc_i, \quad (4.5.2)$$

where $c_m = 1 - \sum_{i=1}^{m-1} c_i$. Integration over a subset of the simplex $0 < c_m < \dots < c_1 < 1$ is certainly cumbersome; simpler densities involving beta and Dirichlet functions are extremely difficult, and general formulas are out of any consideration; only small cases or some recurrence

relations have been published.

For the defined Davis' condition number and associated VaR, we study the first two stages. If $m = 2$ and $c_1 = \frac{t_1}{tr(A)}$ the pdf of the condition number is given by:

$f(c_1) = 2^{n-2}(n-1)(1-2c_1)[c_1(1-c_1)]^{\frac{1}{2}(n-3)}$ where $0 < c_1 < \frac{1}{2}$ and $n > 2$. Now, we can define a Value-at-Risk $0 < v < \frac{1}{2}$ for the condition number c_1 , in this case $\int_0^v f(c_1)dc_1 = 2^{n-1}((1-v)v)^{\frac{n-1}{2}}$ for $n > 2$. And for a probability $0 < \alpha < 1$, $\int_0^v f(c_1)dc_1 = \alpha$ and explicit formula for the VaR can be obtained as: $v = \frac{1}{2} \left[1 - \sqrt{1 - \alpha^{\frac{2}{n-1}}} \right]$, with $n > 2$.

For $m = 3$, an explicit formula for even $n = 2r \geq 4, r = 2, 3, \dots$ can be simplified as follows: $\frac{4^{r-1}(3r-1)!}{(r-1)r(r-1)!(2r-3)!} (1-2c_1)^r c_1^{2r-2}$, where $0 < c_2 < c_1 < \frac{1}{2}$.

$\int_0^v f(c_1)dc_1 = \frac{(3r-1)!B_{2v}(2r-1,r+1)}{(r+1-1)!(2r-1-1)!}$. where $B_x(a, b)$ gives the incomplete beta function and the VaR can be evaluated.

Now, a hidden relation of the condition number (and VaR) with shape theory can be proposed next. After some analysis, which we will study in a future work, the joint distribution of the ratios c_i 's can be seen as the well known invariant shape disk density. However, as a number of results wrong stated in literature and revisited here as corollaries, the shape disk density also needed revisions; for example, when $m = 2$ the above pdf for c_1 can be rewritten in polar coordinates as: $2(n-2) \cos(2\theta) \sin^{n-3}(2\theta)$ with $0 < \theta < \frac{\pi}{4}$, $n > 2$. This density was studied by Goodall and Mardia [1993] (eq.7.2) in the current central Gaussian case and by Díaz-García et al. [2003] (corollary 2.2) in the central elliptical case. The correction and generalization of the shape disk density for any m and non central elliptical models was given by Díaz-García and Caro-Lopera [2012].

Relation of VaR of condition number with shape theory open a perspective for application in finance and economy sciences and allows a simplification of the elusive densities, studied in the Euclidean space, by using the algebraical and topological properties of the quotient shape spaces.

For example, a useful related density with (4.5.2) emerges from the old result about the joint density function of the latent roots $0 < u_m < \dots < u_1 < 1$ of a matrix variate beta $\mathbf{U} \sim Beta_m \left(\frac{n_1}{2}, \frac{n_2}{2}, \mathbf{I}_m \right)$:

$$\frac{\pi^{\frac{m^2}{2}} \Gamma_m \left[\frac{1}{2}(n_1 + n_2) \right]}{\Gamma_m \left[\frac{m}{2} \right] \Gamma_m \left[\frac{n_1}{2} \right] \Gamma_m \left[\frac{n_2}{2} \right]} \prod_{i=1}^m u_i^{\frac{n_1-m-1}{2}} (1-u_i)^{\frac{n_2-m-1}{2}} \prod_{i<j}^m (u_i - u_j), \quad (4.5.3)$$

see for example Muirhead [2005].

We are interested in the bivariate VaR v , for $m = 2$ and $n_2 = 3$, such that $0 < u_2 < u_1 < v < 1$. Then the distribution of $\mathbf{U} \sim Beta_2 \left(\frac{n_1}{2}, \frac{3}{2}, \mathbf{I}_2 \right)$ can be written as the following sum of a product of two univariate beta distributions: $R(n_1, u_1, u_2) = n_1 B(n_1 + 1, 2, u_1) B(n_1 - 1, 2, u_2) - n_1 B(n_1 - 1, 2, u_1) B(n_1 + 1, 2, u_2)$, where $B(a, b, x)$ denotes a univariate beta distribution of parameters a and b in the random variable x . Then the corresponding bivariate VaR with probability α_2 follows from $\int_0^v \int_{u_2}^v R(n_1, u_1, u_2) du_1 du_2 = \alpha_2$, which is conveniently reduced to $\int_0^v B(2n_1, 2, u_2) du_2 = \alpha_2$ and the known VaR of the univariate beta implies that the bivariate VaR is given by $v = \alpha_2^{1/n_1}$. Now, the univariate VaR v with probability α_1 of $B(n_1, 2, x)$ is

given by $v = \alpha_1^{2/n_1}$, then taking $\alpha_2 = \alpha_1$, both bivariate and univariate VaR are related as: $VaR_{uni} = VaR_{biv}^2$.

Finally, the above computation of VaR via (4.5.3) can be connected with the VaR of $\mathbf{U} \sim Beta_2\left(\frac{n_1}{2}, \frac{3}{2}, \mathbf{I}_2\right)$ by using directly the density of \mathbf{U} . In this particular case the series terminates and simplifies into $|v\mathbf{I}_2|^{n_1/2} = \alpha_2$, and the bivariate VaR results again.

The challenging connection between the above different methods for VaR in beta matrices with the untractable direct integration of condition number in (4.5.2) and the corresponding VaR will be part of a future work.

4.6 Concluding Remarks

In the last decade, many extensions to risk measures multivariate have been investigating, and some papers suggest alternative ways of measuring risk for multivariate portfolios. When we search for a matrix variate extension for VaR, the literature in finance does not provide any approach. However, from the mathematical point of view, the risk measures just require meaningful percentiles of finance in the context of matrix cumulative distribution functions. In this paper, first, we propose some generalized integrals over positive definite matrices that will then allow us to calculate $P(\mathbf{X} < VaR_\alpha(\mathbf{X}))$ and $P(\mathbf{X} > VaR_\alpha(\mathbf{X}))$ when \mathbf{X} follows matrix-variate generalized Wishart, F and beta distributions and then we introduce two alternative extensions of the classical univariate VaR for generalized Wishart, F and beta distributions, connecting the theory of hypergeometric functions of matrix argument and integration over positive definite matrices.

As particular cases of the VaR Wishart, we provide the VaR for exponential, Erlang, and chi-square distributions for the univariate setting, and the VaR for gamma, beta, and F distributions in the matrix-variate setting. We also obtained closed expressions to calculate the matrix Value-at-Risk when $\mathbf{X} \sim Beta_m(a, \frac{m+1}{2})$ and when $\mathbf{X} \sim W_2(n, \mathbf{I})$.

As future research, it is expected to consider a generalization of the triangular distribution to the matrix-variate case and consider it as a proxy to the matrix-variate beta distribution, and discuss its role in matrix-variate risk measures, and then carry out a comparative study between the two distributions in terms of accuracy, ease of computation, and implementation.

Also, we expected to advance in the solution of the matrix equations proposed for the calculation of the matrix upper VaR and the matrix lower VaR. It is expected to obtain other closed solutions for the matrix-variate VaR such as the one obtained in the Corollary 4.4.2 and 4.4.3. However, given the complexity of the hypergeometric functions of matrix argument, we hope to at least find numerical solutions using algorithm 4.2 of Koev and Edelman [2006].

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Declarations of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

4.7 Appendix: Proofs

Proof 4.7.1 *Theorem 4.2.1.* As usual, the integral is reduced over $0 < \mathbf{X} < \mathbf{I}$ upon substituting $\mathbf{A} = \mathbf{V}^{\frac{1}{2}} \mathbf{X} \mathbf{V}^{\frac{1}{2}}$, with Jacobian $(d\mathbf{A}) = |\mathbf{V}|^{\frac{m+1}{2}} (d\mathbf{X})$, so the integral in 4.2.2 can be written as

$$|\mathbf{V}|^a \int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} \text{etr} \left(-\frac{1}{2} \mathbf{Z} \mathbf{V} \mathbf{X} \right) |\mathbf{X}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{X}|^{b - \frac{m+1}{2}} (d\mathbf{X}),$$

integrating term by term,

$$|\mathbf{V}|^a \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} |\mathbf{X}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{X}|^{b - \frac{m+1}{2}} C_{\kappa} \left(-\frac{1}{2} \mathbf{Z} \mathbf{V} \mathbf{X} \right) (d\mathbf{X})$$

where we have used the fact that

$$\text{etr}(R) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} C_{\kappa}(R)$$

Using, Muirhead [2005] [Th.7.2.10] to evaluate this last integral we get

$$\frac{\Gamma_m(a) \Gamma_m(b)}{\Gamma_m(a+b)} |\mathbf{V}|^a \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \frac{(a)_{\kappa}}{(a+b)_{\kappa}} C_{\kappa} \left(-\frac{1}{2} \mathbf{Z} \mathbf{V} \right).$$

Finally, by definition of Confluent function, concludes the proof.

Proof 4.7.2 *Theorem 4.2.2.* Take $\mathbf{A} = \mathbf{V}^{\frac{1}{2}} (\mathbf{I} + \mathbf{X}) \mathbf{V}^{\frac{1}{2}}$, with $(d\mathbf{A}) = |\mathbf{V}|^{\frac{m+1}{2}} (d\mathbf{X})$, then the integrate in 4.2.3 is given by

$$|\mathbf{V}|^{a+b - \frac{(m+1)}{2}} \int_{\mathbf{0} < \mathbf{X}} \text{etr} \left(-\frac{1}{2} \mathbf{V}^{\frac{1}{2}} (\mathbf{I} + \mathbf{X}) \mathbf{V}^{\frac{1}{2}} \mathbf{Z} \right) |\mathbf{X}|^{a - \frac{m+1}{2}} |\mathbf{I} + \mathbf{X}^{-1}|^{a+b - (m+1)} (d\mathbf{X}).$$

As $r = a + b - (m + 1)$, is a positive integer, then hypergeometric series representation of $|\mathbf{I} + \mathbf{X}^{-1}|^{a+b - (m+1)}$ is a polynomial of mr degree (Corollary 7.3.5) Muirhead [2005]. Then we have:

$$\begin{aligned} & |\mathbf{V}|^{a+b - \frac{m+1}{2}} \text{etr} \left(-\frac{1}{2} \mathbf{V} \mathbf{Z} \right) \sum_{k=0}^{mr} \frac{(-1)^k}{k!} \sum_{\kappa}^* (-r)_{\kappa} \\ & \times \int_{\mathbf{0} < \mathbf{X}} \text{etr} \left(-\frac{1}{2} \mathbf{V}^{\frac{1}{2}} \mathbf{Z} \mathbf{V}^{\frac{1}{2}} \mathbf{X} \right) |\mathbf{X}|^{a - \frac{m+1}{2}} C_{\kappa}(\mathbf{X}^{-1}) (d\mathbf{X}), \end{aligned}$$

where the summation runs over partitions $\kappa = (k_1, \dots, k_m)$ of k such that with $k_1 \leq r$ if r is a positive integer or $k_2 \leq r$ if r is a positive half-integer. By using [Th.7.2.13] in Muirhead [2005] we have the required result.

Proof 4.7.3 Theorem 4.2.3. To prove (4.2.4) expand

$$|\mathbf{I} - \mathbf{WA}|^{b - \frac{m+1}{2}} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \left(-b + \frac{m+1}{2} \right)_{\kappa} C_{\kappa}(\mathbf{WA})$$

Taking $b = \frac{m+1}{2}$, and integrate term by term using Muirhead [2005] [Th 7.2.10].

To prove (4.2.5), the integral is reduced over $0 < \mathbf{X} < \mathbf{I}$, substituting $\mathbf{A} = \mathbf{V}^{\frac{1}{2}} \mathbf{X} \mathbf{V}^{\frac{1}{2}}$, with Jacobian $(d\mathbf{A}) = |\mathbf{V}|^{\frac{m+1}{2}} (d\mathbf{X})$, then by (4.2.4) we have the result.

Proof 4.7.4 Theorem 4.2.4. The probability density function of \mathbf{A} is given by:

$$f(\mathbf{A}) = c^{-1} \text{etr} \left(-\frac{1}{2} \mathbf{ZA} \right) |\mathbf{A}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1} \mathbf{A}|^{b - \frac{m+1}{2}},$$

with

$$c = \frac{\Gamma_m(a) \Gamma_m(b)}{\Gamma_m(a+b)} |\mathbf{V}|^a {}_1F_1 \left(a; a+b; -\frac{1}{2} \mathbf{ZV} \right).$$

Then

$$P(\mathbf{A} < \mathbf{G}) = c^{-1} \int_{\mathbf{0} < \mathbf{A} < \mathbf{G}} \text{etr} \left(-\frac{1}{2} \mathbf{ZA} \right) |\mathbf{A}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1} \mathbf{A}|^{b - \frac{m+1}{2}} (d\mathbf{X}),$$

Taking $\mathbf{A} = \mathbf{G}^{\frac{1}{2}} \mathbf{X} \mathbf{G}^{\frac{1}{2}}$, with Jacobian $(d\mathbf{A}) = |\mathbf{G}|^{\frac{m+1}{2}} (d\mathbf{X})$, then

$$c^{-1} \int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} \text{etr} \left(-\frac{1}{2} \mathbf{ZG}^{\frac{1}{2}} \mathbf{X} \mathbf{G}^{\frac{1}{2}} \right) |\mathbf{G}|^{a - \frac{m+1}{2}} |\mathbf{X}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1} \mathbf{G}^{\frac{1}{2}} \mathbf{X} \mathbf{G}^{\frac{1}{2}}|^{b - \frac{m+1}{2}} |\mathbf{G}|^{\frac{m+1}{2}} (d\mathbf{X})$$

Using $\text{etr}(R) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} C_{\kappa}(R)$ and integrating term to term, we have

$$= c^{-1} |\mathbf{G}|^a \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} C_{\kappa} \left(-\frac{1}{2} \mathbf{ZG}^{\frac{1}{2}} \mathbf{X} \mathbf{G}^{\frac{1}{2}} \right) |\mathbf{X}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1} \mathbf{G}^{\frac{1}{2}} \mathbf{X} \mathbf{G}^{\frac{1}{2}}|^{b - \frac{m+1}{2}} (d\mathbf{X})$$

Using $|\mathbf{I} - \mathbf{A}|^{b - \frac{m+1}{2}} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa} \left(-b + \frac{m+1}{2} \right)_{\kappa} C_{\kappa}(\mathbf{A})$, we have

$$\begin{aligned} &= c^{-1} |\mathbf{G}|^a \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!} \sum_{\kappa} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\lambda} \left(b - \frac{m+1}{2} \right)_{\lambda} \times \\ &\int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} |\mathbf{X}|^{a - \frac{m+1}{2}} C_{\kappa} \left(\mathbf{ZG}^{\frac{1}{2}} \mathbf{X} \mathbf{G}^{\frac{1}{2}} \right) C_{\lambda} \left(\mathbf{V}^{-1} \mathbf{G}^{\frac{1}{2}} \mathbf{X} \mathbf{G}^{\frac{1}{2}} \right) (d\mathbf{X}) \end{aligned}$$

It is equivalent to

$$\begin{aligned} &= c^{-1} |\mathbf{G}|^a \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!} \sum_{\kappa} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\lambda} \left(b - \frac{m+1}{2} \right)_{\lambda} \times \\ &\int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} |\mathbf{X}|^{a - \frac{m+1}{2}} \int_{O(m)} C_{\kappa} \left(\mathbf{G}^{\frac{1}{2}} \mathbf{ZG}^{\frac{1}{2}} \mathbf{H}' \mathbf{X} \mathbf{H} \right) C_{\lambda} \left(\mathbf{G}^{\frac{1}{2}} \mathbf{V}^{-1} \mathbf{G}^{\frac{1}{2}} \mathbf{H}' \mathbf{X} \mathbf{H} \right) (d\mathbf{H}) (d\mathbf{X}) \end{aligned}$$

Then, by equation 1.2 in Davis [1980]

$$= c^{-1} |\mathbf{G}|^a \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k}{k!} \sum_{\kappa} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\lambda} \left(b - \frac{m+1}{2} \right)_{\lambda}$$

$$\times \int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} |\mathbf{X}|^{a - \frac{m+1}{2}} \sum_{\phi \in \kappa, \lambda} \frac{C_{\phi}^{\kappa, \lambda}(\mathbf{Z}\mathbf{G}, \mathbf{V}^{-1}\mathbf{G}) C_{\phi}^{\kappa, \lambda}(\mathbf{X}, \mathbf{X})}{C_{\phi}(\mathbf{I})} (d\mathbf{X})$$

By equation 2.1 in Davis [1980], it is equivalent to

$$= c^{-1} |\mathbf{G}|^a \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k}{k!} \sum_{\kappa} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\lambda} \left(b - \frac{m+1}{2} \right)_{\lambda}$$

$$\times \sum_{\phi \in \kappa, \lambda} \Theta_{\phi}^{\kappa, \lambda} \frac{C_{\phi}^{\kappa, \lambda}(\mathbf{Z}\mathbf{G}, \mathbf{V}^{-1}\mathbf{G})}{C_{\phi}(\mathbf{I})} \int_{\mathbf{0} < \mathbf{X} < \mathbf{I}} |\mathbf{X}|^{a - \frac{m+1}{2}} C_{\phi}(\mathbf{X}) (d\mathbf{X}).$$

Finally, taking $b = \frac{m+1}{2}$ by Theorem (7.2.10) in Muirhead [2005], we have the required result.

Proof 4.7.5 Theorem 4.2.5. Using the $W_m(n, \Sigma)$ density function for \mathbf{A} , it follows that

$$P(\mathbf{A} > \mathbf{V}) = \frac{1}{2^{\frac{mn}{2}} \Gamma_m(\frac{n}{2}) |\Sigma|^{\frac{n}{2}}} \int_{\mathbf{A} > \mathbf{V}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} \mathbf{A} \right) |\mathbf{A}|^{\frac{n-m-1}{2}} (d\mathbf{A})$$

Take $\mathbf{A} = \mathbf{V}^{\frac{1}{2}} (\mathbf{I} + \mathbf{X}) \mathbf{V}^{\frac{1}{2}}$, with $(d\mathbf{A}) = |\mathbf{V}|^{\frac{m+1}{2}} (d\mathbf{X})$, we have

$$P(\mathbf{A} > \mathbf{V}) = \frac{\text{etr} \left(-\frac{1}{2} \Sigma^{-1} \mathbf{V} \right) \left| \frac{1}{2} \Sigma^{-1} \mathbf{V} \right|^{\frac{n}{2}}}{\Gamma_m(\frac{n}{2})} \int_{\mathbf{X} > \mathbf{0}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} \mathbf{V} \mathbf{X} \right) |\mathbf{I} + \mathbf{X}|^{\frac{n-m-1}{2}} (d\mathbf{X}) \quad (4.7.1)$$

As $r = \frac{n-m-1}{2}$, is a positive integer, then hypergeometric series representation of $|\mathbf{I} + \mathbf{X}|^{\frac{n-m-1}{2}}$ is a polynomial of mr degree (Corollary 7.3.5) Muirhead [2005]. Then we have:

$$\frac{\text{etr} \left(-\frac{1}{2} \Sigma^{-1} \mathbf{V} \right) \left| \frac{1}{2} \Sigma^{-1} \mathbf{V} \right|^{\frac{n}{2}}}{\Gamma_m(\frac{n}{2})} \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* (-r)_{\kappa} \int_{\mathbf{X} > \mathbf{0}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} \mathbf{V} \mathbf{X} \right) C_{\kappa}(-\mathbf{X}) (d\mathbf{X}) \quad (4.7.2)$$

Taking $a = \frac{m+1}{2}$, $\mathbf{Y} = -\mathbf{I}$ and $\mathbf{Z} = \frac{1}{2} \Sigma^{-1} \mathbf{V}$ in Muirhead [2005] [Th 7.2.7], we have the required result.

Proof 4.7.6 Corollary 4.2.6. Using the $W_m(n, \Sigma)$ density function for \mathbf{A} , it follows that

$$P(\mathbf{A} < \mathbf{V}) = \frac{1}{2^{\frac{mn}{2}} \Gamma_m(\frac{n}{2}) |\Sigma|^{\frac{n}{2}}} \int_{\mathbf{0} < \mathbf{A} < \mathbf{V}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} \mathbf{A} \right) |\mathbf{A}|^{\frac{n-m-1}{2}} (d\mathbf{A}) \quad (4.7.3)$$

Take $b = \frac{m+1}{2}$, $a = \frac{n}{2}$ and $\mathbf{Z} = \Sigma^{-1}$ in Theorem (4.2.1), we have the required result.

Proof 4.7.7 Corollary 5.2.5. Using the $W_m(n, \Sigma)$ density function for \mathbf{A} , it follows that

$$P(\mathbf{A} > \mathbf{V}) = \frac{1}{2^{\frac{mn}{2}} \Gamma_m(\frac{n}{2}) |\Sigma|^{\frac{n}{2}}} \int_{\mathbf{A} > \mathbf{V}} \text{etr} \left(-\frac{1}{2} \Sigma^{-1} \mathbf{A} \right) |\mathbf{A}|^{\frac{n-m-1}{2}} (d\mathbf{A}) \quad (4.7.4)$$

Take $b = \frac{m+1}{2}$, $a = \frac{n}{2}$ and $\mathbf{Z} = \Sigma^{-1}$ in Theorem (4.2.2), we have the required result.

Proof 4.7.8 Corollary 4.2.8. Note that the inequality $l_m > x$ is equivalent to $\mathbf{A} > x\mathbf{I}$, and put $\mathbf{V} = x\mathbf{I}$ in Lemma 4.2.5, we have the required result.

Proof 4.7.9 Corollary 4.2.9. Note that the inequality $l_1 < x$ is equivalent to $\mathbf{A} < x\mathbf{I}$, and put $\mathbf{V} = x\mathbf{I}$ in Corollary 4.2.6, we have the required result.

Proof 4.7.10 Corollary 4.2.10. Note that the inequality $l_m > x$ is equivalent to $\mathbf{A} > x\mathbf{I}$, and put $\mathbf{V} = x\mathbf{I}$ in Corollary 5.2.5, we have the required result.

Proof 4.7.11 Corollary 4.2.11. Using the $\text{Beta}_m(a, b, \mathbf{W})$ density function for \mathbf{U} , it follows that

$$P(\mathbf{U} < \mathbf{V}) = \frac{\Gamma_m^{-1}(a)\Gamma_m^{-1}\left(\frac{m+1}{2}\right)\Gamma_m\left(a + \frac{m+1}{2}\right)}{{}_2F_1\left(a, \frac{m+1}{2} - b; a + \frac{m+1}{2}; \mathbf{W}\right)} \int_{\mathbf{0} < \mathbf{U} < \mathbf{V}} |\mathbf{U}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{W}\mathbf{U}|^{b - \frac{m+1}{2}} (d\mathbf{U}) \quad (4.7.5)$$

Then by Theorem 4.2.3 (4.2.5) we have the required result.

Proof 4.7.12 Corollary 4.2.12. Using the $\text{Beta}_m(a, b)$ density function for \mathbf{U} , it follows that

$$P(\mathbf{U} > \mathbf{V}) = \frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} \int_{\mathbf{0} < \mathbf{V} < \mathbf{U} < \mathbf{I}} |\mathbf{U}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{U}|^{b - \frac{m+1}{2}} (d\mathbf{U}) \quad (4.7.6)$$

put $\mathbf{U} = \mathbf{V}^{\frac{1}{2}}(\mathbf{I} + \mathbf{X})\mathbf{V}^{\frac{1}{2}}$, then $\mathbf{V}^{-1} - \mathbf{I} > \mathbf{X} > \mathbf{0}$, with Jacobian $(d\mathbf{U}) = |\mathbf{V}|^{\frac{m+1}{2}}(d\mathbf{X})$ then

$$P(\mathbf{U} < \mathbf{V}) = \frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^a \int_{\mathbf{0} < \mathbf{X} < \mathbf{V}^{-1} - \mathbf{I}} |\mathbf{I} + \mathbf{X}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}(\mathbf{I} + \mathbf{X})|^{b - \frac{m+1}{2}} (d\mathbf{U}) \quad (4.7.7)$$

Let, $\mathbf{X} = (\mathbf{V}^{-1} - \mathbf{I})^{\frac{1}{2}}\mathbf{Y}(\mathbf{V}^{-1} - \mathbf{I})^{\frac{1}{2}}$; with Jacobian $(d\mathbf{X}) = |\mathbf{V}^{-1} - \mathbf{I}|^{\frac{m+1}{2}}(d\mathbf{Y})$ then the left hand of 4.7.7

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^a \int_{\mathbf{0} < \mathbf{Y} < \mathbf{I}} |\mathbf{I} + (\mathbf{V}^{-1} - \mathbf{I})\mathbf{Y}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V} - \mathbf{Y} + \mathbf{V}\mathbf{Y}|^{b - \frac{m+1}{2}} |\mathbf{V}^{-1} - \mathbf{I}|^{\frac{m+1}{2}} (d\mathbf{Y}) \quad (4.7.8)$$

This is equivalent to

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}|^b \int_{\mathbf{0} < \mathbf{Y} < \mathbf{I}} |\mathbf{I} + (\mathbf{V}^{-1} - \mathbf{I})\mathbf{Y}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{Y}|^{b - \frac{m+1}{2}} (d\mathbf{Y}) \quad (4.7.9)$$

By Corollary (7.3.5) in Muirhead [2005], if $r = a - \frac{m+1}{2}$ is a real positive

$$|\mathbf{I} + (\mathbf{V}^{-1} - \mathbf{I})\mathbf{Y}|^{a - \frac{m+1}{2}} = {}_1F_0\left(\frac{m+1}{2} - a; (\mathbf{I} - \mathbf{V}^{-1})\mathbf{Y}\right) \quad (4.7.10)$$

$$\sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa^*} (-r)_{\kappa} C_{\kappa}(\mathbf{I} - \mathbf{V}^{-1})\mathbf{Y} \quad (4.7.11)$$

Then, 5.3.6 is equivalent to

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} |\mathbf{V}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}|^b \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa^*} (-r)_{\kappa} \int_{\mathbf{0} < \mathbf{Y} < \mathbf{I}} |\mathbf{I} - \mathbf{Y}|^{b - \frac{m+1}{2}} C_{\kappa}((\mathbf{I} - \mathbf{V}^{-1})\mathbf{Y})(d\mathbf{Y}), \quad (4.7.12)$$

Take $a = \frac{m+1}{2}$ and $\mathbf{Z} = \mathbf{I} - \mathbf{V}^{-1}$ in Theorem (7.2.10) in Muirhead [2005], we have the required result.

Proof 4.7.13 Corollary 4.2.13. Note that the inequality $l_1 < x$ is equivalent to $\mathbf{U} < x\mathbf{I}$, and put $\mathbf{V} = x\mathbf{I}$ in Corollary 4.2.11, we have the required result.

Proof 4.7.14 Corollary 4.2.14. Note that the inequality $l_m > x$ is equivalent to $\mathbf{U} > x\mathbf{I}$, and put $\mathbf{V} = x\mathbf{I}$ in Corollary 4.2.12, we have the required result.

Proof 4.7.15 Corollary 4.2.15. The result is immediate by Corollary 4.2.11, using the fact that $P(\mathbf{F} < \mathbf{V}) = P(\mathbf{U} < \mathbf{V}(\mathbf{I} + \mathbf{WV})^{-1})$.

Proof 4.7.16 Corollary 4.2.16. The result is immediate by Corollary 4.2.12, using the fact that $P(\mathbf{F} > \mathbf{V}) = P(\mathbf{U} > \mathbf{V}(\mathbf{I} + \mathbf{WV})^{-1})$.

Proof 4.7.17 Corollary 4.2.17. Note that the inequality $l_1 < x$ is equivalent to $\mathbf{F} < x\mathbf{I}$, and put $\mathbf{V} = x\mathbf{I}$ in Corollary 4.2.15, we have the required result.

Proof 4.7.18 Corollary 4.2.18. Note that the inequality $l_m > x$ is equivalent to $\mathbf{F} > x\mathbf{I}$, and put $\mathbf{V} = x\mathbf{I}$ in Corollary 4.2.16, we have the required result.

Proof 4.7.19 Corollary 4.3.1. By Corollary 4.2.9

$$f_{l_m}(x) = \frac{\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n}{2} + \frac{m+1}{2}\right)} \left| \frac{1}{2}\boldsymbol{\Sigma}^{-1} \right|^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{x^{\frac{mn}{2}+k}}{k!} \sum_{\kappa} \frac{\binom{n}{2}_{\kappa}}{\binom{n+m+1}{2}_{\kappa}} C_{\kappa} \left(-\frac{1}{2}\boldsymbol{\Sigma}^{-1} \right),$$

then deriving term to term, we have the required result.

Proof 4.7.20 Corollary 4.3.2. By Corollary 4.2.10

$$f_{l_1}(x) = e^{-\frac{1}{2}x\text{tr}(\boldsymbol{\Sigma}^{-1})} \sum_{k=0}^{mr} \frac{\left(\frac{x}{2}\right)^k}{k!} \sum_{\kappa}^* C_{\kappa}(\boldsymbol{\Sigma}^{-1}),$$

then deriving term to term, we have the required result.

Proof 4.7.21 Corollary 4.3.4. The proof is immediate deriving 4.2.13 and using

$$C_{\kappa}(\mathbf{I}_m) = 2^{2\kappa} \kappa! \binom{m}{2}_{\kappa} \frac{\prod_{i<j}^p (2\kappa_i - 2\kappa_j - i + j)}{\prod_{i<j}^p (2\kappa_i + p - i)!}$$

Proof 4.7.22 Corollary 4.3.5. The proof is immediate deriving 4.2.14 and using

$$C_{\kappa}(\mathbf{I}_m - x^{-1}\mathbf{I}_m) = (-1)^k (1-x)^k x^{-k} 2^{2\kappa} \kappa! \binom{m}{2}_{\kappa} \frac{\prod_{i<j}^p (2\kappa_i - 2\kappa_j - i + j)}{\prod_{i<j}^p (2\kappa_i + p - i)!}$$

Proof 4.7.23 Corollario 4.4.3. The Definition 4.4.1 states that the matrix upper $\text{VaR } \overline{\text{VaR}}_{\alpha}(\mathbf{X}) > \mathbf{0}$ of $\mathbf{X} \sim W_2(n, I)$, satisfies the equation

$$\frac{2^{-n}\Gamma_2\left(\frac{3}{2}\right)}{\Gamma_2\left(\frac{n}{2} + \frac{3}{2}\right)} \left| \overline{\text{VaR}}_{\alpha}(\mathbf{X}) \right|^{\frac{n}{2}} {}_1F_1\left(\frac{n}{2}; \frac{n}{2} + \frac{3}{2}; \frac{-\overline{\text{VaR}}_{\alpha}(\mathbf{X})}{2}\right) = \alpha. \quad (4.7.13)$$

As $\overline{\text{VaR}}_{\alpha}(\mathbf{X})$ is positive definite, by the spectral theorem, Muirhead [2005], $\overline{\text{VaR}}_{\alpha}(\mathbf{X}) = \mathbf{H}\mathbf{D}\mathbf{H}'$, where \mathbf{H} is an orthogonal matrix and $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2)$, with λ_1, λ_2 being the latent roots of \mathbf{V} ,

then $|\overline{\mathbf{VaR}}_\alpha(\mathbf{X})| = |\mathbf{HDH}'| = |\mathbf{D}| = \lambda_1, \lambda_2$. Of the above, and applying the definition of the multivariate Gamma function, the equation (4.7.13) is equivalent to

$$\frac{2^{-n-1}\pi}{\Gamma_2\left(\frac{n}{2} + \frac{3}{2}\right)} (\lambda_1 \lambda_2)^{\frac{n}{2}} {}_1F_1\left(\frac{n}{2}; \frac{n}{2} + \frac{3}{2}; \frac{-\overline{\mathbf{VaR}}_\alpha(\mathbf{X})}{2}\right) = \alpha.$$

Now, by definition of hypergeometric function

$$\frac{2^{-n-1}\pi}{\Gamma_2\left(\frac{n}{2} + \frac{3}{2}\right)} (\lambda_1, \lambda_2)^{\frac{n}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\left(\frac{n}{2}\right)_{\kappa}}{\left(\frac{n}{2} + \frac{3}{2}\right)_{\kappa}} C_{(\kappa_1, \kappa_2)}\left(\frac{-\overline{\mathbf{VaR}}_\alpha(\mathbf{X})}{2}\right) = \alpha.$$

But, $C_\kappa(\mathbf{V}) = C_\kappa(\mathbf{HDH}') = C_\kappa(\mathbf{HH}'\mathbf{D}) = C_\kappa(\mathbf{I}_2\mathbf{D}) = C_\kappa(\mathbf{D})$, and by definition of generalized hypergeometric coefficient $(a)_\kappa$, it follows that

$$\frac{2^{-n-1}\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)} (\lambda_1, \lambda_2)^{\frac{n}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma\left(\frac{n}{2} + \lambda_1\right)\Gamma\left(\frac{n}{2} + \lambda_1 - \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{2} + \lambda_1\right)\Gamma\left(\frac{n}{2} + \lambda_1 + 1\right)} C_\kappa\left(\frac{-\lambda_1}{2}, \frac{-\lambda_2}{2}\right) = \alpha. \quad (4.7.14)$$

Now, by equation (10) in Caro-Lopera et al. [2007],

$$\begin{aligned} \frac{C_{(\kappa_1, \kappa_2)}(\mathbf{Y})}{C_{(\kappa_1, \kappa_2)}(\mathbf{I}_2)} &= (y_1 y_2)^{\frac{\kappa_1 + \kappa_2}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)} {}_2F_1\left(\frac{-n}{2}, \frac{n}{2} + \frac{1}{2}; \frac{1}{2}; \frac{(y_1 + y_2)^2}{4y_1 y_2}\right) + \\ &\frac{(y_1 y_2)^{\frac{\kappa_1 \kappa_2 - 1}{2}}}{2(y_1 + y_2)^{-1}} \frac{\Gamma\left(\frac{-1}{2}\right)}{\Gamma\left(\frac{-n}{2}\right)\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)} {}_2F_1\left(\frac{1}{2} + \frac{1+n}{2}, \frac{1}{2} - \frac{n}{2}; \frac{3}{2}; \frac{(y_1 + y_2)^2}{4y_1 y_2}\right). \end{aligned}$$

By Muirhead [2005],

$$C_{(\kappa_1, \kappa_2)}(\mathbf{I}_2) = 2^{2\kappa}(\kappa)!(1)_\kappa \frac{\prod_{t < j}^p (2\kappa_i - 2\kappa_j - i + j)}{\prod_{t < i}^p (2\kappa_i + p - i)!}$$

Then, $C_{(\kappa_1, \kappa_2)}\left(-\frac{\lambda_1}{2}, -\frac{\lambda_2}{2}\right)$ is given by

$$\begin{aligned} &\left(\frac{\lambda_1 \lambda_1}{4}\right)^{\frac{\kappa}{2}} 2^{2\kappa}(\kappa)!(1)_\kappa \frac{\Gamma\left(\frac{1}{2}\right) \prod_{t < j}^p (2\kappa_i - 2\kappa_j - i + j)}{\Gamma\left(\frac{1}{2} + \frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right) \prod_{t < i}^p (2\kappa_i + p - i)!} {}_2F_1\left(\frac{-n}{2}, \frac{n}{2} + \frac{1}{2}; \frac{1}{2}; \frac{(\lambda_1 + \lambda_1)^2}{4\lambda_1 \lambda_1}\right) + \\ &\frac{\left(\frac{\lambda_1 \lambda_1}{4}\right)^{\frac{\kappa-1}{2}} 2^{2\kappa}(\kappa)!(1)_\kappa}{2\left(-\frac{\lambda_1}{2}, -\frac{\lambda_2}{2}\right)^{-1}} \frac{\Gamma\left(\frac{-1}{2}\right) \prod_{t < j}^p (2\kappa_i - 2\kappa_j - i + j)}{\Gamma\left(\frac{-n}{2}\right)\Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \prod_{t < i}^p (2\kappa_i + p - i)!} {}_2F_1\left(\frac{1}{2} + \frac{1+n}{2}, \frac{1}{2} - \frac{n}{2}; \frac{3}{2}; \frac{(\lambda_1 + \lambda_1)^2}{4\lambda_1 \lambda_1}\right). \end{aligned}$$

Finally, replacing in (4.7.14), concludes the proof.

Proof 4.7.24 Colollary 4.4.4. As $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$ is positive definite, by the spectral in Muirhead [2005], $\overline{\mathbf{VaR}}_\alpha(\mathbf{X}) = \mathbf{HDH}'$, where \mathbf{H} is an orthogonal matrix and $\mathbf{D} = \text{diag}(\zeta_1, \zeta_2, \dots, \zeta_m)$, with $\zeta_1, \zeta_2, \dots, \zeta_m$ being the latent roots of $\overline{\mathbf{VaR}}_\alpha(\mathbf{X})$. Now, we can order the latent roots as $\zeta_1 > \dots > \zeta_m > 0$. Then

$$\overline{\mathbf{VaR}}_\alpha(\mathbf{X}) = \mathbf{HDH}' < H\zeta_1\mathbf{I}H' = \zeta_1\mathbf{I},$$

Similarly

$$\overline{\mathbf{VaR}}_\alpha(\mathbf{X}) = \mathbf{HDH}' > H\zeta_m\mathbf{I}H' = \zeta_m\mathbf{I},$$

Proof 4.7.25 Corollary 4.4.5. The result is followed because

$$m\zeta_m = \zeta_m + \zeta_m + \dots + \zeta_m < \text{tr}(\overline{\mathbf{VaR}}_\alpha(\mathbf{X})) = \zeta_1 + \zeta_2 + \dots + \zeta_m < \zeta_1 + \zeta_1 + \dots + \zeta_1 = m\zeta_1.$$

Proof 4.7.26 Corollary 4.4.6. The result is followed because

$$\zeta_m^m = \zeta_m \zeta_m \dots \zeta_m < |\overline{\mathbf{VaR}}_\alpha(\mathbf{X})| = \zeta_1 \zeta_2 \dots \zeta_m < \zeta_1 \zeta_1 \dots \zeta_1 = \zeta_1^m.$$

Chapter 5

Appendix: Conference Papers and Working Paper

5.1 APPENDIX 1: Conference Papers

1. M. A. Arias-Serna, F. J. Caro-Lopera, J. A. Echeverri-Arias, J. G. Murillo-Gómez, and L. C. Franco-Arbeláez, "Information system for the quantification of operational risk in financial institutions," 2016 11th Iberian Conference on Information Systems and Technologies (CISTI), Las Palmas, 2016, pp. 1-7. doi: 10.1109/CISTI.2016.7521570, Arias-Serna et al. [2016].
2. M. A. Arias-Serna, F. J. Caro-Lopera, J. A. Echeverri-Arias, D. A. Castañeda-Palacio and J. G. Murillo-Gómez, "Information system for the quantification of financial risk," 2017 12th Iberian Conference on Information Systems and Technologies (CISTI), Lisbon, 2017, pp. 1-6, doi: 10.23919/CISTI.2017.7975680, URL: <https://ieeexplore.ieee.org/document/7975680>, Arias-Serna et al. [2017].
3. M. A. Arias-Serna, F. J. Caro-Lopera, J. A. Echeverri-Arias, J. G. Murillo-Gómez, and L. C. Franco-Arbeláez, Arquitectura de software para la medición del riesgo operacional en entidades del sector financiero. International Journal of Information Systems and Software Engineering for Big Companies (IJISEBC), 2017, 4(1), pp. 21-32, Arias-Serna et al. [2017a].
4. M. A. Arias-Serna, F. J. Caro-Lopera, D. A. Castaneda-Palacio, J. G. Murillo-Gomez and L. T. Toro, "Software suite for the measurement of financial risk," 2018 13th Iberian Conference on Information Systems and Technologies (CISTI), Caceres, 2018, pp. 1-6, doi: 10.23919/CISTI.2018.8399143, URL: <https://ieeexplore.ieee.org/abstract/document/8399143>, Arias-Serna et al. [2018].
5. Arias Serna, M. A. and Caro Lopera, F. J. (2019). Expected loss management. Software, Arias-Serna and Caro-Lopera [2019].

Extreme latent roots, Condition number, Value-at-Risk and matrix barycenter of certain distributions based on positive definite matrix

5.2 Some Probabilistic Results

In this section, we propose some integrals that will allow us to compute probabilities in the matrix case.

Before starting, we want to give sense to the notion of order in the matrix framework. For this we will use the Loewner order, see for example Sharon and Itai [2013]. Let A and B two positive definite matrices, $\mathbf{A} < \mathbf{\Omega}$ means that $\mathbf{\Omega} - \mathbf{A}$ is positive definite. This order will be denoted " $<$ ".

Theorem 5.2.1 *Let \mathbf{Z} be a complex symmetric $m \times m$ matrix, with $\text{Re}(\mathbf{Z}) > \mathbf{0}$, and \mathbf{V}, \mathbf{A} be positive definite $m \times m$ matrices, such that $\mathbf{0} < \mathbf{A} < \mathbf{V}$. If $h(y)$ is a measurable function, then*

$$\begin{aligned} \int_{\mathbf{0} < \mathbf{A} < \mathbf{V}} h(\text{tr} \mathbf{Z} \mathbf{A}) |\mathbf{A}|^{a - \frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1} \mathbf{A}|^{b - \frac{m+1}{2}} (d\mathbf{A}) \\ = \frac{\Gamma_m(a) \Gamma_m(b)}{\Gamma_m(a+b)} |\mathbf{V}|^a {}_1P_1 \left(h^k(0); a; a+b; \mathbf{Z} \mathbf{V} \right), \end{aligned} \quad (5.2.1)$$

where $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$, $h: \mathfrak{R} \rightarrow [0, \infty)$, termed the generator function, is such that $\int_0^\infty u^{p-1} h(u^2) du < \infty$. ${}_pP_q(h^k(0); a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{A}) = \sum_{k=0}^\infty \frac{h^k(0)}{k!} \sum_{\kappa} \frac{(a_1)_{\kappa} \dots (a_p)_{\kappa}}{(b_1)_{\kappa} \dots (b_q)_{\kappa}} C_{\kappa}(\mathbf{A})$ is the generalized hypergeometric function, $p \geq 0$ and $q \geq 0$ be integers, \sum_{κ} denotes summation over all partitions $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_m)$ of k , with $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 0$, $(a)_{\kappa} = \prod_{i=1}^m (a - i + 1)_{\kappa_i}$ and $C_{\kappa}(\mathbf{A})$ denote the Zonal polynomial in the latent roots of the positive definite matrix \mathbf{A} (see Caro-Lopera et al. [2010]).

Theorem 5.2.2 *Let \mathbf{Z} be a complex symmetric $m \times m$ matrix, with $\text{Re}(\mathbf{Z}) > \mathbf{0}$, and \mathbf{V}, \mathbf{A} be positive definite $m \times m$ matrices, such that $\mathbf{0} < \mathbf{V} < \mathbf{A}$. If $h(y)$ is a measurable function, then,*

$$\begin{aligned} \int_{\mathbf{0} < \mathbf{V} < \mathbf{A}} h(\text{tr} \mathbf{A} \mathbf{Z}) |\mathbf{A}|^{a+b-(m+1)} |\mathbf{V}^{-1} \mathbf{A} - \mathbf{I}|^{-b + \frac{m+1}{2}} (d\mathbf{A}) \\ = \frac{|\mathbf{V}|^{b - \frac{m+1}{2}}}{|\mathbf{Z}|^a} \sum_{k=0}^{mr} \frac{(-1)^k}{k!} \sum_{t=0}^{\infty} \frac{\text{tr}^t(\mathbf{V} \mathbf{Z}) \int_0^\infty h(t)(w) w^{ma-k-1} dw}{t! \Gamma(ma-k)} \\ \times \sum_{\kappa}^* (-r)_{\kappa} \Gamma_m(a, -\kappa) C_{\kappa}(\mathbf{V} \mathbf{Z}), \end{aligned} \quad (5.2.2)$$

where $r = a + b - (m + 1)$ is a positive integer or a positive half-integer, $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$, $\Gamma_m(a, -\kappa) = \frac{(-1)^k \Gamma_m(a)}{(-a + \frac{m+1}{2})_{\kappa}}$ and \sum_{κ}^* denotes summation over those partitions $\kappa = (k_1, \dots, k_m)$ of k with $k_1 \leq r$ if r is a positive integer or $k_2 \leq r$ if r is a positive half-integer.

When $b = \frac{m+1}{2}$ the above expression is related with a polemic result back to sixties. Li [1997] tried it by using Muirhead [2005] [Lem. 7.2.12], and then a number of papers used the result. Later the lemma was revised by Díaz-García and Gutiérrez-Jáimez [2010] and proposed

a new version in Díaz-García and Gutiérrez-Jáimez [2011] and applications in a series of articles. Finally, Caro-Lopera et al. [2016] gave the right form of the referred lemma and the evaluation of the general integral with consonant corollaries involving the well known particular cases available in the works of A.T. James, A. G. Constantine, C. G. Khatri since the 60's.

Note that integrands of (4.2.1) and (4.2.2) generate two new parametric matrix variate distribution which generalize the Wishart distribution when $b = \frac{m+1}{2}$, however for general b , the constant integration of each case requires the use of A. W. Davis's invariant polynomials of several matrix arguments. This occurs after expanding the exponential and determinant, then products of two zonal polynomials appear, forcing the rising of Davis' polynomials. The polynomials are extensions of zonal polynomials of one matrix argument. Zonal polynomials are eigenfunctions of the Laplace-Beltrami operator, then they can be computed by recurrence relations (James [1968]). For years, Davis conjectured that the new polynomials can be constructed in a similar way (Davis [1980] and Davis [2006]), unfortunately, Caro-Lopera [2016] proved that the recurrence construction is not possible, then computation is only possible for the first small degrees. Literature collects dozens of papers involving Davis' polynomials, but they are so far to be applied. That why we will focus on computable densities in terms of zonal polynomials, which we can handle.

In the next section, we will use the two previous theorems to compute $P(\mathbf{A} < \mathbf{V})$ and $P(\mathbf{A} > \mathbf{V})$ when $\mathbf{A} \sim W_m(n, \Sigma)$ and \mathbf{V} is an $m \times m$ positive definite matrix.

5.2.1 Generalized Wishart cumulative distributions

Matrix variate Wishart distribution has played an important role in the Gaussian distribution theory profusely studied and applied by Muirhead [2005] and the references therein.

The generalized Elliptical Wishart distribution allows multiple extensions to robust elliptical models when the Gaussian assumption is difficult to keep. The addressed distribution was derived by Caro-Lopera et al. [2016] and Caro-Lopera et al. [2014] and it is given as follows:

The $m \times m$ matrix $\mathbf{A} = \mathbf{Z}'\mathbf{Z}$, is said to have a matrix variate Elliptical Wishart distribution with n degrees of freedom and covariance matrix Σ , if its p.d.f is given by

$$f_{\mathbf{A}}(\mathbf{A}) = \frac{\pi^{\frac{mn}{2}}}{\Gamma_m(\frac{n}{2})|\Sigma|^{\frac{n}{2}}} |\mathbf{A}|^{\frac{n-m-1}{2}} h(\text{tr}\Sigma^{-1}\mathbf{A}), \quad (5.2.3)$$

where the $n \times m$ matrix \mathbf{Z} is $E_{n \times m}(\mathbf{0}, \mathbf{I}_n \otimes \Sigma, h)$ with $n \geq m$ and $\Gamma_m(a) = \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^n \Gamma[a - \frac{1}{2}(i-1)]$, with $\Re(a) > \frac{1}{2}(m-1)$. This distribution is denoted by $EW_m(n, \Sigma, h)$.

Except by a normalization constant, when $b = \frac{m+1}{2}$, the theorem 4.2.1 reduces to a cumulative density function of the Elliptical Wishart matrix. However, for a general b , the integral can provide the normalization constant of a new distribution for the positive definite random matrix \mathbf{A} defined in $\mathbf{0} < \mathbf{A} < \mathbf{V}$. If the right hand side of (4.2.2) is denoted by c , then the probability density function of \mathbf{A} is given by:

$$f(\mathbf{A}) = c^{-1} h(\text{tr}(\mathbf{Z}\mathbf{A})) |\mathbf{A}|^{a-\frac{m+1}{2}} |\mathbf{I} - \mathbf{V}^{-1}\mathbf{A}|^{b-\frac{m+1}{2}}, \quad (5.2.4)$$

with

$$c = \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} |\mathbf{V}|^a {}_1P_1 \left(h^k(0); a; a+b; -\frac{1}{2}\mathbf{Z}\mathbf{V} \right). \quad (5.2.5)$$

Then we can further compute the following cumulative density function of \mathbf{A} . Unfortunately, the addressed invariant polynomials of several matrix polynomials appear. We include the computation for completeness and connection with the old results resulting as corollaries.

Theorem 5.2.3 *Let \mathbf{Z} be a complex symmetric $m \times m$ matrix, with $\text{Re}(\mathbf{Z}) > \mathbf{0}$, and $\mathbf{V}, \mathbf{A}, \mathbf{G}$ be positive definite $m \times m$ matrices, such that $\mathbf{0} < \mathbf{G} < \mathbf{V}$, then*

$$P(\mathbf{0} < \mathbf{A} < \mathbf{G}) = c^{-1} |\mathbf{G}|^a \sum_{k=0}^{\infty} \frac{h^k(0)}{k!} \sum_{\kappa} \sum_{l=0}^{\infty} \frac{h^l(0)}{l!} \sum_{\lambda} \left(b - \frac{m+1}{2} \right)_{\lambda} \\ \times \sum_{\phi \in \kappa \cdot \lambda} \theta_{\phi}^{\kappa, \lambda} \frac{(a)_{\phi}}{(a + \frac{m+1}{2})_{\phi}} C_{\phi}^{\kappa, \lambda}(\mathbf{Z}\mathbf{G}, \mathbf{V}^{-1}\mathbf{G}), \quad (5.2.6)$$

where $\text{Re}(a) > \frac{1}{2}(m-1)$, $\text{Re}(b) > \frac{1}{2}(m-1)$, κ, λ, ϕ are partitions of $k, l, k+l$, respectively; $\phi \in \kappa \cdot \lambda$ represents $2\phi \in 2\kappa \otimes 2\lambda$; $\theta_{\phi}^{\kappa, \lambda}$ comes from $C_{\kappa}(\mathbf{X})C_{\lambda}(\mathbf{Y}) = \sum_{\phi \in \kappa \cdot \lambda} \theta_{\phi}^{\kappa, \lambda} C_{\phi}^{\kappa, \lambda}(\mathbf{X}, \mathbf{Y})$, and $C_{\phi}^{\kappa, \lambda}(\mathbf{X}, \mathbf{Y})$ denote the invariant polynomials of two matrix arguments, extending the zonal polynomials, (see Davis [1980] and the references therein).

First, we provide a new way of computing and important c.d.f, by using an old integral due to James and Constantine.

Proof 5.2.1 *Lemma 4.2.5. Using the $W_m(n, \Sigma)$ density function for \mathbf{A} , it follows that*

$$P(\mathbf{A} > \mathbf{V}) = \frac{\pi^{\frac{mn}{2}}}{\Gamma_m(\frac{n}{2}) |\Sigma|^{\frac{n}{2}}} \int_{\mathbf{A} > \mathbf{V}} h(\Sigma^{-1}\mathbf{A}) |\mathbf{A}|^{\frac{n-m-1}{2}} (d\mathbf{A})$$

Take $\mathbf{A} = \mathbf{V}^{\frac{1}{2}}(\mathbf{I} + \mathbf{X})\mathbf{V}^{\frac{1}{2}}$, with $(d\mathbf{A}) = |\mathbf{V}|^{\frac{m+1}{2}} (d\mathbf{X})$, we have

$$P(\mathbf{A} > \mathbf{V}) = \frac{\pi^{\frac{mn}{2}} |\Sigma^{-1}\mathbf{V}|^{\frac{n}{2}}}{\Gamma_m(\frac{n}{2})} \int_{\mathbf{X} > \mathbf{0}} h(\text{tr}(\Sigma^{-1}\mathbf{V}\mathbf{X} + \Sigma^{-1}\mathbf{V})) |\mathbf{I} + \mathbf{X}|^{\frac{n-m-1}{2}} (d\mathbf{X})$$

As $r = \frac{n-m-1}{2}$, if r is a positive integer, then hypergeometric series representation of $|\mathbf{I} + \mathbf{X}|^{\frac{n-m-1}{2}}$ is a polynomial of mr degree (Corollary 7.3.5) Muirhead [2005]. Then we have:

$$\frac{\pi^{\frac{mn}{2}} |\Sigma^{-1}\mathbf{V}|^{\frac{n}{2}}}{\Gamma_m(\frac{n}{2})} \sum_{k=0}^{mr} \frac{1}{k!} \sum_{\kappa}^* (-r)_{\kappa} \int_{\mathbf{X} > \mathbf{0}} h(\text{tr}(\Sigma^{-1}\mathbf{V}\mathbf{X} + \Sigma^{-1}\mathbf{V})) C_{\kappa}(-\mathbf{X}) (d\mathbf{X})$$

Taking $a = \frac{m+1}{2}$, $\mathbf{Y} = -\mathbf{I}$ and $\mathbf{Z} = \frac{1}{2}\Sigma^{-1}\mathbf{V}$ in Muirhead [2005] [Th 7.2.7], we have the required result.

The most important fact to note here comes from the fact that a ${}_2F_0(\cdot, \cdot; \cdot)$ diverges unless the series terminates. But it exactly happens when $r = \frac{1}{2}(n-m-1)$ ranges in the natural domain of the degrees of freedom n and the matrix order m .

Now, the well known results of upper and lower probabilities in Wishart matrices, full derived in [Muirhead, 2005, Th. 9.7.2, Th. 9.7.3] (and references therein), are consequences of Theorems 5.2.1 and 5.2.2.

Corollary 5.2.4 *If $\mathbf{A} \sim EW_m(n, \Sigma, h)$, $n > m-1$ and \mathbf{V} is an $m \times m$ positive definite matrix ($\mathbf{V} > \mathbf{0}$) then*

$$P(\mathbf{A} < \mathbf{V}) = \frac{\pi^{\frac{mn}{2}} \Gamma_m(\frac{m+1}{2})}{\Gamma_m(\frac{n}{2} + \frac{m+1}{2})} |\Sigma^{-1}\mathbf{V}|^{\frac{n}{2}} {}_1P_1 \left(h^{(k)}(0) : \frac{n}{2}; \frac{n}{2} + \frac{m+1}{2}; \Sigma^{-1}\mathbf{V} \right). \quad (5.2.7)$$

Recall that $P(\mathbf{A} > \mathbf{V}) \neq 1 - P(\mathbf{A} < \mathbf{V})$ does not hold for $m \geq 2$, then both probabilities must be computed by different methods.

As we quoted before, a correct prove of [Muirhead, 2005, th. 7.2.13] was provided recently, but curiously the integral was unperturbed, because the implicit symmetry of the involved polynomials, see Caro-Lopera et al. [2016] for all the details. The final consequence remains as follows:

Corollary 5.2.5 *Let $\mathbf{A} \sim EW_m(n, \mathbf{\Sigma}, h)$, $n > m - 1$ and V is an $m \times m$ positive definite matrix ($\mathbf{V} > \mathbf{0}$).*

$$P(\mathbf{A} > \mathbf{V}) = \frac{\pi^{\frac{mn}{2}}}{\Gamma_m(\frac{n}{2})} \sum_{k=0}^{mr} \frac{(-1)^k}{k!} \sum_{t=0}^{\infty} \frac{tr^t(\mathbf{V}\mathbf{\Sigma}^{-1}) \int_0^{\infty} h^{(t)}(w) w^{ma-k-1} dw}{t! \Gamma(ma - k)} \\ \times \sum_{\kappa}^* (-r)_{\kappa} \Gamma_m\left(\frac{n}{2}, -\kappa\right) C_{\kappa}(\mathbf{V}\mathbf{\Sigma}^{-1}),$$

Where $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer.

Again some interesting hints appear in the history of these particular results. For example, the above corollary was bounded by [Muirhead, 2005, Th. 9.7.3] only for positive integers r , however it is also true for positive half-integers.

As usual, setting $\mathbf{V} = x\mathbf{I}$, upper and lower probabilities for extreme latent roots distributions can be obtained.

We start with a new expression for the c.d.f of a the smallest latent root of a Wishart matrix is given next.

As alternatives, the classical results for c.d.f of extreme Wishart latent roots follows from the main integrals derived in Theorems 5.2.1 and 5.2.2

Corollary 5.2.6 *If l_1 is the largest latent root of $\mathbf{A} \sim W_m(n, \mathbf{\Sigma}, h)$, $n > m - 1$, then its distribution function can be expressed in the form*

$$P(l_1 < x) = \frac{\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n}{2} + \frac{m+1}{2}\right)} \left| \frac{1}{2} x \mathbf{\Sigma}^{-1} \right|^{\frac{n}{2}} {}_1P_1\left(h^k(0); \frac{n}{2}; \frac{n+m+1}{2}; -\frac{1}{2} x \mathbf{\Sigma}^{-1}\right). \quad (5.2.8)$$

Corollary 5.2.7 *If l_m is the smallest latent root of $\mathbf{A} \sim W_m(n, \mathbf{\Sigma})$, $n > m - 1$ and $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer, then*

$$P(l_m > x) = \frac{\pi^{\frac{mn}{2}}}{\Gamma_m(\frac{n}{2})} \sum_{k=0}^{mr} \frac{(-1)^k}{k!} \sum_{t=0}^{\infty} \frac{tr^t(x\mathbf{\Sigma}^{-1}) \int_0^{\infty} h^{(t)}(w) w^{ma-k-1} dw}{t! \Gamma(ma - k)} \\ \times \sum_{\kappa}^* (-r)_{\kappa} \Gamma_m\left(\frac{n}{2}, -\kappa\right) C_{\kappa}(x\mathbf{\Sigma}^{-1}),$$

Where $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer.

5.2.2 Extreme Latent Root Distributions

Distributions of the extreme latent roots of positive definite matrices are useful in a number of modern applied areas because they can define bounds for random processes, laws and principles. They for example provide information about computational problems of inversions of matrices

and risks, or survival phenomena based on positive definite random matrices or singular value distributions.

Curiously, the implicit power of cumulative density functions for matrices is not used rarely, and the required univariate distributions for extreme latent roots are obtained by multiple integration of the joint latent root distribution.

In this section we take advance of the c.d.f before derived, and the extreme latent root distribution follows by simple differentiation. In the sequel we just inform the results which are straightforwardly obtained.

The previous corollaries allow to calculate the probability density function of extreme latent roots of A .

Corollary 5.2.8 *The pdf of the largest latent roots l_1 of $\mathbf{A} \sim W_m(n, \Sigma)$, $n > m - 1$ is given by*

$$f(l_1) = \frac{\Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n}{2} + \frac{m+1}{2}\right)} \left| \frac{1}{2} \Sigma^{-1} \right|^{\frac{n}{2}} \sum_{k=0}^{\infty} \frac{\left(k + \frac{mn}{2}\right)}{k!} l_1^{k + \frac{mn}{2} - 1} \sum_{\kappa} \frac{\left(\frac{n}{2}\right)_{\kappa}}{\left(\frac{n+m+1}{2}\right)_{\kappa}} C_{\kappa} \left(-\frac{1}{2} \Sigma^{-1} \right). \quad (5.2.9)$$

Note that this result coincides with the result obtained in the proof of Theorem 1. in cite Shinozaki (2018), they were based on the results obtained in Caro-Lopera et al. [2016] and then used the results of Sugiyama [1967] to find the marginal of l_1 using the traditional methodology of multiple integration with respect to the other latent roots. We propose we obtain the same result but simply deriving.

Corollary 5.2.9 *The pdf of the smallest latent roots l_m of $\mathbf{A} \sim W_m(n, \Sigma)$, $n > m - 1$ and $r = \frac{1}{2}(n - m - 1)$ is a positive integer is or a positive half-integer is given by*

$$f(l_m) = \frac{1}{2} \text{tr}(\Sigma^{-1}) e^{-\frac{1}{2} x \text{tr}(\Sigma^{-1})} \sum_{k=0}^{mr} \frac{\left(\frac{x}{2}\right)^k}{k!} \sum_{\kappa^*} C_{\kappa}(\Sigma^{-1}) - \frac{1}{2} e^{-\frac{1}{2} x \text{tr}(\Sigma^{-1})} \sum_{k=0}^{mr-1} \frac{\left(\frac{x}{2}\right)^k}{k!} \sum_{\kappa^*} C_{\kappa}(\Sigma^{-1}). \quad (5.2.10)$$

5.3 Matrix Value-at-Risk Of Certain Distributions Based On Positive Definite Matrix

From the usual definition in the univariate setting Cousin and Di Bernardino [2013], the Value-at-Risk is the minimal amount of the loss which accumulates a probability α to the left tail and $1 - \alpha$ to the right tail. Then, if $F_X(x) = P(X \leq x)$ denotes the cumulative distribution function associated with risk X , and $\overline{F}_X(x) = P(X > x)$ its associated survival function, then the Value-at-Risk ($V_{\alpha}(X)$) at the confidence level $\alpha \in (0, 1)$ is a real number such that

$$V_{\alpha}(X) = \sup\{u \mid \overline{F}(u) \geq 1 - \alpha\}.$$

and equivalently,

$$V_{\alpha}(X) = \min\{u \mid F(u) \geq \alpha\},$$

If F is strictly increasin then the Value-at-Risk means a single real value, in the matrix variate case a set of values satisfice the above definitions. For this set we use the term Matrix Value-at-Risk.

Since in the case matrix variate the two previous definitions are not equivalent because $F(X) + \overline{F}_X(X) = 1$ is not true in general, it is necessary to define a Value at Risk for each case.

Corollary 5.3.1 *The matrix upper Value-at-Risk $\overline{\mathbf{V}}_\alpha(\mathbf{X}) > \mathbf{0}$ of $\mathbf{X} \sim W_m(n, \boldsymbol{\Sigma}, h)$, $n > m - 1$, with significance $\alpha \in (0, 1)$, satisfies the following equation*

$$\frac{\pi^{\frac{mn}{2}} \Gamma_m\left(\frac{m+1}{2}\right)}{\Gamma_m\left(\frac{n}{2} + \frac{m+1}{2}\right)} |\boldsymbol{\Sigma}^{-1} \overline{\mathbf{V}}_\alpha(\mathbf{X})|^{\frac{n}{2}} {}_1F_1\left(h^{(k)}(0) : \frac{n}{2}; \frac{n}{2} + \frac{m+1}{2}; \boldsymbol{\Sigma}^{-1} \overline{\mathbf{V}}_\alpha(\mathbf{X})\right) = \alpha. \quad (5.3.1)$$

Numerical Solution. To find the upper matrix Value-at-Risk the Algorithm 4.2 of Koev and Edelman [2006] can be used. They have made an algorithm that efficiently approximate the hypergeometric function of a matrix argument through its expansion as a series of zonal polynomials. The implementation of the algorithms in MATLAB (Current version: 1.5, February 12, 2018) is available in Koev [2021]. Basically the algorithm computes the truncated Hypergeometric function ${}_pF_q$ as a series of zonal polynomials, truncated for partitions of size not exceeding M .

Remark. Only exist a formula for the second order zonal polynomials; the real case was due to James [1968], then complex case was provided by Caro-Lopera et al. [2007], finally, those results were unified by a formula given for Jack polynomials of second order in Caro-Lopera et al. [2016]. This formula allows us to derive in an exact way the VaR of a two order Wishart matrix, as we shall see below.

Corollary 5.3.2 *The matrix upper Value-at-Risk $\overline{\mathbf{V}}_\alpha(\mathbf{X}) > \mathbf{0}$ of $\mathbf{X} \sim W_2(n, I)$, with significance $\alpha \in (0, 1)$ satisfies the equation*

$$\frac{2^{-n-1} \sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n-1}{2}\right)} (\lambda_1, \lambda_2)^{\frac{n}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma\left(\frac{n}{2} + \lambda_1\right) \Gamma\left(\frac{n}{2} + \lambda_1 - \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{2} + \lambda_1\right) \Gamma\left(\frac{n}{2} + \lambda_1 + 1\right)} * \quad (5.3.2)$$

$$\left(\frac{\lambda_1 \lambda_1}{4}\right)^{\frac{\kappa}{2}} 2^{2\kappa} (\kappa)! (1)_{\kappa} \frac{\Gamma\left(\frac{1}{2}\right) \prod_{t < j}^p (2\kappa_i - 2\kappa_j - i + j)}{\Gamma\left(\frac{1}{2} + \frac{1+n}{2}\right) \Gamma\left(\frac{1-n}{2}\right) \prod_{i < j}^p (2\kappa_i + p - i)!} {}_2F_1\left(\frac{-n}{2}, \frac{n}{2} + \frac{1}{2}; \frac{1}{2}; \frac{(\lambda_1 + \lambda_1)^2}{4\lambda_1 \lambda_1}\right) + \quad (5.3.3)$$

$$\frac{\left(\frac{\lambda_1 \lambda_1}{4}\right)^{\frac{\kappa-1}{2}} 2^{2\kappa} (\kappa)! (1)_{\kappa} \Gamma\left(\frac{-1}{2}\right) \prod_{t < j}^p (2\kappa_i - 2\kappa_j - i + j)}{2\left(-\frac{\lambda_1}{2}, -\frac{\lambda_2}{2}\right)^{-1} \Gamma\left(\frac{-n}{2}\right) \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \prod_{i < j}^p (2\kappa_i + p - i)!} {}_2F_1\left(\frac{1}{2} + \frac{1+n}{2}, \frac{1}{2} - \frac{n}{2}, \frac{3}{2}; \frac{(\lambda_1 + \lambda_1)^2}{4\lambda_1 \lambda_1}\right) = \alpha. \quad (5.3.4)$$

Corollary 5.3.3 *When $r = \frac{1}{2}(n - m - 1)$ is a positive integer or a positive half-integer, the matrix lower Value-at-Risk $\underline{\mathbf{V}}_\alpha(\mathbf{X}) > \mathbf{0}$ of $\mathbf{X} \sim W_m(n, \boldsymbol{\Sigma}, h)$, with significance $\alpha \in (0, 1)$ is obtained solving the following equation*

$$\frac{\pi^{\frac{mn}{2}}}{\Gamma_m\left(\frac{n}{2}\right)} \sum_{k=0}^{mr} \frac{(-1)^k}{k!} \sum_{t=0}^{\infty} \frac{tr^t (\underline{\mathbf{V}}_\alpha(\mathbf{X}) \boldsymbol{\Sigma}^{-1}) \int_0^{\infty} h^{(t)}(w) w^{ma-k-1} dw}{t! \Gamma(ma - k)} \quad (5.3.5)$$

$$\times \sum_{\kappa}^* (-r)_{\kappa} \Gamma_m\left(\frac{n}{2}, -\kappa\right) C_{\kappa}(\underline{\mathbf{V}}_\alpha(\mathbf{X}) \boldsymbol{\Sigma}^{-1}) = 1 - \alpha.$$

5.3.1 Bounds for the matrix Value-at-Risk

First we establish a useful result for the bounds for the Matrix Value-at-Risk, which we will apply later.

Theorem 5.3.4 *Let ζ_1 and ζ_m are the largest and smallest latent root of $\overline{\mathbf{V}}_\alpha(\mathbf{X})$ respectively, then $\zeta_m \mathbf{I} < \overline{\mathbf{V}}_\alpha(\mathbf{X}) < \zeta_1 \mathbf{I}$.*

Note that the same result is obtained for $\underline{\mathbf{V}}_\alpha(\mathbf{X})$, i.e $\iota_m \mathbf{I} < \underline{\mathbf{V}}_\alpha(\mathbf{X}) < \iota_1 \mathbf{I}$, where ι_m and ι_1 are the smallest and larger latent roots of $\underline{\mathbf{V}}_\alpha(\mathbf{X})$ respectively.

Corollary 5.3.5 *Let ζ_1 and ζ_m are the largest and smallest latent root of $\overline{\mathbf{V}}_\alpha(\mathbf{X})$ respectively, then $m\zeta_m < \text{tr}(\overline{\mathbf{V}}_\alpha(\mathbf{X})) < m\zeta_1$.*

Corollary 5.3.6 *Let ζ_1 and ζ_m are the largest and smallest latent root of $\overline{\mathbf{V}}_\alpha(\mathbf{X})$ respectively, then $\zeta_m^m < |(\overline{\mathbf{V}}_\alpha(\mathbf{X}))| < \zeta_1^m$.*

5.3.2 Matricial barycenter

In this section we propose a definition of the barycenter for a matrix sample in the context of shape theory.

First we study the well known Wasserstein barycenter.

5.3.3 The problem

Following the exposition of Cuturi [2014], the well known Wasserstein barycenter can be seen as the mean of a set of empirical probability measures under the optimal transport metric. The barycenter corresponds to the measure that minimizes the sum of its Wasserstein distances to each element in the set. Cuturi [2014] smooth the Wasserstein distance used in the definition of Wasserstein barycenters with an entropic regularizer and recover in doing so a strictly convex objective whose gradients can be computed for a considerably cheaper computational cost using matrix scaling algorithms. The algorithms are very useful to visualize a large family of images and to solve a constrained clustering problem.

In order to extend the barycenter from the vectorial to the matrix case, we consider first the behavior of the above Wasserstein barycenter in some difficult situation.

Consider the small mice vertebra data, see for example Johnson et al. [1988]. After digitalizing the first 20 images, we find the following figure:



Figure 5.1 – Images from the original small mouse vertebra data sample

The Wasserstein Barycenter computed by the fast algorithm of Cuturi [2014] is given by:
Which is a plausible approximation for the mean.

Now, if we rotate the same data with an angle of 45° , the new sample is:

And the Wasserstein Barycenter behaves well:

Now, if we collect the 40 images of Figures 5.1 and 5.2 into a unified group and we compute again the Wasserstein Barycenter, the result is the following:

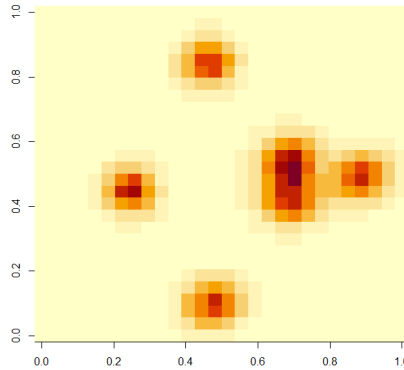


Figure 5.2 – Wasserstein Barycenter of the images of Figure 5.1



Figure 5.3 – 45° rotated images from the original small mouse vertebra data sample of Figure 5.1

Which is not the expected mean shape. Unfortunately, the above counterexample with mouse vertebra shows that the Wasserstein Barycenter is not invariant under rotation, then an alternative definition for the matricial barycenter is claimed in order that the shape of the object is preserved.

However, the mean shape as a simplex of the join rotated and non rotated sample under certain matrix models given by Díaz-García and Caro-Lopera [2017] can be used for a robust definition of the barycenter.

5.3.4 Barycenter based on Pseudo-Wishart Kotz distributions

For a self-content definition of a consistent barycenter, we provide all the required elements in this section, based on Díaz-García and Caro-Lopera [2017] and the references therein.

First we start with a class of matrix variate that generalize the Gaussian case. They include the contaminated Gaussian, Pearson type II and VII, Kotz, Jensen-Logistic, power exponential and Bessel distributions, among others; and these distributions have tails that are more or less weighted, and/or present a greater or smaller degree of kurtosis than the Gaussian distribution.

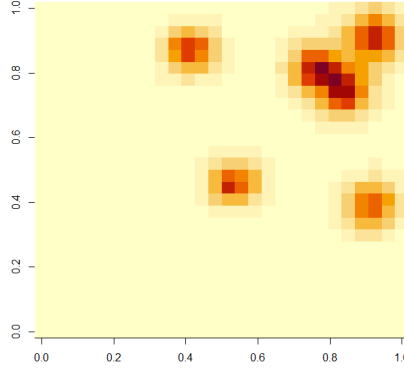


Figure 5.4 – Wasserstein Barycenter of rotated images of Figure 5.3

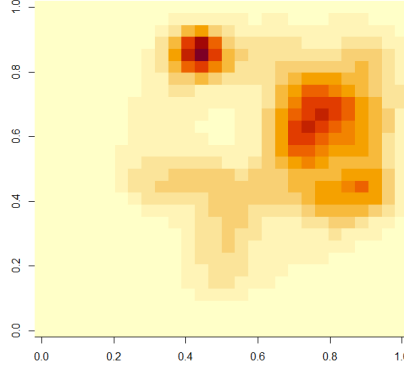


Figure 5.5 – Wasserstein Barycenter of joint rotated and non-rotated images of Figures 5.1 and 5.3

Definition 5.3.1 The $K \times D$ random matrix \mathbf{Y} is said to have a matrix variate elliptical distribution, with location parameter $\boldsymbol{\mu} \in \mathfrak{R}^{K \times D}$ and scale parameter $\boldsymbol{\Sigma} \otimes \boldsymbol{\Theta} \in \mathfrak{R}^{KD \times KD}$, if its density function with respect to the Lebesgue measure ($d\mathbf{Y}$) is given by

$$dF_{\mathbf{Y}}(\mathbf{Y}) = |\boldsymbol{\Sigma}|^{-D/2} |\boldsymbol{\Theta}|^{-K/2} h[\text{tr} \boldsymbol{\Theta}^{-1} (\mathbf{Y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{Y} - \boldsymbol{\mu})] (d\mathbf{Y}). \quad (5.3.6)$$

Here $\boldsymbol{\Sigma} \in \mathfrak{R}^{K \times K}$ and $\boldsymbol{\Theta} \in \mathfrak{R}^{D \times D}$ are positive definite matrices, denoted by $\boldsymbol{\Sigma} > 0$ and $\boldsymbol{\Theta} > 0$. The function $h : \mathfrak{R} \rightarrow [0, \infty)$ such that $\int_0^\infty u^{KD/2-1} h(u) du < \infty$ is termed the density generator. In notation, the elliptical matrix will be expressed as $\mathbf{Y} \sim \mathcal{E}_{K \times D}(\boldsymbol{\mu}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Theta}, h)$.

When the columns and/or rows of $\mathbf{Y} \sim \mathcal{E}_{K \times D}(\boldsymbol{\mu}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Theta}, h)$ are linearly dependent, the matrix \mathbf{Y} is said to have a *singular matrix variate elliptical distribution*. In this case \mathbf{Y} has a density respect to the *Hausdorff measure*. Moreover, such dependence is summarized in the rank of the matrices $\boldsymbol{\Sigma}$ and/or $\boldsymbol{\Theta}$. This will be denoted by $\mathbf{Y} \sim \mathcal{E}_{K \times D}^{s,r}(\boldsymbol{\mu}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Theta}, h)$, where $s = (\boldsymbol{\Sigma}) \leq K$ and $r = (\boldsymbol{\Theta}) \leq D$. We also highlight that the maximum likelihood estimators under singular matrix variate elliptical models follow the same rules of the singular Gaussian case.

In general, the estimation of Σ or Θ is not possible, but the Kronecker product $\Sigma \otimes \Theta = \text{cov}(\text{Vec}\mathbf{Y}^T)$ or $\Theta \otimes \Sigma = \text{cov}(\text{Vec}\mathbf{Y})$ are identifiable; here Vec denotes the vectorization operator

Now, assume that our data consist of a sample of independent matrices $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ from a given population.

Defining

$$\bar{\mathbf{y}} = \frac{1}{n} \sum_{i=1}^n \text{Vec}\mathbf{Y}_i^T, \text{ and } \mathbf{S} = \sum_{i=1}^n (\text{Vec}\mathbf{Y}_i^T - \bar{\mathbf{y}})(\text{Vec}\mathbf{Y}_i^T - \bar{\mathbf{y}})^T,$$

and given that $h(\cdot)$ is nonincreasing and continuous, then the *maximum likelihood estimate* of $(\text{Vec}\boldsymbol{\mu}, \Sigma \otimes \Theta)$ is

$$\left(\widetilde{\text{Vec}\boldsymbol{\mu}}, \widetilde{\Sigma \otimes \Theta} \right) = (\bar{\mathbf{y}}, \lambda_{\max} \mathbf{S}).$$

Here, λ_{\max} is the critical point where $h^*(\lambda)$ attains its maximum, and

$$h^*(\lambda) = \lambda^{-KDn/2} h(KD/\lambda).$$

Then the estimator of $\boldsymbol{\mu}$ is

$$\tilde{\boldsymbol{\mu}} = \bar{\mathbf{Y}} = \frac{1}{n} \sum_{i=1}^n \mathbf{Y}_i.$$

If $h(\cdot)$ is nonincreasing and continuous, and \mathbb{Y} has a finite 2nd moment, then

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{Y}} \quad \text{and} \quad \widehat{\Sigma \otimes \Theta} = \frac{1}{2(1-n)\psi'(0)} \mathbf{S},$$

are *unbiased estimators* of $\boldsymbol{\mu}$ and $\Sigma \otimes \Theta$.

Here, the characteristic function $\psi_{\mathbf{Y}}(\mathbf{T})$ of \mathbf{Y} is given by

$$\psi_{\mathbf{Y}}(\mathbf{T}) = \text{etr}(i\boldsymbol{\mu}^T \mathbf{T}) \phi(\text{tr} \mathbf{T} \Theta \mathbf{T}^T \Sigma), \quad (5.3.7)$$

with $i = \sqrt{-1}$, $\phi : [0, \infty) \rightarrow \Re$ and $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$.

The estimators also hold the preferable properties of *sufficiency, completeness, consistency and unbiasedness*.

Now we introduce the perturbation model which supports the barycenter in the context of shape theory.

Assume that the random matrix $\mathbf{X} \in \Re^{K \times D}$ represents a geometrical figure comprising K landmarks in D dimensions, with $K > D$.

Consider an independent sample of landmark coordinate matrices $\mathbf{X}_i \in \Re^{K \times D}$, $i = 1, 2, \dots, n$, from a given population.

If $\boldsymbol{\mu} \in \Re^{K \times D}$ is the corresponding mean form, then perturbation model is given by

$$\mathbf{X}_i = (\boldsymbol{\mu} + \mathbf{E}_i) \boldsymbol{\Gamma}_i + \mathbf{t}_i, \quad , i = 1, 2, \dots, n, \quad (5.3.8)$$

where $\mathbf{E}_i \sim \mathcal{E}_{K \times D}(\mathbf{0}, \Sigma_K \otimes \Sigma_D, h)$. The orthogonal matrices $\boldsymbol{\Gamma}_i \in \Re^{D \times D}$ are rotation and/or reflection of $(\boldsymbol{\mu} + \mathbf{E}_i)$. Meanwhile, the matrices $\mathbf{t}_i \in \Re^{K \times D}$ ($\mathbf{t}_i = \mathbf{1}_k \mathbf{a}_i^T$) represent translations with some $\mathbf{a}_i \in \Re^D$. Then we have that

$$\mathbf{X}_i \sim \mathcal{E}_{K \times D}(\boldsymbol{\mu} \boldsymbol{\Gamma}_i + \mathbf{t}_i, \Sigma_K \otimes \boldsymbol{\Gamma}_i^T \Sigma_D \boldsymbol{\Gamma}_i, h), \quad i = 1, 2, \dots, n. \quad (5.3.9)$$

The parameters of interest are $(\boldsymbol{\mu}, \Sigma_K \otimes \Sigma_D)$ and the nuisance parameters are $(\boldsymbol{\Gamma}_i^T, \mathbf{t}_i)$ $i = 1, 2, \dots, n$.

We can remove the nuisance parameters by using a simple transformation. Setting $\mathbf{X}_i^c = \mathbf{H}_K \mathbf{X}_i$ and using $\mathbf{H}_K \mathbf{1}_K = \mathbf{0}_k$ and $\mathbf{1}_K^T \mathbf{H}_K = \mathbf{0}_k^T$, we obtain

$$\mathbf{X}_i^c \sim \mathcal{E}_{K \times D}^{(K-1), D}(\boldsymbol{\mu}^* \boldsymbol{\Gamma}_i, \boldsymbol{\Sigma}_K^* \otimes \boldsymbol{\Gamma}_i^T \boldsymbol{\Sigma}_D \boldsymbol{\Gamma}_i, h), \quad i = 1, 2, \dots, n, \quad (5.3.10)$$

where $\boldsymbol{\mu}^* = \mathbf{H}_K \boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_K^* = \mathbf{H}_K \boldsymbol{\Sigma}_K \mathbf{H}_K$. Note that $\mathbf{H}_K \mathbf{t}_i = \mathbf{H}_K \mathbf{1}_K \mathbf{a}_i^T = \mathbf{0}$ for all $i = 1, 2, \dots, n$, then the summation of the columns of $\boldsymbol{\mu}^*$ is zero, which means that it is a centered matrix .

Recall that $K > D$ and $\text{rank}(\boldsymbol{\Sigma}_K^*) = K - 1$, then we get

$$\mathbf{B}_i = \mathbf{X}_i^c (\boldsymbol{\Gamma}_i^T \boldsymbol{\Sigma}_D \boldsymbol{\Gamma}_i)^{-1} (\mathbf{X}_i^c)^T \sim \mathcal{GPW}_K^q(D, \boldsymbol{\Sigma}_K^*, \boldsymbol{\Sigma}_D, \boldsymbol{\Omega}, h), \quad i = 1, 2, \dots, n. \quad (5.3.11)$$

where

$$\boldsymbol{\Omega} = (\boldsymbol{\Sigma}_K^*)^{-1} \boldsymbol{\mu}^* \boldsymbol{\Gamma}_i (\boldsymbol{\Gamma}_i^T \boldsymbol{\Sigma}_D \boldsymbol{\Gamma}_i)^{-1} \boldsymbol{\Gamma}_i^T (\boldsymbol{\mu}^*)^T = (\boldsymbol{\Sigma}_K^*)^{-1} \boldsymbol{\mu}^* \boldsymbol{\Sigma}_D^{-1} (\boldsymbol{\mu}^*)^T,$$

$q = \min((K - 1), D)$ and \mathbf{A}^- is any symmetric generalized inverse of \mathbf{A} such that $\mathbf{A} \mathbf{A}^- \mathbf{A} = \mathbf{A} = \mathbf{A}^T$. By definition, \mathbf{B}_i is said to have a *generalized singular pseudo-Wishart distribution*, which is independent of the nuisance parameters.

Observe that \mathbf{B}_i can be written as

$$\mathbf{B}_i = \mathbf{X}_i^c (\boldsymbol{\Gamma}_i^T \boldsymbol{\Sigma}_D \boldsymbol{\Gamma}_i)^{-1} (\mathbf{X}_i^c)^T = \mathbf{X}_i^c \boldsymbol{\Gamma}_i^T \boldsymbol{\Sigma}_D^{-1} \boldsymbol{\Gamma}_i (\mathbf{X}_i^c)^T = \mathbf{Y}_i \boldsymbol{\Sigma}_D^{-1} \mathbf{Y}_i^T$$

where $\mathbf{Y}_i = \mathbf{X}_i^c \boldsymbol{\Gamma}_i^T$ and

$$\mathbf{Y}_i \sim \mathcal{E}_{K \times D}^{(K-1), D}(\boldsymbol{\mu}^*, \boldsymbol{\Sigma}_K^* \otimes \boldsymbol{\Sigma}_D, h), \quad i = 1, 2, \dots, n.$$

Behind the common assumption of independent landmarks along the D axes, we are formally considering $\boldsymbol{\Sigma}_D = \mathbf{I}_D$ in the context of a matrix variate Gaussian model. However, in the elliptical case the perspective is wider and we have two possible scenarios:

1. Independence and non correlated landmarks
2. Probabilistic dependence and non correlated landmarks.

In both cases $\boldsymbol{\Sigma}_D = \mathbf{I}_D$, but the moments of the matrix \mathbf{B} are different.

Taking $\boldsymbol{\Sigma}_D = \mathbf{I}_D$ and

$$\mathbf{X}_i^c = (\mathbf{X}_{1,i}^c | \mathbf{X}_{2,i}^c | \dots | \mathbf{X}_{D,i}^c),$$

with

$$\mathbf{X}_{d,i}^c \sim \mathcal{E}_K^{(K-1)}(\boldsymbol{\mu}^* \boldsymbol{\Gamma}_i \mathbf{e}_d^K, \boldsymbol{\Sigma}_K^*, h), \quad d = 1, 2, \dots, D; \quad i = 1, 2, \dots, n,$$

then we have,

$$\mathbf{B}_i = \mathbf{X}_i^c (\mathbf{X}_i^c)^T = \sum_{d=1}^D \mathbf{X}_{d,i}^c (\mathbf{X}_{d,i}^c)^T;$$

furthermore,

$$\mathbf{B}_i \sim \mathcal{GPW}_K^q(D, \boldsymbol{\Sigma}_K^*, \mathbf{I}_D, \boldsymbol{\Omega}, h), \quad i = 1, 2, \dots, n, \quad (5.3.12)$$

where $\boldsymbol{\Omega} = (\boldsymbol{\Sigma}_K^*)^{-1} \boldsymbol{\mu}^* (\boldsymbol{\mu}^*)^T$.

Here we only consider the first two moments of

$$\mathbf{B} = \mathbf{Y} \mathbf{Y}^T = \sum_{d=1}^D \mathbf{y}_d \mathbf{y}_d^T, \quad (5.3.13)$$

under independence, i.e. the \mathbf{y}_d 's are independent and uncorrelated.

We return to the original notation $\Theta = \Sigma_D$, $\Sigma = \Sigma_K^*$ and $\mu = \mu^*$.

The first two sample moment estimators of \mathbf{B} under independent and dependent cases are given by

$$\widetilde{E}(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^n \mathbf{B}_i = \bar{\mathbf{B}} = (\bar{b}_{ij}), \quad i, j = 1, \dots, K,$$

and

$$\text{cov}(\widetilde{Vec}\mathbf{B}) = \frac{1}{n} \sum_{i=1}^n (Vec\mathbf{B}_i^T - Vec\widetilde{E}(\mathbf{B})) (Vec\mathbf{B}_i^T - Vec\widetilde{E}(\mathbf{B}))^T = \mathbf{S},$$

where $\mathbf{S} = (s_{tr})$, $t, r = 1, 2, \dots, K^2$. Moreover, for $i \leq j$, $\mathbf{M} = \mu^* \mu^{*T} = (m_{ij}) = \mathbf{M}^T$ and $\Sigma_K^* = (\sigma_{ij})$.

Then finally we have:

5.3.5 Matrix Barycenter

We have finally arrive at the called mean form in shape theory, but under the conditions given in Cuturi [2014] it can be seen as the matrix barycenter $\tilde{\mu}^*$ obtained by the method-of-moments estimator $\tilde{\mathbf{M}}$. In this case the barycenter has the remarkable properties of invariance under translation, rotation, and reflection transformations.

Lemma 5.3.7 (Matrix barycenter) *Let $\tilde{\mathbf{M}}$ be the method-of-moments estimator of $\mathbf{M} = \mu^* \mu^{*T}$ (for dependent or independent cases). Let $\tilde{\mathbf{M}} = \mathbf{V}_1 \mathbf{L} \mathbf{V}_1^T$ the nonsingular part of the corresponding spectral decomposition, where \mathbf{V}_1 is a semiorthogonal matrix. Here $\mathbf{V}_1 \in \mathfrak{R}^{K \times D}$, $\mathbf{V}_1^T \mathbf{V}_1 = \mathbf{I}_D$ and $\mathbf{L} = \text{diag}(\lambda_1, \dots, \lambda_D)$, where D is the rank of \mathbf{M} . Then the method-of-moments estimator of μ^* is*

$$\tilde{\mu}^* = \mathbf{V}_1 \mathbf{W},$$

with $\mathbf{W} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_D})$.

The barycenter also holds that:

Lemma 5.3.8 *Let $(\tilde{\mu}^*, \tilde{\Sigma}_K^*)$ be the method-of-moments estimators of*

$$(\mu^*, \Sigma_K^*).$$

Then as $n \rightarrow \infty$

$$(\tilde{\mu}^*, \tilde{\Sigma}_K^*) \rightarrow (\mu^*, \Sigma_K^*) \quad \text{in probability.}$$

5.3.6 The matrix barycenter in the class of Kotz distributions

We end this section by proposing a useful class of distributions including Gaussian for studying the matrix barycenter. Following again from Díaz-García and Caro-Lopera [2017], the Singular Pseudo-Wishart Gaussian distribution takes the form:

Corollary 5.3.9 (Singular Pseudo-Wishart Gaussian distribution) *Assume that $\mathbf{Y} \sim \mathcal{N}_{K \times D}^{K-1, D}(\mu, \Sigma \otimes \Theta)$, and let $q = \min(K-1, D)$; then the density of $\mathbf{B} = \mathbf{Y} \Theta^{-1} \mathbf{Y}^T$ is given by*

$$= \text{Cetr}(-\text{Half}(\Sigma^{-1} \mathbf{B} - \Omega)) {}_0F_1 \left(\text{Half} D; \frac{1}{4} \Omega \Sigma^{-1} \mathbf{B} \right) (d\mathbf{B}). \quad (5.3.14)$$

Here

$$C = \frac{\pi^{D(q-(K-1))/2} |\mathbf{L}|^{(D-K-1)/2}}{2^{D(K-1)/2} \Gamma_q[D/2] \left(\prod_{i=1}^{K-1} \lambda_i^{D/2} \right)},$$

where ${}_0F_1(\cdot)$ is a hypergeometric function with a matrix argument, see Muirhead [2005].

5.3.7 Singular Pseudo-Wishart Kotz distribution.

For the general class of distribution we recall that the $K \times D$ random matrix \mathbf{X} is said to have a *singular matrix variate symmetric Kotz type distribution* with parameters $N, r, s \in \mathfrak{R}$, $\boldsymbol{\mu} : K \times D$, $\boldsymbol{\Sigma} : K \times K$, of rank $K - 1$, $\boldsymbol{\Theta} : D \times D$ with $r > 0$, $s > 0$, $2N + (K - 1)D > 2$, $\boldsymbol{\Sigma} > \mathbf{0}$, and $\boldsymbol{\Theta} > \mathbf{0}$ if its density is

$$\frac{s r^{(2N+(K-1)D-2)/2s} \Gamma[(K-1)D/2]}{\pi^{(K-1)D/2} \Gamma[(2N+(K-1)D-2)/2s] \left(\prod_{i=1}^{K-1} \lambda_i^{D/2} \right) |\boldsymbol{\Theta}|^{(K-1)/2}} \\ \times [\text{tr} \boldsymbol{\Theta}^{-1} (\mathbf{Y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{Y} - \boldsymbol{\mu})]^{N-1} \exp \left\{ -r t r^s \boldsymbol{\Theta}^{-1} (\mathbf{Y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{Y} - \boldsymbol{\mu}) \right\}.$$

When $T = s = 1$, and $R = 1/2$ we get the singular matrix variate gaussian distribution.

Note that particular singular Pseudo-Wishart distributions just depend on the general derivative $h^{(2t)}(\cdot)$ of the elliptical generator function; it seems a trivial fact, but the general formulae involves cumbersome expressions indexed by partitions, see Caro-Lopera et al. [2010].

When $s = 1$, the Kotz type models and their general derivative simplify substantially. Thus, the following expressions applies for Gaussian, Kotz 1, and Kotz 2 models, with parameters $N = 1, s = 1, r = 1/2$; $N = 2, s = 1, r = 1/2$; $N = 3, s = 1, r = 1/2$; respectively. The generator model is given by

$$h(y) = \frac{r^{N-1+(K-1)D/2} \Gamma[(K-1)D/2]}{\pi^{(K-1)D/2} \Gamma[N-1+(K-1)D/2]} y^{N-1} \exp\{-ry\},$$

And, the corresponding k -th derivative of h , follows from

$$\frac{d^k}{dy^k} y^{N-1} \exp[-ry],$$

which is given by

$$(-r)^k y^{N-1} \exp[-ry] \left\{ 1 + \sum_{v=1}^k \binom{k}{v} \left[\prod_{i=0}^{v-1} (N-1-i) \right] (-ry)^{-v} \right\},$$

where $k = 2t$.

For the remaining models of the example, the termed Kotz 3, Kotz 4 and Kotz 5, have parameters $N = 2, s = 2, r = 1/2$; $N = 2, s = 3, r = 1/2$ and $N = 20, s = 20, r = 1/2$, respectively. The generator function is given by:

$$h(y) = \frac{s r^{(2N+(K-1)D-2)/2s} \Gamma[(K-1)D/2]}{\pi^{(K-1)D/2} \Gamma[(2N+(K-1)D-2)/2s]} y^{N-1} \exp(-ry^s).$$

The required k -th derivative of h , follows from $\frac{d^k}{dy^k} \exp(-ry^s)$, which is given by

$$y^{T-1} e^{-Ry^s} \left\{ \sum_{\kappa \in P_k} \frac{k! (-R)^{\sum_{i=1}^k v_i} \prod_{j=0}^{k-1} (s-j)^{\sum_{i=j+1}^k v_i}}{\prod_{i=1}^k v_i! (i!)^{v_i}} y^{\sum_{i=1}^k (s-i)v_i} \right. \\ \left. + \sum_{m=1}^k \binom{k}{m} \left[\prod_{i=0}^{m-1} (T-1-i) \right] \right. \\ \left. \times \sum_{\kappa \in P_{k-m}} \frac{(k-m)! (-R)^{\sum_{i=1}^{k-m} v_i} \prod_{j=0}^{k-m-1} (s-j)^{\sum_{i=j+1}^{k-m} v_i}}{\prod_{i=1}^{k-m} v_i! (i!)^{v_i}} y^{\sum_{i=1}^{k-m} (s-i)v_i - m} \right\},$$

where $\sum_{\kappa \in P_k}$ denotes the summation over all the partitions

$$\kappa = (k^{v_k}, (k-1)^{v_{k-1}}, \dots, 3^{v_3}, 2^{v_2}, 1^{v_1})$$

of k , with $\sum_{i=1}^k i v_i = k$, i.e. κ is a partition of k consisting of v_1 ones, v_2 twos, v_3 threes, etc. It is important to quote that all the singular Pseudo-Wishart distributions based on Kotz type kernels can be computed by some modifications of the Gaussian version in Koev and Edelman [2006].

5.3.8 Application

We are now in position of given an invariant barycenter under rotation, translation and reflection of the landmark data studied at the beginning of this section under the usual Wasserstein barycenter.

We have noted that if we collect compute the Wasserstein Barycenter of the 40 images of Figures 5.1 and 5.2, the result cannot obtained the expected mean shape of the vertebrae. However, our definition of the matrix barycenter based on the mean shape form, allows mixtures of rotated and non rotated figures, moreover it depends on the complexity of the selected model.

We compute in this sample the barycenter under six different models with independent landmarks. Moment-method estimates of mean shape by using the classical Gaussian model is shown in figure 5.6.

In the Gaussian case the estimate is unrealistic, we expect a mean form similar to the ideal vertebra. Note that each point of the polygon represents a landmark of the bone. In the Gaussian case, the assumption of landmark independence explains the broken symmetry in the estimate. However, if we consider more robust isotropic models than Gaussian, the estimation tends to recover the symmetry of the vertebra. Indeed, this is a surprising and interesting aspect to research, because the independence of the landmarks is neglected by the complexity of the model and the mean form gets closer to the ideal vertebra.

Note that the Kotz-5 model provides a better estimation of the barycenter, even under independence.

The addressed heuristic evolution suggests also the role of the analysis in shape data. First note that the mouse vertebra data is based on non anatomical landmarks, then a number of models have equal chance to estimate the mean form. In this case, we have noticed that the classical isotropic Gaussian model of the literature is not appropriate. Then, we can propose robust laws and a selection criteria.

Finally note that we have also an estimation of the correlation structure of the matrix barycenter. And this fact is usually out of the classical exposition of the barycenter based on Wasserstein barycenter.

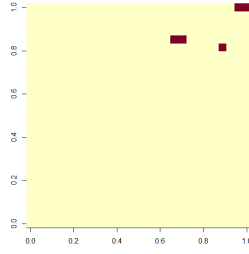


Figure 5.6 – Moment method estimates under independence: Gaussian model

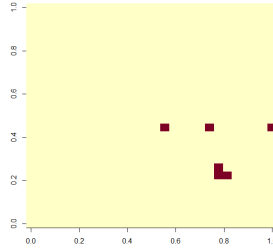


Figure 5.7 – Moment method estimates under independence: Kotz 1 model

5.4 Elliptical Beta Distribution

The classical two-parameter probability density function of the Beta distribution with shape parameters a and b is given by

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}, \quad 0 \leq x \leq 1, \quad a > 0, \quad b > 0 \quad (5.4.1)$$

The Beta distribution is a useful model for a number of phenomena governed by random variables restricted to finite length domains Wang [2005]. It is one of the most used in statistics since it models a wide class of data with different shapes in closed domains. The distribution can be strongly right-skewed or less skewed as the parameters approach each other, also the distributions would be left-skewed if the parameters values were switched, as can see in Figure 1.

Various generalizations of the Beta distribution have been considered see, for instance, Wang [2005] and reference therein, but all these techniques are based on the hypothesis that some variables A and B are independent with Chi-Square distributions in the scalar case or Wishart distribution in the matrix case. In this section, we are generalizing these results, assuming the variables have an Elliptical Wishart distribution.

The generalized Elliptical Wishart distribution allows multiple extensions to robust Elliptical models when the Gaussian assumption is difficult to keep. The addressed distribution was derived by Caro-Lopera et al. [2014] and it is given as follows.

Definition 5.4.1 *The variable $X = Z'Z$, is said to have a Elliptical Wishart distribution with*

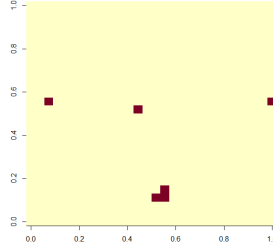


Figure 5.8 – Moment method estimates under independence: Kotz 2 model

n degrees of freedom $EW_1(n, \Sigma, h)$, if its p.d.f is given by

$$f_X(x) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n}{2}\right) \Sigma^{\frac{n}{2}}} x^{\frac{n}{2}-1} h_1\left(\Sigma^{-1}x\right), \quad (5.4.2)$$

where Z is a variable with Elliptically contoured $E(0, 1, h)$, with p.d.f given in Gupta et al. [2013] by $f_Z(Z) = \Sigma^{-\frac{n}{2}} h(\Sigma^{-1}ZZ')$, where the function $h: \Re \rightarrow [0, \infty)$, called the generator function, is such that $\int_0^\infty u^{p-1} h(u^2) du < \infty$.

Now, we developed a result, analogous to those of Muirhead [2005], for introducing the Beta distribution, under elliptical models.

Proposition 5.4.1 Let A and B be independent univariate elliptical Wishart distributions, where $A \sim EW_1(n_1, \Sigma, h_1)$ and $B \sim EW_1(n_2, \Sigma, h_2)$ with $\Sigma > 0, n_1 > 0, n_2 > 0$. Put $A + B = T^2$ where $T > 0$ and let X be defined by $A = T^2 X$, then the density function of X is

$$\frac{\pi^{\frac{n_1+n_2}{2}}}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} x^{-\frac{n_2}{2}-1} (1-x)^{\frac{n_2}{2}-1} \sum_{k=0}^{\infty} \frac{h_2^{(k)}(0)}{k! x^k (1-x)^{-k}} \int_0^\infty h_1(y) y^{\frac{n_1}{2} + \frac{n_2}{2} + k - 1} dy. \quad (5.4.3)$$

with $0 \leq x \leq 1$. We will said that X has an Elliptical Beta distribution indexed by the generators h_1 and h_2 with suitable existence conditions.

The generalized Beta will be denoted by given $EB(n_1, n_2; h_1, h_2)$.

Remark The well known univariate Beta distribution is derived as follows: Take the gaussian generators $h_1(y) = \frac{1}{(2\pi)^{\frac{n_1}{2}}} e^{-\frac{y}{2}}$, $h_2(y) = \frac{1}{(2\pi)^{\frac{n_2}{2}}} e^{-\frac{y}{2}}$ and $h_2^{(k)}(0) = \frac{(-1)^k}{(2\pi)^{\frac{n_2}{2}} 2^k}$. Then,

$$\int_0^\infty h_1(y) y^{\frac{n_1}{2} + \frac{n_2}{2} + k - 1} dy = \frac{2^{\frac{n_2}{2} + k}}{\pi^{\frac{n_1}{2}}} \Gamma\left(\frac{n_1}{2} + \frac{n_2}{2} + k\right)$$

For the summation, use $\Gamma(z+k) = \Gamma(z)(z)_k$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k (1-x)^k}{k! x^k} \Gamma\left(\frac{n_1}{2} + \frac{n_2}{2} + k\right) = \Gamma\left(\frac{n_1}{2} + \frac{n_2}{2}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{n_1}{2} + \frac{n_2}{2}\right)_k}{k!} \left(1 - \frac{1}{x}\right)^k.$$

Now, by zonal polynomial theory, $|\mathbf{I}_m - \mathbf{X}|^{-a} = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_{\kappa}}{k!} C_{\kappa}(\mathbf{X})$, $\|\mathbf{X}\| < 1$, see for example Muirhead [2005]. Applying this for $m = 1$, the summation just becomes $\left(\frac{1}{x}\right)^{-\left(\frac{n_1}{2} + \frac{n_2}{2}\right)}$.

Then the distribution $\frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}x^{\frac{n_1}{2}-1}(1-x)^{\frac{n_2}{2}-1}$ is obtained.

Remark A number of generalized Beta distributions can be derived by using the following classical generators of elliptical distributions. As you can see, these generalizations are closely related to the hypergeometric Gauss function and the generalized hypergeometric Gauss function are defined in Caro-Lopera et al. [2010] respectively.

5.4.1 Beta-Pearson model

An interesting generalized Beta distribution can be derived from the Proposition 5.4.1 in terms of the Pearson VII model, in this case,

$$h(y) = \frac{\Gamma(s)}{(\pi R)^{\frac{n}{2}} \Gamma(s - \frac{n}{2})} \left(1 + \frac{y}{R}\right)^{-s}$$

and

$$h^{(k)}(0) = \frac{\Gamma(s)(-1)^k (s)_k}{R^k (\pi R)^{\frac{n}{2}} \Gamma(s - \frac{n}{2})}.$$

Then $\int_0^\infty h_1(y) y^{\frac{n_1}{2} + \frac{n_2}{2} + k - 1} dy = \frac{R_1^{\frac{n_2}{2} + k} \Gamma(\frac{n_1}{2} + \frac{n_2}{2} + k) \Gamma(s_1 - \frac{n_1}{2} - \frac{n_2}{2} - k)}{\pi^{\frac{n_1}{2}} \Gamma(s_1 - \frac{n_1}{2})}$ and the distribution can be written as:

$$\begin{aligned} & \frac{\Gamma(s_2) x^{-\frac{n_2}{2}-1} (1-x)^{\frac{n_2}{2}-1}}{\Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2}) \Gamma(s_2 - \frac{n_2}{2}) \Gamma(s_1 - \frac{n_1}{2})} \\ & \times \sum_{k=0}^{\infty} \frac{(s_2)_k \Gamma(\frac{n_1}{2} + \frac{n_2}{2} + k) \Gamma(s_1 - \frac{n_1}{2} - \frac{n_2}{2} - k)}{(-1)^k k! x^k (1-x)^{-k} \left(\frac{R_2}{R_1}\right)^{\frac{n_2}{2} + k}} \end{aligned}$$

Now, using $\Gamma(z+k) = (z)_k \Gamma(z)$ and $\Gamma(z-k) = \frac{\Gamma(z)\Gamma(-z+1)}{(-1)^k \Gamma(-z+k+1)}$, we have:

$$\begin{aligned} & \frac{\Gamma(\frac{n_1}{2} + \frac{n_2}{2}) \Gamma(s_2) \Gamma(s_1 - \frac{n_1}{2} - \frac{n_2}{2}) x^{-\frac{n_2}{2}-1} (1-x)^{\frac{n_2}{2}-1}}{\Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2}) \Gamma(s_1 - \frac{n_1}{2}) \Gamma(s_2 - \frac{n_2}{2})} \\ & \times \sum_{k=0}^{\infty} \frac{(s_2)_k \left(\frac{n_1}{2} + \frac{n_2}{2}\right)_k}{k! x^k (1-x)^{-k} \left(\frac{R_2}{R_1}\right)^{\frac{n_2}{2} + k} \left(-s_1 + \frac{n_1}{2} + \frac{n_2}{2} + 1\right)_k} \end{aligned}$$

And using the generalized hypergeometric function, we have:

$$\begin{aligned} & \frac{\Gamma(\frac{n_1}{2} + \frac{n_2}{2}) \Gamma(s_2) \Gamma(s_1 - \frac{n_1}{2} - \frac{n_2}{2}) x^{-\frac{n_2}{2}-1} (1-x)^{\frac{n_2}{2}-1}}{\Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2}) \Gamma(s_1 - \frac{n_1}{2}) \Gamma(s_2 - \frac{n_2}{2})} \\ & \times {}_2P_1 \left(\left(\frac{R_1}{R_2} \right)^{\frac{n_2}{2} + k} : s_2, \frac{n_1}{2} + \frac{n_2}{2}; -s_1 + \frac{n_1}{2} + \frac{n_2}{2} + 1; x^{-1}(1-x) \right) \end{aligned}$$

When $R_1 = R_2$, the Beta-Pearson VII simplifies to:

$$\begin{aligned} & \frac{\Gamma(\frac{n_1}{2} + \frac{n_2}{2}) \Gamma(s_2) \Gamma(s_1 - \frac{n_1}{2} - \frac{n_2}{2}) x^{-\frac{n_2}{2}-1} (1-x)^{\frac{n_2}{2}-1}}{\Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2}) \Gamma(s_1 - \frac{n_1}{2}) \Gamma(s_2 - \frac{n_2}{2})} \\ & \times {}_2F_1 \left(s_2, \frac{n_1}{2} + \frac{n_2}{2}; -s_1 + \frac{n_1}{2} + \frac{n_2}{2} + 1; x^{-1}(1-x) \right) \end{aligned}$$

If $a = s_2, b = \frac{n_1}{2} + \frac{n_2}{2}, c = -s_1 + \frac{n_1}{2} + \frac{n_2}{2} + 1$, the Kummer relations of Erdelyi, p. 105, can provide a polynomial when $c - a$ or $c - b$ is a negative integer.

5.4.2 Beta-Kotz model

Note that, among any other restrictions, the Proposition 5.4.1 can be applied if $h_2^{(k)}(0)$ exists and it is different from zero. If we consider two Kotz type I models,

$$h(y) = \frac{r^{t_1 + \frac{n_2}{2} - 1}}{\pi^{\frac{n_1}{2}} \Gamma(t_1 + \frac{n_2}{2} - 1)} y^{t_1 - 1} e^{-ry} \quad (5.4.4)$$

Then h_1 and h_2 , depending on parameters $t_i \neq 1, s_i = 1, r_i = R, i = 1, 2$, then $h_2^{(k)}(0) = 0$ and the distribution of the associated Beta must be derived by another method. To get to this end, we just use the Taylor expansion in the joint density function of T^2 and X , so we have

$$\frac{\pi^{\frac{n_1}{2} + \frac{n_2}{2}} x^{\frac{n_1}{2} - 1} (1-x)^{\frac{n_2}{2} - 1} (t^2)^{\frac{n_1}{2} + \frac{n_2}{2} - 1}}{\Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2}) \Sigma^{\frac{n_1}{2} + \frac{n_2}{2}}} h_1(\Sigma^{-1} t^2 x) h_2(\Sigma^{-1} t^2 (1-x)) dt^2 dx.$$

Replacing the generator of the Kotz function $h_i(y) = \frac{R^{t_i + \frac{n_i}{2} - 1} \Gamma(\frac{n_i}{2})}{\pi^{\frac{n_i}{2}} \Gamma(t_i + \frac{n_i}{2} - 1)} y^{t_i - 1} e^{-Ry}$, $i = 1, 2$, we obtain

$$\frac{r^{t_1 + \frac{n_1}{2} - 1} r^{t_2 + \frac{n_2}{2} - 1}}{\Gamma(t_1 + t_2 + \frac{n_1}{2} + \frac{n_2}{2} - 2) \Sigma^{(t_1 + t_2 + \frac{n_1}{2} + \frac{n_2}{2} - 2)}} t^{2(t_1 + t_2 + \frac{n_1}{2} + \frac{n_2}{2} - 3)} e^{-(r_1 + r_2) \Sigma^{-1} t^2} \times$$

$$\frac{\Gamma(t_1 + t_2 + \frac{n_1}{2} + \frac{n_2}{2} - 2)}{\Gamma(t_1 + \frac{n_1}{2} - 1) \Gamma(t_2 + \frac{n_2}{2} - 1)} x^{(t_1 + \frac{n_1}{2} - 1) - 1} (1-x)^{(t_2 + \frac{n_2}{2} - 1) - 1} dx dt^2$$

Then $T^2 = A + B$ is independent of X , and the distribution of X is

$$\frac{\Gamma(t_1 + t_2 + \frac{n_1}{2} + \frac{n_2}{2} - 2)}{\Gamma(t_1 + \frac{n_1}{2} - 1) \Gamma(t_2 + \frac{n_2}{2} - 1)} x^{(t_1 + \frac{n_1}{2} - 1) - 1} (1-x)^{(t_2 + \frac{n_2}{2} - 1) - 1} \quad (5.4.5)$$

Note that if $r_1 = r_2 = R$ then $T^2 \sim EW_1(n_1 + n_2, \Gamma, h(t_1 + t_2 - 1, R, s = 1))$.

Let us denote the Beta-Kotz distribution $KBeta(t_1 + \frac{n_1}{2} - 1, t_2 + \frac{n_2}{2} - 1)$ as the distribution of the random variable X previously defined. Note also that if $t_1 = t_2 = 1$, then $X \sim Beta(\frac{n_1}{2}, \frac{n_2}{2})$.

Remark: We check a corollary of the above general Beta-Kotz distribution. We can mix for example a Gaussian class model $h_2(y) = \frac{r^{\frac{n_2}{2}}}{\pi^{\frac{n_2}{2}}} e^{-ry}$ with exponential scale r instead of $\frac{1}{2}$ and non vanishing $h_2^{(k)}(0) = \frac{(-1)^k r^{\frac{n_2}{2} + k}}{\pi^{\frac{n_2}{2}}}$, with a Kotz type I model, $h_1(y) = \frac{r^{t_1 + \frac{n_2}{2} - 1}}{\pi^{\frac{n_1}{2}} \Gamma(t_1 + \frac{n_2}{2} - 1)} y^{t_1 - 1} e^{-ry}$.

After simplification the summation in (5.4.3) goes to

$\frac{\Gamma(\frac{n_1}{2}) \pi^{-\frac{n_1}{2} - \frac{n_1}{2}} x^{-1 + \frac{n_1}{2} + \frac{n_2}{2} + t_1}}{\Gamma(t_1 + \frac{n_1}{2} - 1)} \Gamma(t_1 + \frac{n_1}{2} + \frac{n_2}{2} - 1)$, and then the associated Beta-Kotz is obtained as follows

$$\frac{\Gamma(t_1 + \frac{n_1}{2} + \frac{n_2}{2} - 1)}{\Gamma(\frac{n_2}{2}) \Gamma(t_1 + \frac{n_1}{2} - 1)} x^{t_1 + \frac{n_1}{2} - 2} (1-x)^{\frac{n_2}{2} - 1} \quad (5.4.6)$$

This coincides with (5.4.5) when $t_2 = 1$, as we expect.

5.5 Value-at-Risk using the Beta-Kotz Distribution

In this section we are studying Value-at-Risk, when the data come from the family of Beta-Kotz distributions introduced early.

Theorem 5.5.1 *Let $X \sim KBeta(n_1, n_2, t_1, t_2)$ with $n_1 > 0, n_2 > 0, t_1 + \frac{n_1}{2} - 1 > 0$ and $t_2 + \frac{n_2}{2} - 1 > 0$, and, F_X denote the cumulative distribution function. The Beta-Kotz Value-at-Risk of X at probability level $\alpha \in (0, 1)$ is obtained solving the hypergeometric equation:*

$$C \frac{VaR_{\beta_\alpha}(X)^{t_1 + \frac{n_1}{2} - 1}}{(t_1 + \frac{n_1}{2} - 1)} {}_2F_1\left(t_1 + \frac{n_1}{2} - 1, -t_2 - \frac{n_2}{2} + 2; t_1 + \frac{n_1}{2}, VaR_{\beta_\alpha}(X)\right) - \alpha = 0, \quad (5.5.1)$$

where $C = \frac{\Gamma(t_1 + t_2 + \frac{n_1}{2} + \frac{n_2}{2} - 2)}{\Gamma(t_1 + \frac{n_1}{2} - 1)\Gamma(t_2 + \frac{n_2}{2} - 1)}$.

From now on, to simplify the notation, let $a = t_1 + n_1/2 - 1$ and $b = t_2 + n_2/2 - 1$, then the equation (5.5.1) is equivalent to

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{VaR_{\beta_\alpha}(X)^a}{a} {}_2F_1(a, 1-b; a+1; VaR_{\beta_\alpha}(X)) - \alpha = 0 \quad (5.5.2)$$

Then the Beta-Kotz Value-at-Risk of X at probability level $\alpha \in (0, 1)$ is obtained solving the hypergeometric equation:

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(1-b)_k}{(a+k)k!} VaR_{\beta_\alpha}(X)^{a+k} - \alpha = 0$$

5.6 Switch random fit approach for certain class of univariate distributions

Consider an i.i.d. random sample $\mathbf{z} = (z_1, \dots, z_n)$ of an univariate distribution with pdf $w = c(\boldsymbol{\theta})f(\mathbf{z}; \boldsymbol{\theta})$ indexed by an unknown vector of p parameters. Here $c(\boldsymbol{\theta})$ is the normalization constant.

Assume a multiplicative error continuous model $w_i = f(z_i; \boldsymbol{\theta})\epsilon_i$, $i = 1, \dots, n$, such that its logarithm is well defined and the corresponding linear model $\ln(w_i) = \ln(f(z_i; \boldsymbol{\theta})) + \ln \epsilon_i$ satisfies the Gauss-Markov theorem and estimates the following parameters $\beta_i(\boldsymbol{\theta}), i = 0, 1, \dots, p$, via least squares. Here $\ln f(z; \boldsymbol{\theta})$ accepts the expansion $\beta_0 + \beta_1(\boldsymbol{\theta})x_1(\mathbf{z}) + \dots + \beta_p(\boldsymbol{\theta})x_p(\mathbf{z})$ in order that the $\beta_i(\boldsymbol{\theta})$'s provides a consistent system of equations for a unique solution $(\boldsymbol{\theta})$.

This assumption seems to restrictive but it can be checked that most of the well known univariate models can fit in such way.

By another hand, it is known that the random sample z_1, \dots, z_n of i.i.d $w = f(z; \boldsymbol{\theta})$ can provide under some restrictions an estimate of $\boldsymbol{\theta}$ via likelihood, but in general the estimation has not a closed form and must be solved by using numerical methods. We now propose an approach which exploits the simplicity of the least square estimation assuming that the Gauss-Markov theorem holds. Both methods seem to be different in nature, because z_1, \dots, z_n are random variables in the likelihood method, but in our setting their ordering under certain precision are deterministic.

Now, we are interested in applications in finance and other disciplines, so our data has an intrinsic precision and domain, which is known a priori. Moreover, they can not taken real values with infinite number of decimal places or they can not take values in the complete real

line up infinity. It is clear that the simulation algorithm include such restriction also, they have a tolerance for the precision of the probabilities and the random numbers they provide.

For empirical data representing some experiment with certain precision and domain, the method is heuristically constructed as follows:

1. Start with a random sample z_1, \dots, z_n of i.i.d $w = f(z; \boldsymbol{\theta})$.
2. Let $t_1 \leq t_2 \leq \dots \leq t_n$ be the crescent ordering of $z_{(1)} \leq z_{(2)} \leq \dots \leq z_{(n)}$.
3. Let (a, b) be the empirical or theoretical sample domain of the data, which is established by an expert of the experiment. This can be relaxed enough until the experiment has sense.
4. Let k be the precision of the data, which is established by the "instrument" which collects the sample.
5. For numerical performance comparisons with random sample routines, set also (a, b) as their domain of random generation numbers. Set also at k , the precision of the random numbers generated by classical algorithms.
6. With k we can find the number of possible outcomes x 's in a, b . Let m be such number. So the possible empirical outcomes can be listed. Let P be that strict ordered set, say $P = (r_1 = a < r_2 < r_3 < \dots < r_{m-1} < r_m = b)$.
7. Now, the precision k divides the order set $t_1 \leq t_2 \leq \dots \leq t_n$ in q classes c_1, \dots, c_q with equal elements and sizes s_1, \dots, s_q , say: $x_{c_1,1} = x_{c_1,2} = \dots = x_{c_1,s_1}$, $x_{c_2,1} = x_{c_2,2} = \dots = x_{c_2,s_2}$, ..., $x_{c_q,1} = x_{c_q,2} = \dots = x_{c_q,s_q}$; where $x_{c_1,s_1} < x_{c_2,s_2} < \dots < x_{c_q,s_q}$. Let E be such set $x_{c_1,s_1} < x_{c_2,s_2} < \dots < x_{c_q,s_q}$.

Then in the above setting, and assuming that the random generating routines are fair to the distribution that they are modeling, we conjecture that:

1. For large n , and given (a, b) and k , the numbers $x_{c_1,s_1} < x_{c_2,s_2} < \dots < x_{c_q,s_q}$ are deterministic.
2. They switch or transfers the original randomness of $\mathbf{z} = (z_1, \dots, z_n)$ into the q frequencies s_1, s_2, \dots, s_q which are now the random sample.
3. Finally, $s_i = c(\hat{\boldsymbol{\theta}})f(x_{c_i,s_i}; \hat{\boldsymbol{\theta}})$, where $\hat{\boldsymbol{\theta}}$ are the least squares estimators.

Next we apply the method for a number of distributions. We start with the Gaussian case, which the likelihood estimators are known, they are the sample mean and variance.

Given that there are several distributions with two parameters similar to the normal case, so, we will give next the exact LSE of the multiplicative error model, for applications. Start with $w_i = f(z_i; \boldsymbol{\theta})\epsilon_i$, $i = 1, \dots, n$, where $\boldsymbol{\theta} = (\theta_1, \theta_2)$. Let $y_i = \ln f(z_i; \boldsymbol{\theta})$, $\epsilon_i = \ln \epsilon_i$ and assume that $y_i = \beta_0 + \beta_1(\boldsymbol{\theta})x_{1i}(\mathbf{z}) + \beta_2(\boldsymbol{\theta})x_{2i}(\mathbf{z}) + \epsilon_i$ satisfies de Gauss-Markov theorem. Also accept that $\beta_i(\boldsymbol{\theta})$'s provides a consistent system of equations for a unique solution $(\boldsymbol{\theta})$. The LSE of $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ are $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, which requires the full inversion of the 3×3 matrix $\mathbf{X}'\mathbf{X}$. We simplify it with a two block diagonal matrix as follows: write $y_i = \gamma_0 + \beta_1(x_{1i} - \bar{x}_1) + \beta_2(x_{2i} - \bar{x}_2) + \epsilon_i$, where $\gamma_0 = \beta_0 + \beta_1\bar{x}_1 + \beta_2\bar{x}_2$ then the LSE are: $\hat{\boldsymbol{\beta}}_1 = D^{-1} [\sum_{i=1}^n (x_{2i} - \bar{x}_2)^2 \sum_{i=1}^n (x_{1i} - \bar{x}_1)y_i - \sum_{i=1}^n (x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2) \sum_{i=1}^n (x_{2i} - \bar{x}_2)y_i]$, and $\hat{\beta}_2 = D^{-1} [\sum_{i=1}^n (x_{1i} - \bar{x}_1)^2 \sum_{i=1}^n (x_{2i} - \bar{x}_2)y_i - \sum_{i=1}^n (x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2) \sum_{i=1}^n (x_{1i} - \bar{x}_1)y_i]$, where $D = \sum_{i=1}^n (x_{1i} - \bar{x}_1)^2 \sum_{i=1}^n (x_{2i} - \bar{x}_2)^2 - (\sum_{i=1}^n (x_{1i} - \bar{x}_1)(x_{2i} - \bar{x}_2))^2$. Then θ_1, θ_2 can be obtained. The remaining LSE $\hat{\gamma}_0 = \bar{y}$ is understood as $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}_1 + \hat{\beta}_2\bar{x}_2$ and can be used for some real stochastic representation. In the Gaussian case, for example, it gives a rare representation for π .

5.6.1 Gaussian fit

The i.d.d multiplicative error models are for the random sample are $w_i = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2}\left(\frac{z_i-\mu}{\sigma}\right)^2} \epsilon_i$, $i = 1, \dots, n$, then they are transformed into $\ln w_i = \left(-\frac{1}{2} \ln(2\pi\sigma^2) - \frac{\mu}{2\sigma^2}\right) - \frac{1}{2\sigma^2} z_i^2 + \frac{\mu}{\sigma^2} z_i + \ln \epsilon_i$, where $\ln \epsilon_i = \varepsilon_i$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)' \sim N(\mathbf{0}, \tau^2 \mathbf{I}_n)$. Taking $y_i = \ln w_i$, $x_{1i} = z_i^2$, $x_{2i} = z_i$, $\beta_0 = -\frac{1}{2} \ln(2\pi\sigma^2) - \frac{\mu}{2\sigma^2}$, $\beta_1 = -\frac{1}{2\sigma^2}$ and $\beta_2 = \frac{\mu}{\sigma^2}$, we have $\hat{\sigma}^2 = -\frac{1}{2\hat{\beta}_1}$ and $\hat{\mu} = -\frac{\hat{\beta}_2}{2\hat{\beta}_1}$.

In this case the MLE of μ and σ^2 have an exact form, they are \bar{x} and $\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$, however, most of the classical univariate distributions do not give exact expressions and their MLE needs to be computed by numerical methods.

We prove empirically that the LSE's behave in a similar way to the MLE's. Simulations based on $\mu = 300, \sigma^2 = 1$ with 10000 replicates for $n = 30, 300, 3000, 10000$ are presented in figures 5.9 and 5.10.

From $\hat{\beta}_0$ we have the following stochastic representation: $\pi \approx \frac{1}{2\hat{\sigma}^2} e^{\frac{1}{2}(2\hat{\mu}\bar{x}_2 - \bar{x}_1 - \hat{\mu}^2 - 2\hat{\sigma}^2\bar{y})}$. The following Table provides some simulation results for $\mu = 2, \sigma^2 = 2$. Finally, about the illustration of the conjecture, the last column of Table show the slope (correlation coefficient) of the linear model define by the sets E versus P for an empirical verification of the numbers $x_{c_1, s_1} < x_{c_2, s_2} < \dots < x_{c_q, s_q}$ to be deterministic.

Table 5.1 – Pi approximation by LSE and switch random fit comparison

Aproximation	n	Correlation
3.141592653589793115998	exact	
3.141592653589799333247	3	0.98974
3.141592653589798445068	4	0.94735
3.141592653589767358824	5	0.99312
3.141592653589786898749	6	0.97786
3.141592653589797112801	7	0.98349
3.141592653589795780533	8	0.94721
3.141592653589782013768	9	0.96221
3.141592653589793115998	10	0.96791
3.141592653589821093618	11	0.98956
3.141592653589792671909	20	0.98627
3.141592653589801553693	40	0.98264
3.141592653589796224622	80	0.98766
3.141592653589781569679	1000	9.9852
3.141592653589798445068	10000	0.99972
3.141592653589790451463	100000	0.99973
3.141592653589802885961	1000000	9.9997
3.141592653589787342838	2000000	0.99980
3.141592653589795780533	3000000	0.99998
3.141592653589799333247	4000000	0.99999
3.141592653589793115998	exact	

5.6.2 Beta and Wishart Pearson VII type model

The same procedure can be implemented with several distributions. All the trials went into a right conjecture on the switch random fit.

Common methods of estimation of the parameters of the Beta distribution are maximum likelihood and method of moments. The maximum likelihood equations for the Beta distribution

have no closed-form solution; estimates may be found through the use of the iterative method, see for instance Wang [2005] and reference therein. The method of moments estimators have a closed-form solution. We examine both of these estimators here.

5.6.2.1 Maximum likelihood estimators

Assume that an iid sample x_1, x_2, \dots, x_n of size n has been collected for a random variable X which follows Beta distribution.

$$B(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x_i^{a-1} (1-x_i)^{b-1}$$

The logarithm of the likelihood function is given by

$$\log L(a, b, x) = n \log \Gamma(a+b) - n \log \Gamma(a) - n \log \Gamma(b) + (a-1) \sum_{i=1}^n \log(x_i) + (b-1) \sum_{i=1}^n \log(1-x_i).$$

By differentiating with respect to a, b and equating to zero, the likelihood equations can be obtained.

$$\begin{aligned} \frac{\partial \log L}{\partial a} &= n \frac{\Gamma'(a+b)}{\Gamma(a+b)} - n \frac{\Gamma'(a)}{\Gamma(a)} + \sum_{i=1}^n \log(x_i) = 0 \\ \frac{\partial \log L}{\partial b} &= n \frac{\Gamma'(a+b)}{\Gamma(a+b)} - n \frac{\Gamma'(b)}{\Gamma(b)} + \sum_{i=1}^n \log(1-x_i) = 0 \end{aligned}$$

Using the Newton-Raphson, as described below.

$$\hat{\theta}_{i+1} = \hat{\theta}_i - G^{-1}g, \tag{5.6.1}$$

where g is the vector of normal equations for wich

$$g = [g_1, g_2]$$

whit

$$\begin{aligned} g_1 &= n\psi(a+b) - n\psi(a) + \sum_{i=1}^n \log(x_i) \\ g_2 &= n\psi(a+b) - n\psi(b) + \sum_{i=1}^n \log(1-x_i) \\ G &= \begin{bmatrix} \frac{\partial g_1}{\partial a} & \frac{\partial g_1}{\partial b} \\ \frac{\partial g_2}{\partial a} & \frac{\partial g_2}{\partial b} \end{bmatrix} \end{aligned}$$

$\psi(u)$ and $\psi'(u)$ are the di- and tri-gamma functions defined as $\psi(u) = \frac{\Gamma'(u)}{\Gamma(u)}$ and $\psi'(u) = \frac{\Gamma''(u)}{\Gamma(u)} - \frac{\Gamma'(u)^2}{\Gamma(u)^2}$.

5.6.2.2 Method of moments estimators

The method of moments estimators of the Beta-Kotz distribution parameters involve equating of the four moments of the Beta-Kotz distribution with the sample mean, variance, skewness and kurtosis.

The moment generating function for a moment of order t is

$$\begin{aligned} E(x^t) &= \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} dx \\ &= \frac{\Gamma(a+t)\Gamma(a+b)}{\Gamma((a+b+t)\Gamma(a)} \end{aligned}$$

The method of moments estimators are found by setting the sample mean, \bar{X} , the variance, S^2 , skewness α_3 and the kurtosis α_4 the equal to the population mean, variance, skewness and kurtosis,

$$\bar{X} = \frac{a}{a+b}$$

$$S^2 = \frac{ab}{(a+b)^2(a+b+1)}$$

Then

$$\hat{a} = \bar{X} \left(\frac{\bar{X}(1-\bar{X})}{S^2} - 1 \right) \quad (5.6.2)$$

$$\hat{b} = (1-\bar{X}) \left(\frac{\bar{X}(1-\bar{X})}{S^2} - 1 \right) \quad (5.6.3)$$

5.6.3 Regression fit

In the case of the Beta distribution, the MLE's have not exact expressions and require a solution of a non linear system of equations. The LSE are simple to compute and the experiments gave best results for them than the MLE's. A similar conclusion arrive in the context of the Wishart Pearson VII type distribution. The procedure for the Beta distribution is detailed below.

The i.d.d multiplicative error models are for the random sample are

$$w_i = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} z_i^{a-1} (1-z_i)^{b-1} \epsilon_i,$$

for $i = 1, \dots, n$, then they are transformed into $\ln w_i = \ln \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} + a \ln(z_i) + b \ln(1-z_i) + \ln \epsilon_i$, where $\ln \epsilon_i = \varepsilon_i$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)' \sim N(\mathbf{0}, \tau^2 \mathbf{I}_n)$.

Taking $y_i = \ln w_i$, $x_{1i} = \ln(z_i)$, $x_{2i} = \ln(1-z_i)$, $\beta_1 = a-1$, $\beta_2 = b-1$, and $\beta_0 = \ln \left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \right)$, then by Gauss-Markov theorem the least square estimations of the lineal model

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$$

is

$$\hat{a} = \hat{\beta}_1 + 1 \quad (5.6.4)$$

$$\hat{b} = \hat{\beta}_2 + 1 \quad (5.6.5)$$

Table 5.2 – $\frac{1}{\pi}$ approximation by LSE and switch random fit comparison

Aproximation	n	Correlation
0.31830988618379069	exact	
0.31830988618379047	5	0.9717127
0.31830988618379075	9	0.9766007
0.31830988618379058	10	0.9874834
0.31830988618379075	20	0.9845866
0.31830988618379069	40	0.9885051
0.31830988618379058	80	0.9932744
0.31830988618379052	100	0.9953655
0.31830988618379014	1000	0.9999051
0.31830988618379003	10000	0.9999083
0.31830988618379386	100000	0.999034
0.31830988618380268	1000000	0.99945
0.3183098861837878	2000000	0.99991
0.31830988618378758	3000000	0.99997
0.31830988618377259	4000000	0.99999
0.31830988618379069	exact	

We prove empirically that the LSE's behave in a similar way to the MLE's. Simulations based on $a = 17; b = 12$ with 10000 replicates for $n = 30; 300; 3000; 10000$ are presented in figures 3 and 4.

From $\hat{\beta}_0$ we have the following stochastic representation $\frac{1}{\pi} = e^{(\bar{y} - \hat{\beta}_1 \bar{x}_1 - \hat{\beta}_2 \bar{x}_2)}$. The next bable provides some simulation results for $a = 0.5, b = 0.5$.

Similarly, the procedure for the Wishart Pearson VII is detailed below.

The i.d.d multiplicative error models are for the random sample are

$$w_i = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} z_i^{\frac{n}{2}-1} \frac{\Gamma(s)}{(\pi R)^{\frac{n}{2}} \Gamma(s - \frac{n}{2})} \left(1 + \frac{z_i}{R}\right)^{-s} \epsilon_i$$

for $i = 1, \dots, n$,

If $R = 1$ then

$$w_i = \frac{\Gamma(s)}{\Gamma(s - \frac{n}{2}) \Gamma(\frac{n}{2})} z_i^{\frac{n}{2}-1} (1 + z_i)^{-s} \epsilon_i$$

then they are transformed into $\ln w_i = \ln \left(\frac{\Gamma(s)}{\Gamma(s - \frac{n}{2}) \Gamma(\frac{n}{2})} \right) + (\frac{n}{2} - 1) \ln(z_i) - s \ln(1 + z_i) + \ln \epsilon_i$, where $\ln \epsilon_i = \epsilon_i$ and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)' \sim N(\mathbf{0}, \tau^2 \mathbf{I}_n)$.

Taking $y_i = \ln w_i$, $x_{1i} = \ln(z_i)$, $x_{2i} = \ln(1 + z_i)$, $\beta_1 = \frac{n}{2} - 1$, $\beta_2 = -s$, and $\beta_0 = \ln \left(\frac{\Gamma(s)}{\Gamma(s - \frac{n}{2}) \Gamma(\frac{n}{2})} \right)$, then by Gauss-Markov theorem the least square estimations of the linear model

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$$

is

$$\hat{n} = 2(\hat{\beta}_1 + 1) \tag{5.6.6}$$

$$\hat{s} = -\hat{\beta}_2 \tag{5.6.7}$$

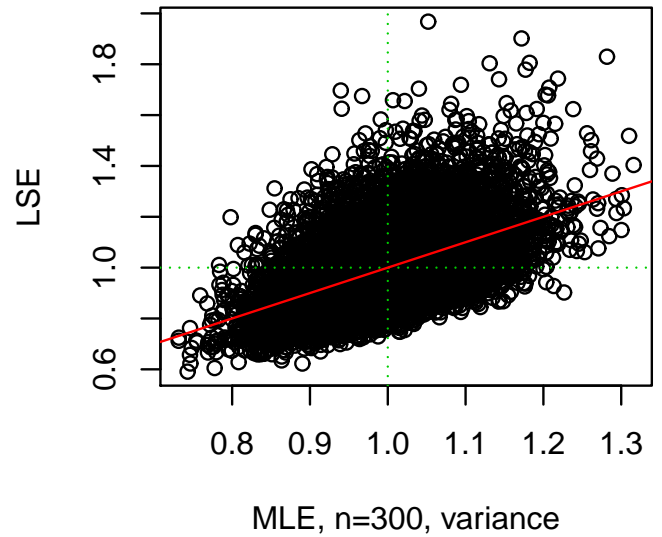
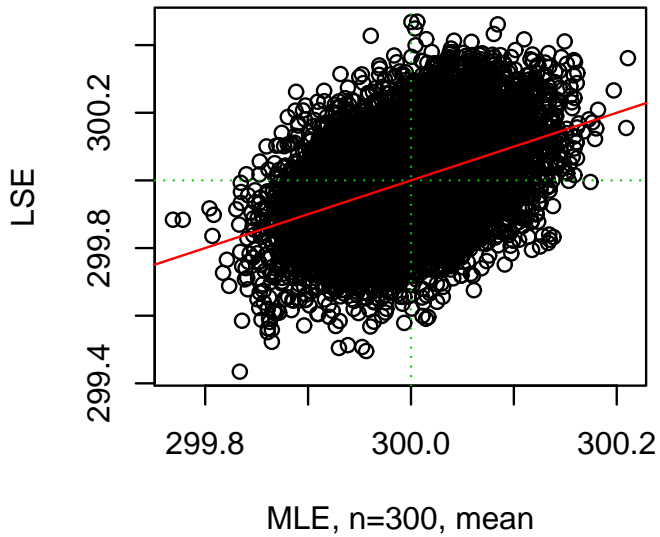
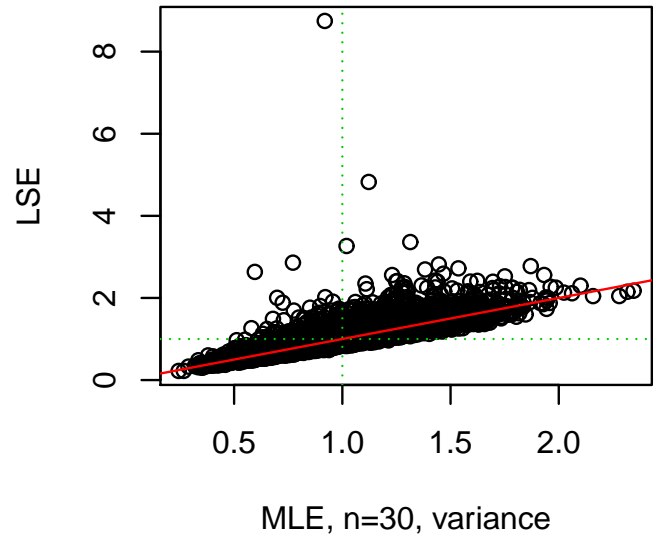
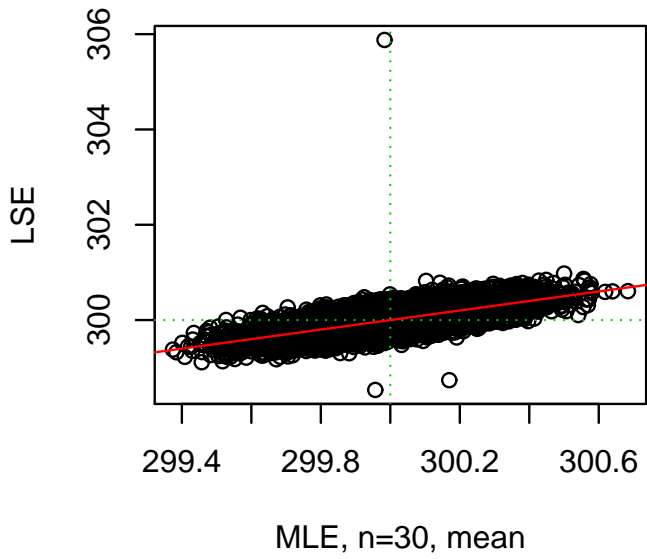


Figure 5.9 – A simulation of LSE's and MLE's for an i.i.d Gaussian

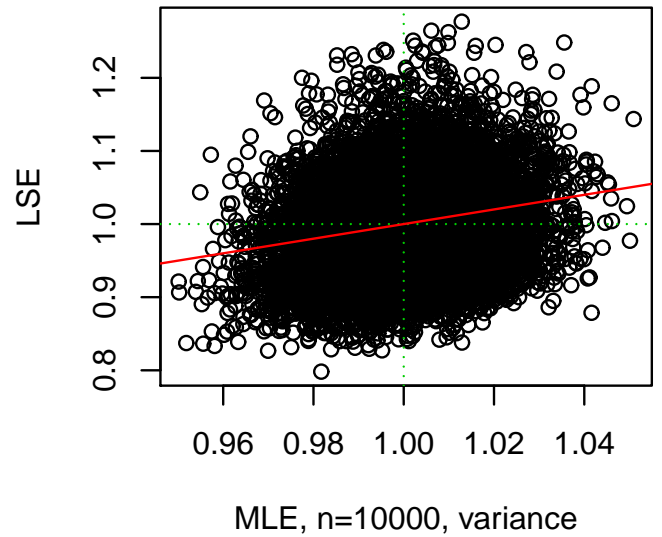
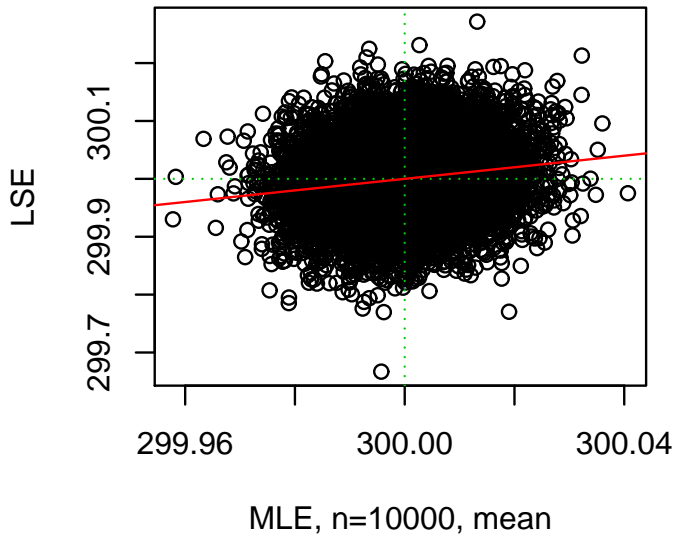
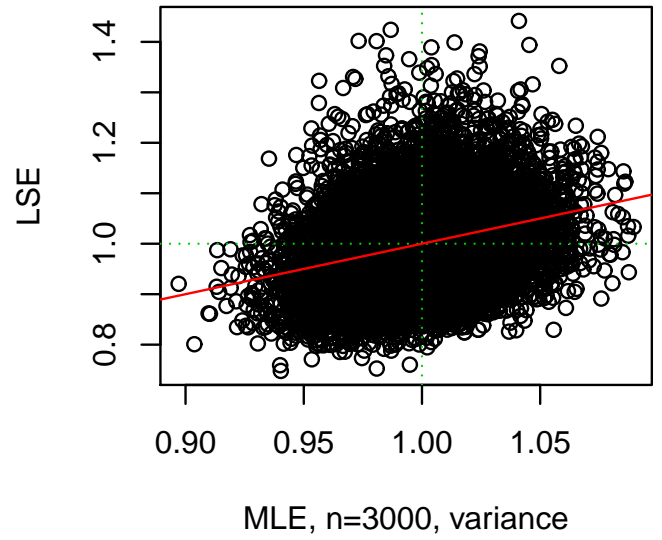
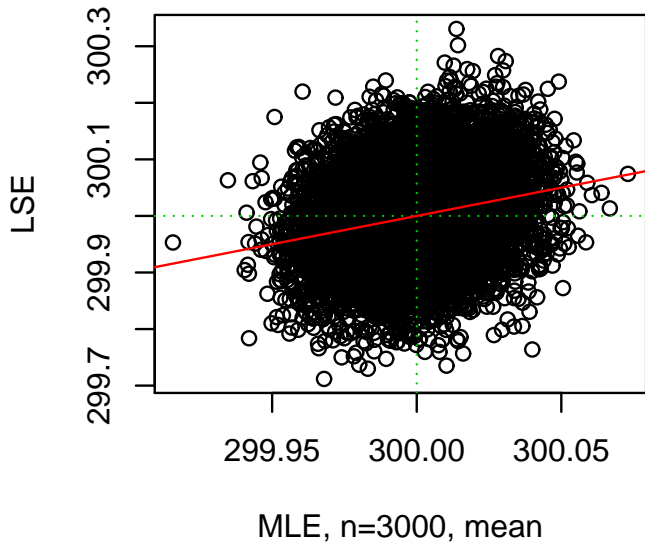
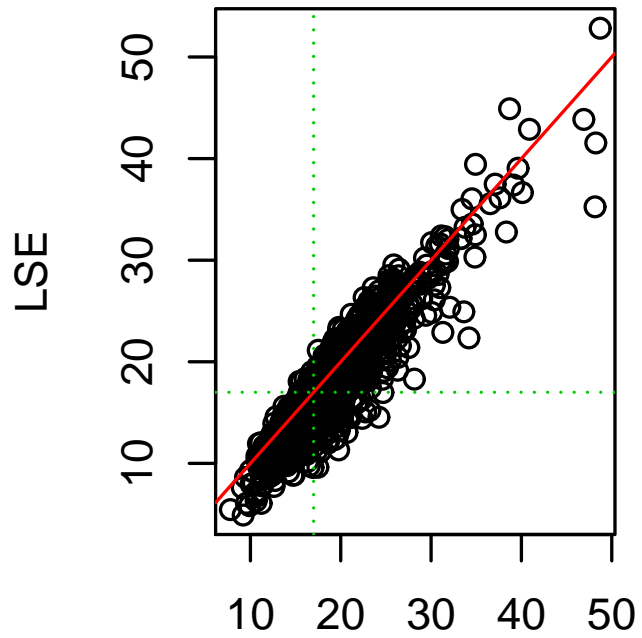
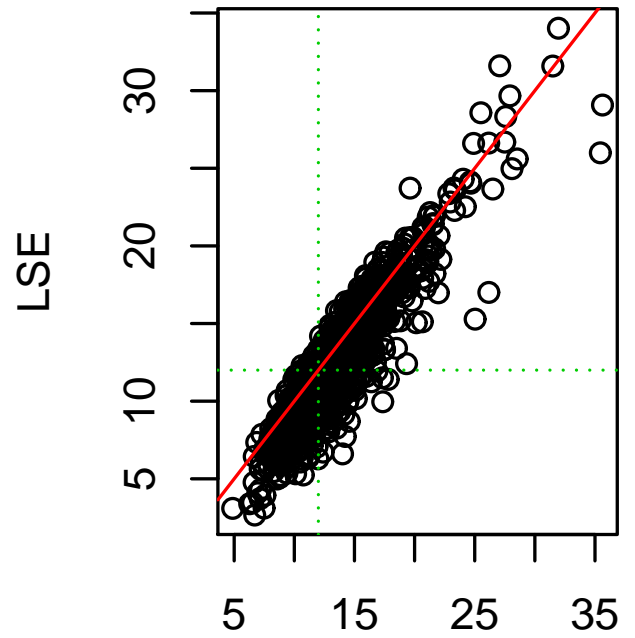


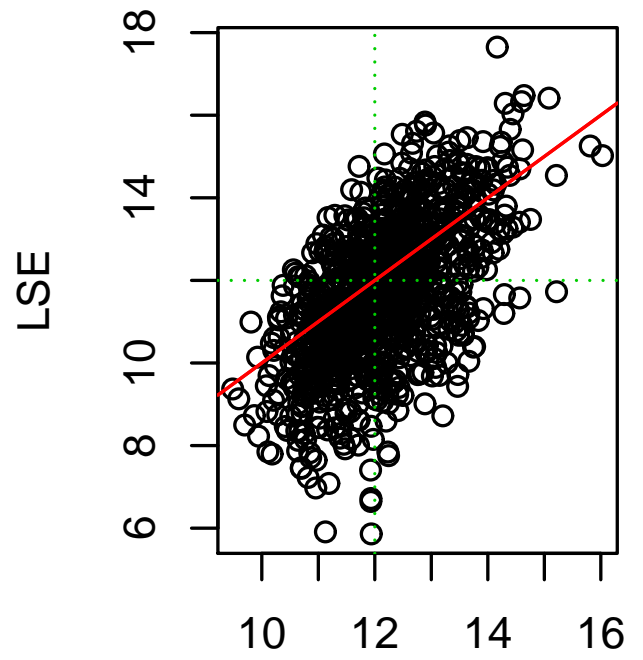
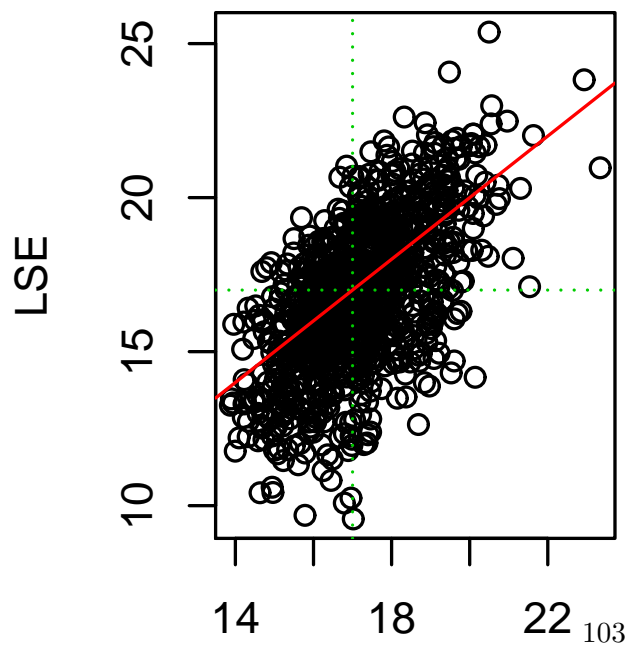
Figure 5.10 – A simulation of LSE's and MLE's for an i.i.d Gaussian

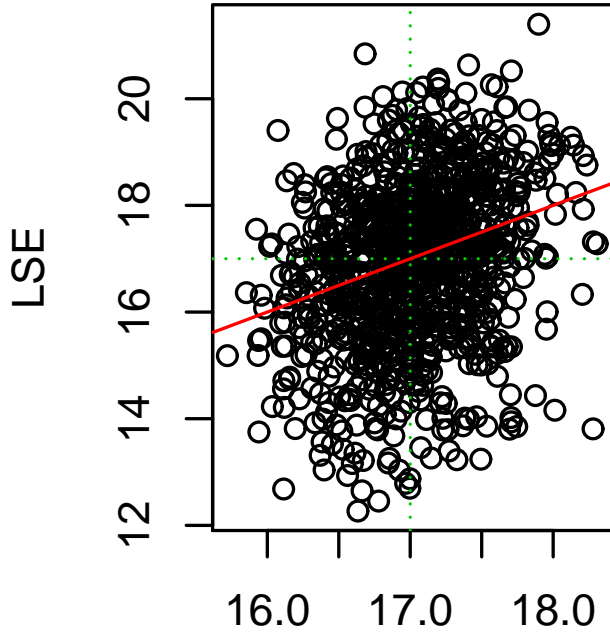


MLE, $n=30$, a

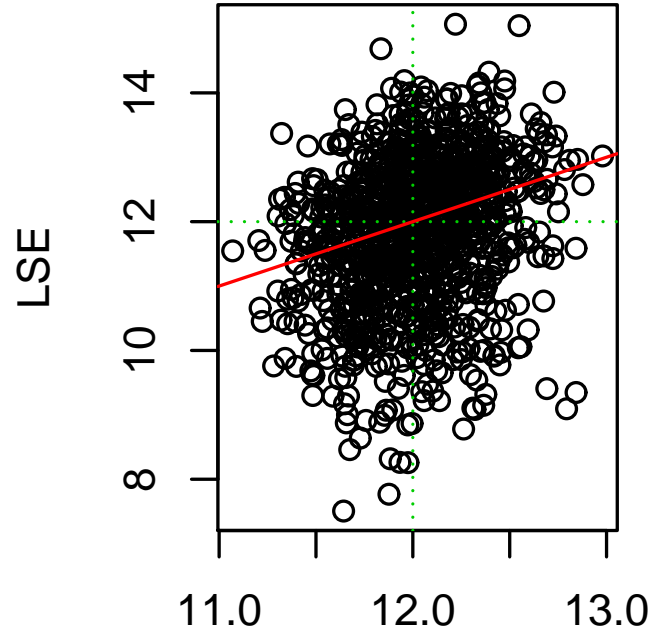


MLE, $n=30$, b

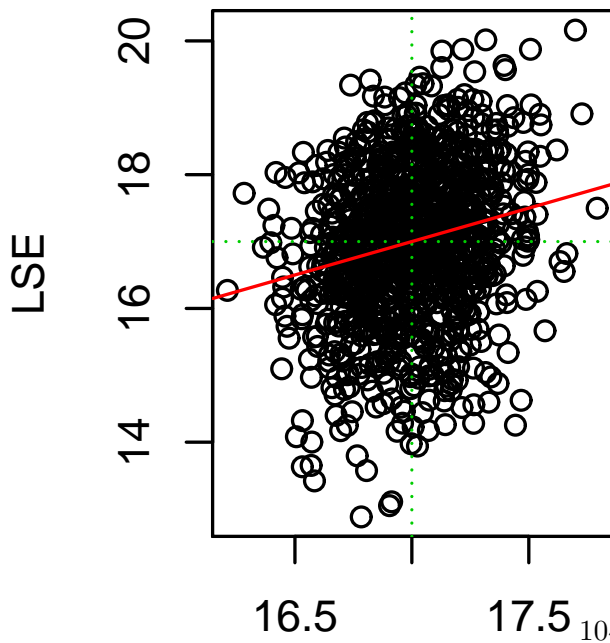




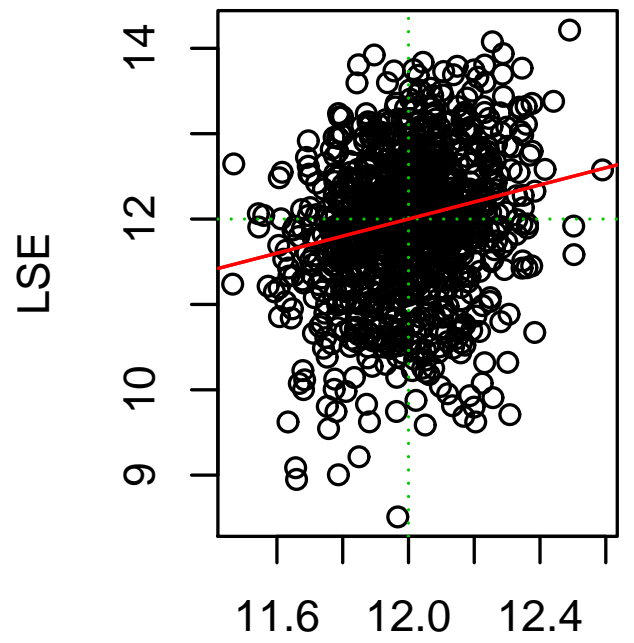
MLE, $n=3000$, a



MLE, $n=3000$, b



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