# PROPOSED CONTROL CHARTS FOR MONITORING CUMULATIVE COUNTS OF CONFORMING ITEMS AND RATIO OF TWO VARIABLES

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**UNIVERSITI SAINS MALAYSIA**

**2021**

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**by**

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**Thesis submitted in fulfilment of the requirements
for the degree of
Doctor of Philosophy**

**September 2021**

## ACKNOWLEDGEMENT

I am greatly indebted to numerous individuals who have assisted me in one way or the other in completing this Ph.D. thesis. First and foremost, I would like to express my heartfelt gratitude to my humble supervisor Prof. Dr. Michael Khoo Boon Chong who has relentlessly guided and encouraged me throughout this journey. I cherish and adore his experience and expertise in Statistical Quality Control (SQC). His wisdom is an epitome of his work and contributions to the field of SQC. He has allotted his precious time to correct and restructure the English language and correct the inconsistencies in my thesis. I thank my supervisor from the bottom of my heart for everything he has done for me.

Next, my deepest appreciation goes to the School of Mathematical Sciences, Universiti Sains Malaysia (USM), for granting me the opportunity to pursue my Ph.D. study. The staff and lecturers in the department have been generous, hospitable and helpful whenever I needed assistance. I will also not forget the Tertiary Education Trust Funds (TETFUND) for sponsoring my Ph.D. study via the Kano University of Science and Technology (KUST), Wudil, Nigeria. I would like to express my profound gratitude to KUST for making my Ph.D. dream a reality. Thirdly, my love and thanks go to my parents, siblings, beloved wife and children for being so caring, supportive and encouraging all the time. I would also like to thank my circle of friends and colleagues for their moral support and advice. Finally, I am no one without the almighty Allah and His infinite favours and blessings. Thank you.

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| | |
|---|---|
| ARL | Average run length |
| $ARL_0$ | In-control ARL |
| $ARL_1$ | Out-of-control ARL |
| ASI | Average sampling interval |
| ATS | Average time to signal |
| $ATS_0$ | In-control ATS |
| $ATS_1$ | Out-of-control ATS |
| CCC | Cumulative count of conforming |
| cdf | Cumulative distribution function |
| $CL$ | Center line |
| CRL | Conforming run length |
| CUSUM | Cumulative sum |
| EARL | Expected average run length |
| $EARL_0$ | In-control EARL |
| $EARL_1$ | Out-of-control EARL |
| EWMA | Exponentially weighted moving average |
| FIR | Fast initial response |
| FSI | Fixed sampling interval |
| icdf | Inverse cdf |
| IF | Improvement factor |
| $LCL$ | Lower control limit |
| $LWL$ | Lower warning limit |
| pdf | Probability density function |
| pmf | Probability mass function |
| ppm | parts-per-million |
| RS | Run sum |
| RZ | Ratio of $Z = X/Y$ |
| SAS | Statistical Analysis Software |
| SH | Shewhart |
| SPC | Statistical Process Control |
| SS | Steady state |
| SYN | synthetic |
| tpm | Transition probability matrix |

| | |
|---|---|
| $UCL$ | Upper control limit |
| $UWL$ | Upper warning limit |
| VSI | Variable sampling interval |
| VSS | Variable sample size |
| $WL$ | Warning limit |
| ZS | Zero state |

## LIST OF NOTATIONS

| | |
|---|---|
| $p_0$ | In-control fraction nonconforming |
| $p_1$ | Out-of-control fraction nonconforming |
| $\boldsymbol{\mu}$ | Mean vector |
| $\mu_0$ | In-control population mean |
| $\mu_1$ | Out-of-control population mean |
| $\sigma$ | Population standard deviation |
| $\boldsymbol{\Sigma}$ | Covariance matrix |
| $n$ | Sample size |
| $\lambda$ | Smoothing constant |
| $t_L$ | Long sampling interval |
| $t_S$ | Short sampling interval |
| $\delta$ | Shift size in the fraction of nonconforming items |
| $\boldsymbol{q}$ | Initial probability vector |
| $\boldsymbol{I}$ | Identity matrix |
| $\boldsymbol{Q}$ | Transition probability matrix |
| $\boldsymbol{t}$ | Vector of sampling intervals |
| $\mathbf{1}$ | Vector of ones |
| $Z_0$ | In-control ratio |
| $Z$ | Ratio of two normal random variables ($Z = X/Y$) |
| $\hat{Z}$ | Ratio of two sample means ($\hat{Z} = \bar{X}/\bar{Y}$) |
| $\rho_0$ | In-control correlation coefficient |
| $\rho_1$ | Out-of-control correlation coefficient |
| $\gamma$ | Coefficient of variation |
| $\omega$ | Ratio of two standard deviations |
| $F_Z$ | cdf of $Z$ |
| $f_Z$ | pdf of $Z$ |

| | |
|---|---|
| $F_Z^{-1}$ | Inverse cdf of $Z$ |
| $\Phi(\cdot)$ | cdf of the standard normal random variable |
| $\phi(\cdot)$ | pdf of the standard normal random variable |
| $\beta$ | Probability of a Type-II error |
| $\tau$ | Shift size in the ratio of two normal random variables |
| $\tau_{min}$ | Lower bound of $\tau$ |
| $\tau_{max}$ | Upper bound of $\tau$ |
| $S_a$ | Score for the $a^{\text{th}}$ region of the RS- $\bar{X}$ chart |
| $S_t$ | Score for the $t^{\text{th}}$ region of the RS-RZ chart |
| $R_N$ | $N^{\text{th}}$ region above the center line of the RS- $\bar{X}$ chart |
| $R_k$ | $k^{\text{th}}$ region above the center line of the RS-RZ chart |
| $R_{Nb}$ | $N^{\text{th}}$ region below the center line of the RS- $\bar{X}$ chart |
| $R_{-k}$ | $k^{\text{th}}$ region below the center line of the RS-RZ chart |
| $p_N$ | Probability of $\bar{X}$ plotting in $R_N$ of the RS- $\bar{X}$ chart |
| $p_k$ | Probability of $\hat{Z}_i$ plotting in $R_k$ of the RS-RZ chart |
| $p_{Nb}$ | Probability of $\bar{X}$ plotting in $R_{Nb}$ of the RS- $\bar{X}$ chart |
| $p_{-k}$ | Probability of $\hat{Z}_i$ plotting in $R_{-k}$ of the RS-RZ chart |
| $\propto$ | Probability of a false alarm |
| $q_j$ | Probability that $X_i$ falls in region $I_j$ of the VSI CCC chart when the nonconforming rate is $p_0$ |
| $q'_j$ | Probability that $X_i$ falls in region $I_j$ of the VSI CCC chart when the nonconforming rate is $p_1$ |

## LIST OF APPENDICES

# CADANGAN CARTA-CARTA KAWALAN UNTUK PEMANTAUAN KIRAAN LONGGOKAN PEMATUHAN BARANG DAN NISBAH DUA PEMBOLEHUBAH

## ABSTRAK

Penyelidikan ini memberikan terbitan formula analitik untuk mengira nilai purata masa untuk memberi isyarat (ATS) bagi carta purata bergerak berpemberat eksponen (EWMA) kiraan longgokan pematuhan (CCC) dengan menggunakan teknik rantai Markov. Dalam literatur sedia ada, carta ini telah dinilai dengan menggunakan simulasi. Selain itu, carta EWMA CCC selang pensampelan boleh berubah (VSI) juga dicadangkan untuk meningkatkan kepekaan carta asas EWMA CCC dalam pengesanan anjakan proses. Reka bentuk optimum carta VSI EWMA CCC diperoleh berdasarkan prosedur rantai Markov dengan meminimumkan jangkaan masa kelewatan dalam pengesanan anjakan proses. Parameter optimum yang meminimumkan kriteria masa untuk memberi isyarat (ATS) diberikan dan boleh digunakan secara langsung dalam praktik. Empat carta, iaitu carta CCC asas, VSI CCC, EWMA CCC dan VSI EWMA CCC dipertimbangkan dalam perbandingan berangka dangan menggunakan kriteria ATS. Carta VSI EWMA CCC mempunyai prestasi yang memberangsangkan berbanding dengan carta CCC asas, VSI CCC dan EWMA CCC. Contoh ilustrasi melalui data sebenar dari suatu proses pengacuan suntikan yang menghasilkan susunan mikro-prisma elemen optik diberikan untuk menunjukkan pelaksanaan carta VSI EWMA CCC dalam praktik. Tambahan pula, kajian ini mencadangkan carta nisbah hasil tambah larian dua sisi untuk memantau nisbah dua pembolehubah rawak normal. Prosedur rantai Markov digunakan untuk menilai prestasi berstatistik carta berdasarkan kriteria panjang larian purata (ARL) dan

jangkaan panjang larian purata (EARL). Perbandingan berangka dengan carta nisbah Shewhart dan nisbah sintetik berdasarkan analisis keadaan sifar menunjukkan bahawa carta nisbah hasil tambah larian mempunyai kepekaan yang lebih baik dalam kebanyakan kes. Khususnya, untuk pekali variasi $(\gamma_X, \gamma_Y) \in \{(0.2, 0.2), (0.2, 0.01)\}$, carta nisbah hasil tambah larian mengatasi dua carta yang disebutkan di atas untuk hampir semua saiz anjakan dalam nisbah dua pembolehubah rawak. Bagi analisis keadaan mantap, keputusan menunjukkan bahawa carta nisbah hasil tambah larian mengatasi carta nisbah sintetik secara seragam. Contoh ilustrasi isu kualiti sebenar dari industri makanan diberikan untuk menunjukkan pelaksanaan carta nisbah hasil tambah larian yang dicadangkan. Program MATLAB digunakan untuk mengira parameter optimum carta-carta yang dicadangkan dan sedia ada manakala Perisian Analisis Berstatistik (SAS) digunakan untuk mengesahkan keputusan yang diperoleh daripada program MATLAB.

**PROPOSED CONTROL CHARTS FOR MONITORING CUMULATIVE COUNTS OF CONFORMING ITEMS AND RATIO OF TWO VARIABLES**

## ABSTRACT

This research provides a derivation of the analytical formulae to compute the average time to signal (ATS) value of the exponentially weighted moving average (EWMA) cumulative count of conforming (CCC) chart using the Markov chain technique. In the current literature, this chart is evaluated using simulation. Additionally, the variable sampling interval (VSI) EWMA CCC chart is also proposed to increase the sensitivity of the basic EWMA CCC chart in detecting process shifts. The optimal designs of the VSI EWMA CCC chart are obtained based on the Markov chain procedure by minimizing the expected delay time in detecting a process shift. The optimal parameters that minimize the average time to signal (ATS) criterion are provided and can be directly used in practice. Four charts, i.e. the basic CCC, VSI CCC, EWMA CCC and VSI EWMA CCC charts are considered in the numerical comparison using the ATS criterion. The VSI EWMA CCC chart has an impressive performance in comparison to the basic CCC, VSI CCC and EWMA CCC charts. An illustrative example via real data from an injection moulding process producing an array of micro-prism of an optical element is given to demonstrate the implementation of the VSI EWMA CCC chart in practice. In addition, this research proposes a two-sided run sum ratio chart to monitor the ratio of two normal variables. A Markov chain procedure is applied to evaluate the statistical performance of the chart based on the average run length (ARL) and expected average run length (EARL) criteria. Numerical comparisons with the Shewhart ratio and synthetic ratio charts based on the zero state analysis reveal that the run sum ratio chart has a better sensitivity in most cases. In

particular, for the coefficients of variation $(\gamma_X, \gamma_Y) \in$ {(0.2, 0.2), (0.2, 0.01)}, the run sum ratio chart outperforms the two aforementioned charts for almost all shift sizes in the ratio of the two variables. As for the steady state analysis, the results show that the run sum ratio chart outperforms the synthetic ratio chart almost uniformly. An illustrative example of a real quality issue from the food industry is presented to demonstrate the implementation of the proposed run sum ratio chart. The MATLAB program is employed in computing the optimal parameters of the proposed and existing charts, while the Statistical Analysis Software (SAS) is used in verifying the results obtained from the MATLAB program.

# CHAPTER 1

# INTRODUCTION

## 1.1 Statistical Process Control (SPC)

Control, as well as improvement of quality, has become an essential part of business strategy for a good number of organisations, such as manufacturing, transportation, financial outfits and so on and so forth. Quality can be defined in several ways. Majority of people have a conceptual comprehension of quality as pertaining to one or more desirable characteristics which a product/service must possess. Quality has remained one of the most crucial consumer decision factors in selecting from many competing products/services. Hence, understanding and improving the quality of the product or service rendered remains a critical success factor for any business (Montgomery, 2019).

SPC which is regarded as a branch of Statistical Quality Control consists of methods for understanding, monitoring and improving the performance of the process over a period of time. Having a clear understanding of the variation pertaining to the values of a quality characteristic is extremely significant as far as SPC is concerned. The two sources of variations are "common cause" and "assignable cause". The former is referred to as natural variation that is inherent in the process where nothing can be done about it, while the latter, also known as special cause of variation comprises disturbances or shocks which can be identified and subsequently eliminated (Woodall, 2000). The main aim of SPC is to identify an assignable cause of variation within a short period of time thereby allowing corrective measures to be taken in order to avoid the production of nonconforming product/item. SPC comprises the following tools for monitoring, namely Pareto diagram, cause-and-effect diagram, check sheet, process flow diagram, scatter diagram, histograms and control chart (Besterfield, 2009).

Organisations are encouraged to use SPC in planning for their long term policies for continuous improvement of product quality and ensuring process stability.

## 1.2 Control Charts

A control chart can be defined as a plot of time sequence with "decision lines" added. The decision lines are used to determine whether or not the process is in-control (Montgomery, 2019). The general idea of a control chart was sketched in a memorandum by Walter Shewhart of Bell Labs on May 16, 1924, for an interesting treatise on the early use of control charts alongside other quality control techniques at AT & T and Bell Labs (Rakitzis et al., 2019).

The construction of a control chart is strictly based on statistical principles, which comprises a centre line ($CL$), upper control limit ($UCL$) and lower control limit ($LCL$). It can be used to ascertain if a process is in a state of statistical control through examining the past data. This step is known as retrospective data analysis or simply Phase I. Additionally, recent data can be used to establish control limits that could be applied in monitoring future data obtained from the process (Phase II). Hence, control charts signify whether statistical control is being maintained or otherwise (Ryan, 2011).

In general, there are two types of control charts, namely control charts for variables and control charts for attributes by considering data characteristics. Variables are quality characteristics that can be expressed in terms of measurement numerically, e.g., dimension, weight or volume. Variable control charts are utilised extensively, especially when dealing with a quality characteristic that is variable in nature. It is usually compulsory to monitor the mean value of a quality characteristic together with its variance. The average quality level of the process being monitored is mostly conducted with the control chart for means, popularly known as the $\bar{X}$ control chart. As for process variability, the monitoring can be done with either of these two charts, namely a control chart for the standard deviation, referred to as the *S* control chart or a control chart for the range, known as the *R* control chart (Montgomery, 2013).

Given that a large number of quality characteristics cannot be represented numerically with ease, sometimes the inspected items are categorised as being conforming or nonconforming and defective or non-defective. Quality characteristics of this nature are regarded as attributes. When dealing with the fraction of nonconforming or defective products obtained from a manufacturing process, a control chart for fraction nonconforming, referred to as the *p*-chart is usually adopted. In some scenarios, it is more appropriate to consider the number of defects or nonconforming items, instead of the fraction of nonconforming items, where in such situations, the *np*-charts are employed. Accordingly, for situations where the interest is on monitoring the average number of nonconformities per unit, we resort to using a control chart for nonconformities per unit, called the *u*-chart (Montgomery, 2019).

With regards to traditional control charts, samples of a fixed size are taken from a process via a fixed sampling interval. However, for both the variable sample size (VSS) and variable sampling interval (VSI) control charts, the sampling rates from the process are varied depending on the position of the control charting statistic plotted on the said charts. If there is an indication of a change in the process, then sampling at a higher rate is required. Generally, adaptive control charts possess the ability of detecting process changes quicker than traditional control charts (Arnold and Reynolds, 2001).

## 1.3 Problem Statement

Since the emergence of the CCC chart, a number of literature on this topic has been suggested with various types of CCC charts being proposed. However, there is no literature on the EWMA CCC chart based on the geometric distribution using the Markov chain procedure. Similarly, the VSI EWMA CCC chart has not been accounted for in the literature of CCC charts.

Furthermore, recently, studies have shown the importance of the role of ratio charts in manufacturing industries and non-manufacturing sectors. Consequently, a good number of ratio charts have been suggested in this regard. However, as far as the run sum control charting procedure is concerned, the literature on ratio charts has yet to witness it.

## 1.4 Research Motivation

Yeh et al. (2008) proposed an exponentially weighted moving average cumulative count of conforming (EWMA CCC) chart to monitor high yield processes, where the performance of the chart was evaluated using simulation. In an attempt to provide the analytical formula for evaluating the performance of the aforementioned chart, Mavroudis and Nicolas (2013) transformed the underlying geometric distribution to its analogous exponential distribution prior to computing the chart's statistic. Motivated by the outstanding performance of the EWMA CCC chart as reported in Yeh et al. (2008), in this thesis, analytical formulae to compute the average time to signal (ATS) criterion of the EWMA CCC chart without having to transform the underlying geometric distribution so that the original form of the EWMA CCC chart's statistic in Yeh et al. (2008) can be retained is suggested. Moreover, in order to boost the sensitivity of the EWMA CCC chart, the VSI EWMA CCC chart is proposed in this study, considering the fact that charts with adaptive features are quicker in detecting shifts compared to their counterparts with fixed sampling intervals.

On the other hand, there are manufacturing environments in which quality practitioners who implement multivariate on-line process monitoring would be interested in the monitoring of the ratio $Z$ of two random variables, say $X$ and $Y$, and not in the monitoring of the process mean vector, $\boldsymbol{\mu}$, and/or the covariance matrix, $\boldsymbol{\Sigma}$, of the bivariate vector $(X, Y)^T$. As a consequence, Celano et al. (2014) proposed the Shewhart-RZ chart for monitoring the ratio of two normal variables. Celano and Castagliola (2016a) extended the aforementioned work by considering a subgroup of $n > 1$ sample units, where each of the sample units is free to change in size from one sample to another. It was shown by Celano and Castagliola (2016b) that the Phase-II synthetic chart for monitoring the ratio of two normal variables exhibited better statistical sensitivity than the Shewhart-RZ chart, in terms of the ARL and EARL criteria.

Since both the Shewhart-RZ and synthetic-RZ are two-sided charts, the research in this thesis proposes a two-sided run sum-RZ chart. The reason for considering a two-sided chart is to simply develop a more sensitive chart than the existing ones in the literature for detecting shifts in the ratio of two random variables of a bivariate normal distribution.

## 1.5 Research Objectives

The main objectives of this thesis are as follows:

(i) To derive analytical formulae and develop optimization algorithm for the EWMA CCC chart without using any transformation technique.

(ii) To develop the VSI EWMA CCC chart for high yield processes.

(iii) To propose the run sum ratio chart for monitoring the ratio of the population means of two variables from a bivariate normal distribution.

## 1.6 Scope and Limitations

The scope of this thesis is to derive the analytical formulae and develop an optimization algorithm for the EWMA CCC chart without using any transformation technique, as well as to propose the VSI EWMA CCC chart. Additionally, the run sum ratio charts are also developed and the charts' performances are investigated.

The limitations of this thesis include the fact that the performance measures in evaluating the charts' performances are limited to only the ATS, ARL and EARL criteria. Moreover, the VSI EWMA CCC chart proposed in this thesis considers only process deterioration, where there is no discussion about process improvement.

## 1.7 Organization of the Thesis

This thesis is organized as follows: In Chapter 1, the concept of SPC and types of control charts are explained. The research motivation, objectives and organization of the thesis are also elucidated in this chapter.

Chapter 2 provides the performance measures and literature review of related control charts, such as the basic CCC, EWMA CCC and adaptive CCC charts. Furthermore, an overview of the VSI EWMA $\bar{X}$ and ratio charts, such as the Shewhart ratio and synthetic ratio charts are given. A table that summarizes the existing CCC and ratio charts is given. Concluding remarks are also drawn at the end of this chapter.

Chapter 3 presents the EWMA CCC chart based on the formulae derived using the Markov chain approach, together with an optimal design in minimizing the out-of-control ATS value. In addition, a flow chart describing the optimization steps of the

EWMA CCC chart is given. Concluding remarks are provided to summarize the discussion in this chapter.

Chapter 4 enumerates the proposed VSI EWMA CCC chart. The derivation of the ATS performance measure, optimal design in minimizing the out-of-control ATS value of the chart and performance evaluation in comparison with other charts are showcased. This chapter also demonstrates the implementation of the VSI EWMA CCC chart. The last section of this chapter gives some concluding remarks.

In Chapter 5, the proposed run sum ratio chart is presented. The derivation of the average run length (ARL) and expected average run length (EARL) performance measures of the proposed chart are explained. The optimal designs in minimizing the out-of-control ARL and EARL values are provided, along with a performance comparison of the proposed chart with other competing charts. Moreover, an illustrative example is given to explain the implementation of the run sum ratio chart. Concluding remarks are also given at the end of this chapter.

Chapter 6 provides the concluding remarks by summarizing the contributions and findings of the thesis. In addition, suggestions for future research are also stated.

Finally, references and appendices are provided. Optimization programs written in the MATLAB software for all the proposed charts are provided in Appendix A, while the simulation programs written in Statistical Analysis Software (SAS) for the proposed charts are presented in Appendix B. Additionally, the MATLAB programs for the competing charts can be found in Appendix C. Appendix D provides the additional optimal scores and parameter of the proposed 4 and 7 regions run sum ratio chart in minimizing the zero state (ZS) and steady state (SS) ARL values. The ZS case assumes that the shift occurs at the beginning of process monitoring, while the SS case assumes that the shift occurs at an unknown random time after process monitoring has commenced (Tran and Knoth, 2018).

# CHAPTER 2

# LITERATURE REVIEW

## 2.1 Introduction

The use of the term attribute in quality control refers to quality characteristics that either conform or do not conform to specifications. Attributes are used when measurements are practically impossible, for instance, for visually inspected items like colour, missing part, damage and scratches. Secondly, attributes are used when measurements can be made but they are not made because of time, cost or needs. For example, the diameter of a hole can be measured with the aid of an inside micrometre but perhaps it is more convenient to use a “go-no-go” gage and see whether it conforms or does not conform to specifications (Besterfield, 2009).

Various terminologies are being used to describe an attribute that does not conform to specifications. Nonconformity can be regarded as a departure of a quality characteristic from its targeted level or state which occurs with severity, sufficient to make an associated product or service in not meeting the specification requirement. The term defect can also be defined in a similar way, except that it is mostly concerned with satisfying the intended conforming standard.

The term defect is appropriate to be used during evaluation in terms of usage, whereas, nonconformity is suitable for conformance to specifications. Similarly, the term nonconforming unit can be used in describing a unit of product or service containing at least one nonconformity (Besterfield, 2009).

The variable control chart is an excellent tool to achieve quality improvement. Variable charts are more informative than the attribute chart because generally more information can be acquired through measurements than in just classifying a unit as being conforming or otherwise.

## 2.2 Description of the Performance Measures

Performance measures of a control chart are the statistical measures used to assess the chart's efficiency. Based on the available literature on control charts, a number of different types of performance measures are used to assess the charts' performances. In this thesis, some of the performance measures adopted in evaluating the performance of the proposed charts are the average run length (ARL), expected average run length (EARL) and average time to signal (ATS) criteria.

### 2.2.1 Average Run Length (ARL)

The average number of sample points plotted on a control chart before an out-of-control signal is issued by the chart is the ARL of the chart. For a Shewhart chart, the ARL is obtained as the reciprocal of the probability of a single point falling beyond the limits of the chart. The in-control ARL, denoted as $\mathrm{ARL}_0$, should be reasonably large so that less frequent false alarms are issued by the chart. On the contrary, the out-of-control ARL, denoted as $\mathrm{ARL}_1$, should be as small as possible so that the process shifts can be detected quicker (Ryan, 2011).

### 2.2.2 Expected Average Run Length (EARL)

By considering the fact that the size of a shift can seldom be predicted with sufficient precision, the quality practitioner may not have enough historical data with regards to the process shifts, in order to fit a distribution to the shift size. Owing to this problem, the EARL is used as a measure of a chart's performance in place of the ARL, as the former requires only a shift interval to be specified, while the latter requires an exact shift size to be specified (Celano and Castagliola, 2016a).

### 2.2.3 Average Time to Signal (ATS)

The ATS as a measure of performance for adaptive charts is defined as the average length of time taken by a control chart to produce a signal. For an in-control process, the larger the ATS value is, the lower the false alarm rate becomes. However, for an out-of-control process, the smaller the ATS value is, the faster it is to detect a process shift (Chen et al., 2011).

## 2.3 CCC Type Control Charts

Many enterprises today can pursue their goal of achieving a high yield process with nearly zero defect due to the rapid growth in automated technologies. The traditional approach monitors the defective rate (fraction nonconforming) for attribute data by using the *p*-chart. However, the use of the *p*-chart is unsuitable for a high yield process due to its very low defective rate. In order to overcome the problem encountered in using the *p*-chart, the CCC chart was developed, where the latter plots the cumulative count of conforming items between two consecutive nonconforming items. Furthermore, the CCC chart has been adjudged to be very useful in monitoring high yield processes. A high yield process refers to a process with a very low in-control fraction of nonconforming ( $p_0$ ), say at most 0.001 or 1000 parts-per-million (ppm) (Chang and Gan, 2001). Control charts for monitoring processes with low defective rates include those by McCool and Joyner-Motley (1998), Wang (2009), Acosta-Mejia (2012) and Abbas et al. (2020), to mention a few. Table 2.1 provides a summary of the existing CCC charts in the literature.

Table 2.1. A summary of existing CCC charts and their descriptions

| Author(s) | | Title and journal information | Description |
|---|---|---|---|
| 1 | Calvin | Title: Quality control technique for zero defect<br><br>Journal: *IEEE Transactions on Components, Hybrid and Manufacturing Technology*, 1983, 6(3), 323-328 | It was pointed out that if attribute control charts are used in achieving zero defect, then the standard *p* and *u* charts are not suitable. A control chart that plots the number of good items between two consecutive defects on a logarithmic scale to accommodate a large number of good items can be employed. This chart is known as the cumulative count of conforming (CCC) control chart for high yield processes and it is based on the geometric distribution. |
| 2 | Goh | Title: A control chart for very high yield processes<br><br>Journal: *Quality Assurance*, 1987, 13(1), 18-22 | This study investigated the properties of a geometric chart (also called the CCC chart) developed by Calvin (1983). |
| 3 | Xie and Goh | Title: The use of probability limits for process control based on geometric distribution<br><br>Journal: *International Journal of Quality and Reliability Management*, 1997, 14(1), 64-73 | It was demonstrated why the use of probability limits is preferred over the traditional *k*-sigma control limits, as far as the geometric distribution is concerned. |
| 4 | Xie et al. | Title: A quality monitoring and decision-making scheme for automated production processes<br><br>Journal: *International Journal of Quality and Reliability Management*, 1999, 16(2), 148-157 | A control scheme was presented for monitoring the cumulative count of items inspected. This procedure limits the consecutive number of nonconforming items to a small value when the process has suddenly deteriorated. |
| 5 | Ohta et al. | Title: A CCC-*r* chart for high yield- processes<br><br>Journal: *Quality and Reliability Engineering International*, 2001, 17(6), 439-446 | The chart is based on the number of items inspected until *r* nonconforming items are observed. The authors demonstrated that as *r* increases, the CCC-*r* chart becomes more sensitive to small changes in upward shifts in the fraction nonconforming. |
| 6 | Chang and Gan | Title: Cumulative sum chart for high yield processes<br><br>Journal: *Statistica Sinica*, 2001, 11(3), 791-805 | The CUSUM chart was proposed for monitoring high yield processes with a sole priority of detecting small and moderate parameter changes based on geometric, binomial and Bernoulli counts. |
| 7 | Ranjan et al. | Title: Optimal control limits for CCC charts in the presence of inspection error<br><br>Journal: *Quality and Reliability Engineering International*, 2003, 19(2), 149-160 | The effect of inspection error on the Shewhart CCC chart and how the chart's control limits can be computed are discussed. |
| 8 | Liu et al. | Title: Cumulative count of conforming with variable sampling intervals<br><br>Journal: *International Journal of Production Economics*, 2006, 101(2), 286-297 | The CCC chart with the variable sampling interval scheme is developed. The average time to signal criterion is used in evaluating the efficiency of the chart. |
| 9 | Yeh et al. | Title: EWMA control charts for monitoring high yield processes based on non-transformed observations<br><br>Journal: *International Journal of Production Research*, 2008, 46(20), 5679-5699 | EWMA control charts were developed based on non-transformed geometric, binomial and Bernoulli counts. These charts were designed using one-sided control limits and the average number of inspected samples criterion was used in evaluating the chart's efficiency. |

Table 2.1 (continued)

| | | | |
|---|---|---|---|
| 10 | Noorossana et al. | Title: Identifying the period of step change in high yield processes<br><br>Journal: *Quality and Reliability Engineering International*, 2009, 25(7), 875-883 | An estimator for a period of time in which a step change has occurred was suggested. The performance of the model was investigated using several numerical examples. |
| 11 | Albers | Title: The optimal choice of negative binomial charts for quality process<br><br>Journal: *Journal of Statistical Planning and Inference*, 2010, 140(1), 214-225 | The chart computes the failure rates by using a negative binomial distribution and demonstrated how the optimal number of failures is related to the degree of an increase in the fraction nonconforming. |
| 12 | Chen et al. | Title: Cumulative conformance control chart with variable sampling intervals and control limits<br><br>Journal: *Applied Stochastic Models in Business and Industry*, 2011, 27(4), 410-420 | This study incorporates the variable sampling interval and variable control limit features to increase the sensitivity of the basic CCC chart. |
| 13 | Amiri and Khosravi | Title: Estimating a change point of the cumulative count of conforming under a drift<br><br>Journal: *Scientia Iranica*, 2012, 19(3), 856-861 | A maximum likelihood estimator for a change point of the nonconforming level of a high quality process with the linear trend was provided. The Monte Carlo simulation was used in evaluating the performance of the estimator. |
| 14 | Amiri and Khosravi | Title: Identifying time of monotonic change in the fraction nonconforming of high quality process<br><br>Journal: *The International Journal of Advanced Manufacturing Technology*, 2013, 68(1-4), 547-555 | An approach for estimating the time of a change which does not require prior knowledge of the change type was suggested. |
| 15 | Mavroudis and Nicolas | Title: EWMA control charts for monitoring high yield process<br><br>Journal: *Communications in Statistics - Theory and Methods*, 2013, 42(20), 3639-3654 | Two one-sided EWMA charts were developed to detect upward and downward shifts in the fraction nonconforming. The charts were developed based on the Markov chain procedure, where the geometric distribution has been transformed to its analogous exponential distribution. |
| 16 | Chen et al. | Title: Economic design of VSI GCCC charts for correlated samples from high yield processes<br><br>Journal: *South African Journal of Industrial Engineering*, 2013, 24(2), 88-101 | An economic design model of the VSI GCCC chart by considering a correlation of the production outputs within the same sample was presented. A cost function that examines the cost of sampling and inspection has been developed. |
| 17 | Khilare and Shirke | Title: The steady-state performance of cumulative count of conforming control chart based on runs rules<br><br>Journal: *Communications in Statistics - Theory and Methods*, 2014, 43(15), 3135-3147 | The SS properties of the *m*-of-*m* runs rules based control chart have been investigated based on the cumulative count of conforming items for high yield processes. |

Table 2.1 (continued)

| | | | |
|---|---|---|---|
| 18 | Bersimis et al. | A compound control chart for monitoring and controlling high quality processes<br><br>Journal: *European Journal of Operational Research,* 2014, 233(3), 595-603 | A compound rule that counts the number of conforming units inspected between the $(i-1)^{th}$ and the $i^{th}$ nonconforming items, as well as the number of conforming units observed between the $(i-2)$ and the $i^{th}$ nonconforming items was proposed. |
| 19 | Zhang et al. | Title: Performance of cumulative count of conforming charts of variable sampling intervals with estimated control limits<br><br>Journal: *International Journal of Production Economics,* 2014, 150, 114-124 | This paper investigated the performance of the CCC chart with variable sampling intervals when the control limits are estimated. The performance of the chart was evaluated using the ATS and SDATS criteria. |
| 20 | Lee and Khoo | Variable sampling interval cumulative count of conforming items with runs rules<br><br>Journal: *Communications in Statistics - Simulation and Computation,* 2015, 44(9), 2410-2430 | A combination of runs rules with the VSI feature by considering the lower sided CCC chart was presented. The findings reveal that the sensitivity of the CCC chart can be improved via the addition of runs rules, as well as varying the sampling intervals. |
| 21 | Ali et al. | An overview of control charts for high quality process<br><br>Journal: *Quality and Reliability Engineering International,* 2016, 32(7), 2171-2189 | An overview of control charts based on the time between events, in which the study takes into account of the cumulative quantity control and CCC charts was provided. |
| 22 | Fallahnezhad and Golbafian | Economic design of cumulative count of conforming control charts based on average number of inspected items<br><br>Journal: *Scientia Iranica,* 2017, 24(1), 330-341 | A mathematical model based on the average number of inspected items for the economic design of the CCC-*r* chart in minimizing the average cost per item was suggested. Additionally, sensitivity analyses with respect to the Type-I and Type-II errors were conducted. |
| 23 | Zhang et al. | CCC-*r* charts' performance with estimated parameter for high quality process<br><br>Journal :*Quality and Reliability Engineering International,* 2019, 35(4), 946-958 | The effect of measurement errors on the CCC-*r* charts' performance based on the expected value of the average number of observations to signal and standard deviation of the average number of observations to signal criteria was studied. |

### 2.3.1 Basic CCC Chart

The CCC chart was originally introduced by Calvin (1983) and was further enhanced by Goh (1987). The CCC chart is based on the time between defects (Albers, 2010).

Let $X$ be the cumulative count of items inspected until a nonconforming item is observed. Then $X$ is said to have a geometric distribution with parameter $p$. The probability mass function (pmf) and cumulative distribution function (cdf) of $X$ are

$$g(x) = p(1-p)^{x-1}, \qquad \text{for} \quad x = 1, 2, ... \tag{2.1}$$

and

$$G(x) = \sum_{i=1}^{x} p\left(1-p\right)^{i-1} = 1-\left(1-p\right)^{x}. \tag{2.2}$$

where $g(x)$ and $G(x)$ are the pmf and cdf of $X$, respectively.

Note that the traditional control limits cannot be applied on the basic CCC chart because the geometric distribution is highly skewed. Instead, the appropriate control limits to be used are the probability control limits (Xie and Goh, 1997). Since the geometric distribution is discrete, the control limits will be rounded to integers and any point that falls on (or beyond) the upper control limit (*UCL*) or lower control limit (*LCL*) will result in an out-of-control signal. Consequently, the *UCL* and *LCL* are obtained by solving the following equation:

$$\Pr\left(X \geq UCL\right) = \Pr\left(X \leq LCL\right) = \frac{\alpha}{2}, \tag{2.3}$$

where $\alpha$ denotes an acceptable probability of a false alarm by the CCC chart. From Equation (2.3), the following is obtained:

$$G\left(UCL-1\right) = 1-\left(1-p\right)^{UCL-1} = 1-\frac{\alpha}{2} \ . \tag{2.4}$$

By solving Equation (2.4), the *UCL* of the CCC chart is obtained as (Liu et al. 2006)

$$UCL = \left( \frac{\log\left(\frac{\alpha}{2}\right)}{\log\left(1-p\right)} + 1 \right). \tag{2.5a}$$

In a similar way, the *LCL* of the CCC chart is obtained as follows:

$$G\left(LCL\right) = 1-\left(1-p\right)^{LCL} = \frac{\alpha}{2}, \text{ i.e.}$$

$$LCL = \left( \frac{\log\left(1-\frac{\alpha}{2}\right)}{\log\left(1-p\right)} \right) \ . \tag{2.5b}$$

It is important to note that the decision procedure with respect to an out-of-control condition on the CCC chart is slightly different from that of the traditional control charts. Thus, if a point in the CCC chart is equal to or exceeds the *UCL*, the process is believed to have improved. Meanwhile, if a point in the CCC chart is equal to or less than the *LCL*, the process is deemed to have deteriorated (Xie et al., 1999).

The ARL of the basic CCC chart is given as (Liu et al., 2006)

$$\begin{aligned}\text{ARL} &= \frac{1}{\Pr\left(X \le LCL\right) + \Pr\left(X \ge UCL\right)} \\ &= \frac{1}{1-\left(1-p\right)^{LCL} + \left(1-p\right)^{UCL-1}},\end{aligned} \tag{2.6}$$

where *UCL* and *LCL* are defined in Equations (2.5a) and (2.5b), respectively. Additionally, the ATS of the basic CCC chart is obtained as (Liu et al., 2006)

$$ATS = \frac{ARL}{p} \times d, \tag{2.7}$$

The in-control and out-of-control ATSs are obtained using Equation (2.7) when $p = p_0$ and $p = p_1$, respectively, where *d* is the sampling interval length of the basic CCC chart. Some CCC type control charts include those proposed by Ohta et al. (2001), Ranjan et al. (2003), Noorossana et al. (2009), Amiri and Khosravi (2012 and 2013), Bersimis et al. (2014), Ali et al. (2016), Fallahnezhad and Golbafian (2017), and Zhang et al. (2019).

### 2.3.2 VSI CCC Chart

The VSI $\bar{X}$ chart was proposed by Reynolds et al. (1998). They demonstrated that the VSI $\bar{X}$ chart was more efficient than the Shewhart $\bar{X}$ chart. Similarly, Prabhu et al. (1993) and Costa (1994) introduced VSS schemes for the $\bar{X}$ chart, where a smaller sample size is used for taking the next sample if the current $\bar{X}$ value is close to the center line, while a larger sample size is considered otherwise.

Additionally, the variable sampling techniques can be used to improve the efficiency of attributes control charts (Vaughan, 1993; Epprecht and Costa, 2001). Epprecht et al. (2003) investigated a general model regarding adaptive *c*, *np*, *u* and *p* charts, where one, two or three design parameters (sample size, sampling interval and control limit width) can be used. The design allows switching between two values. General guidelines for selecting effective design schemes were also provided in the study. Furthermore, Wu and Luo (2004) investigated the optimal design of VSI, VSS and VSIVSS *np* control charts, especially for detecting small or moderate process shifts.

Previous studies on the economic design of control charts indicated that the VSI control chart exhibits a better performance than the fixed sampling interval (FSI) chart in relation to cost. Bai and Lee (1998) conducted a study with respect to a cost model involving the cost of a false alarm, the cost of identifying and eliminating an assignable cause, and the cost of sampling and testing. Their study revealed that with proper design parameters, the VSI $\bar{X}$ chart yielded lower expected cost per unit of time compared with the corresponding FSI $\bar{X}$ chart. Also, Chen (2004) provided an extension to this study by investigating the VSI $\bar{X}$ chart with non-normal data. They provided an alternative cost model that uses the Burr distribution in the economic design of the VSI $\bar{X}$ chart.

Liu et al. (2006) proposed a VSI CCC chart to increase the insensitive nature of the basic CCC chart. They considered Equation (2.1) and therefore, suggested that, since the control limits given in Equations (2.5a) and (2.5b) are rounded to integers, the true-false alarm rate $\alpha'$ is unlikely to be exactly equal to the given value of $\alpha$, hence,

$$\alpha' = (1 - p_0)^{UCL-1} + 1 - (1 - p_0)^{LCL} \tag{2.8}$$

is mainly utilised for the CCC chart in high-quality environments, as $p_0$ is very small. Note that $p_0$ is the fraction of nonconforming items when the process is in-control.

The VSI CCC chart was designed to detect an increase in the nonconforming rate. The chart operates in such a way that the sampling interval varies with the cumulative count of conforming items. The sampling interval length, $L_i$ that is used for inspection between the $(i-1)^{th}$ and $i^{\text{th}}$ nonconforming items strictly depends on the value of $X_{i-1}$, where $X_{i-1}$ refers to the cumulative count of items inspected before the $i^{\text{th}}$ conforming item.

For the implementation purpose, only finite number of sampling interval lengths $d_1, d_2, ..., d_n$ with $d_n < d_{n-1} < ... < d_2 < d_1$ are considered. The region between *UCL* and *LCL* of the VSI CCC chart is divided into sub-regions, which are defined by the interval limits $IL_1, IL_2, ..., IL_{n-1}$, computed as (Liu et al., 2006)

$$IL_1 = \left( \frac{\ln\left(\frac{\alpha}{2} + q_1\right)}{\ln(1 - p_0)} \right) \tag{2.9a}$$

$$IL_2 = \left( \frac{\ln\left(\frac{\alpha}{2} + q_1 + q_2\right)}{\ln(1 - p_0)} \right) \tag{2.9b}$$

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.

$$IL_{n-1} = \left( \frac{\ln\left(\frac{\alpha}{2} + q_1 + q_2 + \ldots + q_{n-1}\right)}{\ln(1 - p_0)} \right), \tag{2.9c}$$

where $q_1 = q_2 = \ldots = q_{n-1} = \frac{1-\alpha}{n}$, $n$ is the number of sampling intervals, and $\alpha$ is the false alarm rate.

Having defined the control limits, interval limits and ARL for the VSI CCC chart, as in Equations (2.5a) – (2.5b), (2.9a) – (2.9c) and (2.6), respectively, the ATS for the VSI CCC chart is defined as (Liu et al., 2006)

$$ATS = \frac{ARL}{p} \times \frac{d_1 q_1 + d_2 q_2 + \ldots + d_n q_n}{q_1 + q_2 + \ldots + q_n}. \tag{2.10}$$

The in-control ATS is obtained using Equation (2.10) when $p = p_0$, while the out-of-control ATS is obtained using the same equation when $p = p_1$ and $q_j$ is replaced by $q'_j$, for $j$ =1, 2, …, $n$, where $q'_1 = (1-p_1)^{IL_1} - (1-p_1)^{UCL-1}$, $q'_2 = (1-p_1)^{IL_2} - (1-p_1)^{IL_1}$, …, $q'_{n-1} = (1-p_1)^{IL_{n-1}} - (1-p_1)^{IL_{n-2}}$, and $q'_n = (1-p_1)^{LCL} - (1-p_1)^{IL_{n-1}}$. Other researchers that investigated the VSI CCC charts include Chen et al. (2011), Chen et al. (2013), Zhang et al. (2014), and Lee and Khoo (2015).

### 2.3.3 EWMA CCC Chart

The EWMA chart is another type of control chart which is more efficient than the Shewhart chart in detecting small changes in the population parameters. When the smoothing constant of the EWMA chart is carefully selected, the EWMA chart has the potential of outperforming the cumulative sum (CUSUM) chart (Yeh et al., 2008).

Roberts (1959) was the first to introduce the EWMA chart. Since then, many research articles on the EWMA-type charts were published (for example, see Borror et al., 1998; Castagliola, 2005; Zhang et al., 2009; Huwang et al., 2010; Castagliola et al., 2011; Tran et al., 2016a; and Tran and Knoth, 2018). It should be noted that Chang and Gan (2001) proposed CUSUM charts for high yield processes, while Khilare and Shirke (2014) investigated the performance of CCC charts based on runs rules.

Yeh et al. (2008) proposed the EWMA CCC chart for monitoring high-yield processes based on non-transformed observations. Their study is based on the pmf described in Equation (2.1) and the following EWMA statistic:

$$Z_i = \lambda X_i + (1-\lambda) Z_{i-1} \text{, for } i = 1, 2, \ldots, \tag{2.11}$$

where $0 < \lambda \leq 1$ is a pre-determined smoothing constant. $Z_0 = \frac{1}{p_0}$, where $p_0$ is the in-control fraction of nonconforming items, while $i$ represents the observation number and $X_i$ is the CCC observation. Hence, when the process is in-control and by using the properties of the geometric series, it can be shown that (Yeh et al., 2008)

$$E(Z_i) = \frac{1}{p_0}\left[\sum_{j=0}^{i-1} \lambda (1-\lambda)^j + (1-\lambda)^i\right] = \frac{1}{p_0} \tag{2.12a}$$

and

$$\text{Var}(Z_i) = \frac{1-p_0}{p_0^2} . \frac{\lambda\left[1-(1-\lambda)^{2i}\right]}{(2-\lambda)} . \tag{2.12b}$$

To compute the control limit of the EWMA CCC chart, Yeh et al. (2008) considered only the following lower control limit.

$$LCL = \frac{1}{p_0} - L\sqrt{\frac{\lambda(1-p_0)}{(2-\lambda)\, p_0^2} \times \left(1-(1-\lambda)^{2i}\right)}, \tag{2.13}$$

where $L$ is a predetermined multiplier that depends on the desired in-control performance of the EWMA CCC chart, as well as the choice of $\lambda$. To implement the EWMA CCC chart, one collects the CCC observations $X_i$ (for $i = 1, 2, \ldots$) and then updates the calculation of $Z_i$. An out-of-control condition indicating a process deterioration is detected as soon as $Z_i < LCL$.

Although Yeh et al. (2008) evaluated the performance of the EWMA CCC chart using the Monte Carlo simulation, in this thesis, analytical formulae for evaluating the performance of the EWMA CCC chart proposed by Yeh et al. (2008) will be provided in Chapter 3.

## 2.4 VSI EWMA $\bar{X}$ Chart

In using VSI control schemes, the short sampling interval is employed when the control charting statistic exhibits a probable out-of-control situation. Otherwise, the long sampling interval is adopted (Saccucci et al., 1992). Following the pioneering work of Roberts (1959) on the EWMA $\bar{X}$ chart, Waldman (1986) and Crowder (1987) evaluated the ARL properties of the EWMA charts using numerical computations. In another development, Saccucci et al. (1992) examined the properties of an EWMA chart when the time between samples is not constant but is dependent on the current value of the EWMA statistic. If the control charting statistic is close to the target value, then a long sampling interval is used, whereas, if the control charting statistic is near to one of the control limits, then a short sampling interval is employed. An out-of-control signal is issued if the control charting statistic plots beyond the control limits.

The significance of varying the sampling intervals is that it allows information about the process to be acquired quickly when there is a process shift, thereby preventing the continued production of poor quality products. Hence, the use of VSI control schemes can drastically improve the overall quality of the process.

The VSI EWMA $\bar{X}$ chart is based on the EWMA model presented in Equation (2.11), by replacing $X_i$ in Equation (2.11) with $\bar{X}_i$, where $\bar{X}_i$ represents the sample mean of the process (Saccucci et al., 1992). The EWMA model considers both the current and past observations. The starting value $Z_0$ is mostly regarded as the process target value. Contrary to FSI control schemes, the sampling interval between the sample means, $\bar{X}_i$ and $\bar{X}_{i+1}$ depends on the current value of $Z_i$. A long sampling interval $t_L$ is employed when the control charting statistic, i.e. $Z_i$ falls in the interval [*LWL*, *UWL*]. Conversely, a short sampling interval $t_S$ is needed if $Z_i$ falls in the warning region given by the intervals [*LCL*, *LWL*] or [*UWL*, *UCL*]. Note that *LWL* and *UWL* are the lower and upper warning limits, respectively, given as follows:

$$LWL = \mu_0 - w\sigma_0\sqrt{\frac{\lambda}{n(2-\lambda)}} \tag{2.14a}$$

and

$$UWL = \mu_0 + w\sigma_0\sqrt{\frac{\lambda}{n(2-\lambda)}}, \tag{2.14b}$$

Additionally, *LCL* and *UCL* are the lower and upper control limits, respectively, given as

$$LCL = \mu_0 - L\sigma_0\sqrt{\frac{\lambda}{n(2-\lambda)}} \tag{2.15a}$$

and

$$UCL = \mu_0 + L\sigma_0\sqrt{\frac{\lambda}{n(2-\lambda)}}. \tag{2.15b}$$

Note that in Equations (2.14a), (2.14b), (2.15a) and (2.15b), *w* and *L* are the parameters controlling the width of the warning limits and control limits, respectively. The parameters *w* and *L* are functions of the in-control process standard deviation and sample size, and $0 < w < L$.

An appropriate performance metric for evaluating the VSI EWMA $\bar{X}$ chart's performance is the ATS criterion, given in Equation (2.16) (Saccucci et al., 1992).

$$\text{ATS}(\delta) = \boldsymbol{q}^T \left(\boldsymbol{I} - \boldsymbol{Q}\right)^{-1} \boldsymbol{t} - \boldsymbol{q}^T \boldsymbol{t}, \tag{2.16}$$

where $\delta$ is the standardized shift size, $\boldsymbol{q}^T$ is the initial probability vector, $\boldsymbol{I}$ is the $k \times k$ identity matrix, $\boldsymbol{Q}$ is the $k \times k$ transition probability matrix (tpm) and $\boldsymbol{t}$ is the $k \times 1$ vector of sampling interval lengths, whose values are either $t_S$ or $t_L$. Some recent research works on VSI charts include Amdouni et al. (2017), Khoo et al. (2019) and Tran et al. (2019).

## 2.5 Ratio Type Control Charts

When process monitoring involves *p* correlated variables that are to be monitored over time, the Hotelling's $T^2$ chart proposed by Hotelling (1947), which is a multivariate counterpart of the univariate Shewhart chart is used. Specifically, quality practitioners who are saddled with the responsibility of implementing online multivariate process monitoring may be interested in the continuous monitoring of the ratio, *Z* of *p* = 2 random variables, say *X* and *Y*. Hence, the interest is not in monitoring the stability of the $p\times1$ process mean vector $\boldsymbol{\mu}$ or the $p\times p$ covariance matrix $\boldsymbol{\Sigma}$. Table 2.2 provides a summary of the existing ratio charts in the literature.

Table 2.2. A summary of existing ratio charts and their descriptions

| | Author(s) | Title and journal information | Type of chart | Description |
|---|---|---|---|---|
| 1 | Spisak | Title: A control chart for ratios<br><br>Journal: *Journal of Quality Technology*, 1990, 22(1), 34-37 | Shewhart | The chart's statistic is $\hat{r} = \frac{\sum_{h=1}^{H} y_h}{\sum_{h=1}^{H} x_h}$ , where $(y_h, x_h)$ are the values of the random variables (*Y*, *X*) and *H* is the total number of $(y_h, x_h)$ pairs. An out-of-control is issued when $\hat{r}$ > *UCL* or $\hat{r}$ < *LCL*, where *UCL* and *LCL* are the upper and lower control limits, respectively. |
| 2 | Öksoy et al. | Title: Statistical process control by the quotient of two correlated normal variables<br><br>Journal: *Quality Engineering*, 1994, 6(2), 179-194 | Shewhart | The chart's statistic is $\xi' = \frac{\bar{\bar{X}}_1}{\bar{\bar{X}}_2}$ , where $\bar{\bar{X}}_1 = \sum_{i=1}^{N} \frac{\bar{X}_{1i}}{N}$ , $\bar{\bar{X}}_2 = \sum_{i=1}^{N} \frac{\bar{X}_{2i}}{N}$ , $\bar{X}_{1i} = \sum_{j=1}^{n} \frac{X_{1ij}}{n}$ and $\bar{X}_{2i} = \sum_{j=1}^{n} \frac{X_{2ij}}{n}$ are computed for the data of *N* periods (i.e. *i* = 1, 2,…, *N*). Here, the $(X_{1ij}, X_{2ij})$ pair, for *i* = 1, 2, …, *N* and *j* = 1, 2, …, *n*, are obtained at equal periods (hourly, daily, etc.). The chart issues an out-of-control when $\xi'$ > *UCL* or $\xi'$ < *LCL*. |
| 3 | Celano et al. | Title: Statistical performance of a control chart for individual observations monitoring the ratio of two normal variables<br><br>Journal: *Quality and Reliability Engineering International*, 2014, 30(8), 1361-1377 | Shewhart | The chart's statistic is $Z_i = \frac{X_i}{Y_i}$ , for *i* = 1, 2, …, where $(X_i, Y_i)$ are pairs of individual observations. The chart issues an out-of-control when $Z_i$ > *UCL* or $Z_i$ < *LCL*. |
| 4 | Celano and Castagliola | Title: Design of phase II control chart for monitoring the ratio of two normal variables<br><br>Journal: *Quality and Reliability Engineering International*, 2016a, 32(1), 291-308 | Shewhart | The chart's statistic is $\hat{Z}_i = \frac{\bar{X}_i}{\bar{Y}_i}$ , for *i* = 1,2, …, where $\bar{X}_i$ and $\bar{Y}_i$ are the sample means of variables *X* and *Y*, respectively. The chart issues an out-of-control when $\hat{Z}_i$ > *UCL* or $\hat{Z}_i$ < *LCL*. |
| 5 | Celano and Castagliola | Title: A synthetic control chart for monitoring the ratio of two normal variables<br><br>Journal: *Quality and Reliability Engineering International*, 2016b, 32(2), 681-696 | Synthetic | The ratio sub-chart's statistic is $\hat{Z}_i = \frac{\bar{X}_i}{\bar{Y}_i}$ , for *i* = 1,2, … . If *LCL* ≤ $\hat{Z}_i$ ≤ *UCL*, the ratio at sample *i* is conforming, otherwise, it is non-conforming. When the sample ratio is nonconforming, the conforming run length is computed as the number of conforming sample ratios between two nonconforming ones, including the current nonconforming sample ratio. If CRL > *H* in the CRL sub-chart, the process is in-control, otherwise, it is out-of-control, where *H* is the lower limit of the CRL sub-chart. |
| 6 | Tran et al. | Title: Monitoring the ratio of two normal variables using EWMA type control charts<br><br>Journal: *Quality and Reliability Engineering International*, 2016a, 32(2), 1853-1869 | Two one-sided EWMA | The statistics for the upward and downward EWMA charts are $Y_i^+ = \max\left(Z_0, (1-\lambda^+)Y_{i-1}^+ + \lambda^+\hat{Z}_i\right)$ and $Y_i^- = \min\left(Z_0, (1-\lambda^-)Y_{i-1}^- + \lambda^-\hat{Z}_i\right)$ , respectively, for *i* = 1,2, …, where $\hat{Z}_i = \frac{\bar{X}_i}{\bar{Y}_i}$ , and $\lambda^+$ , $\lambda^-$ $\in$ (0,1] are the charts' smoothing parameters, while $Y_0^+ = Y_0^-$ takes the value of the in-control ratio. An upward shift is signalled when $Y_i^+$ > *UCL* and a downward shift is signalled when $Y_i^-$ < *LCL*. |