

USING COMPUTER TO DISCOVER SOME THEOREMS IN GEOMETRY

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Dedicated to the memory of Professor Péter Kiss

Abstract. By means of Buchberger's algorithm for computing Groebner bases of ideals some theorems from elementary geometry are proved. Besides the well-known formula of Heron for the calculation of the area of a triangle analogical formulas and relations for planar quadrangles are derived. It is shown that with the help of a common software (Maple, Mathematica) formulas from elementary geometry can be proved and even discovered.

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1. Introduction

It is likely known that after the year 1960 Buchberger and Hironaka discovered a new algorithm for solving a system of algebraic equations. A great interest in this area of mathematics and a general using computers and mathematical software, which makes possible not only numerical computation but computations with symbols, caused big changes in commutative algebra and algebraic geometry. Nowadays is so called Buchberger's algorithm for computing of a Groebner basis of an ideal implemented even in some calculators (models TI 89 or TI 92).

Example. Solve the system of equations

$$(1) \quad \begin{aligned} x^2 + y^2 + z^2 &= 6, \\ x^3 + y^3 + z^3 - xyz &= -4, \\ xy + xz + yz &= -3. \end{aligned}$$

Solution. First we shall "prove" that even calculators mentioned above are able to solve such a quite difficult system of equations. Calculators mentioned above are equipped with the command `4: solve(and, (x, y))` for solving system of equations. We write

`solve(x^2 + y^2 + z^2 = 6 and x^3 + y^3 + z^3 - xyz = -4 and
xy + xz + yz = -3, {x, y, z})`

and in a while the result appears on the screen. On the screen we see only a part of the result, but a display is rolling and by means of arrows we can shift it. We can read

$x=1$ and $y=1$ and $z=-2$ or $x=1$ and $y=-2$ and $z=1$ or $x = -2$ and $y = 1$ and $z=1$

i.e., the solution P of the given system (1) is

$$P = \{[1, 1, -2], [1, -2, 1], [-2, 1, 1]\}.$$

In manuals of calculators there is explained, that in the case of solving a system of algebraic equations the Buchberger's algorithm for computing Groebner bases of an ideal by so called lexicographic order of variables is used. If the user wants to know more it is suitable to take advantage of some commercial programs of a computer algebra (Maple, Mathematica) or free to download programs from Internet (Singular, CoCoA). In Mathematica /version 4/ we can write

```
poly = {x^2 + y^2 + z^2 == 6, x^3 + y^3 + z^3 - x * y * z == -4,
x * y + x * z + y * z == -3}
```

and after that we put `GroebnerBasis[poly, {x, y, z}]` and obtain

```
-12z + 9z^2 + 4z^3 - 6z^4 + z^6 + 4, 49y^2 + 12yz^5 - 16yz^4 - 18yz^3 + 72yz^2 - 37yz + 36y -
16z^5 + 54z^4 + 24z^3 - 145z^2 + 180z - 195, 49x + 49y + 12z^5 - 16z^4 - 18z^3 + 72z^2 - 37z + 36.
```

To comment the situation better denote the above polynomials by

$$\begin{aligned} g_1(z) &= -12z + 9z^2 + 4z^3 - 6z^4 + z^6 + 4, \\ g_2(y, z) &= 49y^2 + 12yz^5 - 16yz^4 - 18yz^3 + 72yz^2 - 37yz + 36y - 16z^5 \\ &\quad + 54z^4 + 24z^3 - 145z^2 + 180z - 195, \\ g_3(x, y, z) &= 49x + 49y + 12z^5 - 16z^4 - 18z^3 + 72z^2 - 37z + 36. \end{aligned}$$

Solutions of the system (1) are certain ordered triples of real (or complex) numbers $[x, y, z]$. How to find them? It is obvious that instead of the system (1) it is better to solve the equivalent system

$$(2) \quad g_1(z) = 0, \quad g_2(y, z) = 0, \quad g_3(x, y, z) = 0,$$

where the polynomials $g_1(z), g_2(y, z), g_3(x, y, z)$ form a Groebner basis of the ideal $I = \langle f_1, f_2, f_3 \rangle$ by the lexicographic order $<_L$, where $z <_L y <_L x$. To find the solution of (2) the simplest way is first to solve the equation $g_1(z) = 0$, which contains only one unknown z . Although the polynomial $g_1(z)$ doesn't look simple, we can use the function `Factor[expr]`, which decomposes a given expression `expr`. In this case we have $g_1(z) = (z-1)^4(z+2)^2$ and see that the equation $g_1(z) = 0$ has two real (and multiple) roots $z_1 = 1, z_2 = -2$. Setting e.g. the value $z_1 = 1$ into the equation $g_2(y, z) = 0$ we obtain $g_2(y, 1) = 49y^2 + 49y - 98 = 0$, with the roots

$y_1 = 1, y_2 = -2$. If we substitute all these values into the last equation $g_3(x, y, z) = 0$, we arrive at the solution P of (2), which is the same as the solution of (1) $P = \{[1, 1, -2], [1, -2, 1], [-2, 1, 1]\}$. Behind the result which the calculator yielded not only new technology is hidden but also a big progress, which commutative algebra and algebraic geometry achieved in the last third of the last century. The extent of this paper doesn't allow us to write more about these issues, we prefer rather non formal, intuitive approach. For detailed information see the books [4], [2], or [6] on Internet or [3], [9], [7].

We could notice that in the course of solving the system of equations (1) by the lexicographic order $<_L$, where $z <_L y <_L x$ the variable x is first eliminated and then y . In the end in the Groebner basis of the ideal $I = \langle f_1, f_2, f_3 \rangle$, where $f_i, i = 1, 2, 3$ denote the polynomials which form the system (1) the polynomial g_1 occurs, which is a function only of one variable z . The equation $g_1(z) = 0$ is not a problem to solve.

The elimination of variables, which is realized in programs of computer algebra using Groebner bases can also bring the method of proving and discovering theorems. In the next part of this paper we would want to give non traditional proofs of some theorems from elementary geometry. In these proofs we shall take advantage of the elimination of variables. To do this first we have to look at the elimination closely.

Definition. Let $I = \langle f_1, f_2, \dots, f_s \rangle \subset k[x_1, x_2, \dots, x_n]$ be an ideal. The r^{th} elimination ideal I_r is the ideal of the domain of integrity $k[x_1, x_2, \dots, x_n]$ which fulfils

$$I_r = I \cap k[x_{r+1}, x_{r+2}, \dots, x_n].$$

In general the following theorem about elimination holds, see [4].

Theorem. Let $I \subset k[x_1, x_2, \dots, x_n]$ be an ideal and G the Groebner basis of the ideal I with respect to lexicographic order, where $x_1 >_L x_2 >_L \dots >_L x_n$. Then for every $r, 0 \leq r \leq n$, the set $G_r = G \cap k[x_{r+1}, x_{r+2}, \dots, x_n]$ is a Groebner basis of the r^{th} elimination ideal I_r .

Example 1. Find the formula of Heron for the area F of a triangle ABC with sides a, b, c . Give "a computer proof".

Solution. Choose the coordinate system so that coordinates of the vertices of a triangle ABC are $A = [0, 0], B = [c, 0], C = [x, y]$ and $|AB| = c, |BC| = a, |AC| = b$. Let us construct the ideal $I = \langle a^2 - (c-x)^2 - y^2, b^2 - x^2 - y^2, F - \frac{1}{2}cy \rangle$ in the ring $R[a, b, c, x, y, F]$. We try to obtain a formula, which describes a relation between the lengths of sides a, b, c of a triangle ABC and its area F . Such a polynomial should belong into the elimination ideal $I \cap R[a, b, c, F]$. In this example the whole computation can be performed not only with the software specialized on ideals but even with such a common software like Mathematica /version 4/. We write

$$\text{Eliminate}[\{(c-x)^2 + y^2 == a^2, x^2 + y^2 == b^2, F == 1/2c * y\}, \{x, y\}]$$

and obtain

$$16F^2 == -a^4 + 2a^2b^2 - b^4 + 2a^2c^2 + 2b^2c^2 - c^4$$

the result. The next command

`Factor[-a^4 + 2a^2b^2 - b^4 + 2a^2c^2 + 2b^2c^2 - c^4]`

gives

$$-(a - b - c)(a + b - c)(a - b + c)(a + b + c).$$

It is easy to see that the last relation is the same as

$$F = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a + b + c)$. We get the formula of Heron.

Now we will investigate the area of a quadrangle in a plane.

Example 2. Let $ABCD$ be a planar quadrangle with sides a, b, c, d and diagonals e, f . Find the formula of the area F of a quadrangle $ABCD$. Give a “computer proof”.

Solution. Let the coordinates of the vertices of a quadrangle $ABCD$ be $A = [a, 0]$, $B = [x, y]$, $C = [z, v]$, $D = [0, 0]$ and $a = |DA|$, $b = |AB|$, $c = |BC|$, $d = |CD|$, $e = |BD|$, $f = |AC|$. It is easy to see that for the area of a quadrangle $F = \frac{1}{2}(xv - zy + ay)$ holds. By means of Mathematica we write

`Eliminate[{(x - a)^2 + y^2 == b^2, (x - z)^2 + (y - v)^2 == c^2, z^2 + v^2 == d^2, x^2 + y^2 == e^2, (z - a)^2 + v^2 == f^2, 2F == x * v - z * y + a * y}, {x, y, z, v}]`

which gives

$$\begin{aligned} & e^4 f^2 + e^2(a^2 b^2 - a^2 c^2 - b^2 d^2 + c^2 d^2 - a^2 f^2 - b^2 f^2 - c^2 f^2 - d^2 f^2 + f^4) == \\ & - a^4 c^2 + a^2 b^2 c^2 - a^2 c^4 + a^2 b^2 d^2 - b^4 d^2 + a^2 c^2 d^2 + b^2 c^2 d^2 - b^2 d^4 + a^2 c^2 f^2 \\ (3) \quad & - b^2 c^2 f^2 - a^2 d^2 f^2 + b^2 d^2 f^2 \&\& 16F^2 == -a^4 + 2a^2 b^2 - b^4 - 2a^2 c^2 + 2b^2 c^2 + 2b^2 c^2 d^2 - \\ & - c^4 + 2a^2 d^2 - 2b^2 d^2 + 2c^2 d^2 - d^4 + 4e^2 f^2. \end{aligned}$$

It seems that the second equality is the relation we are looking for. We can simplify it by

`FullSimplify[16F^2 == -a^4 + 2a^2b^2 - b^4 - 2a^2c^2 + 2b^2c^2 - c^4 + 2a^2d^2 - 2b^2d^2 + 2c^2d^2 - d^4 + 4e^2f^2]`

and get

$$(a^2 - b^2 + c^2 - d^2)^2 + 16F^2 == 4e^2 f^2,$$

which is a desired result.

Remark. This formula by means of which we can express the area of a quadrangle by the all six distances between the four vertices has often been given in the form

$$(4) \quad 16F^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2.$$

The formula (4) was published by Staudt [11]. Notice that if we set e.g. $d = 0$ into (3) we obtain the formula of Heron.

The first equality in (3) is related to the so called Euler's four points relation, see [5], which expresses the dependence of six distances a, b, c, d, e, f between four vertices of a quadrangle. Euler's four points relation follows from the Cayley–Menger determinant for the volume V of a tetrahedron with edges of lengths a, b, c, d, e, f

$$(5) \quad 288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & b^2 & f^2 & a^2 \\ 1 & b^2 & 0 & c^2 & e^2 \\ 1 & f^2 & c^2 & 0 & d^2 \\ 1 & a^2 & e^2 & d^2 & 0 \end{vmatrix}$$

if we put $V = 0$. We will compare the equation $V = 0$ from (5) with the first equality in (3). Denote by m the determinant above

$$m = \{\{0, 1, 1, 1, 1\}, \{1, 0, b^2, f^2, a^2\}, \{1, b^2, 0, c^2, e^2\}, \{1, f^2, c^2, 0, d^2\}, \{1, a^2, e^2, d^2, 0\}\}.$$

Then the command

Det [m]

gives $-2a^4c^2 + 2a^2b^2c^2 - 2a^2c^4 + 2a^2b^2d^2 - 2b^4d^2 + 2a^2c^2d^2 + 2b^2c^2d^2 - 2b^2d^4 - 2a^2b^2e^2 + 2a^2c^2e^2 + 2b^2d^2e^2 - 2c^2d^2e^2 + 2a^2c^2f^2 - 2b^2c^2f^2 - 2a^2d^2f^2 + 2b^2d^2f^2 + 2a^2e^2f^2 + 2b^2e^2f^2 + 2c^2e^2f^2 + 2d^2e^2f^2 - 2e^4f^2 - 2e^2f^4$.

We see that the condition $V = 0$ is the same as the first condition in (3).

Now we will investigate the case of a cyclic quadrangle, i.e., a quadrangle which is inscribed into the circle. Suppose we are given a cyclic quadrangle A, B, C, D with the sides $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$ and the radius r of the circumscribed circle. The well-known formula of Brahmagupta for the evaluation of the area F of a cyclic convex quadrangle with the sides a, b, c, d is as follows:

$$(6) \quad F = \sqrt{\frac{(-a+b+c+d)}{2} \frac{(a-b+c+d)}{2} \frac{(a+b-c+d)}{2} \frac{(a+b+c-d)}{2}}.$$

Example 3. Find the formula of Brahmagupta. Give “a computer proof”.

Solution. Choose the Cartesian coordinate system so that $A = [r, 0]$, $B = [x, y]$, $C = [u, v]$, $D = [z, w]$ and place the origin into the center of the circumscribed circle with radius r . To express the area F of a quadrangle A, B, C, D we use the following formula for evaluating the oriented area of a n -gon A_1, A_2, \dots, A_n with coordinates $A_i = [x_i, y_i]$. Then

$$(7) \quad F = \frac{1}{2} \sum_{i=1}^n \begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix}$$

holds. Using Mathematica we enter

```
Eliminate[{x^2 + y^2 == r^2, u^2 + v^2 == r^2, z^2 + w^2 == r^2, (x - r)^2 + y^2 == a^2, (u - x)^2 + (v - y)^2 == b^2, (z - u)^2 + (w - v)^2 == c^2, (r - z)^2 + w^2 == d^2, 2F == y * r - u * y + x * v - z * v + u * w - r * w}, {x, y, u, v, z, w, r}]
```

and get

$$(32a^4 - 64a^2b^2 + 32b^4 - 64a^2c^2 - 64b^2c^2 + 32c^4 - 64a^2d^2 - 64b^2d^2 - 64c^2d^2 + 32d^4)F^2 + 256F^4 == -a^8 + 4a^6b^2 - 6a^4b^4 + 4a^2b^6 - b^8 + 4a^6c^2 - 4a^4b^2c^2 - 4a^2b^4c^2 + 4b^6c^2 - 6a^4c^4 - 4a^2b^2c^4 - 6b^4c^4 + 4a^2c^6 + 4b^2c^6 - c^8 + 4a^6d^2 - 4a^4b^2d^2 - 4a^2b^4d^2 + 4b^6d^2 - 4a^4c^2d^2 + 40a^2b^2c^2d^2 - 4b^4c^2d^2 - 4a^2c^4d^2 - 4b^2c^4d^2 + 4c^6d^2 - 6a^4d^4 - 4a^2b^2d^4 - 6b^4d^4 - 4a^2c^2d^4 - 4b^2c^2d^4 - 6c^4d^4 + 4a^2d^6 + 4b^2d^6 + 4c^2d^6 - d^8.$$

After the command

```
FullSimplify[%]
```

we obtain

$$(8) \quad \begin{aligned} & ((a - b - c - d)(a + b + c - d)(a + b - c + d)(a - b + c + d) + 16F^2) \\ & ((a + b - c - d)(a - b + c - d)(a - b - c + d)(a + b + c + d) + 16F^2) == 0. \end{aligned}$$

From (8) we get two relations. The first one

$$16F^2 = (-a + b + c + d)(a + b + c - d)(a + b - c + d)(a - b + c + d)$$

gives the Brahmagupta's relation (6).

Remark. The second relation which follows from (8) is

$$(9) \quad 16F^2 = (-a - b + c + d)(a - b + c - d)(a - b - c + d)(a + b + c + d).$$

It is easy to show that F from (9) is the (oriented) area of a non convex quadrangle with the sides a, b, c, d which is inscribed into the circle, whereas the Brahmagupta's formula (6) holds for cyclic *convex* quadrangles. We could arrive at it from the Brahmagupta's formula writing $-b$ instead of b .

In the last example we will deal with the well-known Ptolemy's formula. We won't be able to do "a computer discovery" but we will be successful in proving it.

Example 4. Let A, B, C, D be a quadrangle with lengths of sides $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$, $|BD| = e$, $|AC| = f$. The necessary and sufficient condition for the points A, B, C, D to be on a circle is, see [8]

$$(10) \quad \begin{vmatrix} 0 & a^2 & f^2 & d^2 \\ a^2 & 0 & b^2 & e^2 \\ f^2 & b^2 & 0 & c^2 \\ d^2 & e^2 & c^2 & 0 \end{vmatrix} = 0.$$

Give “a computer proof” of (10).

Solution. Let us evaluate the determinant in the equation (10).

By the command

$$\text{Det}[\{0, a^2, f^2, d^2\}, \{a^2, 0, b^2, e^2\}, \{f^2, b^2, 0, c^2\}, \{d^2, e^2, c^2, 0\}] \\ == 0$$

we get

$$a^4c^4 - 2a^2b^2c^2d^2 + b^4d^4 - 2a^2c^2e^2f^2 - 2b^2d^2e^2f^2 + e^4f^4 == 0,$$

and after

$$\text{Factor}[a^4c^4 - 2a^2b^2c^2d^2 + b^4d^4 - 2a^2c^2e^2f^2 - 2b^2d^2e^2f^2 + e^4f^4]$$

we obtain

$$(11) \quad (ac - bd - ef)(ac + bd - ef)(ac - bd + ef)(ac + bd + ef) = 0.$$

From (11) we could derive various types of Ptolemy’s formula in accordance with the order of the vertices A, B, C, D of a quadrangle on the circle. First we will try “to discover” (11) in a similar way we did it in previous examples. Suppose we have chosen the same coordinate system as in the Example 3. We put

$$\text{Eliminate}[\{x^2 + y^2 == r^2, u^2 + v^2 == r^2, z^2 + w^2 == r^2, (x - r)^2 + y^2 == a^2, (u - x)^2 + (v - y)^2 == b^2, (z - u)^2 + (w - v)^2 == c^2, (r - z)^2 + w^2 == d^2, (x - z)^2 + (y - w)^2 == e^2, (u - r)^2 + v^2 == f^2\}, \{x, y, u, v, z, w, r\}]$$

and get

$$a^4b^2c^2 + a^2(-b^4d^2 - c^4d^2 - 2b^2c^2e^2 + 2b^2d^2e^2 + 2c^2d^2e^2 - d^2e^4) == b^2c^2(-d^4 + 2d^2e^2 - e^4) \&\& e^2(b^2 + c^2 - e^2)f^2 == -a^2b^2c^2 + a^2c^4 + b^4d^2 - b^2c^2d^2 - a^2c^2e^2 + 2b^2c^2e^2 - b^2d^2e^2 \&\& e^2(a^2 + d^2 - e^2)f^2 == a^4c^2 - a^2b^2d^2 - a^2c^2d^2 + b^2d^4 - a^2c^2e^2 + 2a^2d^2e^2 - b^2d^2e^2 \&\& (a^2b^2 - c^2d^2 + c^2e^2 + d^2e^2 - e^4)f^2 == a^2c^4 - a^2c^2d^2 - b^2c^2d^2 + b^2d^4 - a^2c^2e^2 + b^2c^2e^2 + a^2d^2e^2 - b^2d^2e^2 \&\& (-a^2c^2 - b^2d^2 + 2c^2d^2 - c^2e^2 - d^2e^2)f^2 + e^2f^4 == -a^2c^4 + a^2c^2d^2 + b^2c^2d^2 - b^2d^4.$$

We obtained a Groebner basis of the elimination ideal $I \cap R[a, b, c, d, e, f]$ but the relation (11) is not involved in it. To prove (11) we will try to find out whether the polynomial given by (11) belongs to the ideal $I \cap R[a, b, c, d, e, f]$. It suffices to prove that the remainder on division of the polynomial from (11) by the elements of a Groebner basis of $I \cap R[a, b, c, d, e, f]$ is zero, see [4]. The remainder is often called normal form. Since the command `normalf` for evaluating of the remainder is not present in Mathematica (but is available in Maple) we will use the command `PolynomialReduce` instead. The syntax of this command is as follows:

```
In[1]:=PolynomialReduce[f,polylist,varlist,options]
```

This command computes the quotients and remainder of f on division by the polynomials in `polylist` using monomial order specified by `varlist` and `MonomialOrder` in option. If we do not type this option, Mathematica will use default order, which is `Lexicographic`. The output is a list of two entries: the first is the list of quotients and the second the remainder. We type

```
PolynomialReduce[a^4c^4 - 2a^2b^2c^2d^2 + b^4d^4 - 2a^2c^2e^2f^2 - 2b^2
d^2e^2f^2 + e^4f^4, a^4b^2c^2 + a^2(-b^4d^2 - c^4d^2 - 2b^2c^2e^2 + 2b^2
d^2e^2 + 2c^2d^2e^2 - d^2e^4) - (b^2c^2(-d^4 + 2d^2e^2 - e^4)), e^2(b^2 + c
^2 - e^2)f^2 - (-a^2b^2c^2 + a^2c^4 + b^4d^2 - b^2c^2d^2 - a^2c^2e^2 +
2b^2c^2e^2 - b^2d^2e^2), e^2(a^2 + d^2 - e^2)f^2 - (a^4c^2 - a^2b^2d^2 - a
^2c^2d^2 + b^2d^4 - a^2c^2e^2 + 2a^2d^2e^2 - b^2d^2e^2), (a^2b^2 - c^2d^2 +
c^2e^2 + d^2e^2 - e^4)f^2 - (a^2c^4 - a^2c^2d^2 - b^2c^2d^2 + b^2d^4 - a
^2c^2e^2 + b^2c^2e^2 + a^2d^2e^2 - b^2d^2e^2), (-a^2c^2 - b^2d^2 + 2c^2d
^2 - c^2e^2 - d^2e^2)f^2 + e^2f^4 - (-a^2c^4 + a^2c^2d^2 + b^2c^2d^2 - b
^2d^4)}, {a, b, c, d, e, f}]
```

and the result

$$\{\{0, -d^2, -c^2, 0, e^2\}, 0\}$$

immediately appears. It means that the remainder is zero and (11) holds. In addition we know that the polynomials needed for multiplying of the elements of a Groebner basis above to arrive at the Ptolemy's formula (11) are $0, -d^2, -c^2, 0, e^2$.

Remark. Using the same method to investigate properties of planar n -gons for $n > 4$ fails for the present. To improve the process of elimination perhaps it could be helpful to use special monomial orders of r -elimination type [1].

References

- [1] BAYER, D., STILLMAN, M., A theorem on refining division orders by the reverse lexicographic order, *Duke J. Math.* **55** (1987), 321–328.
- [2] BECKER, TH., WEISPFEIMING, V., KREDEL, H., *Groebner Bases: a Computational Approach to Commutative Algebra*, Graduate Text in Mathematics, Vol. **141** Springer-Verlag, New York, 1993.

- [3] BUCHBERGER, B., Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems, *Aegn. Math. H.* (1970), 374–383.
- [4] COX, D., LITTLE, J., O'SHEA, D., *Ideals, varieties and algorithms*, Second Edition, Springer-Verlag, 1997.
- [5] DÖRRIE, B. H., *Triumph der Mathematik*, Breslau 1933.
- [6] HECK, A., A Bird's-Eye View of GrBases,
In: <http://www.can.nl/~heck/AIHENP96/AIHENP96.html>
- [7] HORA, J., *O některých otázkách souvisejících s využíváním programu počítačové algebry ve škole, II. díl*, Pedagogické centrum, Plzeň, 1998.
- [8] KOWALEWSKI, G., *Einführung in die Determinantentheorie*, Veit & Comp., Leipzig, 1909.
- [9] PETHŐ, A., *Algebraische Algorithmen*, Vieweg, Braunschweig, Wiesbaden, 1999.
- [10] ROBBINS, D. P., Areas of polygons inscribed in a circle, *Discrete Comput. Geom.* **12** (1994), 223–236.
- [11] STAUDT, CH. R., Über die Inhalte der Polygone und Polyeder, *Journal für die reine und angewandte Mathematik* **24** (1842), 252–256.

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