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ON SEPARATELY CONTINUOUS FUNCTIONS $f: \ell^2 \to \mathbf{R}$

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Abstract. In this paper the notions of separately continuous and strongly separately continuous functions $f: l^2 \to \mathbf{R}$ are introduced and properties of such functions are investigated. The obtained results are compared with the corresponding known results for functions defined on $\mathbf{R}^m$ $(m \geq 2)$. It is shown that there are several interesting and essential differences between properties of (strongly) separately continuous functions defined on ℓ^2 and properties of (strongly) separately continuous functions defined on $\mathbf{R}^m$.

Introduction

Separately continuous functions $f: \mathbf{R}^m \to \mathbf{R}$ were investigated in several papers (see e.g. [2], [4], [8], [11]). Recall that a function $f: \mathbf{R}^m \to \mathbf{R}$ is said to be *separately continuous at a point* $x_0 = (x_1^0, \dots, x_m^0) \in \mathbf{R}^m$ provided that for each $k = 1, 2, \dots, m$ the function $\varphi_k: \mathbf{R} \to \mathbf{R}$ defined by $\varphi_k(t) = f(x_1^0, \dots, x_{k-1}^0, t, x_{k+1}^0, \dots, x_m^0)$ is continuous at x_k^0. It is well known that a function can be separately continuous at x^0 without being continuous at x^0. The standard example illustrating this phenomenon is the function $f: \mathbf{R}^2 \to \mathbf{R}$ given by $f(x_1, x_2) = 0$ if $x_1 \cdot x_2 \neq 0$, while $f(x_1, x_2) = 1$ if $x_1 \cdot x_2 = 0$. This function is separately continuous at $(0,0)$ without being continuous at $(0,0)$. On the other hand, if a function $f: \mathbf{R}^m \to \mathbf{R}$ is continuous at x^0 then it is separately continuous at x^0 as well.

In the paper [4] the author introduced the notion of strongly separately continuous function $f: \mathbf{R}^m \to \mathbf{R}$ at x^0 and obtained the following result: A function $f: \mathbf{R}^m \to \mathbf{R}$ is continuous at a point x^0 if and only if it is strongly separately continuous at x^0 (see [4; Theorem 2.1])

In this paper we extend the notions of separately continuous function and strongly separately continuous function to the functions defined defined on the space ℓ^2 and prove several basic results about functions. We show that there are essential differences between some properties of (strongly) separately continuous functions $f: \mathbf{R}^m \to \mathbf{R}$ and the corresponding properties of functions $f: \ell^2 \to \mathbf{R}$.

The paper consists of three sections. In the first section we introduce the notions of separately and strongly separately continuous function for the functions $f: \ell^2 \to \mathbf{R}$ and prove some basic results. In the second section we will investigate some properties of limit functions with respect to pointwise and weakly locally uniform convergence of sequences of (strongly) separately continuous functions $f: \ell^2 \to \mathbf{R}$ and also with respect to pointwise convergence of transfinite sequences of (strongly) separately continuous functions $f\colon \ell^2 \to \mathbf{R}$. In the third section we will study determining sets for the class of (strongly) separately continuous functions on ℓ^2.

In this paper we, as usually, denote by ℓ^2 the metric space consisting of all sequences $x = (x_j)_{j=1}^{\infty}$ of real numbers such that $\sum\limits_{k=1}^{\infty} x_k^2 < +\infty$ endowed with the metric ϱ defined by

$$\varrho(x,y) = \sqrt{\sum_{k=1}^{\infty}(x_k - y_k)^2}$$

for all $x, y \in \ell^2$.

If $x^0 \in \ell^2$ and $\delta > 0$, then $B(x^0,\delta)$ denotes the set $\{x \in \ell^2 : \varrho(x^0, x) < \delta\}$.

1. Separately and strongly separately continuous functions

The definitions of separate and strong separate continuity of functions $f\colon \mathbf{R}^m \to \mathbf{R}$ can be in a natural way extended to the case of functions $f\colon \ell^2 \to \mathbf{R}$.

Definition 1.1.

(a) A function $f\colon \ell^2 \to \mathbf{R}$ is said to be *separately continuous at a point* $x^0 = (x_j^0)_{j=1}^{\infty} \in \ell^2$ *with respect to a variable* x_k provided that the function $\varphi_k\colon \mathbf{R} \to \mathbf{R}$ defined by $\varphi_k(t) = f(x_1^0, \dots, x_{k-1}^0, t, x_{k+1}^0, \dots)$ is continuous at x_k^0. If f is separately continuous at x^0 with respect to x_k for all $k \in \mathbf{N}$, then f is said to be *separately continuous at* x^0. If f is separately continuous at every point $x^0 \in \ell^2$, then f is said to be *separately continuous on* ℓ^2.

(b) A function $f\colon \ell^2 \to \mathbf{R}$ is said to be *strongly separately continuous at a point* $x^0 = (x_j^0)_{j=1}^{\infty} \in \ell^2$ *with respect to a variable* x_k provided that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ holds for each $x = (x_j)_{j=1}^{\infty} \in B(x^0,\delta)$, and $x' = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots)$. If f is strongly separately continuous at x^0 with respect to x_k for all $k \in \mathbf{N}$, then f is said to be *strongly separately continuous at* x^0. The function $f\colon \ell^2 \to \mathbf{R}$ is said to be *strongly separately continuous on* ℓ^2 provided that it is strongly separately continuous at every $x^0 \in \ell^2$.

Remark. Observe that in Definition 1.1 (b) $\varrho(x^0, x') \le \varrho(x^0, x)$. Hence, if $x \in B(x^0,\delta)$, then $x' \in B(x^0,\delta)$ as well. It is also obvious that a function $f\colon \ell^2 \to \mathbf{R}$ is strongly separately continuous at $x^0 = (x_j^0)_{j=1}^{\infty}$ with respect to x_k if only if for any sequence $(x^{(n)})_{n=1}^{\infty}$ in ℓ^2 which converges to x^0 we obtain that $\lim\limits_{n\to\infty}(f(x^{(n)}) - f(x^{(n)'})) = 0$, where $x^{(n)} = (x_j^{(n)})_{j=1}^{\infty}$ and $x^{(n)'} = (x_1^{(n)}, \dots, x_{k-1}^{(n)}, x_k^0, x_{k+1}^{(n)}, \dots)$ for all $n \in \mathbf{N}$.

From the above definition it follows the following:

Proposition 1.2.

(a) If a function $f:\ell^2 \to \mathbf{R}$ is continuous at x^0, then f is strongly separately continuous at x^0.

(b) If a function $f:\ell^2 \to \mathbf{R}$ is strongly separately continuous at x^0, then f is separately continuous at x^0.

Proof. (a) Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in ℓ^2 which converges to x^0, $x^{(n)} = (x_j^{(n)})_{j=1}^{\infty}$. Then, obviously, $\lim\limits_{n\to\infty} f(x^{(n)}) = f(x^0)$. Let $k \in \mathbf{N}$. For every $n \in \mathbf{N}$ put $x^{(n)'} = (x_1^{(n)},\ldots,x_{k-1}^{(n)},x_k^0,x_{k+1}^{(n)},\ldots)$. Since $\varrho(x^{(n)'},x^0) \le \varrho(x^0,x^{(n)})$ for all $n \in \mathbf{N}$ we obtain that $\lim\limits_{n\to\infty} x^{(n)'} = x^0$ and it follows that $\lim\limits_{n\to\infty} f(x^{(n)'}) = f(x^0)$. Hence, $\lim\limits_{n\to\infty}(f(x^{(n)})-f(x^{(n)'})) = 0$ and this yields that f is strongly separately continuous at x^0 with respect to x_k for arbitrary $k \in \mathbf{N}$.
(b) Similarly to (a).

In the paper [4] the following result was proved.

Theorem A. *A function $f:\mathbf{R}^m \to \mathbf{R}$ is continuous at x^0 if and only if f is strongly separately continuous at x^0.*

In the case of functions $f:\ell^2 \to \mathbf{R}$ only the implication presented in Proposition 1.2 (a) is valid and we show that there exist strongly separately continuous functions $f:\ell^2 \to \mathbf{R}$ (on ℓ^2) which are discontinuous at every point of the space ℓ^2. For defining such functions the following notion seems to be useful. A subset $\mathcal{S}$ is said to be a set of type (P_1) provided the following holds: If $x = (x_j)_{j=1}^{\infty} \in \mathcal{S}$, $y = (y_j)_{j=1}^{\infty} \in \ell^2$ and the set $\{j \in \mathbf{N}; x_j \ne y_j\}$ contains at most one element, then $y \in \mathcal{S}$. Next we present some examples of subsets $\mathcal{S} \subseteq \ell^2$ such that $\mathcal{S}$ is a set of type(P_1) and $\mathcal{S}$ as well as $\ell^2 \setminus \mathcal{S}$ are dense in ℓ^2.

Example 1.3.

(a) $\mathcal{S} = \{x = (x_j)_{j=1}^{\infty} \in \ell^2 : j \in \mathbf{N};\quad x_j$ is a rational (irrational, algebraic, transcendent) number$\}$ is a finite set (see [14]).

(b) $\mathcal{S}' = \left\{ x = (x_j)_{j=1}^{\infty} \in \ell^2 : \sum\limits_{j=1}^{\infty} x_j < +\infty \right\}$

Theorem 1.4. *There exists a function $g:\ell^2 \to \mathbf{R}$ such that g is strongly separately continuous on ℓ^2 and g is discontinuous at every point of ℓ^2.*

Proof. Let $\mathcal{S} \subseteq \ell^2$ be a set of type (P_1) such that $\mathcal{S}$ and $\ell^2 \setminus \mathcal{S}$ are dense in ℓ^2 (we can take some of the sets from Examples 1.3). Let $c \in \mathbf{R}$, $c \ne 0$. Define a function $g:\ell^2 \to \mathbf{R}$ by $g(x) = c$ for all $x \in \mathcal{S}$ and $g(x) = 0$ otherwise. If $x^0 \in \ell^2$, then for every neighbourhood U of x^0 we have $U \cap \mathcal{S} \neq \emptyset$, $U \cap (\ell^2 \setminus \mathcal{S}) \neq \emptyset$, and this yields that g is discontinuous at x^0. On the other hand, let $k \in \mathbf{N}$ and $x^0 = (x_j^0)_{j=1}^{\infty}$, $x = (x_j)_{j=1}^{\infty}$, $x' = (x'_j)_{j=1}^{\infty}$ be arbitrary points of ℓ^2 such that for all $j \neq k, x_j = x'_j$ and $x_k^0 = x'_k$. It is obvious that if $x \in \mathcal{S}$, then also $x' \in \mathcal{S}$ and if $x \notin \mathcal{S}$, then also $x' \notin \mathcal{S}$. Hence we always obtain $|g(x) - g(x')| = 0$ so that for each $x^0 \in \ell^2$ and each $k \in \mathbf{N}$ the function g is strongly separately continuous at x^0 with respect to x_k.

Remark. While all separately continuous functions $f\colon \mathbf{R}^m \to \mathbf{R}$ belong to the first Baire class $\mathcal{B}_1$, Theorem 1.4 shows that neither strongly separately continuous nor separately continuous functions $f\colon \ell^2 \to \mathbf{R}$ have this property. The function $g\colon \ell^2 \to \mathbf{R}$ defined in the proof of Theorem 1.4 does not belong to $\mathcal{B}_1$ because the set of all discontinuity points of g is a set of the second Baire category.

We close this section with two examples. The function $f\colon \ell^2 \to \mathbf{R}$ define by $f(x_1, x_2, \ldots) = 1$ if $\sum\limits_{k=1}^{\infty} x_k^2 \in \mathbf{Q}$, $\mathbf{Q}$ being the set of all rationals, and $f(x_1, x_2, \ldots) = 0$ otherwise is an example of a function which is nowhere separately continuous. The function $g\colon \ell^2 \to \mathbf{R}$ given by $g(x_1, x_2, \ldots) = 0$ if $x_1 \cdot x_2 \neq 0$ while $g(x_1, x_2, \ldots) = 1$ in the opposite case is separately continuous at $(0, 0, \ldots)$ without being strongly separately continuous at this point.

2. Limit functions of sequences of separately continuous functions $f\colon \ell^2 \to \mathbf{R}$

If a sequence $(f_n\colon \ell^2 \to \mathbf{R})_{n=1}^{\infty}$ converges pointwise to a function $f\colon \ell^2 \to \mathbf{R}$ and all f_n are (strongly) separately continuous, then the function f need not be separately continuous.

Theorem 2.1. *There exists a sequence $(f_n\colon \ell^2 \to \mathbf{R})_{n=1}^{\infty}$ of functions each of which is continuous on ℓ^2 such that it converges pointwise to a function $f\colon \ell^2 \to \mathbf{R}$ which is not separately continuous on ℓ^2.*

Proof. For each $n \in \mathbf{N}$ define a function $g_n\colon \mathbf{R} \to \mathbf{R}$ by $g_n(x) = \sin \frac{1}{x}$ for all $x \in \langle \frac{1}{(n+1)\pi}, \frac{1}{\pi} \rangle$ and $g_n(x) = 0$ otherwise. It is clear that all g_n are continuous functions on $\mathbf{R}$ and the sequence $(g_n)_{n=1}^{\infty}$ converges pointwise to the function $g\colon \mathbf{R} \to \mathbf{R}$ given by $g(x) = \sin \frac{1}{x}$ for all $x \in (0, \frac{1}{\pi}\rangle$ and $g(x) = 0$ otherwise. Obviously, g is discontinuous at 0. For each $n \in \mathbf{N}$ define a function $f_n\colon \ell^2 \to \mathbf{R}$ by $f_n(x_1, x_2, \ldots) = g_n(x_1)$ and let $f\colon \ell^2 \to \mathbf{R}$ be the function given by $f(x_1, x_2, \ldots) = g(x_1)$. It is evident that for all $n \in \mathbf{N}$, f_n is a continuous function on ℓ^2 ($f_n = g_n \circ p_1$, where $p_1\colon \ell^2 \to \mathbf{R}$ is the first projection) and f is not separately continuous at the point $(0, 0, \ldots)$ with respect to x_1. Clearly, the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise to f.

It is natural to ask whether some of various types of convergence of functions which are stronger than the pointwise convergence can guarantee that the limit function of a sequence of (strongly) separately continuous functions on ℓ^2 with respect to this type of convergence is also a (strongly) separately continuous function on ℓ^2. Next we show that there is a weaker type of locally uniform convergence (see [14], [5; p. 149]) which fulfills this requirement in the case of strongly separately continuous functions on ℓ^2.

Definition 2.2. Let X be a topological space, $(f_n\colon X\to\mathbf{R})_{n=1}^{\infty}$ be a sequence of functions and $x^0\in X$. A sequence $(f_n)_{n=1}^{\infty}$ is said *to converge weakly locally uniformly to a function* $f\colon X\to\mathbf{R}$ *at* x^0 if for every $\varepsilon>0$ there exist $\delta>0$ and $p\in\mathbf{N}$ such that $|f_n(x)-f(x)|<\varepsilon$ holds for each $n\in\mathbf{N}$ with $n\geq p$ and each $x\in B(x^0,\delta)$.

If a sequence $(f_n)_{n=1}^{\infty}$ converges weakly locally uniformly to a function f at every point $x^0\in X$, then it is said *to converge weakly locally uniformly to* f *on* X.

Theorem 2.3. *If a sequence* $(f_n\colon\ell^2\to\mathbf{R})_{n=1}^{\infty}$ *converges weakly locally uniformly to* $f\colon\ell^2\to\mathbf{R}$ *at* $x^0\in\ell^2$ *and for each* $n\in\mathbf{N}$ *the function* f_n *is strongly separately continuous at* x^0*, then the function* f *is also strongly separately continuous at* x^0.

Proof. Let $k\in\mathbf{N}$. We will prove that f is strongly separately continuous at x^0 with respect to x_k. Let $\varepsilon>0$. Since $(f_n)_{n=1}^{\infty}$ converges weakly locally uniformly to f at x^0 there exist an open ball $B(x^0,\delta_1)$ and $p\in\mathbf{N}$ such that $|f_n(x)-f(x)|<\frac{\varepsilon}{3}$ holds for all $n\geq p$ and $x\in B(x^0,\delta_1)$. The function f_p is strongly separately continuous at x^0 with respect to x_k and it follows that there exists $\delta_2>0$ such that $|f_p(x)-f_p(x^{'})|<\frac{\varepsilon}{3}$ holds for each $x=(x_j)_{j=1}^{\infty}\in B(x^0,\delta_2)$ and $x^{'}=(x_1,\ldots,x_{k-1},x_k^0,x_{k+1},\ldots)$. Put $\delta=\min\{\delta_1,\delta_2\}$. Then for each $x\in B(x^0,\delta)$ we obtain that $|f_p(x)-f_p(x^{'})|<\frac{\varepsilon}{3}$, $|f_p(x)-f(x)|<\frac{\varepsilon}{3}$ and because $\varrho(x^{'},x^0)\leq\varrho(x^0,x)<\delta$ we have also $|f_p(x^{'})-f(x^{'})|<\frac{\varepsilon}{3}$. Hence, for all $x\in B(x^0,\delta)$ we obtain $|f(x)-f(x^{'})|\leq|f(x)-f_p(x)|+|f_p(x)-f_p(x^{'})|+|f_p(x^{'})-f(x^{'})|<\varepsilon$ and this yields that f is strongly separately continuous at x^0 with respect to x_k.

In the rest of this section we will investigate some properties of limit functions of convergent transfinite sequences of (strongly) separately continuous functions. Recall that a transfinite sequence $(x_\xi)_{\xi<\Omega}$ (Ω is the first uncountable ordinal) in a metric space (X,σ) converges to a point $x\in X$ (we write $x_\xi\to x$) if for every $\varepsilon>0$ there exists $\xi_0<\Omega$ such that $\sigma(x_\xi,x)<\varepsilon$ holds for each ξ, $\xi_0\leq\xi<\Omega$. It is well known (see e.g. [9]) that if $x_\xi\to x$ in a metric space (X,σ), then there exists $\xi_0<\Omega$ such that $x_\xi=x$ holds for each $\xi\geq\xi_0$. A transfinite sequence $(f_\xi\colon M\to\mathbf{R})_{\xi<\Omega}$ of functions, M is a set, converges pointwise to a function $f\colon M\to\mathbf{R}$ (we write $f_\xi\to f$) on M, if for each $x\in M$ we have $f_\xi(x)\to f(x)$ in $\mathbf{R}$. In the next theorem we show that the pointwise convergence of transfinite sequences of functions preserves (strong) separate continuity.

Theorem 2.4. *Let $(f_\xi\colon \ell^2 \to \mathbf{R})_{\xi<\Omega}$ be a transfinite sequence of functions which converges pointwise to a function $f\colon \ell^2 \to \mathbf{R}$ on ℓ^2. If for all $\xi < \Omega$ the function f_ξ is (strongly) separately continuous at x^0, then the function f is also (strongly) separately continuous at x^0.*

Proof. Let for each $\xi < \Omega$ the function f_ξ be strongly separately continuous at x^0 with respect to x_k. We show that f is strongly separately continuous at x^0 with respect to x_k. Let $(x^{(n)})_{n=1}^\infty$ be a sequence in ℓ^2 which converges to x^0, $x^{(n)} = (x_j^{(n)})_{j=1}^\infty$. For each $n \in \mathbf{N}$ put $x^{(n)'} = (x_1^{(n)}, \dots, x_{k-1}^{(n)}, x_k^0, x_{k+1}^{(n)}, \dots)$. It suffices to check that $\lim\limits_{n\to\infty}(f(x^{(n)}) - f(x^{(n)'})) = 0$. Let $n \in \mathbf{N}$. For every $\xi < \Omega$ we have $\lim\limits_{n\to\infty}(f_\xi(x^{(n)}) - f_\xi(x^{(n)'})) = 0$. Since $f_\xi \to f$ on ℓ^2 we obtain $f_\xi(x^{(n)}) \to f(x^{(n)})$ and $f_\xi(x^{(n)'}) \to f(x^{(n)'})$. Then there exists $\xi_n < \Omega$ such that $f_\xi(x^{(n)}) = f(x^{(n)})$ and $f_\xi(x^{(n)'}) = f(x^{(n)'})$ holds for all $\xi \geq \xi_n$. We can choose $\xi_0 < \Omega$ such that for all $n \in \mathbf{N}$ we have $\xi_n \leq \xi_0$. Then for all $n \in \mathbf{N}$ $f_{\xi_0}(x^{(n)}) = f(x^{(n)})$ and $f_{\xi_0}(x^{(n)'}) = f(x^{(n)'})$. Clearly, $\lim\limits_{n\to\infty}(f(x^{(n)}) - f(x^{(n)'})) = \lim\limits_{n\to\infty}(f_{\xi_0}(x^{(n)}) - f_{\xi_0}(x^{(n)'})) = 0$. Hence, the function f is strongly separately continuous at x^0 with respect to x_k. The case of separate continuity immediately follows from the known fact that a limit of a transfinite sequence$(f_\xi\colon \mathbf{R} \to \mathbf{R})_{\xi<\Omega}$ of continuous functions is a continuous function (see e. g. [10], [9]).

3. Determining sets for separately continuous functions $f\colon \ell^2 \to \mathbf{R}$

If $\mathcal{F}$ is a class of (real) functions defined on a set X and $M \subseteq X$, then the set M is said to be a determining set for $\mathcal{F}$ provided that any functions $f, g \in \mathcal{F}$ satisfying $f|_M = g|_M$ are coincidental on X. For the class $\mathcal{G}$ of all separately continuous function of two variables the following result was proved (see [13], [11], [8]).

Theorem B. *Let $\mathcal{G}$ be the class of all separately continuous functions defined on $\mathbf{R}^2$. Then a set $M \subseteq \mathbf{R}^2$ is a determining set for the class $\mathcal{G}$ if and only if M is dense in $\mathbf{R}^2$.*

Obviously, this result can be extended to the class of all separately continuous functions defined on $\mathbf{R}^m$,$m \geq 2$. On the other hand, from Theorem 1.4 it follows that there exist dense subsets of the space ℓ^2, e. g. $\mathcal{S}, \ell^2 \setminus \mathcal{S}, \mathcal{S}', \ell^2 \setminus \mathcal{S}'$ where $\mathcal{S}, \mathcal{S}'$ are presented in Example 1.3, that are not determining sets for the class of all (strongly) separately continuous functions on ℓ^2. Another example is given in the next theorem.

Theorem 3.1. *There exists a strongly separately continuous function $h:\ell^2 \to \mathbf{R}$ and a residual (and, consequently, dense) set E in ℓ^2 such that $h(x) = 0$ for all $x \in E$ and $h(y) \neq 0$ for some $y \in \ell^2 \setminus E$.*

Proof. Denote by H the set of all $x = (x_j)_{j=1}^{\infty} \in \ell^2$ for which $\sum\limits_{j=1}^{\infty} x_j$ converges. Put E $= \ell^2 \setminus$ H and define $h:\ell^2 \to \mathbf{R}$ by $h(x) = \sum\limits_{j=1}^{\infty} x_j$ for all $x \in$ H and $h(x) = 0$ otherwise. According to [7; Theorem 3.1.] (it suffices to put $\alpha_n = 1$ for all $n = 1, 2, \dots$ and $p = q = 2$) the set E is residual in ℓ^2. To complete the proof it suffices to show that h is strongly separately continuous on ℓ^2. Let $x^0 = (x_j^0)_{j=1}^{\infty} \in \ell^2$ and $k \in \mathbf{N}$. We show that h is strongly separately continuous at x^0 with respect to x_k. Let $\varepsilon > 0$. If $x = (x_j)_{j=1}^{\infty} \in B(x^0, \varepsilon)$, then also $x' = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots) \in B(x^0, \varepsilon)$. If $x \in$ H and $h(x) = \sum\limits_{j=1}^{\infty} x_j$, then $|h(x) - h(x')| = |x_k - x_k^0| \leq \varrho(x, x^0) < \varepsilon$. If $x \notin$ H, then $h(x) = h(x') = 0$ and we have $|h(x) - h(x')| = 0 < \varepsilon$. This yields that h is strongly separately continuous at x^0 with respect to x_k.

In connection with determining sets for strongly separately continuous functions on ℓ^2 the following observation seems to be useful. Let M be a subset of ℓ^2 and $\widetilde{M}$ is the set of all $y = (y_j)_{j=1}^{\infty} \in \ell^2$ such that there exists $x = (x_j)_{j=1}^{\infty} \in M$ for which the set $\{j \in \mathbf{N} : x_j \neq y_j\}$ is finite. It is obvious, that $M \subseteq \widetilde{M}, \widetilde{\widetilde{M}} = \widetilde{M}$ and $\widetilde{M}$ is a set of type (P_1). Similarly to the proof of Theorem 1.4 it can be checked that for any subset $M \subseteq \ell^2$ the function $g:\ell^2 \to \mathbf{R}$ given by $g(x) = 0$ for all $x \in \widetilde{M}$ and $g(x) = 1$ otherwise is strongly separately continuous. Hence, we obtain:

Proposition 3.2. *If M is a subset of ℓ^2 such that $\widetilde{M} \neq \ell^2$, then M is not a determining set for the class of all (strongly) separately continuous functions on ℓ^2.*

It is easy to see that if $M \subseteq \ell^2$ and card $M < \mathbf{c}$, $\mathbf{c}$ being the cardinality of continuum, then $\widetilde{M} \neq \ell^2$ (evidently, there exists $y = (y_j)_{j=1}^{\infty} \in \ell^2$ such that for each $x = (x_j)_{j=1}^{\infty} \in M$, $\{j \in \mathbf{N} : x_j = y_j\} = \emptyset$). Hence, as a consequence of Proposition 3.2 we obtain.

Proposition 3.3. *If $M \subseteq \ell^2$ is a determining set for the class of all (strongly) separately continuous functions on ℓ^2, then card $M = \mathbf{c}$.*

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