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STUDYING THE MULTICHROMATIC NUMBER OF THE ALMOST s-STABLE KNESER GRAPHS

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Abstract
In the early 1970's Gilbert introduced n-tuple colorings of graphs motivated by practical problems. After this Saul Stahl studied the properties of these colorings and formulated the conjecture on the multichromatic number of the Kneser graphs. Motivated by Stahl's conjecture we will investigate the multichromatic number of the almost s-stable Kneser graphs.

1 Introduction

In 1978, László Lovász [6] proved the famous Kneser Conjecture -concerning the chromatic number of the Kneser graphs $KG_{m,n}$- introducing the neighborhood complex. In the same year, Alexander Schrijver [11] defined certain induced subgraphs of $KG_{m,n}$ - called the stable Kneser graphs $SG_{m,n}$ - and showed that they are vertex-critical. Schrijver used another method, Bárány's method to obtain the chromatic number of the stable Kneser graphs. Frédéric Meunier [7] generalized Schrijver's construction and formulated the conjecture on the chromatic number of the s-stable and almost s-stable Kneser graphs. After the advances of Meunier, several papers were devoted to the study of the chromatic number of the s-stable Kneser graphs [4, 2, 14]. Recently, Peng-An Chen [1] studied the multichromatic number of the s-stable Kneser graphs. We will investigate the multichromatic number of the almost s-stable Kneser graphs.

2 The n-tuple colorings

The idea of n-tuple colorings was introduced by Gilbert [3] in connection with the mobile radio frequency assignment problem. Other applications of n-tuple colorings including fleet maintenance, task assignment, and traffic phasing are discussed in [9]. The graph-theoretical formulation of these problems is the following: there is a graph G, and there is an assignment on G which assigns a set of n colors to each vertex of G so that the sets of colors assigned to adjacent vertices are disjoint. It is easy to see that this is a generalization of the usual coloring of G.

In this section we recall some basic facts on sets, graphs and chromatic number to fix notation.

2.1 The Kneser graphs

Hereafter, the symbol $[m]$ stands for the set $\{1,\dots,m\}$. For positive integers $m \geq n$, let $2^{[m]}$ denote the collection of all subsets of $[m]$ and let $\binom{[m]}{n}$ denote the collection of all n-subsets of $[m]$. For an integer $s \geq 2$ a subset $S \subseteq [m]$ is s-*stable* if any two of its elements are at least "at distance s apart" on the m-cycle, that is $s \leq |i-j| \leq m-s$ for distinct $i,j \in S$. A subset $S \subseteq [m]$ is *almost*

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s-*stable*, if $s \le |i-j|$ for distinct $i, j \in S$. Hereafter, the symbols $\binom{[m]}{n}_s$, $\binom{[m]}{n}_{s\sim}$ stand for the collection of all s-stable n-subsets of $[m]$ and the collection of all almost s-stable n-subsets of $[m]$, respectively.

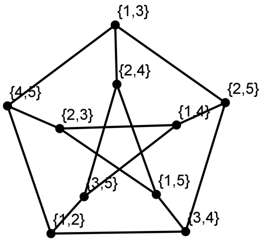

Figure 1. The Kneser graph $KG_{5,2}$.

Let $\mathcal{F} \subseteq 2^{[m]}$ be a system of sets. The *Kneser graph of $\mathcal{F}$* has $\mathcal{F}$ as the vertex set, and two sets $A, B \in \mathcal{F}$ are adjacent iff $A \cap B = \emptyset$. Let $KG_{m,n}$ denote the Kneser graph of $\binom{[m]}{n}$. The *stable* (or 2-*stable*) *Kneser graph* $SG_{m,n}$ was obtained by restricting the vertex set of $KG_{m,n}$ to the 2-stable n-subsets, that is, the Kneser graph of $\binom{[m]}{n}_2$. The s-*stable Kneser graph*, denoted as $SG^s_{m,n}$, is the Kneser graph of $\binom{[m]}{n}_s$ for positive integers $n, s \ge 2$ and $m \ge sn$. The *almost s-stable Kneser graph*, denoted as $SG^{s\sim}_{m,n}$, is the Kneser graph of $\binom{[m]}{n}_{s\sim}$ for positive integers $n, s \ge 2$ and $m \ge s(n-1)+2$.

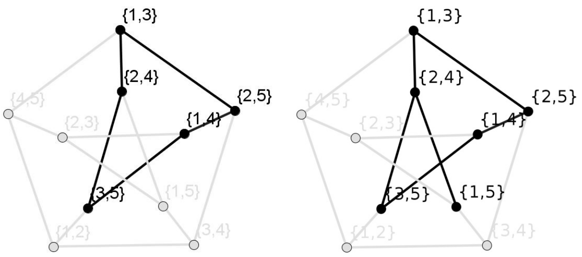

Figure 2. The 2-stable Kneser graph $SG^2_{5,2}$, and $SG^{2\sim}_{5,2}$.

2.2 The multichromatic numbers

For any positive integers m and n, an n-*tuple coloring* of a graph G with m colors is an assignment of n distinct colors to each vertex of G in such a way that no two adjacent vertices share a color. Equivalently, such a coloring is a graph homomorphism from G to $KG_{m,n}$. $\chi_n(G)$, the n*th multichromatic number* of G, is the least integer m such that G has n-tuple coloring with m colors.

Stahl studied these multichromatic numbers and proved the following general properties of χ_n.

Theorem 1 (Stahl [12])**.** *For all positive integers n, q, r and simple graph G*

$$\chi_{nq+r}(G) \le q\chi_n(G) + \chi_r(G). \tag{1}$$

Theorem 2 (Stahl [12])**.** *If G has an edge, then $\chi_n(G) \ge 2 + \chi_{n-1}(G)$ for all $n > 1$.*

Theorem 3 (Stahl [12])**.** *If there is a graph homomorphism from G to H, then*

$$\chi_n(G) \le \chi_n(H). \tag{2}$$

In this paper Stahl calculated the multichromatic numbers of various classes of graphs.

- complete n-partite graphs,
- cycles,
- $KG_{2n+1,n}$.

2.3 The multichromatic numbers of $KG_{m,n}$

The Kneser graphs $KG_{m,n}$ play the same role for the n-tuple coloring that the complete graph K_m for the conventional coloring. The multichromatic numbers of the Kneser graph $KG_{m,n}$ are trivial for $n = 1$ and Stahl computed them for $n = 2$, $n = 3$ and $m = 2n + 1$ in [12, 13]. Moreover he formulated the following conjecture.

Conjecture 4 (Stahl [12]). *If $l = nq + r$, where $0 \leq q$ and $0 < r \leq n$, then*

$$\chi_l(KG_{m,n}) = mq + m - 2n + 2r. \tag{3}$$

2.4 The multichromatic numbers of $SG^s_{m,n}$

Chen studied the multichromatic numbers of the s-stable Kneser graphs $SG^s_{m,n}$ in [1] and formulated the following conjecture.

Conjecture 5. *Let m, n, s, r positive integers such that $n, s \geq 2$, $m \geq sn$ and $n \geq r$. Then*

$$\chi_r(SG^s_{m,n}) = m - sn + sr. \tag{4}$$

Chen [1] have concluded that Conjecture 5 is true for the following cases:

- s is even,
- $r = n$,
- $m \leq sn + 1$.

3 The multichromatic number of $SG^{s\sim}_{m,n}$

In this section we will study the multichromatic numbers of the almost s-stable Kneser graphs $SG^{s\sim}_{m,n}$. Stahl proved that there is a graph homomorphism $KG_{m,n} \to KG_{m-2,n-1}$. We prove a similar proposition about the almost s-stable Kneser graphs $SG^{s\sim}_{m,n}$.

Proposition 6. *Let m, n, s positive integers such that $s \geq 2$, $n \geq 3$ and $m \geq sn$, then there exists a graph homomorphism*

$$\eta : SG^{s\sim}_{m,n} \to SG^{s\sim}_{m-s,n-1} \tag{5}$$

Proof. Let $A = \{a_1 < a_2 < \ldots < a_n\}$ a vertex of $SG^{s\sim}_{m,n}$, then

$$a_k \geq a_{k-1} + s \tag{6}$$

for $k = 2, \ldots, n$. Now we define the map η as follows:

$$\eta(A) = \{a_1, a_2, \ldots, a_{n-1}\}. \tag{7}$$

It can easily be seen that η is in fact a mapping into $V(SG^{s\sim}_{m-s,n-1})$. Now it remains to show that η is in fact a graph homomorpism of $SG^{s\sim}_{m,n}$ into $SG^{s\sim}_{m-s,n-1}$. We must show that if A and B are adjacent vertices of $SG^{s\sim}_{m,n}$, then $\eta(A)$ and $\eta(B)$ are adjacent vertices of $SG^{s\sim}_{m-s,n-1}$. In other words, if $A \cap B = \emptyset$ then $\eta(A) \cap \eta(B) = \emptyset$, as $\eta(A) \subset A$ and $\eta(B) \subset B$. □

First, we prove an upper bound for the multichromatic numbers of almost s-stable Kneser graph $SG^{s\sim}_{m,n}$:

Proposition 7. *Let m, n, s, r be positive integers such that $n, s \geq 2$, $m \geq sn$ and $n \geq r$. Then*

$$\chi_r(SG^{s\sim}_{m,n}) \leq m - sn + sr. \tag{8}$$

Proof. From the fact that the almost s-stable Kneser graph $SG^{s\sim}_{m,n}$ is a subgraph of the Kneser graph $KG_{m,n}$ we obtain an upper bound for the nth multichromatic number of $SG^{s\sim}_{m,n}$:

$$\chi_n(SG^{s\sim}_{m,n}) \leq m. \tag{9}$$

Let $1 \leq r < n$. Then we have

$$SG^{s\sim}_{m,n} \to SG^{s\sim}_{m-s,n-1} \to \cdots \to SG^{s\sim}_{m-s(n-r),r} \tag{10}$$

by Proposition 6. The almost s-stable Kneser graph $SG^{s\sim}_{m-sn+sr,r}$ is a subgraph of the Kneser graph $KG_{m-sn+sr,r}$, thus there is a graph homomorphism

$$SG^{s\sim}_{m,n} \to KG_{m-sn+sr,r}, \tag{11}$$

that is,

$$\chi_r(SG^{s\sim}_{m,n}) \leq m - sn + sr. \tag{12}$$

□

Recently, Peng-An Chen [2] and the author [10] have determined the chromatic number for all almost s-stable Kneser graph $SG^{s\sim}_{m,n}$

$$\chi_1(SG^{s\sim}_{m,n}) = m - sn + s. \tag{13}$$

The main result of this note is the weaker form of Conjecture 5.

Theorem 8. *Let m, n, s, r be positive integers such that $n, s \geq 2$, s is even, $m \geq sn$ and $n \geq r$. Then*

$$\chi_r(SG^{s\sim}_{m,n}) = m - sn + sr. \tag{14}$$

Proof. We have the upper bound for the rth multichromatic number of $SG^{s\sim}_{m,n}$

$$\chi_r(SG^{s\sim}_{m,n}) \leq m - sn + sr \tag{15}$$

by Proposition 7, now we need prove that this is the lower bound for the rth multichromatic number of $SG^{s\sim}_{m,n}$ for the case when s is even. But this is an easy consequence of the fact that the s-stable Kneser graph $SG^{s}_{m,n}$ is a subgraph of the almost s-stable Kneser graph $SG^{s\sim}_{m,n}$, so we have

$$\chi_r(SG^{s\sim}_{m,n}) \geq \chi_r(SG^{s}_{m,n}) = m - sn + sr. \tag{16}$$

This proves our statement. □

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