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Young, timid, and risk takers

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A paper in honor of Mark H.A. Davis

Abstract

Time-varying asset returns lead highly risk-averse investors to choose market-timing exposures that increase in their horizon, in agreement with the common advice to reduce risk with age, but in contrast to theoretical work that prescribes constant portfolio weights. In a market where an investor with constant absolute risk aversion and finite horizon trades an asset with temporary fluctuations, we find asymptotically optimal investment strategies that are independent of the asset's average return and decline over time with a power of the remaining horizon, with the exponent determined by the curvature of mean reversion. For long-term safe assets, which have a zero average return, the investor's certainty equivalent declines over time at a lower rate, implying that a nonzero average return is negligible for asymptotically optimal strategies but critical to their performance.

KEYWORDS

exponential utility, long-term safe assets, mean-reversion, portfolio choice

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G11, G12, 91G10, 91B16

1 INTRODUCTION

According to modern retirement planners, "as investors age, they should start cutting back on riskier investments" (Malkiel, 1999, p. 361). According to modern portfolio theory, they should not. Instead—the theory prescribes—investors' portfolios should be insensitive to their investment horizons (Markowitz, 1952; Merton, 1969).

This tension between theory and practice hinges on implicit assumptions on the statistical properties of stock returns. While theoretical results supporting fixed portfolio weights assume that stock prices have independent, identically distributed returns, financial planning advice posits mean-reversion: as Siegel (1998) puts it, "the actual risk of average stock returns declines far faster than predicted by the random walk hypothesis because of the mean reversion of equity returns." Malkiel (1999) chimes in that "A substantial amount (but not all) of the risk of common-stock investment can be eliminated by adopting a program of long-term ownership".

Although mean-reverting returns (and stochastic investment opportunities in general) yield portfolio weights that jointly depend on investors' horizons and a market's states, they also typically imply that, as the horizon increases, weights converge to a fixed limit known as the *turnpike*.[1] As a result, portfolio weights change mainly in the late stages of the investment period, but otherwise remain close to constant.

This paper offers a rather different result, focusing on an asset with mean-reverting fluctuations and on an investor with exponential utility, which represents the high risk-aversion limit of isoelastic utilities (Nutz, 2012, Theorem 3.2). In this setting, our main result (Theorems 2.1 and 2.3) demonstrates that: (i) the portfolio weight declines over time with a power of the remaining horizon; (ii) at the leading order, the trading strategy does not depend on the average asset return or mean-reversion speed, but only on mean-reversion curvature; and (iii) the resulting certainty equivalent is also proportional to a power of the horizon that depends only on mean-reversion curvature. These observations motivate the paper's title: although our investors are timid (i.e., highly risk-averse), it is optimal for them to take significant risk while young, because mean-reversion entails that early shocks subside over time.

The significance of this result is fourfold. First, it reveals the central role of mean-reversion curvature for highly risk-averse investors, both in trading strategies and their performance. This quantity has eluded so far the attention of researchers, possibly because it is absent from popular models based on the Ornstein-Uhlenbeck and Feller diffusions, in which mean-reversion is linear for the sake of tractability. By contrast, we consider a class of models with nonlinear mean-reversion that nests linearity as a special case, and find that the curvature parameter alone identifies both the asymptotically optimal strategy and its performance.

Second, as the trading strategy grows with the horizon without bounds, it is clear that no turnpike exists in the model, and that no time-homogeneous strategy can be asymptotically optimal. Put differently, although the asset price dynamics is described by a time-homogeneous and ergodic Markov process, ergodicity fails for the optimal strategy, which does not recover time-homogeneity even in the long-horizon limit. In fact, the certainty-equivalent of any time-homogeneous strategy could only be a finite, fixed annuity, while the certainty-equivalent of the optimal (and time-inhomogeneous) strategy is an annuity payment that grows with a power of time.

Third, we identify two separate regimes with different characteristics. The nonstationary regime corresponds to assets whose prices have nonzero average growth rate, such as stocks and bonds, which have a consistent historical record of exceeding inflation. (Also cash, which consistently underperforms inflation, falls into this case.) By contrast, the stationary regime corresponds to an asset, such as gold, whose price tends to grow at the inflation rate without exceeding it (Siegel, 1998, Figure 5.4). Such assets, which have considerable short term fluctuations, but no significant long-term trend in real terms, are *long-term safe* assets. In the nonstationary case, the certainty-equivalent corresponds to an annuity that grows with time raised to twice the curvature of mean-reversion. By contrast, in the stationary case of a long-term safe asset, the equivalent annuity grows with time at the power of the curvature of mean-reversion (i.e., not twice). In other words, though average growth does not enter the optimal investment strategy or its performance explicitly, it does affect both of them, insofar as the asset's growth rate is zero or not.

Fourth, this paper also offers a novel methodological approach to portfolio choice problems that do not admit an explicit solution, such as the nonlinear mean-reversion class considered here. Indeed, these models lead to Hamilton-Jacobi-Bellman equations that are impervious to typical techniques: On one hand, the finite-horizon problems cannot be tackled with a quadratic guess due to the nonlinear term. On the other hand, one cannot sidestep time-dependence by focusing on the ergodic limit—because ergodicity fails. Nevertheless, combining duality estimates with educated guesses, we are able to identify asymptotically optimal strategies in closed form.

The voluminous literature on continuous-time portfolio choice with stochastic investment opportunities begins with the work of Merton (1973) on multiple fund separation. Subsequent research has sought to bring tractability to stochastic investment opportunities, either through convex duality (Cox & Huang, 1989; He & Pearson, 1991; Karatzas et al., 1987, 1991), or stochastic control (Duffie et al., 1997; Kim & Omberg, 1996; Liu, 2007; Wachter, 2002; Zariphoupoulou, 2001).

Long-horizon asymptotics, first considered in the risk-sensitive control literature (Bielecki & Pliska, 2000; Fleming & McEneaney, 1995; Fleming & Sheu, 2002; Hata & Sekine, 2005; Kuroda & Nagai, 2002; Kaise & Sheu, 2006; Nagai & Peng, 2002; Nagai, 2003) are a powerful approach to bring tractability to problems with stochastic investment opportunities, as their focus on ergodic Hamilton-Jacobi-Bellman equations yields long-horizon limits of optimal portfolios even when finite-horizon solutions are unavailable.

Yet, the conditions under which optimal portfolios converge in the long horizon can be delicate. Guasoni et al. (2012) identify joint restrictions on market and preference parameters under which convergence holds. In a model of commodity futures with linear mean-reversion and power utility, Guasoni et al. (2019) obtain convergence only for sufficiently low relative risk aversion, raising the question of whether high risk aversion may subvert the result.

This paper shows that, in the presence of mean-reversion, high-risk aversion investors, as described by exponential utility, exhibit a qualitatively different sensitivity to the investment horizon, in comparison to power utility. Far from converging, optimal portfolios actually diverge, and do so at a speed that is determined by the curvature of mean reversion. Thus, for a given horizon the riskiness of an optimal portfolio tends to decline over time. Altogether, the results suggest that financial planners' recommendations for gradual risk reduction are most relevant for highly risk-averse investors.

The rest of the paper is organized as follows. Section 2 contains the main results, and its discussion, which is further developed in Section 3, where the special case of the Ornstein-Uhlenbeck process is solved explicitly. Section 4 offers a limit argument from discrete-time models that motivates the results in the paper. Concluding remarks are in Section 5. Proofs are in the Appendix.

2 | A NONLINEAR MEAN-REVERSION MODEL

The market includes a safe asset earning zero interest rate and a risky asset with price S_t that follows the diffusion

$$dS_t = \mu\, dt + dX_t, \qquad S_0 = 0, \tag{1}$$

$$dX_t = -\alpha\, \mathrm{sgn}(X_t)|X_t|^{\beta} dt + dB_t, \qquad X_0 = 0. \tag{2}$$

where $\beta \geq 1, \alpha > 0, \mu \in \mathbb{R}$, while $(B_t)_{t\geq 0}$ is a standard, one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ and endowed with the augmented natural filtration $(\mathcal{F}_t)_{t\geq 0}$. The stochastic differential equation (2) has a unique strong solution (see, e.g., Krylov (1999, Theorem 1.2)). Note that the model assumes, for economy of notation, that the asset has unit volatility. The general case of a volatility σ is reduced to the present one by replacing the average growth rate μ with μ/σ (cf. Section 3 below). Diffusion parameters are typically estimated either through the generalized method of moments (Hansen, 1982) or approximate maximum likelihood (Aït-Sahalia, 2002).

The case $\beta = 1$ recovers the familiar, linear mean-reversion of the Ornstein-Uhlenbeck process. Higher values of β generate nonlinear mean-reversion, in that the mean-reverting drift is not proportional to the current distance from the mean, but a power thereof. Likewise, the case $\alpha = 0$ recovers the familiar Bachelier model of a linear Brownian motion with constant drift, in which all shocks to the asset price are permanent, that is, they accumulate over time.

As this paper focuses on $\alpha > 0, \beta > 1$, the central idea of the model is that mean-reversion is imperceptible when deviations from average long-term growth are minimal ($X_t \ll 1$, hence $|X_t|^{\beta} \approx 0$), so that price dynamics resembles a random walk. By contrast, when deviations are large ($X_t \gg 1$), price shocks strongly revert to the mean, and departures from random-walk dynamics become apparent.

The deviation process X_t has a stationary distribution with density

$$\frac{e^{-\frac{\alpha}{\beta+1}|x|^{\beta+1}}}{2(\frac{\alpha}{\beta+1})^{-\frac{1}{1+\beta}}\Gamma(\frac{2+\beta}{1+\beta})}$$

where Γ denotes the usual Gamma function $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$.[2] Such a distribution is qualitatively similar to a normal ($\beta = 1$), but is flatter near the mean and has thinner tails, consistently with the observation that small deviations are virtually ignored but large deviations elicit a swift mean-reverting response.

Also, note that the asset price is itself stationary only if $\mu = 0$, whence $S_t = X_t$. In general, $S_t = X_t + \mu t$, which means that the price is the sum of a linear trend and a stationary process, but is not itself stationary. (For example, the linear mean-reversion case $\beta = 1$ leads to the trending Ornstein-Uhlenbeck process in Grundy (1991) and Lo & Wang (1995).)

Finally, observe that the model in (1) assumes an arithmetic drift, but it could be equivalently formulated in terms of a geometric drift, that is, as $dS_t/S_t = \mu\, dt + dX_t$. Indeed, both formulations lead to the same set of attainable payoffs, as the corresponding replicating strategies are in a one-to-one correspondence (that is, $\int_0^T H_t dS_t = \int_0^T \tilde{H}_t dS_t/S_t$ for $\tilde{H}_t = H_t S_t$). In particular, although prices may become negative in the arithmetic setting (1), they remain strictly positive in the equivalent geometric setting, as $S_t = S_0 e^{(\mu-\frac{1}{2})T+X_T}$.

If an investor holds H_t shares of the risky asset at time t, the overall gain (or loss) at time $T \geq 0$ is described by the stochastic integral $(H \cdot S)_T$, defined for all S-integrable processes $H_t, t \in [0,T]$. For a fixed horizon T, the investor seeks to maximize the exponential utility

$$E\left[-e^{-(H\cdot S)_T}\right] \to \max. \tag{3}$$

(For economy of notation, the discussion focuses on unit risk aversion. An arbitrary risk aversion reduces the certainty-equivalent proportionally, cf. Section 3.) Here H varies among the class of *admissible strategies*, defined as the set $\mathcal{X}_T$ of S-integrable processes H_t, $t \in [0,T]$ such that $(H \cdot S)_t, t \in [0,T]$ is a Q_T-martingale, where Q_T denotes the unique risk-neutral measure, whose existence and explicit expression is in Proposition A.1 below. ($\mathcal{X}_T$ is a natural choice for the domain of the optimization in (3), cf. Delbaen et al. (2002); Kabanov & Stricker (2002).)

Thus, the value function of the portfolio optimization problem on $[0,T]$ is

$$u_T := \sup_{H\in\mathcal{X}_T} E\left[-e^{-(H\cdot S)_T}\right],$$

for all $T > 0$. The corresponding *certainty-equivalent*, $c_T := -\ln(-u_T)$, represents the investor's opportunity cost of trading, that is, the amount of money the investor would accept as compensation for not being able to trade in the interval $[0,T]$.[3]

For a family of strategies $\mathfrak{H} := (H(T))_{T>0}$ with $H(T) \in \mathcal{X}_T$, $T > 0$, define the *order* of the certainty-equivalent as

$$C(\mathfrak{H}) := \sup\left\{\theta : \liminf_{T\to\infty} \frac{-\ln(E\left[e^{-(H(T)\cdot S)_T}\right])}{T^\theta} > 0\right\}.$$

For example, with a nonzero average return ($\mu \neq 0$) but in the absence of mean reversion ($\alpha = 0$), the certainty-equivalent c_T is proportional to T, which means that its order is one, and the investor is indifferent between trading optimally and receiving a constant annuity (as the annuity is the rate of change of the certainty-equivalent, its order equals the order of the certainty-equivalent, minus one).

The first part of the main result asserts that, when $\mu \neq 0$ and $\alpha > 0$, the optimal certainty-equivalent c_T grows at rate $T^{2\beta+1}$ as $T \to \infty$ and the intuitive strategies defined in (4) below attain such performance.

Theorem 2.1. *If $\alpha > 0$ and $\mu \neq 0$, then $c_T \leq C_{\beta,\mu,\alpha} T^{2\beta+1}$ for some constant $C_{\beta,\mu,\alpha} > 0$, and each family of strategies $\mathfrak{H}$ satisfies $C(\mathfrak{H}) \leq 2\beta + 1$. The family of strategies $\mathfrak{H}_\beta = (H(\beta,T))_{T>0}$ defined by*

$$H_t(\beta,T) := (\beta+1)(T-t)^\beta \operatorname{sgn}(S_t)|S_t|^\beta, \; t \in [0,T] \tag{4}$$

satisfies $C(\mathfrak{H}_\beta) = 2\beta + 1$.

This result highlights a few unusual features: First, as the order of the certainty-equivalent is greater than one, no fixed annuity is sufficient to compensate for the loss of the trading opportunity over an arbitrarily long amount of time. Instead, an acceptable annuity would have to grow with the horizon with the power of 2β.

Second, and contrary to what intuition may suggest, asymptotic optimality is not achieved by time-homogeneous strategies that buy the asset when it lies below its long-term trend and sell it otherwise. Instead, it is sufficient for the asymptotically optimal strategy to time the market around its starting point, that is, zero, thereby neglecting the drift. The success of such an ostensibly myopic approach is explained by the sheer size of early bets compared to later ones. Because early bets are so large, and the initial value of S_t is zero, it turns out that it is sufficient to calibrate the trading strategy around its initial target to capture optimality at the leading order. Put differently, the increasing importance of the drift in the long run is offset by the concentration of risk of the trading strategy on the early stage of the trading interval.

Importantly, the mean-reversion curvature β controls both the trading response to the asset price and its time-dependent magnification. The higher the curvature, the more sensitive the strategy to price changes and the higher its concentration in the early part of the trading period. Higher curvature also leads to higher performance, equivalent to a variable annuity proportional to $t^{2\beta}$ at time t, which accumulates to an overall certainty-equivalent of order $T^{2\beta+1}$.

Remark 2.2. The question arises whether the strategy (4) may be optimal for some *other* utility function U. Indeed, under certain conditions, an investor's utility function can be recovered from the optimal strategy (Cox et al., 2014). In the present setting, in general there is no utility U for which the strategy (4) is exactly optimal. This fact is proved in the appendix in the linear case of $\beta = 1$.

The next result shows that the stationary case $\mu = 0$ is substantially different:

Theorem 2.3. *If $\alpha > 0$ and $\mu = 0$, then $c_T = C_\beta T^{1+\beta}$ for some constant $C_\beta > 0$, and each family of strategies $\mathfrak{H}$ satisfies $C(\mathfrak{H}) \leq 1 + \beta$. If $\beta > 1$, then for each $1 < \gamma < \beta$, the family of strategies $\mathfrak{H}_\gamma = (H(\gamma, T))_{T>0}$ defined by*

$$H_t(\gamma, T) := -2(T - t)^\gamma S_t, \ t \in [0, T] \tag{5}$$

satisfies $C(\mathfrak{H}_\gamma) \geq 1 + \gamma$. If $\beta = 1$ then $C(\mathfrak{H}_1) = 2 = 1 + \beta$.

In the case $\mu = 0$, which identifies long-term safe assets, the certainty-equivalent c_T is of order $T^{1+\beta}$, hence the mean-reversion curvature adds only once, rather than twice, to the order of the certainty-equivalent. Counterintuitive at first, this result is best understood in the context of the Ornstein-Uhlenbeck process, solved in closed form in the next section. In that setting, it becomes clear that the term of order $T^{1+2\beta}$ results from the interaction between average growth and mean-reversion, hence is lost when either of them vanishes. By contrast, the term of order $T^{1+\beta}$ results from mean-reversion alone, and persists even in the absence of average growth. In other words, $T^{1+\beta}$ is the highest order that can be achieved purely through market-timing.

Note also that the statement of Theorem 2.3 is slightly weaker than that of Theorem 2.1 because the strategies defined in (5) below achieve the performance $T^{1+\beta}$ *asymptotically* as $\gamma \to \beta$. This family of strategies entails the same decline in risk-taking over time as in the case of $\mu \neq 0$. However, the absence of drift implies that the overall certainty-equivalent has a smaller order.

3 | THE ORNSTEIN-UHLENBECK MODEL IN CLOSED FORM

This section derives in closed form the value function and the optimal strategy in the case of linear mean reversion, that is, $\beta = 1$. In this case, the value function is an exponential quadratic function of the state variable X_t with time-varying coefficients, which are determined by a system of differential equations. Here is reported the derivation of the solution and the discussion of its implications. The corresponding verification theorem follows from similar arguments as in Kraft (2005). This section also includes the two additional parameters σ, for the volatility of the asset price, and γ for the investor's risk aversion.

Let the asset price S_t satisfy the dynamics

$$dS_t = \mu\, dt + dX_t \tag{6}$$

$$dX_t = -\alpha X_t dt + \sigma dB_t \tag{7}$$

where X_t is an Ornstein-Uhlenbeck process. The investor maximizes exponential utility of terminal wealth, that is,

$$\max_H E\left[\frac{e^{-\gamma(H\cdot S)_T}}{-\gamma}\right] \tag{8}$$

where $\gamma > 0$ is the absolute risk aversion and H varies in the set of admissible strategies. Denoting the investor's wealth by $W_t = (H \cdot S)_t$, the optimization problem's value function is $V(t, w, x) = \sup_H E[\frac{e^{-\gamma W_T}}{-\gamma} | X_t = x, W_t = w]$, and follows the dynamics (henceforth omitting the arguments of V)

$$\begin{aligned} dV(t, W_t, X_t) = &\left(V_t + (\mu - \alpha X_t)H_t V_w - \alpha X_t V_x + \frac{\sigma^2}{2}H_t^2 V_{ww} + \sigma^2 H_t V_{wx} + \frac{\sigma^2}{2}V_{xx}\right)dt \\ &+ V_w \sigma H_t dB_t + V_x \sigma dB_t. \end{aligned}$$

The martingale principle of optimal control of Davis & Varaiya (1973) posits the value function to be a supermartingale for all admissible strategies and a martingale for the optimal strategy, thereby requiring that its maximal drift over all strategies is zero, and leading to the Hamilton-Jacobi-Bellman (HJB) equation

$$V_t + \frac{\sigma^2}{2}V_{xx} - \alpha x V_x + \sup_h\left((\mu - \alpha x)hV_w + \frac{\sigma^2}{2}h^2 V_{ww} + \sigma^2 h V_{wx}\right) = 0. \tag{9}$$

The first-order condition for the drift leads to the candidate optimal strategy

$$\hat{h} = -\frac{(\mu - \alpha x)V_w}{\sigma^2 V_{ww}} - \frac{V_{wx}}{V_{ww}}, \tag{10}$$

and replacing this expression in the drift, the HJB equation reduces to:

$$V_t + \frac{\sigma^2}{2}V_{xx} - \alpha x V_x - \frac{\left((\mu - \alpha x)V_w + \sigma^2 V_{wx}\right)^2}{2\sigma^2 V_{ww}} = 0. \tag{11}$$

To solve this equation, guess a solution of exponential-quadratic form

$$V(t,w,x) = \frac{e^{-\gamma\left(w+\frac{a(t)}{2}x^2+b(t)x+c(t)\right)}}{-\gamma}, \tag{12}$$

which eliminates the variable w from the HJB equation, reducing it to

$$\left(-\frac{\alpha^2}{2\sigma^2}-\frac{\gamma}{2}a'(t)\right)x^2+\left(\frac{\alpha\mu}{\sigma^2}+\gamma\mu a(t)-\gamma b'(t)\right)x+\left(-\frac{\mu^2}{2\sigma^2}-\frac{\sigma^2}{2}\gamma a(t)+\gamma\mu b(t)-\gamma c'(t)\right)=0. \tag{13}$$

Because the equation must hold for all t, x, each of the coefficients of x^2, x, and 1, must be zero, hence the system of differential equations holds:

$$\frac{\alpha^2}{\sigma^2}+\gamma a'(t)=0, \tag{14}$$

$$\frac{\alpha\mu}{\sigma^2}+\gamma\mu a(t)-\gamma b'(t)=0, \tag{15}$$

$$\frac{\mu^2}{2\sigma^2}+\frac{\sigma^2}{2}\gamma a(t)-\gamma\mu b(t)+\gamma c'(t)=0. \tag{16}$$

Solving these equations from the top down, it follows that

$$a(t)=\frac{\alpha^2}{\gamma\sigma^2}(T-t), \tag{17}$$

$$b(t)=-(T-t)\frac{\alpha\mu}{\gamma\sigma^2}-(T-t)^2\frac{\alpha^2\mu}{2\gamma\sigma^2}, \tag{18}$$

$$c(t)=\frac{\mu^2}{2\gamma\sigma^2}(T-t)+\frac{\alpha\left(2\mu^2+\alpha\sigma^2\right)}{2\gamma\sigma^2}(T-t)^2+\frac{\alpha^2\mu^2}{6\gamma\sigma^2}(T-t)^3, \tag{19}$$

whence the expression for the optimal strategy is

$$H_t=\frac{\mu-\alpha X_t}{\gamma\sigma^2}+\frac{\mu-\alpha X_t}{\gamma\sigma^2}\alpha(T-t)+\frac{\mu\alpha^2}{2\gamma\sigma^2}(T-t)^2 \tag{20}$$

and the certainty-equivalent, that is, $C(t,w,x)=\frac{\log(-\gamma V(t,w,x))}{-\gamma}$ is

$$C(t,w,x)=w+\frac{(\mu-\alpha x)^2}{2\gamma\sigma^2}(T-t)+\left(\frac{\alpha\mu(\mu-\alpha x)}{2\gamma\sigma^2}+\frac{\alpha^2}{4\gamma}\right)(T-t)^2+\frac{\mu^2\alpha^2}{6\gamma\sigma^2}(T-t)^3. \tag{21}$$

In particular, the leading order of the certainty-equivalent is

(i) $(T-t)^3$ if $\alpha,\mu\neq 0$;
(ii) $(T-t)^2$ if $\mu=0$ but $\alpha\neq 0$;

(iii) $(T-t)^1$ if $\mu \neq 0$ but $\alpha = 0$;
(iv) $(T-t)^0$ if $\alpha = \mu = 0$. (In this case the certainty-equivalent is merely w.)

These observations help understand the significance of the main results in Theorem 2.1 and 2.3. Recall that, because in this setting $\beta = 1$, it follows that $1 + 2\beta = 3$ and $1 + \beta = 2$, which identifies the relevant terms in the formula for the certainty-equivalent in (21).

First, the term of order $(T-t)^3$ originates from the interaction between the average return μ and the mean-reversion α, and such interaction would thus suggest that any strategy capable to generate cubic growth must be based on the knowledge of both μ and α. However, Theorem 2.1 demonstrates that it does not: in fact, a strategy that at time t depends only on t, T, β, and the observed price S_t, is sufficient to generate cubic growth without the need to know the values of α and μ (provided that $\mu \neq 0$, $\alpha > 0$). This fact is of practical relevance because it implies that, without estimating the exact parameter values of α and μ, it is possible to capture the leading order of the certainty-equivalent with a strategy that responds appropriately to price changes and the passing of time.

Second, the exact finite-horizon optimal strategy in (20) does depend on all model parameters, but Theorems 2.1 and 2.3 show that not all terms are equally important in determining the order of the certainty-equivalent. In particular, the first term $\frac{\mu-\alpha X_t}{\gamma\sigma^2}$ is negligible, as it depends only on the state variable X_t but not on time, and only generates a certainty-equivalent of order one. The last term $\frac{\mu\alpha^2}{2\gamma\sigma^2}(T-t)^2$ and the first part of the second term $\frac{\mu}{\gamma\sigma^2}\alpha(T-t)$ are also negligible because they do not involve the state variable X_t. As the theorems show, the dominant term is $-\frac{\alpha^2 X_t}{\gamma\sigma^2}(T-t)$, which combines sensitivity to both time and state to generate the dominant order in the certainty-equivalent.

Third, all terms except the initial capital in the certainty-equivalent in (21) are inversely proportional to the risk-aversion γ, as expected for exponential utility, and justifying the paper's focus in the previous section on $\gamma = 1$. By contrast, all terms in the certainty-equivalent are inverse in the variance, with the exception of $\frac{\alpha^2}{4\gamma}(T-t)^2$, which captures the effects of mean-reversion in the absence of an average return μ. This term is insensitive to σ^2 because a higher variance generates more frequent deviations, thereby creating trading opportunities, but also increases the risk of strategies that attempt to exploit such opportunities, and the two effects offset each other. Instead, an increase in the mean-reversion curvature β, which changes the shape (rather than the scale) of the stationary density of X_t, always results in an increase of the certainty-equivalent.

4 | DISCRETE-TIME LIMIT

First consider the discrete-time analogue of (2) when $\beta = 1$ and $\mu = 0$, that is, in the case of the Ornstein-Uhlenbeck process satisfying

$$dS_t = -\alpha S_t\, dt + dB_t.$$

Let η_k, $k \in \mathbb{N}$ be an i.i.d. sequence of standard Gaussian random variables. Recursively define an autoregressive process R_k, $k \in \mathbb{N}$ with mean reversion parameter $\nu_0 \in (-1, 1)$ as follows:

$$R_0 := 0,\ R_k := \nu_0 R_{k-1} + \eta_k,\ k \geq 1.$$

Now rewrite the above, using the parameter $\nu := \nu_0 - 1$, as

$$R_k - R_{k-1} = \nu R_{k-1} + \eta_k,\ k \geq 1, \tag{22}$$

and interpret R_k as the discounted price of a risky asset at time k.

Define the filtration $\mathcal{G}_k := \sigma(R_0, \dots, R_k)$, $k \in \mathbb{N}$. Portfolio strategies are identified with processes ϕ_k, $k \geq 1$ where ϕ_k is the number of shares of the risky asset in the portfolio, assumed $\mathcal{G}_{k-1}$-measurable. Fix a time horizon $N \geq 1$. The investor aims at maximizing $E[-e^{-L_N^\phi}]$, where

$$L_N^\phi := \sum_{j=1}^{N} \phi_j (R_j - R_{j-1}) \tag{23}$$

denotes wealth at time N (assuming, without loss of generality, a zero initial position). In this setting, laborious calculations lead to the the following result:

Theorem 4.1 (Deák & Rásonyi (2015), Theorem 2.1)**.** *For each $N \geq 1$, the optimal strategy for time horizon N is*

$$\bar{\phi}_k(N) := \mathrm{g}_k^N(R_{k-1}),\ 1 \leq k \leq N,$$

where

$$\mathrm{g}_k^N(z) = \nu z[1 - (N-k)\nu] \qquad \textit{for all } 1 \leq k \leq N \textit{ and } z \in \mathbb{R}. \tag{24}$$

These strategies yield the maximal expected utilities

$$r(N;\nu) := \sup_\phi \mathbb{E}\left[-e^{-L_N^\phi}\right] = \mathbb{E}\left[-e^{-L_N^{\bar{\phi}(N)}}\right] = -\bar{\gamma}(N;\nu)^{-\frac{1}{2}}, \tag{25}$$

where $\bar{\gamma}(N;\nu) := \nu^{2N}\Gamma(1/\nu^2 + N)\Gamma(1/\nu^2)^{-1}$ and Γ is the Gamma function. □

This result highlights some striking features: First, it is easy to check (see Remark 2.3 of Deák & Rásonyi (2015)) that the certainty-equivalent $-\ln(-r(N,\nu))$ is of the order $N\ln(N)$ as $N \to \infty$. In the case of a random walk, where $R_k - R_{k-1} = \mu + \eta_k$ with some constant $\mu \neq 0$, the corresponding certainty-equivalent grows only at a rate N, which is hence outperformed in the autoregressive case by a factor tending to infinity. This phenomenon already suggests that mean-reverting models behave rather differently from martingale-like models.

Second, as R_k tends to a stationary law for $k \to \infty$, it is heuristically clear (assuming that an appropriate upper large deviation estimate holds) that homogeneous Markov strategies (where $\phi_k = g(R_{k-1})$ with some fixed measurable g) can provide a certainty-equivalent rate of N only. Hence the temporal structure of the optimal strategy is decisive.

Third, the optimal strategy has an intuitive form: mean-reversion is maximally exploited at the beginning (when k is small) and then to a lesser and lesser degree as the factor $(N-k)\nu$ in (24) decreases.

Turning now the attention to the continuous-time model (2) in the case $\beta = 1$, where $S_t = X_t$ is the Ornstein-Uhlenbeck process with mean-reversion parameter α, consider the corresponding

Euler approximations, $(\bar{R}_k)_{k=1}^N$, up to the fixed time horizon T, with a grid of resolution N, that is,

$$\bar{R}_k^{(N)} = \bar{R}_{k-1}^{(N)} - \alpha \frac{T}{N} \bar{R}_{k-1}^{(N)} + \bar{\eta}_k, \tag{26}$$

where $\bar{R}_0^N = 0$ and $(\bar{\eta})_{k=1}^N$ is a standard Gaussian white noise. In this approximation, for every N, $\bar{R}^{(N)}$ is an autoregressive process. Rewrite (26) as

$$\bar{R}_k^{(N)} - \bar{R}_{k-1}^{(N)} = -\alpha \frac{T}{N} \bar{R}_{k-1}^{(N)} + \bar{\eta}_k. \tag{27}$$

Thus, matching the parameter ν in (22) and the asymptotics in (25), in the limit as $N \to \infty$, the mapping

$$T \to r\left(N; -\frac{\alpha T}{N}\right) \tag{28}$$

provides a heuristic for the growth rate of the certainty-equivalent for the limiting process, namely for S_t, $t \in \mathbb{R}_+$ when $\beta = 1$.

Consider $\ln \bar{\gamma}(N; \nu)$. The estimate $\ln(n!) \approx n \ln(n) - n$ yields

$$\begin{aligned}
\ln \bar{\gamma}(N; \nu) \approx & - N \ln(1/\nu^2) + (1/\nu^2 + N) \ln(1/\nu^2 + N) \\
& - (1/\nu^2 + N) - (1/\nu^2) \ln(1/\nu^2) + 1/\nu^2 \\
= & (1/\nu^2 + N)\big(\ln(1/\nu^2 + N) - \ln(1/\nu^2)\big) - N \\
= & (1/\nu^2 + N)\big(\ln(1 + \nu^2 N)\big) - N.
\end{aligned}$$

Substituting $\nu = -\frac{\alpha T}{N}$, algebraic manipulation and Taylor's expansion yield

$$\begin{aligned}
\ln \bar{\gamma}\left(N; -\frac{\alpha T}{N}\right) & \approx \left(\frac{N^2}{\alpha^2 T^2} + N\right) \ln\left(1 + \frac{\alpha^2 T^2}{N}\right) - N \\
& = N\left(\frac{N}{\alpha^2 T^2} \ln\left(1 + \frac{\alpha^2 T^2}{N}\right) - 1\right) + \ln\left(1 + \frac{\alpha^2 T^2}{N}\right)^N \\
& = N\left(\frac{N}{\alpha^2 T^2}\left(\frac{\alpha^2 T^2}{N} - \frac{\alpha^4 T^4}{2N^2} + \mathcal{O}(N^{-3})\right) - 1\right) + \ln\left(1 + \frac{\alpha^2 T^2}{N}\right)^N \\
& = \mathcal{O}(N^{-1}) - \frac{\alpha^2 T^2}{2} + \ln\left(1 + \frac{\alpha^2 T^2}{N}\right)^N.
\end{aligned}$$

Taking limit as $N \to \infty$, the expected utility in continuous-time on the trading interval $[0, T]$ is estimated to be $-e^{-\alpha^2 T^2/4}$. We thus arrive at the surprising conjecture that the optimal growth rate of the certainty-equivalent of an investor in the Ornstein-Uhlenbeck case is of the order T^2.

The main result in the paper rigorously verifies this conjecture and its generalization to nonlinear mean reversion with $\beta > 1$, proposing similarly intuitive, asymptotically optimal strategies as in Theorem 4.1 above.

5 | CONCLUSION

This paper solves a portfolio choice model with temporary price fluctuations and nonlinear mean-reversion for an investor with constant absolute risk aversion. Although investment opportunities are stationary, as the horizon increases the optimal portfolio does not stabilize to a turnpike. Instead, it diverges with a power of the horizon that depends on the curvature of mean reversion. Accordingly, for a fixed horizon portfolio risk declines over time, reproducing the conventional wisdom of financial planners.

Overall, the paper emphasizes the joint role of sensitivity to time and price changes in obtaining portfolios that are optimal at the leading order, as neither time-dependence nor price-dependence alone are sufficient for this purpose. When combined, they can succeed even with strategies that depend on the single parameter describing mean-reversion curvature.

CONFLICT OF INTEREST

The authors do not declare any conflicts of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable no new data generated, as the article describes entirely theoretical research.

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ENDNOTES

[1] See Guasoni et al. (2014) and the references therein for a recent treatment of portfolio turnpikes.

[2] The form of the density follows from the formula $e^{2\int_0^x b(y)dy}$ for the speed measure of a diffusion with unit volatility and drift b, cf. Borodin & Salminen (2012).

[3] Note that this interpretation is specific to the setting of zero initial wealth and exponential utility, any additional initial wealth would add one-to-one to the certainty-equivalent. For other utility functions, the certainty-equivalent represents the wealth that one would exchange for the initial wealth and the opportunity of trading, combined.

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APPENDIX A: PROOFS

A.1 | Preliminary calculations and estimates

Proposition A.1. *The process*

$$\xi_t^* = \exp\left\{-\int_0^t \left(\mu - \alpha\,\mathrm{sgn}(X_u)|X_u|^\beta\right)dB_u - \frac{1}{2}\int_0^t \left(\mu - \alpha\,\mathrm{sgn}(X_u)|X_u|^\beta\right)^2 du\right\},\ t \in \mathbb{R}_+ \quad \text{(A1)}$$

is a P-martingale and $dQ_T/dP := \xi_T^*$ defines a probability $Q_T \sim P$ on $\mathcal{F}_T$ such that S_t, $t \in [0,T]$ is a Q_T-martingale (actually, a Q_T-Brownian motion) and Q_T is the only such equivalent probability.

Proof. By Girsanov's theorem, it suffices to establish that the process ξ^* is a true martingale. Apply Theorem 2.1 of Mijatović & Urusov (2012) with the choice $J = \mathbb{R}$, $Y_t = X_t$, $b(x) := \mu - \alpha\,\mathrm{sgn}(x)|x|^\beta$, $x \in \mathbb{R}$. According to the notation of that paper, $\tilde{\rho}(x) = 1$ for all $x \in \mathbb{R}$, as easily checked. Then the quantity $\tilde{v}(x)$ defined there equals $x^2/2$ for all $x \in \mathbb{R}$ and this satisfies $\tilde{v}(\pm\infty) = \infty$ hence the claim follows from the mentioned theorem. An alternative proof could be obtained from the abstract results in Cheridito et al. (2005) for general jump-diffusions. □

Recall that Q_T (defined in Proposition A.1 above) is the unique martingale measure for the process X. Hence, by the duality theory of optimal investment (see Delbaen et al. (2002); Kabanov & Stricker (2002)), it follows that

$$u_T = -e^{-J}$$

where

$$J := E\left[\frac{dQ_T}{dP}\ln\left(\frac{dQ_T}{dP}\right)\right] = E[\xi_T^* \ln(\xi_T^*)] = E_{Q_T}[\ln(\xi_T^*)] \tag{A2}$$

(provided that the latter quantity exists and is finite). Here E_{Q_T} denotes the expectation under the probability Q_T.

>From (2) and reordering (A1), it follows that

$$\ln({\xi_T}^*) = \frac{1}{2}\int_0^T \left(\mu - \alpha\,\mathrm{sgn}(X_u)|X_u|^\beta\right)^2 du - \int_0^T \left(\mu - \alpha\,\mathrm{sgn}(X_u)|X_u|^\beta\right) dS_u.$$

Because, under the measure Q_T, the process S is a standard Brownian motion on $[0,T]$, the second term in the above expression is a Q_T-martingale and

$$\begin{aligned}
J &= E_{Q_T}\left[\frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T |X_u|^{2\beta}du - \alpha\mu\int_0^T \mathrm{sgn}(X_u)|X_u|^\beta du\right] \\
&\le E_{Q_T}\left[\frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T |X_u|^{2\beta}du + \alpha|\mu|\int_0^T |X_u|^\beta du\right] \\
&= \frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T E_{Q_T}|X_u|^{2\beta}du + \alpha|\mu|\int_0^T E_{Q_T}|X_u|^\beta du.
\end{aligned}$$

Note that under the measure Q_T, the process $X_t = S_t - \mu t$ is a standard Brownian motion with a constant drift on $[0,T]$. Thus, in view of the convexity of the mappings $x \to |x|^\beta$ and $x \to |x|^{2\beta}$,

$$\begin{aligned}
J &\le \frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T E_{Q_T}|S_u - \mu u|^{2\beta}du + \alpha|\mu|\int_0^T E_{Q_T}|S_u - \mu u|^\beta du \\
&\le \frac{\mu^2}{2}T + \frac{\alpha^2}{2}\int_0^T \left(2^{2\beta-1}u^\beta M_{2\beta} + 2^{2\beta-1}|\mu|^{2\beta}u^{2\beta}\right)du + \alpha|\mu|\int_0^T \left(2^{\beta-1}u^{\beta/2}M_\beta + 2^{\beta-1}|\mu|^\beta u^\beta\right)du \\
&= \frac{\mu^2}{2}T + \alpha^2 2^{2\beta-2}\frac{M_{2\beta}T^{\beta+1}}{\beta+1} + \alpha^2 2^{2\beta-2}|\mu|^{2\beta}\frac{T^{2\beta+1}}{2\beta+1} \\
&\quad + \alpha|\mu|2^{\beta-1}\frac{M_\beta T^{\beta/2+1}}{\beta/2+1} + \alpha|\mu|^{\beta+1}2^{\beta-1}\frac{T^{\beta+1}}{\beta+1},
\end{aligned}$$

where $M_\varkappa$ is the $\varkappa$th moment of a standard Gaussian variable. This shows that $c_T \le C_{\beta,\mu,\alpha}T^{2\beta+1}$ where $C_{\beta,\mu,\alpha}$ can be explicitly given and the first statement of Theorem 2.1 is proved.

Now we turn to some estimates familiar in the theory of Markov processes. We have been inspired by Kontoyiannis et al. (2005) in particular. Let $C^2(\mathbb{R})$ denote the family of twice continuously differentiable functions on $\mathbb{R}$. Define the operator $\mathcal{A}$ by

$$\mathcal{A}f := -\alpha\,\mathrm{sgn}(x)|x|^\beta\partial_x f + \frac{1}{2}\partial_{xx}f,\ f \in C^2(\mathbb{R}), \tag{A3}$$

which coincides with the infinitesimal generator associated to the process X on its domain of definition. Define also the operator $\mathcal{H}$ (the "nonlinear generator", see Kontoyiannis et al. (2005)) as

$$\mathcal{H}f := e^{-f}\mathcal{A}e^{f},\ f \in C^2(\mathbb{R}).$$

Now we introduce a condition related to $\mathcal{H}$.

Condition A.2. There is a compact $C \subset \mathbb{R}$, there are $\delta, b > 0$ and functions $V, W : \mathbb{R} \to \mathbb{R}_+$ with W measurable and $V \in C^2(\mathbb{R})$ such that, for all $x \in \mathbb{R}$,

$$\mathcal{H}V(x) \le -\delta W(x) + b\mathbb{1}_C(x). \tag{A4}$$

For a given $\delta, b > 0$, define the process M_t by

$$M_t := \exp\left\{V(X_t) + \int_0^t \left(\delta W(X_u) - b\mathbb{1}_{\{X_u \in C\}}\right)du\right\},\ t \in \mathbb{R}_+. \tag{A5}$$

Lemma A.3. *If Condition A.2 holds then the process M is a supermartingale.*

Proof. Setting $Y_t = \exp\{V(X_t)\}$ and $Z_t = \exp\{\int_0^t (\delta W(X_u) - b\mathbb{1}_{\{X_u \in C\}})du\}$, it follows that $M_t = Y_t Z_t$. Now Ito's formula yields

$$\begin{aligned} dY_t = de^{V(X_t)} &= \mathcal{A}e^V(X_t)dt + \partial_x e^V(X_t)dB_t \\ &= \mathcal{A}e^V(X_t)dt + (e^V\partial_x V)(X_t)\,dB_t \end{aligned}$$

and

$$dZ_t = Z_t\left(\delta W(X_t) - b\mathbb{1}_{\{X_t \in C\}}\right)dt.$$

By the product rule of Ito calculus, using that $[Y, Z]_t \equiv 0$,

$$\begin{aligned} dM_t &= Y_t dZ_t + Z_t dY_t \\ &= Y_t Z_t\left(\delta W(X_t) - b\mathbb{1}_{\{X_t \in C\}}\right)dt + Z_t e^{V(X_t)}\mathcal{H}V(X_t)dt + Z_t(e^V\partial_x V)(X_t)\,dB_t, \\ &= Z_t e^{V(X_t)}\left(\delta W(X_t) - b\mathbb{1}_{\{X_t \in C\}} + \mathcal{H}V(X_t)\right)dt + Z_t(e^V\partial_x V)(X_t)\,dB_t. \end{aligned}$$

Here the last term is a local martingale, the first term is non-increasing by Condition A.2, hence M is a local supermartingale. As M is positive, Fatou's lemma guarantees that it is, in fact, a true supermartingale. □

Corollary A.4. *With Condition A.2 in force for $T > 0$ it follows that*

$$E\left[\exp\left\{\int_0^T \delta W(X_u)du\right\}\right] \le e^{V(0)+bT}.$$

Proof. By the supermartingale property of M, $E[M_T] \leq M_0 = 1$. Since $b\mathbb{1}_C \leq b$, the statement follows. □

Define the functions

$$\bar{V}(x) := \frac{\alpha}{1+\beta}|x|^{1+\beta} \text{ and } \bar{W}(x) := \alpha^2|x|^{2\beta},\ x \in \mathbb{R}.$$

Proposition A.5. *For each $0 < \bar{\delta} < 1/2$, there is an appropriate constant $\bar{b} > 0$ and a compact set $\bar{C}$ such that Condition A.2 is fulfilled with $V = \bar{V}$, $W = \bar{W}$, $b = \bar{b}$, $\delta = \bar{\delta}$ and $C = \bar{C}$.*

Proof. The claim would follow from Proposition 1.3 of Kontoyiannis et al. (2005) but we provide a direct proof. Note that $\partial_x e^V(x) = e^{V(x)}\partial_x V(x)$, $\partial_{xx} e^V(x) = e^{V(x)}(\partial_x V(x))^2 + e^{V(x)}\partial_{xx}V(x)$, $\partial_x V(x) = \alpha\,\mathrm{sgn}(x)|x|^\beta$, and $\partial_{xx}V(x) = \alpha\beta|x|^{\beta-1}$. Thus, (A3) yields

$$e^{-V}\mathcal{A}e^V(x) = -\frac{\alpha^2}{2}|x|^{2\beta} + \frac{\alpha\beta}{2}|x|^{\beta-1}. \tag{A6}$$

The criterion in (A4) then becomes equivalent to

$$\frac{\alpha\beta}{2}|x|^{\beta-1} \leq \left(\frac{1}{2} - \bar{\delta}\right)\alpha^2|x|^{2\beta} + b\mathbb{1}_C(x)$$

which clearly shows that the set C and the constant b can be chosen in such a way that Condition A.2 is fulfilled, provided that $\bar{\delta} < 1/2$. □

Lemma A.6. *There exist constants $\delta_0, c_0, C_0 > 0$ such that*

$$E\left[\exp\left\{\delta_0\int_0^T |X_t|^{2\beta}dt\right\}\right] \leq c_0 e^{C_0 T}.$$

Proof. Corollary A.4, Proposition A.5 and the definitions of $\bar{V}$, $\bar{W}$ immediately yield the upper bound with $\delta_0 := \alpha^2\bar{\delta}$. In fact, $c_0 = 1$ can be chosen as $\bar{V}(0) = 0$. □

A.2 | Asymptotic optimality in the case $\mu \neq 0$

Consider the process $U_t := (T-t)^\beta|S_t|^{\beta+1}$, $t \in [0,T]$. As $U_0 = 0$, Ito's lemma implies that

$$0 = U_T = \int_0^T (\beta+1)(T-t)^\beta\,\mathrm{sgn}(S_t)|S_t|^\beta dS_t + \int_0^T \frac{\beta(\beta+1)}{2}(T-t)^\beta|S_t|^{\beta-1}dt$$
$$-\int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1}dt,$$

which is equivalent to

$$\int_0^T (\beta+1)(T-t)^\beta \operatorname{sgn}(S_t)|S_t|^\beta dS_t = -\int_0^T \frac{\beta(\beta+1)}{2}(T-t)^\beta |S_t|^{\beta-1} dt + \int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1} dt.$$

Note that the above expression is the value of the investor's portfolio utilizing the strategy $H_t(\beta, T) = (\beta+1)(T-t)^\beta \operatorname{sgn}(S_t)|S_t|^\beta$, $t \in [0, T]$. Since S is a Q_T-Brownian motion, clearly $H(\beta, T) \in \mathcal{X}_T$.

First, consider the case $\beta > 1$ and denote $I_1(T) := \int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1} dt$, and $I_2(T) := \int_0^T \frac{\beta(\beta+1)}{2}(T-t)^\beta |S_t|^{\beta-1} dt$. Thus,

$$E\left[-e^{-(H\cdot S)_T}\right] = E\left[-e^{-I_1(T)+I_2(T)}\right]. \tag{A7}$$

Now, define the event $A(T)$ as

$$\Omega \supset A(T) := \left\{ \left| \int_0^{T/2} X_t dt \right| \leq \frac{\mu T^2}{16} \right\}$$

and denote its set-theoretic complement as $\bar{A}(T)$. To obtain a deterministic bound for $I_1(T)$ on the event $A(T)$, first note that

$$I_1(T) = \int_0^T \beta(T-t)^{\beta-1}|S_t|^{\beta+1} dt \geq \beta\left(\frac{T}{2}\right)^{\beta-1} \int_0^{T/2} |S_t|^{\beta+1} dt, \tag{A8}$$

and by Jensen's inequality,

$$\left(\frac{1}{T/2}\int_0^{T/2} |S_t|^{\beta+1} dt\right)^{1/(\beta+1)} \geq \frac{1}{T/2}\left|\int_0^{T/2} S_t dt\right| = \frac{1}{T/2}\left|\frac{\mu T^2}{8} + \int_0^{T/2} X_t dt\right|. \tag{A9}$$

On the event $A(T)$ these yield

$$\int_0^{T/2} |S_t|^{\beta+1} dt \geq 2^{-3(\beta+1)-1}\mu^{\beta+1}T^{\beta+2}. \tag{A10}$$

and in return using (A9) and (A10), it follows that, on the event $A(T)$

$$I_1(T) \geq \beta 2^{-3(\beta+1)-\beta}\mu^{\beta+1}T^{2\beta+1} =: C_{\beta,\mu}T^{2\beta+1}. \tag{A11}$$

Now the expectation in (A7) is estimated by splitting it along the event $A(T)$. First, (A11) implies that

$$E\left[-e^{-I_1(T)+I_2(T)}\mathbb{1}_A\right] \geq -e^{-C_{\beta,\mu}T^{2\beta+1}} E\left[e^{I_2(T)}\right] \geq -e^{-C_{\beta,\mu}T^{2\beta+1}}\left(E\left[e^{2I_2(T)}\right]\right)^{1/2}. \tag{A12}$$

On the other hand, by the Cauchy-Schwartz inequality and recalling that $-e^{-x} \geq -1$ for $x \geq 0$,

$$E[-e^{-I_1(T)}e^{I_2(T)}\mathbb{1}_{\bar{A}}] \geq -\left(E\left[e^{2I_2(T)}\right]\right)^{1/2}\left(P\left(\bar{A}\right)\right)^{1/2}. \tag{A13}$$

Now, to estimate the quantities $P(\bar{A}(T))$ and $E[e^{2I_2(T)}]$, consider a corollary to Lemma A.6 that handles $P(\bar{A}(T))$ and a Lemma bounding $E[e^{2I_2(T)}]$ which is also a consequence of Lemma A.6.

Corollary A.7. *There exist positive constants c_1, C_1 such that*

$$P(\bar{A}(T)) \leq c_1 e^{-C_1 T^{2\beta+1}}.$$

Lemma A.8. *There exist positive constants c_2, C_2 and $q > 0$ such that*

$$E[e^{2I_2(T)}] \leq c_2 e^{C_2 T^{2\beta+1-q}}.$$

Corollary A.7 and Lemma A.8 will be proved shortly.

Proceeding with these results and using (A7), (A12) and (A13), Corollary A.7 and Lemma A.8 it follows that

$$\begin{aligned}
E\left[-e^{-(H\cdot S)_T}\right] &\geq -e^{-C_{\beta,\mu}T^{2\beta+1}}\left(E\left[e^{2I_2(T)}\right]\right)^{1/2} - \left(E\left[e^{2I_2(T)}\right]\right)^{1/2}\left(P\left(\bar{A}\right)\right)^{1/2} \\
&\geq -e^{-C_{\beta,\mu}T^{2\beta+1}}\left(c_2 e^{C_2 T^{2\beta+1-q}}\right)^{1/2} - \left(c_2 e^{C_2 T^{2\beta+1-q}}\right)^{1/2}\left(c_1 e^{-C_1 T^{2\beta+1}}\right)^{1/2} \\
&= -c_2^{1/2} e^{-C_{\beta,\mu}T^{2\beta+1}+\frac{C_2}{2}T^{2\beta+1-q}} - c_1^{1/2}c_2^{1/2} e^{\frac{C_2}{2}T^{2\beta+1-q}-\frac{C_1}{2}T^{2\beta+1}}.
\end{aligned}$$

This completes the proof of Theorem 2.1 when $\beta > 1$. The same calculations can be done when $\beta = 1$: Corollary A.7 holds with $\beta = 1$ as written. The term I_2, being deterministic and of order T^2, shows that the conclusion of Lemma A.8 also remains valid, completing the proof of Theorem 2.1.

It remains to prove Corollary A.7 and Lemma A.8.

Proof of Corollary A.7. By Jensen's inequality,

$$\left|\frac{1}{T}\int_0^T X_t dt\right|^{2\beta} \leq \frac{1}{T}\int_0^T |X_t|^{2\beta} dt. \tag{A14}$$

Lemma A.6 and Markov's inequality lead to

$$P\left(\left|\int_0^T X_t dt\right| \geq \frac{\mu T^2}{16}\right) \leq P\left(\int_0^T |X_t|^{2\beta} dt \geq \mu^{2\beta}2^{-8\beta}T^{2\beta+1}\right) \tag{A15}$$

$$= P\left(\exp\left\{\delta_0 \int_0^T |X_t|^{2\beta} dt\right\} \geq e^{\delta_0 \mu^{2\beta}2^{-8\beta}T^{2\beta+1}}\right) \tag{A16}$$

$$\leq c_0 e^{-\delta_0 \mu^{2\beta}2^{-8\beta}T^{2\beta+1}+C_0 T}. \tag{A17}$$

□

Proof of Lemma A.8. First, note that there exist positive constants c_β and $c_{\beta,\mu}$ such that

$$\begin{aligned} E[e^{2I_2(T)}] &\le E\left[\exp\left\{\beta(\beta+1)T^\beta\int_0^T |S_t|^{\beta-1}dt\right\}\right] \\ &\le E\left[\exp\left\{\beta(\beta+1)T^\beta\int_0^T \left(c_\beta|X_t|^{\beta-1}+c_{\beta,\mu}t^{\beta-1}\right)dt\right\}\right] \\ &= e^{(\beta+1)c_{\beta,\mu}T^{2\beta}}E\left[\exp\left\{\beta(\beta+1)c_\beta T^\beta\int_0^T |X_t|^{\beta-1}dt\right\}\right]. \end{aligned} \tag{A18}$$

By Jensen's inequality,

$$\int_0^T |X_t|^{\beta-1}dt \le T^{1-\frac{\beta-1}{2\beta}}\left(\int_0^T |X_t|^{2\beta}dt\right)^{\frac{\beta-1}{2\beta}}.$$

Denoting $\Xi_T := \int_0^T |X_t|^{2\beta}dt$ and defining $h(x) = h_{\beta,T}(x) := \exp\{\beta(\beta+1)c_\beta T^{\beta+1-\frac{\beta-1}{2\beta}}x^{\frac{\beta-1}{2\beta}}\}$, $x > 0$ the following estimate holds:

$$\begin{aligned} E[e^{2I_2(T)}] &\le e^{(\beta+1)c_{\beta,\mu}T^{2\beta}}E\left[\exp\left\{\beta(\beta+1)c_\beta T^{\beta+1-\frac{\beta-1}{2\beta}}\left(\int_0^T |X_t|^{2\beta}dt\right)^{\frac{\beta-1}{2\beta}}\right\}\right] \\ &= e^{(\beta+1)c_{\beta,\mu}T^{2\beta}}Eh(\Xi_T). \end{aligned} \tag{A19}$$

The estimate in Lemma A.6, along with Markov's inequality, implies that, for all $x > 0$,

$$P(\Xi_T > x) \le c_0\exp\{C_0T-\delta_0x\}, \tag{A20}$$

and also observe that

$$E[h(\Xi_T)] = \int_0^\infty h'(x)P(\Xi_T > x)dx. \tag{A21}$$

Since $h'(x) = \frac{(\beta+1)\beta(\beta-1)}{2\beta}c_\beta T^{\beta+1-\frac{\beta-1}{2\beta}}x^{\frac{\beta-1}{2\beta}-1}\exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}}x^{\frac{\beta-1}{2\beta}}\}$, $x > 0$, (A20) and (A21) yield

$$\begin{aligned} E[h(\Xi_T)] &\le \int_0^\infty \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta}T^{\beta+1-\frac{\beta-1}{2\beta}}x^{\frac{\beta-1}{2\beta}-1}\exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}}x^{\frac{\beta-1}{2\beta}}-\delta_0x+C_0T\}dx \\ &= \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta}T^{\beta+1-\frac{\beta-1}{2\beta}}\int_0^\infty x^{\frac{\beta-1}{2\beta}-1}\exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}}x^{\frac{\beta-1}{2\beta}}-\delta_0x+C_0T\}dx \end{aligned}$$

$$\leq \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} T^{\beta+1-\frac{\beta-1}{2\beta}} \times$$

$$\times \left(e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}}+C_0 T} \int_0^1 x^{\frac{\beta-1}{2\beta}-1} dx + \int_1^\infty e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} -\delta_0 x + C_0 T} dx \right)$$

$$= \tilde{c}_1 T^{\beta+1-\frac{\beta-1}{2\beta}} e^{\frac{c_\beta(\beta+1)\beta}{2} T^{\beta+1-\frac{\beta-1}{2\beta}}+C_0 T}$$

$$+ \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} T^{\beta+1-\frac{\beta-1}{2\beta}} e^{C_0 T} \int_1^\infty e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} -\delta_0 x} dx, \quad \text{(A22)}$$

where $\tilde{c}_1 = \frac{(\beta+1)\beta(\beta-1)c_\beta c_0}{2\beta} \int_0^1 x^{\frac{\beta-1}{2\beta}-1} dx$. To estimate the integral $\int_1^\infty e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} -\delta_0 x} dx$, first define

$$\tilde{C}(T) := \left(\frac{2(\beta+1)\beta c_\beta}{\delta_0} \right)^{\frac{2\beta}{\beta+1}} T^{\frac{2\beta^2+\beta+1}{\beta+1}} =: \tilde{c}_2 T^{\frac{2\beta^2+\beta+1}{\beta+1}}.$$

Note that $c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} = c_\beta(\beta+1)\beta T^{\frac{2\beta^2+\beta+1}{2\beta}} x^{\frac{\beta-1}{2\beta}} \leq \frac{\delta_0}{2} x$ for $x > \tilde{C}(T)$, whence

$$\int_{\tilde{C}(T)}^\infty \exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x\} dx \leq \int_{\tilde{C}(T)}^\infty e^{-\frac{\delta_0}{2}x} dx$$

$$= \frac{2}{\delta_0} e^{-\frac{\delta_0}{2}\tilde{C}(T)} = \frac{2}{\delta_0} \exp\left\{ -\frac{\delta_0}{2} \tilde{c}_2 T^{\frac{2\beta^2+\beta+1}{\beta+1}} \right\} \leq \frac{2}{\delta_0} \exp\left\{ -\frac{\delta_0}{2} \tilde{c}_2 T^2 \right\}, \quad \text{(A23)}$$

where the last step follows from $\frac{2\beta^2+\beta+1}{\beta+1} > 2$. Next, observing that

$$\frac{\partial}{\partial x}\left(c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} - \delta_0 x \right) = \frac{c_\beta(\beta+1)\beta(\beta-1)}{2\beta} T^{\frac{2\beta^2+\beta+1}{2\beta}} x^{-\frac{\beta+1}{2\beta}} - \delta_0,$$

the integrand $x \to e^{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}} x^{\frac{\beta-1}{2\beta}} -\delta_0 x}$ reaches its maximum at

$$x = x_0 := \left(\frac{2\beta\delta_0}{c_\beta(\beta+1)\beta(\beta-1)} \right)^{-\frac{2\beta}{\beta+1}} T^{\frac{2\beta^2+\beta+1}{\beta+1}} =: \tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}},$$

and the value of such maximum is

$$
\begin{aligned}
&\exp\left\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}}x_0^{\frac{\beta-1}{2\beta}}-\delta_0 x_0\right\}\\
&=\exp\left\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}}\left(\tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}}\right)^{\frac{\beta-1}{2\beta}}-\delta_0\tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}}\right\}\\
&=\exp\left\{c_\beta(\beta+1)\beta\tilde{c}_3^{\frac{\beta-1}{2\beta}}T^{\frac{2\beta^2+\beta+1}{\beta+1}}-\delta_0\tilde{c}_3 T^{\frac{2\beta^2+\beta+1}{\beta+1}}\right\}
\end{aligned}
\tag{A24}
$$

Because $\frac{2\beta^2+\beta+1}{\beta+1}<2\beta+1$, there exists $q>0$ such that

$$
\int_1^{\tilde{C}(T)}\exp\{c_\beta(\beta+1)\beta T^{\beta+1-\frac{\beta-1}{2\beta}}x^{\frac{\beta-1}{2\beta}}-\delta_0 x\}dx\le\left(\tilde{C}(T)-1\right)\exp\{c_\beta(\beta+1)\beta\tilde{c}_3^{\frac{\beta-1}{2\beta}}T^{2\beta+1-q}\}.
\tag{A25}
$$

Using (A19), (A22), (A23) and (A25), the proof is complete. □

Proof of Remark 2.2. Recall that optimality requires the marginal utility of the optimal payoff to be proportional to the state-price density, that is, for some constant $y>0$

$$
U'((H\cdot S)_T)=y\frac{dQ_T}{dP}
\tag{A26}
$$

As $\beta=1$, observe that $S_t=X_t$ and $H_t(\beta,T)=2(T-t)X_t$. Note that

$$
\int_0^T X_t dX_t=\int_0^T d\left(\frac{X_t^2}{2}-\frac{1}{2}t\right)=\frac{X_T^2}{2}-\frac{T}{2}
\tag{A27}
$$

$$
\int_0^T tX_t dX_t=\int_0^T\left(d\left(t\frac{X_t^2}{2}\right)-\frac{1}{2}tdt-\frac{X_t^2}{2}dt\right)=T\frac{X_T^2}{2}-\frac{T^2}{4}-\frac{1}{2}\int_0^T X_t^2 dt
\tag{A28}
$$

and therefore

$$
2\int_0^T(T-t)X_t dX_t=-\frac{T^2}{2}+\int_0^T X_t^2 dt
\tag{A29}
$$

Now, in view of the expression for $\frac{dQ_T}{dP}$ in (A1), the first-order condition (A26) is

$$
U'\left(-\frac{T^2}{2}+\int_0^T X_t^2 dt\right)=y\exp\left(-\int_0^T(\mu-\alpha X_t)dB_t-\frac{1}{2}\int_0^T(\mu-\alpha X_t)^2 dt\right)
\tag{A30}
$$

$$=y\exp\left(-\mu X_T-\frac{T}{2}(\alpha+\mu^2)+\frac{\alpha}{2}X_T^2+\frac{\alpha^2}{2}\int_0^T X_t^2dt\right). \tag{A31}$$

Now, such a relation would imply that the right-hand side is a deterministic function of the random variable $\int_0^T X_t^2dt$, hence that $-\mu X_T+\frac{\alpha}{2}X_T^2$ is also a function of $\int_0^T X_t^2dt$. But this implication is false because the pair $(X_T,\int_0^T X_t^2dt)$ has a joint density (Borodin & Salminen, 2012, 1.9.8, p. 526). □

A.3 | The case $\mu=0$

Proceeding as in the case $\mu\neq 0$,

$$\ln(\xi_T^*)=\frac{\alpha^2}{2}\int_0^T |X_u|^{2\beta}du+\alpha\int_0^T \operatorname{sgn}(X_u)|X_u|^\beta dX_u.$$

As the process X is a standard Brownian motion on $[0,T]$ under the measure Q_T, the second term in the above expression is a Q_T-martingale so

$$J=\frac{\alpha^2}{2}\int_0^T E_Q|X_u|^{2\beta}du=\frac{\alpha^2}{2(1+\beta)}T^{1+\beta}M_{2\beta},$$

showing that $c_T=C_\beta T^{1+\beta}$ with $C_\beta=\frac{\alpha^2M_{2\beta}}{2(1+\beta)}$, which proves the first statement of Theorem 2.1. Note that for $\beta=1$ this result confirms the heuristics in Section 4.

Assume $\beta>1$ until further notice. Consider the process $U_t:=(T-t)^\gamma X_t^2,\ t\in[0,T]$ with some $1<\gamma<\beta$. Since $U_0=0$, Ito's lemma implies that

$$0=U_T=\int_0^T 2(T-t)^\gamma X_tdX_t+\int_0^T (T-t)^\gamma dt-\int_0^T \gamma(T-t)^{\gamma-1}X_t^2dt,$$

which is equivalent to

$$\int_0^T -2(T-t)^\gamma X_tdX_t=\frac{1}{\gamma+1}T^{\gamma+1}-\int_0^T \gamma(T-t)^{\gamma-1}X_t^2dt.$$

Note that the above expression is the value of the investor's portfolio using the strategy $H_t(\gamma,T)=-2(T-t)^\gamma X_t,\ t\in[0,T]$ hence

$$E\left[-e^{-(H\cdot S)_T}\right]=-e^{-\frac{1}{\gamma+1}T^{\gamma+1}}E\left[\exp\left\{\int_0^T \gamma(T-t)^{\gamma-1}X_t^2dt\right\}\right]. \tag{A32}$$

Since X is a Q_T-Brownian motion, clearly $H(\gamma,T)\in\mathcal{X}_T$. Denoting

$$G(T):=E\left[\exp\left\{\int_0^T \gamma(T-t)^{\gamma-1}X_t^2dt\right\}\right],$$

the next lemma states that $G(T)$ is negligible in comparison with $e^{T^{\gamma+1}}$, in the following sense:

Lemma A.9. *There exist positive constants* c_1, C_1 *and* $0 < q < 1$ *such that*

$$G(T) \leq c_1 e^{C_1 T^{\gamma+q}}.$$

Now, Lemma A.9 and (A32) implies Theorem 2.3 in the case $\beta > 1$. For the case $\beta = 1$, following an analogous method as above, consider the process $\bar{U}_t := \delta_0(T-t)X_t^2$. Similar calculations to the ones that yield (A32) lead to a strategy

$$\bar{H}_t = -2\delta_0(T-t)X_t,$$

and a portfolio value

$$E\left[-e^{-(\bar{H}\cdot S)_T}\right] = -e^{-\frac{1}{2}T^2} E\left[\exp\left\{\int_0^T \delta_0 X_t^2 dt\right\}\right].$$

Lemma A.6 with $\beta = 1$ immediately yields

$$E\left[-e^{-(\bar{H}\cdot S)_T}\right] \geq -e^{-\frac{1}{2}T^2 + C_0 T},$$

proving the claim for $\beta = 1$.

Proof of Lemma A.9. First note that, by Jensen's inequality,

$$\int_0^T X_t^2 dt \leq T^{1-1/\beta}\left(\int_0^T |X_t|^{2\beta} dt\right)^{1/\beta}.$$

Denoting $\Xi_T := \int_0^T |X_t|^{2\beta} dt$ and defining $h(x) = h_{\gamma,\beta,T}(x) := \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta}\}, x > 0$ yields the estimate

$$G(T) \leq E\left[\exp\left\{\gamma T^{\gamma-1/\beta}\left(\int_0^T |X_t|^{2\beta} dt\right)^{1/\beta}\right\}\right] = Eh(\Xi_T). \tag{A33}$$

The estimate in Lemma A.6 along with Markov's inequality, implies that, for all $x > 0$,

$$P(\Xi_T > x) \leq c_0 \exp\{C_0 T - \delta_0 x\}, \tag{A34}$$

and also observe that

$$E[h(\Xi_T)] = \int_0^\infty h'(x) P(\Xi_T > x) dx. \tag{A35}$$

Since $h'(x) = \frac{\gamma}{\beta} T^{\gamma-1/\beta} x^{1/\beta-1} e^{\gamma T^{\gamma-1/\beta} x^{1/\beta}}$, $x > 0$, (A34) and (A35) yield

$$E[h(\Xi_T)] \leq \int_0^\infty \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} x^{1/\beta-1} e^{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x + C_0 T} dx \tag{A36}$$

$$\leq \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} e^{\gamma T^{\gamma-1/\beta}+C_0 T} \int_0^1 x^{1/\beta-1} dx + \int_1^\infty \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} e^{\gamma T^{\gamma-1/\beta} x^{1/\beta}-\delta_0 x+C_0 T} dx \tag{A37}$$

$$= \tilde{c} T^{\gamma-1/\beta} e^{\gamma T^{\gamma-1/\beta}+C_0 T} + \frac{c_0\gamma}{\beta} T^{\gamma-1/\beta} e^{C_0 T} \int_1^\infty e^{\gamma T^{\gamma-1/\beta} x^{1/\beta}-\delta_0 x} dx, \tag{A38}$$

where $\tilde{c} = \tilde{c}_{\beta,\gamma} = c_0 \frac{\gamma}{\beta} \int_0^1 x^{1/\beta-1}\, dx$. To estimate the integral $\int_1^\infty \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x\} dx$, first define

$$\tilde{C}(T) := \left(\frac{2\gamma}{\delta_0}\right)^{\frac{\beta}{\beta-1}} T^{\frac{\gamma\beta-1}{\beta-1}}.$$

First note that $\gamma T^{\gamma-1/\beta} x^{1/\beta} \leq \frac{\delta_0}{2} x$ for $x > \tilde{C}(T)$, whence

$$\begin{aligned}\int_{\tilde{C}(T)}^\infty \exp\left\{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x\right\} dx &\leq \int_{\tilde{C}(T)}^\infty e^{-\frac{\delta_0}{2} x} dx = \frac{2}{\delta_0} e^{-\frac{\delta_0}{2}\tilde{C}(T)} \\ &= \frac{2}{\delta_0} \exp\left\{-\frac{\delta_0}{2}\left(\frac{2\gamma}{\delta_0}\right)^{\frac{\beta}{\beta-1}} T^{\frac{\gamma\beta-1}{\beta-1}}\right\} \leq \frac{2}{\delta_0} \exp\left\{-\frac{\delta_0}{2}\left(\frac{2\gamma}{\delta_0}\right)^{\frac{\beta}{\beta-1}} T\right\},\end{aligned} \tag{A39}$$

where the last step follows from $\frac{\gamma\beta-1}{\beta-1} > 1$. Second, the integrand $x \to \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x\}$ reaches its maximum at $x = x_0 := (\frac{\delta_0\beta}{\gamma})^{\frac{\beta}{1-\beta}} T^{\frac{\gamma\beta-1}{\beta-1}}$, and the value of such maximum is

$$\exp\left\{\gamma T^{\gamma-1/\beta} x_0^{1/\beta} - \delta_0 x_0\right\} = \exp\left\{\gamma T^{\gamma-1/\beta}\left(\frac{\delta_0\beta}{\gamma}\right)^{\frac{1}{1-\beta}} T^{\frac{\gamma-1/\beta}{\beta-1}} - \delta_0\left(\frac{\delta_0\beta}{\gamma}\right)^{\frac{\beta}{1-\beta}} T^{\frac{\beta\gamma-1}{\beta-1}}\right\}. \tag{A40}$$

Noting that $0 < -1/\beta + \frac{\gamma-1/\beta}{\beta-1} = \frac{\gamma-1}{\beta-1} < 1$, there exists $q < 1$ such that $\gamma - 1/\beta + \frac{\gamma-1/\beta}{\beta-1} = \gamma + \frac{\gamma-1}{\beta-1} < \gamma + q$ which, along with (A40) implies

$$\int_1^{\tilde{C}(T)} \exp\{\gamma T^{\gamma-1/\beta} x^{1/\beta} - \delta_0 x\} dx \leq \left(\tilde{C}(T) - 1\right) \exp\{\gamma\left(\frac{\delta_0\beta}{\gamma}\right)^{\frac{1}{1-\beta}} T^{\gamma+q}\}. \tag{A41}$$

From (A33), (A36), (A39) and (A41) it follows that

$$E[h(\Xi_T)] \leq c_1 e^{C_1 T^{\gamma+q}}$$

for suitable constants c_1 and C_1, and the proof is complete. □