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Extended Graph of the Fuzzy Topographic Topological Mapping Model

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Abstract: Fuzzy topological topographic mapping ($FTTM$) is a mathematical model which consists of a set of homeomorphic topological spaces designed to solve the neuro magnetic inverse problem. A sequence of $FTTM$, $FTTM_n$, is an extension of $FTTM$ that is arranged in a symmetrical form. The special characteristic of $FTTM$, namely the homeomorphisms between its components, allows the generation of new $FTTM$. The generated $FTTM$s can be represented as pseudo graphs. A graph of pseudo degree zero is a special type of graph where each of the $FTTM$ components differs from the one adjacent to it. Previous researchers have investigated and conjectured the number of generated $FTTM$ pseudo degree zero with respect to n number of components and k number of versions. In this paper, the conjecture is proven analytically for the first time using a newly developed grid-based method. Some definitions and properties of the novel grid-based method are introduced and developed along the way. The developed definitions and properties of the method are then assembled to prove the conjecture. The grid-based technique is simple yet offers some visualization features of the conjecture.

Keywords: $FTTM$; graph; pseudo degree; sequence

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1. Introduction

Fuzzy topographic topological mapping ($FTTM$) [1] was introduced to solve the neuro magnetic inverse problem, particularly with regards to the sources of electroencephalography (EEG) signals recorded from epileptic patients. Originally, the model was a 4-tuple of topological spaces and mappings. The topological spaces are the magnetic plane (MC), base magnetic plane (BM), fuzzy magnetic field (FM) and topographic magnetic field (TM). The third component of $FTTM$, FM, is a set of three tuples with the membership function of its potential reading obtained from a recorded EEG. $FTTM$ is defined formally as follows (see Figure 1).

$$
\begin{tikzcd}
\begin{array}{l} MC = \{(x,y,0), \beta_z | x,y,\beta_z \in \mathbb{R}\} \\ \quad = \{(x,y)_0, \beta_z \in \mathbb{R}\} \end{array} \arrow[r] \arrow[d, dotted] & TM = \{(x,y,z) | x,y \in \mathbb{R}, z \in (-h,0)\} \\
\begin{array}{l} BM = \{(x,y,h), \beta_z | x,y,\beta_z \in \mathbb{R}\} \\ \quad = \{(x,y)_h, | x,y,\beta_z \in \mathbb{R}\} \end{array} \arrow[r, dotted] & \begin{array}{l} FM = \{(x,y,h), \mu_\beta | x,y,h \in \mathbb{R}, \mu_\beta \in (0,1)\} \\ \quad = \{(x,y)_h, \mu_\beta | x,y,h \in \mathbb{R}, \mu_\beta \in (0,1)\} \end{array} \arrow[u, dotted]
\end{tikzcd}
$$

Figure 1. The $FTTM$.

Definition 1. *Ref. [1] Let $FTTM_i = (MC_i, BM_i, FM_i, TM_i)$ such that MC_i, BM_i, FM_i, TM_i are topological spaces with $MC_i \cong BM_i \cong FM_i \cong TM_i$. Set of $FTTM_i$ is denoted by*

$FTTM = \{FTTM_i : i = 1,2,3,\ldots,n\}$. Sequence of $nFTTM_i$ of FTTM is $FTTM_1, FTTM_2$, $FTTM_3$,$FTTM_4, \ldots, FTTM_n$ such that $MC_i \cong MC_{i+1}, BM_i \cong BM_{i+1}, FM_i \cong FM_{i+1}$ and $TM_i \cong TM_{i+1}$.

Furthermore, a sequence of *FTTM*, $FTTM_n$, is an extension of *FTTM* and illustrated in Figure 2. It is arranged in a symmetrical form, since the model can accommodate magnetoencephalography (MEG) signals as well as image data due to its homeomorphism.

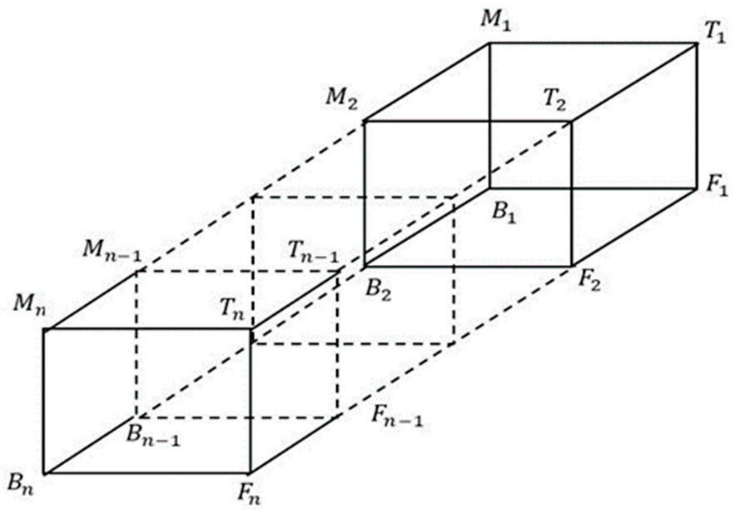

Figure 2. The sequence of $FTTM_n$.

2. Generalized FTTM

Generally, the *FTTM* structure can also be expanded for any n number of components.

Definition 2. *Ref. [2] A FTTM is defined as*

$$FTTM_n = \{\{A_1,\ A_2,\ \ldots,\ A_n\}\ :\ A_1 \cong A_2 \cong\ \ldots\ \cong A_n\} \tag{1}$$

such that $A_1,\ A_2,\ \ldots,\ A_n$ are the components of $FTTM_n$

The same generalization can be applied to any k number of *FTTM* versions as well, denoted as $FTTM_n^k$. Without the loss of generality, the collection of the k version of $FTTM$, in short $FTTM_n^k$, is now simply called as a sequence of $FTTM$ unless otherwise stated.

Definition 3. *Ref. [2] A sequence of k versions of $FTTM_n$ denoted by $*\,FTTM_n^k$ such that*

$$*\ FTTM_n^k = \left\{FTTM_n^1\ , FTTM_n^2,\ \ldots,\ FTTM_n^k\right\} \tag{2}$$

where *$FTTM_n^1$ is the first version of $FTTM_n$, the $FTTM_n^2$ is the second version of $FTTM_n$ and so forth.*

Obviously, a new $FTTM$ can be generated from a combination of components from different versions of $FTTM$ due to their homeomorphisms.

Definition 4. *Ref. [2] A new FTTM generated from $*\,FTTM_n^k$ is defined as*

$$F = \left\{A_1^{m_1}\ , A_2^{m_2}\ ,\ \ldots,\ A_n^{m_n}\right\} \in FTTM \tag{3}$$

where $0\ \leq\ m_1,\ m_2,\ \ldots,\ m_n\ \leq\ k$ and $m_i \neq\ m_j$ for at least one $i,\ j$.

A set of elements generated by $* \ FTTM_n^k$ is denoted by $G\left(* \ FTTM_n^k\right)$. Mukaram et al. [2] showed that the number of $FTTM$ can be determined from $* \ FTTM_4^k$ using the geometrical features of its graph representation.

Theorem 1. *Ref. [2] The number of generated $FTTM$ that can be created from $* \ FTTM_4^k$ is*

$$\left|G\left(* \ FTTM_4^k\right)\right| = k^4 - k. \quad (4)$$

Theorem 1 is then extended to include n number of $FTTM$ components.

Theorem 2. *Ref. [2] The number of generated FTTM that can be created from $* \ FTTM_n^k$ is*

$$\left|G\left(* \ FTTM_n^k\right)\right| = k^n - k. \quad (5)$$

The following example is presented to illustrate Theorem 2.

Example 1. *Consider $* \ FTTM_3^2$, with $FTTM_3^1 = \{A_1^1, A_2^1, A_3^1\}$ and $FTTM_3^2 = \{A_1^2, A_2^2, A_3^2\}$, then $G(* \ FTTM_3^2) = \{\{A_1^1, A_2^2, A_3^1\}, \{A_1^1, A_2^1, A_3^2\}, \{A_1^2, A_2^1, A_3^1\}, \{A_1^2, A_2^2, A_3^1\}, \{A_1^2, A_2^1, A_3^2\}, \{A_1^1, A_2^2, A_3^2\}\}$ that is $\left|G(* \ FTTM_3^2)\right| = 2^3 – 2 = 6$ as given by Theorem 2.*

3. Extended Generalization of FTTM

There are many studies on ordinary and fuzzy hypergraphs available in the literature such as [3,4]. However, $* \ FTTM_n^k$ is an extended generalization of $FTTM$ that is represented by a graph of a sequence of k number of polygons with n sides or vertices. The polygon is arranged from back to front where the first polygon represents $FTTM_n^1$, the second polygon represents $FTTM_n^2$ and so forth. An edge is added to connect $FTTM_n^1$ to the $FTTM_n^2$ component wisely. A similar approach is taken for $FTTM_n^2$, $FTTM_n^3$ and the rest (Figure 3).

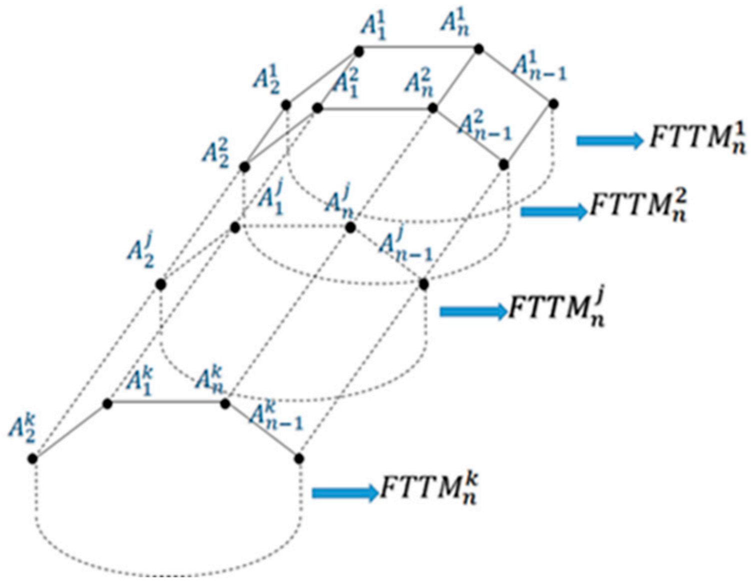

Figure 3. Graph of $* \ FTTM_n^k$.

When a new $FTTM$ is obtained from $* \ FTTM_n^k$, it is then called a pseudo-graph of the generated $FTTM$ and plotted on the skeleton of $* \ FTTM_n^k$. A generated element of a pseudo-graph consists of vertices that signify the generated $FTTM$ and edges which connect the incidence components. Two samples of pseudo-graphs are illustrated in Figure 4.

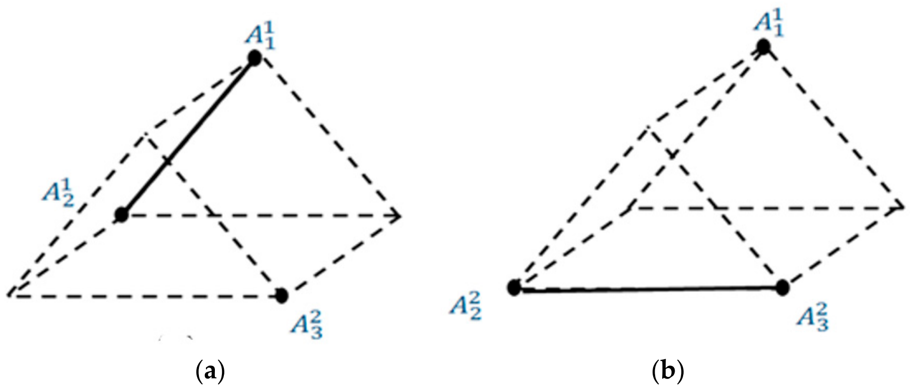

Figure 4. Pseudo graph: (**a**) $\{A_1^1, A_2^1, A_3^2\}$; (**b**) $\{A_1^1, A_2^2, A_3^2\}$ of $*\,FTTM_3^2$.

Another concept related closely to the pseudo-graph is the pseudo degree. It is defined as the sum of the pseudo degree from each component of the $FTTM$. The pseudo degree of a component is the number of other components that are adjacent to that particular component.

Definition 5. *Ref. [2] The* $deg_p : FTTM \to Z$ *defines the pseudo degree of the FTTM component. It maps a component of* $F \in G\left(*\,FTTM_n^k\right)$ *to an integer*

$$deg_p\left(A_j^{m_j}\right) = \begin{cases} 0; & m_{j-1} \neq m_j \neq m_{j+1} \\ 1; & m_{j-1} = m_j \text{ or } m_j = m_{j+1}, \\ 2; & m_{j-1} = m_j = m_{j+1} \end{cases} \quad (6)$$

for $A_j^{m_j} \in FTTM$.

Definition 6. *Ref. [2] The* $deg_p G : G\left(*\,FTTM_n^k\right) \to Z$ *defines the pseudo degree of the FTTM graph. Let* $F \in FTTM$

$$deg_p G(F) = \sum_{i=1}^{n} \deg_p A_i^{m_i} \quad (7)$$

where $F = \{A_1^{m_1}, A_2^{m_2}, \ldots, A_n^{m_n}\} \in G\left(*\,FTTM_n^k\right)$.

Definition 7. *Ref. [2] The set of elements generated by* $*\,FTTM_n^k$ *that have pseudo degree zero is*

$$G_0\left(*\,FTTM_n^k\right) = \left\{F \in G\left(*\,FTTM_n^k\right) \middle| deg_p G(F) = 0\right\} \quad (8)$$

From now on,

1. $G_0\left(*\,FTTM_n^k\right)$ is simply denoted by $G_0\left(FTTM_n^k\right)$.
2. $\#G_0\left(FTTM_n^k\right)$ denotes the cardinality of the set $G_0\left(FTTM_n^k\right)$.

Example 2. (*See Figure 5*).

$$\begin{aligned}
FTTM_4^3 &= \{(A_1, A_2, A_3, A_4),\ (B_1, B_2, B_3, B_4),\ (C_1, C_2, C_3, C_4)\} \\
G_0(FTTM_4^3) &= \{(A_1, B_2, A_3, C_4),\ (A_1, B_2, C_3, B_4),\ (A_1, C_2, A_3, B_4),\ (A_1, C_2, B_3, C_4), \\
&\quad (B_1, A_2, B_3, C_4), (B_1, A_2, C_3, A_4),\ (B_1, C_2, B_3, A_4),\ (B_1, C_2, A_3, C_4), \\
&\quad (C_1, B_2, C_3, A_4),\ (C_1, B_2, A_3, B_4),\ (C_1, A_2, C_3, B_4),\ (C_1, A_2, B_3, A_4)\} \\
G_0(FTTM_4^3) &= 12.
\end{aligned} \tag{9}$$

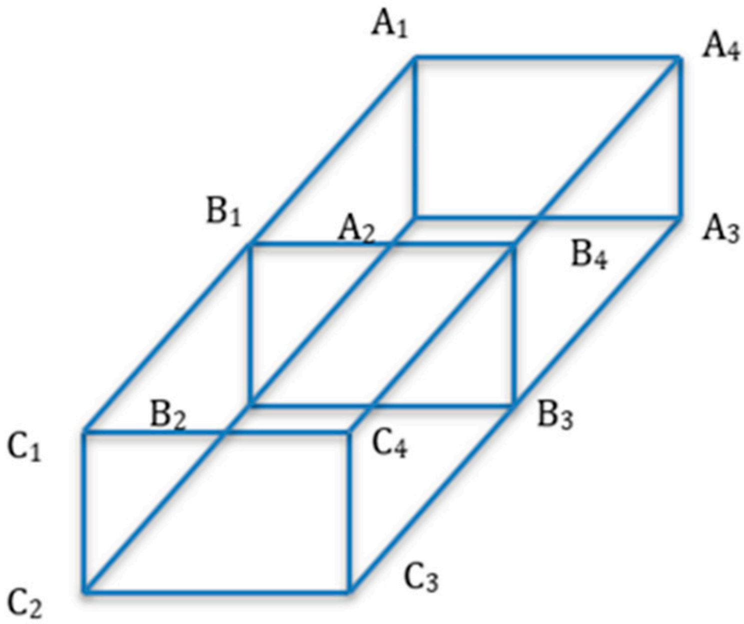

Figure 5. $FTTM_4^3$.

Previously, Elsafi proposed a conjecture in [5] related to the graph of pseudo degree.

Conjecture 1. *Ref. [5]*

$$\left|G_0^3\left(FTTM_n^3\right)\right| = \begin{cases} 4\left|G_0^3(FTTM_{n-2}^3)\right| + 12, & \textit{when } n \textit{ is even} \\ 4\left|G^3{}_0(FTTM_{n-2}^3)\right| + 6, & \textit{when } n \textit{ is odd} \end{cases} \tag{10}$$

In order to observe some patterns that may appear from the proposed conjecture, Mukaram et al. [2] have developed an algorithm to compute $\left| G_0(FTTM_n^3\ )\right|$ in order to prove the conjecture analytically. A flowchart on $\left|G_0(*\ FTTM_3^n)\right|$ is sampled in Figure 6.

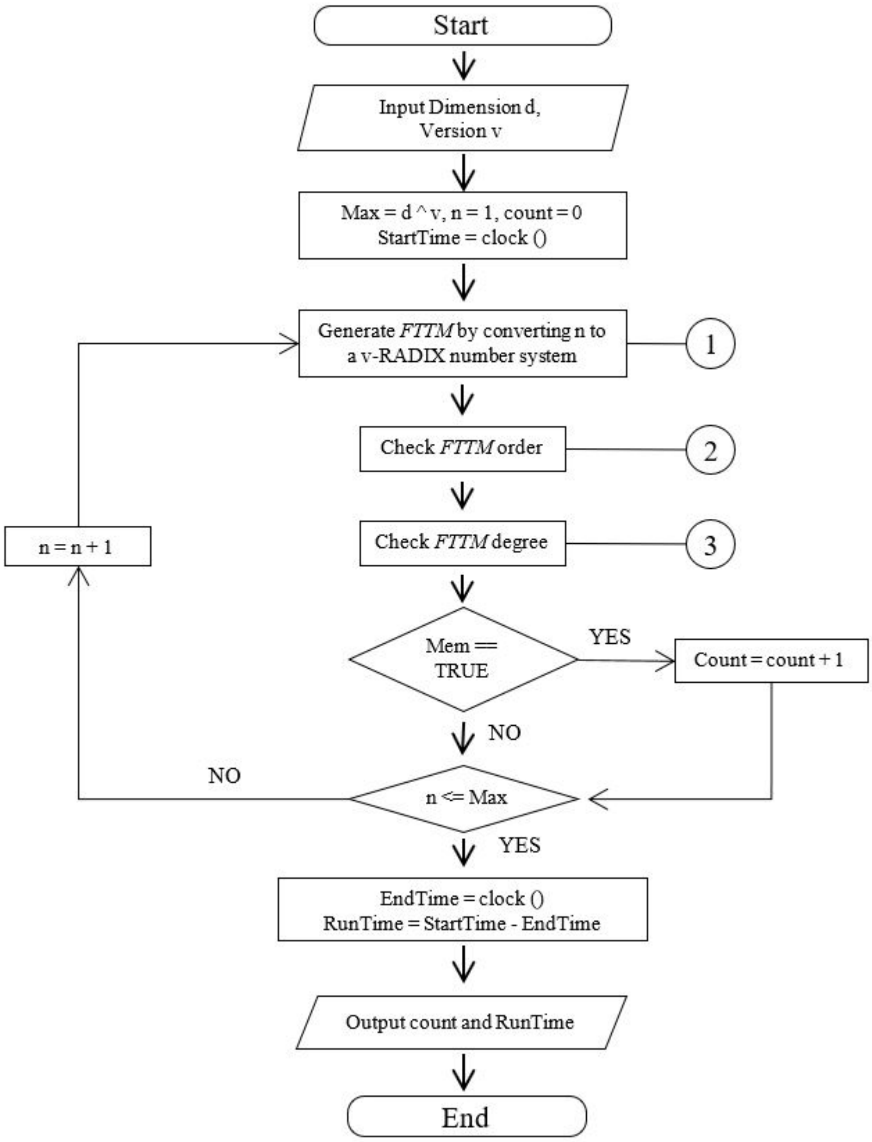

Figure 6. Flowchart for determining $\left|G_0\left(* \, FTTM_3^n\right)\right|$.

The researchers generated all $FTTM$ combinations for $3 \leq k \leq 4$, $4 \leq \ n \ \leq \ 15$ and were able to isolate graphs with pseudo degree zero, which are listed below (Table 1).

Table 1. $\left| \ G_0\left(FTTM_n^k\right)\right|$ for $4 \leq \ n \ \leq \ 15$ and $k = 3, 4$.

n	$\mid G_0(FTTM_n^3) \mid$	$\mid G_0(FTTM_n^4) \mid$
4	12	24
5	30	120
6	60	480
7	126	1680
8	252	5544
9	510	17,640
10	1020	54,960
11	2046	168,960
12	4092	515,064
13	8190	1,561,560
14	16,380	4,717,440
15	32,766	14,217,840

The researchers then simulated $\left|G_0\left(FTTM_n^k\right)\right|$ for some values of k as well [2]. The number of graphs of pseudo degree zero for $2 \leq k \leq 8$ and $2 \leq n \leq 10$ are listed in Table 2.

Table 2. $\left|G_0\left(FTTM_n^k\right)\right|$ for $2 \leq k \leq 8$ and $2 \leq n \leq 10$.

k*/*n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
2	2	0	2	0	2	0	2	0	2
3	0	6	12	30	60	126	252	510	1020
4	0	0	24	120	480	1680	5544	17,640	54,960
5	0	0	0	120	1080	6720	35,280	168,840	763,560
6	0	0	0	0	720	10,080	90,720	665,280	4,339,440
7	0	0	0	0	0	5040	100,800	1,239,840	12,096,000
8	0	0	0	0	0	0	40,320	1,088,640	17,539,200

4. Grid of FTTM

An alternative presentation of a sequence of $FTTM$, called an $FTTM$ grid, is briefly overviewed. It provides a different perspective of the structure of $FTTM$. Instead of a polygon representation for each version of $FTTM$, a straight line is now used. The components of $FTTM_n$ are arranged on a horizontal line of vertices and the lines represent the homeomorphisms between the components of $FTTM_n$. The only exception is the homeomorphism between the first and last components of $FTTM_n$, A_1 and A_n, respectively. Two open segments on the left of A_1 and on the right of A_n are used to represent the homeomorphism between them. A vertical line is added to represent a homeomorphism between two components of different versions; hence, a grid is created (see Figure 7).

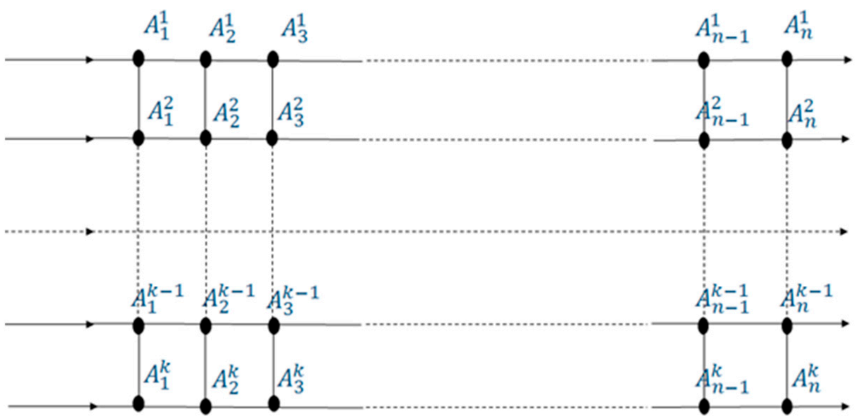

Figure 7. A graph representation of $*\,FTTM_n^k$ as a grid.

There are four advantages when $FTTM$ is represented as a grid instead of a sequence of polygon.

- It is represented in two dimensions; therefore, it reduces the complexity of the structure.
- The process of adding a new component is easier than in a sequence of polygon.
- It can take any number of components by adding the number of vertices at the end of the grid.
- The homeomorphism between two components of the same version is presented as a horizontal edge, whereas the homeomorphism between two components of two different versions is represented by a diagonal edge (see Figure 8). These arrangements are necessary to produce the graph of pseudo degree zero.

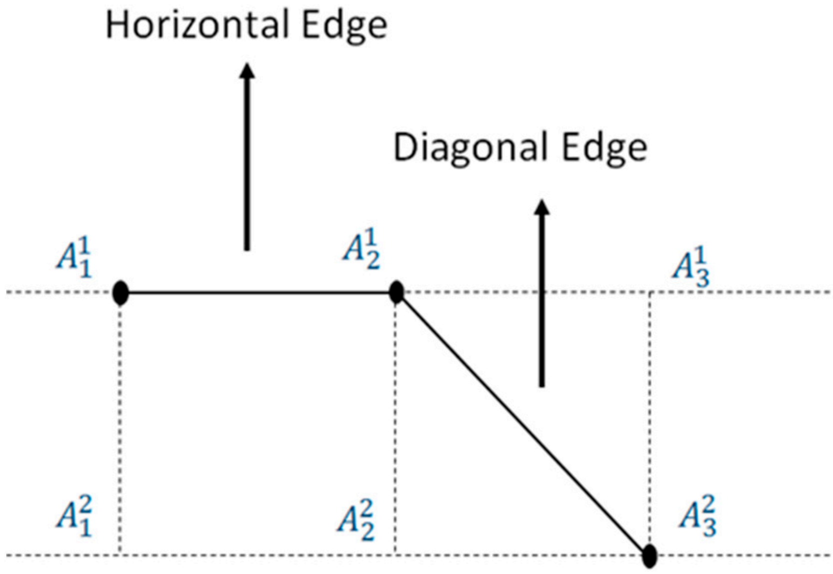

Figure 8. Generated element $\{A_1^1,\ A_2^1,\ A_3^2\}$ on $*\,FTTM_3^2$ grid.

Furthermore, Zilullah et al. [2] introduced some operations and properties with respect to the FTTM grid. They are recalled, summarized and listed below for convenience. Then, we will move on to the next main section of the paper wherein Conjecture 1 is finally proven as a theorem.

Definition 8. *Let* $F \in G\left(*\,FTTM_n^k\right)$ *and* $F = \left\{A_1^{m_1}, A_2^{m_2}, \ldots, A_n^{m_n}\right\}$. *A block* B *, where* $B \subseteq F$ *is defined as*

$$B = \left\{A_i^{m_i}, A_{i+1}^{m_{i+1}}, A_{i+2}^{m_{i+2}}, \ldots,\ A_{i+j}^{m_{i+j}}\right\},\ 1 \le i < n,\ 0 < j \le n-1 \tag{11}$$

$B\left(G\left(*\,FTTM_n^k\right)\right)$ *is the set of FTTM blocks that can be generated from* $G\left(*\,FTTM_n^k\right)$.

Definition 9. *The function* C_i^j *is defined as* $C : G\left(*\,FTTM_n^k\right) \to B\left(G\left(*\,FTTM_n^k\right)\right)$ *for* $F \in G\left(*\,FTTM_n^k\right)$,

$$B = \left\{A_i^{m_i}, A_{i+1}^{m_{i+1}}, A_{i+2}^{m_{i+2}}, \ldots,\ A_{i+j}^{m_{i+j}}\right\},\ 1 \le i < n,\ 0 < j \le n-1 \tag{12}$$

for $1 < i < j < n$, *where* $F = \left\{A_1^{m_1}, A_2^{m_2}, A_3^{m_3}, \ldots,\ A_n^{m_n}\right\}$.

Definition 10. *The operation* $\oplus$ *is a mapping* $\oplus : B\left(G\left(*\,FTTM_n^k\right)\right) \times B\left(G\left(*\,FTTM_n^k\right)\right) \to B\left(G\left(*\,FTTM_n^k\right)\right)$ *such that*

$$\left\{A_i^{m_i}, A_{i+1}^{m_{i+1}}, \ldots, A_k^{m_k}\right\} \oplus \left\{A_p^{m_p}, A_{p+1}^{m_{p+1}}, \ldots, A_j^{m_j}\right\} = \left\{A_i^{m_i}, A_{i+1}^{m_{i+1}}, \ldots, A_j^{m_j}\right\} \tag{13}$$

when $k = p$ *and* $m_k = m_p$, *then* $B_3 = B_1 \oplus\ B_2 = \left\{A_i^{m_i}, A_{i+1}^{m_{i+1}}, \ldots, A_j^{m_j}\right\}$.

Definition 11. *An indexed FTTM* $\underset{j=i}{G}\left(*\,FTTM_n^k\right)$ *is defined as*

$$\underset{m_j=i}{G}\left(*\,FTTM_n^k\right) = \left\{F \in G\left(*\,FTTM_n^k\right) \middle|\ A_j^{m_j} \in F, m_j = i\ \right\} \tag{14}$$

A generated $FTTM$ is then divided into blocks of three components. A set of blocks is defined as follows.

Definition 12. *A set of blocks B_{ijk} is defined as*

$$B_{ijk} = \left\{ B \in G\left(* \, FTTM_n^k \right) \middle| \, B = \left\{ A_p^{m_p}, A_{p+1}^{m_{p+1}}, A_{p+2}^{m_{p+2}} \right\}, m_p = i, m_{p+1} = j, m_{p+2} = k \right\} \tag{15}$$

Since this study is concerned with graphs of pseudo degree zero, the sets that need to be taken into consideration are the ones with diagonal paths, namely, $B_{121}, B_{121}, B_{123}, B_{131}, B_{132}, B_{212}, B_{213}, B_{232}, B_{231}, B_{321}, B_{312}, B_{323}$ and B_{313}.

Lemma 1. *Let $F \in * \, FTTM_n^k$ and $F = \left\{ A_1^{m_1}, A_2^{m_2}, \ldots, A_n^{m_n} \right\}$. For any $A_j^{m_j} \in F, 1 < j < n$, then $\deg_p\left(A_j^{m_j} \right) = 0$ if $A_j^{m_j}$ is connected to $A_{j-1}^{m_{j-1}}$ and $A_{j+1}^{m_{j+1}}$ by a diagonal path.*

Theorem 3. *If $F \in G_d\left(* \, FTTM_n^3 \right)$, where $G_d\left(* \, FTTM_n^3 \right)$ is the set of generated $FTTM$s with a diagonal path, then $\deg_p G(F) = 2$ or 0.*

Corollary 1. *The element of $G_0\left(FTTM_n^k \right)$ has a $FTTM$ path with the following properties*:

1. All the edges connecting the path are diagonal.
2. The starting and the end points of the path belong to different versions of *FTTM*.

Theorem 4. *If $x \in B\left(G_0\left(* \, FTTM_n^k \right) \right)$, then all the paths for x are diagonals.*

Proposition 1. *If $F \in G\left(* \, FTTM_n^k \right)$, then $C_1^{n-2}(F) \in G\left(* \, FTTM_{n-2}^k \right)$.*

Lemma 2. *If $F \in G\left(* \, FTTM_n^k \right)$, then $\exists x, y$ such that $x \in G\left(* \, FTTM_{n-2}^k \right)$, $y \in C_{n-2}^n\left(G\left(* \, FTTM_n^k \right) \right)$ and $F = x \oplus y$.*

Lemma 3. *If $F \in G\left(* \, FTTM_n^k \right)$, then $\exists$ unique tuple (x, y) such that $x \in G\left(* \, FTTM_{n-2}^k \right)$, $y \in C_{n-2}^n\left(G\left(* \, FTTM_n^k \right) \right)$ and $F = x \oplus y$.*

Theorem 5. *If $H \subseteq G\left(* \, FTTM_n^k \right)$ and $K = \{(x, y) | x \oplus y \in H, \, x \in G(* \, FTTM_{n-2}^3), y \in C_{n-2}^n(G(* \, FTTM_n^3))\}$, then $|K| = |C|$.*

Lemma 4.

$$\left(* \, FTTM_n^3 \right) = \underset{m_{n-2}=1}{G}\left(* \, FTTM_n^3 \right) \cup \underset{m_{n-2}=2}{G}\left(* \, FTTM_n^3 \right) \cup \underset{m_{n-2}=3}{G}\left(* \, FTTM_n^3 \right). \tag{16}$$

Lemma 5.

$$\underset{m_{n-2}=a}{G}\left(* \, FTTM_n^3 \right) \cap \underset{m_{n-2}=b}{G}\left(* \, FTTM_n^3 \right) = \varnothing \tag{17}$$

for any $a, b \in \mathbb{Z}$ and $a \neq b$.

Theorem 6.

$$\left| G\left(* \, FTTM_n^3 \right) \right| = \left| \underset{m_{n-2}=1}{G}\left(* \, FTTM_n^3 \right) \right| + \left| \underset{m_{n-2}=2}{G}\left(* \, FTTM_n^3 \right) \right| + \left| \underset{m_{n-2}=3}{G}\left(* \, FTTM_n^3 \right) \right| \tag{18}$$

5. The Theorem

All the materials laid down in previous sections are assembled to produce the analytical proof of Conjecture 1. The first step is to find $\left|G_d\left(*\,FTTM_n^3\right)\right|$ since $G_0\left(*\,FTTM_n^3\right)$ is a subset of $G_d\left(*\,FTTM_n^3\right)$ by Theorem 2.

Theorem 7.

$$\left|G_d\left(*\,FTTM_n^3\right)\right| = \begin{cases} 12\,.\,4^{\frac{n-3}{2}}, & n \text{ is odd},\ n \geq 3 \\ 6\,.\,4^{\frac{n-2}{2}}, & n \text{ is even },\ n \geq 4. \end{cases} \tag{19}$$

Proof of Theorem 7. (By mathematical induction)

Let

$$P(m) = \left|G_d\left(*\,FTTM_n^3\right)\right| = \begin{cases} 12\,.\,4^{\frac{n-3}{2}}, & n \text{ is odd},\ n \geq 3 \\ 6\,.\,4^{\frac{n-2}{2}}, & n \text{ is even },\ n \geq 4 \end{cases} \tag{20}$$

For odd numbers, $P(3) : n = 3$,

$$P(3) = \left|G_d\left(*\,FTTM_3^3\right)\right| = 12\,.\,4^{\frac{3-3}{2}} = 12. \tag{21}$$

There are exactly 12 combinations, namely

$$\{A_1^1, A_2^2, A_3^3\}, \{A_1^1, A_2^2, A_3^1\}, \{A_1^1, A_2^3, A_3^2\}, \{A_1^1, A_2^3, A_3^1\}, \{A_1^2, A_2^1, A_3^3\}, \{A_1^2, A_2^3, A_3^1\},$$
$$\{A_1^2, A_2^1, A_3^2\}, \{A_1^2, A_2^3, A_3^1\}, \{A_1^3, A_2^2, A_3^3\}, \{A_1^3, A_2^2, A_3^1\}, \{A_1^3, A_2^1, A_3^3\}, \{A_1^3, A_2^1, A_3^2\}$$

Now assume $P(m = 2k+1) : n = 2k+1$ is true with

$$P(m) = \left|G_d\left(*\,FTTM_{2k+1}^3\right)\right| = 12\,.\,4^{\frac{2k+1-3}{2}} = 12\,.\,4^{k-1} \tag{22}$$

for $P\begin{pmatrix} m+2 = 2k+1+2 \\ 2k+3 \end{pmatrix}$.

By using Theorem 4, $P(m+1) = \left|G_0\left(*\,FTTM_{2k+3}^3\right)\right| = |K|$ such that

$$K = \left\{(x,y)\middle|x \oplus y \in H,\ x \in G\left(*\,FTTM_{2k+1}^3\right),\ y \in C_{n-2}^n\left(G\left(*\,FTTM_{2k+3}^3\right)\right)\right\}. \tag{23}$$

By using Theorem 5,

$$\begin{aligned} |P(m+1)| &= \left|G_d\left(*\,FTTM_{2k+3}^3\right)\right| \\ &= \left|\underset{m_{n-2}=1}{G_d}\left(*\,FTTM_{2k+3}^3\right)\right| + \left|\underset{m_{n-2}=2}{G_d}\left(*\,FTTM_{2k+3}^3\right)\right| + \left|\underset{m_{n-2}=3}{G_d}\left(*\,FTTM_{2k+3}^3\right)\right| \end{aligned} \tag{24}$$

The set $\underset{m_{n-2}=1}{G_d}\left(*\,FTTM_{2k+3}^3\right)$ can be constructed from (x,y) where $x \in \underset{m_{n-2}=1}{G_d}\left(*\,FTTM_{2k+1}^3\right)$ and $y \in C_{n-2}^n\left(G_d\left(*\,FTTM_{2k+3}^3\right)\right)$. There are four options for y, namely $B_{121}, B_{123},\ B_{131},$ and B_{132}. Hence,

$$\left|\underset{m_{n-2}=1}{G_d}\left(*\,FTTM_{2k+3}^3\right)\right| = 4\left|\underset{m_{n-2}=1}{G_d}\left(*\,FTTM_{2k+1}^3\right)\right|. \tag{25}$$

The same process can be applied to $\left|\underset{m_{n-2}=2}{G_d}\left(*\,FTTM_{2k+3}^3\right)\right|$ and $\left|\underset{m_{n-2}=3}{G_d}\left(*\,FTTM_{2k+3}^3\right)\right|$.

Thus,

$$
\begin{aligned}
&|P(m+1)| \\
&= \left|G_d\left(*\ FTTM^3_{2k+3}\right)\right| \\
&= 4\left|\underset{m_{n-2}=1}{G_d}\left(*\ FTTM^3_{2k+1}\right)\right| + 4\left|\underset{m_{n-2}=2}{G_d}\left(*\ FTTM^3_{2k+1}\right)\right| + 4\left|\underset{m_{n-2}=3}{G_d}\left(*\ FTTM^3_{2k+1}\right)\right| \\
&= 4\left(\left|\underset{m_{n-2}=1}{G_d}\left(*\ FTTM^3_{2k+1}\right)\right| + \left|\underset{m_{n-2}=2}{G_d}\left(*\ FTTM^3_{2k+1}\right)\right| + \left|\underset{m_{n-2}=3}{G_d}\left(*\ FTTM^3_{2k+1}\right)\right|\right) \\
&= 4\left|G_d\left(*\ FTTM^3_{2k+1}\right)\right| = 4\ .\ 12\ .\ 4^{k-1} = 12\ .\ 4^k.
\end{aligned}
\tag{26}
$$

Similarly, the same induction process can be used as proof for even parts. □

The set $G_d\left(*\ FTTM^3_n\right)$ has only two possible subsets, namely $G_0\left(*\ FTTM^3_n\right)$ and $H_n = \left\{x \in G_d\left(*\ FTTM^3_n\right) \middle| \deg_p x = 2\right\}$. To find $G_0\left(*\ FTTM^3_n\right)$, the relation between $G_0\left(*\ FTTM^3_n\right)$, $G_d\left(*\ FTTM^3_n\right)$ and H_n must be investigated.

Lemma 6. *If* $H_n = \left\{x \in G_d\left(*\ FTTM^3_n\right) \middle| deg_p x = 2\right\}$, *then* $|H_n| = \left|G_d\left(*\ FTTM^3_n\right)\right| - \left|G_0\left(*\ FTTM^3_n\right)\right|$.

Proof of Lemma 6. Let $x \in G_d\left(*\ FTTM^3_n\right)$, then $\deg_p(x) = 0$ or $\deg_p(x) = 2$ by Theorem 5. Thus, $x \in G_0\left(*\ FTTM^3_n\right)$ or $x \in H_n$, i.e., $\left|G_d\left(*\ FTTM^3_n\right)\right| = \left|G_0\left(*\ FTTM^3_n\right)\right| + |H_n|$ or $|H_n| = \left|G_d\left(*\ FTTM^3_n\right)\right| - \left|G_0\left(*\ FTTM^3_n\right)\right|$. □

Finally, $\left|\underset{m_{n-2}=i}{G_0}\left(*\ FTTM^3_n\right)\right|$ is determined using Lemma 6 and Theorem 5.

Theorem 8.

$$
\left|\underset{m_{n-2}=i}{G_0}\left(*\ FTTM^3_n\right)\right| = 3\left|\underset{m_{n-2}=i}{G_0}\left(*\ FTTM^3_{n-2}\right)\right| + 2|H_{n-2}|,\ n > 4 \tag{27}
$$

Proof of Theorem 8. By Theorem 5, $\left|\underset{m_{n-2}=i}{G_0}\left(*\ FTTM^3_n\right)\right|$ can be determined by the combination of (x, y) where $x \oplus y \in \underset{m_{n-2}=i}{G_0}\left(*\ FTTM^3_n\right), x \in \underset{m_{n-2}=i}{G}\left(*\ FTTM^3_{n-2}\right)$, $y \in C^n_{n-2}\left(\underset{m_{n-2}=i}{G}\left(*\ FTTM^3_{n-2}\right)\right)$. By Theorem 4, all x edges must be diagonal; hence, $x \in \underset{m_{n-2}=i}{G_d}\left(*\ FTTM^3_{n-2}\right)$. There are two possibilities for the value of x, namely $x \in \underset{m_{n-2}=i}{G_0}\left(*\ FTTM^3_{n-2}\right)$ or $x \in |H_{n-2}|$, where $H_{n-2} = \left\{x \in G_d\left(*\ FTTM^3_{n-2}\right) \middle| \deg_p x = 2\right\}$ from Theorem 3. Case $i = 1$: if $x \in \underset{m_{n-2}=1}{G_0}\left(*\ FTTM^3_{n-2}\right)$, then $A^{m_1}_1 \in x$, $m_1 \neq 1$ which implies $m_1 = 2$ or $m_1 = 3$ by Corollary 1.

Let $X_2 = \left\{x \in \underset{m_{n-2}=1}{G_0}\left(*\ FTTM^3_{n-2}\right) \middle| m_1 = 2\right\}$, $X_3 = \left\{x \in \underset{m_{n-2}=1}{G_0}\left(*\ FTTM^3_{n-2}\right) \middle| m_1 = 3\right\}$, then for any $x \in X_2$, then $y \in B_{121}, B_{123}, B_{131}$ and also for any $x \in X_3$, then $y \in B_{121}, B_{132}, B_{131}$ by Corollary 1. Thus, for $\in \underset{m_{n-2}=1}{G_0}\left(*\ FTTM^3_{n-2}\right)$, there are $3\left|\underset{m_{n-2}=1}{G_0}\left(*\ FTTM^3_{n-2}\right)\right|$ combinations of tuple (x, y).

If $x \in H_{n-2}$, then $A^{m_1}_1 \in x$, $m_1 = 1$ when $x \in H_{n-2}$ and $y \in B_{123}, B_{132}$ by Corollary 1. Thus, there are $3|H_{n-2}|$ combinations of tuple (x, y) Hence, $\left|\underset{m_{n-2}=1}{G_0}\left(*\ FTTM^3_n\right)\right| =$

$3\left|\underset{m_{n-2}=1}{G_0}(*\,FTTM^3_{n-2})\right| + 2|H_{n-2}|,\ n > 4$. Using the same procedure as for $i = 1$, the same result can be obtained for $i = 2, 3$. □

Theorem 9.

$$\left|G_0\left(*\,FTTM^3_n\right)\right| = \begin{cases} \left|G_0(*\,FTTM^3_{n-2})\right| + 3.\,2^{n-2}, & n \text{ is odd},\ n > 3 \\ \left|G_0(*\,FTTM^3_{n-2})\right| + 3.\,2^{n}, & n \text{ is even},\ n > 4 \end{cases} \tag{28}$$

where $\left|G_0(*\,FTTM^3_3)\right| = 6, \left|G_0(*\,FTTM^3_4)\right| = 12.$

Proof of Theorem 9. Using Theorem 6, $\left|G_0(*\,FTTM^3_n)\right| = \left|\underset{m_{n-2}=1}{G_0}(*\,FTTM^3_n)\right| + \left|\underset{m_{n-2}=2}{G_0}(*\,FTTM^3_n)\right| + \left|\underset{m_{n-2}=3}{G_0}(*\,FTTM^3_n)\right|$. From Theorem 8 and Lemma 6,

$$\begin{aligned} &\left|G_0(*\,FTTM^3_n)\right| \\ &= \left|\underset{m_{n-2}=1}{G_0}(*\,FTTM^3_{n-2})\right| + 2\left|\underset{m_{n-2}=1}{G_d}(*\,FTTM^3_{n-2})\right| + \left|\underset{m_{n-2}=2}{G_0}(*\,FTTM^3_{n-2})\right| + 2\left|\underset{m_{n-2}=2}{G_d}(*\,FTTM^3_{n-2})\right| + \left|\underset{m_{n-2}=3}{G_0}(*\,FTTM^3_{n-2})\right| + 2\left|\underset{m_{n-2}=3}{G_d}(*\,FTTM^3_{n-2})\right| \end{aligned}$$

$$\begin{aligned} &= \left(\left|\underset{m_{n-2}=1}{G_0}(*\,FTTM^3_{n-2})\right| + \left|\underset{m_{n-2}=2}{G_0}(*\,FTTM^3_{n-2})\right| + \left|\underset{m_{n-2}=3}{G_0}(*\,FTTM^3_{n-2})\right|\right) + 2\left(\left|\underset{m_{n-2}=3}{G_d}(*\,FTTM^3_{n-2})\right| + \left|\underset{m_{n-2}=1}{G_d}(*\,FTTM^3_{n-2})\right| + \left|\underset{m_{n-2}=2}{G_d}(*\,FTTM^3_{n-2})\right|\right) \\ &= \left|G_0(*\,FTTM^3_{n-2})\right| + 2 * \left|G_d(*\,FTTM^3_n)\right|. \end{aligned} \tag{29}$$

Hence by Theorem 7,

$$\left|G_0\left(*\,FTTM^3_n\right)\right| = \begin{cases} \left|G_0(*\,FTTM^3_{n-2})\right| + 3\,.\,2^{n-2}, & n \text{ is odd},\ n > 3 \\ \left|G_0(*\,FTTM^3_{n-2})\right| + 3\,.\,2^{n}, & n \text{ is even},\ n > 4 \end{cases} \tag{30}$$

such that $\left|G_0(*\,FTTM^3_3)\right| = 6,\ \left|G_0(*\,FTTM^3_4)\right| = 12.$ □

Theorem 9 is another version of the earlier conjecture. A simple algebraic manipulation is needed to show their equivalence. We formally state and prove this as the final theorem.

Theorem 10.

$$\begin{aligned} \left|G^3_0(FTTM^3_n)\right| &= \begin{cases} 4\left|G^3_0(FTTM^3_{n-2})\right| + 12\,, & \text{where } n \text{ is even} \\ 4\left|G^3{}_0(FTTM^3_{n-2})\right| + 6, & \text{where } n \text{ is odd} \end{cases} \\ &= \begin{cases} \left|G_0(*\,FTTM^3_{n-2})\right| + 3\,.\,2^{n-2}, & n \text{ is odd},\ n > 3 \\ \left|G_0(*\,FTTM^3_{n-2})\right| + 3\,.\,2^{n}, & n \text{ is even},\ n > 4 \end{cases} \end{aligned} \tag{31}$$

where $,\left|G_0\left(*\,FTTM^3_3\right)\right| = 6, \left|G_0(*\,FTTM^3_4)\right| = 12.$

Proof of Theorem 10. By Theorem 9,

$$\left|G_0\left(FTTM^3_n\right)\right| = \begin{cases} 4\left|G_0(FTTM^3_{n-2})\right| + 12\,, & \text{where } n \text{ is even} \\ 4\left|G_0(FTTM^3_{n-2})\right| + 6, & \text{where } n \text{ is odd} \end{cases} \tag{32}$$

and $\left|G_0\left(FTTM^3_3\right)\right| = 6,\ \left|G_0\left(FTTM^3_4\right)\right| = 12.$

However, when n is odd,

$$\begin{aligned}
\left|G_0\left(FTTM_5^3\right)\right| &= 4\,.\,6+6 \\
&= 4^1\,.\,6+4^0\,.\,6 \\
\left|G_0\left(FTTM_7^3\right)\right| &= 4(4\,.\,6+6)+6 \\
&= 4^2\,.\,6+4^1\,.\,6+4^0\,.\,6 \\
\left|G_0\left(FTTM_9^3\right)\right| &= 4(4(4\,.\,6+6)+6)+6 \\
&= 4^3\,.\,6+4^2\,.\,6+4^1\,.\,6+4^0\,.\,6 \\
\left|G_0\left(FTTM_{11}^3\right)\right| &= 4(4(4(4\,.\,6+6)+6)+6)+6 \\
&= 4^4\,.\,6+4^3\,.\,6+4^2\,.\,6+4^1\,.\,6+4^0\,.\,6
\end{aligned} \tag{33}$$

Thus, $\left|G_0\left(FTTM_n^3\right)\right| = \sum_{k=0}^{\frac{n-3}{2}} 4^k\,.\,6$.
Notice that

$$\begin{aligned}
\left|G_0\left(FTTM_n^3\right)\right| &= \sum_{k=0}^{\frac{n-3}{2}} 4^k\,.\,6 \\
&= 4^{\frac{n-3}{2}}\,.\,6+\sum_{k=0}^{\frac{n-5}{2}} 4^k\,.\,6 \\
&= 2^{n-3}\,.\,6+\left|G_0\left(FTTM_{n-2}^3\right)\right| \\
&= 2^{n-2}\,.\,3+\left|G_0\left(FTTM_{n-2}^3\right)\right|
\end{aligned} \tag{34}$$

When n is even,

$$\begin{aligned}
\left|G_0\left(FTTM_6^3\right)\right| &= 4\,.\,12+12 \\
&= 4^1\,.\,12+4^0\,.\,12 \\
\left|G_0\left(FTTM_8^3\right)\right| &= 4(4\,.\,12+12)+12 \\
&= 4^2\,.\,12+4^1\,.\,12+4^0\,.\,12 \\
\left|G_0\left(FTTM_{10}^3\right)\right| &= 4(4(4\,.\,12+12)+12)+12 \\
&= 4^3\,.\,12+4^2\,.\,12+4^1\,.\,12+4^0\,.\,12 \\
\left|G_0\left(FTTM_{12}^3\right)\right| &= 4(4(4(4\,.\,12+12)+12)+12)+12 \\
&= 4^4\,.\,12+4^3\,.\,12+4^2\,.\,12+4^1\,.\,12+4^0\,.\,12
\end{aligned} \tag{35}$$

Thus, $\left|G_0\left(FTTM_n^3\right)\right| = \sum\limits_{k=0}^{\frac{n-4}{2}} 4^k\,.\,12$.
Notice that,

$$\begin{aligned}
\left|G_0\left(FTTM_n^3\right)\right| &= \sum_{k=0}^{\frac{n-4}{2}} 4^k\,.\,12 \\
&= 4^{\frac{n-4}{2}}\,.\,12+\sum_{k=0}^{\frac{n-6}{2}} 4^k\,.\,12 \\
&= 2^{n-2}\,.\,3+\sum_{k=0}^{\frac{n-6}{2}} 4^k\,.\,12 \\
&= 2^{n-2}\,.\,3+\left|G_0\left(FTTM_{n-2}^3\right)\right|
\end{aligned} \tag{36}$$

It shows that the equation in Theorem 9 is exactly the statement of the conjecture. In other words, the conjecture is proven by construction. □

The whole process of proving Conjecture 1 is summarized below in Figure 9.

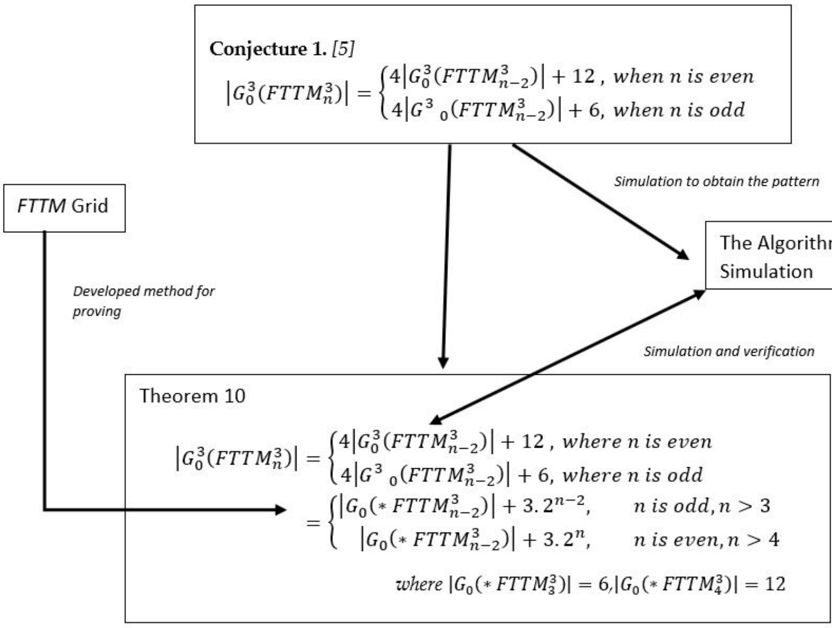

Figure 9. Outline of proving Conjecture 1 by construction.

6. Conclusions

The developed grid-based method of proof is new; some definitions and properties were introduced, whereas others were investigated along the way. The originality and advantages of this method can be summarized in the point forms below.

- It provides a different perspective to the structure of $FTTM$. Instead of a polygon representation for each version of $FTTM$, a straight line is now used. The components of $FTTM_n$ are arranged on a horizontal line of vertices and the lines represent the homeomorphisms between the components of $FTTM_n$.
- A vertical line is added to represent a homeomorphism between two components of different versions; hence, a grid is created.
- It is represented in two dimensions; therefore, it reduces the complexity of the structure.
- The process of adding a new component is easier than in a sequence of polygon.
- It can take any number of components by adding the number of vertices at the end of the grid.
- The homeomorphism between two components of the same version is presented as a horizontal edge, whereas the homeomorphism between two components of two different versions is represented by a diagonal edge (see Figure 8).
- This grid-based technique offers an edge in proving the conjecture; in particular, it enables one to visualize a given problem in a 2-dimensional space.
- Finally, the conjecture that spells the number of the generated $FTTM$ graph of pseudo degree zero with respect to n number of components and k number of versions is proven analytically for the first time using this method.

However, the lengthy computing time for simulation needs to be resolved for larger k and n, accordingly. This may be overcome by employing parallel computing, and the grid-based technique can be very handy for such enumerative combinatorics problems in the near future.

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Abbreviations

The following abbreviations are used in this manuscript.

BM	Base magnetic plane
EEG	Electroencephalography
FM	Fuzzy magnetic field
$FTTM$	Fuzzy topological topographic mapping
$FTTM_n$	Sequence of FTTM
MC	Magnetic plane
MEG	Magnetoencephalography
TM	Topographic magnetic field
$*FTTM_n^k$	Sequence of k versions of $FTTM_n$
$G_0\left(*FTTM_n^k\right)$	Set of elements generated by $*FTTM_n^k$ that have pseudo degree zero
$G_0\left(FTTM_n^k\right)$	Set of elements generated by $*FTTM_n^k$. that have pseudo degree zero

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