Justifying the use of purely formal analogies in physics

Doreen Fraser*
Department of Philosophy
University of Waterloo
dlfraser@uwaterloo.ca

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ABSTRACT: Recent case studies have revealed that purely formal analogies have been successfully used as a heuristic in physics. This is at odds with most general philosophical accounts of analogies, which require analogies to be physical in order to be justifiably used. The main goal of this paper is to supply a philosophical account that justifies the use of purely formal analogies in physics. Using Bartha's (2010) articulation model as a starting point, I offer precise definitions of formal and physical analogies and propose a new submodel of analogical reasoning that accounts for the successful use of purely formal analogies in the development of renormalization group methods for use in particle physics in the early 1970's (Fraser 2020). Two distinctive features of this new *applied mathematics* submodel for analogical reasoning are that the conclusion of the argument from analogy includes both an entire model (and not only a hypothesis or a prediction) and the construction procedure for this model. A third important difference from arguments from physical analogy is that only the *prima facie* plausibility of the conclusion is established, and not stronger types of plausibility associated with confirmation. The use of purely formal analogies is justified because they are suited to supporting conclusions of this sort. Formulating a general philosophical account of analogies that covers purely formal analogies also serves two additional purposes: (1) to highlight the features of this case that are novel in the context of examples of analogies traditionally considered by philosophers and (2) to establish that physicists should not

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automatically dismiss purely formal analogies when evaluating heuristics for the development of new models.

1 Introduction

The goal of this paper is to establish that purely formal analogies can justifiably play a heuristic role in physics. Case studies of analogical reasoning have revealed that purely formal analogies have in fact been successfully used in physics. Examples include the development of the Higgs model for electroweak interactions in the 1960's (Fraser and Koberinski 2016), the construction of new models for particle physics based on renormalization group methods in the 1970's (Fraser 2020), the theory of black body radiation formulated by Einstein in the 1900's (Gingras 2015), and recent analogue bouncing oil droplet models for quantum pilot wave dynamics (Evans and Thébault 2020).[1] There are also possible examples of the successful use of purely formal analogies outside of physics. The Johansen-Ledoit-Sornette model of market crashes in econophysics one potential example,[2] and another is Galton's use of a quincunx as an analogue model to support statistical inferences about inheritance in biology.[3] Standing opposed to case studies of purely formal analogies are philosophical accounts of analogies that argue on general grounds that analogies must be physical (or biological or economic) in order to be justifably used in science, such as Hesse (1966), Bartha (2010), and Norton (2021). I will argue that this conflict between case studies and general accounts should be resolved in the usual way: general philosophical accounts of analogies need to be revised. To demonstrate that formal analogies can be accommodated, one philosophical account of analogies (Bartha's (2010) articulation model) will be revised to cover one case study of purely formal analogies (Wilson and Kogut's (1974) construction of a particle physics model by analogy to a condensed matter physics model). The resulting *applied mathematics* submodel for analogical reasoning supplies a justification for the use of purely formal analogies.

Bartha's articulation model is well-suited to the analysis of formal analogies because, following Hesse, vertical relations play a central role. When an analogy is drawn between

[1]Thanks to Pete Evans and Karim Thébault for this suggestion.

[2]Thanks to Jennifer Jhun and Patricia Palacios for this suggestion. Jhun et al. (2018) argue that the JLS model supports a causal explanation of why stock markets occur because it includes the potential for interventions. An outstanding question is whether the robustness of the model can be established in a way that is compatible with possible interventions; if not, then micro-level causal mechanical explanations may not be supported (Jhun and Palacios 2022)

[3]Nappo (2021) has recently argued that this is an example of a non-causal analogy, but Nappo's examples would not count as purely formal analogies according to my definition if they rely on other physical similarities

a source scientific model and a target scientific model, a *horizontal relation* relates an element of the source model to a similar element of a target model. A *vertical relation* relates elements within a single (source or target) model (e.g., causal or mathematical relations). (See Fig. 1 for examples of horizontal and vertical relations.) Physical and formal analogies can be distinguished using this schema. An analogy is *physical* if at least one of the horizontal relations is a physical similarity or if vertical relations (e.g., causal relations) are physical relations of the same type.[4] For Bartha, "[t]wo features are formally similar if they occupy corresponding positions in formally analogous theories" (195). My interpretation of this definition is that an analogy is *formal* if at least one of the horizontal relations is a formal similarity (e.g., a mathematical similarity) or if vertical relations are formal relations (e.g., mathematical relations) of the same type. These definitions are not mutually exclusive; an analogy may be both physical and formal. Indeed, one way in which formal analogies can serve as a heuristic is to direct a theoretical or experimental search for accompanying physical analogies.

Bartha also offers well-developed arguments for the claim that I reject: that physical analogies are needed to support any argument from analogy in science. He defends a Requirement of Physical Analogy: "[a] necessary condition for a composite analogical argument in the empirical sciences to be plausible is that in at least one of the component analogies the relevant similarities that constitute the basis for the argument have known physical significance." (221). The main line of argument offered in support of this requirement is that the plausibility of the conclusion of an argument from analogy is underwritten by the potential for generalization of the analogy by the discovery of a "common pattern" shared by the source and target (105). In science this means that, at a minimum, there must be the possibility of the unification of the theories for or the identification of phenomena in the source and target domains. In contrast, in mathematics arguments from analogy require mere formal unification of the source and target (Bartha 2010, Chapter 5). Bartha's other line of argument for the Requirement of Physical Analogy is that purely formal analogies would make the applicability of mathematics in science a miracle, as Steiner (1998) argues. I will not address Steiner's arguments directly here, but the account of the application of purely formal analogies in the Wilson-Kogut example set out in Sec. 3 does include a naturalistically acceptable explanation for the applicability of mathematics in this case.[5]

The example of purely formal analogies that will be the focus of this paper is Wilson and Kogut's (1974) construction of a renormalized particle physics model by analogy to

[4]More carefully, for complex examples with multiple vertical relations: there are mappings between vertical relations in the source and target models and a pair of mapped vertical relations are physical relations of the same type. And mutatatis mutandis for the definition of a formal analogy.

[5]Bartha is also unconvinced by Steiner's argument, but he takes issue with Steiner's classification of his case studies as formal analogies.

a condensed matter physics model for critical phenomena (see Fraser (2020) for analysis). In particle physics, renormalization is introduced as a mathematical technique for extracting predictions from a model. Wilson and Kogut's goal was to develop a new renormalization technique that is more broadly applicable and has a sounder motivation than the existing perturbative renormalization techniques. Their strategy was to draw analogies to recently formulated models for the critical phase transition in ferromagnets: at some critical temperature, an iron bar will gain the capacity for spontaneous magnetization (i.e., to be magnetized without being in a magnetic field). Essentially, these purely formal analogies facilitated the transfer of renormalization group (RG) methods from the condensed matter to the particle physics model. While the details of this example of the use of analogies are complex, the main features needed to analyze the underlying pattern of reasoning can be explained with a minimum of technicality. The presentation in Sec. 3 aims to be self-contained.

Bartha's articulation model is a general-level account of analogies that is supplemented by concrete submodels. Sec. 2 presents Bartha's analysis of Priestley's analogy between gravity and electrostatics. This is an instructive example to compare to the Wilson-Kogut example because it involves both formal and physical analogies, and also because it instantiates the deductive/abductive submodel. The dedudctive/abductive submodel is the one of Bartha's submodels that comes closest to capturing the use of purely formal analogies in the Wilson-Kogut example. Sec. 3 analyzes the Wilson-Kogut example using Bartha's articulation model and contrasts the pattern of reasoning with that of the Priestley example. I argue that the Wilson-Kogut case study furnishes an example of a new submodel of analogical reasoning that is not in the taxonomy in Bartha (2010). Two distinctive features of this new *applied mathematics* submodel for analogical reasoning are that the conclusion of the argument from analogy includes both an entire model (and not only a hypothesis or a prediction) and the construction procedure for this model. A third important difference from arguments from physical analogy is that only the *prima facie* plausibility of the conclusion is established, and not stronger types of plausibility associated with confirmation. The applied mathematics submodel of the articulation model thus serves to precisely indicate the respects in which the RG methods case study is novel. Moreover, philosophical accounts of analogies such as Bartha's are valuable because they provide normative guidance about which kinds of similarities are relevant for evaluating arguments from analogy of different types. In this case, the heuristic advice to physicists using analogies to develop new theories or models is that purely formal analogies should not be automatically dismissed. In some circumstances, the use of purely formal analogies is warranted.

Figure 1: Analogies in the Priestley Case

Source: Gravitation	Target: Static Electricity
✓ C: spherical shell of uniform density	✓ C*: uniformly charged spherical shell
✓ E: absence of (gravitational) force inside shell	✓ E*: absence of (electrostatic) force inside shell
✓ Q: inverse square law for gravitation: $F_g \propto \frac{1}{r^2}$	?: Q*: inverse square law for electrostatics: $F_e \propto \frac{1}{r^2}$
✓ $Q,\ C \Rightarrow E$	✓ $Q^*,\ C^* \Rightarrow E^*$

2 A case of a formal and a physical analogy: Priestley's gravity-electrostatics analogy

Bartha posits a general model of analogies that is supplemented by concrete submodels. The general-level articulation model[6] identifies two key features of analogical arguments: a prior association and the potential for generalization. The prior association is a known vertical relation (e.g., a causal relation) between elements P and Q of the source domain. P and Q are horizontally related to P^* and Q^*, respectively, in the target domain, which are conjectured or known to stand in the same vertical relation as P and Q. There is the potential for generalization when the analogy supports the possibility of unification of the source and target domains by the discovery of a "common pattern" (105). Concrete submodels of the articulation model fill in details such as the type of vertical relation and the nature of the potential generalization that are necessary to evaulate a given argument from analogy.

In the eighteenth century, Priestley's goal was to formulate a hypothesis about static electricity by drawing analogies to the better-developed theory for the source domain of gravitation. Fig. 1 illustrates the application of the articulation model to the Priestley example following Bartha (2010, 95, 122-124). The target domain is static electricity. This is an example of the deductive-abductive submodel of analogical reasoning: the prior association is the vertical relation of explanation and the conclusion of the argument from analogy is an explanatory hypothesis about the target domain. An explanation of observable result E is supplied by laws Q combined with initial and boundary conditions and auxiliary hypotheses C when C and Q entail E. In the Priestley example, C is the presence of a hollow spherical shell with uniform mass

[6] I am following Bartha's terminology here. The term *model* will also be used to refer to scientific models (e.g., an Ising model for a ferromagnet).

density. The observation that there is zero gravitational force inside the shell is explained by the inverse square law of gravitation. Priestley argued for an inverse square law for electrostatics (Q^*) by recognizing that in similar circumstances of a uniformly charged spherical shell (C^*) no electrostatic force is observed inside the shell (E^*), and that C^* and Q^* entail E^*.

According to Bartha, the conclusion of a good argument from analogy is at a minimum *prima facie* plausible. This means that the conclusion Q^* "should be taken seriously" because it might be true (101). The modal notion of *prima facie* plausibility is weaker than the probabilistic notions of qualitative plausibility (i.e., that one hypothesis is more plausible than another) or quantitative plausibility (e.g., Bayesian approaches to confirmation). Bartha regards the modal concept as primary in arguments from analogy, but specifies supplementary criteria that can be used to assess qualitative plausibility. He also connects plausibility to Bayesian confirmation theory by arguing that analogical reasoning should inform prior probability judgments. In the Wilson-Kogut case, the argument from analogy supports only the *prima facie* plausibility of the conclusion, so the stronger conceptions of plausibility will be set aside here.

According to the articulation model, Priestley's argument from analogy is a good argument because it satisfies the requirements of prior association and potential for generalization particularized to the deductive-abductive submodel. The check marks in Fig. 1 indicate the positive analogies. Included are all of the relevant elements for the deductive-abductive submodel: Q, C, and E and the explanatory deductive relation $Q,C \Rightarrow E$ in the source domain as well as the analogues C^*, E^*, and deductive relation $C^*,Q^* \Rightarrow E^*$ (i.e., an inverse square law for electrostatics would entail the absence of force in a uniformly charged spherical shell) in the target domain. The requirement of potential for generalization is satisfied because, in Priestley's day, there was the possibility of a future unified physical theory that includes both gravitational and electrostatic forces as forces of the same kind. Bartha notes that the facts that both gravitational and electrostatic forces were regarded as acting at a distance and that there was such a perfect correspondence between the gravitational and electrostatic analogues (e.g., complete absence of force) supported the possibility of a unified physical theory (127). On more general grounds, there was also the hope that Newton's theory of mechanics could supply a template for a unified theory of forces. By Bartha's lights, the physical hypothesis that electrostatic force obeys an inverse square law should be taken seriously on the basis of Priestley's argument from analogy because there is the potential for development of a unified physical theory of forces that portrays gravitational and electrostatic forces as being of the same physical kind.Bartha takes Kitcher (1989) and Morrison (2000) as model accounts of unification for scientific theories (105, 196).

Priestley's argument from analogy incorporates both formal and physical analogies. That the inverse square law for gravity and the proposed inverse square law for electrostatics take the same mathematical form is a formal analogy. This formal analogy plays a central role in the argument because the mathematical form of the inverse square law is what entails that force is zero within a uniform spherical shell. As Bartha emphasizes, there are also physical analogies that are essential to the argument. More specifically, "in empirical settings, there are two basic patterns associated with formal analogies: [*predictive*][7] analogies, used to argue from similar causes to similar effects, and *abductive* analogies, used to argue from formally similar effects to formally similar causes" (209). Priestley's argument is an example of the latter. Gravitational and electrostatic forces are both forces, which have causal efficacy. Furthermore, at this point in history, both forces were commonly regarded as action-at-a-distance forces. It is these physical analogies that lend credence to the supposition that gravitational and electrostatic force can one day be treated within a single theory in which they appear as forces of the same physical kind.

3 A case of a purely formal analogy: Wilson and Kogut's statistical mechanics-QFT analogy

Our example of the successful use of purely formal analogies is also historically situated: Wilson and Kogut (1974)'s use of the classical Ising model for criticial phenomena for a ferromagnet as the source domain for the purpose of the construction by analogy of a renormalized, continuum model for a scalar QFT with a ϕ^4 self-interaction. Wilson had recently developed renormalization group (RG) methods to treat the critical phenomenon of spontaneous magnetization in ferromagnets (i.e., magnetization in the absence of an external magnetic field). Essentially, the formal analogies between the Ising model and the ϕ^4 QFT model facilitate the application of RG methods to the QFT model. This is a complex example of analogical reasoning. Only the most salient details will be recounted here; see Fraser (2020) for a full account. Furthermore, analogies between critical phenomena in condensed matter physics and renormalization in QFT have been fruitfully applied for different purposes over the course of the historical development of QFT. Here I will focus on the argument from analogy in Wilson and Kogut (1974, Sec. 10-12).

At a general level, Wilson and Kogut recognize that critical phenomena and the renormalization of QFTs pose mathematical problems of the same type: there are infinitely many variables, which leads to divergences in quantities that need to be calculated. The starting point for the set up of the analogical mappings is the

[7] Bartha uses the term *mathematical* for submodels of this type because both empirical and pure mathematical instantiations rely on mathematical derivations (97–98). I have changed the label because, for my purposes, it is important to distinguish empirical and pure mathematical instantiations of this submodel.

Figure 2: Analogies in the Wilson-Kogut Case

	Source: Ising model	Target: ϕ^4 QFT model	
initial model: small distance or high energy scale	✓ spin field $s_{m,n}$ ✓ space x_d ✓ correlation functions $\Gamma_{n,m}$	✓ quantum field on a lattice $\phi_m(t)$ ✓ spacetime (x_{d-1}, t) ✓ propagators $D_m(t)$	
	$\Downarrow$ RG transformation, limit of infinite correlation length ξ	$\Downarrow$ RG transformation, continuum limit	?
output model: large distance or low enegy scale	✓ $Q,\ C \Rightarrow E$: Ising model near critical point	$Q^*,\ C^* \Rightarrow E^*$: renormalized continuum ϕ^4 model	

recognition of formal similarities between the Ising model for a ferromagnet and the representation of a scalar quantum field on a spatial lattice. (See the initial models in Fig. 2). The Ising model represents the ferromagnet by a field $s_{n,m}$ of spin values of atoms on an m by n dimensional spatial lattice. In the QFT model, cutoffs are initially introduced as a regularization technique to make the model calculationally tractable. When a small distance cutoff is used, the quantum field $\phi_m(t)$ is also a field on a spatial lattice (where m represents the spatial lattice and t is the continuous time variable). In both models the field values fluctuate over spacetime. The quantity of interest in the Ising model is the correlation function $\Gamma_{n,m}$, which represents the correlation between the values of the spin field at spatially separated lattice points (n, m) and $(0, 0)$. The formally analogous quantity in the QFT model is the propagator $D_m(t)$ that (roughly speaking) represents the correlation between the quantum field $\phi_m(t)$ at spatially separated lattice points (m) and (0) in the vacuum state. Even more roughly, $D_m(t)$ is interpreted as representing the probability for a particle to propagate from spatial location 0 to spatial location m. Equating the spin field $s_{n,m}$ and the quantum field $\phi_m(t)$ (up to a scale factor) and implementing Wick rotation $t \to -it$ on the QFT side allows $\Gamma_{n,m}$ and $D_m(-it)$ to be formally identified.[8]

The initial Ising model and lattice ϕ^4 QFT model are set up as small distance scale and high energy scale representations, respectively. Solving problems raised by critical

[8] $\Gamma_{n,m} = \zeta^2 D_m(-in\tau)$ where ζ is the scale factor and $n\tau$ serves to identify the continuous time QFT propagators at spatial lattice points in the n dimension of the correlation functions in the Ising model.

phenomena (e.g., evaluation of critical exponents) involves modeling the system at the thermodynamic limit, which involves assuming that the lattice of atoms is infinite in spatial extent. The strategy involves taking limit in which the system approaches the critical point (correlation length $\xi \to \infty$) and applying the RG transformation, which rescales the representation so that it is callibrated to large distance scales. (See Fig. 2.) On the target side, the analogue of the critical limit is the continuum limit that takes the lattice spacing to zero and the analogue of the RG transformation that rescales distance is an RG transformation that rescales energy. The continuum, renormalized QFT model is constructed by making use of the formal identifications with the Ising model to translate the results of iteratively approaching the thermodynamic limit and applying the RG transformation into the terms of the QFT model—that is, iteratively approaching the continuum limit and applying the RG transformation on energy scale.

The set of analogies in the Wilson-Kogut case includes both formal horizontal relations and mapped formal vertical relations (see Fig. 2). For example, there are formal horizontal relations between elements of the initial Ising and ϕ^4 QFT models that play similar formal roles (e.g., spin field $s_{m,n}$ and quantum field $\phi_m(t)$). An example of a formal (and physical) vertical relation within the source domain is the critical limit in which the correlation length ξ diverges. In the target domain, the continuum limit is a formal (and physical) vertical relation. There is a formal analogical mapping between these limits, which are formally—but not physically—similar. Likewise, the respective RG transformations are each formal (and perhaps physical) vertical relations within each domain, and the respective RG transformations are formal analgoues.

The argument that these formal analogies are not also physical analogies (i.e., that they are purely formal) requires more detailed consideration of the models and the analogical mappings than I have given here. (See Fraser (2020) for the complete argument.) In lieu of the full argument, consider a feature of this set of mappings that is suggestive of physical disanalogies: space in the Ising model gets mapped to spacetime in the QFT model. This basic 'mismatch' between space and time leads to fundamental differences in physical interpretation between the elements of the models mapped by the formal analogies, including the vertical relations. For example, the causal structure of the Ising model representation of the critical behaviour of the ferromagnet does not get mapped to causal structure in the QFT model by the set of analogical mappings in Fig. 2.

There are important points of contrast with the Priestley example. Comparing Fig. 1 and 2, the conclusion of the Wilson-Kogut argument includes everything in the box in Fig. 2: the continuum, renormalized ϕ^4 model *and* the RG transformation and continuum limit used to produce this model. The ϕ^4 model is not only a hypothesis, Q^*: it (roughly speaking) corresponds to the entire model $Q^*, C^* \Rightarrow E^*$ in Bartha's standard representational scheme. Q^* includes the renormalized field equations with renormalized parameters and E^* is the set of propagators.

To see that the RG transformation and continuum limit are contained in the conclusion, consider how Bartha's criteria of prior association and potential for generalization apply. There are two main vertical relations in the source that have analogues in the target: the relations among quantities in the initial models (e.g., the correlation functions $\Gamma_{n,m}$ are sums of weighted products of spin fields) and the relations between the models at different scales (RG transformations, limits). The plausible conclusion of the argument is that the construction procedure succeeds in generating a renormalized, continuum ϕ^4 model. This conclusion was significant to Wilson and Kogut (and others) not for the sake of obtaining the ϕ^4 model, but because the construction procedure provided a template for applying RG methods to renormalize other types of interactions, especially those that could not be perturbatively renormalized. This is the principle of generalization: that RG methods can be generalized from their application in statistical mechanics to form the basis of a mathematical model construction technique that is generally applicable to both statistical mechanics and QFT. The "common pattern" is a mathematical or formal theory, not a physical theory. This directly contradicts the Requirement of Physical Analogy.

Under what circumstances would the plausible conclusion of this argument from analogy be falsified?[9] The test of whether the construction procedure is successful is whether it yields a continuum model with a finite set of propagators, which is what constitutes successful renormalization. Wilson and Kogut (1974) supplies an argument that the construction procedure is successful for the ϕ^4 interaction.[10] If it had turned out to be impossible to use this analogy to construct a renormalized, continuum ϕ^4 model, then the conclusion would have been falsified. The continuum, renormalized ϕ^4 model can itself be tested by comparing its predictions with observations. However, because the conclusion is not only the predictions (E^*), but the entire model (Q^*,$C^*$$\Rightarrow$$E^*$), disconfirming observations would not necessarily falsify the conclusion of the argument. The usual caveats about testing hypotheses apply. For example, incompatible observations could be attributed to the false background assumption that the system probed is actually represented by the ϕ^4 model (C^*). Assessment of whether the ϕ^4 model applies to a given system might involve evaluation of the appropriateness of the initial QFT modeling assumptions, but these assumptions are not justified by appeal to the analogy to ferromagnets. Moreover, the ϕ^4 model is an abstract model that could in

[9] Thanks to Paul Bartha for urging me to focus on this question.

[10] Interestingly, the construction procedure may well fail for the ϕ^4 interaction in the physically realistic case of four spacetime dimensions in QFT due to the likely nonexistence of a suitable fixed point. However, if this is the case, then there would not exist a fixed point for the analogue statistical mechanical model either, and a critical factor would be missing fom both source and target. This would disqualify the conclusion from counting as *prima facie* plausible according to Bartha's articulation model.

principle be applied to represent various concrete systems in the same manner that in classical physics a single Hamiltonian or Lagrangian could be used to represent various concrete systems. As a result, recalcitrant observations for one system would not undermine the construction procedure for the ϕ^4 model or the resultant ϕ^4 model if (correctly) applied to other systems.

This analysis of disconfirmation also sheds light on the reason that *prima facie* plausibility in the minimal sense of "should be taken seriously" is the appropriate epistemic attitude to adopt towards the conclusion of this type of argument from analogy, and not a stronger type of plausibility.[11] The argument supports taking the renormalized, continuum ϕ^4 model seriously, which does involve testing its predictions for systems to which it is believed to apply. However, for a given system, judgments about the likelihood that the predictions are true of a given system, such as their relative plausibility or their prior probability, are not licensed by the argument from analogy. The argument from analogy is at two removes from judgments about confirmation. First, the conclusion concerns the inference to the model $Q^*, C^* \Rightarrow E^*$, and not observation E^* alone. Second, the ϕ^4 model is abstract and the argument from analogy does not supply reasons to believe it applies to a given system. Bartha worries that *prima facie* plausibility is not a meaningful concept unless it can be tied to the assignment of non-negligible probabilities (19, 279). However, this is not a worry in this case. The primary goal of Wilson and Kogut's argument from analogy is not to support the application of the ϕ^4 model to a specified type of system. This is another contrast with the Priestley example: Priestley's aim was to generate a law for the specific case of electrostatics. Wilson and Kogut's purpose is to generate a model construction procedure that plausibly works (at least) for the ϕ^4 case. This is a worthwhile goal because perturbative renormalization was of limited applicability and lacked principled justification.The object was to produce *some* renormalized, continuum model using the RG procedure, not to determine which of a set of candidate models is most likely to be true; confirmation is not the aim. Mere *prima facie* plausibility is an appropriate goal, and constitutes an important discovery in this context.

In contrast to the Priestley example, then, the conclusion of the argument is not a hypothesis Q^* such as the inverse square law, but a complete model $Q^*, C^* \Rightarrow E^*$ *and* a construction procedure for this model. There is a set of vertical relations (RG transformation, continuum limit), the analogues of which are known to hold in the source (RG transformation, critical limit). There are no vertical relations contained in the conclusion of the Priestley example. In that case the key vertical relation of

[11] Nyrup (2020) has recently argued that the conclusion of the liquid drop model has a similar status of being pursuit-worthy, but for (at least apparently) different reasons than I have given for the RG methods case. For example, on my understanding of the RG methods case, the distinction between purely formal and physical anlogies and details contained in Bartha's submodels are both important.

explanation ($Q(*)$,$C(*)\Rightarrow E(*)$) is among the premises of the argument, known to hold in both the source and target. This is because the absence of force in a spherical shell is, in Priestley's time, known to be a straightforward mathematical consequence of the inverse square law. In contrast, the procedure for mathematically deriving a renormalizable, continuum model for the ϕ^4 theory was not transparent to Wilson and Kogut. Perturbative renormalization methods had been used to construct a renormalized ϕ^4 model, but Wilson and Kogut were seeking a more principled construction technique. They recognized this as a difficult applied mathematics problem.

That the Wilson-Kogut example includes an entire model and a model construction procedure puts this example outside of Bartha's taxonomy of submodels for analogical reasoning. Of Bartha's submodels for empirical science,[12] this argument comes closest to instantiating the deductive-abductive submodel applied to the Priestley case. However, as we have seen, the Wilson-Kogut example does not instantiate this submodel. What about the other submodels? A scientific example of the application of the deductive-predictive submodel[13] is the use of a tabletop hydrodynamic model to determine stress in a twisted bar (Bartha 2010, 2). The equations take the same form, which allows the measured value of the fluid velocity to be used to evaluate the stress in the bar. Naturally, this argument from analogy is only useful in practice when one is unable to calculate the value for stress directly from the equation for the bar. The ϕ^4 model that Wilson and Kogut construct by analogy is not a prediction, but a hypothesis, like the hypothesis of the inverse square law for electrostatics. Furthermore, the calculational techniques for the ϕ^4 model are central to the analogy, not avoided by the analogy. Consequently, the Wilson-Kogut case fits best into Bartha's abductive category, rather than the predictive category that is characterized by the conclusion of the analogical argument being an observation. Furthermore, the Wilson-Kogut case does not fit the probabilistic-abductive submodel because the deductive relationship that the RG transformation establishes between models at different scales is an important component of the conclusion of the analogical argument.[14]

Appreciation of the distinctive features of the new submodel of analogical reasoning

[12]One might wonder whether the Wilson-Kogut example instantiates the deductive-predictive submodel, but applied to pure mathematics rather than empirical science (see Bartha (2010, Sec. 4.4 and Chapter 5)). The Wilson-Kogut argument from analogy does not fit into this category either because the conclusion is not a mathematical theorem and the deductive relation is not provability. However, the new submodel instantiated by the Wilson-Kogut example does seem aptly characterized as a submodel for applied mathematics.

[13]Bartha uses the term *mathematical* for submodels of this type because both empirical and pure mathematical instantiations rely on mathematical derivations (97–98). For my purposes, it is important to distinguish empirical and pure mathematical instantiations of this submodel.

[14]Which should of course not be taken to entail that statistical reasoning is divorced from RG methods, particularly in condensed matter phyiscs (Morrison 2014; Jona-Lasinio 2010).

instantiated by the Wilson-Kogut example helps to explain why formal analogies are sufficient to establish the *prima facie* plausibility of the conclusion. The distinctive features are that the conclusion of the argument includes an entire model, that the conclusion also includes an inference (i.e., the model construction procedure), that the model is at a higher level of abstraction (e.g., compared to Priestley's models of electromagetism or gravity), and that only the *prima facie* plausibility of the conclusion is supported. The problem that the analogy is introduced to solve is how to derive propagators for the ϕ^4 model at the low-energy scale, given the modeling assumptions at the high-energy scale. This is an applied mathematics problem, thus formal similarities between the initial Ising model and the ϕ^4 models and the derivational techniques that worked for critical phenomena and are proposed for QFT are the only relevant similarities for evaluating this argument from analogy. An apt name for this new submodel of analogical reasoning is therefore the *applied mathematics* submodel. Physical similarities between the source and target are not relevant to evaluating the plausibility of either the model construction technique or the model produced for the target. This is fortunate because there are substantial physical dissimilarities between the two cases. In particular, analogical mappings do not map causal structure in the source to causal structure in the target. Bartha's deductive-abductive submodel and deductive-predictive submodel applied to empirical science both rely on causal structure. In the Wilson-Kogut case, the fact that the formal analogical mappings do not preserve causal structure does not undermine the analogy.

4 Conclusion

From a general philosophical perspective, Wilson and Kogut's use of a purely formal analogy was justified. Application of Bartha's articulation model to this case reveals two important features that distinguish this pattern of analogical reasoning from other patterns of deductive analogical reasoning in science that do rely on combinations of formal and physical analogies: the conclusion of the argument from analogy includes an entire model, not only a hypothesis or a prediction; the conclusion also includes the technique for deriving the model; and only the *prime facie* plausibility of the conclusion is established. A purely formal analogy grounds the *prima facie* plausibility of a conclusion of this sort because the mathematical similiarities between the initial models and derivational techniques are the only relevant considerations for determining whether the derivation in the target and the model that it produces should be taken seriously. In this context, being taken seriously means applying the resultant QFT model to particle physics systems to which it is believed to apply and attempting to extend the inferential strategy to other types of interactions. Physical analogies are not needed to substantiate the conclusion because the conclusion does not concern whether the model accurately represents a specified physical system; the conclusion is an abstract one about a generic model for the ϕ^4 interaction and techniques for constructing abstract models for QFT. When compared to other forms of analogical reasoning in science, these characteristic features of the applied mathematics submodel might seem to be limitations. However, this impression is not accurate. The use of purely formal analogies following this pattern is a heuristic for making genuine discoveries in science. As the Wilson-Kogut example illustrates, the introduction of new techniques for model construction can be a ground-breaking development in a field.

The pattern of analogical reasoning exhibited by the Wilson-Kogut case falls outside of Bartha's taxonomy of submodels of analogical reasoning. Adding this applied mathematics submodel to Bartha's taxonomy could be considered a friendly amendment. However, it is important to recognize that this is not a minor revision to the articulation model account of analogies because the implications of this new submodel are significant. First, this applied mathematics submodel is a counterexample to Bartha's Requirement of Physical Analogy by presenting a type of analogical reasoning that has been successfully used in physics but does not require physical analogies. This moral has important implications for physicists: when analogies are used for heuristic purposes, arguments based on purely formal analogies should not be dismissed out of hand. In some contexts, purely formal analogies can justifiably be used as heuristics. Naturally, which contexts these are depends on the purpose for which an argument from analogy is being used. The distinctive features of the applied mathematics submodel help to identify suitable contexts. Second, Bartha's account is informed by a comprehensive survey of examples of analogies in science that have received philosophical attention. The Wilson-Kogut case study is novel in the respect that it does not fit the pattern of other cases that have informed philosophical accounts of analogies in science. In particular, it is different in kind from examples that combine formal and physical analogies, such as Priestley's. This does not imply that the Wilson-Kogut example is the first example in the history of physics of the applied mathematical pattern of analogical reasoning.[15] The point is rather that now that philosophers have been alerted to one example of the applied mathematics submodel, we should look for other examples of this pattern of reasoning in physics and other sciences. Whether other examples of purely formal analogies instantiate the applied mathematics submodel will, of course, have to be determined on a case-by-case basis. Another open question is whether philosophical accounts of analogies other than Bartha's articulation model can be revised to include cases of purely formal analogies.[16]

[15]In fact, the argument from analogy used in the construction of the Higgs model is an apparent counterexample. See Fraser and Koberinski (2016).

[16]Since Norton (2021) treats analogies as a species of induction to which the material theory of induction

I have argued that in the Wilson-Kogut example purely formal analogies are sufficient for justifying the conclusion of the argument from analogy. One might object that there *are* physical analogies in this example that could be invoked.[17] While the Ising model and the QFT model are given different concrete physical interpretations, they do share more abstract physical similarities. For example, an array of classical spins differs from a scalar quantum field, but both could be considered physical fields. More importantly, there are physical similarities between the vertical relations that drive the argument from analogy. For example, scale is a physical concept. Whether the relevant scale is energy or distance is a model-dependent matter of interpretation, but both are physical scales, abstractly conceived. Furthermore, some sort of physical invariance with respect to scale is an important property in both domains. Critical phenomena are scale-invariant in their manifestation of self-similar behaviour. In particle-physics, the propagators are (approximately) scale-invariant. Why do these abstract physical similarities play no role in my account of how this argument from analogy is justified?

In short, the answer is that these abstract physical similarities are not needed to justify the conclusion of the argument from analogy due to the nature of the conclusion: an abstract applied mathematical model and a model construction technique that are worth taking seriously in the target domain. The most direct way to see this is to revisit Bartha's general account of justification in the articulation model. In all submodels, the potential for generalization is what justifies the conclusion of an argument from analogy. Strictly speaking, the general theory that underlies the Wilson-Kogut analogy need only cover statistical mechanics (applied to condensed matter physics) or QFT (applied to particle physics). One might be able to imagine a future general theory of RG methods and limits that is applicable to both condensed matter and particle physics and includes an overarching concept of scale with a physical interpretation common to both domains.[18] However, the application of RG methods by analogy has extended far beyond condensed matter and particle physics. In cases such as econophysics, the Wick rotation ($t \to it$) does not apply and there may be no formal identification between domains, but there are formal similarities. When one considers applications in economics or biology, it seems unlikely that any physical (or biological or economic) property or relation can figure in the general theory of RG methods, no matter how abstract. In other words, it is even clearer that the general theory that underpins the argument from analogy is a formal theory (i.e., mathematical).

is applicable and in the Wilson-Kogut example the conclusion is not confirmed by the analogy, presumably this case (and the general pattern of reasoning it invokes) falls outside the scope of Norton's account.

[17] Thank you to Patricia Palacios and Juha Saatsi for raising this argument in slightly different forms. Again, for the arguments that the analogies in the Wilson-Kogut case are purely formal, see Fraser (2020).

[18] But even this is more difficult than one might imagine due to some important formal differences. See Koberinski and Fraser (2022).

That the justification of the conclusion of arguments from analogy in the applied mathematics category rests entirely on formal analogies is compatible with physical (or biological or economic) interpretations of the source and target models being relied on for other scientific purposes. For example, intuitions about what counts as a physical (or biological or economic) scale could inform strategies for drawing new analogies to new domains, in the hope of successfully applying RG methods. And, of course, the application of RG methods has been used for explanatory and predictive purposes. Most famously, the explanation of universal behaviour in critical phenomena inokes RG methods. However, this explanation is based on the physical interpretation of the formalism applied to condensed matter physics. It does not rely on the analogy to particle physics (or any other domain). When RG methods are deployed for explanatory or predictive purposes in a domain, the domain-specific interpretation is invoked. The purely formal status of the analogies to other domains does not undermine these domain-specific uses of RG methods. However, purely formal analogies cannot be used to infer physical, biological, or economic facts about the target domain. Purely formal analogies are useful for the purpose of devising new model construction techniques and constructing new models, but caution must be exercised when a purely formal analogy is used as a basis for explanation.

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