

Delorme's intertwining conditions for sections of homogeneous vector bundles on two and three dimensional hyperbolic spaces

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Abstract

The description of the Paley-Wiener space for compactly supported smooth functions $C_c^\infty(G)$ on a semi-simple Lie group G , involves certain intertwining conditions, that are difficult to handle. In the present paper, we make them completely explicit for $G = \mathbf{SL}(2, \mathbb{R})^d$ ($d \in \mathbb{N}$) and $G = \mathbf{SL}(2, \mathbb{C})$. Our results are based on a defining criterion for the Paley-Wiener space, valid for general groups of real rank one, that we derive from Delorme's proof of the Paley-Wiener theorem. In a forthcoming paper, we will show how these results can be used to study solvability of invariant differential operators between sections of homogeneous vector bundles over the corresponding symmetric spaces.

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1 Introduction

Consider a Riemannian symmetric space of non-compact type $X = G/K$, where G is a real connected semi-simple Lie group with finite center of non-compact type and $K \subset G$ its maximal compact subgroup. Arthur [Art83] as well as Delorme ([Del05], Thm. 2) established a Paley-Wiener theorem for (K -finite) compactly supported smooth functions on G . Their results involved the so-called Arthur-Campolli and Delorme conditions respectively. Later van den Ban and Souaifi [vdBS14] proved, without using the proof nor validity of any associated Paley-Wiener theorems of Arthur or Delorme, that the two compatibility conditions are equivalent.

In [OlPa22-1], we proved a topological Paley-Wiener(-Schwartz) theorem for sections of homogeneous vector bundles by adapting Delorme's intertwining conditions for our purposes. We considered the intertwining conditions in three levels, namely ([Level 1](#)) referred to Delorme's condition in the setting of van den Ban and Souaifi [vdBS14], ([Level 2](#)) corresponded to the conditions for sections over homogeneous vector bundles and ([Level 3](#)) stood for spherical functions.

However, these intertwining conditions are very difficult to check in practice, even for special K -types. The most important source of such conditions are the Knapp-Stein ([KSt71] & [KSt80]) and Želobenko [Zelo76] intertwining operators, as well as the embedding of discrete series into principal series $H_\infty^{\sigma, \lambda}$, for $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_\mathbb{C}^*$.

Therefore, in this article, we rewrite them in a more accessible way involving such intertwining operators and the Harish-Chandra c -functions, which we introduce in the first [Section 2](#). Moreover, we show that only a part of them is already sufficient for G of real rank one ([Section 3](#), [Thm 3](#)). This is already implicitly contained in Delorme's proof of the Paley-Wiener theorem ([Del05], Thm. 2). For our proof, we essentially use an intermediate result of Delorme ([Del05], Prop. 1) and his induction procedure ([Del05], Prop. 2) on the length of minimal K -types of a generalized principal series representation.

To apply it, one has to know more or less the complete composition series of reducible principle series representations, which is the case for the two special examples, $\mathbf{SL}(2, \mathbb{R})$ and $\mathbf{SL}(2, \mathbb{C})$, which are in the

focus of the present paper.

In fact, for $G = \mathbf{SL}(2, \mathbb{R})$, in Section 4, by drawing its principal series representations $H_\infty^{\pm, \lambda}$ by 'box-pictures' (Fig. 1), we can see in which closed G -submodule of $H_\infty^{\pm, \lambda}$ there is an intertwining condition in (Level 2) (Thm. 6). Afterwards, we can deduce the corresponding results also for the other levels (Thms. 7 & 5) and even go beyond by illustrating the intertwining conditions for finite products of $\mathbf{SL}(2, \mathbb{R})$ (Thm. 8). As last example, in Section 5, we consider $G = \mathbf{SL}(2, \mathbb{C})$. The description of its intertwining conditions (Thms. 10, 11 & 14) are more difficult than for the previous examples. We remark that $\mathbf{SL}(2, \mathbb{R})$ and $\mathbf{SL}(2, \mathbb{C})$ are locally isomorphic to $\mathbf{SO}(1, n)$, $n = 2, 3$ respectively. In fact, based on Thm. 3, we have already checked that it is possible to obtain analogous results for all n . We are grateful to P. Delorme for mentioning to us that the results for odd n (including $\mathbf{SL}(2, \mathbb{C})$) can also be derived from his earlier Paley-Wiener theorem for groups with only one conjugacy class of Cartan subgroups [Del82]. This work is part of the second author's doctoral dissertation [Pa22]. In fact, our results can be used to study solvability of invariant differential operators between sections of homogeneous vector bundles on the corresponding symmetric spaces. The way we expressed the intertwining conditions in (Level 3) is particularly adapted to the solvability questions. In an upcoming paper [OIPa22-3], these questions will be discussed in detail.

2 Intertwining conditions and operators

We adopt the same notations as in [OIPa22-1]. Let G be a real connected semi-simple Lie group with finite center of non-compact type with Lie algebra \mathfrak{g} . Consider its Iwasawa decomposition $G = KAN$, where $K \subset G$ its maximal compact subgroup with Lie algebra \mathfrak{k} , $A = \exp(\mathfrak{a})$ abelian and N nilpotent. The quotient $X = G/K$ is a Riemannian symmetric space of non-compact type.

Let $M = Z_K(\mathfrak{a})$ be the centralizer of A in K . Then $P = MAN$ is a minimal parabolic subgroup of G . Let $(\sigma, E_\sigma) \in \widehat{M}$ be a finite-dimensional irreducible representation of M and $\lambda \in \mathfrak{a}_\mathbb{C}^*$, the complexified dual of the Lie algebra of A . For $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_\mathbb{C}^*$, consider $H_\infty^{\sigma, \lambda}$ the space of smooth vectors of the principal series representations of G induced from the P -representation σ_λ on the vector space E_σ ([Kna86], p. 168). By H_∞^σ we denote its representation space in the compact picture, which is given by functions on K and is independent of λ . Given $\sigma \in \widehat{M}$, let us consider the map

$$\pi_{\sigma, \cdot} : G \rightarrow (\mathfrak{a}_\mathbb{C}^* \rightarrow \text{End}(H_\infty^\sigma)), g \mapsto (\lambda \mapsto \pi_{\sigma, \lambda}(g)).$$

In [OIPa22-1], we introduce the m -th derived principal series representation of G , which is a very similar definition of derived G -representation ([Del05], Déf. 3 (4.4) & [vdBS14], Sect. 4.5).

Definition 1 (m -th derived representation, [OIPa22-1], Def. 2). *For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, let Hol_λ be the set of germs at λ of \mathbb{C} -valued holomorphic functions $\mu \mapsto f_\mu$ and $m_\lambda \subset \text{Hol}_\lambda$ the maximal ideal of germs vanishing at λ .*

Denote by $H_{[\lambda]}^\sigma$ the set of germs at λ of H_∞^σ -valued holomorphic functions $\mu \mapsto \phi_\mu \in H_\infty^\sigma$ with G -action

$$(g\phi)_\mu = \pi_{\sigma, \mu}(g)\phi_\mu, \quad g \in G.$$

For $m \in \mathbb{N}_0$, it induces a representation $\pi_{\sigma, \lambda}^{(m)}$ on the space

$$H_{\infty, (m)}^{\sigma, \lambda} := H_{[\lambda]}^\sigma / m_\lambda^{m+1} H_{[\lambda]}^\sigma, \quad (2.1)$$

which is equipped with the natural Fréchet topology. We call this representation the m -th derived principal series representation of G .

Let us briefly recall the intertwining conditions in the three levels, for more details we refer to [OIPa22-1].

Intertwining conditions and Paley-Wiener theorem in (Level 1). We denote by $\text{Hol}(\mathfrak{a}_\mathbb{C}^*)$ the space of holomorphic functions on $\mathfrak{a}_\mathbb{C}^*$ and by $\text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{End}(H_\infty^\sigma))$ the space of maps $\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto \phi(\lambda) \in \text{End}(H_\infty^\sigma)$ such that for $\varphi \in H_\infty^\sigma$, the function $\lambda \mapsto \phi(\lambda)\varphi \in H_\infty^\sigma$ is holomorphic.

Definition 2 (Delorme's intertwining condition in (Level 1), [OIPa22-1], Def. 3). *Let Ξ be the set of all 3-tuples (σ, λ, m) with $\sigma \in \widehat{M}$, $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $m \in \mathbb{N}_0$. Consider the m -th derived G -representation $H_{\infty, (m)}^{\sigma, \lambda}$ defined in (2.1). For every finite sequence $\xi = (\xi_1, \xi_2, \dots, \xi_s) \in \Xi^s$, $s \in \mathbb{N}$, we define the G -representation*

$$H_\xi := \bigoplus_{i=1}^s H_{\infty, (m_i)}^{\sigma_i, \lambda_i}.$$

We consider proper closed G -subrepresentations $W \subseteq H_\xi$.

Such a pair (ξ, W) with $\xi \in \Xi^s$ and $W \subset H_\xi$ as above, is called an intertwining datum. Every function $\phi \in \prod_{\sigma \in \widehat{M}} \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{End}(H_\infty^\sigma))$ induces an element

$$\phi_\xi \in \bigoplus_{i=1}^s \text{End}(H_{\infty, (m_i)}^{\sigma_i, \lambda_i}) \subset \text{End}(H_\xi).$$

(D.a) We say that ϕ satisfies Delorme's intertwining condition, if $\phi_\xi(W) \subseteq W$ for every intertwining datum (ξ, W) .

Moreover, for $r > 0$, one can introduce a corresponding Paley-Wiener space

$$PW_r(G) \subset \prod_{\sigma \in \widehat{M}} \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{End}(H_\infty^\sigma)),$$

which is characterised by the usual growth condition ([Del05], D ef. 3 (4.3) and [OlPa22-1], Def. 4 (1.ii)_r) and intertwining condition (D.a). Let $\overline{B}_r(o)$ be the preimage of the closed ball of radius $r > 0$ centered at $o \in X$ under the projection $p : G \rightarrow X$. Write by $C_r^\infty(G)$ the space of smooth complex functions on G with support in $\overline{B}_r(o)$. Then, by taking the union over all $r > 0$, and considering the Fourier transform of $f \in C_c^\infty(G)$:

$$\widehat{M} \times \mathfrak{a}_\mathbb{C}^* \ni (\sigma, \lambda) \mapsto \mathcal{F}_{\sigma, \lambda}(f) := \pi_{\sigma, \lambda}(f) = \int_G f(g) \pi_{\sigma, \lambda}(g) dg \in \prod_{\sigma \in \widehat{M}} \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{End}(H_\infty^\sigma)),$$

we have the following isomorphism.

Theorem 1 (Delorme's Paley-Wiener Theorem in (Level 1), [Del05], Thm. 2). *For any $r > 0$, the Fourier transform*

$$C_r^\infty(G) \ni f \mapsto \mathcal{F}_{\sigma, \lambda}(f) \in PW_r(G), \quad (\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_\mathbb{C}^*$$

is a topological isomorphism between the two Fr echet spaces $C_r^\infty(G)$ and $PW_r(G)$. \square

Intertwining conditions and Paley-Wiener theorems in (Level 2) and (Level 3). Next, consider two finite-dimensional, not necessary irreducible, K -representations (τ, E_τ) and (γ, E_γ) . We consider the space of smooth compactly supported sections of homogeneous vector bundles \mathbb{E}_τ over X by

$$C_c^\infty(X, \mathbb{E}_\tau) = \bigcup_{r>0} C_r^\infty(X, \mathbb{E}_\tau) \cong \bigcup_{r>0} \{f : G \xrightarrow{C^\infty} E_\tau \mid f(gk) = \tau^{-1}(k)(f(g)), \forall g \in G, k \in K \text{ \& } \text{supp}(f) \subset \overline{B}_r(0)\}.$$

The group G acts on $C_c^\infty(X, \mathbb{E}_\tau)$ by left translations, $(g \cdot f)(g') = f(g^{-1}g'), \forall g, g' \in G$. It is not difficult to see that we have the G -isomorphisms $C_c^\infty(X, \mathbb{E}_\tau) \cong C_c^\infty(G, E_\tau)^K \cong [C_c^\infty(G) \otimes E_\tau]^K$. Moreover, we also consider the space of (γ, τ) -spherical functions on G

$$\begin{aligned} C_c^\infty(G, \gamma, \tau) &= \bigcup_{r>0} C_r^\infty(G, \gamma, \tau) \\ &:= \bigcup_{r>0} \{f : G \rightarrow \text{Hom}(E_\gamma, E_\tau) \mid f(k_1 g k_2) = \tau(k_2)^{-1} f(g) \gamma(k_1)^{-1}, \forall k_1, k_2 \in K \text{ \& } \text{supp}(f) \subset \overline{B}_r(0)\}. \end{aligned}$$

Note that by taking topological linear duals, we obtain the spaces of distributional sections $C^{-\infty}(X, \mathbb{E}_\tau)$ and $C^{-\infty}(G, \gamma, \tau)$. We are particularly interested in distributional sections with compact support $C_c^{-\infty}(X, \mathbb{E}_\tau)$ and $C_c^{-\infty}(G, \gamma, \tau)$.

The Fourier transform for (distributional) sections of homogeneous vector bundles in (Level 2) and (Level 3) is given in the following definition.

Definition 3 (Fourier transforms, [OlPa22-1], Def. 5 & 6). *Let $g = \kappa(g)a(g)n(g) \in KAN = G$ be the Iwasawa decomposition. For fixed $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $k \in K$, we define the function $e_{\lambda, k}^\tau$ by*

$$\begin{aligned} e_{\lambda, k}^\tau : G &\rightarrow \text{End}(E_\tau) \cong E_{\overline{\tau}} \otimes E_\tau \\ g &\mapsto e_{\lambda, k}^\tau(g) := \tau(\kappa(g^{-1}k))^{-1} a(g^{-1}k)^{-(\lambda + \rho)}, \end{aligned} \tag{2.2}$$

where ρ is the half sum of the positive roots of $(\mathfrak{g}, \mathfrak{a})$.

(a) (*Level 2*) For $f \in C_c^\infty(X, \mathbb{E}_\tau)$, the Fourier transformation is given by

$$\mathcal{F}_\tau f(\lambda, k) = \int_G e_{\lambda, k}^\tau(g) f(g) dg = \int_{G/K} e_{\lambda, k}^\tau(g) f(g) dg, \quad (2.3)$$

where the last equality makes sense, since the integrand is right K -invariant. Similar, for distributional sections $T \in C_c^{-\infty}(X, \mathbb{E}_\tau)$: $\mathcal{F}_\tau T(\lambda, k) := \langle T, e_{\lambda, k}^\tau \rangle = T(e_{\lambda, k}^\tau) \in E_\tau$, with $(\lambda, k) \in \mathfrak{a}_\mathbb{C}^* \times K/M$.

(b) (*Level 3*) The Fourier transformation for $f \in C_c^\infty(G, \gamma, \tau)$ is given by

$$\gamma \mathcal{F}_\tau f(\lambda) := \int_G e_{\lambda, 1}^\tau(g) f(g) dg, \quad \lambda \in \mathfrak{a}_\mathbb{C}^*. \quad (2.4)$$

Similar, for $T \in C_c^{-\infty}(G, \gamma, \tau)$, we have $\gamma \mathcal{F}_\tau T(\lambda) := \langle T, e_{\lambda, 1}^\tau \rangle$.

Note that $\mathcal{F}_\tau(f), \mathcal{F}_\tau(T) \in \text{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{\tau|M})$, where $H_\infty^{\tau|M} := \{f : K \xrightarrow{C^\infty} E_\tau \mid f(km) = \tau(m)^{-1}f(k)\}$ and that $\gamma \mathcal{F}_\tau(f), \gamma \mathcal{F}_\tau(T) \in \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{Hom}_M(E_\gamma, E_\tau))$. We also remark that $\text{Hom}_K(E_\gamma, H_\infty^{\tau|M}) \cong \text{Hom}_M(E_\gamma, E_\tau)$, by Frobenius reciprocity. We will also consider $H_{\infty, (m)}^{\tau|M, \lambda}$ defined as in Def. 1 with σ replaced by $\tau|_M$.

Definition 4. We define

$$\text{Hom}_M(E_\tau, E_\sigma)_{(m)}^\lambda := \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{Hom}_M(E_\tau, E_\sigma)) / m_\lambda^{m+1} \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{Hom}_M(E_\tau, E_\sigma))$$

as in (2.1). For $\tau \in \widehat{K}$ and an intertwining datum (ξ, W) , we set

$$\begin{aligned} D_W^\tau &:= \left\{ t \in \bigoplus_{i=1}^s \text{Hom}_M(E_\tau, E_{\sigma_i})_{(m_i)}^{\lambda_i} \mid T = \widetilde{\text{Frob}}^{-1}(t) \in \text{Hom}_K(E_\tau, W) \subset \text{Hom}_K(E_\tau, H_\xi) \right\} \\ &\subset \bigoplus_{i=1}^s \text{Hom}_M(E_\tau, E_{\sigma_i})_{(m_i)}^{\lambda_i}. \end{aligned} \quad (2.5)$$

We have shown ([OIPa22-1], Def. 7+8, Prop. 7 & Thm. 2) that Delorme's intertwining condition (D.a) correspond to the following intertwining conditions in (Level 2) and (Level 3).

Definition 5 (Intertwining conditions in (Level 2) and (Level 3), [OIPa22-1], Thm. 2). Let Ξ be as in Def. 2 and let $\bar{\Xi}$ the set of all tuples (λ, m) with $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $m \in \mathbb{N}_0$. We define a map

$$\Xi \longrightarrow \bar{\Xi}, \quad \xi = (\sigma, \lambda, m) \mapsto \bar{\xi} = (\lambda, m).$$

For $s \in \mathbb{N}$ and $\xi \in \Xi^s$, we have the corresponding element $\bar{\xi} \in \bar{\Xi}^s$.

(D.2) (*Level 2*) We say that $\psi \in \text{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{\tau|M})$ satisfies the intertwining condition, if for each intertwining datum (ξ, W) and each non-zero $t = (t_1, t_2, \dots, t_s) \in D_W^\tau$, the induced element $\psi_{\bar{\xi}} \in \bigoplus_{i=1}^s H_{\infty, (m_i)}^{\tau|M, \lambda_i} =: H_{\bar{\xi}}^{\tau|M}$ satisfies

$$t \circ \psi_{\bar{\xi}} = (t_1 \circ \psi_1, \dots, t_s \circ \psi_s) \in W.$$

(D.3) (*Level 3*) We say that $\varphi \in \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{Hom}_M(E_\gamma, E_\tau))$ satisfies the intertwining condition, if for each intertwining datum (ξ, W) and each non-zero $t = (t_1, t_2, \dots, t_s) \in D_W^\tau$, the induced element $\varphi_{\bar{\xi}} \in \bigoplus_{i=1}^s \text{Hom}_M(E_\gamma, E_{\sigma_i})_{(m_i)}^{\lambda_i} =: H_{\bar{\xi}}^{\gamma, \tau}$ satisfies

$$t \circ \varphi_{\bar{\xi}} = (t_1 \circ \varphi_1, \dots, t_s \circ \varphi_s) \in D_W^\gamma.$$

Example 1 ([OIPa22-1], Example 1). Consider now $s = 2$ and $m_1 = m_2 = 0$. Let

$$L : H_\infty^{\sigma_1, \lambda_1} \longrightarrow H_\infty^{\sigma_2, \lambda_2}$$

be an intertwining operator between the two principal series representations.

The corresponding intertwining data is given by $\xi := ((\sigma_1, \lambda_1, 0), (\sigma_2, \lambda_2, 0)) \in \Xi^2$ and $W = \text{graph}(L) \subset H_\infty^{\sigma_1, \lambda_1} \oplus H_\infty^{\sigma_2, \lambda_2}$. Moreover, define $l^\tau : \text{Hom}_M(E_\tau, E_{\sigma_1}) \longrightarrow \text{Hom}_M(E_\tau, E_{\sigma_2})$ by

$$l^\tau(t)(v) = L(t\tau(\cdot)^{-1}v)(e) = L(\phi_v(t))(e),$$

where the element $\phi_v(t) \in H_\infty^\sigma$ is given by the function $\phi_v(t)(k) := t\tau(k^{-1})v$, for $k \in K, v \in E_\tau$ and $t \in \text{Hom}_M(E_\tau, E_{\sigma_1})$. Then

$$\begin{aligned} D_W^\tau = \{(t_1, t_2) \mid t_2 = l^\tau(t_1)\} &= \{(t, l^\tau(t)) \mid t \in \text{Hom}_M(E_\tau, E_{\sigma_1})\} \\ &\subset \text{Hom}_M(E_\tau, E_{\sigma_1}) \oplus \text{Hom}_M(E_\tau, E_{\sigma_2}). \end{aligned}$$

In this situation, we get the following intertwining conditions.

(D2.L) (Level 2) For each $t \in \text{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\psi(\lambda_i, \cdot) \in H_\infty^{\tau|M^*}$, $i = 1, 2$

$$L(t \circ \psi(\lambda_1, \cdot)) = l^\tau(t) \circ \psi(\lambda_2, \cdot). \quad (2.6)$$

(D3.L) (Level 3) For each $t \in \text{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\varphi(\lambda_i) \in \text{Hom}_M(E_\gamma, E_\tau)$, $i = 1, 2$

$$l^\gamma(t \circ \varphi(\lambda_1)) = l^\tau(t) \circ \varphi(\lambda_2). \quad (2.7)$$

Similar as in (Level 1), for $r > 0$, we introduce a Paley-Wiener space in:

- (Level 2) $PW_{r,\tau}(\mathfrak{a}_\mathbb{C}^* \times K/M) \subset \text{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{\tau|M^*})$,
- (Level 3) ${}_\gamma PW_\tau(\mathfrak{a}_\mathbb{C}^*) \subset \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{Hom}_M(E_\gamma, E_\tau))$,

which is characterised by the usual growth condition ([OlPa22-1], Def. (2.ii)_r resp. (3.ii)_r) and the corresponding intertwining condition (D.2) and (D.3), respectively. The Paley-Wiener-Schwartz space for distributional sections is denoted by $PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ respectively ${}_\gamma PWS_\tau(\mathfrak{a}_\mathbb{C}^*)$. It is characterised in the same way as above except that we replace the growth condition by a weaker one ([OlPa22-1], Def. (2.iis)_r resp. (3.iis)_r). As in ([OlPa22-1], Sect. 6) we equip the Paley-Wiener(-Schwartz) space with the inductive limit topology and the space of distributional sections $C_c^{-\infty}(X, \mathbb{E}_\tau)$ and $C_c^{-\infty}(G, \gamma, \tau)$ with the strong dual topology. In [OlPa22-1], we derived the following.

Theorem 2 (Topological Paley-Wiener(-Schwartz) theorem for sections in (Level 2) and (Level 3), [OlPa22-1] Thms. 3 & 4). *The Fourier transform*

$$C_c^{\pm\infty}(X, \mathbb{E}_\tau) \ni \psi \mapsto \mathcal{F}_\tau(\psi)(\lambda, k) \in PW(S)_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M), \quad (\lambda, k) \in \mathfrak{a}_\mathbb{C}^* \times K$$

is a topological isomorphism between $C_c^{\pm\infty}(X, \mathbb{E}_\tau)$ and $PW(S)_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$.

Moreover, by considering an additional K -representation (γ, E_γ) with associated homogeneous vector bundle \mathbb{E}_γ , then the Fourier transform

$$C_c^{\pm\infty}(G, \gamma, \tau) \ni \varphi \mapsto {}_\gamma \mathcal{F}_\tau(\varphi)(\lambda) \in {}_\gamma PW(S)_\tau(\mathfrak{a}_\mathbb{C}^*), \quad \lambda \in \mathfrak{a}_\mathbb{C}^*$$

is a topological isomorphism between $C_c^{\pm\infty}(G, \gamma, \tau)$ and ${}_\gamma PW(S)_\tau(\mathfrak{a}_\mathbb{C}^*)$. □

In Example 1, we worked out the intertwining conditions coming from intertwining operators between principal series representations of G . We want to make them more explicit for the most important case, the Knapp-Stein intertwining operators, which we now recall.

Knapp-Stein intertwining operator. Let $W_A := N_K(\mathfrak{a})/M$ be the Weyl group. Note that W_A acts on $\mathfrak{a}_\mathbb{C}^*$ as well as on \widehat{M} . Let $w \in W_A$ be represented by $m_w \in M' := N_K(\mathfrak{a})$ and $\sigma \in \widehat{M}$. We realise σ on the vector space E_σ . We define a new representation $w\sigma \in \widehat{M}$ of M acting on E_σ by

$$w\sigma(m) := \sigma(m_w^{-1}mm_w), \quad m \in M.$$

Its equivalence class only depends on $w \in W_A$ and not on the choice of m_w .

Definition 6 (Knapp-Stein intertwining operator, [KSt71], [KSt80] & [Kna02], Chap. V). *Let $\Delta_\mathfrak{a}^+$ be the positive root system of $(\mathfrak{g}, \mathfrak{a})$ corresponding to N . Let $\Delta_\mathfrak{a}^- := -\Delta_\mathfrak{a}^+$ and \overline{N} be the unipotent subgroup coming from the associated Iwasawa decomposition corresponding to $\Delta_\mathfrak{a}^-$. Write $N_w := N \cap w\overline{N}w^{-1}$. For $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_\mathbb{C}^*$ with $(\text{Re}(\lambda), \alpha) > 0$, for all $\alpha \in \Delta_\mathfrak{a}^+ \cap w^{-1}\Delta_\mathfrak{a}^-$ and for a fixed representative $m_w \in M'$, we define the intertwining operator*

$$J_{w,\sigma,\lambda} : H_\infty^{\sigma,\lambda} \longrightarrow H_\infty^{w\sigma,w\lambda}$$

by the convergent integral

$$J_{w,\sigma,\lambda}(\varphi(g)) := \int_{N_w} \varphi(gnm_w) dn, \quad g \in G, \varphi \in H_\infty^{\sigma,\lambda},$$

which depends holomorphically on $\lambda \in \mathfrak{a}_\mathbb{C}^*$. This operator has a meromorphic continuation to $\mathfrak{a}_\mathbb{C}^*$.

For later reference, let us state the intertwining conditions coming from the Knapp-Stein operators.

Example 2 (Knapp-Stein intertwining condition in (Level 1)). For $w \in W_A$ and fixed $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$, we consider the Knapp-Stein intertwining operator $J_{w,\sigma,\lambda}$ as in Def. 6. Let $\phi \in \prod_{\sigma \in \widehat{M}} \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{End}(H_{\infty}^{\sigma}))$. Then, the condition

$$J_{w,\sigma,\lambda} \circ \phi(\sigma, \lambda) = \phi(w\sigma, w\lambda) \circ J_{w,\sigma,\lambda}, \quad (2.8)$$

is a special intertwining condition of (D.a) in Def. 2. Note that the corresponding intertwining datum (ξ, W) is given by $\xi = ((\sigma, \lambda, 0), (w\sigma, w\lambda, 0))$ and $W = \text{graph}(J_{w,\sigma,\lambda}) \subset H_{\infty}^{\sigma,\lambda} \oplus H_{\infty}^{w\sigma,w\lambda}$, (compare Example 1).

Definition 7 (Harish-Chandra \mathbf{c} -function, (e.g. [Olb95], Def. 3.8)). Let $\tau \in \widehat{K}$, $w \in W_A$, $\overline{N}_w := \overline{N} \cap w^{-1}Nw$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $(\text{Re}(\lambda), \alpha) > 0$, for all $\alpha \in \Delta_{\mathfrak{a}}^+ \cap w^{-1}\Delta_{\mathfrak{a}}^-$. The \mathbf{c} -function is defined by

$$\mathbf{c}_{w,\tau}(\lambda) := \int_{\overline{N}_w} a(\overline{n})^{-(\lambda+\rho)} \tau(\kappa(\overline{n})) d\overline{n} \in \text{End}_M(E_{\tau}),$$

which can be extended to a meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$.

Furthermore, we have $\mathbf{c}_{w,\tau}(\sigma, \lambda) := \text{pr}_{\sigma} \circ \mathbf{c}_{w,\tau}(\lambda) \circ \text{pr}_{\sigma} \in \text{End}_M(E_{\tau}(\sigma))$, where $\text{pr}_{\sigma} : E_{\tau} \rightarrow E_{\tau}(\sigma)$ is the projection on the σ -isotypic component.

Consider now $w \in W_A$ as a Weyl element with maximal length, then, we set

$$\mathbf{c}_{\tau}(\lambda) := \mathbf{c}_{w,\tau}(\lambda) \quad \text{and} \quad \mathbf{c}_{\tau}(\sigma, \lambda) := \mathbf{c}_{w,\tau}(\sigma, \lambda).$$

One can express the Knapp-Stein intertwining operators, by the Harish-Chandra \mathbf{c} -functions ([Olb95], Lem. 3.12 & Satz 3.13):

$$J_{w,\sigma,\lambda}(\phi_v(t)) = \phi_v(t \circ \tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(-w\lambda)), \quad (2.9)$$

where $\phi_v(t)$ is defined as in Example 1 for $v \in E_{\tau}$ and $t \in \text{Hom}_M(E_{\tau}, E_{\sigma})$. Set $J_{w,\tau,\lambda} := \tau(m_w)J_{w,\tau|_M,\lambda}$, which is independent of the choice of $w \in W_A$. Hence, this leads to the following statement.

Proposition 1 (Knapp-Stein intertwining condition in (Level 2) and (Level 3)).

(a) (Level 2) Let $\psi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_{\infty}^{\tau|_M})$ satisfying (D.2). Then, we have

$$J_{w,\tau,\lambda}\psi(\lambda, \cdot) = \mathbf{c}_{w^{-1},\tau}(-w\lambda)\psi(w\lambda, \cdot), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, w \in W_A. \quad (2.10)$$

(b) (Level 3) Let $\varphi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{Hom}_M(E_{\gamma}, E_{\tau}))$ satisfying (D.3). Then, we have

$$\varphi(\lambda)\gamma(m_w^{-1})\mathbf{c}_{w^{-1},\gamma}(-w\lambda) = \tau(m_w^{-1})\mathbf{c}_{w^{-1},\tau}(-w\lambda) \circ \varphi(w\lambda), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, w \in W_A. \quad (2.11)$$

Proof. We consider the operator $l^{\tau} : \text{Hom}_M(E_{\tau}, E_{\sigma}) \rightarrow \text{Hom}_M(E_{\tau}, E_{\sigma})$ as in Example 1 associated by $L = J_{w,\sigma,\lambda}$. Then (2.9) says that

$$l^{\tau}(t) = t \circ \tau(m_w^{-1}) \circ \mathbf{c}_{w^{-1},\tau}(-w\lambda).$$

Now the proposition follows from (2.6) and (2.7), as in Example 1. \square

Example 3. (a) Let τ be a trivial one-dimensional representation. Then, $C_c^{\infty}(X, \mathbb{E}_{\tau}) = C_c^{\infty}(X)$ and $H_{\infty}^{\tau|_M} = C^{\infty}(K/M)$. Helgason showed in ([Hel89], Thm. 5.1.) that $\beta \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, C^{\infty}(K/M))$ belongs to the Paley-Wiener space if, and only, if it satisfies the usual growth condition and the intertwining condition:

$$\int_{K/M} e^{\tau} e_{w\lambda,k}(g)\beta(w\lambda, k)dk = \int_{K/M} e^{\tau} e_{\lambda,k}(g)\beta(\lambda, k)dk, \quad w \in W_A. \quad (2.12)$$

It is not difficult to show that Helgason's intertwining condition is equivalent to (2.10). In fact, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, consider the Poisson transform (e.g. [Hel89] or [Olb95], Def. 3.2) $\mathcal{P}_{\lambda} : C^{\infty}(K/M) \rightarrow C^{\infty}(X)$ given by $\mathcal{P}_{\lambda}(f)(g) := \int_K e^{\tau} e_{\lambda,k}(g)f(k)dk$, for $g \in G$. Then, Helgason's condition (2.12) can be expressed in terms of Poisson transform

$$\mathcal{P}_{\lambda} \circ \beta_{\lambda} = \mathcal{P}_{w\lambda} \circ \beta_{w\lambda}, \quad w \in W_A, \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

where $\beta_{\lambda} := \beta(\lambda, \cdot)$. The result now follows from the functional equation of the Poisson transform ([Olb95], Satz 3.15).

(b) Let τ and γ be two trivial one-dimensional representations. Consider a function $\beta \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*)$ which satisfies the usual growth condition. Helgason and Gangolli ([Gan71] & [Hel20], Thm. 7.1) proved that $\beta \in {}_{\gamma}PW_{\tau}(\mathfrak{a}_{\mathbb{C}}^*)$, if and only, if

$$\beta(\lambda) = \beta(w\lambda), \quad \text{for } \lambda \in \mathfrak{a}_{\mathbb{C}}^*, w \in W_A.$$

This condition is equivalent to (2.11) in the case γ, τ are trivial.

3 Sufficient intertwining conditions for rank one

We want to reduce the amount of intertwining data in (D.a) of Def. 2 to a minimum. In this section, we assume that G has real rank one. In this situation, the set of positive restricted roots $\Delta_{\mathfrak{a}}^+$ consists of at most two elements, namely α and possibly 2α . The Weyl group is reduced to $\{-1, 1\}$ acting on $\mathfrak{a}_{\mathbb{C}}^*$ by multiplication.

We have the following special intertwining condition. An irreducible unitary representation (π, E_{π}) of G is called a representation of the discrete series if there is a G -invariant embedding $E_{\pi} \hookrightarrow L^2(G)$. Here, $L^2(G)$ denote the space of all square integrable functions with respect to invariant measure dg on G . Write \widehat{G}_d the set of equivalence classes of discrete series representations of G . Let H_{π} be any Hilbert space, where the representation $\pi \in \widehat{G}_d$ is realized. Let $H_{\pi}^{\infty} \subset H_{\pi}$ be the corresponding space of smooth vectors. For every representation of discrete series $\pi \in \widehat{G}_d$, we choose an embedding

$$i_{\pi} : H_{\pi}^{\infty} \hookrightarrow H_{\infty}^{\sigma_{\pi}, \lambda_{\pi}}$$

into some principal series representation (Casselman's subrepresentation theorem, [Wal88], Thm. 3.8.3. & Casselman's and Wallach's globalization Thm. [Wal92], Chap. 11) and set

$$W_{\pi} := i_{\pi}(H_{\pi}^{\infty}) \subset H_{\infty}^{\sigma_{\pi}, \lambda_{\pi}}.$$

It is a closed G -invariant subspace. Hence, the condition

$$\phi(\sigma_{\pi}, \lambda_{\pi})(W_{\pi}) \subset W_{\pi}, \quad \pi \in \widehat{G}_d \tag{3.1}$$

is also of the form (D.a), with $s = 1$ and $m = 0$, and it permits us to define that

$$\phi(\pi) := \phi(\sigma_{\pi}, \lambda_{\pi})|_{W_{\pi}} \in \text{End}(W_{\pi}). \tag{3.2}$$

Thus, for $r > 0$, we define the 'special' Paley-Wiener space $PW_r^+(G)$ by replacing Delorme's intertwining condition (D.a) by the conditions (2.8) and (3.1) only.

Let $w \in W_A$ be the non-trivial element. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $(\text{Re}(\lambda), \alpha) > 0$, let $m \in \mathbb{N}_0$ be the maximal order of the zeros of $J_{w, \sigma, \mu}(f_{\mu})$ at $\mu = \lambda$, where $\mu \mapsto f_{\mu} \in H_{\infty}^{\sigma, \mu}$ runs over all germs of holomorphic functions at λ with $f_{\lambda} \neq 0$.

We consider the induced operator

$$J_{w, \sigma, \lambda}^{(m-1)} : H_{\infty, (m-1)}^{\sigma, \lambda} \longrightarrow H_{\infty, (m-1)}^{w\sigma, -\lambda} \tag{3.3}$$

and its kernel $\text{Ker}(J_{w, \sigma, \lambda}^{(m-1)}) \subset H_{\infty, (m-1)}^{\sigma, \lambda}$. By convention, we set $H_{\infty, (-1)}^{\sigma, \lambda} = \{0\}$, for $m = 0$. Notice that due to condition (2.8), we have for $\phi^{(m-1)}(\sigma, \lambda) \in \text{End}(H_{\infty, (m-1)}^{\sigma, \lambda})$

$$\phi^{(m-1)}(\sigma, \lambda)(\text{Ker}(J_{w, \sigma, \lambda}^{(m-1)})) \subset \text{Ker}(J_{w, \sigma, \lambda}^{(m-1)}) \subset H_{\infty, (m-1)}^{\sigma, \lambda}, \quad (\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*.$$

Let $\tau \in \widehat{K}$ with highest weight $\mu_{\tau} \in \mathfrak{t}^*$, where $\mathfrak{t} \subset \mathfrak{k}$ is the Lie algebra of a maximal torus $T \subset K$. We define

$$2\rho_c := \sum_{\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})} \alpha \in \mathfrak{t}^*$$

the sum of all positive roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{k}_{\mathbb{C}}$. For a K -representation V , we denote by $V(\tau)$ its corresponding isotypic component. For $\sigma \in \widehat{M}$ and $\pi \in \widehat{G}_d$, we define $|\sigma|, |\pi| \in [0, \infty]$ by

$$|\sigma| := \min_{\{\tau \mid H_{\infty}^{\sigma}(\tau) \neq \{0\}\}} \|\mu_{\tau} + 2\rho_c\| \quad \text{and} \quad |\pi| := \min_{\{\tau \mid H_{\pi}^{\infty}(\tau) \neq \{0\}\}} \|\mu_{\tau} + 2\rho_c\|$$

i.e. $|\sigma|, |\lambda|$ are the lengths of 'the' minimal K -type τ of H_{∞}^{σ} and H_{π}^{∞} respectively, ([Del05], Sect. 1.3). Denote by $B(\sigma), B(\pi) \subset \widehat{K}$ the finite set of all minimal K -types of $H_{\infty}^{\sigma}, H_{\pi}^{\infty}$.

Example 4. Let $G = SL(2, \mathbb{R})$ and $K = SO(2)$ its maximal compact subgroup. With the notations introduced in Sect. 4 below, we have that $\widehat{K} \cong \mathbb{Z}$ and $\rho_c = 0$.

(i) $M = \{\pm 1\}$, thus

- if σ is trivial, then $B(\sigma) = \{0\} \subset \mathbb{Z}$, (trivial K -type) and $|\sigma| = 0$,
- if σ is non-trivial, then $B(\sigma) = \{+1, -1\} \subset \mathbb{Z}$ and $|\sigma| = 1$.

- (ii) Let $\pi = D_k, k \in \mathbb{Z} \setminus \{0\}$ be the discrete series representation of G parametrized as in Thm. 4 below, then

$$B(D_k) = \begin{cases} \{k+1\}, & k > 0, \\ \{k-1\}, & k < 0 \end{cases}$$

and $|\pi| = k + 1$.

The following result tells us that the intertwining conditions (2.8) and (3.1) with an additional 'vanishing' condition are sufficient for semi-simple Lie group G of real rank one.

Theorem 3. *Let $\text{rk}_{\mathbb{R}}(G) = 1$. For $r > 0$, let \mathcal{A} be a linear closed and $K \times K$ invariant subspace of $PW_r^+(G)$ satisfying $\mathcal{F}_{\sigma,\lambda}(C_r^\infty(G)) \subset \mathcal{A}$ and the following condition:*

(D.b) *Let $\sigma \in \widehat{M}$ and $\phi \in \mathcal{A}$ such that*

(i) *$\phi(\sigma', \lambda) = 0$, for all $\sigma' \in \widehat{M}$ with $|\sigma'| > |\sigma|$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,*

(ii) *$\phi(\pi) = 0$, for all $\pi \in \widehat{G}_d$ with $|\pi| > |\sigma|$.*

Then, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^$ with $(\text{Re}(\lambda), \alpha) > 0$, ϕ induces the zero-operator on $\text{Ker}(J_{w,\sigma,\lambda}^{(m-1)})$:*

$$\phi^{(m-1)}(\sigma, \lambda) \Big|_{\text{Ker}(J_{w,\sigma,\lambda}^{(m-1)})} = 0.$$

Here, $(m-1)$ depends on (σ, λ) as defined above (3.3) and $\phi(\pi)$ is defined in (3.2).

Then,

$$\mathcal{A} = PW_r(G) \cong \mathcal{F}_{\sigma,\lambda}(C_r^\infty(G)).$$

Proof of Thm. 3. By Delorme's Paley-Wiener Thm. 1, we already know that $PW_r(G) \cong \mathcal{F}_{\sigma,\lambda}(C_r^\infty(G))$ is a closed and $K \times K$ invariant subspace of $PW_r^+(G)$. Therefore, it suffices to show that $\mathcal{F}_{\sigma,\lambda}(C_r^\infty(G)) \subset \mathcal{A}$ is dense. Thus for every $K \times K$ -finite element $\phi \in \mathcal{A}$, we need to find a function $f \in C_r^\infty(G)_{K \times K}$ such that

$$\pi_{\sigma,\lambda}(f) = \phi(\sigma, \lambda), \quad \forall (\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*.$$

Let $\phi \in \mathcal{A}_{K \times K}$. It is given by a collection $(\phi_\sigma), \sigma \in \widehat{M}$. By $K \times K$ -finiteness, only finitely many ϕ_σ are non-zero. Similar, by $K \times K$ -finiteness, $\phi(\pi) = 0$, for all but finitely many $\pi \in \widehat{G}_d$. Indeed, for any given K -type τ , there are only finitely many $\pi \in \widehat{G}_d$, with $H_\pi^\infty(\tau) \neq 0$ (e.g. [Wal88], Cor. 7.7.3). We define $l(\phi) \in [0, \infty)$ by

$$l(\phi) := \max\{|\sigma|, |\pi| \mid \sigma \in \widehat{M}, \phi_\sigma \neq 0; \pi \in \widehat{G}_d, \phi(\pi) \neq 0\}.$$

We can now imitate the inductive proof of Prop. 2 in Delorme's paper [Del05].

Assume, as induction hypothesis, that for all $\psi \in \mathcal{A}$ with $l(\psi) < l(\phi)$, there are $f \in C_r^\infty(G)$ with $\mathcal{F}(f) = \psi$.

We enumerate

$$\{\sigma \in \widehat{M} \mid |\sigma| = l(\phi)\} = \{\sigma_1, \dots, \sigma_n\} \cup \{w\sigma_1, \dots, w\sigma_n\}$$

and

$$\{\pi \in \widehat{G}_d \mid |\pi| = l(\phi)\} = \{\pi_1, \dots, \pi_s\}.$$

Condition (ii) together with (2.8) says, in particular, that ϕ_{σ_i} belongs to a space that Delorme denotes by \mathcal{K}_{σ_i} , ([Del05], Def. 1). Strictly speaking Delorme has a condition for $(\text{Re}(\lambda), \alpha) \geq 0$. But if $\text{rk}_{\mathbb{R}}(G) = 1$, only $(\text{Re}(\lambda), \alpha) > 0$ matters. Note, that $\phi(\pi_j)$ belongs automatically to \mathcal{K}_{π_j} . We can apply Prop. 1 together with Eq. (1.38) of Delorme's paper [Del05], to deduce the existence of $f_1, f_2, \dots, f_n \in C_r^\infty(G)$ and $g_1, g_2, \dots, g_s \in C_r^\infty(G)$ with

$$\begin{aligned} \pi_{\sigma_i,\lambda}(f_i) &= \phi(\sigma_i, \lambda), \quad i \in \{1, 2, \dots, n\}, \\ \pi_j(g_j) &= \phi(\pi_j), \quad j \in \{1, 2, \dots, s\}, \end{aligned}$$

for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Moreover, the discussion after Eq. (3.9) in ([Del05], p.1018), makes clear that we can choose the f_i and g_j such that

- (i) $l(\mathcal{F}(f_i)) = l(\mathcal{F}(g_j)) = l(\phi), \forall i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, s\}$ and
- (ii) $\pi_{\sigma_k,\lambda}(f_i) = 0, \forall k \neq i$,
- (iii) $\pi_{\sigma_i,\lambda}(g_j) = 0, \forall i, j$,
- (iv) $\pi_j(f_i) = 0, \forall i, j$,

(v) $\pi_k(g_j) = 0, \forall k \neq j$.

Now, we set

$$\psi := \phi - \sum_{i=1}^n \mathcal{F}(f_i) - \sum_{j=1}^s \mathcal{F}(g_j).$$

Then, by (i)-(v) we have $l(\psi) < l(\phi)$. Thus, by induction hypothesis $\psi = \mathcal{F}(f_0)$. We conclude that $\phi = \mathcal{F}(f)$ with $f = f_0 + f_1 + \cdots + f_n + g_1 + g_2 + \cdots + g_s$. \square

Remark 1. The result above can be extended to higher real rank. The extension involves representations induced from all cuspidal parabolic subgroups P as well as the Knapp-Stein intertwining operator for them.

4 The case $G = \mathbf{SL}(2, \mathbb{R})$ and beyond

We consider $G = \mathbf{SL}(2, \mathbb{R}) = \left\{ g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}(2, \mathbb{R}) \mid \det(g) = 1 \right\}$ the special linear group of \mathbb{R}^2 . It has dimension three. We fix the Iwasawa decomposition $G = KAN$, where

$$K = \mathbf{SO}(2) = \left\{ k_\theta := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}, \quad A = \left\{ a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

$$N = \left\{ n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

G is a connected and simple Lie group with maximal compact subgroup K of dimension one. Clearly, K is isomorphic to the unit circle S^1 . Hence

$$\widehat{K} = \{ \delta_n \mid n \in \mathbb{Z} \} \cong \mathbb{Z}, \quad \delta_n(k_\theta) := e^{in\theta} \in \mathbf{GL}(1, \mathbb{C}) \cong \mathbb{C} \setminus \{0\}$$

is the set of all irreducible representations of K . The representation space E_{δ_n} is one-dimensional and equal to \mathbb{C} . We sometimes denote the K -representation δ_n simply by n . If $H = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{a}$, then the positive root α is given by $\alpha(H) = 2t$ and $\rho(H) = t$, for all $t \in \mathbb{R}$. We identify $\mathfrak{a}_{\mathbb{C}}^*$ isometrically to \mathbb{C} with respect to the norm of the invariant twice trace form on $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, i.e., $z\alpha \mapsto z$ and $\rho \mapsto \frac{1}{2}$.

Since $M = \{\pm \text{Id}\}$, we have $\widehat{M} \cong \mathbb{Z}/2\mathbb{Z}$. Let $+$ be the trivial and $-$ the non-trivial element of \widehat{M} . For $\sigma = \pm \in \widehat{M}$ and $\lambda \in \mathbb{C} \cong \mathfrak{a}_{\mathbb{C}}^*$, we write $(\pi_{\pm, \lambda}, H_\infty^{\pm, \lambda})$ for the principal series representations of G . Its restriction to K is the set of Fourier series on S^1 with only non-zero even or odd Fourier coefficients

$$H_\infty^\pm = \{ f \in C^\infty(K, \mathbb{C}) \mid f \text{ even or odd} \} \cong \bigoplus_{n \text{ even or odd}} \delta_n.$$

In order to write down the composition series of $H_\infty^{\pm, \lambda}$, we will from now on denote, for convenience, an exact, non-splitting, module sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

shortly by a 'boxes-picture'

$$B = \begin{array}{|c|} \hline C \\ \hline A \\ \hline \end{array} :$$

A proof of the following classical result can be found for example in ([Wal88], 5.6) or in ([La75], Ch. VI). Note that the referenced proof is also valid for G -representations of smooth vectors instead of (\mathfrak{g}, K) -modules, if we apply Casselman's and Wallach's globalization theorem ([Wal92], Prop. 11).

Theorem 4 (Structure of principal series representations of $\mathbf{SL}(2, \mathbb{R})$). *The principal series representations $H_\infty^{\pm, \lambda}$ of $\mathbf{SL}(2, \mathbb{R})$ is exactly reducible, if*

$$\lambda \in I_\pm := \begin{cases} \frac{1}{2} + \mathbb{Z}, & \sigma = +, \\ \mathbb{Z}, & \sigma = -. \end{cases}$$

For $\lambda = \frac{k}{2} \in I_\pm, k \in \mathbb{N}$, we have

$$H_\infty^{\pm, -\frac{k}{2}} = \frac{D_{-k} \oplus D_k}{F_k} \quad H_\infty^{\pm, \frac{k}{2}} = \frac{F_k}{D_{-k} \oplus D_k}$$

where $F_k := \bigoplus_{l=-(k-1)}^{k-1} \delta_{2l}$ are the finite-dimensional irreducible $\mathbf{SL}(2, \mathbb{R})$ -representation of dimension k and D_k resp. D_{-k} are smooth vectors of a representation of the discrete series, which are characterized by the K -type decomposition $D_{\pm k} = \bigoplus_{j \geq 0} \delta_{\pm(k+1+2j)}$. Furthermore, for $\lambda = 0$, we have

$$H_\infty^{-, 0} = \frac{D_- \oplus D_+}{}$$

where $D_\pm = \bigoplus_{j \geq 0} \delta_{\pm(1+2j)}$ are the limits of the discrete series. \square

Remark 2. Let W_λ be a proper closed invariant G -submodule of $H_\infty^{\pm, \lambda}$, for $\lambda \in I_\pm$, as in Thm. 4. Then, one can observe that

- for $\lambda > 0$, $W_\lambda \in \{D_{-k}, D_k, D_{-k} \oplus D_k\}$, $k = 2\lambda$,

- for $\lambda < 0$, $W_\lambda \in \left\{ F_k, \frac{D_{-k}}{F_k}, \frac{D_k}{F_k} \right\}$, $k = 2\lambda$,

- while for $\lambda = 0$ and $\sigma = -$, $W_\lambda \in \{D_+, D_-\}$.

To describe the intertwining conditions for $G = \mathbf{SL}(2, \mathbb{R})$ in the three levels, we first need some preparation. The Harish-Chandra \mathbf{c} -function for G is denoted by $\mathbf{c}_n(\lambda)$ for $n \in \mathbb{Z}$. Due Cohn ([Co74], App. 1), it is given explicitly in terms of gamma function $\Gamma(\cdot)$, by the formula (for a suitable normalization of the Haar measure $d\bar{n}$)

$$\mathbf{c}_n(\lambda) = \mathbf{c}_{-n}(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\lambda)\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1+n}{2})\Gamma(\lambda + \frac{1-n}{2})}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \quad (4.1)$$

Let $n \equiv m \pmod{2}$, not necessary distinct, then using the gamma function recurrence formula

$$\Gamma(\lambda + a) = (\lambda + (a-1))\Gamma(\lambda + (a-1)), \quad a \in \mathbb{Z}, \lambda \in \mathfrak{a}_{\mathbb{C}}^* \quad (4.2)$$

repeatedly, the quotient of the \mathbf{c} -functions is given by

$$\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_m(\lambda)} = \frac{\Gamma(\lambda + \frac{1+m}{2})\Gamma(\lambda + \frac{1-m}{2})}{\Gamma(\lambda + \frac{1+n}{2})\Gamma(\lambda + \frac{1-n}{2})} = \begin{cases} 1, & \text{for } |n| = |m|, \\ \frac{(\lambda - \frac{|n|-1}{2})(\lambda - \frac{|n|-3}{2}) \dots (\lambda - \frac{|m|+1}{2})}{(\lambda + \frac{|n|-1}{2})(\lambda + \frac{|n|-3}{2}) \dots (\lambda + \frac{|m|+1}{2})}, & \text{for } |n| > |m|, \\ \frac{(\lambda + \frac{|m|-1}{2})(\lambda + \frac{|m|-3}{2}) \dots (\lambda + \frac{|n|+1}{2})}{(\lambda - \frac{|m|-1}{2})(\lambda - \frac{|m|-3}{2}) \dots (\lambda - \frac{|n|+1}{2})}, & \text{for } |n| < |m|. \end{cases} \quad (4.3)$$

Note that the quotient has zeros $\lambda \in \{\frac{|n|-1}{2}, \frac{|n|-3}{2}, \dots, \frac{|m|+1}{2}\}$ and poles in $\{-\frac{|n|-1}{2}, -\frac{|n|-3}{2}, \dots, -\frac{|m|+1}{2}\}$, for $|n| > |m|$, and inversely for $|n| < |m|$. We know by (2.11) that the matrix coefficient of the Knapp-Stein intertwining operator $J_{w, \pm, \lambda} : H_\infty^{\pm, \lambda} \rightarrow H_\infty^{\pm, -\lambda}$ with respect to the Fourier decomposition of $H_\infty^{\pm, \lambda} \cong \bigoplus_{n \text{ even or odd}} \delta_n$ is given by $c_n(\lambda)$ (up to sign).

Theorem 5 (Intertwining conditions in (Level 1)). *For $r > 0$, let \mathcal{A} be the space of all $\phi \in \prod_{\sigma \in \widehat{M}} \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{End}(H_\infty^\pm))$ such that ϕ satisfies the corresponding growth condition as well as the two intertwining conditions (2.8) and*

(D.b') ϕ leaves every proper closed G -submodule W_λ of H_∞^\pm , listed in Remark 2, invariant.

Then, \mathcal{A} satisfies the conditions of Thm. 3, this means that $\mathcal{A} = PW_r(G)$.

Proof. Note first, that the space \mathcal{A} is $K \times K$ invariant and linear closed, due the intertwining conditions (2.8) and (D.b').

We have that (D.b) of Thm. 3 gives a condition for each $\sigma \in \widehat{M} = \{\pm\}$. Let us first consider $\sigma = \{+\} \in \widehat{M}$. By Example 4, we have $|+\} = 0$ and $|\pi| = k+1 > 0$. Now let $\phi \in \mathcal{A}$ satisfying the assumption (D.b) (ii), i.e., in particular

$$\phi^{(0)} \left(+, \frac{k}{2} \right) \Big|_{D_{-k} \oplus D_k} = 0, \quad k \in 2\mathbb{Z} + 1. \quad (4.4)$$

Let us check that:

- (a) for $\text{Re}(\lambda) > 0$, the intertwining operator $J_{-, +, \lambda}$ has zeros of order at most one.
- (b) the kernel of $J_{w, +, \lambda}$ is equal to 0 or $D_{-k} \oplus D_k$ for $\text{Re}(\lambda) > 0$.

Consider the K -type $n \in 2\mathbb{Z}$ and the Harish-Chandra \mathbf{c} -function \mathbf{c}_n as in (4.1). If $n = 0$, then $\mathbf{c}_0(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})}$ and we see that $\mathbf{c}_0(\lambda)$, for $\text{Re}(\lambda) > 0$, has no zeros and no poles. Thus, we can consider the quotient

$$\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)} = \frac{\Gamma(\lambda + \frac{1}{2})^2}{\Gamma(\lambda + \frac{1+n}{2})\Gamma(\lambda + \frac{1-n}{2})} = \frac{(\lambda - \frac{|n|-1}{2}) \cdots (\lambda - \frac{1}{2})}{(\lambda + \frac{|n|-1}{2}) \cdots (\lambda + \frac{1}{2})}.$$

It has zeros $\lambda \in \{\frac{1}{2}, \dots, \frac{|n|-1}{2}\}$ of first order. Due to (2.9), we know that the intertwining operator $J_{w,+,\lambda}$ is in relation with the \mathbf{c} -function. If on all K -types, we have zeros of first order, then $J_{w,+,\lambda}$ should also have zeros of first order. Hence $J_{w,+,\lambda}$ has zeros of at most order one, this proves the first assertion (a) of the claim.

Concerning (b), we need to check for which K -type n , the quotient $\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)}$ has a zero, for fixed $\text{Re}(\lambda) > 0$. It is clear that, if $\lambda \notin I_+$, then $\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)}$ has no zeros, i.e. that $\text{Ker}(J_{w,+,\lambda}) = 0$. For fixed $\lambda = \frac{k}{2}$, $k \in 2\mathbb{Z} + 1$, the \mathbf{c} -quotient $\frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_0(\lambda)}$ has zeros if, and only if, n is a K -type of D_{-k} and D_k , i.e. $\text{Ker}(J_{w,+,\lambda}) = D_{-k} \oplus D_k$. Thus, this implies (b).

By (b) and (4.4), we have that the operator $\phi^{(0)}(+, \lambda)$ annihilates $\text{Ker}(J_{w,+,\lambda})$ for $\sigma = \{+\} \in \widehat{M}$ and $\text{Re}(\lambda) > 0$. By (a), we have that the order m is equal to the one, thus this condition is sufficient. By arguing in a similar way as above for $\sigma = \{-\} \in \widehat{M}$ with $|-| = 1$, and $\pi = D_k \in \widehat{G}_d$, $k \in 2\mathbb{Z} \setminus \{0\}$, with $|\pi| = |k| + 1$, we can conclude that \mathcal{A} satisfies the condition (D.b) of Thm. 3. \square

Now let us move to (Level 2).

Theorem 6 (Intertwining conditions in (Level 2)). *Let $m \in \mathbb{Z}$ be a K -type, Then, $\psi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_{\infty}^{m|M})$ satisfies the intertwining condition (D.2) of Def. 5 if, and only if,*

(2.a) $J_{w,m,\lambda}\psi(\lambda, \cdot) = \mathbf{c}_m(\lambda)\psi(-\lambda, \cdot)$, for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

(2.b) $\psi(\lambda, \cdot) \in W_{\lambda}$, where W_{λ} is the invariant (colored in blue) G -submodule of $H_{\infty}^{\pm,\lambda}$ represented by the boxes-pictures in Fig. 1. Here the choice of \pm depends on the parity of m .

Notice that if W_{λ} is the whole colored blue box, then there are no intertwining condition.

Proof. We need to show that the conditions (2.a) and (2.b) correspond to the condition (D.2) in Def. 5. In fact, by Prop. 1(a), we have that (2.a) is a special case of (D.2).

Concerning (2.b), condition (D.2) says that for each W we have an intertwining condition corresponding to (D.b') in Thm. 5. Now we need to extract in which of these W_{λ} , there is an intertwining condition. If the K -type m is in a closed G -submodule W_{λ} of $H_{\infty}^{\pm,\lambda}$, then D_W^m is one-dimensional. Hence, by (D.2), ψ has values in this G -submodule W_{λ} . By (D.b') in Thm. 5, we thus take the smallest closed proper invariant G -submodule of them. Otherwise, if the K -type is not in a closed G -submodule W_{λ} of $H_{\infty}^{\pm,\lambda}$, then $D_W^m = \{0\}$ and thus there are no intertwining conditions. Consequently, we obtain the boxes-pictures in Fig. 1. \square

The final step will be to move in (Level 3). Note that $\text{Hom}_M(E_n, E_m) = \{0\}$ if $n \equiv m \pmod{2}$.

Definition 8. *Let $n \equiv m \pmod{2}$ K -types. We define the polynomial $q_{n,m}$ in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with values in $\text{Hom}_M(E_n, E_m) \cong \mathbb{C}$ by*

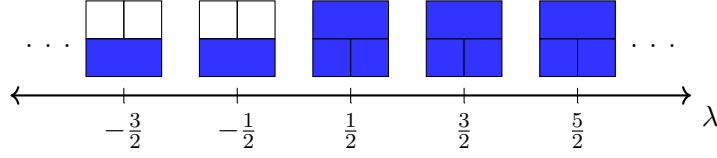
$$q_{n,m}(\lambda) := \begin{cases} 1, & \text{if } n = m, \\ (\lambda + \frac{|m|+1}{2})(\lambda + \frac{|m|+3}{2}) \cdots (\lambda + \frac{|n|-1}{2}), & \text{if } |n| > |m| \text{ and same signs,} \\ (\lambda - \frac{|n|+1}{2})(\lambda - \frac{|n|+3}{2}) \cdots (\lambda - \frac{|m|-1}{2}), & \text{if } |n| < |m| \text{ and same signs,} \\ (\lambda + \frac{|n|-1}{2})(\lambda + \frac{|n|-3}{2}) \cdots (\lambda - \frac{|m|-1}{2}), & \text{else, with different signs.} \end{cases} \quad (4.5)$$

Theorem 7 (Intertwining conditions in (Level 3)). *Let $n \equiv m \pmod{2}$ be two K -types. Then, $\varphi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{Hom}_M(E_n, E_m))$ satisfies the intertwining condition (D.3) of Def. 5 if, and only if, there exists an even holomorphic function $h \in \text{Hol}(\lambda^2)$ such that*

$$\varphi(\lambda) = h(\lambda) \cdot q_{n,m}(\lambda), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \quad (4.6)$$

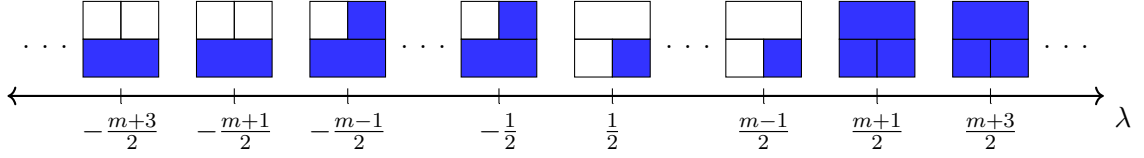
where $q_{n,m}$ is the polynomial (4.5).

- for $m = 0$:

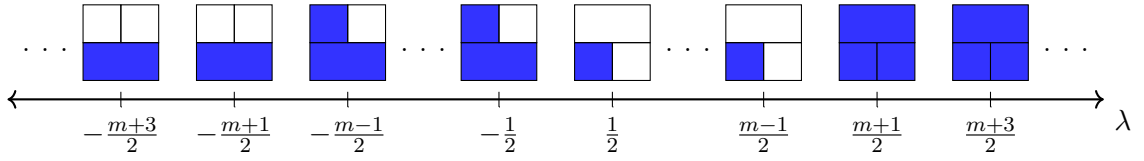


- for $m \in 2\mathbb{Z}$:

- $m > 0$:

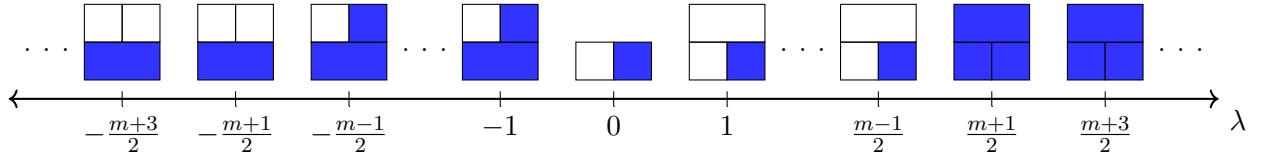


- $m < 0$:



- for $m \in 2\mathbb{Z} + 1$:

- $m > 0$:



- $m < 0$:

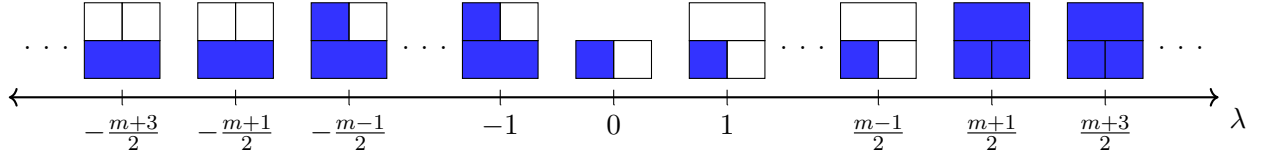


Figure 1: Boxes-pictures for $G = \mathbf{SL}(2, \mathbb{R})$.

Proof. By Thm. 6, it is sufficient to prove that the conditions (2.a) and (2.b) correspond to (4.6). In particular, we want to show that $(\lambda, k_\theta) \mapsto \varphi(\lambda)e^{im\theta}$ satisfies (2.b) if, and only if, φ has zeros at the zeros of the polynomial $q_{n,m}$. From Thm. 6 (2.b), we know that the invariant G -submodule W_λ are represented by the boxes-pictures in Fig. 1. Thus, we need to check, where the K -type n is not in the colored blue invariant G -submodule W_λ . We leave it to the reader to check that this happens exactly at the zeros of $q_{n,m}$. Thus, we can deduce that φ is of the form (4.6) with h an arbitrary holomorphic function.

Concerning the correspondence between the conditions (2.a) and (4.6), by Prop. 1(b), we observe that (2.a) corresponds to (D.3):

$$(-1)^{(m-n)/2} \frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_m(\lambda)} \varphi(\lambda) = \varphi(-\lambda), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, n, m \in \mathbb{Z}. \quad (4.7)$$

By using Def. 8, we observe that $\frac{q_{n,m}(-\lambda)}{q_{n,m}(\lambda)} = (-1)^{(m-n)/2} \frac{\mathbf{c}_n(\lambda)}{\mathbf{c}_m(\lambda)}$, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*, n, m \in \mathbb{Z}$. Hence, we obtain

$$\frac{\varphi(\lambda)}{q_{n,m}(\lambda)} = \frac{\varphi(-\lambda)}{q_{n,m}(-\lambda)}.$$

This means, (4.7) is satisfied if, and only if, $h(\lambda) = h(-\lambda)$, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. \square

We have completely determined the Paley-Wiener-(Schwartz) spaces for $G = \mathbf{SL}(2, \mathbb{R})$ in (Level 2) and (Level 3).

The case $G = \mathbf{SL}(2, \mathbb{R}) \times \mathbf{SL}(2, \mathbb{R})$

Let

$$G := G' \times G' = \mathbf{SL}(2, \mathbb{R}) \times \mathbf{SL}(2, \mathbb{R}) = \{g = (g_1, g_2) \mid g_1, g_2 \in G'\}$$

be the Cartesian product of two copies of semi-simple non-compact connected real Lie group $G' = \mathbf{SL}(2, \mathbb{R})$. Since $K' = \mathbf{SO}(2)$ is the maximal compact subgroup of G' , $K := K' \times K'$ is maximal compact in G .

The irreducible representations of K are given by pairs $(n_1, n_2), n_i \in \widehat{K}' \cong \mathbb{Z}$ ([Wal73], Sect. 2.36). More precisely, we denote by a tuple of integers $n := (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2 \cong \widehat{K}$ the K representations on the vector space $E_n := E_{n_1} \otimes E_{n_2} \in \mathbb{C}$ with action $[n_1, n_2](k_1, k_2) = n_1(k_1) \otimes n_2(k_2)$. The associated homogeneous line bundle over $X := X' \times X'$ is denoted by \mathbb{E}_n . For $l, n \in \widehat{K}$, we observe that $\text{Hom}_M(E_l, E_n) = \{0\}$, if $l_1 \not\equiv n_1 \pmod{2}$ or $l_2 \not\equiv n_2 \pmod{2}$. Note that $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C} \times \mathbb{C}$. By using Def. 8, we define for $l, n \in \mathbb{Z}^2, l_1 \equiv n_1 \pmod{2}, l_2 \equiv n_2 \pmod{2}$, the polynomial $q_{l,n}$ given by

$$q_{l,n}(\lambda_1, \lambda_2) := q_{l_1, n_1}(\lambda_1) \cdot q_{l_2, n_2}(\lambda_2), \quad (\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C} \cong \mathfrak{a}_{\mathbb{C}}^*, \quad (4.8)$$

where $q_{i, n_i}(\lambda_i)$ is the 'intertwining' polynomial (4.5) for $\lambda_i \in \mathbb{C}, i = 1, 2$.

Theorem 8 (Intertwining condition in (Level 3)). *Let $l, n \in \mathbb{Z}^2$ be two tuples of integers. Then, $\varphi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{Hom}_M(E_l, E_n))$ satisfies the intertwining condition (D.3) of Def. 5 if, and only if, there exists an even holomorphic function $h \in \text{Hol}(\lambda_1^2, \lambda_2^2): h(\lambda_1, \lambda_2) = h(-\lambda_1, \lambda_2) = h(\lambda_1, -\lambda_2)$ such that*

$$\varphi(\lambda_1, \lambda_2) := h(\lambda_1, \lambda_2) \cdot q_{l,n}(\lambda_1, \lambda_2), \quad (4.9)$$

where $q_{l,n}$ is the polynomial (4.8) in $(\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^*$.

To prove Thm. 8, we need first a density argument, which permits us to approximate the even holomorphic function h in (4.9) by even polynomials.

Lemma 1. *Consider the subset P of polynomial functions in $A := \{h \in \text{Hol}(\mathbb{C}^2) \mid h(\lambda_1, \lambda_2) = h(-\lambda_1, \lambda_2) = h(\lambda_1, -\lambda_2), \forall (\lambda_1, \lambda_2) \in \mathbb{C}^2\}$. Then, $P \subset A$ is dense with respect to uniform convergence on compact subset of \mathbb{C}^2 .*

Proof. Let $h(\lambda_1, \lambda_2) \in A$ be a holomorphic function. Consider the Taylor series at the point $0 = (0, 0)$ in two variables $(\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^*$:

$$\sum_{\alpha} a_{\alpha} \lambda^{\alpha} = \sum_{\alpha_1, \alpha_2} a_{\alpha_1, \alpha_2} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2},$$

where a_{α} are constants and the sum runs over multi-indices $\alpha = (\alpha_1, \alpha_2), \alpha_j \in \mathbb{N}_0$. Let $|\alpha| = \sum_{j=1}^2 \alpha_j$ be the length. Note that $a_{\alpha_1, \alpha_2} = 0$ if α_1 or α_2 is odd. Thus the corresponding Taylor polynomials $\sum_{|\alpha| \leq k} a_{\alpha} \lambda^{\alpha}$ belong to P . The Taylor polynomials converge uniformly for k going to infinity on each ball $\overline{B}_r(0)$. \square

Next, by using the Iwasawa decomposition of $g = (g_1, g_2) \in G$, the 'exponential' function $e_{\lambda, k}^n$ can be rewritten as follows.

Proposition 2. *For fixed $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^*$ and $k = (k_1, k_2) \in K$, the function $e_{\lambda, k}^n \in C^{\infty}(G)$ defined as in Def. 3, is a product of the corresponding functions on G' :*

$$e_{\lambda, k}^n(g_1, g_2) = e_{\lambda_1, k_1}^{n_1}(g_1) \cdot e_{\lambda_2, k_2}^{n_2}(g_2), \quad (g_1, g_2) \in G.$$

Proof. Consider the Iwasawa decomposition of

$$g = (g_1, g_2) = (n'_1 a_1 k'_1, n'_2 a_2 k'_2) = n' a k' \in G$$

so that $n' = (n'_1, n'_2) \in N, a = (a_1, a_2) \in A$ and $k' = (k'_1, k'_2) \in K$. One can easily deduce that for $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^*$, we get

$$e^{(\lambda+\rho) \log(a)} = e^{(\lambda_1+\rho) \log(a_1) + (\lambda_2+\rho) \log(a_2)} = a_1^{\lambda_1+\rho} \cdot a_2^{\lambda_2+\rho}.$$

Hence, for $g \in G$, we then have

$$\begin{aligned} e_{\lambda,k}^n(g) &= e_{\lambda,k}^n(g_1, g_2) = a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} e_{\lambda_1,1}^n(k_1, k_2) &= a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} n_1(k_1) n_2(k_2) \\ & &= a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} e_{\lambda_1,1}^{n_1}(k_1) e_{\lambda_2,1}^{n_2}(k_2) \\ & \stackrel{(2.2)}{=} e_{\lambda_1,k_1}^{n_1}(g_1) \cdot e_{\lambda_2,k_2}^{n_2}(g_2). \quad \square \end{aligned}$$

Let ${}_lPWS_{n,H}(\mathfrak{a}_{\mathbb{C}}^*) = \{\varphi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{Hom}_M(E_l, E_n)) \mid \varphi \text{ satisfies (D.3)}\}$ be the 'pre'-Paley-Wiener-Schwartz space. Note that ${}_lPW_n(\mathfrak{a}_{\mathbb{C}}^*) \subset {}_lPWS_n(\mathfrak{a}_{\mathbb{C}}^*) \subset {}_lPWS_{n,H}(\mathfrak{a}_{\mathbb{C}}^*)$.

Proof of Thm. 8. It suffices to show that

- (a) every function $\varphi \in {}_lPW_n(\mathfrak{a}_{\mathbb{C}}^*)$ is of the form (4.9) and
- (b) (inversely) if φ is of the form (4.9) then it is in ${}_lPWS_{n,H}(\mathfrak{a}_{\mathbb{C}}^*)$.

Let $\varphi \in {}_lPW_n(\mathfrak{a}_{\mathbb{C}}^*)$. By the Paley-Wiener Thm. 2, there exists $f \in C_c^\infty(G, l, n)$ with ${}_l\mathcal{F}_n(f) = \varphi$. By Fubini's theorem and Prop. 2, we have

$$\begin{aligned} {}_l\mathcal{F}_n(f)(\lambda) &= \int_{G'} e_{\lambda_1,1}^{n_1}(g_1) \left(\int_{G'} e_{\lambda_2,1}^{n_2}(g_2) f(g_1, g_2) dg_2 \right) dg_1 &= \int_{G'} e_{\lambda_1,1}^{n_1}(g_1) \tilde{f}_{\lambda_2}(g_1) dg_1 \\ & &= {}_{l_1}\mathcal{F}_{n_1} \tilde{f}_{\lambda_2}(\lambda_1), \end{aligned} \quad (4.10)$$

where we set $\tilde{f}_{\lambda_2}(g_1) := \int_{G'} e_{\lambda_2,1}^{n_2}(g_2) f(g_1, g_2) dg_2$. Note that $\tilde{f}_{\lambda_2} \in C_c^\infty(X', \mathbb{E}_{n_1})$. Similarly, if we fix $\lambda_1 \in \mathfrak{a}_{\mathbb{C}}^*$, we have $\tilde{f}_{\lambda_1}(g_2) := \int_{G'} e_{\lambda_1,k_1}^{n_1}(g_1) f(g_1, g_2) dg_1 \in C_c^\infty(X', \mathbb{E}_{n_2})$ and ${}_l\mathcal{F}_n(f)(\lambda) = {}_{l_2}\mathcal{F}_{n_2} \tilde{f}_{\lambda_1}(\lambda_2)$. Thus, by Thm. 7, the Fourier transform has the form

$${}_l\mathcal{F}_n(f)(\lambda) = h_{\lambda_2}(\lambda_1) \cdot q_{l_1, n_1} = h_{\lambda_1}(\lambda_2) \cdot q_{l_2, n_2}(\lambda_2), \quad (4.11)$$

where h_{λ_i} is an even holomorphic function in $\lambda_i \in \mathfrak{a}_{\mathbb{C}}^*$ and q_{l_i, n_i} is the 'intertwining' polynomial in $\lambda_i \in \mathfrak{a}_{\mathbb{C}}^*$, defined in (4.5). By (4.8), we deduce that there exists $h \in \text{Hol}(\lambda_1^2, \lambda_2^2)$ such that

$$\varphi(\lambda) = {}_l\mathcal{F}_n f(\lambda) = h(\lambda_1, \lambda_2) \cdot q_{l,n}(\lambda_1, \lambda_2), \quad \lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^*,$$

as desired.

Concerning (b), let φ of the form (4.9), by (4.8), we have that

$$\varphi(\lambda_1, \lambda_2) = h(\lambda_1, \lambda_2) q_{l_1, n_1}(\lambda_1) q_{l_2, n_2}(\lambda_2),$$

for $(\lambda_1, \lambda_2) \in \mathfrak{a}_{\mathbb{C}}^*$. By Lem. 1, we can approximate h by the sum of products of two monomials $\lambda_1^{\alpha_1}$ and $\lambda_2^{\alpha_2}$. By Thm. 7 and the Paley-Wiener-Schwartz Thm. 2:

$$\lambda_i^{\alpha_i} q_{l_i, n_i}(\lambda_i) = {}_{l_i}\mathcal{F}_{n_i}(f_i)(\lambda_i) \in {}_{l_i}PWS_{n_i}(\mathfrak{a}_{\mathbb{C}}^*)$$

is the image of the Fourier transform of a distribution $f_i \in C_c^{-\infty}(G', l_i, n_i)$, $i = 1, 2$. Consider now the tensor product of these two distributions: $f_1 \otimes f_2 \in C_c^{-\infty}(G, l, n)$. By taking the Fourier transform and using the computations involving Fubini's theorem, we obtain that

$${}_l\mathcal{F}_n(f_1 \otimes f_2)(\lambda) = {}_{l_1}\mathcal{F}_{n_1}(f_1)(\lambda_1) \cdot {}_{l_2}\mathcal{F}_{n_2}(f_2)(\lambda_2) = \lambda^\alpha q_{l,n}(\lambda).$$

By using Paley-Wiener-Schwartz Thm. 2, we have ${}_l\mathcal{F}_n(f_1 \otimes f_2) = \lambda^\alpha q_{l,n} \in {}_lPWS_n(\mathfrak{a}_{\mathbb{C}}^*) \subset {}_lPWS_{n,H}(\mathfrak{a}_{\mathbb{C}}^*)$. Since ${}_lPWS_{n,H}(\mathfrak{a}_{\mathbb{C}}^*) \subset \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*)$ is closed with respect to uniform convergence on compact subsets, we conclude that $\varphi = h \cdot q_{l,n} \in {}_lPWS_{n,H}(\mathfrak{a}_{\mathbb{C}}^*)$. \square

By Thm. 8, we have explicitly determined the Paley-Wiener(-Schwartz) spaces for $G = \mathbf{SL}(2, \mathbb{R}) \times \mathbf{SL}(2, \mathbb{R})$ in (Level 3).

Moreover, all the previous results can be generalized to $G = \mathbf{SL}(2, \mathbb{R})^d$, $d \geq 2$.

5 The case $G = \mathbf{SL}(2, \mathbb{C})$

Let $G = \mathbf{SL}(2, \mathbb{C}) = \{g \in \mathbf{GL}(2, \mathbb{C}) \mid \det(g) = 1\}$ be the special linear group of \mathbb{C}^2 with maximal compact subgroup

$$K = \mathbf{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathbf{GL}(2, \mathbb{C}) \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Note that K is homeomorphic to the 3-sphere, therefore simply connected. Furthermore, the one-dimensional abelian subgroup A of G ($A \subset \mathbf{SL}(2, \mathbb{R}) \subset G$) is the same as in Sect. 4 and the nilpotent subgroup N of G is given by $N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{C} \right\}$. We identify $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ by sending λ to $\lambda(H)$, where $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$. Note that $\rho(H) = 2$. For the irreducible complex representations of K , we have (e.g. [Wal88], Sect. 5.7)

$$\widehat{K} = \{\delta_n \mid n \in \mathbb{N}_0\} \cong \mathbb{N}_0 \text{ with } d_{\delta_n} := \dim(\delta_n) = n + 1.$$

The tensor product of two irreducible K -representations decomposes into irreducibles according to the classical Clebsch-Gordan rule (e.g. [Wal88], 5.7.1 (1)):

$$\delta_n \otimes \delta_m = \bigoplus_{0 \leq j \leq \min(n, m)} \delta_{n+m-2j}, \quad n, m \in \mathbb{N}_0. \quad (5.1)$$

In addition, $M = Z_K(A) = \left\{ m_\theta := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$, which is abelian and a maximal torus in K .

We parametrize $\widehat{M} := \{\sigma_l \mid l \in \mathbb{Z}\} \cong \mathbb{Z}$ by the integers with $\sigma_l(m_\theta) = e^{il\theta} \in \mathbb{C}$. Moreover, let $\chi : M \rightarrow \mathbb{C}$ denotes the character of a finite-dimensional irreducible representation (δ_n, E_n) of K . Then, the Weyl character formula for $m_\theta \in M$ (e.g. [Kna02], Chap. V.6)

$$\chi(m_\theta) = \text{Tr}(\delta_n(m_\theta)) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)} = e^{-in\theta} + e^{-i(n-2)\theta} + \dots + e^{in\theta}$$

tells us that the weights of (δ_n, E_n) have the form $-n, -(n-2), \dots, n-2, n$, each with multiplicity one, with $n \in \mathbb{N}_0$ a highest weight. The following important result clarifies the reducibility of the principal series representations of $\mathbf{SL}(2, \mathbb{C})$. We refer for example to Wallach's book ([Wal88], Sect. 5.7) for a proof. Note that Wallach's proof is also valid for G -representation on smooth vectors (see remark before Thm. 4).

Theorem 9 (Structure of principal series representations of $\mathbf{SL}(2, \mathbb{C})$). *The principal series representations $H_\infty^{\sigma, \lambda}$ of $\mathbf{SL}(2, \mathbb{C})$ is exactly reducible, if λ is real and*

$$|\lambda| > |\sigma| \quad \& \quad |\lambda| - |\sigma| \text{ even integer.}$$

In this case, for $\lambda > 0$, there is an unique irreducible subrepresentation $R^{\sigma, \lambda}$ of each $H_\infty^{\sigma, \lambda}$. Then, we have

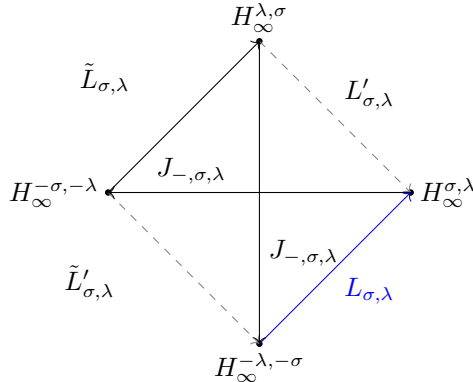
$$H_\infty^{-\sigma, -\lambda} = \frac{R^{\sigma, \lambda}}{F_{m, n}} \quad H_\infty^{\sigma, \lambda} = \frac{F_{m, n}}{R^{\sigma, \lambda}}$$

where $m = \frac{\sigma + \lambda}{2} - 1, n = \frac{\lambda - \sigma}{2} - 1$ and $F_{m, n}$ is an irreducible finite-dimensional G -representation that is isomorphic to $\delta_m \otimes \delta_n$ as a K -representation. Moreover, there is an intertwining operator

$$L_{\sigma, \lambda} : H_\infty^{-\lambda, -\sigma} \longrightarrow H_\infty^{\sigma, \lambda}$$

so that $\text{Ker}(J_{w, \sigma, \lambda}) = \text{Im}(L_{\sigma, \lambda}) = R^{\sigma, \lambda}$. In particular $R^{\sigma, \lambda}$ is isomorphic to $H_\infty^{-\lambda, -\sigma}$. Here, $J_{w, \sigma, \lambda} : H_\infty^{\sigma, \lambda} \longrightarrow H_\infty^{-\sigma, -\lambda}$ denotes the Knapp-Stein intertwining operator defined in Def. 6 with $w = -1$ and $m_w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Furthermore, from the intertwining operator $L_{\sigma, \lambda}$, we can deduce the existence of further intertwining operators:



Note that the operators $L_{\sigma,\lambda}, \tilde{L}_{\sigma,\lambda}, L'_{\sigma,\lambda}$ and $\tilde{L}'_{\sigma,\lambda}$ are precisely the Želobenko operators (also called BGG-operators) for $G = \mathbf{SL}(2, \mathbb{C})$ ([Zelo76], [BGG71], [BGG75] & [BGG76]). In Fig. 2, we illustrate the principal series representations (with regular integral infinitesimal character) in a grid, where the horizontal axis represents the values of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and the vertical one the values of $\sigma \in \widehat{M}$. Note that inside the region $\{\pm|\sigma| > \lambda\}$, we have the irreducible principal series representations $H_{\infty}^{-\lambda, -\sigma}$ respectively $H_{\infty}^{\lambda, \sigma}$ colored in gray and outside the reducible ones, colored in black.

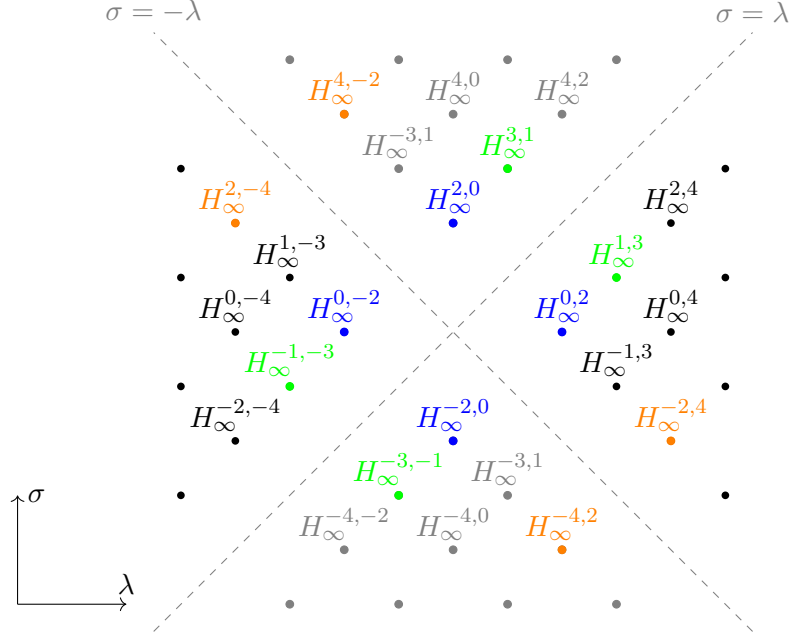


Figure 2: Principal series representations, where the colored one indicate the intertwining relations that occurs between each others with the same colors.

Example 5. Fix $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ such that $\lambda - |\sigma| \in 2\mathbb{N}$, we have the special condition

$$L_{\sigma,\lambda} \circ \phi(-\lambda, -\sigma) = \phi(\sigma, \lambda) \circ L_{\sigma,\lambda}. \quad (5.2)$$

Theorem 10 (Intertwining condition in (Level 1)). *For $r > 0$, let \mathcal{A} be the space of all $\phi \in \prod_{\sigma \in \widehat{M}} \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{End}(H_{\infty}^{\sigma}))$ such that ϕ satisfies the corresponding growth condition as well as the two intertwining conditions (2.8) and (5.2). Then, \mathcal{A} satisfies the conditions of Thm. 3, this means that $\mathcal{A} = PW_r(G)$.*

Before we proceed with the proof of Thm. 10, let us first state the explicit expression of the Harish-Chandra \mathbf{c} -function ([C074], App. 2) for $G = \mathbf{SL}(2, \mathbb{C})$, which is given by the following formula, for $|\sigma| \leq n, \sigma \equiv n \pmod{2}$:

$$\mathbf{c}_{n,\sigma}(\lambda) := \delta_n(\sigma\lambda) = \frac{\Gamma(\frac{1}{2}(\lambda + \sigma))\Gamma(\frac{1}{2}(\lambda - \sigma))}{\Gamma(\frac{1}{2}(\lambda + n + 2))\Gamma(\frac{1}{2}(\lambda - n))}, \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Consider an additional, not necessary distinct, K -type m and fix $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then, by using repeatedly the relation (4.2), we obtain for $n \equiv m \pmod{2}$, the following quotient:

$$\frac{\mathbf{c}_{n,\sigma}(\lambda)}{\mathbf{c}_{m,\sigma}(\lambda)} = \frac{\Gamma(\frac{\lambda}{2} + \frac{m}{2} + 1)\Gamma(\frac{\lambda}{2} - \frac{m}{2})}{\Gamma(\frac{\lambda}{2} + \frac{n}{2} + 1)\Gamma(\frac{\lambda}{2} - \frac{n}{2})} = \begin{cases} 1, & \text{if } n = m \\ \frac{(\lambda+m)(\lambda+m-2)(\lambda+m-4)\cdots(\lambda+n+2)}{(\lambda-m)(\lambda-(m+2))(\lambda-(m+4))\cdots(\lambda-(n+2))}, & \text{if } n < m \\ \frac{(\lambda-n)(\lambda-(n+2))(\lambda-(n+4))\cdots(\lambda-(m+2))}{(\lambda+n)(\lambda+n-2)(\lambda+n-4)\cdots(\lambda+m+2)}, & \text{if } n > m. \end{cases} \quad (5.3)$$

Hence, we can directly see that the quotient has zeros in $\{-m, -m+2, \dots, -n-2\}$ and poles in $\{m, m+2, \dots, n+2\}$ for $n < m$ and inversely for $n > m$.

Proof of Thm. 10. Note that \mathcal{A} is a $K \times K$ invariant closed linear subspace, due the intertwining conditions. We proceed similar as in the proof of Thm. 5. Note that $\widehat{G}_d = \emptyset$. Consider $\phi \in \mathcal{A}$ such that for each $\sigma \in \widehat{M}$, the assumption (D.b)(i) of Thm. 3:

$$\phi_{\sigma'} = 0, \text{ for all } \sigma' \in \widehat{M} \text{ with } |\sigma'| > |\sigma| \quad (5.4)$$

is satisfied. Analogous to the proof of Thm. 5, we need to check that:

- (a) for $\operatorname{Re}(\lambda) > 0$, the intertwining operator $J_{w,\sigma,\lambda}$ has a zero of order at most one.
- (b) the kernel of $J_{w,\sigma,\lambda}$ is equal to 0 or $R^{\sigma,\lambda}$, for $\operatorname{Re}(\lambda) > 0$.

The minimal K -type of the principal series representation $H_\infty^{\sigma,\lambda}$ is $n = |\sigma|$, hence its Harish-Chandra \mathbf{c} -function (5) $\mathbf{c}_{\sigma,\sigma}$ is regular for $\operatorname{Re}(\lambda) > 0$. Let $n \in \hat{K}$, then we see that for $n \geq |\sigma|$ and for $\operatorname{Re}(\lambda) > 0$:

$$\frac{\mathbf{c}_{n,\sigma}(\lambda)}{\mathbf{c}_{\sigma,\sigma}(\lambda)} = \frac{(\lambda - n)(\lambda - (n + 2))(\lambda - (n + 4)) \cdots (\lambda - (\sigma + 2))}{(\lambda + n)(\lambda + n - 2)(\lambda + n - 4) \cdots (\lambda + \sigma + 2)}$$

is also regular and has no poles, but zeros $\lambda \in \{n, n + 2, \dots, \sigma + 2\}$ of first order. Hence due (2.9), $J_{w,\sigma,\lambda}$ has zeros of order one, this proves the first assertion (a) of the claim.

By Thm. 9, we have $\operatorname{Ker}(J_{w,\sigma,\lambda}) = \operatorname{Im}(L_{\sigma,\lambda}) = R^{\sigma,\lambda}$, thus, this implies (b).

Now, by putting everything together and using the intertwining condition (5.2) as well as (5.4), we have, for each $\operatorname{Re}(\lambda) > 0$ with $|\lambda| = |-\lambda| > |\sigma|$, that

$$\phi_\sigma(\lambda) \circ L_{\sigma,\lambda} = L_{\sigma,\lambda} \circ \underbrace{\phi_{-\lambda}(-\sigma)}_{=0} = 0.$$

Thus, by (b) and the assumption, we deduce that the operator $\phi_\sigma^{(0)}(\lambda)$ annihilates $\operatorname{Ker}(J_{w,\sigma,\lambda}) = \operatorname{Im}(L_{\sigma,\lambda}) = R^{\sigma,\lambda}$, for $|\lambda| > |\sigma|$ and $|\lambda| - |\sigma|$ even. Moreover, by (a), this condition is sufficient since the order of m is one. This completes the proof. \square

Now let us move to (Level 2) and state the corresponding intertwining conditions there. In fact, this will determine explicitly the Paley-Wiener(-Schwartz) spaces for $G = \mathbf{SL}(2, \mathbb{C})$ in (Level 2). In order to distinguish between representation spaces of K and M , we denote E_{δ_n} by E_n . while the one-dimensional space E_{σ_l} is denoted by \mathbb{C}_l .

Theorem 11 (Intertwining condition in (Level 2)). *Let $n \in \mathbb{N}_0$ be a K -type and $k, l \in \widehat{M} \cong \mathbb{Z}$ so that $n \geq |k|, |l|, l > |k|$ and $k \equiv l \equiv n \pmod{2}$. Consider the operator*

$$l_{k,l}^n : \operatorname{Hom}_M(E_n, \mathbb{C}_{-l}) \longrightarrow \operatorname{Hom}_M(E_n, \mathbb{C}_k)$$

defined as in Example 1 corresponding to $L = L_{k,l}$ and $\tau = \delta_n$. Then, $\psi \in \operatorname{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{n|M})$ satisfies the intertwining condition (D.2) of Def. 5 if, and only if,

$$(2.a) \quad J_{w,n,\lambda} t \psi(\lambda) = \begin{pmatrix} c_{n,n}(\lambda) & 0 & \cdots & 0 \\ 0 & c_{n,n-2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n,-n}(\lambda) \end{pmatrix} \psi_{-k}(-\lambda), \text{ for all } \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

$$(2.b) \quad L_{k,l}(t \circ \psi(-k)) = l_{k,l}^n(t) \circ \psi(l), \text{ for all } \lambda \in \mathfrak{a}_{\mathbb{C}}^* \text{ and } t \in \operatorname{Hom}_M(E_n, E_{-l}).$$

Proof of Thm. 11. By Prop. 1, we have that (2.a) corresponds to (2.6), hence it corresponds to the intertwining condition (D.2) in Def. 5. Similar for (2.b), which is, by Example 1 (2.6), a special intertwining condition of (D.2). Hence, we have equivalence between the conditions (2.a) & (2.b) and (D.2). \square

In order to move to (Level 3), let us first consider the case where the two K -type (n, E_n) and (m, E_m) are equal and then progress to the case of distinct K -types.

Initial case: The K -types m and n are equal

Definition 9. *Let (m, E_m) be a fixed K -type and $k = -m, -(m - 2), \dots, m - 2, m$.*

We define ${}_m\mathcal{A}_m$ the space of all elements in $\operatorname{Hol}(\mathfrak{a}_{\mathbb{C}}^, \operatorname{End}_M(E_m))$, which are given by holomorphic functions $\varphi_k : \mathfrak{a}_{\mathbb{C}}^* \rightarrow \mathbb{C}$, ordered to a $(m + 1) \times (m + 1)$ matrix with respect to M -weight vectors as a basis of E_m :*

$$\varphi := \begin{pmatrix} \varphi_m(\lambda) & 0 & \cdots & 0 \\ 0 & \varphi_{m-2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_{-m}(\lambda) \end{pmatrix} \in \operatorname{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \operatorname{End}_M(E_m))$$

such that

$$\begin{aligned} \varphi_k(\lambda) &= \varphi_{-k}(-\lambda), \text{ for } \lambda \in \mathfrak{a}_{\mathbb{C}}^* \\ \varphi_k(l) &= \varphi_l(k), \text{ for } k \equiv l \equiv m \pmod{2} \text{ \& } |k|, |l| \leq m. \end{aligned} \tag{5.5}$$

Note that ${}_m\mathcal{A}_m \subset \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{End}_M(E_m))$ is even an algebra. Let ${}_mPWS_{m,H}(\mathfrak{a}_{\mathbb{C}}^*)$ be the pre-Paley-Wiener-Schwartz space such that $\varphi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{End}_M(E_m))$ and satisfies the intertwining condition (D.3) of Def. 5.

Theorem 12. *With the previous notations, we have that*

$${}_m\mathcal{A}_m \cong {}_mPWS_{m,H}(\mathfrak{a}_{\mathbb{C}}^*).$$

Proof. We need to check that the intertwining conditions (5.5) of ${}_m\mathcal{A}_m$ correspond to the intertwining condition (D.3) of ${}_mPWS_{m,H}(\mathfrak{a}_{\mathbb{C}}^*)$. More precisely, by Example 1, it suffices to show that the intertwining condition (2.7) for $L = J_{w,\sigma,\lambda}$ and $L = L'_{\sigma,\lambda}$ corresponds to (5.5).

We know from Prop. 1 with $m = \gamma = \tau \in \mathbb{N}_0$, that for each intertwining datum $((\sigma, \lambda, 0), (-\sigma, -\lambda, 0); W)$ and $t \in \text{Hom}_M(E_m, E_\sigma)$, we have

$$\delta_m(m_w)^{-1}\varphi(\lambda)\delta_m(m_w) = \varphi(-\lambda), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \varphi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{End}_M(E_m)),$$

Note that the complex hull of \mathfrak{a} is the sum of \mathfrak{a} and \mathfrak{m} , where \mathfrak{m} is in the maximal torus M . This means that the Weyl group W_A acts on \mathfrak{a} as well as on \mathfrak{m} by -1 . Thus, also on $i\mathfrak{m}^*$ by -1 . Let $-k, -(k-2), \dots, k-2, k$ be the weights of the representation $m \in \mathbb{N}_0$. Then, by the $\delta_m(m_w)$ -component is sent to $\delta_m(-m_w)$ -component. The matrix $\text{diag}(\varphi_m(\lambda), \dots, \varphi_{-m}(\lambda))$ is reversed by conjugation by $\delta_m(m_w)$, i.e., we get $\text{diag}(\varphi_{-m}(-\lambda), \dots, \varphi_m(-\lambda))$. Hence, $\varphi_{-k}(-\lambda) = \varphi_k(\lambda)$, for all $|k| \leq m$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Let $|l|, |k| \leq m$ and $k \equiv l \equiv m \pmod{2}$. Let $l_{k,l}^m$ as in Thm. 11. We know that for the intertwining datum $((k, l, 0), (l, k, 0); W)$ and $t \in \text{Hom}_M(E_m, \mathbb{C}_{-l})$, we have (2.7) for $m = \gamma = \tau \in \mathbb{N}_0$ and $l > |k|$:

$$l_{k,l}^m(t \circ \varphi(-k)) = l_{k,l}^m(t) \circ \varphi(l).$$

Normalized $t \in \text{Hom}_M(E_m, \mathbb{C}_{-l})$ and $t' \in \text{Hom}_M(E_m, \mathbb{C}_k)$ such that $\varphi_{-l} = t \circ \varphi$ and $\varphi_k = t' \circ \varphi$. We have that $l_{k,l}^m(t) = c \cdot t'$, where $c \in \mathbb{C}$. Note that, $c \neq 0$. In fact, by Thm. 9, the intertwining operator $L_{k,l}$ on the K -type m is not zero, hence $l_{k,l}^m$ too. Consequently, we have

$$l_{k,l}^m(t \circ \varphi(-k)) = l_{k,l}^m(t) \circ \varphi(l) \iff c \cdot \varphi_{-l}(-k) = c \cdot \varphi_k(l) \iff \varphi_{-l}(-k) = \varphi_k(l). \quad (5.6)$$

For $0 \leq l < |k|$, we consider $L_{l,|k|}$ instead. Combined with $\varphi_{-l}(-k) = \varphi_l(k)$, we obtain (5.6) for every pair k, l as above. This completes the proof. \square

We also consider the corresponding situation in polynomial functions:

$${}_m\text{Pol}_m := \{\varphi \in \text{Pol}(\mathfrak{a}_{\mathbb{C}}^*, \text{End}_M(E_m)) \mid \varphi_k \text{ satisfies (5.5)} \forall |k| \leq m\}.$$

Note that ${}_m\text{Pol}_m$ is equal to the vector space ${}_mPWS_{m,0}(\mathfrak{a}_{\mathbb{C}}^*)$. We will sometimes write element of ${}_m\text{Pol}_m$ and ${}_m\mathcal{A}_m$ as functions of two-variables: $\varphi(\lambda, k) = \varphi_k(\lambda)$. We consider the subalgebra of ${}_m\text{Pol}_m$ generated by $\lambda^2 + k^2$. It is the subalgebra generated by the Fourier image of the Casimir operator and isomorphic to $\text{Pol}(\mathbb{C})$. Thus, we can view ${}_m\text{Pol}_m$ as a $\text{Pol}(\mathbb{C})$ -module. Similarly, ${}_m\mathcal{A}_m$ has the structure of $\text{Hol}(\mathbb{C})$ -module. Here, $h \in \text{Hol}(\mathbb{C})$ acts on ${}_m\mathcal{A}_m$ by

$$(h \cdot \varphi)_k(\lambda) = h(\lambda^2 + k^2)\varphi_k(\lambda), \quad h \in \text{Hol}(\mathbb{C}), \varphi \in {}_m\mathcal{A}_m.$$

Theorem 13. *The algebra ${}_m\mathcal{A}_m$ is a free $\text{Hol}(\mathbb{C})$ -module with the $m+1$ generators $(k\lambda)^l \in {}_m\text{Pol}_m \subset {}_m\mathcal{A}_m, l = 0, \dots, m$. Furthermore, we have*

$${}_m\mathcal{A}_m \cong \text{Hol}(\mathbb{C}) \otimes_{\text{Pol}(\mathbb{C})} {}_m\text{Pol}_m. \quad (5.7)$$

Analogously, ${}_m\text{Pol}_m$ is a free $\text{Pol}(\mathbb{C})$ -module with same generators as ${}_m\mathcal{A}_m$.

Note that Thm. 13 also tells that the two elements $\lambda^2 + k^2$ and kl generate ${}_m\text{Pol}_m$ as a algebra. Observe also that ${}_m\text{Pol}_m$ is isomorphic to $\mathcal{D}_G(\mathbb{E}_m, \mathbb{E}_m)$, the set of all invariant differential operators $D : C^\infty(X, \mathbb{E}_m) \rightarrow C^\infty(X, \mathbb{E}_m)$ ([OIPa22-1], Sect. 7).

Proof. Consider $\varphi \in {}_m\mathcal{A}_m$. It is sufficient to show the existence and the uniqueness of holomorphic functions $h_0, \dots, h_m \in \text{Hol}(\mathbb{C})$ so that

$$\varphi_k(\lambda) := \sum_{l=0}^m h_l(\lambda^2 + k^2) \cdot (k\lambda)^l, \quad \text{for } k = -m, \dots, m \quad (5.8)$$

and

$$\varphi \in {}_m\text{Pol}_m \text{ implies } h_l \in \text{Pol}(\mathbb{C}), \quad \text{for } l = 0, \dots, m. \quad (5.9)$$

Then, ${}_m\mathcal{A}_m$ is a free $\text{Hol}(\mathbb{C})$ -module with generators $(k\lambda)^l, l = 0, \dots, m$. Similarly for ${}_m\text{Pol}_m$. Note that $\text{Hol} \otimes_{\text{Pol}} \text{Pol} \cong \text{Hol}$. Since, we have $m + 1$ free generators,

$${}_m\mathcal{A}_m \cong \text{Hol}(\mathbb{C})^{m+1} \text{ and } {}_m\text{Pol}_m \cong \text{Pol}(\mathbb{C})^{m+1}$$

we thus have that (5.8) & (5.9) imply (5.7).

We proceed by a step two induction on $m \in \mathbb{N}_0$, for the existence of $h_l \in \text{Hol}(\mathbb{C}), l = 0, \dots, m$.

For $m = 0$:

$${}_0\mathcal{A}_0 = \{\varphi_0 \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*) \mid \varphi_0(\lambda) = \varphi_0(-\lambda), \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*\}$$

we see immediately that there is exactly one holomorphic function h_0 such that $\varphi_0(\lambda) = h_0(\lambda^2)$. φ_0 is a polynomial function, if and only, if h_0 is one.

For $m = 1$, we have that ${}_1\mathcal{A}_1 = \{\varphi_1, \varphi_{-1} \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*) \mid \varphi_{-1}(\lambda) = \varphi_1(-\lambda), \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*\}$ and $\varphi_1(\lambda)$ can be decomposed into an even and odd part as follows:

$$\varphi_1(\lambda) = \varphi_1^{\text{even}}(\lambda) + \varphi_1^{\text{odd}}(\lambda) = h_0(\lambda^2 + 1) + \lambda h_1(\lambda^2 + 1).$$

Then, $\varphi_{-1}(\lambda) = \varphi_1(-\lambda) = h_0(\lambda^2 + 1) - \lambda h_1(\lambda^2 + 1)$. Hence, this leads us to desired relation $\varphi_k(\lambda) = h_0(\lambda^2 + k^2) + (k\lambda)h_1(\lambda^2 + k^2)$, for all $k = \pm 1$. h_0 and h_1 are polynomials, if and only, if $\varphi_{\pm 1}$ are polynomials as well.

Assume now the existence of h_l satisfying (5.8) & (5.9) for m replaced by $m - 2$. Let $\varphi \in {}_m\mathcal{A}_m$. We have $\tilde{\varphi} := \text{diag}(\varphi_{m-2}(\lambda), \varphi_{m-4}(\lambda), \dots, \varphi_{-(m-4)}(\lambda), \varphi_{-(m-2)}(\lambda)) \in {}_{m-2}\mathcal{A}_{m-2}$ so that, by induction hypothesis, there exists $h_l \in \text{Hol}(\mathbb{C})$ with

$$\varphi_k(\lambda) = \sum_{l=0}^{m-2} h_l(\lambda^2 + k^2)(k\lambda)^l \quad \text{for } |k| \leq m - 2,$$

and $h_l \in \text{Pol}(\mathbb{C})$ if $\varphi \in {}_m\text{Pol}_m$. Consider $\tilde{\varphi} := \text{diag}(\tilde{\varphi}_m(\lambda), \varphi_{m-2}(\lambda), \dots, \varphi_{-(m-2)}(\lambda), \tilde{\varphi}_{-m}(\lambda)) \in {}_m\mathcal{A}_m$ with

$$\tilde{\varphi}_{\pm m}(\lambda) := \sum_{l=0}^{m-2} h_l(\lambda^2 + m^2)(\pm m\lambda)^l.$$

By taking the difference of φ and $\tilde{\varphi}$, we get that $\varphi - \tilde{\varphi} = \text{diag}(\varphi_m^+(\lambda), 0, \dots, 0, \varphi_{-m}^+(\lambda)) \in {}_m\mathcal{A}_m$, where we have set $\varphi_{\pm m}^+(\lambda) := \varphi_{\pm m}(\lambda) - \tilde{\varphi}_{\pm m}(\lambda)$. Notice that $\varphi_{\pm m}^+(l) = 0$ for $|l| \leq m - 2, l \equiv m \pmod{2}$. We introduce the polynomial function:

$$p_m(\lambda, k) = \prod_{\substack{|l| \leq m-2 \\ l \equiv m \pmod{2}}} (k - l)(\lambda - l) \in {}_m\text{Pol}_m.$$

Note that $p_m(\lambda, k) \equiv 0$ for $|k| \leq m - 2, k \equiv m \pmod{2}$. Moreover, if $k = m$, then $p_m(\lambda, m) = c_m \prod_{\substack{|l| \leq m-2 \\ l \equiv m \pmod{2}}} (\lambda - l)$, where c_m is a non-zero constant depending on the integer m . We conclude that there exist $h_0^+, h_1^+ \in \text{Pol}(\mathbb{C})$ if $\varphi \in {}_m\text{Pol}_m$ such that

$$\varphi_m^+(\lambda) = [h_0^+(\lambda^2 + m^2) + h_1^+(\lambda^2 + m^2)(m\lambda)]p_m(\lambda, m).$$

This implies that $(\varphi - \tilde{\varphi})(\lambda, k) = [h_0^+(\lambda^2 + k^2) + h_1^+(\lambda^2 + k^2)(k\lambda)]p_m(\lambda, k)$. In addition, $(k - \lambda)(\lambda - l)(k + l)(\lambda + l) = (k^2 - l^2)(\lambda^2 - l^2) = (k\lambda)^2 - l^2(\lambda^2 + k^2) + l^4$ and thus $p_m(\lambda, k)$ is of the form:

$$p_m(\lambda, k) = (k\lambda)^{m-1} + \sum_{\substack{l=0 \\ l \equiv m-1 \pmod{2}}}^{m-3} p_l^m(\lambda^2 + k^2)(k\lambda)^l, \quad (5.10)$$

where p_l^m are certain polynomials. We obtain

$$\begin{aligned} (\varphi - \tilde{\varphi})(\lambda, k) &= h_0^+(\lambda^2 + k^2)(k\lambda)^{m-1} + h_0^+(\lambda^2 + k^2) \sum_{\substack{l=0 \\ l \equiv m-1 \pmod{2}}}^{m-3} p_l^m(\lambda^2 + k^2)(k\lambda)^l \\ &\quad + h_1^+(\lambda^2 + k^2)(k\lambda)^m + h_1^+(\lambda^2 + k^2) \sum_{\substack{l=0 \\ l \equiv m-1 \pmod{2}}}^{m-3} p_l^m(\lambda^2 + k^2)(k\lambda)^{l+1}. \end{aligned}$$

This implies that $\varphi - \tilde{\varphi}$ is of the desired form, hence φ .

Concerning the uniqueness, we need to show that

$$\sum_{l=0}^m h_l(\lambda^2 + k^2)(k\lambda)^l = 0 \quad \forall |k| \leq m, \lambda \in \mathfrak{a}_{\mathbb{C}}^* \text{ implies } h_l = 0, l = 0, \dots, m. \quad (5.11)$$

We proceed again by a two step induction on $m \in \mathbb{N}_0$. For the initial cases $m = 0, 1$, the assertion is clear, see above. Assume that (5.11) holds true for $m - 2$ and let us prove that it holds for m . Let $\varphi \in {}_m\mathcal{A}_m$. By using the polynomial function (5.10), we have for $|k| \leq m - 2$:

$$\begin{aligned} 0 &= \sum_{l=0}^m h_l(\lambda^2 + k^2)(k\lambda)^l - [h_{m-1}(\lambda^2 + k^2) + (k\lambda)h_m(\lambda^2 + k^2)p_m(\lambda, k)] \\ &= \sum_{l=0}^{m-2} h_l(\lambda^2 + k^2)(k\lambda)^l - h_{m-1}(\lambda^2 + k^2) \sum_{\substack{l=0 \\ l \equiv m-1 \pmod{2}}}^{m-3} p_l^m(\lambda^2 + k^2)(k\lambda)^l \\ &\quad - h_m(\lambda^2 + k^2) \sum_{\substack{l=0 \\ l \equiv m-1 \pmod{2}}}^{m-3} p_l^m(\lambda^2 + k^2)(k\lambda)^{l+1} \end{aligned}$$

in ${}_{m-2}\mathcal{A}_{m-2}$. By induction hypothesis, this implies that for $l \leq m - 2$:

$$h_l(\mu) = \begin{cases} h_{m-1}(\mu)p_l^m(\mu), & l \equiv m - 1 \pmod{2} \\ h_m(\mu)p_{l-1}^m(\mu), & l \equiv m \pmod{2}. \end{cases}$$

Then, for $k = m$, this implies

$$0 = \sum_{l=0}^m h_l(\lambda^2 + m^2)(k\lambda)^l = [h_{m-1}(\lambda^2 + m^2) + h_m(\lambda^2 + m^2)(m\lambda)]p_m(\lambda, m).$$

Since $p_m(\lambda, m)$ is not identical zero on $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we obtain that h_{m-1} and h_m are zero. Hence $h_l = 0$ for $l \leq m$. This completes the proof. \square

General case: The K -types n and m are distinct

Consider now two distinct K -types (n, E_n) and (m, E_m) . Since $\text{Hom}_M(E_n, E_m) = 0$ for $n \not\equiv m \pmod{2}$, we assume throughout the section that $n \equiv m \pmod{2}$. We define

$${}_n\mathcal{A}_m := {}_n\text{PWS}_{m,H}(\mathfrak{a}_{\mathbb{C}}^*) \subset \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{Hom}_M(E_n, E_m))$$

and ${}_n\text{Pol}_m := {}_n\text{PWS}_{m,0}(\mathfrak{a}_{\mathbb{C}}^*)$. Here, ${}_n\text{PWS}_{m,H}(\mathfrak{a}_{\mathbb{C}}^*)$ denotes the pre-Paley-Wiener-Schwartz space such that $\varphi \in \text{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \text{Hom}_M(E_n, E_m))$ and satisfies the intertwining condition (D.3) of Def. 5, same for ${}_n\text{PWS}_{m,0}(\mathfrak{a}_{\mathbb{C}}^*)$ but for polynomial functions $\text{Pol}(\mathfrak{a}_{\mathbb{C}}^*, \text{Hom}_M(E_n, E_m))$. Note that after a choice of basis vectors in E_n, E_m consisting of M -weight vectors, we can write the elements in the following way:

$$\varphi := \begin{cases} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \\ \varphi_n(\lambda) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varphi_{-n}(\lambda) \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{(m+1) \times (n+1)} & \text{if } n < m, \\ \begin{pmatrix} 0 & \cdots & 0 & \varphi_m(\lambda) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \varphi_{-m}(\lambda) & 0 & \cdots & 0 \end{pmatrix}_{(m+1) \times (n+1)} & \text{if } n > m. \end{cases}$$

We start with the case $|n - m| = 2$.

Proposition 3. *Let m be a K -type. There exists a unique, up to normalization, operator of first order q_m^+ in ${}_m\text{Pol}_{m+2}$ (resp. q_m^- in ${}_{m+2}\text{Pol}_m$ by taking the adjoint), which corresponds to*

$$\begin{aligned} & (\lambda + m + 2) \cdot \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}_{(m+2) \times m} \\ & \left(\text{resp. } (\lambda - (m + 2)) \cdot \begin{pmatrix} 0 & d(m, m) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d(m, -m) & 0 \end{pmatrix}_{m \times (m+2)} \right) \end{aligned}$$

under some appropriate basis choice and where $d(m, k) = (m + 2)^2 - k^2$, for all $|k| \leq m$.

Proof. Consider the symmetric algebra $S(\mathfrak{p})$ of $\mathfrak{p} \cong \mathbb{R}^3$ over \mathbb{C} . From ([Ol95], Folgerung 2.5), we deduce that the associated graded module of ${}_m\text{Pol}_{m+2}$ is isomorphic to $[S(\mathfrak{p}) \otimes \text{Hom}_M(E_m, E_{m+2})]^K$. By using Clebsch-Gordan rule (5.1), we have

$$\begin{aligned} [S^{\leq 1}(\mathfrak{p}) \otimes \text{Hom}(E_m, E_{m+2})]^K &\cong [(S^0(\mathfrak{p}) \oplus S^1(\mathfrak{p})) \otimes (E_2 \oplus E_4 \oplus \cdots \oplus E_{2m+2})]^K \\ &\cong [S^1(\mathfrak{p}) \otimes E_2]^K, \end{aligned}$$

where $[S^1(\mathfrak{p}) \otimes E_2]^K$ is one-dimensional. Note that $S^1(\mathfrak{p}) \cong \mathfrak{p}_{\mathbb{C}} \cong E_2$. This means that ${}_m\text{Pol}_{m+2}$ contains exactly (up to normalization) one element q_m^+ of filter degree 1. According to an appropriate choice of basis, we can write $q_{m,n}^+(\lambda)$ in a matrix form as above. Since $q_{m,k}^+(\lambda)$ is of first order (but the individual components could be constants), this means that all elements, under normalization, are of the form

$$q_{m,k}^+(\lambda) = \begin{cases} \tilde{c}(m, k), & |k| \leq m \text{ or} \\ (\lambda + c(m, k)), & |k| \leq m, \lambda \in \mathfrak{a}_{\mathbb{C}}^* \text{ (at least for one } k). \end{cases}$$

Here $\tilde{c}(m, k)$ are constants depending on k and m . Consider $\varphi_k \in {}_m\mathcal{A}_{m+2}$. By Prop. 1 (b) and Harish-Chandra \mathfrak{c} -functions (5.3), we have that

$$\begin{aligned} \varphi_{-k}(-\lambda) &= (-1)^{(m+2-m)/2} \frac{\mathfrak{c}_{m+2,k}(\lambda)}{\mathfrak{c}_{m,k}(\lambda)} \varphi_k(\lambda) = (-1) \frac{\Gamma(\frac{\lambda}{2} + \frac{m}{2} + 1) \Gamma(\frac{\lambda}{2} - \frac{m}{2})}{\Gamma(\frac{\lambda}{2} + \frac{m}{2} + 2) \Gamma(\frac{\lambda}{2} - \frac{m}{2} - 1)} \varphi_k(\lambda) \\ &= (-1) \frac{(\lambda - (m + 2))}{(\lambda + (m + 2))} \varphi_k(\lambda) \\ &= \frac{(-\lambda + (m + 2))}{(\lambda + (m + 2))} \varphi_k(\lambda). \end{aligned}$$

Since φ_{-k} has no singularities, we conclude that φ_k has a zero at $-(m + 2)$. Applying this to $\varphi = q_m^+$, we see that

$$q_{m,k}^+(\lambda) = \begin{cases} 0, \\ \lambda + (m + 2). \end{cases}$$

Analogously, there exists exactly one (up to normalization) $q_m^- \in {}_{m+2}\mathcal{A}_m$ of degree 1. We consider $q_m^+ q_m^- \in {}_{m+2}\mathcal{A}_{m+2}$. Then $q_m^+ q_m^-(\lambda, \pm(m + 2)) = 0$. This implies that $q_m^+ q_m^-(\pm(m + 2), k) = 0$ for $|k| \leq m, k \equiv m \pmod{2}$ of degree 2. Hence, $q_m^+ q_m^-(\lambda, k) = d(m, k)(\lambda^2 - (m + 2)^2)$ for some constants $d(m, k)$. Since $q_m^- = c \cdot (q_m^+)^*$, where $*$ stands for the adjoint $p^*(\lambda) := p(-\bar{\lambda})^*$, we have $d(m, k) \neq 0$ for all $|k| \leq m$. In addition, by (5.5)

$$d(m, k) = \begin{cases} d(m, 0) \cdot \frac{(m+2)^2 - k^2}{(m+2)^2} \neq 0, & m \text{ even} \\ d(m, 1) \cdot \frac{(m+2)^2 - k^2}{(m+2)^2 - 1} \neq 0, & m \text{ odd.} \end{cases}$$

We conclude that $q_m^+(\lambda, k) = \lambda + c(m, k) = \lambda + (m + 2)^2$ and $q_m^-(\lambda, k) = ((m + 2)^2 - k^2)(\lambda - (m + 2))$ (up to normalization). \square

Now we consider general $n, m \in \mathbb{N}_0, n \equiv m \pmod{2}$.

Definition 10. Consider q_m^\pm and q_n^\pm as in Prop. 3. We define the polynomial $q_{n,m} \in {}_n\text{Pol}_m$ in $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by

$$q_{n,m} = \begin{cases} q_{m-2}^+ \cdot q_{m-4}^+ \cdots q_{n+2}^+ \cdot q_n^+, & \text{if } n < m \\ q_m^- \cdot q_{m+2}^- \cdots q_{n-4}^- \cdot q_{n-2}^-, & \text{if } n > m \\ \text{Id}, & \text{if } n = m. \end{cases} \quad (5.12)$$

Finally, the following theorem, gives explicitly the Paley-Wiener(-Schwartz) spaces in (Level 3) for $G = \mathbf{SL}(2, \mathbb{C})$. Note that ${}_n\text{Pol}_m$ is isomorphic to the set of all invariant differential operators $\mathcal{D}_G(\mathbb{E}_n, \mathbb{E}_m)$ ([OLPa22-1], Sect. 7)..

Theorem 14 (Intertwining condition in (Level 3)). *Let $n, m \in \mathbb{N}_0$ be two K -types, which are not necessary distinct, and let $l := \min(n, m)$. Then, ${}_n\text{Pol}_m$ (resp. ${}_n\mathcal{A}_m$) is a free ${}_l\text{Pol}_l$ (resp. ${}_l\mathcal{A}_l$)-module with generator $q_{n,m}$. This means that there exists a unique function $h \in {}_l\mathcal{A}_l$ such that*

$${}_n\mathcal{A}_m \ni \varphi(\lambda) = \begin{cases} h(\lambda)q_{n,m}(\lambda), & \text{if } m < n, \\ q_{n,m}(\lambda)h(\lambda), & \text{if } m > n, \end{cases} \quad \text{for } \lambda \in \mathfrak{a}_\mathbb{C}^*. \quad (5.13)$$

Moreover, if $L = \max(n, m)$, then ${}_n\text{Pol}_m$ (resp. ${}_n\mathcal{A}_m$) is a ${}_L\text{Pol}_L$ (resp. ${}_L\mathcal{A}_L$)-module generated by $q_{n,m}$.

Proof. Consider the case $m < n$, it suffices to prove that

- (a) there exists a unique $h \in {}_m\mathcal{A}_m$ such that $\varphi = h \cdot q_{n,m} \in {}_n\mathcal{A}_m$,
- (b) there exists a $\tilde{h} \in {}_n\mathcal{A}_n$ such that $\varphi = q_{n,m} \cdot \tilde{h} \in {}_n\mathcal{A}_m$.

The polynomial case as well as $m > n$ can be proved in a similar way. Consider

$$q_{m,n} {}_n\mathcal{A}_m \subset \{g \in {}_n\mathcal{A}_n \mid g_k = 0, \forall |k| > m\} \subset {}_n\mathcal{A}_n.$$

We also have zeros between the lines $-m$ and m , this means $g_l(k) = 0, \forall k \equiv \pm(m+2), \pm(m+4), \dots, \pm n$. Let $\varphi \in {}_n\mathcal{A}_m$. Every component of $g = q_{m,n} \cdot \varphi$ has zeros at described above and $q_{m,n}$ has zeros only at negative λ , we have that every component φ has zeros at $m+2, \dots, n$. Hence, there exists a unique $h \in \text{Hol}(\mathfrak{a}_\mathbb{C}^*, \text{End}(E_m))$ so that

$$\varphi = h \cdot q_{n,m}.$$

In fact, since g satisfies the intertwining condition $g_k(l) = g_l(k)$ as well as the symmetry one $g_k(\lambda) = g_{-k}(-\lambda)$, and

$$\begin{aligned} q_{m,n}(k, \lambda)q_{n,m}(k, \lambda) &= q_{m,n}(-k, -\lambda)q_{n,m}(-k, -\lambda) \\ q_{m,n}(k, l)q_{n,m}(k, l) &= q_{m,n}(l, k)q_{n,m}(l, k) \end{aligned}$$

we then have that $h_l(k) = h_k(l)$ and $h_k(\lambda) = h_{-k}(-\lambda)$ for $k \equiv l \equiv m \pmod{2}$, $|k| \leq m$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. This proves (a).

For (b), we know from (a) that $\varphi = h \cdot q_{n,m}$ is in the 'small' space, thus we need to find

$$\tilde{h} = \text{diag}(\tilde{h}_n, \dots, \tilde{h}_{m+2}, \tilde{h}_m, \dots, \tilde{h}_{-m}, \tilde{h}_{-(m-2)}, \dots, \tilde{h}_{-n}) \in {}_n\mathcal{A}_n$$

with $\tilde{h}_k = h_k$ for $|k| \leq m$, so that $\tilde{h}_k(l) = \tilde{h}_l(k) = h_l(k)$ and $\tilde{h}_k(\lambda) = \tilde{h}_{-k}(-\lambda)$ for $\forall l$ and $k > m$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then $\varphi = {}_n q_m \cdot \tilde{h}$. By using the interpolation polynomial for each $|k| > m$, we define recursively for $\lambda \in \mathfrak{a}_\mathbb{C}^*$:

$$\begin{aligned} \tilde{h}_{m+2}(\lambda) &= \sum_{\substack{i=-m \\ i \equiv m \pmod{2}}}^m h_i(m+2) \prod_{\substack{l=-m \\ l \neq i}}^m \left(\frac{\lambda-l}{i-l} \right) \\ \tilde{h}_{m+4}(\lambda) &= \sum_{\substack{i=-(m+2) \\ i \equiv m+2 \pmod{2}}}^{m+2} \tilde{h}_i(m+4) \prod_{\substack{l=-(m+2) \\ l \neq i}}^{m+2} \left(\frac{\lambda-l}{i-l} \right) \\ &\vdots \\ \tilde{h}_{n-2}(\lambda) &= \sum_{\substack{i=-(n-4) \\ i \equiv n-4 \pmod{2}}}^{n-4} \tilde{h}_i(n-2) \prod_{\substack{l=-(n-4) \\ l \neq i}}^{n-4} \left(\frac{\lambda-l}{i-l} \right) \\ \tilde{h}_n(\lambda) &= \sum_{\substack{i=-(n-2) \\ i \equiv n-2 \pmod{2}}}^{n-2} \tilde{h}_i(n) \prod_{\substack{l=-(n-2) \\ l \neq i}}^{n-2} \left(\frac{\lambda-l}{i-l} \right), \end{aligned}$$

and $\tilde{h}_{-k}(\lambda) := \tilde{h}_k(\lambda)$. The intertwining relations are also satisfied for $|l|, |k| > m$ due $h \in {}_m\mathcal{A}_m$. \square

We observe an analogue of Thm. 13, in particular relation (5.7), is also true in general for distinct K -type n and m .

Corollary 1. *With the notations above, ${}_n\mathcal{A}_m \cong \text{Hol}(\mathbb{C}) \otimes_{\text{Pol}(\mathbb{C})} {}_n\text{Pol}_m$.*

Proof. As a consequence of Thm. 14, the $\text{Hol}(\mathbb{C})$ -module ${}_n\mathcal{A}_m$ is isomorphic to ${}_l\mathcal{A}_l$, and ${}_n\text{Pol}_m$ is isomorphic to ${}_l\text{Pol}_l$ as $\text{Pol}(\mathbb{C})$ -module. Then, we can apply Thm. 13 and obtain the desired result. \square

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