5-2022

The Role of Music Context in High School Students' Translations Among Representations in Algebra

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THE ROLE OF MUSIC CONTEXT IN HIGH SCHOOL STUDENTS'

TRANSLATIONS AMONG REPRESENTATIONS IN ALGEBRA

by

Danielle Divis

A dissertation submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Education

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UTAH STATE UNIVERSITY
Logan, Utah

2022

ABSTRACT

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by

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Previous research suggests that music-contextualized mathematics can benefit students' mathematics learning but leaves questions as to exactly how students interact with music context and mathematics simultaneously. This study used a descriptive multiple-case study design to understand the way high school students utilize music while engaging in music-contextualized algebra tasks focused on representations and translations. Eight pairs of high school students participated in this study. The procedures included training four volunteer high school teacher-researchers to carry out three music-contextualized mathematics lessons in their own classes. Data sources for each pair included three student work samples, three audio-video recordings of class sessions, and one audio recording of a semi-structured interview. Qualitative data analysis within and between cases using both open and deductive coding revealed how students utilized music during the three consecutive mathematics lessons. Results revealed five distinct ways the students utilized the music context. Students used the music context (1) as a source of engagement, (2) to label mathematics, (3) to contextualize an answer, (4) to reevaluate the accuracy of answers, and (5) as a fund of knowledge to draw from. However, students did not always utilize the music context in every task problem. Task problems involving physical, verbal, and contextual representations were more often associated with contextualized answers than task problems involving visual and symbolic representations. The findings also showed how students translating from contextual representations engaged in the constructive activities of “articulating” and “modelling.” Students translating to contextual representations from other representation types engaged in processes of “contextualizing.” These findings suggest that when music is purposefully integrated into mathematics lessons, there is potential for students to utilize the music context in a variety of ways. These findings are important to teachers and curriculum writers wishing to understand how they can create music-contextualized mathematics lessons and understand the types of connections their students might make to music during those lessons. Implications suggest that by bridging the two fields of study, mathematical representations and music integration, more students have opportunities to engage fully in high school mathematics.

(202 Pages)

PUBLIC ABSTRACT

The Role of Music Context in High School Students' Translations Among Representations in Algebra

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Previous research suggests that integrating music into mathematics can benefit students' mathematics learning but leaves questions as to exactly how students interact with music context and mathematics simultaneously. This study used a descriptive multiple-case study design to understand the way high school students utilize music while engaging in music-contextualized algebra tasks. Eight pairs of high school students participated in this study. The procedures included training four volunteer high school teacher-researchers to carry out three music-contextualized mathematics lessons in their own classes. Data sources for each pair included three student work samples, three audio-video recordings of class sessions, and one audio recording of a semi-structured interview. Qualitative data analysis included within- and between-cases using both open and deductive coding. Results revealed how students utilized music during the three consecutive mathematics lessons and five distinct ways the students utilized the music context. Students used the music context (1) as a source of engagement, (2) to label mathematics, (3) to contextualize an answer, (4) to reevaluate the accuracy of answers, and (5) as a fund of knowledge to draw from. However, students did not always utilize the music context in every task problem. Task problems involving physical, verbal, and contextual representations were more often associated with contextualized answers than task problems involving visual and symbolic representations. The findings also showed how students translating from contextual representations engage in processes of “articulating” and “modelling.” Students translating to contextual representations from other representation types engaged in processes of “contextualizing.” These findings suggest that when music is purposefully integrated into mathematics lessons, there is potential for students to utilize the music context in a variety of ways. These findings are important to teachers and curriculum writers wishing to understand how they can create music-contextualized mathematics lessons and understand the types of connections their students might make to music during those lessons. Implications from this study help educators understand the importance of bridging the gap between mathematical representations and music in high school mathematics classes.

ACKNOWLEDGMENTS

I would like to thank my co-chairs Dr. Beth MacDonald and Dr. Katherine Vela for their endless patience, guidance, and time during the many edits of this document. Your attention to detail and expertise in qualitative methods were invaluable in my ability to successfully carry out this study.

I am grateful to Dr. Patricia Moyer-Packenham and the other Teacher Education and Leadership faculty members for designing a graduate program in such a thoughtful and carefully designed way, ensuring that each class contributed to my preparation for this study.

To my husband Ryan, I am indebted to you for all your nights alone with the kids and the times I hid myself behind a closed door in the house to work. Thank you for helping me cut and tape hundreds of PVC pipes. Thank you for listening to me talk about weird mathematics ideas and telling me that everything was going to be okay the many times I came to you crying. To my parents and siblings, thank you for all the free babysitting and for listening to all my complaining for four years.

And to Kingsley and Indy, you inspire everything I have done and will do from now to eternity. Never let a potential challenge stop you from getting where you want to go. I hope if you learn one thing from me, it is that that you can do hard things.

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CHAPTER I

INTRODUCTION

Piaget (1972) argued that we look beyond the traditional compartmentalized view of disciplines in education, instead deconstructing the boundaries that separate school subjects in search of their "interactions and common mechanisms" (p. 129). While school disciplines today remain isolated for operational convenience, the possibility for teachers to engage students in mathematics that is purposefully and thoughtfully informed by other disciplines remains compelling. Kaufman et al. (2003) argue that while a single discipline provides only a narrow view of knowledge, it is through the combination of disciplines that we can "elucidate the whole" (p. 7).

It is important that students be given the opportunity to construct deep, meaningful understanding of mathematical topics across various disciplines. There is preliminary evidence that music-contextualization can facilitate students' mathematics learning (An & Tillman, 2014; An et al., 2013; An et al., 2014; An et al., 2015; Brock & Lambeth, 2013; Courey et al., 2012) because it provides students with a unique lens for exploring mathematical content (An & Tillman, 2015). But details regarding how students engage in music-contextualized mathematics are not yet understood. More specifically, how they utilize music as a contextual representation as they work with other mathematical representations remains unknown. Consequently, the purpose of this study was to describe and categorize the role that music context plays in high school students' translations among representations while engaging in music-contextualized algebra instruction.

Background of the Problem

This study sits at the intersection of two areas of study in mathematics education research, namely translations among mathematical representations, and music integration in mathematics. While these research areas have traditionally remained distinct from one another, an understanding of the background of each field of study illuminates the need for their integration.

The first area of research relevant to this study is mathematical representations and students' translations among them. Lesh et al. (1987) defined mathematical representations as "external embodiments of students' internal representations" (p. 33). Since then, the importance of students' ability to use and translate among representations has been emphasized in the *Principles and Standards for School Mathematics* (NCTM, 2000), the Common Core State Standards for Mathematics (NGA & CCSSO, 2010), and the principles of effective teaching (NCTM, 2014). More specifically, students' *translations* among representations, where conversions and interpretations are made across multiple representations (Janvier, 1987; Fonger, 2011) are described as important in students' problem-solving (Gagatsis & Shiakalli, 2004), and specifically in their algebraic reasoning development (Greenes & Findell, 1999).

There is consistency in the way researchers have conceptualized translations in the literature. Students first interpret the information provided by a source translation, and then engage in various processes called *constructive activities* (Bossé et al., 2011a) to create a target representation. Recent research has investigated other forms of connections between representations made by students outside of this uni-directional type, and has found that students also engage in bi-directional, multi-directional, and abstract translations, all of which involve students making connections across representations (Fonger, 2011; Fonger & Altindis, 2019). The research examining these constructive activities and translation types has most often investigated visual, verbal, manipulative, and symbolic representations (Bossé et al., 2011; Fonger, 2011; Fonger & Altindis, 2019; Gulkilik et al., 2020), while another representation type has consistently been neglected.

Situated among the five main types of mathematical representations is what Lesh et al. (1987) defined as "real scripts," more recently referred to as "relevant situations" (Clement, 2004) or "contextual" representations (NCTM, 2014). These contextual representations allow students to draw on their previous knowledge and experience as they solve mathematics problems in a familiar context. There seems to be widespread agreement on the importance of contextualizing mathematics problems for the purpose of engagement and activating prior knowledge (Davis, 2007; Gainsburg, 2008; Heckman & Weissglass, 1994). Despite this agreement, context and its role as one of the five mathematical representation types (NCTM, 2014) is underrepresented in mathematics education research.

The second area of research relevant to this study is music as a context for learning mathematics. The integration of music into mathematics activities has been of recent interest in mathematics education research (An et al., 2015). The research specifically highlights the extraordinary similarities between the content of both subjects and the benefits afforded to students when music is used as a context in mathematics.

The research has shown that there exist remarkable similarities in the content of both music and mathematics. The processes involved in music composition, instrument design, and musical notation have been intricately connected to mathematical content spanning all areas of the Common Core State Standards for Mathematics (An & Tillman, 2014; An et al., 2015). Teachers given the opportunity to create music-integrated mathematics lessons are able to successfully move beyond superficial integration techniques such as using songs as a pneumonic device, and instead build lesson plans that demonstrate deep and meaningful integration of both subjects (An et al., 2013; An et al., 2015). This research has primarily investigated teachers' lesson plans, integration techniques, and self-efficacy. While these content similarities are compelling, the intricacies of the way students interact with music and mathematics components as they engage in both subjects remains unknown.

Other research has shown that when teachers capitalize on these subject matter similarities and use music as a context for teaching mathematics, many benefits are afforded to students. Music-contextualized mathematics instruction can positively influence students' mathematical beliefs and dispositions (An et al., 2014; An et al., 2015; Brock & Lambeth, 2013). There is also preliminary evidence that music-contextualized mathematics can positively influence students' mathematical achievement (An & Tilman, 2015; An et al., 2013; Courey et al., 2012). Researchers often attribute these positive effects to the way music context allows students to make connections to previous mathematical knowledge and to real world experiences (An et al., 2013; Hargreaves & Moore, 2000) and help students learn mathematics from a novel perspective (An et al., 2013; An et al., 2014). But empirical work has not yet investigated exactly *why* and *how* these student learning benefits manifest when mathematics is contextualized with music. Questions remain about students' interactions with the music and mathematic content, and the way those interactions somehow leverage this productive type of mathematics learning demonstrated by the literature. It is also unclear how music interacts with other modes of representations.

When taken together, the literature on representations and music integration overlap, specifically in the area of context, which is one of the five main representation types and the most common way of using music in mathematics. Merging what is known from both research areas reveals the need for a study that simultaneously investigates questions regarding the role of contextual representation in students' translations, and the details of the way students interact simultaneously with music and mathematics.

Statement of the Problem

Research on students' translations among representations in mathematics has not yet investigated how students translate using contextual representations. While it is known that translations consist of students' interpretations of representations and the constructive activities they engage in while forming new representations (Janvier, 1987; Bossé et al., 2011a), these interpretations and constructive activities have not yet been explored when contextual representations are involved. Music is a compelling choice for context when teaching mathematics because of the content similarities between the two subjects established in the literature (An & Tillman, 2014; An et al., 2015). While benefits to student learning seem to result when music is used as a context, the field lacks understanding of how students interact with music and mathematics simultaneously in a way that leverages these benefits. Therefore, the way students use music when it is utilized as a contextual representation is not yet understood.

Significance of Study

This study advances mathematics education research in three ways. First, research has previously examined students' use of visual, manipulative, verbal, and symbolic representations (see Bossé et al., 2011; Fonger, 2011; Fonger & Altindis, 2019; Gulkilik et al., 2020), and a better understanding of the role of context as a representation contributes to the literature by painting a more complete picture of the representation framework and the way students translate among all five representation types. A holistic understanding of the entire representation framework is important to both researchers seeking to further develop translations theory, and teachers wishing to implement instruction focused on representations and translations.

Second, this study contributes to the field by providing a better understanding of how students make use of music as they engage in music-contextualized mathematics. This study builds on the existing research showing the benefits of music integration in mathematics by providing a deeper understanding of why and how those benefits are brought about. This knowledge is important for researchers who wish to further explore music integration in mathematics. Navigating an unconventional terrain such as arts integration can be a daunting task for teachers, but with the support of this research, teachers can have more confidence in designing, implementing, and assessing music-integrated mathematics activities.

Finally, this study establishes the beginnings of a theoretical connection between mathematical representations and music in the field, something that has not yet been attempted. A conceptualization of the constructive activities and translations students make when engaging in music-integrated mathematics helps initial development of this theory and opens the door to an era of research on music as a legitimate mathematical representation.

Study Purpose and Research Questions

The main purpose of this study was to describe and categorize the role that music context plays in high school students' translations among representations while engaging in music-contextualized algebra instruction. The overarching research question and accompanying sub-questions guiding this study are:

What role does music context play in high school students' translations among mathematical representations while engaging in music-contextualized algebra instruction that emphasizes translations?

(1) How do students make use of music context while completing uni-directional, bi-directional, multi-directional, and abstract translations?

(2) How does students' use of music context while completing translations differ according to the representation types involved?

(3) What constructive activities do students engage in when translating to and from a contextual representation?

Summary of Research Study Design

To describe and categorize the ways students utilize music as a context while translating among mathematical representations, this study employed a descriptive multiple-case study (Mills et al., 2009; Yin, 2017), utilizing qualitative coding methods to analyze several data sources. Data was collected over the course of six weeks from eight pairs of high school algebra students enrolled in four public schools in the Mountain

West. Each pair constituted as a "case" in this study. Data sources included audio-video recordings, student work samples, and audio recordings of semi-structured student pair interviews. To describe and categorize students' use of music while translating, open and deductive qualitative coding were used in both within and between cases study analyses.

Assumptions

I made two assumptions prior to conducting this study. I assumed that participating pairs of students would engage actively in mathematical discussions centered around the algebraic task sheets. I also assumed the music context would influence students' active engagement with algebraic task sheets, and their utilization of music could be captured through audio-video recordings and work samples.

Delimitations

I put necessary constraints on this study to limit its scope and provide necessary focus. Selection of student pairs was limited to a particular population. Potential variations between participating pairs of students were outside the scope of this study. In addition, I did not examine the influence of the classroom teacher on students' translation processes. Teacher moves regarding scaffolding students' translations were also outside the scope of this study.

Definition of Terms

The following terms are defined for this study.

Abstract connection: Generalization made across multiple representations without specifically mentioning individual representations (Fonger, 2011).

Bi-directional connection: Two mathematical representations are interpreted simultaneously (Fonger, 2011).

Constructive activities: The various cognitive processes students engage in while completing a translation from a source representation to a target representation (Bossé et al., 2011).

Contextual representation: Also called a "real script" (Lesh et. al., 1987) or "relevant situation" (Clement, 2004); a context for a mathematical task grounded in a real-world situation.

Mathematical representations: The various ways mathematics concepts can be captured and analyzed (NCTM, 2000).

Multi-directional translation: Translating among more than two representations (Fonger & Altindis, 2019).

Music-contextualized mathematics: Properties of musical composition, instrument design and playing, and notation are utilized as a context within mathematical tasks (An & Tillman, 2015).

Physical representation: A mathematical concept captured by a physical object (Lesh et al., 1987).

Symbolic representation: A mathematical concept captured by a number, an equation, a symbol, etc. (Lesh et al., 1987).

Translation: A mathematical process where students interpret a source representation and construct a target representation (Bossé et al., 2011), or connect among invariant features across two or more representations (Fonger, 2011; Fonger & Altindis, 2019).

Uni-directional translation: When a given representation is interpreted based on a target representation (Fonger, 2011).

Verbal representation: A mathematical concept captured by oral language (Lesh et al., 1987).

Visual representation: A mathematical concept captured by a graph, picture, table, diagram, etc. (Lesh et al., 1987).

CHAPTER II

LITERATURE REVIEW

This chapter comprises a review of the theoretical and empirical literature in two main areas of research relevant to this study. Literature focused on mathematical representations and translations among them comprises the first section of this chapter, and a review of the integration of music in mathematics classrooms comprises the second. Following a review of the literature in those two domains, the conceptual framework guiding this study outlines connections between mathematical representations, translations, and music.

Mathematical Representations and Translations

While representations in mathematics have been defined in many ways, in this study mathematical representations were defined as the many ways mathematics concepts can be captured and analyzed by students (NCTM, 2000), and the specific representation types of interest in this study are visual, symbolic, verbal, contextual, and physical (NCTM, 2014). While representations can be internal or external (Goldin, 2014), this study is primarily focused on students' external observable representations. This section provides a critical examination of the theoretical foundations and literature related to representations and translations in mathematics education, the benefits of representations and translations to students' mathematical learning, and finally contextual representations.

Theoretical Foundations of Mathematical Representations and Translations

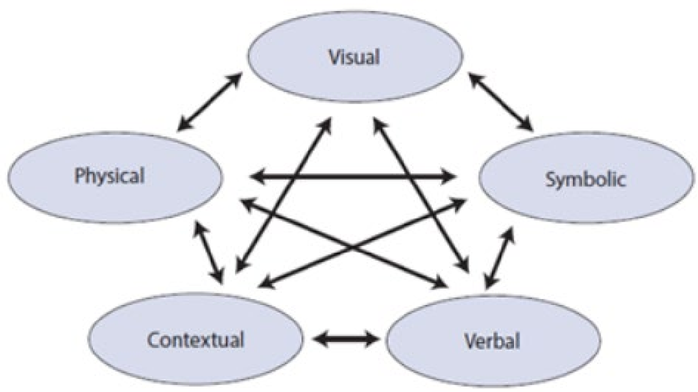
This section provides an overview of both mathematical representations and translations among those representations. These theoretical conceptions of mathematical representations and translations provide the foundation for the study design, analysis, and interpretation of results in this study.

Theoretical Foundations of Mathematical Representations

Mathematical representations are a topic of great interest in mathematics education research and practice (Özmantar et al., 2010), and there is consistency in the way they are defined in the literature. Lesh (1981) first formally conceptualized mathematical representations, defining them as “different contexts in which mathematical ideas may be imbedded” (p. 245-246). Lesh and colleagues later defined representations as “external embodiments of students’ internal representations” (Lesh et al., 1987, p. 33) that could be organized into five categories: 1) real scripts; 2) manipulative models; 3) static pictures; 4) spoken language; and 5) written symbols. Others have since emphasized representations in the era of reform mathematics teaching. The term representation was used in *Principles and Standards* (NCTM, 2000) to encapsulate the many ways mathematics concepts can be captured and analyzed, including but not limited to graphs, tables, symbols, and physical objects (NCTM, 2000). Clement (2004) reintroduced Lesh et al.’s model, altering only the vocabulary slightly, as a mechanism for enhancing teachers’ instruction. These representation types included pictures, written symbols, spoken language, relevant situations, and manipulatives. More recently, the National Council of Teachers of Mathematics (2014) has advocated for the use of and connection between mathematical representations, proposing this practice as one of eight principles that foster high-quality mathematics instruction. This model contains the same five representations and associated descriptions used by Lesh et al. and Clement, though with different terminology (see Figure 1). This definition of representations as observable and external was organized into five types and prevalent in practicum-focused literature. Given these roots, representations have been conceptualized in several other ways in mathematics education.

Figure 1

The Five Representation Types



Note. From NCTM (2014), who adapted from Lesh et al. (1987).

Representations have sometimes been discussed in other ways, such as with a focus on the distinction between external and internal representations, and semiotic vs. non-semiotic representations (e.g., Duval, 1999, 2006, 2017; Goldin & Kaput, 1996; Goldin, 2014; Kaput, 1999). Goldin (2014) defines external representations as “external to the individual who produced them, and accessible to others for observation, discussion, interpretation, and/or manipulation (p. 1) and internal (or mental, cognitive) representations as "a person's mental or cognitive constructs, concepts, or configurations" (p. 1). However, Duval (1999) warns that this dichotomy can be misleading, because internal and external do not refer to the representation type, but rather the manner in which they are produced by individuals. In other words, graphs, tables, and language representations can be both mentally/internally and externally constructed by students. He instead highlights the distinction between semiotic representations, which are intentionally produced, and non-semiotic representations, which are actions taken on a physical object. He argues that semiotic representations are important in students' mathematical learning because mathematical objects cannot be direction accessed, so students need semiotic representations to access them. The theoretical work of Kaput (1999) similarly grappled with the definitions of internal and external representation, also suggesting that they constitute a potentially harmful dichotomy. It is implied by both authors that more important than the internal/external distinction is the representative system or *register* within which they exist, such as visual, or numeric. Representations "should be seen as belonging to structured systems, whether embodied internally or externally" (Goldin & Kaput, 1996, p. 402). In later work, Duval (2006; 2017) equates mathematical representations with *semiotic registers.* These registers allow access to mathematical objects, are intentionally produced, and can be categorized into four types of registers: linguistic, figurative, symbolic, and graphic. These categories bear similarities to Lesh's (1981) early five-node model, though Duval considers students' interactions with physical/manipulative representations as non-semiotic (Duval, 1999), and Duval does not distinguish relevant situations or context (NCTM, 2014) as its own representation type. Given these distinctions, this study recognizes these theoretical contributions but, as discussed in detail later, draws broadly from external representations and externalization of students' internal representations.

The theories above inform the definition of representations utilized in this study. This study is consistent with the theories of Goldin, Kaput, and Duval in its focus on the representational systems or registers within which a representation resides. The distinction between internal and external representations is not a focus of this study, because as a researcher taking on an observational role during data collection, I did not have direct access to students' internal representations. Consequently, I was only able to investigate students' external representations, and their externalizations of their internal representations. Furthermore, while the study inevitably investigated students' interpretations and constructions of both semiotic and non-semiotic representations, this distinction is not a focus of this study. Rather this study aligns with the Lesh (1981) model of representation types, because of its prevalence in the literature, and in particular its application in practical settings.

Building on these theoretical foundations, nationally guiding policies and standards in mathematics education are also focused on mathematical representations. For example, the *Principles and Standards for School Mathematics* (NCTM, 2000) designate the use of representations as one of five "process standards" in which students should be fluent. Representations are emphasized as both a *process* of capturing mathematical ideas, and a *product*, the resulting outcome of that process. Representations play in important role in the content strands of the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). Several of the high school standards call for students to both participate in the process of representing mathematical ideas, as well as examine representations themselves. It is clear from the pervasiveness of representations theory that representations in mathematics are important to both researchers and teachers.

Theoretical Foundations of Students' Translations among Representations

The terms and definitions used in the literature to describe the way students move between representational systems or registers have varied. Lesh et al. (1987) argued that in addition to representations, the *translations* made between and within representations are important in the learning of mathematics. Students translate among representations as they reinterpret the same mathematical concept across differing representations (Cramer et al., 2009, Lesh et al., 1987), and can also transform a mathematical idea within a single representation (Lesh et al., 1987). Duval (2006) uses differing terminology and instead defines *treatments* as "transformations of representations that happen within the same register" (p. 111) and *conversions* as "transformations of representation that consist of changing a register without changing the objects being denoted" (p. 112). For example, a treatment occurs when a student uses information from a system of equations to find a solution (x,y) for that system. Alternately, a conversion occurs when a student constructs a graph given an algebraic equation. Goldin and Kaput (1996) used the term *relations* in a more general sense to describe how students correspond between two representations. The authors do not distinguish between translations based on the representation systems involved. Gagatsis and Shiakalli (2004) defined translations as the psychological processes that students engage in as they move from one representation type to another. This definition takes on a more active view of translations compared to previous definitions, which includes the processes embedded in the translation. NCTM (2014) later argued the importance of students' *connections* among representations, where information provided in two or more differing representation types is coordinated to solidify deep mathematical understanding, again taking a more general stance. While conceptions of students' movement within and among representations have varied in specificity and terminology in the literature, in this study, the word translation is used to encompass these conversions, transformations, relations, or connections made that students' within or across representation systems. Together these publications provide a generalized understanding of what a translation is, but lack details regarding the different types of translations, and students' translations processes.

Uni-Directional Translations. Recent research has investigated the details of the way students translate from one representation to another. Fonger (2011) observed the translations made by algebra students in an effort to create an analytic framework of varying types of translations. The result was the multi-level framework shown in Figure 2. Fonger observed that students often made *uni-directional* translations and connections. In later work Fonger and Altindis (2019) more explicitly defines a uni-directional translation as a translation from one representation type to another, and a uni-directional connection as occurring when students interpret "an invariant feature across multiple representations or types" (p. 4). While Fonger (2011, 2019) theorized that translations and connections consist of students interpreting, creating, and connecting representations, she does not detail the processes involved in creating. Meanwhile, Bossé et al. (2011a) describes translations as including a *source representation* and a *target representation*. During the translation process, a student first interprets the information provided by the source representation, and then engages in some kind of activity to create the target representation. These activities, originally theorized by Lesh (1981) as *translation* *processes* and later by Janvier (1987) as *constructive activities*, include a unique associated "action, technique, or heuristic" that "specifies how constructs or descriptions expressed in the source representation can be directly articulated through structures available in the target representation" (Bossé et al. 2011a, p. 118). For example, when moving from a graphical source representation to a formulaic target representation, a student would engage in the construction activity of "curve fitting" (Janvier, 1987). Goldin and Kaput (1996) later called these activities *representational acts*, emphasizing that these processes are not the same as the representations themselves. Although Goldin and Kaput focus their theory of representational acts on as the processes students engage in while externally representing an internal mental representation. Together these conceptualization of translation processes focuses solely on single-direction translations from one representation to another, lacking a broader view of translations that includes other types of coordinating among translations.

Bi-directional, Multi-directional, and Abstract Translations and Connections. Fonger's (2011, 2019) work examining algebra students' translations extended translation theory from consisting of a uni-directional process of moving from one representation to another to also include other sophisticated translation and connection types. As shown in Figure 2, Fonger (2011) observed that in addition to the uni-directional translations and connections described above, the algebra students in the study also performed bi-directional translations and connections, where students translate "back and forth between two representations" (Fonger, 2011, p. 93), or "recognize invariant features across the two representation types (Fonger & Altindis, 2019, p. 4). Multi-directional translations and connections are similar, but students translate among more than two representations or recognize features across these representations. And finally, abstract connections represent generalizations students make between "underlying mathematical objects/principles" that do not necessarily reference a particular representation (Fonger, 2011, p. 92). The addition of these types of translations to previous translation theories in mathematics helps paint a more complete and detailed picture of what a translation is, and how students' translations vary. Fonger's work also reveals that mathematical tasks can be written in ways that yield a variety of translations in students' solutions, which is an important implication for the design of this study.

Figure 2

Fonger's (2011) Model of Students' Translation Types

Level of Connection	Brief Description
I. Uni-directional Connection	Translation; interprets meaning of a given source representation in reference to a target representation (Janvier, 1987).
IA. Representational Resourcefulness	Uses a representation to overcome a barrier (Jon Davis, personal communication, 11/18/2010).
IB. Uni-directional Justification	"Use representations as justifications for other claims" (Sandoval et al., 2000, p. 6).
II. Bi-directional Connection	Translation and complementary translation (Janvier, 1987).
IIA. Bi-directional Justification	Pair of representations are used to (dis)confirm an approach (Sandoval et al., 2000).
IIB. Bi-directional Reconciling	Coordinated activity; checking the solution between two representations (Kieran & Saldanha, 2008).
IIC. Reflection on Reconciled Objects	Reflection on the compatibility of a result between a pair of representations (Kieran & Saldanha, 2008).
III. Multi-directional Connection	More than two representations are related by translation processes.
IV. Abstract Connection	A generalization is made across different representations.

Note. Taken from Fonger (2011, p. 9)

Summary of Theories on Students Translations. The theories above on the types of translations students engage in and the processes involved in translating are important to this study. The differing translation types described in Fonger's (2011, 2019) provide detail lacking in previous definitions of translations in the literature, particularly with the categorization of translations according to their direction and the number of representations involved. Also, her addition of the concept of connections as a type of translation model align with NCTM's (2014) articulation of students' connections between translations, where students recognize invariant features across representations. However, Fonger (2011; 2019) developed and tested this translations model by observing students' translations among four representation types – verbal, graphic, numeric, and symbolic – and did not investigate translations involving contextual representations. This model of the four translation types has yet to be examined in relation to a contextual representation, leaving questions unanswered as to the way students utilize context as they engage in these different types of translations and connections. Additionally, the translation processes (Lesh, 1981) or constructive activities (Janvier, 1987, Bossé et al. 2011a) students engage in when working with contextual representations have yet to be explored.

The Effects of Mathematics Instruction Focused on Representations

The literature investigating the effects of focusing instruction on representations and translations reveals two overarching benefits to students' learning of mathematics. Mathematics instruction focused on representations and translations has been shown to leverage strong conceptual understanding of topics and improve algebraic reasoning. Existing literature is important in justifying further representations and translations research, and also reveals what is not yet understood about students' translations.

Representations and Conceptual Understanding of Mathematics

Representations play an important role in students' conceptual understanding of mathematics. Students' conceptual understanding should be a fundamental goal of mathematics education (Çıkla, 2004; NCTM, 2014), though facilitating students in developing this understanding is challenging. Mathematical procedures are understood conceptually only when they are connected to the underlying mathematical concepts at play (NCTM, 2014; Skemp, 1976). Using and translating among representations plays an important role in the conceptual understanding of algebraic topics (Fonger & Altindis, 2019; Pape & Tchoshanov, 2001), According to Mofgan et al. (2009), the ability to translate between different representations is essential for a student to truly understand a mathematical idea. Lesh et al. (1987) accompanied his original theory of translations with an argument that if a student understands a mathematical idea, then

> (1) he or she can recognize the idea embedded in a variety of qualitatively different representational systems, (2) he or she can flexibly manipulate the idea within given representational systems, and (3) he or she can accurately translate the idea from one system to another. (p. 36)

Overall, it appears the important role of translations in students' conceptual understanding is pervasive in the literature.

Studies have investigated the relationship between students' translations abilities and their conceptual understanding. Panasuk (2010) developed a three-phase ranking of middle-school students' conceptual understanding of algebra. After testing this ranking system on 24 students, Panasuk concluded that representations and fluency in moving between them was an important component of higher phases of conceptual understanding. The findings suggested that once students can recognize the same mathematical relationships across many representations, they have progressed from a procedural understanding to conceptual understanding. Suh and Moyer-Packenham (2007) investigated the effects of a virtual and physical manipulative environment on students' conceptual understanding of algebra. Findings from a pre-post test revealed that this environment positively influenced student achievement and fluency in translating. Together these studies provide empirical evidence of the important role of translations in students' mathematics learning.

Practical work also stresses the role of representations in conceptual mathematics learning. NCTM (2014) encourages teachers to design classroom activities meant specifically to help students form connections between representation types. Bossé et al. (2011) also emphasized the importance of creating classroom environments that provide frequent opportunities for students to perform translations. This practical and empirical work reinforces the need for a focus on representations and translation in the classroom for facilitating students' learning of mathematics. Other research has investigated representations specifically in relation to algebra learning.

Representations in Algebra Learning

Representations are prevalent throughout algebra and play an important role in students' algebraic reasoning (Greenes & Findell, 1999). Representations are key to algebraic modeling, graphing, manipulating symbols, representing data, working with matrices, and more (NCTM, 2000). Empirical research has shown a positive impact of a focus on representations and translations on algebra performance. Akkus and Cakiroglu (2010) examined how instruction that focused on mathematical representations would impact algebra performance when compared to traditional instruction. The results showed statistically better algebraic performance in the group who received what Akkus and

Cakiroglu calls *multiple representation-based instruction*. This is consistent with findings from the Suh and Moyer-Packenham (2007) study described above, where the authors found that the manipulative representations provide “unique features that encourage relational thinking and promote algebraic reasoning” (p. 171).

Findings on the benefits of instruction focused on representations and translations are important to this study. Because representations and translations play such an important role in student learning, particularly in algebra, there is a need for further investigation into the way students translate, and why and how these processes elicit these benefits to student learning. These investigations have primarily focused on elementary and middle school children’s translations, while studies examining high school students’ translations are less common. These studies also consistently fail to include contextual representations in their theoretical frameworks, warranting a closer look at translations when contextual representations are involved.

Context as a Representation Type

Contextual representations are of particular interest to this study. In the original representations model proposed by Lesh et al. (1987), a representation termed *real scripts* was described as an experienced-based representation used to “serve as general contexts for interpreting and solving” (p. 33). Lesh et al. described how teachers wishing to facilitate students in representation and connecting representations can imbed mathematical ideas in a “familiar situation” (p. 36). The model of representations later proposed by Clement (2004) for practitioners included *relevant situations* as one of the five main mathematical representations. A relevant situation is “any context that involves appropriate mathematical ideas and holds interest for children” (Clement, 2004, p. 99).

These situations are connected to life outside of school and help students "develop an image that they can use in deciding how to solve the problem" (Clement, 2004, p. 99). NCTM (2014) most recently labeled this representation as *contextual*.

It is well agreed upon that context plays an important role in the learning of mathematics (Davis, 2007; Gainsburg, 2008; Heckman & Weissglass, 1994). Outside of the representation literature, context is referred to in a variety of ways in the mathematics education, including analogies, word problems, mathematics in society, ''hands-on'' mathematics, and modeling real phenomena (Gainsburg, 2008, p. 200). When any of these are included in mathematics classrooms, several benefits are afforded to students, including facilitating students' mathematical understanding, engaging them in learning mathematics, and encouraging them to apply mathematics outside of the classroom (Gainsburg, 2008). With the importance of context in mathematics teaching established, context holds an important place among the five representation types.

Despite its inclusion as one of five representation types, little is known about the role of context in translations. In the theoretical literature, Janvier (1987) and Bossé et al. (2011a) intertwine contextual representations with verbal, with both types acting together as source representations. Contextual representations are never viewed as distinct and are not considered as possible target representations. The researchers also do not theorize the possibility of context and its role in students' *constructive activities*, and Duval (2006) does not discuss context as type of semiotic register. As far as empirical investigations, Davis (2007) found context to play an important role in students' translations between representations in algebra as it encouraged the development of students' informal vocabulary. Students in this study made use of their everyday experiences outside of the mathematics classrooms when translating between tables and graphs. The context of the algebra problems was embedded in students' constructive activity of creating a graph from a table or vice versa. Outside of this study, empirical research investigating students' translations has been consistently void of contextual representations (e.g., Akkus & Cakiroglu, 2010; Bossé et al., 2011b; Fonger, 2011, Gagatsis & Shiakalli, 2004). To gain a more complete theoretical understanding of all five representations and students' translation among them, contextual representations must be better understood. Because music has been theorized as a productive source for contextualizing mathematics, the contextual representation type and its role amidst the entire representation model is of particular interest to this study.

Summary of Mathematical Representations and Translations

Mathematical representations are an important component in reform mathematics education (NCTM 2000, 2014). Students' ability to translate between representations is a key signifier of conceptual understanding of a mathematics topic (Lesh & Doerr, 2003; Suh & Moyer-Packenham, 2007; Pape & Tchoshanov, 2001), particularly in algebra (Greenes & Findell, 1999; Akkus & Cakiroglu, 2010). Translations are commonly theorized in the literature as containing a source and target representation, with a construction activity carried out by students in between. Though in a more recent attempt to characterize the various translations students make among representations, Fonger (2011, 2019) provides a useful model that captures four types of translations with varying degrees of sophistication. Though context acts as one of five main representation types (Clement, 2004; Lesh et al., 1987; NCTM, 2014) and contextualized mathematics has been shown beneficial to students' understanding of mathematics (Davis, 2007;

Gainsburg, 2008; Heckman & Weissglass, 1994), the role of context in translations is not yet understood.

Music in Mathematics Learning

Teachers and researchers alike have taken interest in integrating music into mathematics classrooms over the last several decades. This section comprises the theoretical and empirical literature relevant to music as a context in mathematics, specifically the theoretical foundation of music integration in school mathematics, and the practical and empirical literature on music-contextualized mathematics.

Theoretical Foundations of Music Integration in Mathematics

Modern research and practical efforts focused on integrating music into mathematics instruction have been consistently grounded in long-standing theory on the subject-matter connections between the subjects. Music was considered a science in ancient Greece and was respected and studied as such (Fauvel et al., 2006). Music stood alongside arithmetic, geometry, and astronomy as one of four subjects that together comprised the *quadrivium* of liberal arts (Lundy et al., 2010). It was mathematician Pythagoras who first discovered that musical pitch is determined by string length, and that certain harmonious intervals between these pitches occur when the relative lengths represent simple ratios (Harkleroad, 2006). Since then, many have sought to understand the mathematics of musical composition (Fauvel et al., 2006), and multiple books have been written on the various connections and similarities between the subjects. Just a few of these connections include the role of fractions, Fibonacci numbers, and trigonometry in musical tuning systems, composition, and instrument design (Bleiler & Kummel, 2016;

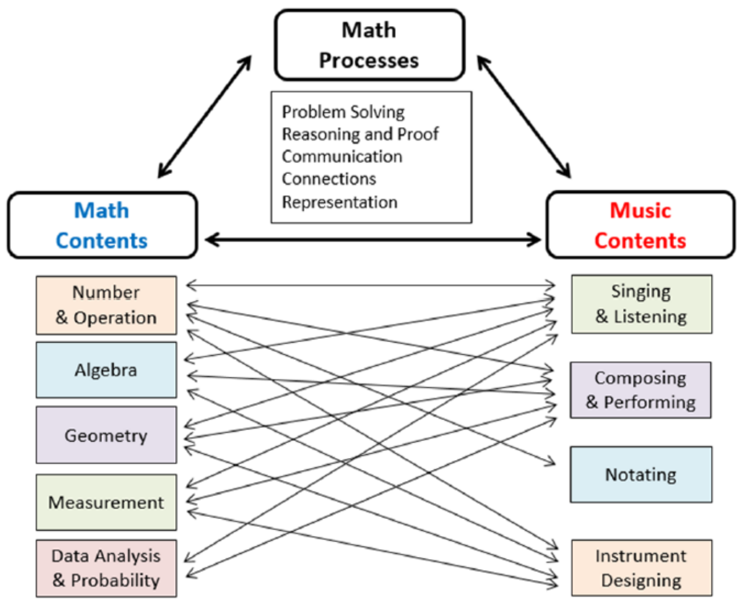
Fauvel et al., 2006; Garland & Kahn, 1995; Link, 1965). Overall, the historical connections between music and mathematics are well-established, and have prompted more modern efforts to examine these connections in relation to school mathematics and education.

Recent studies have sought to establish a theoretical connection between music and Common Core mathematics topics (An & Tillman, 2014; An et al., 2015). In these investigations, An and colleagues conducted a music and mathematics intervention with teachers, and afterwards examined their lesson planning and enactment of music-integrated lessons. And Tillman (2014) found that following an intervention, both pre-service and in-service teachers successfully developed and implemented lesson plans connecting between the musical topics of singing, composing, notating, and instrument design and all five of the mathematical content strands in the current Common Core State Standards for Mathematics (NGA & CCSSO, 2010). An et al., 2015) compared teachers' music-integration strategies before and after an intervention and similarly found that their post-intervention teaching strategies demonstrated deep levels of music integration techniques into each of the five Common Core content strands. Findings from these studies have contributed to the development of the theoretical model shown in Figure 3 establishing these connections between music and mathematics. Cranmore and Turks (2015) interviewed high school students enrolled in music and mathematics to understand what connections they saw between the two subjects. Students primarily mentioned rhythm and counting as the source of connection between music and mathematics. This demonstrates a much narrower view than the connections found by An and colleagues

(2014, 2015) in Figure 3, perhaps suggesting that interventions are a powerful tool for changing perceptions of how music and mathematics are connected.

Figure 3

Subject-Matter Connections Between Music and School Mathematics



Note. From the conceptual framework of An et al.'s (2015, p. 43).

In summary, these studies have built upon historical beliefs about the connections between music and mathematics by establishing theory on the connections between music and school mathematics topics.

Findings on the subject-matter connections between music and mathematics are significant to this study in several ways. First, these studies influenced the creation of lesson plans and student tasks sheets that were utilized in this study, specifically by helping me design quality tasks that would allow students to engage in a variety of mathematics topics while learning about the process of musical instrument design, notation, and composition. Next, these studies demonstrate that music and mathematics interventions are a common methodological technique employed when investigating the integration of music in mathematics, and influenced the design used in this study. Finally, An and Tillman (2014) and An et al., 2015) investigated teachers' creation and implementation of lesson plans, and questions remain about the way *students* connect between music and mathematics. The following section contains a review of what is known about student outcomes when teachers capitalize on these subject matter connections between music and mathematics to implement music-contextualized mathematics lesson.

Music-Contextualized Mathematics

In recent investigations into the effects of music integration in mathematics, researchers have referred to music as a context for mathematics exploration (An & Tillman, 2015). These studies capitalize on the subject-matter connections described above to contextualize mathematics with music. This section reviews the practical suggestions and empirical findings related to contextualizing mathematics with music.

Practical Suggestions for Music-Contextualized Mathematics

The literature is ripe with practical suggestions for teachers using music as a context in school mathematics. For example, Johnson and Edelson (2003) outline using musical instruments and music notation to teach patterns, ordering, sorting, and ratios. More recently, An and Capraro (2011) outline 50 suggested activities for contextualizing mathematics with music in their book *Music-Math Integrated Activities for Elementary and Middle School Students*. These suggestions are primarily for integration at the elementary mathematics level. At the secondary level, Barger and Haehl (2007) propose using the proportions on strings of a guitar to explore geometric sequences. Similarly, Robertson and Lesser (2013) also detail how string instrument construction and performance involves advanced secondary mathematics topics including rational exponents and logarithms. Divis (2021) outlines a lesson where two musical tuning systems, Pythagorean tuning and equal temperament, are examined while students compare linear and exponential growth in multiple mathematical representations. Because Pythagorean tuning is grounded in ratios between common musical intervals, and equal temperament is based on an irrational exponential increase in frequency between adjacent notes, learning about musical tuning systems naturally engages students in a variety of high school mathematics topics. These practical suggestions for contextualizing mathematics with music all demonstrate how musical instrument construction and tuning are governed by mathematical properties found in secondary algebra Common Core content. The pedagogical techniques suggested by these practical works influenced the design of the teacher training, lesson plans, and student work samples that were used in this study. However, these publications provide only anecdotal evidence of the benefits of using music to contextualize mathematics. Further examination of the literature on outcomes on student learning is needed to better understand music-contextualized mathematics.

Studies Examining the Outcome of Music-Contextualized Mathematics on Student Learning

Students' beliefs about their mathematical abilities and their dispositions towards mathematics are important in their mathematics learning (Hoffman, 2010). Several studies have focused on the influence of music-integrated mathematics instruction on student beliefs and dispositions (An et al., 2014; An et al., 2015; Brock & Lambeth, 2013). An et al., 2014) used a series of independent and paired samples t-tests to investigate how the beliefs of students from two third-grade classes would differ when one teacher participated in an interdisciplinary music and mathematics intervention and the other did not. Throughout a nine-week period, the teacher leading the control classroom taught lessons using a traditional method from a textbook provided by the district, while the teacher leading the experimental classroom created and implemented 14 music-integrated mathematics lessons with the help of the authors. A pre-test and post-test assessing mathematical beliefs and confidence showed significant improvement in dispositions of the treatment group. In a nearly identical study, An et al. (2015) saw similar significant improvement in the mathematical beliefs of 71 Hispanic students following a music-contextualized mathematics intervention.

Brock and Lambeth (2013) also saw similar statistically significant improvements in students' attitudes towards and confidence in mathematics following their teacher's participation in a music-integrated mathematics intervention when compared to a control group of students. Findings from these studies are consistent and suggest powerful implications for the influence of music-contextualized mathematics on students' mathematical beliefs, and possibly in turn their mathematical ability. However, while mathematical beliefs are important in mathematics learning, an evaluation of literature assessing mathematical achievement is necessary to better understand the full effect of music-contextualized mathematics.

There is preliminary evidence that music-contextualized mathematics can positively influence students' mathematical achievement (An & Tilman, 2015; An et al., 2013; Courey et al., 2012). An et al. (2013) conducted a music integration intervention with two teachers, who then implemented a series of 10 music-contextualized mathematics lessons with 46 elementary students. The students demonstrated statistically significant improvement between a pre and post-test designed to assess whether they could apply their mathematical knowledge to the real world. An et al. (2013) attributed the findings from their study and success of these students to specific qualities of the lessons, concluding:

> Students can communicate mathematics ideas with their peers, represent mathematics concepts with multiple forms, connect mathematics content with different real life situations, think mathematics meanings from reasonable and logical perspectives and solve mathematics problems by using various problem solving strategies. (p. 15)

An and Tillman (2015) found similar results. In this study, students showed statistically significant improvement on a mathematics assessment given before, during, and after an intervention of music-integrated mathematics lessons. This assessment was a more generalized evaluation of students' modeling, strategizing, and application, while Courey et al. (2012) sought to understand how music and mathematics lessons can specifically strengthen students' understanding of fractions. They found that after receiving 12 40-minute mathematics lessons that incorporated the ideas of fractions in music composition, a classroom of third-grade students performed significantly better on a fractions worksheet than the control group who received traditional instruction. In fact, the instruction seemed to particularly help struggling students who entered the intervention at a low level (as evidenced by a pre-test). Together these studies provide beginning evidence that music-contextualization can positively influence students' mathematical understanding. However, while these highly quantitative studies successfully show correlations between the use of a music-contextualized mathematics intervention and student learning outcomes, they fail to investigate the details of students' mathematical processes taking place during the intervention that leverage these benefits. The way students interact with music and mathematic simultaneously has yet to be categorized and understood.

Conclusion of Literature on Music in Mathematics Learning

Findings from the literature related to music in mathematics learning are important to this study. The theoretical work regarding connections between music and mathematics and the practical suggestions for contextualizing mathematics using music profoundly influenced the design of this study, including the teacher training, the lesson plans, and the data sources. The outcomes for student learning when mathematics is contextualized with music are compelling. These studies are few in number, and while they provide preliminary evidence that music-contextualized mathematics yields benefits to student learning, no investigations to present have sought to categorize students' mathematical processes and the role of music in those processes as they engage in music-contextualized activities. It is not yet known *why* and *how* music-contextualization influences students' mathematical learning.

The present study coordinates findings from the literature on both representations and music integration in mathematics to examine the role of music context in high school students' translations between representations. The following section presents the conceptual framework guiding this work.

Conceptual Framework for Examining Music as a Contextual Representation in Mathematics

This section brings together the theoretical work and empirical findings on mathematical representations, translations, and music-contextualized mathematics presented above. A conceptual framework for the proposed study is presented, outlining the various relationships between concepts and laying the groundwork necessary to justify and guide this research. This framework draws from three main theories in the literature: 1) the five types of mathematical representations; 2) the varying types of translations made among algebraic representations by students engaging in the learning of algebra, and 3) music-contextualized mathematics.

Explanation of Underlying Theoretical Constructs

The first theoretical construct underlying the conceptual framework is the five-node representations model (NCTM, 2014) that includes visual, manipulative, symbolic, verbal, and contextual representation types. This model has been consistent across practical work for decades. Contextual representations, previously defined as "real scripts" (Lesh et al., 1987) or "relevant situations" (Clement, 2004), are of particular interest to this study. This node of representation is one that captures student interest and is connected to life outside of school (Clement, 2004) and can "serve as general contexts for interpreting and solving" (Lesh et al., 1987, p. 33). While some representation models are void of this representation type, the established importance of contextualizing mathematics in student learning (Davis, 2007; Gainsburg, 2008; Heckman & Weissglass, 1994) and its inclusion in the reform mathematics movement (NCTM, 2014), warrant its inclusion as a representation in the conceptual framework for this study.

The second theoretical construct guiding the conceptual framework for this study is students' translations (Janvier, 1987) among mathematical representations and the varying processes and translation types involved. Janvier (1987) and Bossé et al. (2011a) describe how translations begin with a source representation, from which students can draw the information necessary to construct a target representation. In this translation theory, students engage in a variety of mathematical processes called *translation processes* (Janvier, 1987) or *constructive activities* (Bossé et al., 2011). Later theoretical work pertaining to translations does not contradict this theory, but rather adds to the nature of the directionality of students' translations. Fonger (2011, 2019) conceptualizes bi-directional (back and forth between two modes), multi-directional (among more than two modes) and abstract (generalization across many modes) translations, referred to in generality as *connections*. These four connection types align well with modern reform mathematics theories, which advocate for the importance of students' making connections among representations (NGAC & CCSSO, 2010; NCTM, 2000; NCTM, 2014).

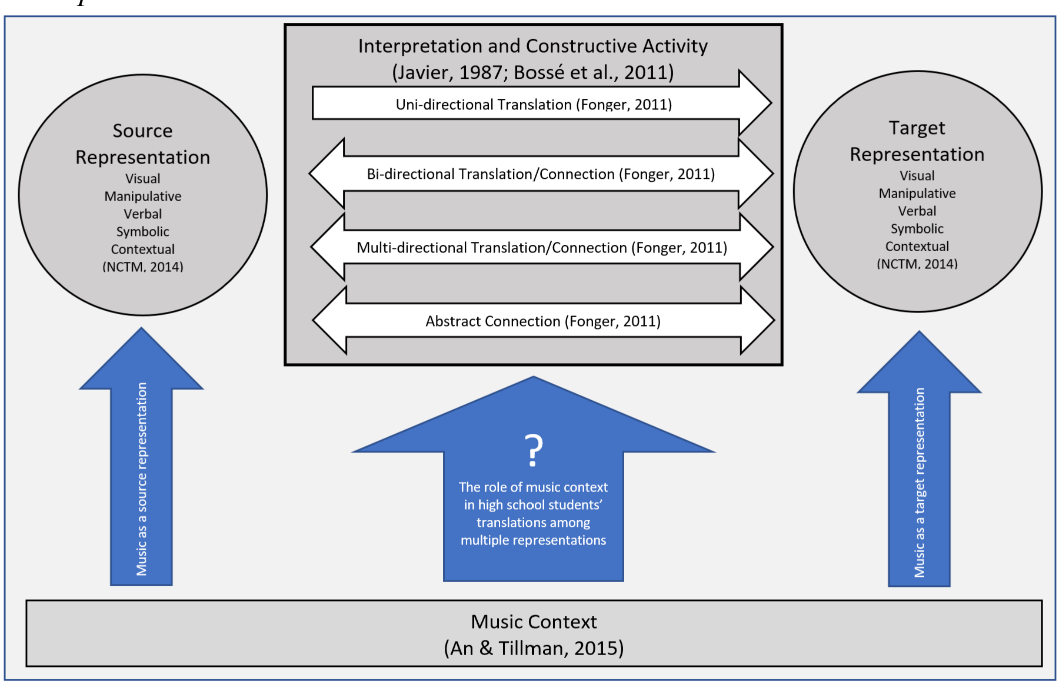
The third theoretical construct guiding the conceptual framework is music-contextualized mathematics (An & Tillman, 2015; An et al., 2015). This theory draws on previous empirical work that establishes the subject area similarities existing between music and mathematics (An & Tillman, 2014; An et al., 2015). Because the subject matter of certain musical structures and mathematics content areas is similar, music provides a viable choice for contextualizing mathematics, and wide-ranging benefits to students' mathematical learning are afforded (An & Tillman, 2014; An et al., 2013; An et al., 2014; An et al., 2015; Brock & Lambeth, 2013; Courey et al., 2012).

Relationships Between Constructs within the Conceptual Framework

Taken together, these three theoretical constructs above form a concept map (see Figure 4) that framed this study. The conceptual framework centers around the foundational translation theories of Janvier (1987), and later exemplified Bossé et al.'s (2011), emphasizing that the students' translations consist of their interpretation of representations and constructive activities. Also at the center of this framework is the modern translation theory of Fonger (2011, 2019), consisting of four varying types of translations. This suggests that students not only interpret given information from representations and engage in constructive activities to create representations, but can also make connections across representations, recognizing invariant features within two or more representations. These constructs are the primary focus of this study and inform the research questions.

Figure 4

Conceptual Framework



It is meaningless to discuss translation types and constructive activities void of the representation types involved. This model forms a necessary connection between translation theory and NCTM's (2014) five-node representations model. In doing so, this framework applies the modern translation theory of Fonger (2011, 2019) to the practitioner-based model of representations, suggesting that students' engagement in mathematical tasks focused on representations can yield the interpretations and constructions of all five representation types, as well as a variety of translation types. It is especially important that contextual representations be included as a possible source and target representation, and as connected to all translation types, because they have been consistently neglected in translation research and are a focus of the research questions in this study.

Finally, at the base of the model, acting as a foundation for this study, is An et al.'s (2015) conceptualization of music context. The traditionally distinct research areas of mathematical representations and music integration in mathematics intersect in the area of context, because context is both a representation type, and an effective way to utilize music in mathematics. This framework connects An et al.'s (2015) conceptualization of music context to representations by viewing it as both a possible source and target contextual representation, as shown by the outer blue arrows in this model. This suggests that students can interpret and construct contextual (music) representations and engage is differing types of translations involving contextual (music) representations. And finally, the large blue arrow in this model reflects the need for investigation into the unknown role of music in students' translations, which is the purpose of this study.

CHAPTER III

METHODS

The main purpose of this study was to describe and categorize the role that music context plays in high school students' translations among representations while engaging in music-contextualized algebra instruction.

Research Design

To answer the research questions, this study used a descriptive multiple-case study design (Mills et al., 2009; Yin, 2017), a type of design where qualitive data is gathered from multiple cases to develop detailed descriptions of a phenomenon, especially when there is no pre-existing framework for describing the phenomenon. This allowed me to describe and categorize students' utilization of music as a context for mathematical tasks. A multiple-case study design was most appropriate for this study because there was no pre-existing model for examining students' utilization of music context in mathematical tasks. According to Baxter and Jack (2008), case studies are appropriate "when the focus of the study is to answer 'how' and 'why' questions" (p. 545). Qualitative data was gathered using four data sources. In this study, a case was defined as a single pair of students engaging in music-contextualized algebra lessons conducted by their regular classroom teacher. According to Miles et al. (2014), cross-case analysis in multiple-case studies can both "enhance generalizability" and "deepen understanding and explanation" (p. 95) of the phenomenon, because "the goal is to replicate findings across cases" (Baxter & Jack, 2008, p. 548). Furthermore, descriptive case studies allow a researcher to describe the data within the nature of the phenomenon as it occurs (Baxter & Jack, 2008; Zainal, 2007). A descriptive case study begins with a "reference theory or model that directs data collection and case description" (Scholz & Tietje, 2002, p. 12), and therefore is especially appropriate for this study, given the existing theoretical models of mathematical representations (Lesh et al., 1987; NCTM, 2014), translations among representations (Fonger, 2011), and constructive activity types (Bossé et al., 2011) that informed the analysis. The research question and accompanying sub-questions that guided this study are:

> What role does music context play in high school students' translations among mathematical representations while engaging in music-contextualized algebra instruction that emphasizes translations?
>
> > (1) How do students make use of music context while completing uni -directional, bi-directional, multi-directional, and abstract translations?
> >
> > (2) How does students' use of music context while completing translations differ according to the representation types involved?
> >
> > (3) What constructive activities do students engage in when translating to and from a contextual representation?

Setting, Teacher-Researchers, and Participants

Four high schools, located in the western portion of the United States, served as the data collection sites for this study. All schools were located in middle to upper-middle class suburban neighborhoods. Two of the schools were public schools, in different but geographically adjacent districts. The other two schools were nearby public charter schools and not part of any district. Four eligible teacher-researchers willing to implement the music-integrated lessons were first recruited to become teacher researchers by word of mouth, after which student participants were recruited from those four teachers' classrooms. These teachers were chosen based on the following criteria: (1) willing to implement the music-integrated mathematics lessons, (2), self-identified as a "task-based" mathematics teacher; and (3) teaching a Core Mathematics Secondary I, II, or III course that would at some point in the year focus on exponential and linear growth.

Once the four teacher-researchers had been identified and the teacher-training phase (described below) was complete, eight pairs of students (two pairs per teacher) were recruited according to a criterion-based purposive sample (Miles et al., 2014; Onwuegbuzie & Collins, 2007). I chose eight cases because this number falls within Stake's (2005) recommendation for the number of cases needed in a multiple-case study in order achieve saturation of the phenomenon observed. The students were selected by their teacher as likely to engage in thoughtful mathematical discussions and persevere through difficult mathematical tasks with a classmate. This purposeful sample helped simplify implementation of the music-integrated lessons and ability to control for additional variables because students had experienced solving task-based activities and exponential growth tasks. Five females and 13 males who elected to participate in this study. Fourteen of these students identified as white, and 2 as Hispanic or Latino. All participants were enrolled in 9^{th}-11^{th} grade and were between the ages of 14 and 17. This age group was appropriate for recruitment because the focus of the algebraic content in this study aligns with the Common Core State Standards (NGAC & CCSSO, 2010) traditionally taught to this age of students.

Case Profiles

The following subsections describe the settings for the four data sites, the role each teacher played as a teacher-researcher implementing the music-integrated mathematics lessons, and a brief description of background knowledge for each of the eight student pairs.

Mr. J's Secondary Mathematics II Classroom

Mr. J acted as teacher-researcher at one of the four data collection sites, implementing the music-contextualized mathematics lessons in each of his Secondary Mathematics II classes over the course of three consecutive days. Mr. J's school is a public charter high school in the Mountain West serving students in grades 9-12, with a focus on STEM education. Prior to implementation, Mr. J had completed a unit on comparing linear and exponential growth in the two classes selected for data collection. This unit, part of a task-based curriculum recently adopted at the school, highlighted multiple representation types.

Mr. J described these classes overall as demonstrating average proficiency in mathematics. Both classes had an established classroom culture of partner and group collaboration. The desks in Mr. J's classroom were arranged in groupings to facilitate these discussions, and a smart television at the front of the room was utilized to display the lesson slides throughout the implementation. Two pairs of students (Vincent and Ethan, Cameron and Eric) were from two different classes taught by Mr. J and assented to participate.

Case I: Vincent and Ethan. Vincent and Ethan were one of the pairs of students observed in Mr. J's Secondary II classes. Vincent and Ethan's conversations did not show any evidence that either of them had any previous formal experience with music theory or training with musical instruments. Vincent and Ethan were both confident in their mathematics abilities and were identified by Mr. Johnson as high-achieving students.

Case II: Cameron and Eric. The other pair of students from Mr. J's Secondary II classes, Cameron and Eric, did not show any evidence that either of them had any previous experience as musicians or training with musical instruments. Like Vincent and Ethan, Cameron and Eric were also confident in their mathematics abilities and were identified by Mr. Johnson as high-achieving students.

Ms. K's Secondary Mathematics I Classroom

Ms. K acted as teacher-researcher at one of the four data collection sites, implementing the music-contextualized mathematics lessons in each of her Secondary Mathematics I classes over the course of three consecutive days. Ms. K's school is a public high school in the Mountain West serving students in grades 10-12. Prior to implementation, Ms. K had completed a unit on general equations in the class selected for data collection. While students in this class had learned about linear equations in previous school years, they had not yet covered linear or exponential growth in Ms. K's class. This set apart Ms. K's classroom from the other three data collection sights. She planned to use these lessons as an opportunity to review linear growth and introduce exponential growth for the first time.

Ms. K described this class overall as being fairly remedial, with below-average proficiency in mathematics. Ms. K commented during the teacher training phase that this class required more teacher-centered learning. However, this class had an established classroom culture of partner and group collaboration, with the desks arranged in groupings to facilitate these discussions. Ms. K projected the lesson slides from her computer onto a white canvas screen at the front of the room throughout the implementation. Two pairs of students (Elise and Logan, Rachel and Rebecca) from Mr. K's class agreed to participate.

Case III: Elise and Logan. Elise and Logan were one of the pairs of students observed in Ms. K's Secondary I class. Elise mentioned that she had some experience playing the clarinet, though she hadn't played in over a year, while Logan said he knew little about music. Elise and Logan were identified by their teacher as students who worked hard in mathematics, were willing to engage in rich discussions, and were average achieving in class.

Case IV: Rachel and Rebecca. Rachel and Rebecca were the other pair of students observed in Ms. K's Secondary I class. It was evident from their conversations throughout the lessons that Rachel had a small amount of experience with music theory. There was no evidence that Rebecca had any prior music experience. Rachel and Rebecca were identified by Ms. K as high-achieving and hard working in her class and likely to engage in mathematical discussions.

Ms. S's Secondary Mathematics I Classroom

Ms. S acted as teacher-researcher at the third of the four data collection sites, implementing the music-contextualized mathematics lessons in each of her honors Secondary Mathematics III classes over the course of the consecutive days. Ms. S's school is a public high school in the Mountain West serving students in grades 10-12. Prior to implementation, Ms. S had completed a unit on polynomials in the class selected for data collection, and had previously gone over linear, quadratic, and exponential equations earlier in the year. She used these lessons as an opportunity to review linear and exponential growth.

Ms. S described this class overall as demonstrating very high mathematical proficiency. Like the other data collection sites, Ms. S's class had an established classroom culture of partner and group collaboration, with the desks arranged in groupings to facilitate these discussions. Ms. S projected the lesson slides from her computer onto an interactive SmartBoard screen at the front of the room throughout the implementation. Two pairs of students (Aaron and Sam, Mitch and Paul) from Mr. S's class agreed to participate.

Case V: Aaron and Sam. Aaron and Sam were one of the pairs of students observed in Ms. S's honors Secondary III class. It was unclear from their conversations throughout the lessons that either boy had previous experience with music theory. Aaron and Sam were in an honors mathematics track, and therefore taking a class that was one grade level ahead of most other students at the school. Aaron and Sam were identified by their teacher as loving mathematics, high-achieving, and hard working in class.

Case VI: Mitch and Paul. Mitch and Paul were the second pair of students observed in Ms. S's honors Secondary III class. It was evident from their conversations throughout the lessons that Mitch had significant previous experience with music theory, though Paul's music experience was unclear. Like Aaron and Sam, Mitch and Paul were also in the honors mathematics track at their school, were high-achieving, hard-working, and enjoyed mathematics.

Ms. W's Secondary Mathematics I Classroom

Ms. W acted as a teacher-researcher at the final collection site, implementing the music-contextualized mathematics lessons in each of her Secondary Mathematics I classes over the course of three consecutive days. Ms. W's school is a public charter school in the Mountain West serving students in grades 7-12. Prior to implementation, Ms. W had completed a unit on arithmetic and geometric sequences in the class selected for data collection and had previously gone over linear equations earlier in the year. She used these lessons as an opportunity to review linear growth and exponential growth.

Ms. W described this class overall as demonstrating very high mathematical proficiency. Like the other data collection sites, Ms. W's class had an established classroom culture of partner and group collaboration, with the desks arranged in groupings to facilitate these discussions. Ms. W projected the lesson slides from her computer onto the whiteboard at front of the room throughout the implementation. Two pairs of students (Emily and Amber, Jason and Gabe) from Mr. W's class assented to participate.

Case VII: Emily and Amber. Emily and Amber were one of the pairs of students observed in Ms. W's Secondary I class. It was evident from their conversations throughout the lessons that Amber had previously taken piano lessons. Like Ms. S's students, Emily and Amber were also a grade level ahead of the rest of the student population in mathematics, taking a high school mathematics course while still in eighth grade. Emily's musical experience was unclear. Emily and Amber kept studious notes in a composition notebook in Ms. W's class. These notes contained big ideas, definitions, formulas, etc. from past lessons. Ms. W identified the pair as high-achieving, hardworking, and willing to engage in mathematical discussions.

Case VIII: Jason and Gabe. Jason and Gabe were the second pair of students observed in Ms. W's Secondary I class. It was unclear from their conversations during the three tasks whether Jason or Gabe had previous experience with music or instruments. Like Emily and Amber, Jason and Gabe were a year ahead of a traditional mathematics track, took detailed mathematics notes, were high-achieving, and hardworking students willing to have mathematical discussions.

Procedures

The following subsections outline procedural details regarding the three phases of this study, including the teacher training phase, the lesson implementation phase, and the follow-up semi-structured interview phase. Table 1 shows the three phases of this study with their associated time frame.

Table 1
Study Phases and Timeline

Phase	**Semester**	**Duration at Each Data Site**	**Phase Duration Across Sites**
Teacher Training	Early Fall 2021	One Day	One Day
Lesson Implementation	Mid Fall 2021	Three 80-minutes class periods over three days	Six weeks
Follow-up Interviews	Mid Fall 2021	30 minutes	Six weeks

Teacher Training Phase

Prior to collecting data, I conducted a training with the four volunteer teacher-researchers where I provided opportunities for them to actively engage with three music-integrated mathematics lessons (see Appendix A for details). This training took place in Fall at the beginning of the school year in which data was collected, inside a high school classroom located a convenient distance from each of the four teachers. The training was conducted over the course of five hours, with a lunch break in between. The training was offered both in person and virtually, but all four teachers elected to attend in person. Prior to lunch, I conducted two of the three lessons with the teachers. Although the lessons were designed for 80-minutes in high school classroom, the lessons were completed in 60 minutes each when conducted with the teachers. Following lunch, I conducted the final lesson, followed by a 30-minute group discussion to answer any questions, review material, and alleviate any hesitation in their future implementation. The group discussion also allowed me to highlight the key mathematical ideas in each lesson and the types of representations involved. During the last hour of training, the teachers completed three CITI modules in accordance with IRB requirements. This allowed for the teachers to take an active role of teacher-researcher in the study and safely implement the music-integrated mathematics lessons in their classrooms.

Appendix A contains the lesson plans I used for this teacher training. Table 2 outlines the structure of the training. Following participation in the training, teachers were given the materials needed to carry out the lessons themselves with their regular classroom students. Each of the four participating teachers was given access to an online shared folder containing the electronic materials needed to conduct the three lessons, including detailed lesson plans, student tasks sheets, lesson slides, and a bin of physical manipulatives.

Table 2

Teacher Training Structure

Activity	Duration
Lesson 1	60 Minutes
Lesson 2	60 Minutes
Lunch	30 Minutes
Lesson 3	60 Minutes
Discussion and CITI Training	90 Minutes

Lesson Implementation Phase

During the Lesson Implementation phase of the study, the teachers implemented the three lessons during three consecutive regular class days within a six-week period across all data collection sites. Each of the three lessons was composed on two launch-explore-discuss cycles (Bahr, 2018). During the launch portion of the cycle, the teacher launched the lesson in an engaging manner. During the explore portion, students engaged in the task exploration and completed the majority of the task sheet problems. During the discuss portion, the teacher brought the class back together for a whole-class discussion (see lesson plans, Appendix A). Table 3 shows the activities and content focus of each of the three lessons. I silently observed the entirety of both lessons, remaining in proximity to the participating pairs as much as possible in order to maintain detailed field-notes (described later). Although the entire class took part in the lesson, I only gathered a student task sheet from the participating students at the conclusion of the lessons. I electronically scanned the tasks sheets to allow for coding.

Table 3

Content Focus for the Two Music-integrated Mathematics Lessons

Lesson	Activities	Content Focus
Lesson 1	PVC pipe exploration Piano exploration Complete Task Sheet 1	Ratios Exponents as repeated multiplication
Lesson 2	Introduce equal-temperament tuning Desmos graphing Completing Task Sheet 2	Geometric sequences Discrete vs. Continuous Exponential growth Exponential equations Domain and range
Lesson 3	Interpret 1.059 Introduce "linear" tuning Tone generator	Rational exponents Slope Comparing exponential to linear growth

Prior to data collection, I developed the following materials for the study: an electronic lesson plan, three electronic student task sheets, lesson slides, and a bin filled with physical manipulatives. These materials are described in the sections that follow.

Lesson Plans

I created a three-part lesson plan for implementing the music-contextualized algebra lessons (see Appendix A). These lesson plans were created in collaboration with Dr. Song An at University of Texas at El Paso, an expert in music-integrated mathematics and author of several works sited in this study, and Dr. Sara Bakker, a music theory assistant professor at Utah State University. This lesson plan was designed with specific goals in mind addressing the Common Core State Standards (NGAC & CCSSO, 2010) algebra strand, specifically the exponential growth strand. Prior to this study, I piloted the lesson plan several times as a classroom mathematics teacher, as well as part of a project to develop a practitioner article (see Divis & Johnson, 2021). An adaptation of the lesson plan can also be found on NCTM's *Illuminations* website (see NCTM, 2021). These lesson plans were provided electronically to each teacher in an online shared folder.

Student Task Sheets

I created three student tasks sheets (one for each of the lessons). These task sheets (see Appendix B) served as one of the data-collection instruments and were designed to purposefully engage students in a variety of different translation types, and translating among all five types of representations, in light of the first and second research sub-questions. The translation framework provided by Fonger (2011) in Figure 2 and the five NCTM (2014) representation types show in Figure 1 served as the primary tools in developing the student task sheets. Table 4 shows several examples of problems from the tasks and the type of translation among representations, while Appendix C shows the complete list of problems and associated representation and translation types. The representation and translation types do not represent exactly what the students actually did with each problem, but only represent what was initially *expected* based on what is explicitly written in the problem.

Table 4

Example Problems from Student Task Sheets and Associated Anticipated Representations and Translation Type

Question	**Representations Involved**	**Translation Type (Fonger, 2011)**
Write an equation that fits the graph of the data points.	Visual Symbolic	Uni-directional Translation
After your teacher passes back your PVC pipe instrument to you, explain how the pipes of the instrument would be sized	Physical Symbolic Verbal	Multi-directional Connection

differently if they were tuned according to this linear progression of frequencies?	Contextual	
Verify the accuracy of your equation by demonstrating it produces at least two sets of data points found in the table.	Symbolic Visual	Bi-directional

The first task sheet consisted of 8 problems, the second task sheet consisted of 13 problems, and the third task sheet consisted of 10 problems. The content focus of each task sheet is shown in Table 3 above.

Lesson Slides

I created a set of Power Point slides to help teachers conduct the three lessons. These slides reminded teachers when to employ a launch, explore, or discuss portion of the lesson. They also contained important tables for referencing, such as the standard Western tuning frequencies from C4 to C5. Finally, the slides contained copies of the task sheet problems so that during the whole class discussing portion of the lessons, the teacher could write answers on the board for each problem. All four teachers used these slides with fidelity with no major edits.

Manipulatives Bin

I compiled a group of materials that were provided in a bin of physical manipulatives to each teacher. Each bin contained enough pre-sanitized sets of pre-cut ½ inch PVC pipes for each pair of students. These pipes were used during the first and third lessons and acted as a physical manipulative for the students (see Figure 5). The teacher and researcher ensured that each pair of students used the same set of manipulatives each day by labeling each set of pipes with the pairs' names on a piece of duct tape. I cut PVC pipes prior to study implementation, according to the lengths shown in Table 5. The eight pipes were cut purposely to play the notes of the major scale between C4 to C5 (Divis & Johnson, 2021, p. 49). The manipulatives bin also contained a classroom set of measuring tapes needed for measuring the PVC pipes, and a roll of duct tape for taping together the pipes to form an instrument.

Figure 5

Example of the PVC Pipes Instrument

Note. From Divis and Johnson (2021, p. 48).

Table 5

Lengths Used When Cutting Pipes for Major Scale Between C4 and C5

C4	D4	E4	F4	G4	A4	B4	C5
33 cm	29.3 cm	26.4 cm	24.8 cm	22 cm	19.8 cm	17.6 cm	16.5 cm

Follow-up Semi-Structured Interview Phase

Following the conclusion of all three lessons and within one week of the final lesson implementation to avoid any loss in student recollection of the lesson, I conducted one semi-structured interview with each pair of participating students. I interviewed the pair together, to gain insights from both students and help them feel more comfortable in the interview setting, and because they completed the task sheet together in class. During the interview, I placed the students' three task sheets in proximity to the students so they can be referenced by the students and myself during the interview. The interviews took place in an empty classroom inside the school where the lesson took place, in online using a conferencing software, and at the end of the school day. Interviews lasted 30 minutes, allowing students to discuss each of the problems on the task sheet for roughly one minute.

Data Sources and Instruments

The data set for the study included 24 student work samples, approximately 32 hours of audio/video data with accompanying fieldnotes, and 4 hours of transcribed audio data. I used four main data sources in this study to effectively answer the three research sub-questions: student work samples, video recordings of students completing the tasks, field notes, and audio/transcriptions of student interviews. According to Baxter and Jack (2007), documentation, direct observation, and interviews are all appropriate data sources for conducting a case study, and these three types of data are popular choices in qualitative research in general (Merriam, 2009). The following subsections outline the details of each data source.

Student Work Samples

The first source of data was student work samples. Each of the eight pairs of students were given one task sheet to complete for each of the three lessons, making a total of 24 task-sheets that I gathered. Because the task sheet problems were specifically designed to engage students in a variety of the Fonger (2011) translation types and NCTM (2014) representation types, the student work samples allowed me to address the research sub-questions in the following ways: (1) First, the student work samples allowed me to examine students' written work when completing all translation types and representation types, specifically examining their use of music in their written work. This allowed me to develop accompanying codes for their music utilization in order to answer the first and second research sub-questions. (2) Second, the student work samples allowed me to examine students' written work when translating to and from a contextual representation, specifically examining any "constructive activities" (Bossé et al., 2011a) they engaged in while doing so. This allowed me to develop accompanying codes for their constructive activities in order to answer the third research sub-question. I electronically scanned the work samples following their collection and stored them on a password-protected external hard drive to allow for coding following the completion of all three phases of the study.

Audio-Video Recordings

Audio-video recordings were used as a second data source in each of the four classrooms. Video recordings can be especially effective in qualitative social science research (Heath et al., 2010), allowing researchers to make sense of the details observed as people interact with their environment. Most importantly to this study, Pirie (1996) suggests that video allow us "the facility through which to re-visit the aspect of the classroom that we have recorded, granting us greater leisure to reflect on classroom events and pursue the answers we seek" (p. 3). In conjunction with the student work samples, the video recordings afforded me several opportunities: First, these recordings allowed me to capture students' use of verbal representations. Second, these recordings allowed me to glean additional understanding of students' translations among representations and constructive activities as they wrote out answers to the tasks, as students were likely to elaborate on their translation processes in a way not captured on paper. This allowed for triangulation between video recordings and student work in answering all three research sub-questions.

At three of the data sites, two video cameras were placed on a tripod near each of the two pairs of participating students in each classroom, allowing me to see visually which task problem the students are working on as they conversed with their partner. However, in Mr. J's classroom, only one camera was needed because the participating pairs were in two separate classes. I set up the video cameras prior to the class period of lesson implementation and took down the video camera following each lesson. Recordings commenced at the beginning of the launch and concluded at the end of the whole class discussion, making a total of 24 audio-video recordings, three from each of the eight participating pairs of students. Each recording was approximately 80 minutes long, the duration of the full lesson, so audio-video data comprised approximately 32 hours data. I stored the audio-video recordings on a password protected external hard drive and coded them in Maxqda during the data analysis phase.

Field Notes

In conjunction with the video recordings, I maintained detailed fieldnotes for the third data source in this study. Much like the video data, these fieldnotes allowed me to capture students' utilization of music and their constructive activities as they translated. According to Tessier (2012), field notes work well in conjunction with recordings, allowing a researcher the ability to capture important "ideas and memories from interviews will most likely be lost further down in the research process" (p. 448). These field notes were later coordinated with the video data to examine what was missing or erroneous in the field notes, and also connect my ideas that occurred in the moment with the video data to assist with quality analysis. Most importantly, the field notes informed the interview questions used in the semi-structured interview phase. Unanswered questions I wrote in the field notes pertaining to the students' music utilization and constructive activities were then revisited in the interviews.

Follow-up Semi-Structured Interviews

The fourth and final data source was transcriptions from audio-recorded follow-up interviews with participating students. Semi-structured interviews are a common type of interviews in social science studies (Brinkmann, 2014), and are appropriate in this study because they allow the interviewees "a degree of freedom to explain their thoughts" and "to be questioned in greater depth" (Horton et al., 2004, p. 340). I conducted semi-structured interviews with a focus on the student work samples to better understand students' use of the music context as they completed the task. The semi-structured nature of the interview allowed me to focus on specific aspects of the student work samples I deemed especially important (Brickmann, 2014; Horton et al., 2004) for answering the first and second research sub-questions. The interview transcriptions provided additional information on how students used music as they completed different types of translations, translated between different representations, and engaged in constructive activities. The interview protocol with guiding questions that I used can be found in Appendix D. I created this protocol through a series of informal pilots of this study. The protocol questions were designed to elicit further information from the student pair on their answers to each task problem. The instrument used to record the semi-structured interviews (Horton et al., 2004) in six of the eight cases was a handheld audio recorder. However, due to student availability, two of the eight interviews were conducted and recorded via Zoom with video turned off. Following the interview, I uploaded the audio files to a password-protected computer and later transcribed them electronically in a word processor following the completion of all three phases of the study. Due to issues with student availability outside of class time, the interviews lasted roughly 30 minutes each, and there was a total of eight interviews transcribed for the data set, one from each pair of students. Interview data comprised approximately four hours of data.

Data Analysis

Data analysis procedures in this descriptive multiple-case study aimed to describe and categorize the various ways students utilized a music context when conducting different types of translations among mathematical representations in algebra, translating among different representation types, and engaging in constructive activities. Data analysis in this study began in conjunction with the data collection phase, allowing me to prepare for the follow-up semi-structured interviews with questions informed by initial coding. Data analysis was conducted in two phases.

The first phase of analysis began following the conclusion of each lesson taught. Following the lesson, I compiled the student work samples, field notes, and audio/video recordings for each pair of students. I reviewed the video data in conjunction with the field notes and made written analytic memos specifying initial ideas and emerging codes regarding their music utilization and constructive activities. I then identified questions about their music utilization and constructive activities that remained unclear from these three data sources and when needed, altered my interview questions accordingly to address these questions or further explore initial ideas or themes arising from the data.

The second phase of data analysis occurred after the conclusion of the semi-structured interviews. During this phase, I compiled data from each of the four sources (student work, video, field notes, interview) for each of the eight cases. Following the compilation of data sources, both a within-case and across-case analysis was conducted to answer each research question. Table 6 shows the relationship between these research questions and the associated data sources and analysis procedures.

Table 6

Research Questions with Data Source and Analysis Technique

Research Sub-Question	Data Source	Within-case Analysis	Cross-Case Analysis
(1) How do students make use of a music context while completing a uni-directional, bi-directional, multi-directional, and abstract translations?	Student work samples Video recordings Field Notes Follow-up interviews	Open qualitative coding and theming (Saldaña, 2021), using Fonger's (2011) categories.	Coordination of themes across the eight cases to form generalizations of music utilization categorizations (Creswell & Poth, 2016).
(2) How does students' use of a music context while completing translations differ according to the representation types involved?	Student work samples Video recordings Field Notes Follow-up interviews	Open qualitative coding and theming (Saldaña, 2021), using NCTM's (2014) five representation types	Coordination of themes across the eight cases to form generalizations of music utilization related to NCTM's (2014) representation types (Creswell & Poth, 2016).
(3) What constructive activities do students engage in when	Student work samples Video recordings Field Notes	Open qualitative coding and theming (Saldaña, 2021),	Coordination of themes across the eight cases to form

translating to and from a contextual representation?	Follow-up interviews	informed by Bossé et al. (2011)	generalizations of students' constructive activities (Creswell & Poth, 2016).

Within-Case Analysis

To answer the research sub-questions and understand the various categorizations of students' utilization of the music context while engaging in different translation types among the five representation types, I first coded the student work samples, the video recordings, field notes, and the follow-up interviews using process codes (Miles et al., 2014). This coding is appropriate for observable actions, which in the case of this study are the students' use of music context. Because there is no existing set of a priori codes to capture the process of students' utilization of the music context, I used open coding (Saldaña, 2021).

The unit of analysis (Miles et al., 2014) for coding student work in conjunction with the video data and follow-up interviews was a single problem from the student task sheet. I used the following procedures to examine/code each answer written by a student:

(1) examined the students' written answer

(2) examined the video data and field notes data associated with that answer,

(3) examined any interview transcription associated with that answer,

(4) assigned codes (from Fonger (2011)) based on the types of translations used

(5) assigned codes (from NCTM (2014)) based on the types of representations used,

(6) developed codes based on the utilization of music, and

(7) developed codes based on any constructive activities students engaged in (if a contextual representation was involved).

Thus, a single problem from the student task sheet received multiple codes. Table 7 shows an example of coding for two task problems for one pair in this study. The complete tables of codes for all task problems and all eight pairs can be found in Appendix E.

Table 7

Example Table of Codes for Two Task problem for Aaron and Sam

Task Sheet	Problem	Translation Type (Fonger, 2011)	Representations (NCTM, 2014)	Or Connected Representations	Music Utilization (Inductive)	Constructive Activities (Bossé et al. 2011) and Inductive
1	5	multi-directional (M)	physical (P) to contextual (C) to verbal (V)	N/A	contextualized answer (CA)	articulating
1	6	uni-directional	physical to symbolic	N/A	none (N)	N/A

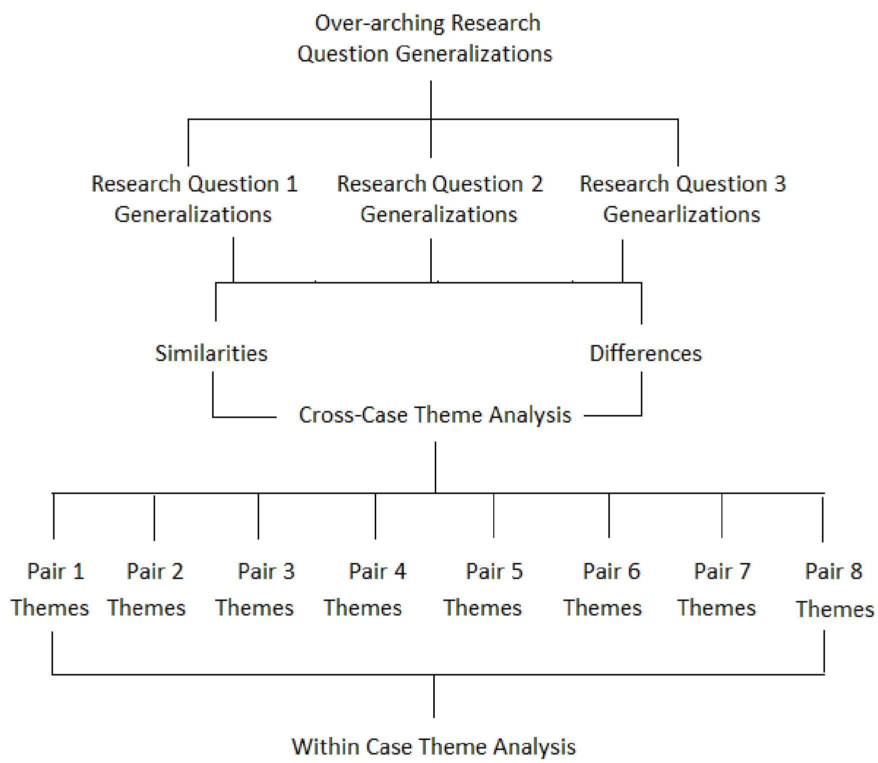
These codes were then used to develop larger themes for each pair of students in relation to each research sub-question. The codes for each sub-question were organized into categories and assigned themes according to generalizations made across codes. These themes represent "broad units of information that consist of several codes aggregated to form a common idea" (Creswell, 2013, p. 186). For example, during analysis, I observed that music context engaged students in the mathematics activities in a variety of ways, such as holding their attention while they persevered through a task problem or giving them a sense of enjoyment during a task problem. While these forms of engagement were different, they could be categorized into a single theme labeled as *engagement*. After themes were generated within all eight cases, a cross-case analysis was conducted.

Cross-Case Analysis

Following completion of the within-case coding and theming, a cross-case analysis was conducted across all eight pairs of students. Examining similarities and differences across the themes from the eight pairs allowed me to develop generalizations in the categories of music utilization across all pairs (Creswell & Poth, 2016; Miles et al., 2014). Hlava and Elfers (2014) describe this process as "identifying experiences common to all participants" (p. 438). In general, I followed the coding procedures for multiple-case studies I adapted from Creswell and Poth (2016) shown in Figure 6. The resulting generalizations from across all eight pairs was then used to develop emerging relationships between the theoretical constructs of music, translations, and representations, further informing the original conceptual framework shown in Figure 4.

Figure 6

Multiple-Case Study Coding Procedures



Note. Adapted from Creswell and Poth's "Template for Coding a Multiple-Case Study" (2016).

Validity and Reliability

There are four main criteria used to assess the rigor of qualitative case studies, namely "internal validity, construct validity, external validity, and reliability" (Gibbert et al., 2008, p. 1466). To address internal and construct validity and reliability, I engaged in two measures. First, I aimed to be as explicit as possible about methods for developing codes, providing necessary detail in their development and using existing frameworks to assist in coding (Meyer, 2001). Second, after I developed the initial set of internal codes for the types of music utilization, I provided a fellow peer researcher with the list of music utilization codes and assigned her roughly 12.5% of data to recode (approximately 4.5 hours of data and 3 work samples). We then conversed about the meaning and use of each code until there was at least an 85% match between coders for each code and inter-rater reliability was achieved (Cho, 2008). Qualitative research, particularly case studies, are not generalizable across large populations (Ali & Yusof, 2011). However, the use of eight cases across four data sites and in five classes (because Mr. J's pairs were in two separate classes) strengthened the external validity of the theoretical results produced from this study. In addition, using triangulation methods to coordinate between the four sources strengthened the external validity of this study (Jonsen & Jehn, 2009). Together, these measures strengthened the validity and reliability of this study.

Researcher Positionality

I acknowledge my own contributions as a researcher in this study. Although I acted as an observer and not a participant during the data collection phase of this study, my experience as a high school teacher for nine years has inevitably influenced my interest in and perceptions of music and translations in the mathematics classroom. The lesson plans and student task sheets developed for this study are largely influenced by my own experiences implementing music activities in my high school algebra and pre-calculus classes for many years. Consequently, I must acknowledge the possibility that these experiences had an unconscious influence on data analysis.

CHAPTER IV

RESULTS

The purpose of this study was to describe and categorize the role that music context plays in high school students' translations among representations while engaging in music-contextualized algebra instruction. To accomplish this purpose, I collected data to answer the following main research question and sub-questions:

What role does music context play in high school students' translations among mathematical representations while engaging in music-contextualized algebra instruction that emphasizes translations?

> (1) How do students make use of music context while completing uni-directional, bi-directional, multi-directional, and abstract translations?
>
> (2) How does students' use of music context while completing translations differ according to the representation types involved?
>
> (3) What constructive activities do students engage in when translating to and from a contextual representation?

This chapter presents the results from data collection. First, within-case results are presented, specifically highlighting the overarching themes that emerged for each of the eight pairs of students. The final section connects themes from across cases, including similarities and differences across cases, specifically in relation to each research sub-question.

Within Case Results

The following subsections describe the themes and patterns observed within each of the eight cases regarding the ways students utilized the music context as they engaged in various translations involving all representation types. The themes presented are not necessarily the only codes that were applied to the case during analysis, nor codes that were exclusive to only that case. Instead, they are themes that were especially prevalent and/or best exemplified in the data for the case, allowing all the themes that emerged during qualitative analysis in this study to be introduced. Prior to a discussion of themes, each subsection first begins with a general description of the student pairs gleaned from observations. I observed a total of 12 classroom lessons (3 for each of the 4 teachers) and observed two pairs of students in each teachers' class. Table 8 summarizes the number of each type of translation completed by the pairs. There were 12 different themes developed from within-case analysis. These themes are shown in Table 9, along with the abbreviated code associated with that theme, a description of the theme, and an example of the theme from this study. Each theme will be discussed in further detail in the following subsections. The eight subsections are ordered according to the order they were observed, which was as follows: Mr. J and his two student pairs (1) Vincent and Ethan, and (2) Cameron and Eric; Ms. K and her two student pairs (3) Elise and Logan, and (4) Rachel and Rebecca; Ms. S and her two student pairs (5) Aaron and Sam, and (6) Mitch and Paul; and Ms. W and her two student pairs (7) Emily and Amber and (8) Jason and Gabe.

Table 8

The Total Number of Translations Completed by Each Pair of Students

	Uni-Directional Translations	Bi-Directional Connections	Multi-Directional Translations	Abstract Connections
Vincent Ethan	18	5	6	1
Cameron Eric	16	6	7	1
Elise Logan	12	2	5	0
Rachel Rebecca	13	2	9	0
Aaron Sam	13	5	9	1
Mitch Paul	12	5	13	1
Emily Amber	17	5	8	1
Jason Gabe	16	5	9	1

Table 9

Descriptions and Examples of Twelve Identified Ways Students Used Music

Theme	Code	Description	Example
None	N	Student makes no use of music context in their mathematics.	A task problem prompts students to determine whether a data set is discrete or continuous. A student writes that the data is continuous because "the points continue on forever."
Engage	E	Student is entertained or engaged by the music context.	A task problem prompts students to find the ratio between the lengths of two PVC pipes. A student exclaims "I was playing with the pipes and I got really entertained."
Label Construction	LC	Student uses music context to label their mathematics during construction.	A task problem prompts students to make a table of all the ratios between the lengths of PVC pipes. While completing the table, a student uses their calculator to divide two numbers and says, "C4 divided by C5 is…"
Label Answer	LA	Student uses music context to label their mathematics answer.	A task problem requires students to find the multiplicative changes in frequency between C_1 and C_8 in two different ways. A student uses the terms "octave" and "fifths" to label their two answers, without indicating they understand how each number relates to octaves and fifths.
Contextualized Answer with Prompt	CA	Student's constructed target representation is contextualized with music after being prompted by their teacher during class.	A task problem prompts students to find the common ratio between consecutive values in their graph or table. A student at first writes only 1.059. When asked later by the teacher what 1.059 represents, the student replies that

			it is the "multiplicative change between frequencies."
Contextualized Answer without Prompt	CA	Student's constructed target representation is contextualized with music without a prompt from their teacher.	A task problem prompts students to determine whether a data set is discrete or continuous. A student says, "It is discrete because there are only so many notes on a piano."
Contextualized Answer during Interview	CA	Student's constructed target representation is contextualized with music only during the follow-up interview.	A task problem requires students to write out the exponents representing fifth and octave intervals from C_1 to C_8 . The student only labels their answer using music but does not explain during class how they understand their work in the context of music. Later in the interview they explain their answer in the context of music using the words notes, frequencies, pitch, etc.
Contextualized Answer during Construction	CAC	Student contextualizes their answer during the construction process.	A task problem requires students to find the frequency of G_4 given C4. A student recognizes before finding the answer that the frequency of G_4 will be higher than that of C4.
Contextualized Incorrect Answer	CIA	Student contextualizes an incorrect mathematics answer.	A task problem requires students to determine if a graph is discrete or continuous. A student incorrectly determined that the data is continuous because "musical frequencies can go on forever, getting higher and higher."
Contextualized Connection	CC	Student incorporates the music context into their bi-directional or abstract connection across representations.	A task problem requires students to makes general interpretations across all representations about the relationship between music and mathematics. A student explains that as long as the frequency of Middle C is known, the hertz of any frequency can be found.
Contextualized Reevaluation	CR	Student reevaluates their mathematics because of their understanding of the music context.	A task problem prompts students to decide what type of relationship exists between data points. A student first decides it is quadratic because a quadratic regression fits fairly well. The student then changes their mind and decides it's exponential because musical frequencies wouldn't "go back up" like a quadratic.
Past Experience	PE	Student draws on past experiences with music outside of class while working on mathematics.	A task problem requires students to listen to a series of musical scales and respond with their thoughts. One student comments that a scale sounds like the "blues scale," something recalled from previous music experience outside of class.

Mr. J's Pairs of Students

The two pairs of participants in Mr. J's two Secondary II mathematics classes remained actively engaged in mathematical discussions throughout the three lessons.

Vincent and Ethan worked independently most of the time and did not request help while working. However, on three occasions, Mr. J noticed incorrect mathematics on their task sheets or that Vincent and Ethan had misunderstood what a problem was asking and stepped in to help with scaffolding questions. Cameron and Eric worked mostly independently but requested help from Mr. J on two to three problems on each task because they were unsure how to proceed. Mr. J was diligent in following the provided lesson slides, sending the class into the “explore” phase to complete the problems in partners before bringing them back together for a whole-class discussion. Only one of the 31 task problems was completed during whole-class discussion rather than in partner discussion. Consequently, 30 of 31 task problems were coded for these two pairs.

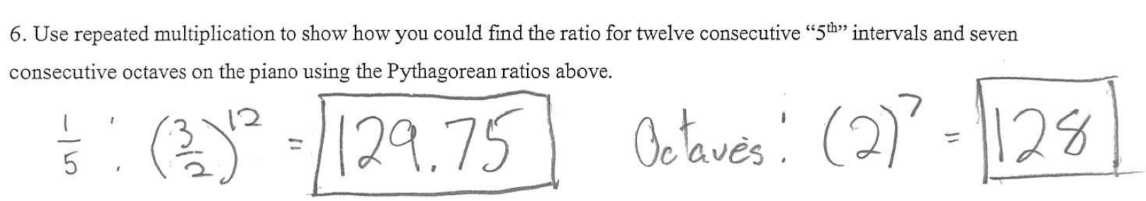
The Case of Vincent and Ethan

Vincent and Ethan remained actively engaged in discussions throughout Mr. J’s three lessons, though Ethan did a majority of the talking and appeared more confident than Vincent, usually making the final say on what answer they would write. Vincent and Ethan completed all 31 task problems, 30 of which were completed in partner discussions and coded. A complete table of codes applied to Vincent and Ethan’s task problems can be found in Appendix E. In short, the pair carried out 18 uni-directional translations, 5 bi-directional connections, 6 multi-directional translations, and 1 abstract connection. Two themes were exemplified by Vincent and Ethan’s work as they completed these varying types of translations, namely (1) using music to *label answers* (LA), and (2) *contextualizing answers* (CA) with music when explicitly prompted.

Using Music to Label Mathematics. A theme that arose from analysis of Vincent and Ethan’s data was using a musical component as a *label* for their mathematics answers, such as a note or other music theory term. This theme, label answers or LA, was evident in Vincent and Ethan's work on Task 1 Problem 7. This task problem required students to calculate the multiplicative change in frequency from C_1 to C_8 on the piano using intervals of fifths and octaves. The pair completed a multi-directional translation from a contextual representation (the given problem) to a symbolic representation (exponent expressions) to another symbolic representation (simplified number). In their final answer, they labeled $\left(\frac{3}{2}\right)^{12}$ and 2^7 as fifths and octaves, respectively (see Figure 7). Although they referenced a musical component, their conversation did not include a contextual interpretation of the meaning of the 129.73 and 128, which are the multiplicative changes in frequency from C_1 to C_8. This was confirmed in the follow-up interview with Vincent and Ethan, where they again emphasized that the numbers represented the "fifths and the octaves," without making mention of hertz, frequency, or the notes C_1 or C_8. Thus, the terms fifths and octaves, given to them by the teacher during their field trip to the piano, served merely as language for labeling their answers.

Figure 7

Ethan's Work on Task 1 Problem 7

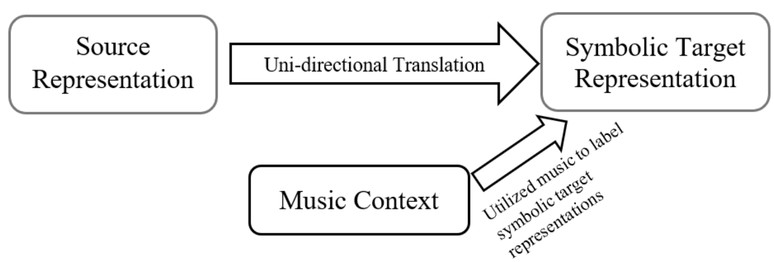

6. Use repeated multiplication to show how you could find the ratio for twelve consecutive "5th" intervals and seven consecutive octaves on the piano using the Pythagorean ratios above.

$\frac{1}{5}$: $\left(\frac{3}{2}\right)^{12}$ = 129.75 Octaves: $(2)^7$ = 128

The code LA was developed to capture instances similar to these where students used components of the music context as labels, but where there was no clear evidence that students truly understood the meaning of their answers within music theory. Thus, just because students used a music label, they were not necessarily drawing from music concepts when representing their solutions for these problems. As illustrated in Figure 8, this theme was most often present in Vincent and Ethan's uni-directional translations when they were translating to symbolic target representations.

Figure 8

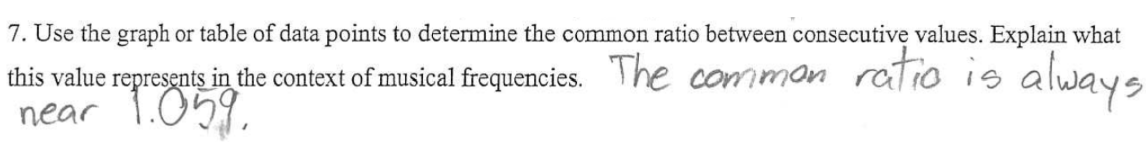
Illustration of Vincent and Ethan's LA Theme.



Contextualizing Answers when Prompted. Another theme that emerged in analysis of Vincent and Ethan's work was contextualizing an answer in music. For Vincent and Ethan, these target representations were most often numerical answers that they had calculated, and when prompted by a task problem or by Mr. J, they could interpret that number in the context of music. For example, this CA theme was present in Vincent and Ethan's work on Task 2 Problem 7. The full text for this task problem can be found in Figure 9 below. The students first worked for several minutes determining how to calculate the common ratio between consecutive values. This source representation provided by the task gave no contextual information. The words "consecutive values" were chosen purposefully for this problem instead of "consecutive frequencies" to give participants opportunities to make connections to the music context and thus translate to their own contextual representations without explicit prompts from the task. At first Vincent and Ethan's interpretation of their symbolic target representation seemed entirely mathematical, as shown in their work in Figure 9. They write the words "common ratio" and then provide the numerical answer 1.059, all free of music context. However, a short exchange between the pair and their teacher regarding their answer revealed they were capable of interpreting the symbolic representation 1.059 contextually. When Mr. J asked, "What was the 1.059?" Vincent replied, "It was the… it's like the change in hertz." Thus, Vincent and Ethan translated from their symbolic representation to a meaningful contextual and then verbal representation as they articulated their understanding of 1.059 to their teacher. The code CA was developed to capture instances like this where the students evidenced understanding of their target representation in connection to music and/or music theory.

Figure 9

Ethan's Work on Task 2 Problem 7

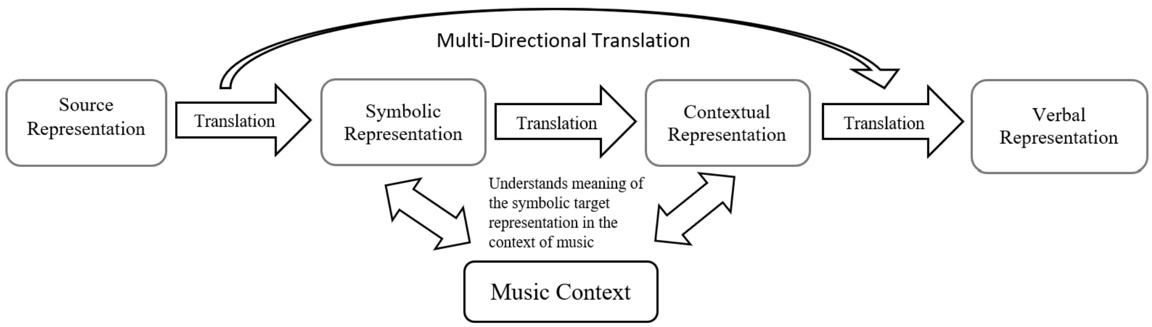


7. Use the graph or table of data points to determine the common ratio between consecutive values. Explain what this value represents in the context of musical frequencies. The common ratio is always near 1.059.

Often the code CA was applied to instances where students translated from a symbolic target representation, like a number, to a contextual representation, and then verbal representation as they articulated their contextual understanding. This is illustrated in Figure 10.

Figure 10

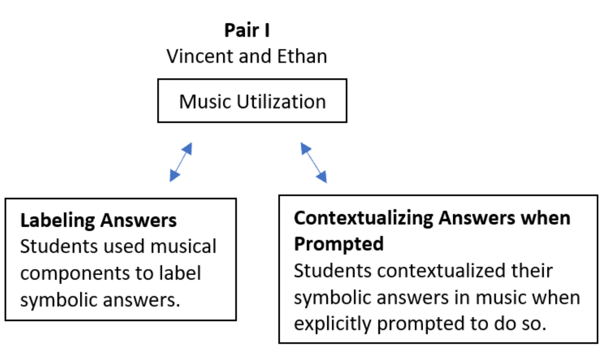
Illustration of Vincent and Ethan's CA Code



In summary, two themes that stood out in the qualitative analysis of Vincent and Ethan's task sheets, video data, and interview transcripts was their propensity for using the music context to label their mathematics as well as attributing the music context to their answer when prompted. These themes are illustrated in Figure 11. These interpretations were more surface level understanding of the music theory and were only present when their teacher or the interviewer explicitly asked Vincent and Ethan to make connections to music theory.

Figure 11

Within-case Themes for Vincent and Ethan



The Case of Cameron and Eric

Cameron and Eric contributed equally to the discussions, though Eric made the final say on what answer the pair would write. Cameron and Eric completed all 31 task problems in their pair discussions, 30 of which were discussed in partners and coded. A complete table of codes of Cameron and Eric's representations and music utilizations can be found in Appendix E. In short, the pair carried out 16 uni-directional translations, 6 bi-directional connections, 7 multi-directional translations, and 1 abstract connection. Two themes surfaced from within-case analysis of Cameron and Eric's work: (1) contextualizing answers during constructive activity (CAC), and (2) contextualizing bi-directional connections (CC).

Contextualizing Answers During Constructive Activity. An interesting theme that emerged in Cameron and Eric's data was their ability to contextualize their answers while in the process of constructing (CAC). Consider this excerpt from Cameron and Eric's conversation around Task 1 Problem 3, which read, "The song 'Twinkle Twinkle Little Star' begins with the interval formed by playing C4 and then G_4. If the frequency of C4 is ~261.6, use the ratios to find the frequency of G_4." In this task, the students were given a contextual source representation and asked to translate to a symbolic representation. The students engaged in a constructive activity of modeling the contextual representation, extrapolating important information from the source, such as the word "ratio" and "frequency" and then mapping that information to associated numbers.

Eric: I think we could use the ratios. 'Cause the lengths can determine the speed. The frequency.

Cameron: Okay. Wait, so, oh yeah so it says use the ratios to find the frequency. So how could we use the ratios? So C4 is 1:1. What is G_4?

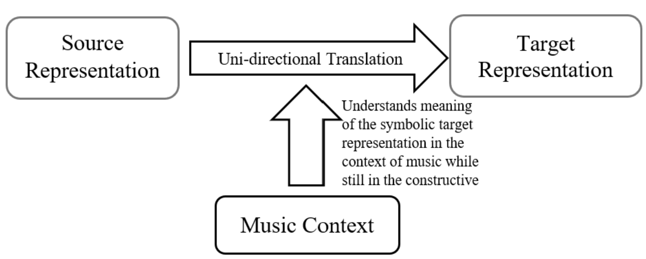
Eric: 3 over 2.

Cameron: So that's just over 1, to 1.
Eric: Well, I guess it would be a bit larger. The frequency.
Cameron: Yeah, it would be larger. But how would we find it?

As they worked, it was clear from their conversation that Cameron and Eric had a strong understanding of their mathematics in the context of musical frequencies. For example, this was seen when Eric pointed out that the number they were trying to find would be larger because of the 3:2 relationship. Even before finding the exact value, Eric mentioned the "frequency," signifying he understands that the desired target representation is a musical frequency. This except showed evidence of the code CAC because the final target representation was contextualized while in the process of constructing it. This theme was common in Cameron and Eric's translations from contextual representations to other representation types. These translations were both uni-directional and multi-directional. Figure 12 illustrates this theme in Cameron's and Eric's uni-directional translations.

Figure 12

Illustration of Cameron and Eric's CAC Theme



Contextualizing Bi-Directional Connections. A second prominent theme in Cameron and Eric's work was their ability to incorporate the music context into their bi-directional connections between two representation types. Fonger (2011) described one type of bi-directional connection as students' ability to use two representations to confirm an approach or check a solution. Later, Fonger and Altindis (2019) described a bi-directional connection as a students' ability to "recognize invariant features across the two representation types" (p. 4). Thus, a bi-directional connection differs from a bi-directional translation, which is two uni-directional translations back and forth between two representation types. Some task problems were purposefully designed to elicit bi-directional connections from students by explicitly asking students to use one representation type to verify another type or recognize invariant features across two representation types. For example, Task 2 Problems 10 and 11 required students to verify that their exponential equation for musical frequencies correctly produced frequencies listed in the given table of frequencies and identify features common in both their equation and table. As Cameron and Eric worked on Task 2 Problems 10 and 11, they successfully conducted bi-directional connections. As shown in Cameron's work in Figure 13, the pair decided to test input values 1 and 2, where 1 and 2 are exponents inserted into their exponential equation $y = 246.9089*1.0595^{x}$. When questioned further in the follow-up interview about their decision-making and this process, Cameron and Eric confirmed that they used their table to verify values in the equation:

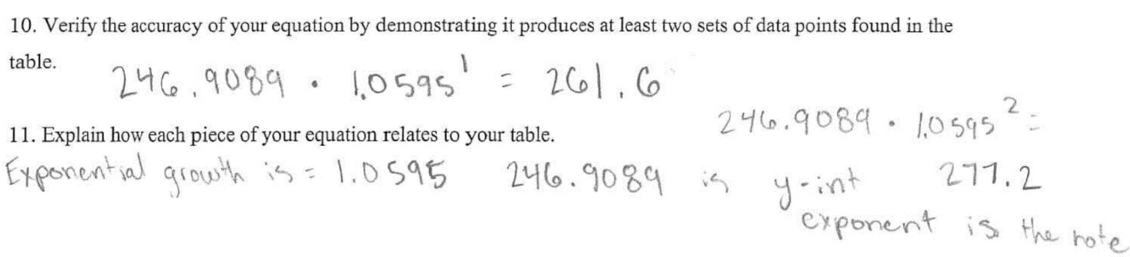
Interviewer:	On the question where you had to test your equation to make sure it works and you had to try two different points, once you plugged in a number and got an output, how did you know that it worked?
Eric:	Because we had a formula that we used, and we plugged in the numbers and got the correct output every time.
Interviewer:	And how did you know that the correct output was the correct output? Like once you got 261.62, how did you know that was right?
Eric:	Because like if we go back to C4 and we put C4 into the equation…

Cameron:	Yeah, C4 was 1 and C4# was two and we kind of just went up
Eric:	And when that got us the right answer on multiple occasions, we thought it was the right formula.

In this exchange, Cameron and Eric reference C4 and C4#, illustrating their understanding of the inputs of their equation as musical notes on the piano. Although the task problem did not specifically call for contextual applications, during the interview, the pair nevertheless drew from the music context in their explanation for verifying their equation.

Figure 13

Cameron's Work on Task 2 Problems 10 and 11


10. Verify the accuracy of your equation by demonstrating it produces at least two sets of data points found in the table.

$246.9089 \cdot 1.0595^1 = 261.6$

$246.9089 \cdot 1.0595^2 =$ 277.2

11. Explain how each piece of your equation relates to your table.

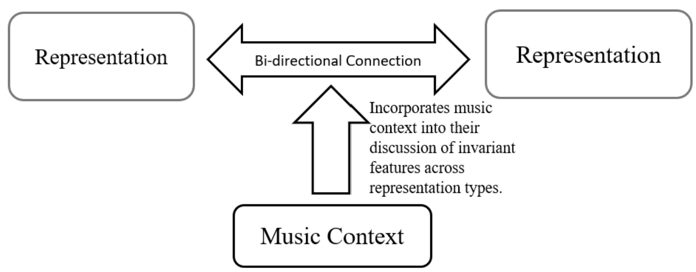
Exponential growth is = 1.0595 246.9089 is y-int

exponent is the note

Similarly, in Problem 11, as the pair of students made connections between their equation and table, they show that they understand that the exponent in their equation indicated the note that was listed in the table in their final statement "exponent is the note." The code *contextualized connection* (CC) was applied in similar instances where students made bi-directional connections across two representations in ways that utilized the music context. These were similar to contextualized answers (CA); however, no particular target representation was contextualized but rather the overall connection between both connected representations. This theme is illustrated in Figure 14.

Figure 14

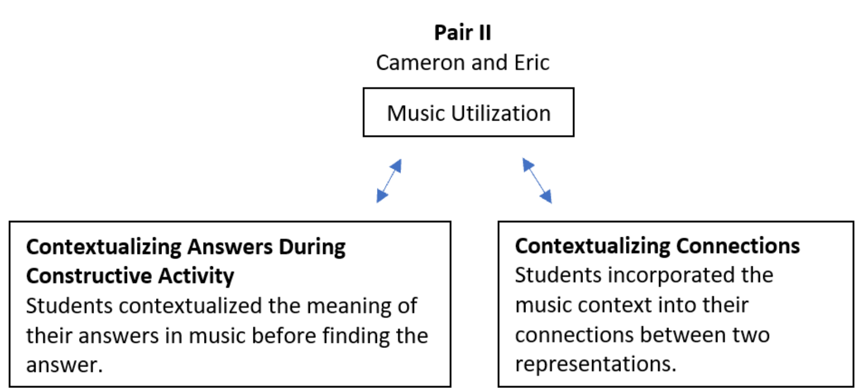
Illustration of Cameron and Eric's CC Theme



In summary, Cameron and Eric made musical interpretations of target representations while constructing them. This suggested it was possible for participants to contextualize an answer while still on the process of constructing it. They also made connections to the music context while making bi-directional connections. These themes are illustrated in Figure 15. This meant that students could contextualize not only an answer but could also understand the invariant features they found across multiple representations in the connection to music theory.

Figure 15

Within-Case Themes for Cameron and Eric



Ms. K's Pairs of Students

The two pairs of students in Ms. K's remedial Secondary I Mathematics class had a difficult time accessing the mathematics in the three tasks. It became clear throughout the lessons that a large portion of the mathematics were outside the zone of proximal development (Vygotsky, 1978) for these students given the topics they had covered in their class and in pervious school years. Two of the task problems were completed during a whole-class discussion and could not be coded. The second half of the second task sheet proved too difficult for the pairs and was not completed in class. However, in the follow-up interview, Rachel and Rebecca were able to discuss some of these problems in a way that could be coded. Despite the difficulty of the task problems, both pairs remained actively engaged in mathematical discussions throughout the lessons. They requested teacher intervention on nearly every problem and were hesitant to work independently without reassurance from their teacher. Ms. K utilized the lesson slides, but rather than large "explore" portions where students worked in partners on many consecutive problems, she preferred to frequently bring the class together for whole-class discussions. A larger portion of class time was spent in whole class discussions facilitated by the teacher than in other data sites.

The Case of Elise and Logan

Elise and Logan spoke equally, though Elise appeared more confident and had the final say on the answers they wrote. Elise and Logan completed 20 of the task problems in their pair discussions. A complete table of codes applied to Elise and Logan's task problems can be found in Appendix E. In short, the pair carried out 12 uni-directional translations, 2 bi-directional connections, 5 multi-directional translations, and no abstract connections. One theme was derived from qualitative analysis of Elise and Logan's work as they completed these varying types of translations: the ability to draw from their *past experiences* (PE) with music outside of class.

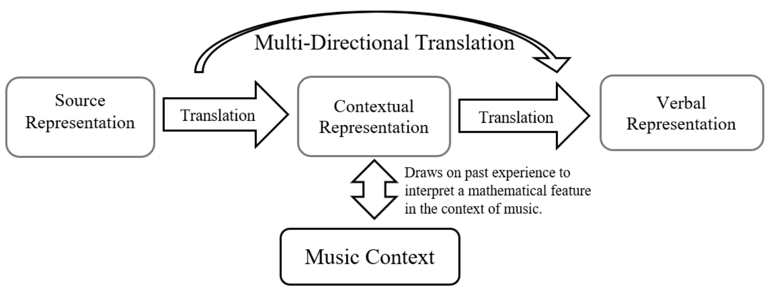
Drawing from Past Music Experience. Although Elise and Logan made few connections between the mathematics and music context overall, during one particular instance they connected the mathematics with their past experiences (PE) with music. Consider an excerpt from Task 2 Problem 5, which required students to interpret a physical space between data points on a graph in the context of music. In this problem, Elise and Logan translated from the given visual representation to a contextual representation and then verbal representation as they articulated their understanding of the space between the discrete data points on the graph of piano frequencies from C4 to C5. In this exchange, the pair grappled with trying to recall a word they remembered from their experiences with music outside of class. Elise later said she could not remember the word because it had been over a year since she had played the clarinet.

Logan:	I just put the time it takes you to get to the next note.
Elise:	I don't remember the phrase.
Logan:	You know what I'm talking about right?
Elise:	Yes! I know what you're talking about.
Ms. K	What do we think about the space in between the points?
Elise:	I think it's just a pause.

Logan and Elise attempted to explain the space by drawing on their previous understanding of music. This was seen in Logan's statement "the time it takes you to get to the next note" and Elise's reply "I don't remember the phrase." It is likely the pair was attempting to think of the word "rest" in music theory, though they were unable to recall the word, and consequently they settled on "pause" seen in Elise's last comment. Elise and Logan's interpretation of the space between data points was faulty, as it represents the frequencies that lie between the adjacent keys on a piano, and they interpreted this space as if the *x*-axis measured time. In any case, the code (PE) was applied to similar situations where students drew on their past experiences with music theory. This theme was present as Elise and Logan interpreted a mathematical feature in a graph using their previous knowledge, and in doing so constructed a new contextual representation. This is illustrated in Figure 16.

Figure 16

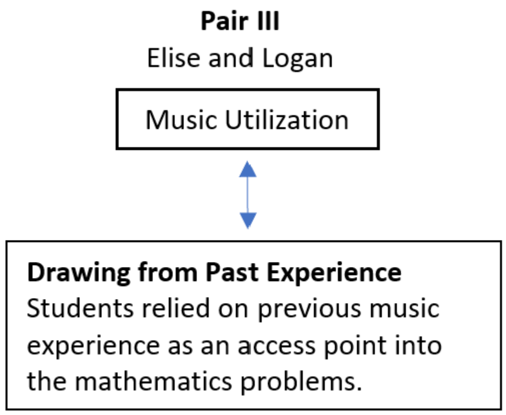
Illustration of Elise and Logan's PE Theme



Elise and Logan struggled through some of the mathematics in the three tasks, only completing 20 of the 31 task problems. While other codes were present in Elise and Logan's work, one notable theme found in their work was the ability to draw on previous experience with mathematics outside of class while working on task problems. This theme is illustrated in Figure 17.

Figure 17

Within-Case Theme for Elise and Logan



The Case of Rachel and Rebecca

Despite the difficulty of the mathematics, Rachel and Rebecca remained notably positive throughout the tasks, wanting to be successful and trying to make connections to music whenever possible. Even though both students were actively engaged in discussion, Rachel had more input on their final answer choice. These conversations were sometimes cut short by the whole class discussions. Rachel and Rebecca discussed 24 problems between their pair discussions and the follow-up interview. A complete table of codes applied to Rachel and Rebecca's work on the task problems can be found in Appendix E. In short, the pair carried out 13 uni-directional translations, 2 bi-directional connections, 9 multi-directional translations, and no abstract connections. Two themes arose from Rachel and Rebecca's data while they completed these varying types of translations: (1) their ability to *contextualize incorrect answers* (CIA) with music, and (2) their *engagement* (E) with the music as they worked.

Contextualizing Incorrect Mathematics Answers. A theme in Rachel and Rebecca's data was the use of music context even when either the mathematics or the musical interpretation was not entirely accurate. For example, in completing Task 2 Problem 7, which requires the students to find the common multiplier in the geometric series of musical frequencies, Rachel and Rebecca incorrectly found the additive changes between each pair of consecutive frequencies. Some of their work can be seen in Figure 18. In addition to checking the first two notes C4 and C#4 (261.6 hz and 277.2 hz), the students also checked the next pair C#4 and D_4 (277.2 and 293.7) and noticed that the results were slightly different.

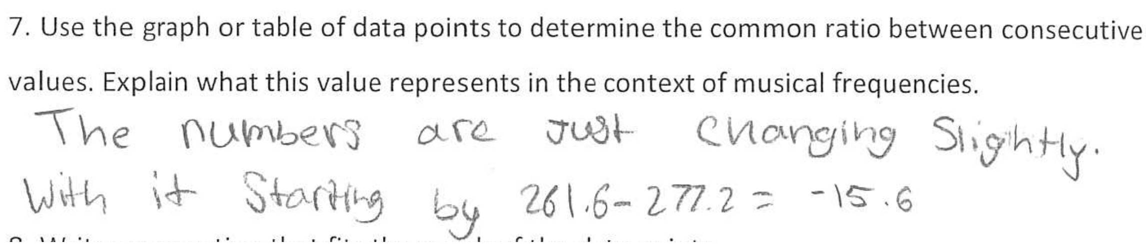
> Rachel: So, it says determine the common ratio. Couldn't the ratio be slowly and slightly… 'cause I said the numbers are just changing slightly. 'Cause with it starting with 261.6-277.2 is 15.6. And then you take the 277.2 and you subtract it by the next one, it goes down to -16.5.

When questioned about this work later in our interview, Rachel referenced the notes C4 and C#4, and tried to make sense of her findings in terms of music.

> Rachel: I remember exactly what I was doing. I was taking the first key, like what's that called… Okay, basically I can see the graph chart because [Ms. K] got it out for us. I took C4 and $C4^{\#}$ and I subtract those numbers below. And I ended up with that -15.6. So, I took the next one and I took the C4 sharp and the D_4 and I subtracted that. And I went all the way to the end, and I got almost 15.6 every time.
>
> Interviewer: Okay, perfect. Yeah, and when you kept going and you do subtract like the next two and the next two, you do actually get a different change each time. So, what do you think that means?
>
> Rachel: That it could just be like changed because of the sound, like in order with the tuning you have to have the right same exact numbers, but if you have the same exact numbers, it's the same exact key, it's not different.

Figure 18

Rachel's Work on Task 2 Problem 7

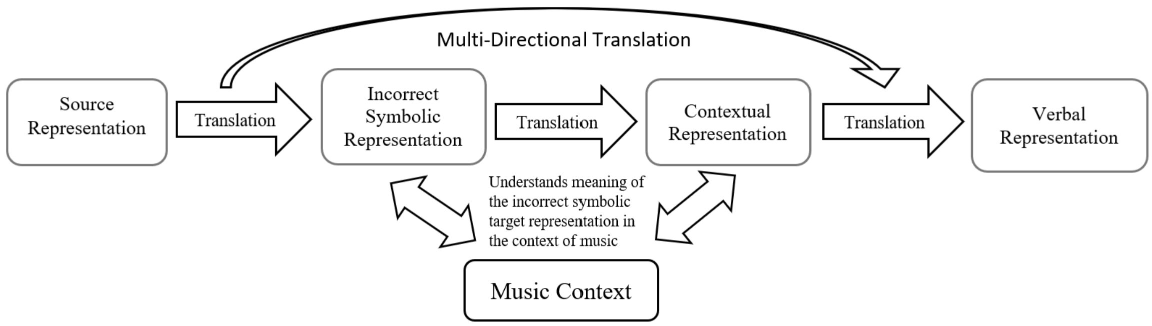

7. Use the graph or table of data points to determine the common ratio between consecutive values. Explain what this value represents in the context of musical frequencies.

The numbers are just changing slightly. With it starting by 261.6 − 277.2 = −15.6

While completing this translation from the visual representation of the graph to the symbolic representation of the common multiplier, Rachel continuously tried to make sense of their mathematical findings within the context of music. As she articulates her work, she translated to a contextual representation and then verbal representation, signifying a multi-directional translation was completed during this problem. She recognized the table as consecutive musical notes, specifically mentioning C4 and C4#, and also attempts to explain the varying answers using the terms "sound", "tuning," and "key." She did this although both the mathematical processes and their musical interpretation of their findings are somewhat flawed. To determine the common multiplier between musical frequencies, Rachel and Rebecca should have divided consecutive frequencies rather than subtracted. Rachel also incorrectly conflates the increasing rate of change with the difference in pitch of consecutive notes. This episode and other work from Rachel and Rebecca indicated that contextualized answers were present even in cases where the mathematical work was somewhat flawed. This theme sometimes found in Rachel and Rebecca's multi-directional translations, as shown in Figure 19.

Figure 19

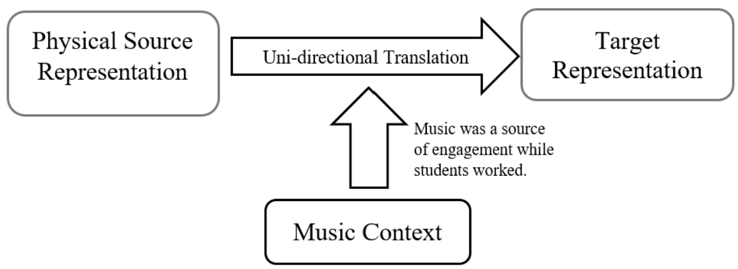
Illustration of Rachel and Rebecca's CA Theme



Engagement with Music. A second theme found in the work of Rachel and Rebecca was the way the music context provided a source of *engagement* (E) for them as they worked. For example, as the pair solved the first and second task problems, they kept the PVC pipe instrument in hand the entire time, tapping on them as they tried to figure out the relationship between the lengths of the pipes and the pitches they made. Video data revealed that as Rachel and Rebecca worked on these problems, they frequently had smiles on their faces, and on some occasions laughed. This evidenced the pair's enjoyment of working with the PVC pipes. In addition to physical examples of their engagement, verbal statements revealing their engagement with creating sounds on the pipes were made as well. While working on Task 1 Problem 1, Rachel laughed and said to Rebecca, "Sorry, I was playing with the pipes and I got really entertained!" Rachel had ignored a question from Rebecca regarding the task, and in this apology revealed that playing with the pipe instrument had captured her attention. The code engage (E) was applied to task problems like this where students either verbally or physically demonstrated a level of joy or engagement as they worked that could be attributed to a musical component of the lesson. As shown in Figure 20, this theme was present during Rachel and Rebecca's construction of representations, usually with physical source representations.

Figure 20

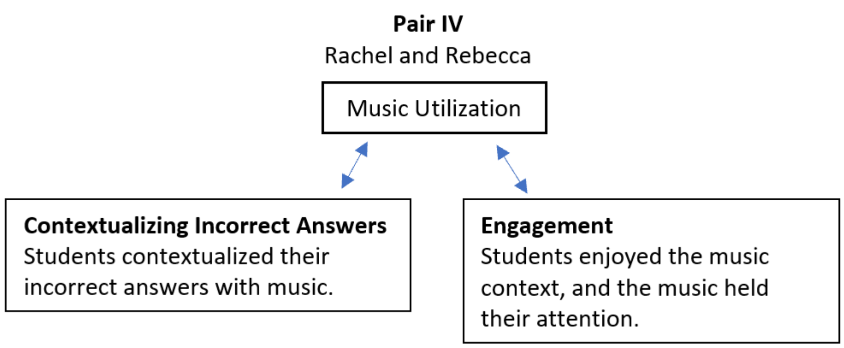
Illustration of Rachel and Rebecca's E Theme



In summary, Rachel and Rebecca had difficulties accessing some of the mathematics topics in the three lessons, which resulted in them constructing mathematically incorrect target representations contextualized with music. Despite this, Rachel and Rebeca still expressed physical and verbal enjoyment with the music context. Overall, this showed that students with limited knowledge of mathematical content being present in a lesson can still make use of and find enjoyment in a music context. These themes are illustrated in Figure 21.

Figure 21

Within-Case Theme for Rachel and Rebecca



Ms. S's Pairs of Students

The two pairs of students in Ms. S's Honors Secondary Mathematics III class were the most mathematically proficient in the content areas of the task worksheets out of all the pairs. They also spent the most minutes engaged in partner discussion. Aaron and Sam worked independently through all three tasks without teacher help, while Mitch and Paul got stuck on one problem in the third task. Ms. S helped them on Task 3 Problem 3 through a series of scaffolding questions. Ms. S used the lesson slides diligently, only bringing the class together for whole-class discussions once or twice each lesson.

The Case of Aaron and Sam

Aaron and Sam worked confidently and remained actively engaged in conversation throughout the lessons, though Sam did a majority of the talking and had the final say on most of their answers. Aaron and Sam completed 29 of the 31 task problems in their pair discussions, 28 of which were discussed by the pairs and coded. A complete table of codes applied to Aaron and Sam's task problems can be found in Appendix E. In short, the pair carried out 13 uni-directional translations, 5 bi-directional connections, 9 multi-directional translations, and 1 abstract connection. One major theme was derived from qualitative analysis of Aaron and Sam's work as they completed these varying types of translations: *contextualizing answers* (CA) without a prompt.

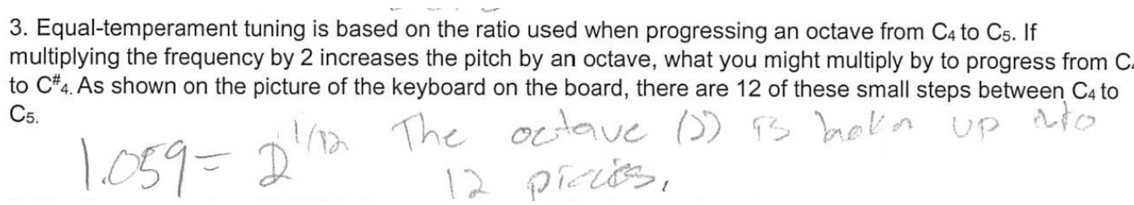
Contextualizing Answers Without a Prompt. A theme that emerged from the culmination of Aaron and Sam's data was their ability to explain the meaning of target representations in the context of music, without being asked by the teacher and with no prompt from within the task problem. After solving Task 3 Problem 3, Aaron interpreted the number $\sqrt[12]{2}$ they had found in the following way:

> The octave, 2, is broken up into 12… so getting from one octave to the next octave, you times the hertz by 2. And that whole octave has 12 half notes in it. I think that's how you can derive it.

Aaron and Sam's work for this problem can be seen in Figure 22. This problem provided students with a contextual representation and asked for a symbolic representation. Not only did Aaron and Sam complete that translation, but also then translated to a contextual representation and then verbal representation as they articulated the meaning of $\sqrt[12]{2}$ or $2^{1/12}$ in their verbal statement above. Their interpretation of this number is contextualized with their understanding of the multiplicative change in frequencies between octaves, as evidenced by their Aaron's verbal statement, "the octave, 2, is broken up into 12" and this same statement in their written work. The statement shows how their understanding of $2^{1/12}$ is intertwined with music context, especially in their choice to use the words "octave," "hertz," and "notes." This represented an example of the code contextualized answer (CA), where students contextualized their answer when it wasn't required by the task.

Figure 22

Aaron's Work on Task 3 Problem 3

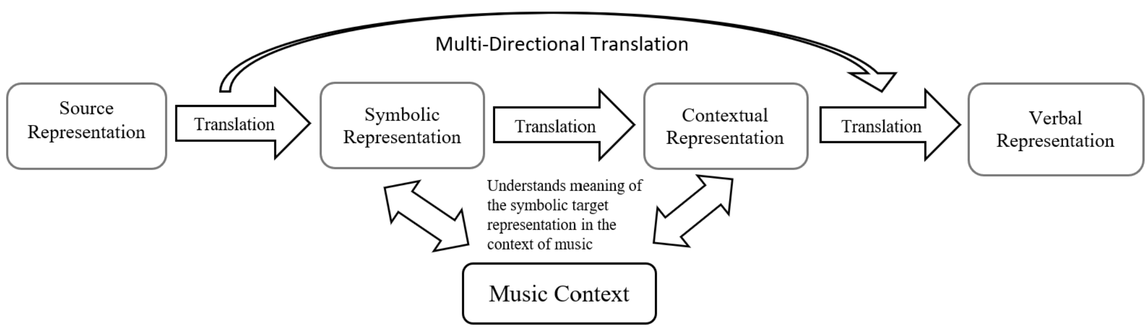

3. Equal-temperament tuning is based on the ratio used when progressing an octave from C_4 to C_5. If multiplying the frequency by 2 increases the pitch by an octave, what you might multiply by to progress from C_4 to $C^{\#}_4$. As shown on the picture of the keyboard on the board, there are 12 of these small steps between C_4 to C_5.

$1.059 = 2^{1/12}$ The octave (2) is broken up into 12 pieces.

This theme presented similarly to how it presented with Vincent and Ethan, though without a prompt. As shown in Figure 23, Aaron and Sam interpreted the meaning of a symbolic representation in the context of music, thus constructing a new contextual representation and overall completing a multi-directional translation.

Figure 23

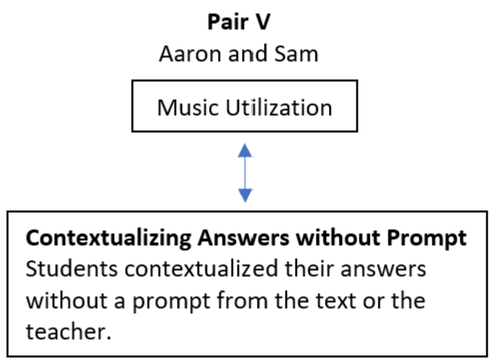
Illustration of Aaron and Sam's CA Theme



To summarize, in this problem and others, Aaron and Sam were able to make interpretations of their answers in the context of music theory. These were sometimes symbolic representations and were present in task problems that did not explicitly ask the students to explain the meaning of their answer in the context of music. This showed that Aaron and Sam aimed to find meaning in their mathematics and articulate those meanings within music on their task sheet even if it was not required. This theme is illustrated in Figure 24.

Figure 24

Within-Case Theme for Aaron and Sam



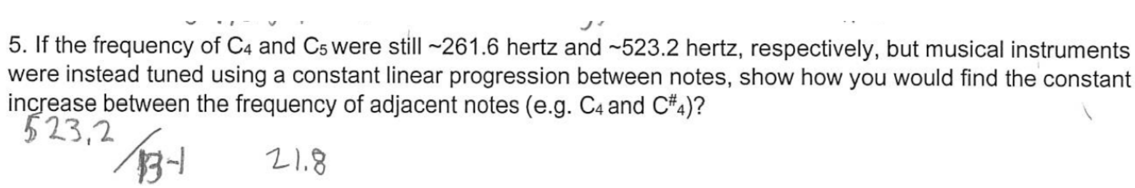
The Case of Mitch and Paul

Mitch and Paul were noticeably interested in music-contextualized mathematics throughout the lessons, often discussing connections between music and mathematics that extended beyond the scope of the lessons and continuing these discussions outside of class time. Mitch and Paul did equal amounts of talking and contributing to answers and engaged in the most minutes of partner discussion time of all eight pairs. Mitch and Paul completed all 31 task problems. A complete table of codes applied to Mitch and Paul's task problems can be found in Appendix E. In short, the pair carried out 12 uni-directional translations, 5 bi-direction connections, 13 multi-directional translations, and 1 abstract connection. Two themes were developed from qualitative analysis of Mitch and Paul's work as they completed these varying types of translations: (1) using music as a label during the construction (LC) of a target representation, and (2) conducting a *contextualized reevaluation* (CR) of the accuracy of their mathematics.

Using Music to Label Mathematics. One theme that manifested in Mitch and Paul's work was the use of a musical component not necessarily as a written label for an answer but instead as a verbal reference used to organize thoughts or procedures while carrying out mathematical operations. For example, this occurred as Mitch and Paul worked on Task 3 Problem 5, as shown in Figure 25. As Mitch worked on this problem, he said he needed to subtract "C4 from C5" and "divide by 12," and then did so on his calculator. Some of this work is seen in Figure 25, though Paul neglected to write the full *523.2-261.6* on the top of his fraction. During this constructive activity of modeling the given contextual representation as a symbolic representation, C4 and C5 were simply used as a reference point to verbally label the two numbers they needed to subtract. However, the students did not discuss the meaning of the resulting answer 21.8 in terms of the music context, which was the constant change in frequency between consecutive notes. As Mitch and Paul described their processes in the follow-up interview, they again did not interpret the meaning of 21.8, discussing it as the "difference between the numbers."

Figure 25

Paul's Work on Task 3 Problem 5

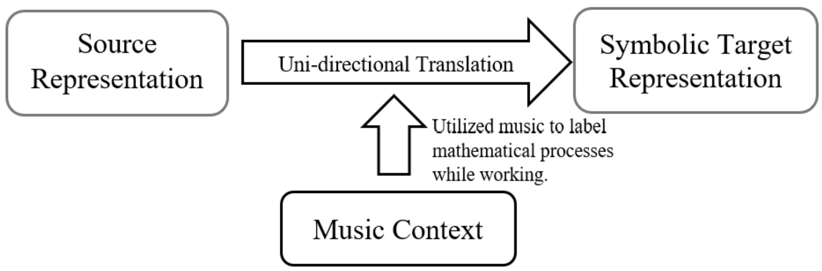

5. If the frequency of C_4 and C_5 were still ~261.6 hertz and ~523.2 hertz, respectively, but musical instruments were instead tuned using a constant linear progression between notes, show how you would find the constant increase between the frequency of adjacent notes (e.g. C_4 and $C^{\#}_4$)?

523.2 / 13-1 21.8

Consequently, the code contextualized answer (CA) was not present here, and the code *label construction* (LC) was extended to include instances like this where students verbally labeled mathematical components using a musical component as they worked. As shown in Figure 26, this theme was present across many source representation types, but always involved a target symbolic representation.

Figure 26

Illustration of Mitch and Paul's LC Theme



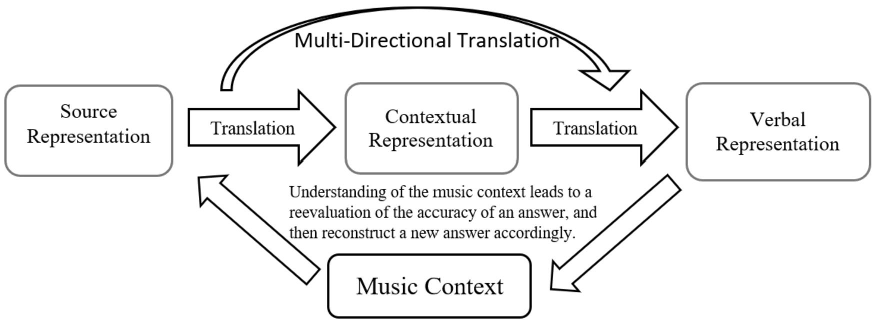
Reevaluating Mathematical Answers Because of Music. A second theme derived from Mitch and Paul's data was the use of music to reevaluate the accuracy of a mathematical answer. This theme was evident in Mitch and Paul's work on Task 1 Problem 3, which required students to find the frequency of G_4 given C4. Mitch first multiplied 261.6 (C4) by 2/3 in a calculator only to arrive at 174.4 hertz. He then commented to Paul that the frequency should have been higher because G_4 is a higher note. This caused him to reevaluate his mathematics, instead multiplying by 3/2 in his calculator to get the correct frequency for G_4. As Mitch completed this translation from the given contextual representation to a symbolic representation, the music context caused Mitch to reevaluate his initial mathematical answer. Later that lesson, as Mitch and Paul worked on Task 1 Problem 7, which required the students to write exponential expressions to represent the change in frequency from C_1 to C_8, Mitch said the following:

Mitch: So, the last C has 128 times more frequency than the first one I suppose. (*Types 3/2 to the 12^{th} power in calculator*). So that'd be 129.75 to 1. And that would just be 128 to 1. So, they're pretty close actually. Closer than I thought. But that's kind of weird that it's not exact, you know, cause they both eventually get to the same note. So, I'm wondering if I did something wrong. (*Redoes the operation in calculator*). But it doesn't look like I did.

From this comment it is clear that Mitch not only arrived at the correct two numbers 128 and 129.75 and recognized that it was strange they were different, but also correctly interpreted the numbers as the multiplicative difference in frequency between C_1 and C_8. This is evidenced by his choice to place 129.75 and 128 in ratios over 1, showing that he understands that C_8 will have 129.75 or 128 times the frequency of C_1. Later in this comment, he shows that his understanding of the experience he had with the piano where the fifths and octaves started and ended on the same notes caused him to second guess his mathematical answer and make sure it was correct. He says "I'm wondering if I did something wrong" after noticing his answers were "kind of weird" and tries the calculation again. The code contextualized reevaluation (CR) was applied to episodes like this where students understanding of music cause them to reevaluate the accuracy of an answer. This theme always involved contextual representation types during multi-directional translations, as shown in Figure 27.

Figure 27

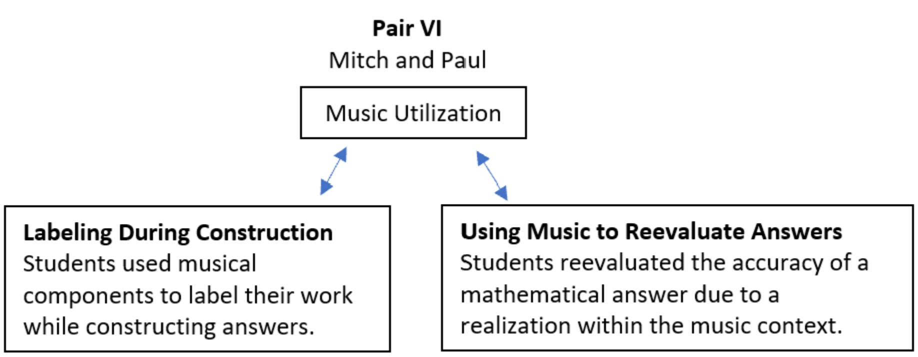
Illustration of Mitch and Pauls' CR Theme



In summary, Mitch and Paul used music components to verbally label mathematical operations they took as they worked. This meant that music could not only be used as a label for an answer but also as a way for students to label ideas and processes while they worked. Mitch and Paul used their knowledge of mathematics to reevaluate the correctness of their mathematics answers. Their understanding of music allowed them to see a flaw (or potential flaw) in their mathematics they might not otherwise have seen and make changes to (or gain confidence in) their answer accordingly. These themes are illustrated in Figure 28.

Figure 28

Within-Case Themes for Mitch and Paul



Ms. W's Two Pairs of Students

The two pairs of students in Ms. W's Secondary I Mathematics class engaged in discussions throughout all three lessons. They also utilized notebooks with mathematics notes from previous lessons in the year to look up vocabulary or confirm a mathematical process. The students worked confidently and quickly, engaging in the least number of minutes of partner discussion time out of the four data sites.

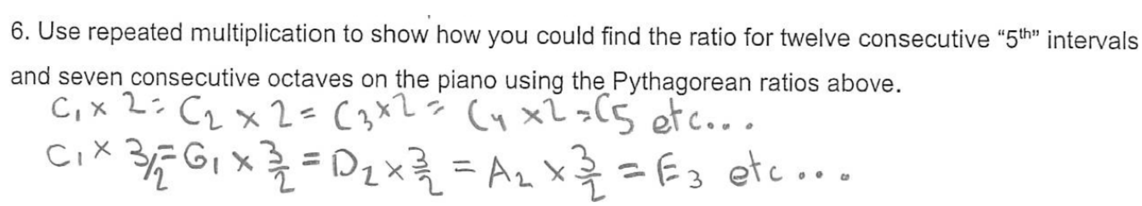
The Case of Emily and Amber

Emily and Amber did equal amounts of talking and contributing to answers. They completed all 31 task problems. A complete table of codes applied to Emily and Amber's task problems can be found in Appendix E. In short, the pair carried out 17 uni-directional translations, 5 bi-directional connections, 8 multi-directional translations, and 1 abstract connection. One theme that arose from qualitative analysis of Emily and Amber's work as they completed these varying types of translations was revealing their contextualized understanding of an answer (CA) only in the follow-up interview.

Contextualized Understanding of Answers in the Interview. A theme found in Emily and Amber's work during analysis was that their understanding of target representations in relation to the music context sometimes only became clear after the lessons in the follow-up interview. For example, in my field notes I noticed that Emily and Amber's way of writing out the progression of fifths and octaves from C_1 to C_8 on the piano was particularly interesting. As shown in Figure 29, the pair wrote the repeated multiplication with each note in the intervals embedded in their answer, thus modeling a given contextual source representation as a symbolic representation. Specifically they wrote $C_1 * 2 = C_2 * 2 = C_3$ etc. to capture the progression by octaves from the bottom of the piano to the top. Their written work and conversation did not reveal if they were using the musical notes solely to label their mathematics, or if they truly understood that each multiplication by 2 represented a multiplicative change in the frequency. However, in the follow-up interview when asked about this problem, the students made frequent mentions of the words "key," "keyboard," and "frequency" while explaining their work. In doing so, they showed that their understanding of the repeated multiplication (a symbolic target representation) was contextualized with music and the multiplicative changes in frequency while progressing up the piano. The code contextualized answer (CA) was still used to capture these instances where students revealed their contextual understanding in only the follow-up interview.

Figure 29

Emily's Work on Task 1 Problem 6


6. Use repeated multiplication to show how you could find the ratio for twelve consecutive "5th" intervals and seven consecutive octaves on the piano using the Pythagorean ratios above.

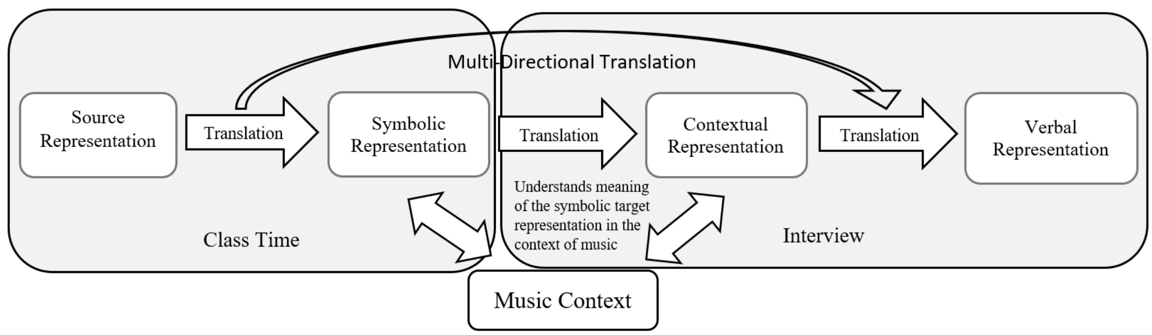
$C_1 \times 2 = C_2 \times 2 = C_3 \times 2 = C_4 \times 2 = C_5$ etc...

$C_1 \times 3/2 = G_1 \times \frac{3}{2} = D_2 \times \frac{3}{2} = A_2 \times \frac{3}{2} = E_3$ etc...

This code presented similar to other pairs, involving contextual and verbal representations. Although the later part of the translation was completed in the follow-up interview, these episodes from Emily and Amber still consisted of a single multi-directional translation, just spread out over a longer time period. This is illustrated in Figure 30.

Figure 30

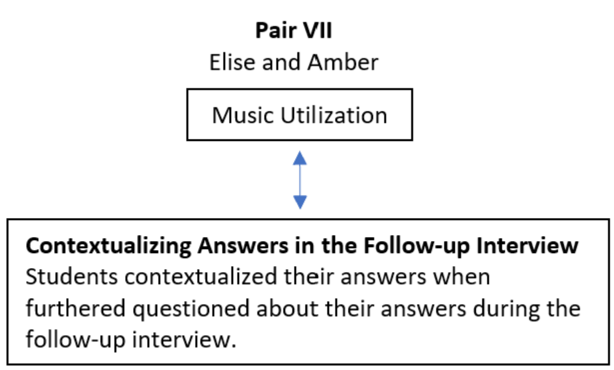
Illustration of Emily and Amber's CA Theme.



In summary, while sometimes Emily and Amber made verbal contextualized interpretations of their answers during class, there were occasions where their contextual interpretations were only made known during the follow-up interview. The follow-up interview was an important tool for assessing Emily and Amber's understandings, especially given the speed at which they worked and their limited minutes of in-class discussion time. This theme is illustrated in Figure 31.

Figure 31

Within-Case Theme for Emily and Amber



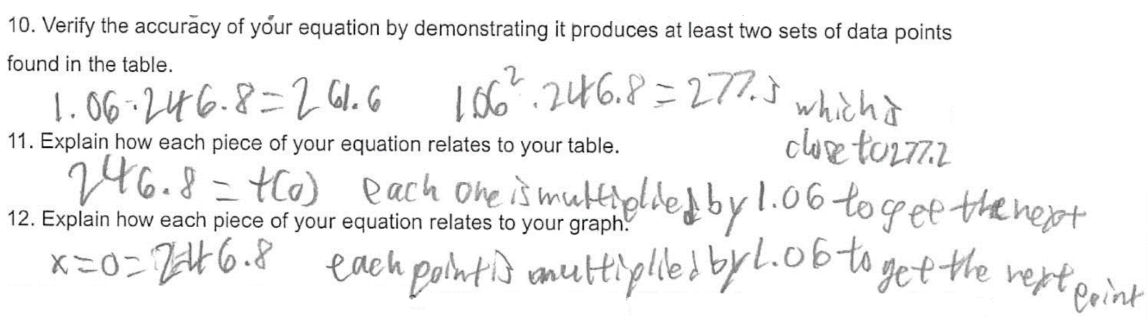
The Case of Jason and Gabe

Jason and Gabe worked very quickly through the tasks, with the least amount of partner discussion minutes across all eight pairs. Their written answers were short, lacking context and complete sentences. The pair was prone to making small mathematical errors on the tasks, relying slightly more on teacher intervention than most other participating pairs. Ms. W frequently pointed out errors in their work and helped them make corrections using scaffolding questions. Jason and Gabe completed all 31 task problems. A complete table of codes applied to Jason and Gabe's task problems can be found in Appendix E. In short, the pair carried out 16 uni-directional translations, 5 bi-directional connections, 9 multi-directional translations, and 1 abstract connection. One theme that was prominent in Jason and Gabe's work as they completed these varying types of translations was their tendency to not utilize the music context (N) in their mathematics.

Lack of Music Utilization. A theme that arose from qualitative analysis of Jason and Gabe's data was their tendency to not use the music context while they completed the mathematics problems. In over half of the task problems, Jason and Gabe did not mention any musical component while constructing or connecting representations and did not interpret their answer in the context of music during class or in the follow-up interview. In these problems, the pair used solely mathematical terms. For example, consider Jason and Gabe's work on Task 2 Problems 10, 11, and 12 in Figure 32. As the students conducted these three bi-directional connections, their choice of wording lacked musical terms. When answering problem 10, the pair wrote out their calculations for testing two different input values, and then wrote "261.6" and "277.3" for their answers without writing the units "hertz" afterwards and without verbally confirming that they understood these values were the correct frequencies of C4 and C#4. Rather than refer to the inputs as musical notes and the outputs as musical frequencies, Jason chooses the words "one" in problem 11 and "point" in problem 12. When I asked "Okay and then in your equation, you have your common multiplier 1.06. What do you think the 246.8 means?" in the follow-up interview, Gabe again said "T of 0." Although I gave Gabe an opportunity to interpret the initial value as the frequency of B_3, his understanding of 246.8 was entirely mathematical. The code none (N) was created to capture task problems where students did not discuss the music context as they worked, nor make musical interpretations of their answers after finding them, even when given opportunities to do so in the follow-up interview.

Figure 32

Jason's Work on Task 2 Problems 10-12


10. Verify the accuracy of your equation by demonstrating it produces at least two sets of data points found in the table.

1.06·246.8=261.6   1.06²·246.8=277.3 which is close to 277.2

11. Explain how each piece of your equation relates to your table.

246.8 = t(0) each one is multiplied by 1.06 to get the next

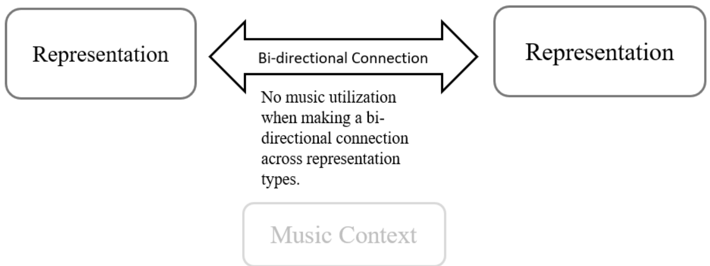
12. Explain how each piece of your equation relates to your graph.

x=0=246.8 each point is multiplied by 1.06 to get the next point

Jason and Gabe were more likely to not utilize music on task problems involving symbolic representations that did not explicitly call for connections to music. As shown in Figure 33, this theme was present in their bi-directional connections.

Figure 33

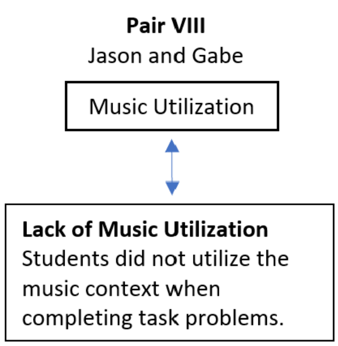
Illustration of Jason and Gabe's N Theme



In summary, although Jason and Gabe had opportunities to make contextual interpretations of their answers, they often did not. Results from this pair highlighted their tendency to choose mathematical words over music theory vocabulary. This was perhaps exacerbated by their propensity for writing short answers without complete sentences and completing each problem quickly in comparison to the other pairs. This theme is illustrated in Figure 34.

Figure 34

Within-Case Theme for Jason and Gabe



Cross-Case Results

The purpose of this section is to present themes that arose during cross-case analysis. Following the production of themes that emerged within each case, a cross-case comparison of themes was conducted to identify similarities and differences. Similar themes from across all cases were extrapolated, specifically in light of each of the three research sub-questions. The following subsections present results from the cross-case analysis with regards to each research sub-question.

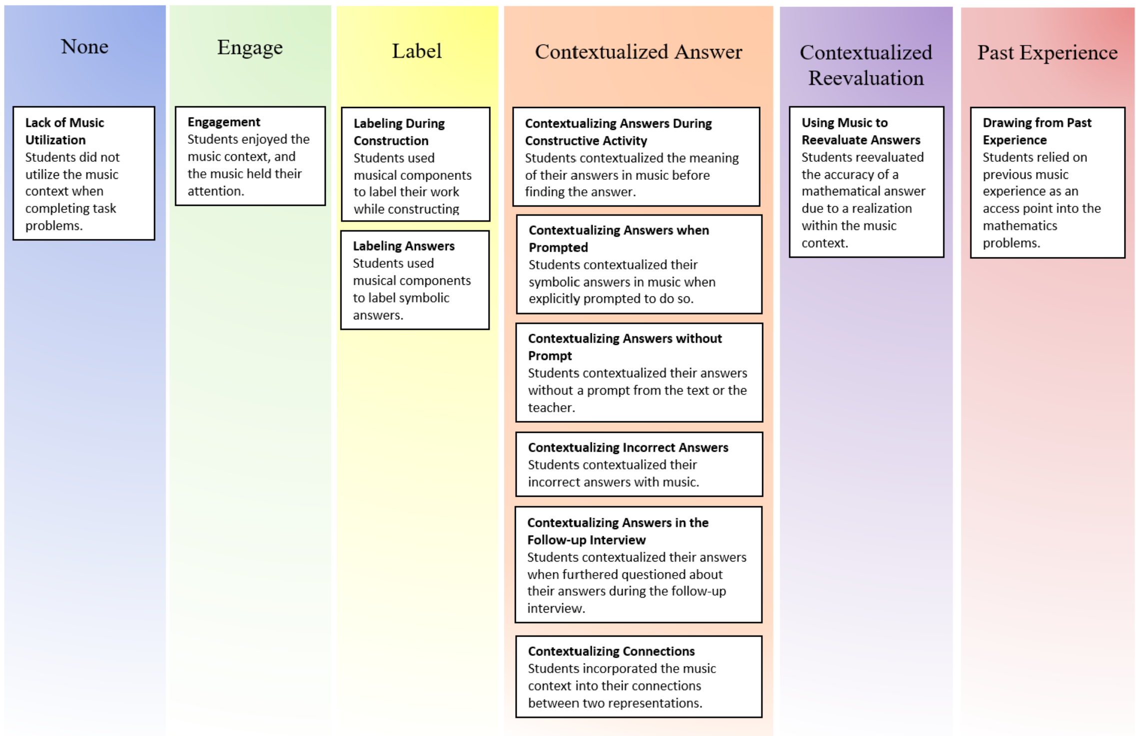
Research Sub-Question 1: Types of Music Utilization While Translating

The purpose of this section is to present results relating to the first research sub-question: How do students make use of music context while completing uni-directional, bi-directional, multi-directional, and abstract translations? While previous research found that students engage in these four types of translations while working with mathematical representations, or make connections across them (Fonger, 2011, 2019), no previous research had examined how students utilize music context while translating. Inductive codes were developed during within-case analysis to capture the various ways the eight pairs of student participants utilized the music context as they completed the three mathematical task sheets. A description of the many ways the students utilized music are shown in Table 8, along with the abbreviated code used and an example of each use from the present study.

During cross-case analysis, six utilizations of music were produced by examining similarities and differences in the 12 within-case themes and coordinating them into larger themes, as illustrated in Figure 35. The six overarching ways the students utilized the music context that were identified during cross-case analysis were none (N), engage (E), label (L), contextualized answer (CA), contextualized reevaluation (CR), and past experience (PE). The subsections that follow include detailed descriptions for each music utilization type that was identified, and how that theme was evidenced similarly or differently across all eight cases.

Figure 35

Six Cross-Case Themes for Music Utilization

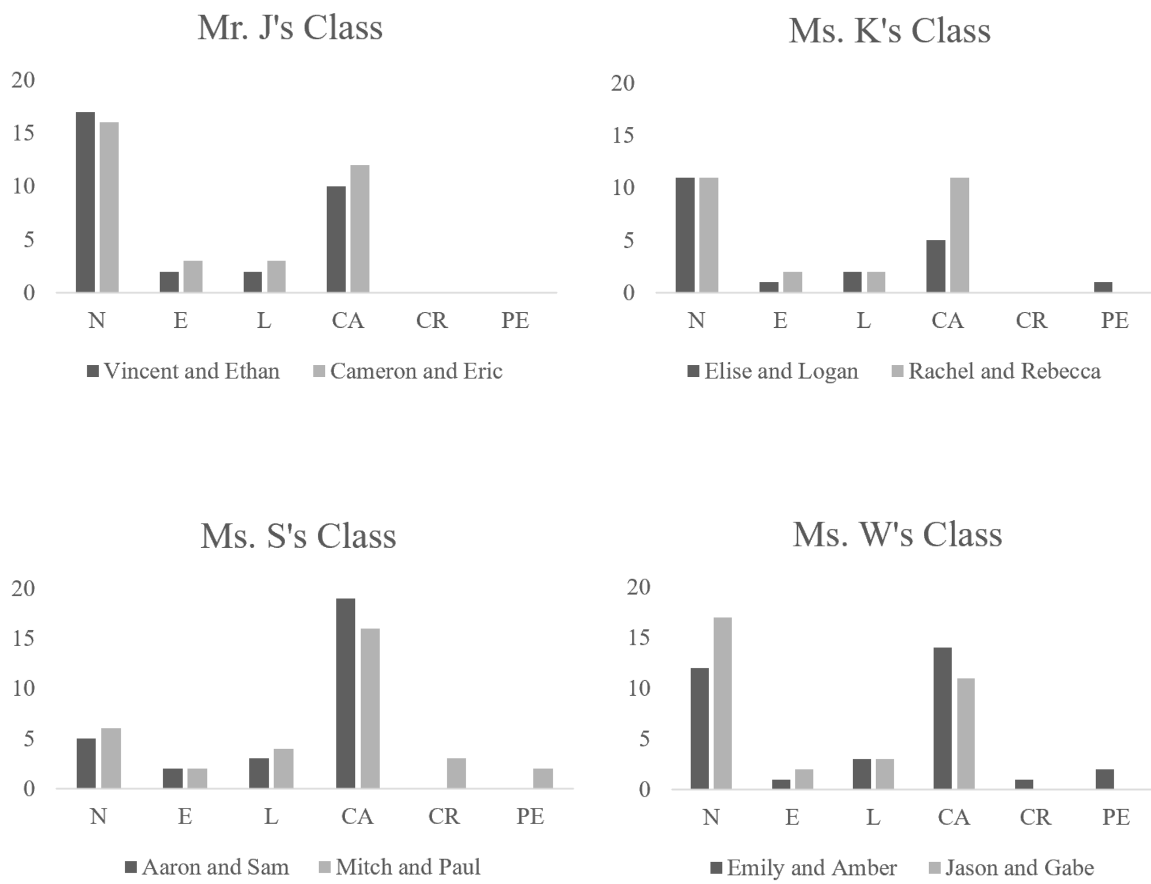


Observed Types of Music Context Use

While six themes for music utilization emerged across all pairs during cross-case analysis, the pairs manifested these themes in different ways and frequencies. Figure 36 shows the music utilization types for each pair of students by frequency. The following sections discuss similarities and differences in the way students evidenced these six cross-case themes.

Figure 36

Total Count for Each Type of Music Utilization for the Eight Pairs of Students



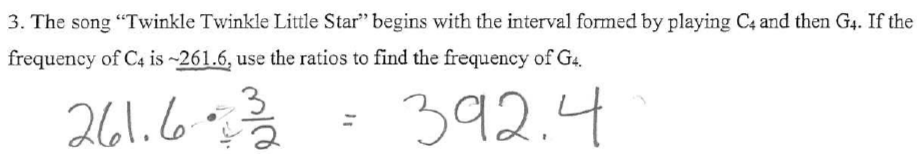
Note. Counts of music utilization types across cases. N = none. E = engage. L= label. CA = contextualized answer. CR = contextualized reevaluation. PE = past experience.

None. The theme none (N) was present in all eight cases, though in different quantities. For example, Jason and Gabe did not utilize the music context in 17 of their 31 completed task problems, while Aaron and Sam used the music context in all but 5 task problems. Thus, there was great variation in the presence of this theme. It is interesting to note that Ms. S's two pairs of very mathematically proficient students in the Honors Secondary Mathematics III class evidenced this theme the least across the eight cases, while the four pairs of students in Ms. K's and Ms. W's Secondary Mathematics I classes evidenced this theme the most. This suggested that the pairs in lower-level mathematics classes were less likely to utilize the music context than those in upper-level mathematics courses.

However, regardless of what class they were taking, the way this theme was manifested across the eight cases was similar. Students frequently failed to label their mathematical answers with music units, or verbally interpret their answer in the context of music. This was seen in Vincent's work on Task 1 Problem 3 shown in Figure 37. The number 392.4 was not labeled as a hertz value, and neither boy discussed the meaning of 392.4 as the frequency of G_4 during class or the interview.

Figure 37

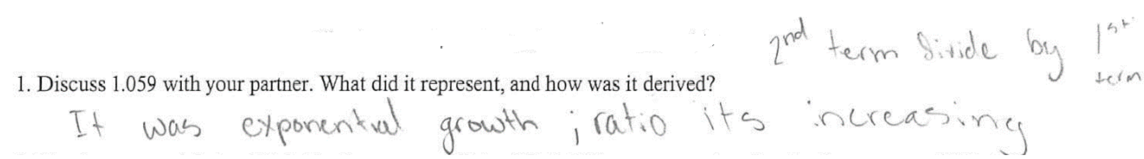
Vincent's Work on Task 1 Problem 3



In other instances, students chose mathematical terms in place of music theory terms. Cameron and Eric's answer to Task 3 Problem 1 is shown in Figure 38. Their answer mentioned 1.059 as the "exponential growth" or "ratio," and they wrote that it was derived by "2nd term divide [sic] by 1st term" rather than referring to the 2nd note, or frequency divided by the first.

Figure 38

Cameron's Work for Task 3 Problem 1



In summary, it was common for the participants to make no use of the music context while completing translations among mathematical representations. In these instances, the students discussed their mathematics using purely mathematical terms, without connecting those terms to their meaning within the music context. The code none (N) was the most common code applied for four of the eight pairs and was more common in lower-level mathematics classes.

Engage. The theme engage (E) was also present across all eight cases, and in nearly equal amounts. This code captured the instances where the students expressed enjoyment in the music context, or it was evident from observation that the music context engaged the students, capturing their attention in a unique way that the mathematics alone did not. Cross-case analysis revealed that this code was most evident in task problems where students worked with the musical pipes. In fact, 7 of the 8 pairs demonstrated engagement with the music while solving Task 1 Problem 1, which required students to compare the lengths of two pipes. This theme was manifested by students' facial expressions or verbal statements that evidenced enjoyment in the music. Sometimes this was just a smile or laugh, or an audible phrase such as "that's cool" while playing the pipes or discovering a new connection between the music and mathematics. This code was also present when the pairs tapped on the pipes trying to play familiar songs, even if they did not necessarily express verbally that they found the music entertaining.

The code engage (E) was applied 1-3 times to every pair of students. While not particularly common compared to other music utilization types, this code was necessary to capture these unique situations where the music context served as something that captured students' attention and provided enjoyment.

Label. The theme label (L) was also present across data from all eight pairs of students. This code captured occasions where the students briefly mentioned or wrote down a musical component, such as a note or other music theory term, as a means of labeling their mathematics. Cross-case analysis revealed two main ways that students evidenced this code in their work, namely labeling their written mathematical answers with a musical component (see Vincent and Ethan's work on Task 1 Problem 7) or verbally labeling their mathematics with a musical component as they worked. Verbal labeling was very common across cases on Task 1 Problem 2, which required students to fill out a table of frequencies. Students read aloud the notes C_1, C_2, C_3, etc. as they filled out their table of ratios.

Labeling was distinguished from contextualized answer (CA) because students briefly mentioned or wrote down a musical component but did not make their understanding of their answer in terms of the music context known. For example, when Mitch and Paul worked on Task 3 Problem 5 discussed previous (and shown in Figure 25) and Mitch said "subtract C4 from C5" and divide by 12, and then did so on his calculator without discussing the meaning of the resulting answer in the context of music.

The code label (L) was applied 2-4 times with each pair, thus utilized relatively consistently across cases. While the students mentioned the music context in these cases, it was unclear whether a deep contextual understanding of their mathematics in terms of the music context was present, and the musical component served as a way to label their mathematics as they carried out calculations or wrote answers.

Contextualized Answer. The most common theme overall was contextualized answers (CA), which captured the all the themes found in within-case analysis that were related to contextualizing answers shown in Figure 35 above, including contextualizing an answer with a prompt, without a prompt, contextualizing during construction, etc. The appearance of this cross-case theme varied greatly among the eight pairs. Nineteen of Aaron and Sam's task problems were coded as CA, while only 5 of Elise and Logan's task problems received this code. Cross-case analysis showed that this theme manifested in many several ways. Some examples of the theme CA were only found in one or two cases. For example, Rachel and Rebecca and Elise and Logan were the only pairs to make contextual connections with an incorrect mathematics answer. And Ms. S's honors students Aaron and Sam and Mitch and Paul were the only pairs able to correctly interpret the meaning of $2^{1/12}$ in the music context. However, four sub-types of CA were common across all pairs, including: 1) interpreting a symbolic target representation in a contextualized way, 2) contextualizing a mathematical parameter or variable, 3) showing contextualized understandings of a future answer during construction, and 4) contextualizing during a bi-directional connection.

Contextualizing a Symbolic Target Representation. Most often, this type of contextualized answer (CA) appeared when students explained the meaning of a numerical answer in terms of music. For example, When Mr. J asked Vincent and Ethan what 1.059 represented, Vincent replied, "It's like the change in hertz." The symbolic target representation is contextualized as a hertz value. As another example, a conversation that transpired with Rachel and Rebecca and their teacher as they worked on Task 3 Problem 4 went as follows:

Ms. K: So why did you put a number right here?
Rachel: Because that is the next key, and if we're trying to figure out the one after that, you need this number….
Ms. K: So, this isn't the next key?

Rachel: No, no, that is the equation to get to the next key.
Ms. K. Nice.
Rachel: So, I need this number, and I need to figure out this square root to get to the next number.
Rebecca: I timesed it. So, do I just write this down?
Ms. K: Yeah, so what did we just find?
Rachel: It's the next key, the C…
Ms. K: The C sharp!

Their written answer can be seen in Figure 39. This exchange shows clear evidence that Rachel recognized the goal of the problem is related to finding the "next key," though it was unclear whether she understood that it was specifically the frequency of the next key. She also knew that the function of the equation they wrote was to get to the next key, as evidenced by her statement "that is the equation to get to the next key." After completing the problem with a calculator, Rachel was able to tell her teacher that the number they found corresponds to a note, and with her teacher's help, correctly identified this note as C#4 on her task sheet (shown in Figure 39). These instances where a symbolic target representation was interpreted in the musical context were present across all pairs and were coded as CA during cross-case analysis.

Figure 39

Rebecca's Work on Task 3 Problem 4

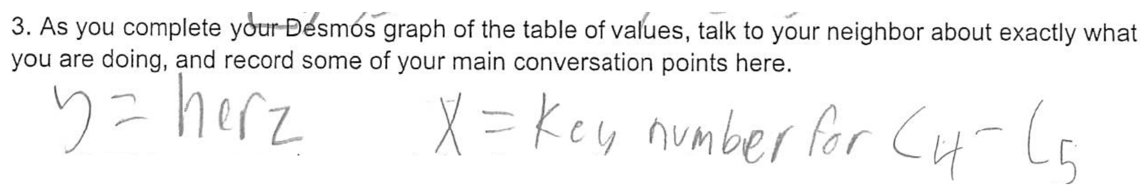

4. Use the exact value of 1.059 that we arrived upon following our class discussion to find the frequency of $C^{\#}_5$ given that the frequency of C_5 is ~523.3 hertz.

$523.3 \cdot \sqrt[12]{2} = 554.417$ $C^{\#}_5$

Contextualizing a Mathematical Parameter or Variable. Another type of CA found across cases was mathematical parameters or variables that were contextualized, such as the work shown in Figure 40 by Jason and Gabe. The students recognized that their x and y axes on their Desmos graph represented the key number and hertz, respectively. These were not contextualized numbers, but rather contextualized interpretations of variables. In fact, six of the eight pairs of students successfully interpreted the contextual meaning of the input and output of their exponential equation on Task 2 Problem 9 and were given the code CA on that problem.

Figure 40

Jason's Work on Task 2 Problem 3



3. As you complete your Desmos graph of the table of values, talk to your neighbor about exactly what you are doing, and record some of your main conversation points here.

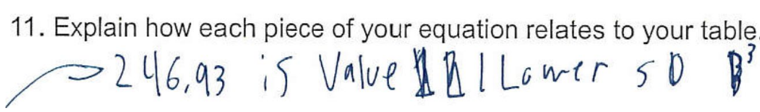
y = herz X = key number for C4 - C5

Contextualizing an Answer During Construction. A final type of CA found across cases was musical interpretations of a target representation from students while still in the process of constructing it. For example, while working on Task 2 Problem 1, Aaron and Sam discussed the graph they aimed to construct of the frequencies of C4 through C5. While discussing, Sam said, "should we do frequency as x and wavelength as *y* or should we do notes as *x*?" In this example of a contextualized answer (CA), the pair incorporates the music context into their decision-making around constructing the target representation, signifying that their understanding of the graph is connected to the music context. In another case, while trying to find mathematical ratios between the PVC pipe lengths, Elise and Logan played with the pipes, and after some time Logan said, "I think the shorter they get the higher the pitch is." Thus, in completing this physical to contextual to verbal multi-directional translation, the pair noticed some emerging relationships between the sounds of the pipes depending on length, making early connections between the musical component of sound and the mathematical component of measurement before arriving at their final answer.

Contextualizing During a Bi-Directional Connection. Finally, students showed contextual understandings of their mathematics as they made bi-directional connections. For example, Task 2 Problem 11 require students to make connections between an equation and table bi-directionally. While making these connections, Aaron pointed out that 246.93 represented B_3 , as shown in Figure 41. He writes "246.93 is value 1 lower, so B_3," presumably meaning one lower than C4. In doing so he connects a value from his equation to a value from his table, all the while connecting to the music context. When students incorporated musical understanding into their bi-directional connections, the cross-case theme CA was present. Although there was no presence of a clear target representation, these instances of contextualized connections (originally coded as CC) were enveloped under the larger theme of CA because they manifested similarly and were only present 1-2 times for each pair.

Figure 41

Aaron's Work on Task 2 Problem 11


11. Explain how each piece of your equation relates to your table.

→246.93 is Value 1 Lower so B^3

In summary, the cross-case theme contextualized answer (CA) was the most common theme overall and ranged in quantity across all eight pairs of students. This theme presented in many ways across cases, including when students interpreted a mathematical number or parameter in the context of music, while students were in the process of constructing their target representation, and as students made bi-directional connections.

Contextualized Reevaluation. Cross-case analysis revealed that only two of the eight pairs of students used their understanding of the music context as a means for reevaluated their mathematics. The theme contextualized reevaluation (CR) appeared in three of Mitch and Paul's task problems, and one of Emily and Amber's. A notable instance of CR was in Mitch and Paul's work on Task 2 Problem 6, which required students to determine the relationship between data points on their Desmos graph. Mitch and Paul used Desmos' regression capabilities to construct a quadratic equation that best fit their data. At first, they were confident in the accuracy of their equation, but then discuss how their quadratic doesn't fit "when you go down to 0 hertz." The students later tried a regression with an exponential structure and Mitch said, "I think the exponential works better because when you go with lower notes the frequencies don't start going back up, so I might actually change my answers." Later on in the follow-up interview, Mitch and Paul revealed more about this thought process.

Interviewer: Okay, talk me through, as you were building your graph on Desmos, how did you rely on the music context to help you construct your graph and interpret your graph? Was the music helpful as you did that?
Mitch: Well, we at first were trying to do quadratic. And we were thinking it was fitting pretty good.
Paul: But then the back, like the left side, it started going up.
Mitch: So then that's how we knew, 'cause we know that in music when you get lower notes, you don't have a higher frequency 'cause that just doesn't make any sense. You don't get higher when you get lower.
Interviewer: So, would it be fair to say that your background in understanding music and frequencies helped you realized that what you'd come up with mathematically was maybe not correct?

Mitch:	Yeah, because if I didn't know anything about musical frequencies, I would've just been like, yeah, that's fine.
Paul:	Yeah, if we were just given these values and put those in, we'd be like oh yeah that seems…
Mitch:	Yeah, that works! But with music, it made us know that it wasn't right.

In Mitch's statement beginning with "So then that's how we knew," he clearly demonstrates how their knowledge of music and the behavior of frequencies ultimately led to their decision that a quadratic was incorrect. They even go as far as to recognize how their music knowledge assisted them in Mitch's final statement, "But with music, it made us know that it wasn't right." While solving this problem, Mitch and Paul reevaluated a previous mathematical decision because of their understanding of the music context. While only four instances of contextualized reevaluation occurred, a unique code was necessary to capture these powerful examples of the pairs using the music context to reexamine their mathematical work. The theme contextualized reevaluation (CR) was absent from the work of six of the eight pairs. Most pairs of students immediately recognized that the graph of data points was exponential, possibly because of their lack of experience, at least in recent past, with quadratic functions. Although all pairs corrected their answers as they worked, this was often done because of teacher intervention, and without connection to the music context.

Past Experiences. A final theme of past experiences (PE) was present in three of the eight cases. This theme constituted a few instances where the student pairs drew from their knowledge of music outside of class to make connections between the music context and their mathematical work. This was evident in Elise and Logan's struggle to recall the word "rest" mentioned in the previous section. Emily and Amber also had previous experience with music that they drew on during the lessons. While working on Task 3 Problems 9 and 10, Emily and Amber discussed the size of PVC pipes and the sound of a musical scale in a hypothetical situation where music is tuned according to a constant linear progression rather than exponential. In their discussion of both problems, Emily and Amber described the sound of the scale as being "flat," a term not previously mentioned by the teacher in the lesson or on the task sheet. In doing so, they drew on previous experience (PE) from music outside of class. Similarly, when listening to the hypothetical "linear scale" played by Ms. S on her computer, Mitch commented that it sounded like the "blues scale," a term he knew from outside of class in his own experiences with music.

Though only prevalent in the work of three of the eight pairs of students, the code previous experience (PE) was applied to five task problems where it was clear the pair was making connections with their experiences with music outside of class and their mathematics. With the exception of Rachel and Rebecca, who mentioned some previous music experience outside of class but did not draw from it during the lessons, the pairs of students who drew from previous music experience were students who verbally confirmed that they had previously played an instrument. Elise had experience with the clarinet, and Mitch, Paul, Emily and Amber with the piano.

Conclusion

Results from the first research sub-question reveal that across all participants, there were a total of six distinct ways students used the music context while completing the three task sheets. These included no music utilization (N), finding the music context engaging or entertaining (E), using the music context to label their mathematics (L), creating a contextualized answer (CA), using the music context to reevaluate their mathematics (CR), or finally drawing on past experiences with music (PE). The themes N, E, L, and CA were prevalent across all eight student pairs; however, the themes CR and PE were only present in the work of two and three pairs respectively. There was significant variation in the frequency of each music utilization type among the eight pairs, some using the music often and others rarely. More investigation was needed into the second research sub-question to determine what types of task problems and representation types that seemed to elicit these different music utilization types.

Research Sub-Question 2: Music Utilization According to Representation Type

The purpose of this section is to present results related to the second research sub-question: “How does students’ use of music context while completing translations differ according to the representation types involved?” Practical suggestions in mathematics education focus on five distinct representation types, namely contextual, physical, verbal, visual, and symbolic (NCTM, 2014). However, no previous research had examined how students use music context while translating and connecting among these five representation types. Results from the previous section revealed that there were six distinct ways the students used the music context. Table 10 shows the frequency of each music utilization type according to the type of representation that was involved in the translation or connection.

Table 10

Music Utilization Types According to Representation Type Across All Participants

	Code System	N	E	L	CA	CR	PE
Contextual	From	16	2	6	54	4	3
	To	3	3	1	37	3	1

	Connection	1	6		13		1
	Total	20	11	7	**104**	7	4
Physical	From	5	9	9	16		
	To						
	Connection	1	6		13		1
	Total	6	15	9	**29**	0	1
Verbal	From						
	To	29	5	5	51	3	3
	Connection				6		
	Total	29	5	5	**57**	3	3
Visual	From	41	2	7	29	2	1
	To	13		3	7		
	Connection	13			1		
	Total	**67**	2	10	39	2	1
Symbolic	From	38	4	1	36	2	
	To	46	7	16	34	1	
	Connection	13			1		
	Total	**97**	11	17	71	3	0

Note. The six music utilization types none (N), engage (E), label (L), contextualized answer (CA), contextualized reevaluation, and past experience (PE) represent the columns. Bolded numbers represent the most frequent music utilization type overall for each representation type.

The bolded totals in Table 10 show the total count for the most common music utilization type when that representation type was involved. Within-case results for each of the eight student pairs were similar, with little variation in which music utilization type was most common for each representation type. The tables with these counts for each of the eight pairs can be found in Appendix F. The following subsections detail results for the way students used music while working with each of the five representation types.

Contextual Representations and Music Utilization

While completing the three task sheets, the participants often translated to and from contextual representations, and made connections with contextual representations and other representations. Some task problems gave contextual situations and asked students to make mathematical meaning from them, while others gave mathematical information and explicitly called for the students to interpret that mathematics in the context of music. However, in other instances students created or derived contextual meaning in mathematics without explicitly being asked. As shown in the Table 10 above, a cross-case comparison of representation type and music utilization type revealed that when contextual representations were involved, students were more likely to create contextualized answers (CA) than any other type of music utilization. There were 104 task problems coded as both involving a contextual representation and CA simultaneously. This signifies that when contextual representations were given or created by the students, creating a target representation that they understood in the context of music was common.

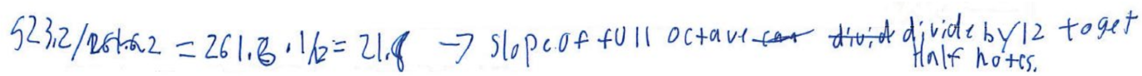
Consider the following illustration of a task problem where a contextual representation was explicitly provided to the students in a task problem that was coded as CA. Task 3 Problem 5 read:

> If the frequency of C4 and C5 were still ~261.6 and ~523.2, respectively, but musical instruments were instead tuned using a constant linear progression between notes, show how you would find the constant increase between the frequency of adjacent notes (e.g., C4 and C#4)?

This problem presented students with a contextual source representation and prompted a translation to a symbolic target representation. Aaron explained to Sam how they needed to "divide by 12 to get the half notes." Aaron's work can be seen in Figure 42. His written answer shows his work of 523.2 divided by 261.6, and then multiplied by 1/12 to arrive at 21.8. He then also explains his understanding of 261.6 as the "slope of the full octave," and implies that he understands 21.8 as the change in frequency between "half notes." In problems like this one where contextual information was provided to students, it was common for students to provide answers contextualized with music like Aaron's.

Figure 42

Aaron's Work on Task 3 Problem 5

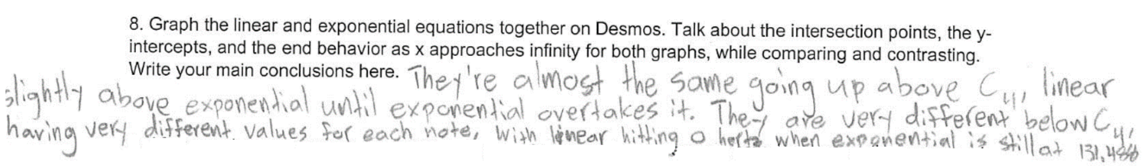


Other task problems explicitly asked students to interpret their mathematics in the context of music. Task 2 Problem 5 read, "What does the space between data points represent in terms of musical frequencies?" It is interesting to note that in this example where the question requires students to interpret the space on the graph "in terms of musical frequencies," all eight pairs of students were given the code CA.

Sometimes no explicit contextual information was provided or required, but students constructed a contextual representation of their own accord. These were made known during observation because the student verbalized their contextual representation aloud. Task 3 Problem 8 read, "Graph the linear and exponential equations together on Desmos. Talk about the intersection points, the y-intercepts, and the end behavior as x approaches infinity for both graphs, while comparing and contrasting. Write your main conclusions here." When completing this problem, Mitch and Paul translated from a symbolic representation (the equations) to a visual representation (the Desmos graph) to a contextual representation to a verbal representation. Paul's written work is shown in Figure 43. As the students discussed their Desmos graph, they made conclusions about their line and curve within the context of music, concluding that a linear musical progression would reach 0 on the y-axis (0 hertz) while the exponential does not. This can be seen in Paul's final statement "with linear hitting 0 hertz when exponential is still at 131.488." This conclusion includes an interpretation of the shapes of the two graphs in relation to music, denoted by their choice of the word "hertz." They also used the note C4 as an important point of comparison between the two equations. Paul says, "they're almost the same going up above C4," and then later "they are very different below C4." This occurred despite the original problem not mentioning C4 at all. Although this problem did not require using the music context in any way, Mitch and Paul translated to and from a contextual representation, and because they constructed a target verbal representation contextualized with music, this problem was coded as CA.

Figure 43

Paul's Work on Task 3 Problem 8



8. Graph the linear and exponential equations together on Desmos. Talk about the intersection points, the y-intercepts, and the end behavior as x approaches infinity for both graphs, while comparing and contrasting. Write your main conclusions here. They're almost the same going up above C_4, linear slightly above exponential until exponential overtakes it. They are very different below C_4, having very different values for each note, with linear hitting 0 hertz when exponential is still at 131.488

Participants translated to and from contextual representations often. There were 104 task problems across participants coded as both involving contextual representations and contextualized answer (CA).

Physical Representations and Music Utilization

While there were no task problems where students translated to a physical representation, students translated *from* physical representations while working with the

PVC pipes and observing the piano activity. Table 10 shows that when completing task problems where they translated from a physical representation to other representations, or made connections with physical representations, students most often created contextualized answers (CA). A total of 29 task problems were coded as involving a physical representation together with CA, revealing that when the participants worked with physical representations, they were likely to also create a target representation understood in the context of music.

Consider Rachel and Rebecca's work on a task problem involving a physical representation that was coded as CA. While handling the PVC pipes at the very beginning of Lesson 1, Rachel said to Rebecca, "C5 makes higher pitch noise because it's shorter. It doesn't have to go through as long of a tube… Yeah C4 is longer, C5 is shorter. C4 makes a lower sound 'cause it has to go through more tubing." In Rachel's insightful discovery, she made inferences about the physical attributes of the PVC pipes, and directly connected those to the musical context. Most of the participating pairs made some of these inferences about the relationship between pipe lengths and the relative pitches, however Rachel was the only participant to go as far as trying to explain the reason for the difference in pitches, which she attributes to how much tubing the sound must travel through.

This particularly interesting solution from Rachel and Rebecca illustrates how students' made connections between physical objects like the PVC pipes or the piano and the music context. While translations with physical representations were the least common compared to the other four representation types, every pair of students translated from physical representations and were likely to create contextualized answers (CA) when they did.

Verbal Representation and Music Utilization

As shown in Table 10, there were no translations from verbal representations to another representation because the written task problems were not the students' verbalizations and therefore were not considered verbal. However, there were many occasions where students verbalized their thinking aloud, which were coded as being a translation to a verbal representation. There were also 6 instances where an abstract connection was made across all representation types. These were general comments made by students when asked about their overall thoughts about mathematics and music connections on Task 2 Problem 13. Because students verbalized these connections across all representations, these were also coded as verbal. Table 10 shows that contextualized answer (CA) was the music utilization type students were most likely to engage in when verbal representations were involved. As students verbalized their mathematics or their contextual understandings, the target representations they constructed in their final answers were likely to be contextualized with music.

Consider these comments between Emily and Amber as they discussed the meaning of the space between data points on Task 2 Problem 5

Amber: The notes you're not playing? Because here's a note, and here's a note… (draws picture shown in Figure 44)

Emily: It's frequencies between the common notes. It's a frequency that's in between the traditional note. Because wouldn't that be a musical frequency that's outside the normal range? Well, it wouldn't be range if we're using normal math terms. It'd be a note outside of what is usually used in music.

As the students tried to make meaning of the space between points on their Desmos graph, they verbalized their interpretations of the space in terms of the music context. Amber's statement "here's a note, and here's a note" shows that she understood the points on the mathematical graph as directly related to musical notes, and the space between them as notes between those. Her drawing in Figure 45 illustrates this thinking, with the outer two piano keys representing points on the graph, and the inner three notes representing potential notes between those two notes. Language played an important role in this interaction between the students, as they struggled to decide how to articulate their understanding of the space as a musical frequency that is not traditionally used in Western tuning. It was common for instances like this where students were required to verbalize their interpretation of mathematics to be coded as CA because their verbalizations were deeply contextualized.

Figure 44

Amber's Drawing of Task 2 Problem 5

Students did not translate from verbal representations, but all participants translated to verbal representations. When students constructed verbal representations, their articulations were commonly contextualized.

Visual Representations and Music Utilization

As participants completed the three task sheets, there were frequent opportunities for translating to and from visual representations and making connections with visual representations. Sometimes these translations were explicitly called for in the wording of the task problem, and other times without being prompted to. Table 10 reveals that when completing task problems involving visual representations, it was most common for students to not utilize the music context (N).

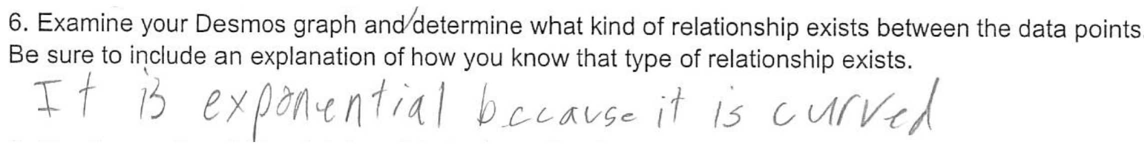
For example, as Jason and Gabe discussed their Desmos graph during Task 2 Problem 6 to determine what type of relationship existed between the data points, Gabe wrote "It is exponential because it is curved," as shown in Figure 45. In this translation from a visual representation to a verbal representation, Gabe made no connections to the music context. When I questioned further in the follow-up interview to see if at that point the students would make any connections to the music context, the result was similar:

> Interviewer: Okay so when you first graphed these points and you saw the 13 points for the first time, was it what you expected? Were you surprised? What were your first thoughts when you saw those points?
> Jason: I was a bit surprised it was exponential and not linear
> Gabe: Yeah, I also thought it'd be linear
> Interviewer: You thought it'd just be a straight line? Okay. So, right when you graphed it, you thought this shape looks exponential?
> Jason: Yeah.
> Interviewer: And why did you think it was exponential?
> Gabe: Because it was curved.

When given an opportunity to talk about the graph and their inferences, the students still made no references of the music context. In was common across all participants to make inferences like this about a graph or table that were entirely mathematical in nature and void of context. The exception to this was task problems that explicitly asked for an interpretation of a visual representation in the context of music, as discussed earlier.

Figure 45

Gabe's Work on Task 2 Problem 6


6. Examine your Desmos graph and determine what kind of relationship exists between the data points. Be sure to include an explanation of how you know that type of relationship exists.

It is exponential because it is curved

There were 67 translation involving visual representation coded with no use of music (N). This occurred when students interpreted information in a graph or table verbally or created a symbolic representation like a number or equation from a graph or table, without mentioning any musical components in their mathematics or in the follow-up interview.

Symbolic Representations and Music Utilization

Symbolic representations were the most prevalent representation type to occur in the participants' translations overall. Symbolic representations were common in task problems that required students to interpret or find mathematical quantities or parameters, or look for connections between a symbolic representation and other representation types. Table 10 illustrates how task problems coded as involving symbolic representations were also likely to be coded with no music utilization (N). In these instances, students created symbolic representations or used symbolic representations to construct tables or graphs. Task 3 Problem 6 required that students complete a table of frequencies using a constant linear change of 21.8 (see Figure 46). All eight pairs completed this table, translating from a symbolic to visual representation, without using the music context in any way.

Each pair utilized their calculator to add 21.8 repeatedly and wrote down the results in the table, exactly like what is shown in Cameron's work in Figure 46. Similarly, every pair that completed Task 3 Problem 7 (Ms.'s K's class did not do this problem) created an equation, and thus a symbolic representation, without using the music. Their answers looked similar to Victor's work in Figure 47. Victor simply writes the equation *y=21.8+239.8* without any musical components or discussion involving music. These two problems illustrate what it looked like when students used symbolic representations void of any contextual connections.

Figure 46

Cameron's Work on Task 3 Problem 6

6. Use the constant rate of change you found to construct this table will all the frequencies between C_4 and C_5.

"Linear" musical frequencies from C_4 to C_5												
C_4	$C^{\#}_4$	D_4	$D^{\#}_4$	E_4	F_4	$F^{\#}_4$	G_4	$G^{\#}_4$	A_4	$A^{\#}_4$	B_4	C_5
261.6	283.4	305.2	327	348.8	370.6	392.4	414.2	436	457.8	479.6	501.4	523.2

Figure 47

Victor's Work on Task 3 Problem 7

7. Write an equation that fits the data set in the table.

$y = 21.8 + 239.8$

There were 97 total translations across all participants coded as involving symbolic representations and no music utilization (N). If a symbolic representation was given or required by students, it was likely that no connections with the music context were made. Some problems on the task sheets requiring symbolic representations were completed by all eight pairs without any music context use. When students used symbolic representations without musical context, their answers were entirely mathematical, such as a symbolic equation, or a table of numbers.

Conclusion

The way students used the music context differed depending on what representation types were involved. Cross-case analysis revealed that when contextual, physical, or verbal representations were utilized in a translation, it was common for the associated task problem to also be coded as CA, meaning a contextualized answer was constructed. When visual or symbolic representations were involved in a translation, it was likely that that task problem was coded as having no music utilization (N). While some questions explicitly gave or require contextual applications, students used contextual representations in times when it was not required. More investigation into the third research sub-question was needed to illuminate what processes the students engaged in while translating to and from contextual representations.

Research Sub-Question 3: Constructive Activities with Contextual Representations

The purpose of this section is to present results regarding the third and final research sub-question: What constructive activities do students engage in when translating to and from a contextual representation? Previous research suggests that students translating among multiple representations make interpretations of a source representation and engage in certain constructive activities to create a target representation (Bossé et al., 2011a; Janvier, 1987). Previous models showing constructive activity types fail to consider contextual representations as a unique representation type. For example, Janvier (1987) and Bossé et al. (2011a) intertwine what they call "situations" with verbal representations in their models. These models also did not include physical representations. Prior to this study, no research investigated the constructive activities involved when translating between a contextual representation and all four of the other representation types in the standard representation model of practical mathematics education (NCTM, 2014).

Across all participants, a total of 82 translations were coded as involving a contextual source representation, and 44 were coded as involving a contextual target representation. With influence from Bossé et al.'s (2011a) constructive activity model, codes were created to capture the constructive activities students engaged in when contextual representations were involved in their translation. Table 11 shows the counts for the number of times overall that students made translation to or from contextual representations, and Table 12 shows the various constructive activities codes.

Table 11

Total Number of Translations to and from Contextual Representations

	Physical	Verbal	Visual	Symbolic
From Contextual	N/A	36	N/A	31
To Contextual	10	N/A	12	17

Note. N/A is used to indicate translations that were not present in this study and therefore not codable for constructive activities.

Table 12

Constructive Activities when Translating to and from Contextual Representations

	Physical	Verbal	Visual	Symbolic
From Contextual	N/A	Articulating (A)	N/A	Modeling* (M)
To Contextual	Contextualizing (C)	N/A	Contextualizing (C)	Contextualizing (C)

Note. The * indicates that a term was from Bossé et al. (2011a) representation model was used.

Three distinct constructive activity types were observed in this study: articulating (A), modeling (M), and contextualizing (C). A description of these three constructive activities are shown in Table 13, along with the abbreviated code used and an example of each use from the study. The constructive activity modeling (M), first introduced by Janvier (1987) and adapted by Bossé et al. (2011a), was confirmed in observation, while the other two constructive activities were developed inductively. In the following subsections, examples from the work of participating pairs are used to create illustrations of the constructive activity types.

Table 13

Descriptions and Examples of Three Identified Constructive Activities

Constructive Activity Code	Description	Example
A	Student articulates a contextual understanding.	A task problem prompts students to describe what they notice while listening to two different scales. A student comments that the scale tuned using a constant rate of change sounds "out of tune."
M	Student models a contextual situation using mathematical symbols.	A task problem prompts students to create an equation that fits the points in a table. A student first identifies 21.8 as the rate of change and B_3 as the y-intercept, and then tests the equation to confirm it produces the correct frequencies.
C	Student contextualizes a physical, visual, or symbolic representation through a process of identification, description, and mapping.	A task problem prompts students to interpret the meaning of the number 1.059. A student identifies 1.059 and the common multiplier, and consequently also must be the multiplicative change in hertz between notes.

Note. Identified music utilization types. A = articulating. M = modeling. C = contextualizing.

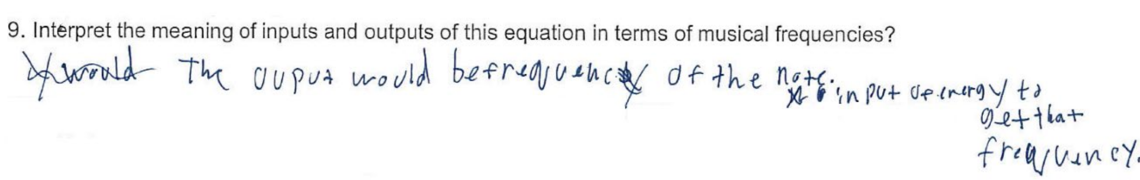
Constructive Activities Used when Translating from Contextual Representations

I observed students enact two distinct constructive activities when translating from a contextual representation: articulating (A) and modeling (M). The participants translated from a contextual to verbal representation 36 times, and from a contextual representation to a symbolic representation 31 times.

The term articulating (A) was used to capture the process students engaged in when translating from contextual to verbal representations. To illustrate, consider Aaron and Sam's answer to Task 2 Problem 9 in Figure 48. The students first translated from a symbolic representation (the equation) to a contextual representation. However, to answer the question, their contextual interpretation of the equation was said aloud to each other, and then written down on their task sheet. Aaron's work in Figure 48 shows that the students interpreted the output of their equation as frequency in the statement "the output would be frequency of the note, and the input as the "energy" to get to that frequency in the statement "x input of energy to get to that frequency." When questioned further in the follow-up interview about their thought process, Aaron and Sam explained that energy was their way of describing how many times the frequency had been multiplied by the common multiplier 1.059. This example illustrates how Aaron and Sam chose language that best represented their unique understanding of the way the musical frequencies progressed with each consecutive note. The word articulating (A) was chosen to represent the process involved in translations like this completed by the students where a contextual understanding was simply verbalized aloud or on paper.

Figure 48

Aaron's Work on Task 2 Problem 9


9. Interpret the meaning of inputs and outputs of this equation in terms of musical frequencies?

The oupu would befrequency of the note. x input of energy to get that frequency.

In Bossé et al.'s (2011a) diagram of constructive activities, which they adapted from Janvier (1987), Bossé and colleagues define modeling as the constructive activity students engage in when translating from a contextual or verbal representation to a symbolic representation. Janvier (1987) describes modeling as a series of processes where students identify useful information in the source representation, create a mapping between the source and target representation (the model), test the model, and identify if the model is useful. Results from this study confirmed that students translating from a contextual to symbolic representation used a modeling process to do so. Justification for the code modeling (M) can be seen in Mitch and Paul's process of completing Task 3 Problem 3:

Mitch:	(*Reading the text for Task 3 Problem 3*). "Equal-temperament tuning is based on the ratio used when progressing an octave from C4 to C5. If multiplying the frequency by 2 increases the pitch by an octave, what you might multiply by to progress from C4 to C#4? As shown on the picture of the keyboard on the board, there are 12 of these small steps between C4 to C5."
Mitch:	That would probably just be the 2 times ratio, but divided by 12, right?
Paul:	So that'd just be 1/6th?
Mitch:	Yeah. So, then you're increasing by 1/6. So, would it be multiplied by 7/6ths? That's not quite our 1.059 though. It's about a .1 off. Hold on. So, if we multiply 261.6 by 1.059 we get that (*points to calculator*). And if we multiply 261.6 by that we get a completely different number. So, wait, based on that it would by 1/6 of C4 times C4 would be C#4. But that doesn't work. Based on that C#4 would be 305 hertz.
Teacher:	How's it going?
Mitch:	Is there supposed to be a discrepancy between this one and 1.059?
Teacher:	Okay so if that's not working, then maybe we need to think about a different way to derive that number.
Mitch:	Okay, so it should be equal. What do you think Paul?
Paul:	I'm not sure.
Mitch:	What about 2 to the 1/12th? (*Types in calculator*). Woah! If you get 2 to the 1/12th you get the number! That doesn't make any sense though! Why does that work? I did 2 to the 1/12 and it worked but I don't understand why!

Teacher: Well, what are you doing when you're going up each time?
Mitch: Well, you're multiplying by… Oh. Okay. Okay. I understand. (*Laughs*). So, I guess that makes sense, cause you're just multiplying every time.
Paul: Oh, cause they're 12 steps, and…
Mitch: And it's an exponential.

In this exchange between the pair and their teacher, Mitch and Paul performed a series of modeling processes similar to that described by Janvier (1987). Mitch first identified the multiplier "2" and the number of notes "12" in the contextual source representation as the relevant information in his statement "That would probably just be the 2 times ratio, but divided by 12, right?" Mitch and Paul then developed 7/6 as their model for finding the frequency change between notes. The students then test 7/6 only to find that it predicts a C#4 value of 305 hertz, which they know is not the correct frequency. They recognize this as a "discrepancy." This discrepancy caused them to reevaluate their model and eventually find the symbolic representation that accurately models the contextual source representation. The statement "If you get 2 to the 1/12th you get the number!" shows their test of the new model and their realization that it is correct. These modeling processes were not unique to Mitch and Paul, but rather common across the participants when translating from contextual to symbolic representations.

The examples of student work above illustrate the constructive activities students engaged in while translating from contextual representations to verbal or symbolic representations. No participants in this study translated from contextual representations to physical or visual representations, so those constructive activities remain unknown.

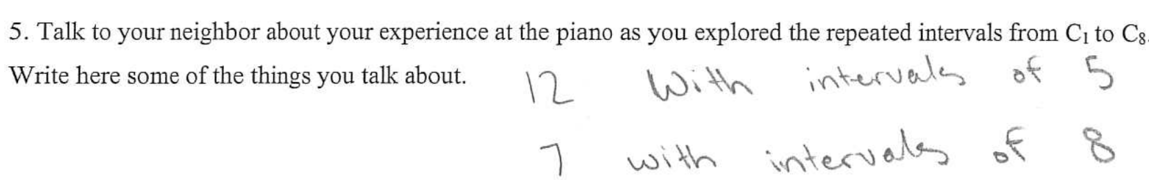
Constructive Activities Used when Translating to Contextual Representations

Students not only translated from contextual representations but also to them. The participants translated from a physical to contextual representation 10 times, from a visual to contextual representation 12 times, and from a symbolic to contextual representation 17 times. A coordination of the data from student observations, task sheet work, and follow-up interviews revealed that the processes students engaged in while completing each of these three translation types were similar and therefore not differentiable. Consequently, the term *contextualizing* (C) was developed to capture the process of translating to a contextual representation from a physical, visual, or symbolic representation. Data from across the pairs revealed that students constructing contextual representations would often 1) identify an important or relevant feature or features of the source representation, 2) describe that feature, and 3) map that description to a contextual interpretation.

These processes are illustrated in the students' work while translating from physical to contextual representations. Whether writing about their experiences with the piano or the PVC pipes, observations revealed how the students identified physical properties of the objects, described them, and made connections between those physical properties and the music context. For example, Elise and Logan and Rachel and Rebecca from Ms. K's class discussed the lengths of the PVC pipes, specifically which ones were shorter, and which were longer. This was followed by comments such as, "I think the shorter they get, the higher the pitch." The students recognized the importance of length with the pipes, describe them as shorter or longer, and then connect that with the music through their use of the word "pitch." Similarly, when asked to talk to their partner about their experience working with the piano on Task 1 Problem 5, Cameron said to Eric, "It started at the same point" and "It got to the same point" and later wrote the answer shown in Figure 49, where they describe the 12 intervals of 5 and 7 intervals of 8 that they observed. Later in the follow-up interval, Cameron and Eric further explained that their answer was referring to "fifths" and "octaves." Overall, Cameron and Eric recognize that the piano intervals begin and end of the same note, describe that verbally, and then connect that observation to musical components, namely fifths and octaves. Other pairs used musical terms such as fifths and octaves. When asked more about these experiences with the piano and PVC pipes in the follow-up interview, the pairs often talked about things they "noticed," such as noticing that the 12 5^{th} intervals and 7 octaves started and ended on the same note. Overall, coordinating written work, observations, and the follow-up interviews revealed that in these translations to contextual representations, the students described the features of physical representations and then made sense of those properties in the context of music.

Figure 49

Cameron's Work on Task 1 Problem 5


5. Talk to your neighbor about your experience at the piano as you explored the repeated intervals from C_1 to C_8. Write here some of the things you talk about. 12 With intervals of 5
7 with intervals of 8

Contextualization (C) processes could also be seen in students' work while translating from visual to contextual representations. In these translations, students observed a graph or table and then contextualized that visual element. For example, students contextualized the space between data points while solving Task 2 Problem 5 and interpreted the linear and exponential graphs solving Task 3 Problem 8. Vincent and

Ethan's interpretation of the empty space between data points on their graph can be seen in this excerpt:

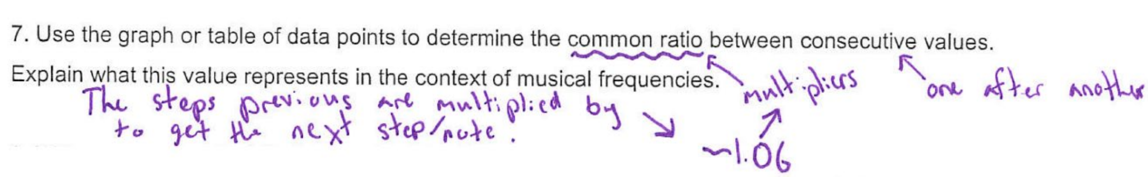
Ethan:	(*Reads task problem and then looks at the graph on his screen*). It represents the amount of waves that happens between each one, right?
Vincent:	Yeah, I think so and the change in hertz, right? Between the frequencies?
Ethan:	It shows the difference in hertz?
Vincent:	Yeah.

Vincent and Ethan observed a physical space of their graph and interpreted that space in the context of music, specifically seen in Vincent's statement, "the change in hertz" and Ethan's similar statement, "the difference in hertz."

Finally, this contextualizing process was observed in translations from symbolic to contextual representations. Task 2 Problem 7 asked students to interpret the value 1.059 (a symbolic representation) in the context of music. Emily and Amber's work, seen in Figure 50, illustrates how they first identified 1.059 as the "multiplier" as shown by Amber's arrow pointing towards the words "common ratio." They then describe 1.059 as the number by which a previous step is "multiplied to get to the next step." In a subtle way, the students then map their mathematical description to a contextual one in writing "/note" as another more musical interpretation of step.

Figure 50

Amber's Work on Task 2 Problem 7



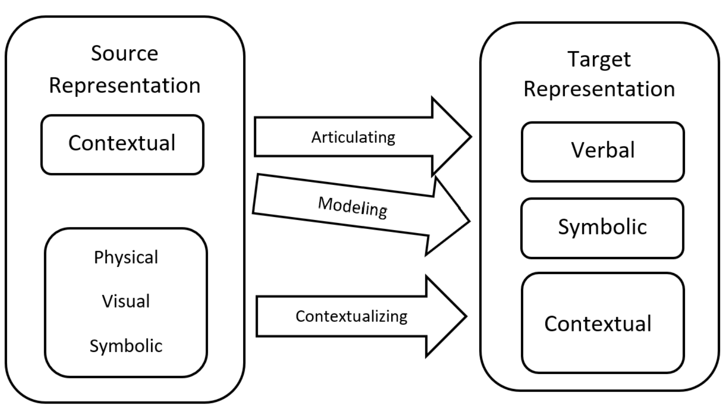
Regardless of what the source representation type was, students translating to a contextual representation engaged in a constructive activity labeled as contextualizing (C), which included several sub-processes of identifying, describing, and mapping. A total of 39 translations across the participants were coded as contextualized (C).

Conclusion

Participants engaged in three distinct constructive activities while translating with contextual representations. As shown in Figure 51, students engaged in a process of articulating (A) when translating from a contextual to verbal representation, and a process of modeling (M) when translating from a contextual to symbolic representation. When translating to a contextual representation from other representations, students enacted a series of sub-processes encapsulated by the term contextualizing (C). The following chapter presents a discussion of the main overarching research question while interpreting the results of this study in light of previous literature and concludes with limitations and suggestions for future research.

Figure 51

Diagram Illustrating the Results from the 3rd Research Sub-question



CHAPTER V

DISCUSSION

Previous research suggests that music-contextualization can be beneficial to students in mathematics. However, no previous studies examined the way students utilize music when translating among mathematical representations. Therefore, the purpose of this study was to describe and categorize the role that music context plays in high school students' translations among representations while engaging in music-contextualized algebra instruction. This chapter consists of five sections. The first section provides a discussion of the results and conceptual takeaways associated with the overarching research question. The final four sections present the study limitations, implications, recommendations for future research, and conclusions.

Overarching Research Question and Emerging Conceptual Takeaways

The purpose of this section is to discuss the results in light of the overarching research question: What role does music context play in high school students' translations among mathematical representations while engaging in music-contextualized algebra instruction that emphasizes translations? The following subsections discuss findings from the overarching research question in relation to the literature.

Students' Utilization of Music While Translating

No previous research has categorized the way students utilize music context while doing mathematics. Results from this study identified six distinct ways the participants used the music context while translating among mathematical representations, providing further insight into the ways students interact with music and mathematics simultaneously. The six ways the students utilized the music context help illuminate the role that music context played in the students' translations.

First, this study revealed that participants did not always use the music context. The code none (N) was present across all cases, and even comprised the majority of codes in three of the eight cases. This result suggests that merely the presence of music context in mathematics lessons does not guarantee that students will draw on the context while they solve mathematics problems. There is minimal previous research on the positive effects of music-contextualization in students' mathematical achievement (An & Tilman, 2015; An et al., 2013; Courey et al., 2012). This study compliments these studies by examining up close how students can interact with music and mathematics simultaneously when given opportunities to do so. To maximize potential benefits to student achievement argued in the literature, mathematics lessons with music integration must be carefully designed with purposeful task problems meant to elicit contextual connections. This will be further discussed with the second research sub-question.

Second, participants in this study relied on music as a source of engagement as they completed mathematics problems. Participants expressed their enjoyment in the music context verbally and physically, and the physical PVC pipes held their attention often as they did mathematics. This finding confirmed results from previous studies that also found how using a music context in mathematics lessons can be more engaging for students than traditional instruction (An et al., 2013; An et al., 2014). This is important because, as An and colleagues found, these findings reinforce how music-contextualized mathematics can potentially reduce students' mathematics anxiety and promote better dispositions towards mathematics. Overall, students' engagement with the music in this study furthers understanding of the potential benefits of music-contextualized mathematics.

Third, in addition to being a source of engagement, results from this study also showed that students used the music context to label their mathematics, make sense of answers, evaluate the correctness of answers, and connect to knowledge outside of class. These results confirm past findings from Barrera-Mora and Reyes-Rodriguez (2019), who also found that students drew on the problem context while completing number sense tasks. However, this study illuminates in greater detail exactly how students draw on music context while translating among mathematical representations. This study is important in furthering understanding in the field of mathematics education of exactly what it looks like when students use context. Within and cross-case analysis showed specificity and depth in the ways students utilized the context that has been missing from previous work. Previous conceptual frameworks have made claims that music integration in mathematics allows students to connect to previous knowledge (An et al., 2013; Barrera-Mora & Reyes-Rodriquez, 2019). This study confirmed these claims because students found contextual meanings in their answers and drew from music knowledge outside of class. However, this result presented differently across the pairs. For example, the students in Ms. S's Honors Secondary Mathematics III class who were mathematically proficient in the content areas covered by the lessons were much more likely to understand their answers in the context of music theory than the students who struggled through the mathematical content in Ms. K's remedial Secondary Mathematics I class. This suggests that the role of music in students' translations may differ according to their mathematical content knowledge. Students with previous music or musical instrument experience drew from knowledge funds outside of class, while those without previous experience were not able to draw on past experiences. Overall, these findings suggest that students' previous knowledge is an important factor in music integration in mathematics lessons.

Fourth, it is notable that the music context influenced both the students' translation processes and the target representations they constructed. Students used the music to label mathematical processes, evaluate the accuracy of their mathematics, and draw from previous funds of knowledge as they worked. They also used music to label their mathematical answers, make sense of their answers, and access funds of knowledge from outside of class. This finding suggests that the role of music in students' translations is twofold, influencing both students' mathematical processes and students' understanding of their mathematical answers. This is important in the conceptual understanding of the relationships between mathematical translations and music integration discussed in the conceptual framework later in this chapter.

Finally, it is interesting to note the prevalence of uni-directional and multi-directional translations made by the students. These translation types far outnumbered the bi-directional and abstract connections made. This was likely influenced by the task questions themselves. The task questions coded as bi-directional and abstract connections were task questions written specifically to elicit these translation types, as predicted in Appendix C. This is consistent with findings from Fonger and colleagues (2011, 2019) suggesting that task questions can be designed to elicit certain types of translations. These results suggest that bi-directional and abstract connections can be elicited from students by task questions that require students to look for commonalities across representation types.

Students' Utilization of Music According to Representation Type

Previous research on representations and translations has consistently failed to include contextual representations as a distinct representation type (Bossé et al., 2011; Fonger, 2011; Fonger & Altindis, 2019; Gulkilik et al., 2020), despite their inclusion in modern mathematics practicum texts (NCTM, 2014). Research on translations has more heavily investigated symbolic, tabular, and graphical representations. Results from this study illuminated that the role the music context played in students' translations varied depending on the representation types involved.

A main finding was that it was much more likely for translations involving contextual, physical, and/or verbal translations to result in a contextualized answer than translations involving visual and/or symbolic representations. Task problems that elicited more contextualized answers were problems that explicitly provided or called for musical interpretations, encouraged students to write out their thinking, and/or required students to work with the PVC pipes or piano. Task problems that required filling out tables, interpreting graphs, or writing equations without explicitly asking students to think about music theory were more likely to not generate any musical interpretations from the students. Findings from this study highlighted both the existence of contextual representations and also their influence on students' likelihood to generate mathematics that was understood in terms of music theory. More specifically, music-contextualized mathematics lessons that span across all five representation types can be created. These results do not suggest that visual and symbolic representations should not be emphasized, but rather that any task question with any representation type can present students with more opportunities to use music. Simply writing a mathematical task that requires students to conceptualize their answer and its meaning in the context of music is more likely to generate students' connections with music. Overall, these findings suggest that music utilization can differ according to mathematical representation type, and that the wording of task problems plays an important role in students' understandings.

Students' Constructive Activities with Contextual Representations

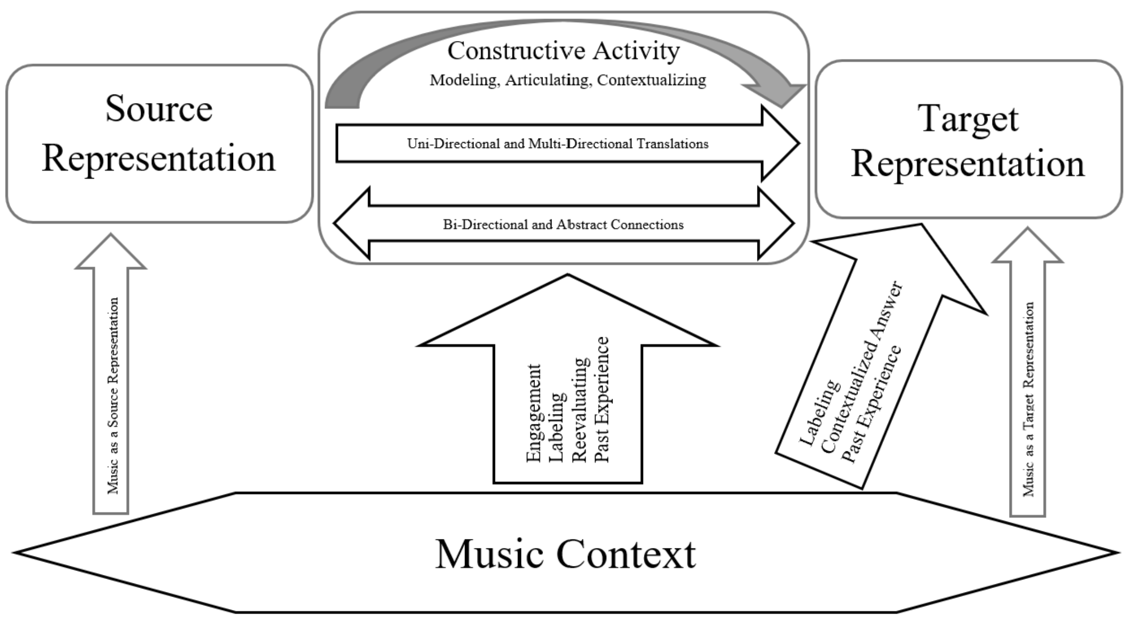
Finally, results from this study showed that while using music context as a source and target contextual representation, students will engage in certain constructive activities. Previous theoretical work presenting models for the constructive activities students enact when translating has not included contextual representations as a unique representation type (Janvier, 1987; Bossé et al., 2011a). Findings from this study provide further insight into the kinds of constructive activities that students engage in when contextual representations are present in the translation. Specifically, students in this study *articulated* their understandings when translating from the music context to a verbal representation. They also engaged in *modeling* when translating from the music context to symbolic representation. This finding confirmed the argument from NCTM (2000) that working with representations is important for students to acquire the skill of modeling. And finally, the students *contextualized* while translating to the music context from physical, visual, or symbolic source representations. These findings suggests that students are able to both extract mathematical meaning from real-world situations and construct real-world applications from mathematics when in a music-contextualized mathematics environment. This is important in developing a greater conceptual understanding of the role of music context acting as both a source and target representation.

The Role of Music Context in High-School Students' Translations

Music context plays an expansive and diverse role in students' translations among representations. Figure 52 illustrates the emerging understanding of the relationships between concepts explored in this case study. This conceptual framework differs from the original framework from this study in several ways.

Figure 52

Revisited Conceptual Framework



First, results from this study showed that when task problems are purposefully designed to encourage students to draw connections between music theory and mathematics, music context influences students' mathematical processes. While it was previously unknown what role music context could play in translations, the center vertical arrow now captures four ways the music context was involved in those processes. The role of music while students construct representations can be (1) a source of engagement, (2) a way to label mathematics, (3) a means of reevaluating mathematical accuracy, and (4) a fund of outside knowledge.

Second, results from this study showed that the music context was also present in students' interpretation of answers. The diagonal arrow connecting music context to target representations illustrates three ways students used the context in their answers that were not present in the original conceptual framework. The role of music in students' mathematical answers was (1) a way to label, (2) a way to make sense of answers, and (3) a fund of outside knowledge from outside of class to which they could connect their answers in a meaningful way.

Finally, in the original conceptual framework, it was not yet known what constructive activities students engaged in while translating to and from contextual representations. Results from this study showed that students engaged in modeling, articulating, and contextualizing while translating to and from contextual representations. These three constructive activities can be seen at the top of the revisited framework In Figure 52.

Limitations

There were several limitations in this descriptive multiple-case study. The main limitations were the number of participants, the number of task problems, and the duration of the study.

Like most multiple-case studies, this study gathered data from a small group of participants. With only 16 student participants, it is likely that this sample is not representative of all high school aged students in general. A large-scale study with a random sampling of more participants would be needed to affirm whether the music utilizations found in this study extend to larger populations of high school students.

Another limitation of this study was the number of task sheets and problems completed by students. To make data analysis manageable for qualitative analysis, students in this study completed at most 31 task problems, sometimes less, across just three lessons. This limited the pool of task problems to analyze for potential utilizations of music, possibly limiting the types of codes developed and the relative quantity of each code.

Finally, data collection for this study was completed in 3-4 consecutive days at each site. Students' mathematics, music utilization, and conversations were likely influenced by the time of the year and the mathematics content covered in class just before data collection. A longitudinal study would be necessary to determine students' utilization of music across an entire high school course.

Educational Implications

Three main implications for mathematics classrooms can be drawn from this study. First, teachers and other educational stakeholders wishing to heighten students' engagement in and enjoyment of mathematics should consider the potential value of music integration in high school mathematics courses.

Second, teachers wishing to implement music-contextualized mathematics should be cognizant of the important role of students' previous music and mathematical knowledge in the way students will utilize the music context. Teachers should make use of students' previous knowledge and incorporate it into the lesson design.

Finally, teachers integrating music into their mathematics lessons who wish to maximize the number of contextual connections students make with mathematics can ensure there are problems that (1) provide contextual information, (2) explicitly require a contextualized answer, (3) work with physical representations like instruments, and (4) require them to articulate their understandings. These features can be present alongside any representation type. Teachers can also create variety in their tasks by encouraging more bi-directional and abstract translations. Teachers can choose wording in task problems that encourages students to find common invariant features throughout two or more different representations.

Future Research

This study examined students' utilization of music during lessons focused primarily on algebraic topics. However, subject matter connections between music theory and mathematics expand across all of the five common core mathematics content strands (An et al., 2015). A similar study to this one could be conducted with mathematics lessons focusing on music integration in geometry, trigonometry, measurement, and/or data and analysis. This would provide insight into whether the students' utilizations of music in this study are exclusive to algebra content or extend across all mathematics content areas.

Investigating the teacher's role in students' translations was outside the scope of this study. However, teacher knowledge and choices are important in students' translations among mathematical representations is important (Dreher & Kuntze, 2015; Samsuddin & Retnawati, 2018). This study did suggest that in the follow-up interview, questioning students allowed them to further articulate their contextual understandings beyond what they discussed with their partner in class. Further research could investigate the influence of teachers' questioning, scaffolding, and in-class decision-making on students' utilization of music in music-contextualized mathematics lessons. This research could offer teachers a better understanding of how they can better assist students in making connections to music.

Finally, few studies have investigated contextual, "real-world," or "situational" representations as a legitimate representation type. This study confirmed the presence of this representation type and helped illustrate what translations with contextual representations look like. However, these contextual representations were limited to the narrow context of music. Future research should investigate translations and constructive activities with contextual representations in other real-world contexts outside of music. This research could further the field of study on mathematical representations and translations by illuminating conceptual relationships not just with music source and target representations, but with contextual source and target representations in general.

Conclusion

This study confirmed that music context plays an intricate role in students' translations when integrated into high-school mathematics algebra lessons emphasizing representations. The key takeaways in this study are the various ways students utilize a music context as they complete translations. These utilizations are influenced by the wording of task problems and the representation types involved. This study also provides further understanding of what contextual representations look like, and how students translate to and from them. Findings from this study illuminate new conceptual relationships between music context and mathematical translations, bridging these two distinct and seemingly unrelated fields of study. These findings are important in supporting teachers wishing to integrate music into their mathematics classrooms, and in extending conceptual understanding of the relationships between music, representations, and translations.

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Appendices

Appendix A. Teacher Lesson Plan

(Adapted from NCTM, 2021)

Grade Band: 10th/11th

Time: 240 Minutes across 3 days (80 minutes each day)

Materials

- Set of 8 pre-cut ½ inch PVC pipes for each group
- Measuring tapes/ruler for each group
- Duct Tape
- Accompanying Lesson Worksheet
- Computer with internet access for each partnership
- Table of equal tempered musical frequencies
- Table of "linear" musical frequencies
- Piano/keyboard

Instructional Plan

DAY I (80 Minutes)

PART I (35 Minutes)

Launch (5 Minutes)

To launch the lesson, the teacher will pass out the sets of PVC pipes to each group and let the students play with them briefly. The teacher will then pass out the measuring tools and ask the students to start comparing each pipe to the C4 pipe, asking students to make inferences about the ratios between them. The teacher will then pass out the worksheet that accompanies this lesson and tell students to begin working on it in partners.

Image Source: Divis & Johnson, 2021, p. 49

Explore (20 Minutes)

Students will be given 20 minutes to explore the lengths of the pipes and fill out the accompanying table on Task Sheet 1 Problem 1. The following tables show the lengths of the pre-cut pipes, and the correct ratios.

Lengths used when cutting pipes for major scale between C4 and C5

C4	D4	E4	F4	G4	A4	B4	C5
33 cm	29.3 cm	26.4 cm	24.8 cm	22 cm	19.8 cm	17.6 cm	16.5 cm

Table Source: Divis & Johnson, 2021, p. 49

Students will then complete Problems 2-4 on Task Sheet 1 in partners.

Pythagorean ratios for C Major Scale

C4	D4	E4	F4	G4	A4	B4	C5
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Table Source: Divis & Johnson, 2021, p. 49

Discuss (10 Minutes)

After giving the students sufficient time to fill out the table with some of the ratios, the teacher will bring the class back together to complete the table with the teacher's assistance. The teacher will encourage students to use duct tape to construct instruments with the pipes and allow them to play with the instruments. The teacher will play some familiar songs for the students that illustrate these common intervals. Examples include:

Perfect 5th

- Elvis, Wise Men Say: https://www.youtube.com/watch?v=vGJTaP6anOU

- Mary Poppins, Chim Chim Cheree: https://www.youtube.com/watch?v=kG6O4N3wxf8
- Star Wars Theme song: https://www.youtube.com/watch?v=_D0ZQPqeJkk&t=13s (after the pickup)
- Flintstones Theme song: https://www.youtube.com/watch?v=UL7beNWNLEQ
- Superman Theme: https://www.youtube.com/watch?v=e9vrfEoc8_g#t=43s

Perfect 4th

- Here Comes the Bride: https://www.youtube.com/watch?v=lgh9XTkQTDI

Perfect 6th

- NBC Today's Show theme: https://youtu.be/ZsXN_GaPL9U

FIVE MINUTE BREAK

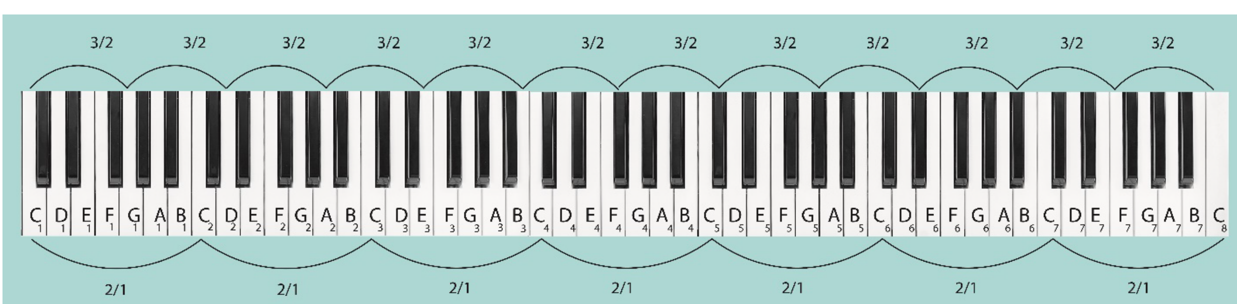
PART II (40 Minutes)

Launch (5 Minutes)

The teacher will then explain that these ratios were first discovered by Pythagoras, and that a system of tuning known as "Pythagorean Tuning" was then used to develop and tune instruments, until problems with it were later discovered. The teacher will explain that while the lengths of the pipes decrease as the pitch increases, the frequency increases as the pitch increases. If possible, at this point the class will travel together to a piano. If that is not possible, a large screen display of a full piano will suffice.

Explore (30 Minutes)

The teacher will ask the students to begin on C_1 near the bottom of the piano and count up by perfect fifths (7 half-steps, or steps between notes that are adjacent with no note between them) until they reached C_8. The teacher will then ask students to begin on C1 again and this time count up by octaves until they reached C_8. The students will conclude that both twelve intervals of perfect fifths and eight intervals of octaves landed on the same note (C_8)



The class will then return to their normal classroom and seats to write out their work for Task Sheet 1 Problems 5-8. The students will mathematically discover the inconsistency in the Pythagorean tuning system shown here.

$$\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}=\left(\frac{3}{2}\right)^{12}=\frac{531{,}441}{4096}\cong 129.75$$

$$\frac{2}{1}\times\frac{2}{1}\times\frac{2}{1}\times\frac{2}{1}\times\frac{2}{1}\times\frac{2}{1}\times\frac{2}{1}\times\frac{2}{1}=\left(\frac{2}{1}\right)^{7}=\frac{128}{1}=128$$

Discuss (5 Minutes)

The teacher will bring the class back together and have a student share their ratios on the board. The teacher will explain that this discrepancy in the ratios between frequencies is called “the Pythagorean Comma,” and prompted the creation of a more mathematically consistent system for tuning.

DAY II (80 Minutes)

PART I

Launch (5 Minutes)

The teacher will show students a table with the standard “equal-tempered” frequencies between C4 to C5 on the board. The teacher will explain that mathematical principles still govern the tuning of instruments today, but in a different way than the Pythagorean system.

C4	C#4	D4	D#4	E4	F4	F#4	G4	G#4	A4	A#4	B4	C5
261.6	277.2	293.7	311.1	329.6	349.2	370	392	415.3	440	466.2	493.9	523.3

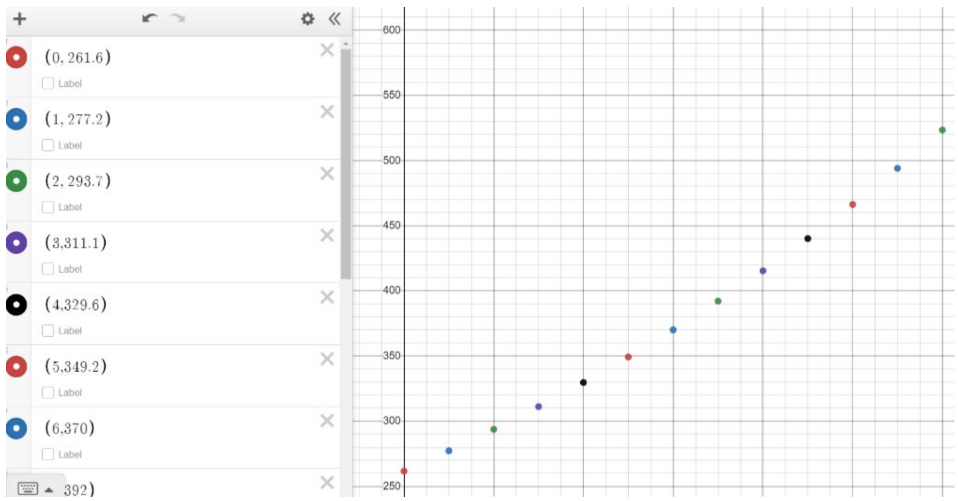
Explore (55 Minutes)

The teacher will ask the students to open the Desmos Graphing Calculator and discuss Task Sheet 2 Problems 1-12.
The teacher will walk around the room and use scaffolding questions to encourage student discussions of domain, range, scale, what to use for the input values, etc. Students

will enter in the 13 data points and observe the outcome. Example questions might include:

1. If we want to view all of our data points at once in our window, what might be a good choice for our input values?

2. What are our lowest and highest output values?



FIVE MINUTE BREAK

Discuss (20 Minutes)

The teacher will bring the whole class back together for a follow-up discussion of the graph. The teacher will highlight the relationship between the table, graph, and equation with the assistance of the student presentations.

DAY III (80 Minutes)

PART I

Launch (5 minutes)

The teacher will explain that today the focus will be investigating the mysterious number 1.059 from the previous day. The teacher will conduct a brief conversation with students about their thoughts about the number. Where might it come from? Why is it such an "ugly" and irrational number that governs modern musical tuning?

Explore (30)

Students will work in pairs on answering Problems 1-4 on Task Sheet 3 to try to derive the meaning and exact value of 1.059.

Discuss (15 Minutes)

Teacher will bring the class back together to talk about what they discovered about the number. The teacher will have a student present their work on how they found $\sqrt[12]{2}$. The teacher will take this opportunity to discuss rational exponents in general and their properties.

FIVE MINUTE BREAK

PART II

Launch (5 minutes)

The teacher will ask the students to imagine that musical frequencies instead progressed in a constant linear manner and ask the students to think about how that would change the progression of pitches and what the resulting scale might sound like.

Explore (15 Minutes)

C4	C#4	D4	D#4	E4	F4	F#4	G4	G#4	A4	A#4	B4	C5
261.6	283.4	305.2	327	348.8	370.6	392.4	414.2	436	457.8	479.6	501.4	523.2

The students will work with partners to construct the table above with “linear” pitches, completing Problems 5-9 on Task Sheet 3.

Discuss (10 Minutes)

The teacher will then assign eight of the students a note in the major scale (no sharps) from C4 to C5 and asked them to enter their associated frequency into an online tone generator, asking them to listen carefully to the “linear scale” as they each played their note in turn. Students will then repeat this activity with the correct equal tempered eight-note scale to hear that it is more agreeable to our ears. The audio files provided in the accompanying MT: PK-12 article can also be used to shorten and simplify this activity. Students will then complete Problem 10 in partners. If time, the teacher can play some song examples illustrating the difference between tuning systems.

- https://open.spotify.com/track/1WbfAfJrkvIPOBDi9h7Yo7?si=20965d3078884d12&nd=1
- https://www.youtube.com/watch?v=apxFH6CDBMU

Standards

Main Content Standards

CCSS.MATH.CONTENT.HSF.LE.A.1
Distinguish between situations that can be modeled with linear functions and with exponential functions.
CCSS.MATH.CONTENT.HSF.LE.A.2
Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).
CCSS.MATH.CONTENT.HSF.LE.B.5

Interpret the parameters in a linear or exponential function in terms of a context.

Related Resources

Desmos: https://www.desmos.com/calculator

Link to article: https://pubs.nctm.org/view/journals/mtlt/114/1/article-p47.xml?tab_body=Audio

Full list of frequencies: https://pages.mtu.edu/~suits/notefreqs.html

Appendix B. Student Task Sheets

Music and Mathematics Task Sheet 1

1. Compare the C4 pipe to the C5 pipe. What ratio do you notice about their comparative lengths?

2. What other common ratios do you notice when examining the lengths of the pipes?

C4	D4	E4	F4	G4	A4	B4	C5

3. The song "Twinkle Twinkle Little Star" begins with the interval formed by playing C4 and then G_4. If the frequency of C4 is ~261.6, use the ratios to find the frequency of G_4.

4. Write down the names of other commonly known songs whose intros can be played on your pipes instrument, with the associated ratios between the first two notes.

5. Talk to your neighbor about your experience at the piano as you explored the repeated intervals from C_1 to C_8. Write here some of the things you talk about.

6. Use repeated multiplication to show how you could find the ratio for twelve consecutive "5th" intervals and seven consecutive octaves on the piano using the Pythagorean ratios above.

7. Simplify these repeated multiplications using exponents.

8. Compare the final resulting ratios. What do you notice?

Music and Mathematics Task Sheet 2

1. Using the list of common Western frequencies shown in the table on the board, and at https://pages.mtu.edu/~suits/notefreqs.html we are going to construct a graph. First, what would be a good choice for a window for viewing all the data points in the table?

2. If we assign each value in the table an input value of 1-13 because there are 13 values, what will be the domain and range of our graph?

3. As you complete your Desmos graph of the table of values, talk to your neighbor about exactly what you are doing, and record some of your main conversation points here.

4. Does the resulting graph represent discrete or continuous data?

5. What does the space between data points represent in terms of musical frequencies?

6. Examine your Desmos graph and determine what kind of relationship exists between the data points. Be sure to include an explanation of how you know that type of relationship exists.

7. Use the graph or table of data points to determine the common ratio between consecutive values. Explain what this value represents in the context of musical frequencies.

8. Write an equation that fits the graph of the data points.

9. Interpret the meaning of inputs and outputs of this equation in terms of musical frequencies?

10. Verify the accuracy of your equation by demonstrating it produces at least two sets of data points found in the table.

11. Explain how each piece of your equation relates to your table.

12. Explain how each piece of your equation relates to your graph.

13. Based on what we've discovered with this graph, table, and equation, what are some generalizations you can make about the mathematical properties of musical tuning?

Music and Mathematics Task Sheet 3

1. Discuss 1.059 with your partner. What did it represent, and how was it derived?

2. The frequency of C4 is ~261.6. The frequency of C5 is ~523.2. Write an expression showing how you would find the frequency of C5 by taking the frequency of C4 and using the Pythagorean ratio between octaves.

3. Equal-temperament tuning is based on the ratio used when progressing an octave from C4 to C5. If multiplying the frequency by 2 increases the pitch by an octave, what you might multiply by to progress from C4 to C#4? As shown on the picture of the keyboard on the board, there are 12 of these small steps between C4 to C5.

4. Use the exact value of 1.059 that we arrived upon following our class discussion to find the frequency of C#5 given that the frequency of C5 is ~523.3.

5. If the frequency of C4 and C5 were still ~261.6 and ~523.2, respectively, but musical instruments were instead tuned using a constant linear progression between notes, show how you would find the constant increase between the frequency of adjacent notes (e.g., C4 and C#4)?

6. Use the constant rate of change you found to construct this table will all the frequencies between C4 and C5.

"Linear" musical frequencies from C4 to C5												
C4	C#4	D_4	$D^{\#}_4$	E_4	F_4	$F^{\#}_4$	G_4	$G^{\#}_4$	A_4	$A^{\#}_4$	B_4	C5
261.6												523.2

7. Write an equation that fits the data set in the table.

8. Graph the linear and exponential equations together on Desmos. Talk about the intersection points, the y-intercepts, and the end behavior as x approaches infinity for both graphs, while comparing and contrasting. Write your main conclusions here.

9. After your teacher passes back your PVC pipe instrument to you, explain how the pipes of the instrument would be sized differently if they were tuned according to this linear progression of frequencies?

10. Listen to your teacher play the audio files for the equal-tempered and "linear" scales. What do you notice about the sounds of these?

Appendix C. List of Task problems

Complete List of Problems with Expected Representation and Translation

Location	Problem	Expected Representations Involved (NCTM, 2014)	Expected Translation Type (Fonger, 2011)
Task Sheet 1	Problem 1	Physical/Symbolic	Uni-directional
Task Sheet 1	Problem 2	Physical/Visual	Uni-directional
Task Sheet 1	Problem 3	Contextual/Symbolic	Multi-directional
Task Sheet 1	Problem 4	Contextual/Physical/Symbolic	Multi-directional
Task Sheet 1	Problem 5	Physical/Verbal	Uni-directional
Task Sheet 1	Problem 6	Symbolic/Contextual/Physical	Multi-directional
Task Sheet 1	Problem 7	Symbolic	Uni-directional
Task Sheet 1	Problem 8	Symbolic/Verbal	Uni-directional
Task Sheet 2	Problem 1	Visual	Multi-directional
Task Sheet 2	Problem 2	Visual/Symbolic	Multi-directional
Task Sheet 2	Problem 3	Visual/Verbal	Multi-directional
Task Sheet 2	Problem 4	Visual/Verbal	Uni-directional
Task Sheet 2	Problem 5	Visual/Contextual/Verbal	Multi-directional
Task Sheet 2	Problem 6	Visual/Symbolic	Uni-directional
Task Sheet 2	Problem 7	Visual/Verbal/Contextual/Symbolic	Multi-directional
Task Sheet 2	Problem 8	Visual/Symbolic	Uni-directional
Task Sheet 2	Problem 9	Symbolic/Contextual	Multi-directional
Task Sheet 2	Problem 10	Visual/Symbolic	Bi-directional
Task Sheet 2	Problem 11	Visual/Symbolic	Bi-directional
Task Sheet 2	Problem 12	Visual/Symbolic	Bi-directional
Task Sheet 2	Problem 13	Visual/Symbolic/Verbal/Contextual	Abstract
Task Sheet 3	Problem 1	Symbolic/Visual/Verbal	Multi-directional
Task Sheet 3	Problem 2	Symbolic/Contextual	Uni-directional
Task Sheet 3	Problem 3	Symbolic/Contextual	Uni-directional
Task Sheet 3	Problem 4	Symbolic	Uni-directional
Task Sheet 3	Problem 5	Contextual/Symbolic	Uni-directional
Task Sheet 3	Problem 6	Symbolic/Tabular	Uni-directional
Task Sheet 3	Problem 7	Visual/Symbolic	Uni-directional
Task Sheet 3	Problem 8	Symbolic/Physical/Verbal/Contextual	Multi-directional
Task Sheet 3	Problem 9	Physical/Contextual	Multi-directional
Task Sheet 3	Problem 10	Contextual/Verbal	Uni-directional

Total Minimum Anticipated* Symbolic	20
Total Minimum Anticipated Visual	15
Total Minimum Anticipated Physical	6
Total Minimum Anticipated Verbal	10
Total Minimum Anticipated Contextual	12
Total Minimum Anticipated Uni-directional	15
Total Minimum Anticipated Bi-directional	3
Total Minimum Anticipated Multi-directional	12
Total Minimum Anticipated Abstract	1

**Note*. These are minimum anticipated number of representations based on the content of the task problem and the representations that are explicitly called for. However, the actual numbers varied based on students' processes.

Appendix D. Follow-up Semi-Structured Interview Protocol with Guiding Questions

Beginning Script:
This interview is to help me gain further understanding of your thought processes as you completed the music and mathematics tasks with your teacher last week. I have your completed worksheets here and am going to ask you to elaborate on your thinking for each question and talk to me about what you wrote on your worksheet. Depending on your answers, I might ask you to further elaborate on something you say. I am particularly interested in the way you utilized the music context while answering these questions. I am conducting this interview with both of you because you completed the task together, and either of you is welcome to answer my questions or elaborate on your partner's answers. This interview is expected to last 30 minutes. Let me know at any time if you need a break or would like to discontinue the interview.

General Guiding Questions for each Task problem:

1. Tell me more about what the two of you discussed as you worked on this question.
2. Can you explain to me this part of your answer right here?
3. How did you arrive at this answer?
4. Why did you decide to do _____ instead of ______?
5. I notice you didn't complete this question. Did you get stuck? If so, can you tell me more about that?

Questions Specific to Research Question 1:

1. In what ways did you rely on the music context as you constructed this ______.
2. What aspects of the music context did you notice as you interpreted this original representation given to you in this problem?
3. Were you thinking about the (music notes/ pipe instrument/ frequencies/ etc.) when you worked on this problem?

Questions Specific to Research Question 2:

1. What were you creating here in this problem?
2. What information were you given in this problem?
3. Did you go straight from this _____ you were given to this _____ you created or was there anything that you created in between?
4. In this problem, you were asked to create a ____ given a _____. As you did that, how did you rely on the music context?

Questions Specific to Research Question 3:

1. In this problem, you were asked to create a __________ given ________. Can you tell me the processes you went through to create that?
   a. What were you thinking in your head when you did this part?
   b. I noticed you didn't write down your work. Can you show me what you did in your head here on paper?

Ending Script:

Thank you so much for participating in this interview. Your answers will be very helpful to me in understanding the way you used music in the lessons your teacher taught last week. I have everything I need, so I am going to end the audio recording, and you are free to go.

Appendix E. Complete List of Each Pair's Translations and Music Utilizations

Vincent and Ethan's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical >Symbolic>Symbolic>Verbal		Multi	CA, E
1	2	Visual>Symbolic>Symbolic		Multi	N
1	3	Contextual>Symbolic>Symbolic		Uni	N
1	4				
1	5	Physical>Contextual>Verbal		Uni	CA
1	6	Physical>Symbolic		Uni	L
1	7	Contextual>Symbolic>Symbolic		Uni	L
1	8	Symbolic>Verbal		Uni	N
2	1	Visual>Symbolic>Visual		Multi	N
2	2	Visual>Symbolic		Uni	N
2	3	Visual>Visual		Uni	N
2	4	Visual>Verbal		Uni	N
2	5	Visual>Contextual>Verbal		Multi	CA
2	6	Visual>Verbal		Uni	N
2	7	Visual>Symbolic>Contextual>Verbal		Multi	CA
2	8	Visual>Symbolic		Uni	N
2	9	Symbolic>Contextual		Uni	CA
2	10		Visual/Symbolic	Bi	N
2	11		Visual/Symbolic	Bi	N
2	12		Visual/Symbolic	Bi	N
2	13		All	Abs	CA
3	1	Symbolic>Verbal		Uni	CA
3	2	Contextual>Symbolic		Uni	CA
3	3	Contextual>Symbolic>Symbolic>Contextual		Multi	CA
3	4	Contextual>Symbolic		Uni	N
3	5	Contextual>Symbolic>Symbolic		Uni	N
3	6	Symbolic>Visual		Uni	N
3	7	Visual>Symbolic		Uni	N
3	8	Symbolic>Visual>Verbal		Multi	N
3	9		Contextual/Physical	Bi	E
3	10	Contextual>Verbal		Uni	CA

Cameron and Eric's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical>Symbolic		Uni	L, E
1	2	Physical>Verbal		Uni	E, L
1	2	Visual>Symbolic>Symbolic		Multi	N
1	3	Contextual>Symbolic>Symbolic		Multi	CA
1	4		Physical/Contextual	Bi	E
1	5	Physical>Contextual>Verbal		Multi	L
1	6	Physical>Symbolic		Uni	N
1	7	Symbolic>Symbolic		Uni	N
1	8	Symbolic>Verbal		Uni	CA
2	1	Visual>Symbolic>Visual		Multi	CA
2	2	Visual>Symbolic		Uni	N
2	3	Visual>Visual		Uni	CA
2	4	Visual>Verbal		Uni	N
2	5	Visual>Contextual>Verbal		Multi	CA
2	6	Visual>Verbal		Uni	N
2	7	Visual>Symbolic>Contextual>Verbal		Multi	N
2	8	Visual>Symbolic		Uni	N
2	9	Symbolic>Contextual>Verbal		Multi	CA
2	10		Symbolic/Visual	Bi	CA
2	11		Symbolic/Visual	Bi	CA
2	12		Symbolic/visual	Bi	N
2	13		All	Abstract	CA
3	1	Symbolic>Verbal		Uni	N
3	2	Contextual>Symbolic		Uni	N
3	3	Contextual>Symbolic		Uni	N
3	4	Contextual>Symbolic		Uni	CA
3	5	Contextual>Symbolic>Symbolic		Multi	N
3	6	Symbolic>Visual		Uni	N
3	7	Visual>Symbolic		Uni	N
3	8	Visual>Verbal		Uni	CA
3	9		Contextual/Physical	Bi	N
3	10	Contextual>Verbal		Uni	CA

Elise and Logan's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical>Contextual>Verbal		Multi	E, CA
		Physical>Symbolic>Visual>Symbolic		Multi	L
1	2				
1	3		Physical/Contextual	Bi	CA
1	4				
1	5	Physical>Contextual>Verbal		Multi	CA
1	6				
1	7	Symbolic>Symbolic		Uni	N
1	8	Symbolic>Verbal		Uni	L
2	1	Visual>Verbal		Uni	N
2	2	Visual>Symbolic		Uni	N
2	3				
2	4	Visual>Verbal		Uni	N
2	5	Visual>Contextual>Verbal		Multi	PE
2	6	Visual>Verbal		Uni	N
2	7				
2	8	Visual>Symbolic		Uni	N
2	9				
2	10				
2	11				
2	12				
2	13				
3	1	Symbolic>Verbal		Uni	N
3	2	Contextual>Symbolic		Uni	N
3	3				
3	4	Contextual>Symbolic		Uni	CA
3	5	Contextual>Symbolic>Symbolic		Multi	N
3	6	Symbolic>Visual		Uni	N
3	7				
3	8				
3	9		Physical/Contextual	Bi	N
3	10	Contextual>Verbal		Uni	CA

Rachel and Rebecca's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical>Contextual>Verbal		Multi	CA, E
1	2	Visual>Symbolic		Uni	E, L
1	3	Contextual>Symbolic>Symbolic		Multi	N
1	4				
1	5	Physical>Contextual>Verbal		Multi	CA
1	6	Visual>Verbal		Uni	CA
1	7				
1	8	Symbolic>Verbal		Uni	N
2	1	Visual>Symbolic>Visual		Multi	N
2	2	Visual>Symbolic		Uni	N
2	3	Visual>Visual		Multi	CA
2	4	Visual>Verbal		Uni	N
2	5	Visual>Contextual>Verbal		Multi	CA
2	6				
2	7	Visual>Symbolic>Contextual>Verbal		Uni	CA
2	8	Visual>Symbolic>Verbal		Multi	N
2	9	Visual>Verbal		Uni	N
2	10	Symbolic>Contextual>verbal		Multi	CA
2	11		Visual/Symbolic	Bi	N
2	12				
2	13				
3	1	Symbolic>Verbal		Uni	N
3	2	Contextual>Symbolic		Uni	L
3	3				
3	4	Contextual>Symbolic>Symbolic> Contextual		Multi	CA
3	5	Symbolic>Symbolic>Contextual>Ver bal		Multi	CA
3	6	Symbolic>Visual		Uni	N
3	7				
3	8	Visual>Verbal		Uni	N
3	9		Contextual/Physical	Bi	CA
3	10	Contextual>Verbal		Uni	CA

Aaron and Sam's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical>Symbolic>Symbolic		Multi	CA
1	2	Physical>Symbolic>Symbolic		Multi	L, E
1	3	Contextual>Symbolic>Symbolic		Multi	CA
1	4				
1	5	Physical>Contextual>Verbal		Multi	CA
1	6	Physical>Symbolic		Uni	N
1	7	Symbolic>Symbolic		Uni	N
1	8	Symbolic>Contextual>Verbal		Uni	E, CA
2	1	Visual>Symbolic>Visual		Bi	CA
2	2	Visual>Symbolic		Uni	CA
2	3	Visual>Visual		Uni	L
2	4	Visual>Verbal		Uni	CA
2	5	Visual>Contextual>Verbal		Uni	CA
2	6	Visual>Verbal		Uni	CA
2	7				
2	8	Visual>Symbolic		Uni	L
2	9	Symbolic>Contextual>Verbal		Multi	CA
2	10		Symbolic/Visual	Bi	N
2	11		Symbolic/Visual	Bi	CA
2	12		Symbolic/Visual	Bi	CA
2	13		All	Abstract	CA
3	1	Symbolic>Verbal		Uni	CA
3	2	Contextual>Symbolic>Symbolic		Multi	N
3	3	Contextual>Symbolic>Contextual>Verbal		Uni	CA
3	4	Contextual>Symbolic>Symbolic		Multi	CA
3	5	Contextual>Symbolic>Symbolic		Multi	CA
3	6	Symbolic>Visual		Uni	N
3	7	Visual>Symbolic		Uni	N
3	8	Symbolic>Visual>Contextual>Verbal		Multi	CA
3	9				
3	10		Contextual/Verbal	Bi	CA

Mitch and Paul's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical>Symbolic>Symbolic		Multi	E, CA
1	2	Physical>Symbolic>Symbolic		Multi	L
1	3	Contextual>Symbolic>Symbolic		Multi	CR
1	4		Physical/Contextual	Bi	PE, E, CA
1	5	Physical>>Contextual>Verbal		Multi	CA
1	6	Physical>Symbolic		Uni	N
1	7	Symbolic>Symbolic>Contextual>Verbal		Uni	CA
1	8	Symbolic>Contextual>Verbal		Multi	CR
2	1	Visual>Symbolic>Visual		Multi	L
2	2	Visual>Symbolic		Uni	N
2	3	Visual>Visual		Uni	CA
2	4	Visual>Verbal		Uni	N
2	5	Visual>Contextual>Verbal		Multi	CA
2	6	Visual>Contextual>Verbal		Uni	CR
2	7	Visual>Symbolic		Uni	CA
2	8	Visual>Symbolic		Uni	CA
2	9	Symbolic>Contextual>Verbal		Uni	CA
2	10		Visual/Symbolic	Bi	N
2	11		Symbolic/Visual	Bi	N
2	12		Symbolic/Visual	Bi	N
2	13		All	Abstract	CA
3	1	Symbolic>Contextual>Verbal		Multi	CA
3	2	Symbolic>Contextual>Verbal		Multi	CA
3	3	Contextual>Symbolic>Symbolic		Multi	CA
3	4	Contextual>Symbolic>Symbolic		Multi	L
3	5	Contextual>Symbolic>Symbolic		Multi	L
3	6	Symbolic>Visual		Uni	N
3	7	Visual>Symbolic		Uni	N
3	8	Symbolic>Visual>Contextual>Verbal		Multi	CA
3	9		Physical/Contextual	Bi	CA
3	10	Contextual>Verbal		Uni	CA, PE

Emily and Amber's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical>Symbolic		Uni	CA
1	2	Visual>Symbolic>Symbolic		Multi	L
1	3	Contextual>Symbolic>Symbolic		Multi	CA
1	4		Physical/Contextual	Bi	E
1	5	Physical>Contextual>Verbal		Multi	CA
1	6	Physical>Symbolic		Uni	CA
1	7	Symbolic>Symbolic		Uni	CA
1	8	Symbolic>Verbal		Uni	N
2	1	Visual>Symbolic		Uni	CA
2	2	Visual>Symbolic		Uni	N
2	3	Visual>Visual		Uni	CA
2	4	Visual>Verbal		Uni	N
2	5	Visual>Contextual>Verbal		Multi	CA
2	6	Visual>Verbal		Uni	CR
2	7	visual>Symbolic>Contextual		Multi	CA
2	8	Visual>Symbolic		Uni	N
2	9	Symbolic>Contextual>Verbal		Multi	CA
2	10		Visual/Symbolic	Bi	N
2	11		Visual/Symbolic	Bi	N
2	12		Visual/Symbolic	BI	N
2	13		All	Abstract	N
3	1	Symbolic>Verbal		Uni	CA
3	2	Contextual>Symbolic		Uni	L
3	3	Contextual>Symbolic		Uni	N
3	4	Contextual>Symbolic>Symbolic		Multi	N
3	5	Contextual>Symbolic>Symbolic		Multi	N
3	6	Symbolic>Visual		Uni	N
3	7	Visual>Symbolic		Uni	CA
3	8	Visual>Verbal		Uni	L
3	9		Physical/Contextual	Bi	PE, CA
3	10	Contextual>Verbal		Uni	PE, CA

Jason and Gabe's Translations and Music Utilization for Each Problem

Task	Problem	Representation	or Connection	Type	Music
1	1	Physical>Symbolic		Uni	CA
1	2	Visual>Symbolic>Symbolic		Multi	L
1	3	Contextual>Symbolic>Symbolic		Multi	CA
1	4		Physical/Contextual	Bi	E
1	5	Physical>Contextual>Verbal		Multi	CA
1	6	Physical>Symbolic		Uni	CA
1	7	Symbolic>Symbolic		Uni	CA
1	8	Symbolic>Verbal		Uni	N
2	1	Visual>Symbolic		Uni	CA
2	2	Visual>Symbolic		Uni	N
2	3	Visual>Visual		Uni	CA
2	4	Visual>Verbal		Uni	N
2	5	Visual>Contextual>Verbal		Multi	CA
2	6	Visual>Verbal		Uni	CR
2	7	visual>Symbolic>Contextual		Multi	CA
2	8	Visual>Symbolic		Uni	N
2	9	Symbolic>Contextual>Verbal		Multi	CA
2	10		Visual/Symbolic	Bi	N
2	11		Visual/Symbolic	Bi	N
2	12		Visual/Symbolic	BI	N
2	13		All	Abstract	N
3	1	Symbolic>Verbal		Uni	CA
3	2	Contextual>Symbolic		Uni	L
3	3	Contextual>Symbolic		Uni	N
3	4	Contextual>Symbolic>Symbolic		Multi	N
3	5	Contextual>Symbolic>Symbolic		Multi	N
3	6	Symbolic>Visual		Uni	N
3	7	Visual>Symbolic		Uni	CA
3	8	Visual>Verbal		Uni	L
3	9		Physical/Contextual	Bi	PE, CA
3	10	Contextual>Verbal		Uni	PE, CA

Appendix F. Music Utilizations by Representation Type for Each Pair

Vincent and Ethan's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From	2			4		
	To				4		
	Connection		1		1		
	Total	2	1		**9**	0	0
Physical	From	2	1	1	2		
	To						
	Connection		1		1		
	Total	2	2	1	**3**	0	0
Verbal	From						
	To	4	1	1	5		
	Connection				1		
	Total	4	1	1	**6**	0	0
Visual	From	4	1		2		
	To	2					
	Connection	1			1		
	Total	**7**	1	0	3	0	0
Symbolic	From	5	1	1	4		
	To	6	1	1	3		
	Connection	1			1		
	Total	**12**	2	2	8	0	0

Cameron and Eric's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From	3		1	3		
	To	1		1	2		
	Connection	1	1		1		
	Total	5	1	2	**6**	0	0
Physical	From	1	1	2			
	To						
	Connection	1	1		1		
	Total	2	2	2	**1**	0	0
Verbal	From						
	To	2	1	2	4		
	Connection				1		
	Total	2	1	2	**5**	0	0
Visual	From	6			3		
	To	1			1		
	Connection	1			1		
	Total	**8**	1	3	5	0	0
Symbolic	From	5			4		
	To	7	1	1	3		
	Connection	1			1		
	Total	**13**	1	1	8	0	0

Elise and Logan's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From	2			4		1
	To		1		2		1
	Connection				1		
	Total	2	1	0	**7**	0	2
Physical	From		1	1	2		
	To						
	Connection				1		
	Total	0	1	1	**3**	0	0
Verbal	From						
	To	4	1	1	3		1
	Connection						
	Total	4	1	1	**3**	0	1
Visual	From	4		1			1
	To	1		1			
	Connection						
	Total	**5**	0	2	3	0	1
Symbolic	From	4		2			
	To	4		1	1		
	Connection						
	Total	**8**	0	3	1	0	0

Rachel and Rebecca's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From	1	1	1	6		
	To		1		5		
	Connection				1		
	Total	1	2	1	**12**	0	0
Physical	From		1		2		
	To						
	Connection				1		
	Total	0	1	0	**3**	0	0
Verbal	From						
	To	4	1		6		
	Connection						
	Total	4	1	0	**6**	0	0
Visual	From	4	1	1	4		
	To	2					
	Connection	1					
	Total	**7**	1	1	4	0	0
Symbolic	From	4			2		
	To	3	1	2	3		
	Connection	1					
	Total	**8**	1	2	5	0	0

Aaron and Sam's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From	1	1		6		
	To		1		4		
	Connection				1		
	Total	1	2	0	**11**	0	0
Physical	From	1	1	1	2		
	To						
	Connection				1		
	Total	1	1	1	**3**	0	0
Verbal	From						
	To		1		5		
	Connection				1		
	Total	0	1	0	**6**	0	0
Visual	From	1		1	2		
	To	1		1	1		
	Connection	1			1		
	Total	**3**	0	2	4	0	0
Symbolic	From	3	2	1	5		
	To	3	1	2	3		
	Connection	1			1		
	Total	**7**	3	3	9	0	0

Mitch and Paul's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From	1		1	5	3	1
	To	2			5	2	
	Connection		1		3		1
	Total	3	1	1	**13**	5	2
Physical	From	1	1	1	3		
	To						
	Connection		1		3		1
	Total	1	2	1	**6**	0	1
Verbal	From						
	To	3			6	2	1
	Connection				1		
	Total	3	0	0	**7**	2	1
Visual	From	2		1	2	1	
	To	2		1	1		
	Connection	2			1		
	Total	**6**	0	2	4	1	0
Symbolic	From	3	1	2	4	2	
	To	3	1	3	4	1	
	Connection	2			1		
	Total	**8**	2	5	9	3	0

Emily and Amber's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From			1	7	1	1
	To				5	1	
	Connection		1		2		
	Total	0	1	0	**14**	2	1
Physical	From		1		2		
	To						
	Connection		1		2		
	Total	0	2	0	**4**	0	0
Verbal	From						
	To	2			6	1	1
	Connection				1		
	Total	2	0	0	**7**	1	1
Visual	From	3		1	4	1	
	To	1			2		
	Connection	2			1		
	Total	**6**	0	1	9	0	0
Symbolic	From	2		2	5		
	To	2	1	2	6		
	Connection	2			1		
	Total	**6**	1	4	12	0	0

Jason and Gabe's Music Utilization Types According to Representation Type

	Code System	N	E	L	CA	CR	PE
Contextual	From	1			4		
	To				2		
	Connection		2		3		
	Total	1	2	0	**9**	0	0
Physical	From		1	1	1		
	To						
	Connection		2		3		
	Total	0	3	1	**4**	0	0
Verbal	From						
	To	3		1	3		
	Connection				1		
	Total	3	0	1	**4**	0	0
Visual	From	2		1	2		
	To	2			1		
	Connection	1			1		
	Total	**5**	0	1	4	0	0
Symbolic	From	3			2		
	To	3	1	1	2		
	Connection	1			1		
	Total	**7**	1	1	5	0	0

Appendix G. IRB Approval Letter.

OFFICE OF RESEARCH
UtahStateUniversity.

Institutional Review Board

Expedite #7
Letter of Approval

From: Melanie Domenech Rodriguez, IRB Chair

Nicole Vouvalis, IRB Director

To: ***Beth MacDonald***

Date: ***September 16, 2021***

Protocol #: ***11995***

Title: ***The Role of Music Context in High-School Students' Translations Among Representations in Algebra***

Your proposal has been reviewed by the Institutional Review Board and is approved under Expedite procedure #7 (based on the Department of Health and Human Services (DHHS) regulations for the protection of human research subjects, 45 CFR Part 46, as amended to include provisions of the Federal Policy for the Protection of Human Subjects, January 21, 2019):

> *Research on individual or group characteristics or behavior (including, but not limited to, research on perception, cognition, motivation, identity, language, communication, cultural beliefs or practices, and social behavior) or research employing survey, interview, oral history, focus group, program evaluation, human factors evaluation, or quality assurance methodologies.*

This approval applies only to the proposal currently on file for the period of approval specified in the protocol. You will be asked to submit an annual check in around the anniversary of the date of original approval. As part of the IRB's quality assurance procedures, this research may also be randomly selected for audit. If so, you will receive a request for completion of an Audit Report form during the month of the anniversary date of original approval. If the proposal will be active for more than five years, it will undergo a full continuation review every fifth year.

Any change affecting human subjects, including extension of the expiration date, must be approved by the IRB **prior** to implementation by submitting an Amendment request. Injuries or any unanticipated problems involving risk to subjects or to others must be reported immediately to the Chair of the Institutional Review Board. If Non-USU Personnel will complete work on this project, they may not begin until an External Researcher Agreement or Reliance Agreement has been fully executed by USU and the appropriate Non-USU entity, regardless of the protocol approval status here at USU.

Prior to involving human subjects, properly executed informed consent must be obtained from each subject or from an authorized representative, and documentation of informed consent must be kept on file for at least three years after the project ends. Each subject must be furnished with a copy of the informed consent document for their personal records.

Upon receipt of this memo, you may begin your research. If you have questions, please call the IRB office at (435) 797-1821 or email to irb@usu.edu.

The IRB wishes you success with your research.

435.797.1821 | 1450 Old Main Hill | Logan, UT 84322 | MAIN 155 | irb@usu.edu | FWA#00003308

CURRICULUM VITA

Danielle Rose Divis, M.A.
14787 Academy Parkway Herriman, UT 84096
Tel: 801-694-7295 | Email: danielledivis21@gmail.com

EDUCATION

Ph.D., **Education (Curriculum & Instruction Specialization)** **2022**
Area of Concentration: Mathematics Education & Leadership
Utah State University – Logan, UT

M.A., **Mathematics Education** **2016**
Brigham Young University, Provo, UT
Thesis: *The Principles of Effective Teaching Student Teachers have the Opportunity to Learn in an Alternative Student Teaching Structure* (Advisor: Blake Peterson).

B.S., **Mathematics Education** **2012**
Brigham Young University, Provo, UT
Level 2 Teaching License: Secondary Mathematics with Level 4 Math Endorsement

EMPLOYMENT HISTORY

BRIGHAM YOUNG UNIVERSITY

Part-Time Adjunct **Summer 2020-Present**

Teacher Education Department
College of Education, Brigham Young University
El Ed 447: *Teaching Mathematics in Grades K-6*. Taught online and blended course to undergraduate elementary majors at the end of their program. Content focused on pedagogical techniques, and emphasized facilitating meaningful mathematical discourse, using the CMI teaching framework, and using NCTM's 8 principles of effective teaching.

Part-Time Adjunct **Spring 2020**
Mathematics Education Department
College of Physical and Mathematical Sciences, Brigham Young University
Mthed 305: *Basic Concepts in Mathematics*. Taught a face-to class to 26 undergraduate elementary, early childhood, and special education students. Content focused on strengthening pre-service teachers' conceptual understanding of early mathematics topics, including whole number operations and geometry, as well as examining children's intuitive strategies.

UTAH STATE UNIVERSITY

Graduate Research Assistant **2019 – 2020**

School of Teacher Education and Leadership

College of Education and Human Services, Utah State University

Worked under the guidance of university faculty supervisor Dr. Diana Moss (Utah State University), and Dr. Dov Zazkis, (Arizona State University). **Responsibilities included** analysis and coding of data and preparing materials for collaborative presentations and publications. Research project focused on the use of "lesson plays" as a tool for mathematics teacher education.

Course Developer **2020**

TEAL/TEPD 6524: Mathematics for Teaching K-8: Geometry and Measurement. Redeveloped an online graduate level class on Canvas to meet university and department standards.

Course Developer **2019**

TEAL/TEPD 6523: Mathematics for Teaching K-8: Algebraic Reasoning. Redeveloped an online graduate level class on Canvas to meet university and department standards.

SECONDARY SCHOOL TEACHING EXPERIENCE

Secondary Mathematics Teacher/ Math Department Head **2018 – present**

Real Salt Lake Academy, UT

Responsibilities include teaching AP Calculus to sophomores, juniors, and seniors, and Secondary Mathematics 1 to freshman, as well as choosing and overseeing curriculum for the entire mathematics department. Hybrid instruction online and in-person utilized throughout 2020 and 2021 using Canvas.

Secondary Mathematics Teacher **2017 – 2018**

Ascent Academies of Utah – Lehi, UT

Responsibilities included teaching Mathematics 8 to eight-grade students.

Secondary Mathematics Teacher **2015 – 2016**

Ascent Academies of Utah – West Jordan, UT

Responsibilities included teaching Mathematics 8 and Secondary Mathematics 1 Honors to eight and ninth graders.

Secondary Mathematics Teacher **2012 – 2015**

Paradigm High School, UT

Responsibilities included teaching Secondary Mathematics 1,2,3, and Precalculus to high school students, as well as Precalculus curriculum development.

RESEARCH

Research Interests

- interdisciplinarity (specifically music integration) in mathematics education for the purpose of promoting student self-efficacy, positive attitudes, higher achievement, and gender inclusion
- music integration to support teachers in using and connecting multiple representations
- mathematics teacher training program improvement through lesson plays

Research Projects

Utah State University **2019 – 2020**

Exploring Prospective Teachers' Pedagogical Understandings in Mathematics Methods Courses with Lesson Plays. Utah State University (with Dr. Diana Moss, USU and Dr. Dov Zazkis, ASU). My role: meet weekly with research team, analyze and code data (qualitatively code transcripts of written lesson plays; prepare materials for collaborative presentations and publications).

PUBLICATIONS

JOURNAL ARTICLES

1. **Divis, D**. (in press). A review of "Humanizing disability in mathematics education." M*athematics Education Research Journal*.
2. Moss, D., Wilson, R., & **Divis, D.** (accepted). Pre-Service elementary school teachers' perceptions of themselves as learners of mathematics and science. J*ournal on Empowering Teaching Excellence.*
3. An, S. A., Hachey, A., Tillman, D., **Divis, D.** & Birdwell, B. (2021). Larger versus Luckier: preservice teachers' exploration of probabilistic reasoning through an aleatoric music activity. *International Journal of Mathematical Education in Science and Technology,* 1-26.
4. **Divis, D.R.**, Johnson, T. (2021). Pythagoras, PVC Pipes, and Pianos. *Mathematics Teacher: Learning and Teaching PK-12, 114*(1), 47-54.
5. **Divis, D**. (2021). Ensuring success in completing the square. *The Australian Mathematics Education Journal*, *2*(4), 31-35.
6. **Divis, D.** (2019). Interdisciplinary music and mathematics instruction: A review of the literature. *Utah Mathematics Teacher*, *12*, 32-46.

PRESENTATIONS

NATIONAL PRESENTATIONS – SCHOLARSHIP

Divis, D. (2021, November). *The role of music context in high-school students' translations among representations in algebra*. Three-minute thesis presentation conducted at the Annual Convention of the School Science and Mathematics Association (SSMA), online presentation.

Divis, D. (2020, November). *Exploring multiple representations of exponential growth with Pythagoras, PVC pipes, and pianos*. Presentation conducted at the Annual Convention of the School Science and Mathematics Association (SSMA), Minneapolis, MN.

Moss, D., Zazkis, D., & **Divis, D**. (2020, February). *Using Lesson Plays to Assess Prospective Teachers' Pedagogical Understandings in Mathematics Methods Courses*, Presentation conducted at the 24th Annual Conference of the Association of Mathematics Teacher Educators (AMTE), Phoenix, AZ.

Divis, D. (2019, November). *Making a Case for the Use of Interdisciplinary Curriculum to Foster Gender Equity in Mathematics.* Oral presentation at the School Science and Mathematics Association Conference (SSMA), Salt Lake City, Utah.

UNIVERSITY PRESENTATIONS

Divis, D.R. (2015, March). *The Principles of Effective Teaching Student Teachers have the Opportunity to Learn in an Alternative Student Teaching Structure*. Presentation conducted at Brigham Young University Student Research Conference, Provo, UT.

Divis, D. (2014, March). *The Elements of Impactful Teaching and Meaningful Learning Student Teachers Have the Opportunity to Learn in an Alternative Structure*. Presentation conducted at Brigham Young University Student Research Conference, Provo, UT.

Divis, D. (2014, March). *Math Musicians*. Workshop conducted at Salt Lake Community College Expand Your Horizons STEM Conference, West Valley, UT.

GRANT PROPOSALS (FUNDED)

Graduate Research Assistant ($9,000). *Exploring Prospective Teachers' Pedagogical Understandings in Mathematics Methods Courses*. (2019-2021). Academic and Instructional Services' Excellence in Teaching and Learning Grant, Utah State University. Project goal: Investigate lesson play as a tool for improving mathematics methods courses and as a lens into how prospective teachers envision their teaching. (with PI – Diana Moss, Utah State University)

Graduate Research and Creative Opportunities ($1,000). Dissertation research funding from the Utah State University Graduate School. Funds were used to compensate four teacher-researchers for volunteering to contribute to my study.

GRANT PROPOSALS (UNFUNDED)

Graduate Research Assistant ($50,000). *Developing Pre-Post Lesson Plays as a Tool for Pre-Service Teacher Improvement*. (2019). Spencer Foundation. Project goal: Investigate lesson play as a tool for improving mathematics methods courses and as a lens into how prospective teachers envision their teaching (with PI – Diana Moss, Utah State University, and Co-PI – Dov Zazkis, Arizona State University).

SERVICE

NATIONAL LEADERSHIP & SERVICE

Reviewer (2021)
Practitioner Manuscript, *Mathematics Teacher: Learning and Teaching PK-12.*

Reviewer (2020)
Presentation proposals, *North American Chapter of the International Group for the Psychology of Mathematics Education*

Reviewer (2019)
Research manuscript, *Journal for Research in Mathematics Education*.

Volunteer (2019)
Conference volunteer, *National Council of Teachers of Mathematics Regional Conference*, Salt Lake City, UT

STATE LEADERSHIP & SERVICE

Utah
Assistant Journal Editor, Utah Council of Teachers of Mathematics (2019 - Present)
Assistant editor for the annual UCTM Journal. Responsible for choosing submissions and compiling the journal biannually.

PROFESSIONAL SERVICE - INSTITUTIONAL

BRIGHAM YOUNG UNIVERSITY 2014
STEM Fair Booth, Brigham Young University, Provo, UT
Designing and manning a booth at the STEM entitled The Golden Ratio for children grades 6-8

UTAH STATE UNIVERSITY 2019 – 2020

Institutional Service – Department Level
School of Teacher Education and Leadership, Utah State University
Re-developed two online courses:
TEAL/TEPD 6523: Mathematics for Teaching K-8: Algebraic Reasoning and
TEAL/TEPD 6524: Mathematics for Teaching K-8: Geometry and Measurement

- With input from the mathematics education faculty, updated video lectures and slides to include relevant research
- Updated 9 modules to explicitly demonstrate a flow of read, do mathematics, watch, demonstrate, reflect
- Integrated research from the Compendium of Mathematics Education Research (Cai, 2019) into each module
- Integrated opportunities for students to do the elementary level mathematics they are learning about

CONTINUOUS LEARNING & SELF-DEVELOPMENT

PROFESSIONAL MEMBERSHIPS

National Council of Teachers of Mathematics (since 2010)
Utah Council of Teachers of Mathematics (since 2010)
Association of Mathematics Teacher Educators (since 2019)
School Science and Mathematics Association (since 2019)